Isomorphisms between complements of plane curves

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Chapter 1

Introduction

1.1 Isomorphisms between complements

Let X be an irreducible algebraic variety, defined over an algebraically closed field k. Let $\Gamma, \Delta \subsetneq X$ be closed irreducible subvarities and let $\varphi\colon X \setminus \Gamma \to X \setminus \Delta$ be an isomorphism. What can we then say about Γ and Δ? The following questions naturally arise and are the main topic of this thesis.

(1) Does φ extend to an automorphism of X?

(2) Are Γ and Δ equivalent by an automorphism of X?

(3) Are Γ and Δ isomorphic?

The first thing we notice is that φ (as well as its inverse) defines an isomorphism between two open dense subsets of X and thus induces a birational map $X \dashrightarrow X$. If the group $\mathrm{Bir}(X)$ of birational transformations of X is trivial, then the questions above can all trivially be affirmatively answered. It is thus more interesting to consider varieties that have a large group of birational transformations. In this thesis, we are only concerned with rational varieties, whose groups of birational transformations (called *Cremona groups*) are very rich and have been intensely studied for many years. In fact, we restrict our study to projective space $\mathbb{P}^n$ and affine space $\mathbb{A}^n$, where $n \geq 1$. We observe moreover that it is most interesting to study complements in codimension 1.

Lemma 1.1.1. *Let $\varphi\colon \mathbb{P}^n \setminus \Gamma \to \mathbb{P}^n \setminus \Delta$ be an isomorphism, where $\Gamma, \Delta \subset \mathbb{P}^n$ are subvarieties of codimension ≥ 2. Then φ extends to an automorphism of $\mathbb{P}^n$.*

Proof. Consider φ and φ^{-1} as birational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then φ and φ^{-1} each are given componentwise by homogeneous polynomials of the same degree with no common factors. This description is moreover unique, up to multiplication by scalars. By substitution we obtain an expression

$$\varphi^{-1}(\varphi([x_0 : \ldots : x_n])) = [fx_0 : \ldots : fx_n],$$

for some $f \in \mathrm{k}[x_0, \ldots, x_n] \setminus \{0\}$. The map φ thus sends the set $\{f = 0\}$ to the base locus of φ^{-1} and hence φ cannot be extended to an isomorphism along $\{f = 0\}$. The set $\{f = 0\}$ is either empty (if f is constant) or of codimension 1 in $\mathbb{P}^n$ and hence the claim follows. □

Using the standard open embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, given by

$$(x_1, \ldots, x_n) \hookrightarrow [1 : x_1 : \ldots : x_n],$$

we can also obtain the corresponding result for $\mathbb{A}^n$.

We further observe that complements of hypersurfaces in projective space are actually affine.

Lemma 1.1.2. *Let $\Gamma \subset \mathbb{P}^n$ be a hypersurface. Then $\mathbb{P}^n \setminus \Gamma$ is an affine variety.*

Proof. Let $f = 0$ be an equation of Γ, where f is homogeneous of degree $d \geq 1$. We consider the standard d-Veronese embedding $\varphi \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ with $m = \binom{n+d}{n} - 1$, where the components of φ are given by the monomials of degree d in the variables $x_0, \ldots, x_n$. Composing with an automorphism $\alpha \in \mathrm{PGL}_{m+1}(\mathrm{k})$, we can achieve that the last component of $\psi \coloneqq \alpha \circ \varphi$ is equal to f. Since ψ is a closed embedding, it follows that $\mathbb{P}^n \setminus \Gamma \simeq \psi(\mathbb{P}^n \setminus \Gamma) \subset \{x_m \neq 0\} \simeq \mathbb{A}^m$ is closed and thus $\mathbb{P}^n \setminus \Gamma$ is affine. □

In this thesis, we are mainly concerned with isomorphisms between complements of curves in $\mathbb{P}^2$ and $\mathbb{A}^2$ respectively. The fundamental tool in our study is the following foundational result from the birational geometry of surfaces: given a birational map $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, there exists a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

where π and η are compositions of blow-ups. This allows us to study isomorphisms between complements of curves via blow-ups and their configurations of exceptional curves. This turns out to be a surprisingly effective tool throughout this thesis.

1.2 Summary of main results

In Chapter 2, we study isomorphisms between complements of irreducible curves in the projective plane. In [Yos84], it was conjectured that if two irreducible curves $C, D \subset \mathbb{P}^2$ have isomorphic complements, then they are projectively equivalent (Yoshihara's conjecture). The first counterexample was given in [Bla09]. In particular, the construction given there yields a pair of non-isomorphic curves of degree 39 that have isomorphic complements. Later on, a counterexample of degree 9 was found in [Cos12]. We study in detail isomorphisms between complements of irreducible curves of degree ≤ 8 (Theorem 2) and give a new counterexample to Yoshihara's conjecture of degree 8 (Theorem 3), which has moreover the lowest degree possible (Corollary 2.1.2). Furthermore, we show that Yoshihara's conjecture holds if $C \subset \mathbb{P}^2$ admits a line $L \subset \mathbb{P}^2$ such that $C \setminus L \simeq \mathbb{A}^1$ (Theorem 1). This generalizes a Theorem from [Yos84], proven over the complex numbers, to algebraically closed fields of arbitrary characteristic.

Chapter 3 is a joint work with Jérémy Blanc and Jean-Philippe Furter on isomorphisms between complements of irreducible curves in the affine plane ([BFH16]). In [Kra96], the following question was posed:

> Complement Problem. Given two irreducible hypersurfaces $E, F \subset \mathbb{A}^n$ and an isomorphism of their complements, does it follow that E and F are isomorphic?

We construct non-isomorphic curves $C, D \subset \mathbb{A}^2$ that have isomorphic complements (Theorem 6). These curves yield the first counterexample to the complement problem in dimension 2. Using these curves, we can also construct counterexamples to the complement problem in any dimension ≥ 3 (Corollary 3.6.2). In dimension ≥ 3, counterexamples had previously been found in [Pol16]. We show moreover that for any irreducible curve $C \subset \mathbb{A}^2$ that is not isomorphic to an open subset of $\mathbb{A}^1$, any open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$ (Theorem 4). This gives in particular a positive answer to the complement problem for such curves. Finally, we show that Theorem 4 is sharp, by giving a construction, for any proper open subset of $\mathbb{A}^1$, of two non-equivalent closed embeddings in $\mathbb{A}^2$ whose images have isomorphic complements (Theorem 5).

Chapter 4 is a short note summarizing some known results concerning embeddings of the affine line in the affine plane. We study the following problem, found in [Sat76]: given a polynomial $f \in \mathrm{k}[x, y]$ that defines a line in $\mathbb{A}^2$, does it follow that $f - \lambda$ defines a line for all $\lambda \in \mathrm{k}$? The answer is well known if the characteristic of the base-field k is 0, by the theorem of Abhyankar-Moh-Suzuki ([AM75], [Suz74]), but is still open in positive characteristic. We show that the claim holds for lines of degree ≤ 11 (Proposition 4.3.4), in any characteristic. In the proof, we study multiplicity sequences at infinity and use some results developed in the previous chapters (Proposition 3.3.16, Lemma 2.4.16).

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Chapter 2

Isomorphisms between complements of projective plane curves

Abstract. In this article, we study isomorphisms between complements of irreducible curves in the projective plane $\mathbb{P}^2$, over an arbitrary algebraically closed field. Of particular interest are rational unicuspidal curves. We prove that if there exists a line that intersects a unicuspidal curve $C \subset \mathbb{P}^2$ only in its singular point, then any other curve whose complement is isomorphic to $\mathbb{P}^2 \setminus C$ must be projectively equivalent to C. This generalizes a result of H. Yoshihara who proved this result over the complex numbers. Moreover, we study properties of multiplicity sequences of irreducible curves that imply that any isomorphism between the complements of those curves extends to an automorphism of $\mathbb{P}^2$. Using these results, we show that two irreducible curves of degree ≤ 7 have isomorphic complements if and only if they are projectively equivalent. Finally, we describe new examples of irreducible projectively non-equivalent curves of degree 8 that have isomorphic complements.

Contents

2.1 Introduction

Throughout this article, we fix an algebraically closed field k of arbitrary characteristic. Curves in $\mathbb{P}^2$ will always be assumed to be closed. Let $C, D \subset \mathbb{P}^2$ be two irreducible curves. We then call C and D *projectively equivalent* if there exists an automorphism of $\mathbb{P}^2$ that sends C to D. Our aim is to study isomorphisms $\mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ and properties of the curves C and D, given such an isomorphism. In 1984, H. Yoshihara stated the following conjecture.

Conjecture 2.1.1 ([Yos84]). *Let $C, D \subset \mathbb{P}^2$ be irreducible curves and $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism between their complements. Then C and D are projectively equivalent.*

A counterexample to Conjecture 2.1.1 was given in [Bla09]. The construction given there yields non-isomorphic (and hence projectively non-equivalent) rational curves C_0 and D_0 of degree 39 that have isomorphic complements. Both curves have a unique singular point $p_0 \in C_0$ and $q_0 \in D_0$ respectively, such that $C_0 \setminus \{p_0\}$ and $D_0 \setminus \{q_0\}$ are isomorphic to open subsets of $\mathbb{P}^1$, each with 9 complement points. To see that C_0 and D_0 are not isomorphic, it is shown that the two sets of 9 complement points, corresponding to C_0 and D_0, are non-equivalent by the action of $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1$.

It is a general fact that if there exists an isomorphism $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ that does not extend to an automorphism of $\mathbb{P}^2$, then C and D are of the same degree (Lemma 2.2.1) and there exist points $p \in C$ and $q \in D$ such that each $C \setminus \{p\}$ and $D \setminus \{q\}$ are isomorphic to complements of $k \geq 1$ points in $\mathbb{P}^1$ (Proposition 2.2.6). Moreover, when the number k of complement points is ≥ 3, the isomorphism φ is uniquely determined, up to a left-composition with an automorphism of $\mathbb{P}^2$ (Proposition 2.2.8).

The case of unicuspidal rational curves (i.e. when the number k of complement points is 1) is of particular interest since the rigidity of Proposition 2.2.8 does not hold there. Indeed, by a result of P. Costa ([Cos12], [BFH16, Proposition A.3.]), there exists a family of irreducible rational unicuspidal curves $(C_\lambda)_{\lambda \in \mathrm{k}^*}$ in $\mathbb{P}^2$ that are pairwise projectively non-equivalent, but all have isomorphic complements. The first main result of this article shows that a unicuspidal curve C cannot be part of such family if there exists a line L that intersects C only in its singular point.

Theorem 1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve and $L \subset \mathbb{P}^2$ a line such that $C \setminus L \simeq \mathbb{A}^1$. Let $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ be an isomorphism, where $D \subset \mathbb{P}^2$ is some curve. Then C and D are projectively equivalent.*

This theorem was already proven by H. Yoshihara [Yos84] over the field of complex numbers. His proof relies on the theorem of Abhyankar-Moh-Suzuki ([AM75], [Suz74]) and also uses some analytic tools. We give a purely algebraic proof that works over arbitrary algebraically closed fields.

The counterexamples to Conjecture 2.1.1 given by P. Costa are of degree 9 and it is thus natural to ask what happens in lower degrees. This is the second main result of this article. For the definition of multiplicity sequence used below, see Definition 2.4.2.

Theorem 2. *Let $C, D \subset \mathbb{P}^2$ be irreducible curves of degree ≤ 8 and $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism that does not extend to an automorphism of $\mathbb{P}^2$. Then C and D both are either:*

(*i*) *lines;*

(*ii*) *conics;*

(*iii*) *nodal cubics;*

(*iv*) *projectively equivalent rational unicuspidal curves;*

(*v*) *projectively equivalent curves of degree 6 with multiplicity sequence $(3, 2_{(7)})$;*

(*vi*) *curves of degree 8 with multiplicity sequence $(3_{(7)})$ such that*

$$C \setminus \mathrm{Sing}(C) \simeq D \setminus \mathrm{Sing}(D) \simeq \mathbb{A}^1 \setminus \{0\}.$$

In the proof, we study the diagrams of exceptional curves in the resolutions of the birational transformations of $\mathbb{P}^2$ that are induced by the isomorphisms between the complements, for all types of multiplicity sequences that can occur. We also use Theorem 1 as an important tool.

As an immediate consequence of Theorem 2, we get the following corollary.

Corollary 2.1.2. *Conjecture 2.1.1 holds for all irreducible curves of degree ≤ 7.*

Finally, we show that Corollary 2.1.2 is sharp by giving a counterexample of degree 8. The construction is based on a configuration of conics and is given in Section 2.4.5.

Theorem 3. *There exist irreducible projectively non-equivalent curves $C, D \subset \mathbb{P}^2$ of degree 8 with multiplicity sequence $(3_{(7)})$ that have isomorphic complements.*

2.2 Preliminaries

The following lemma is a well known fact, but included for the sake of completeness.

Lemma 2.2.1. *Let $C, D \subset \mathbb{P}^2$ be irreducible curves and $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism. Then $\deg(C) = \deg(D)$.*

Proof. Consider the following exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{\alpha} \mathrm{Pic}(\mathbb{P}^2) \xrightarrow{\beta} \mathrm{Pic}(\mathbb{P}^2 \setminus C) \to 0$$

where α sends 1 to the class of C in $\mathrm{Pic}(\mathbb{P}^2)$ and β is induced by the map that sends a curve $E \subset \mathbb{P}^2$ to the restriction $E \cap (\mathbb{P}^2 \setminus C)$. The exactness at $\mathrm{Pic}(\mathbb{P}^2)$ follows from the irreducibilty of C. Since the class $[C]$ equals $\deg(C)[L]$, where L is a line in $\mathbb{P}^2$, we obtain that $\mathrm{Pic}(\mathbb{P}^2 \setminus C) \simeq \mathbb{Z}/\deg(C)\mathbb{Z}$. The isomorphism $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ induces an isomorphism on the corresponding Picard groups and hence the claim follows. □

Remark 2.2.2. The claim of Lemma 2.2.1 is false for reducible curves. As an example, consider the curves given by the equations $yz = 0$ and $(x^2 - yz)z = 0$. They have isomorphic complements via the automorphism of $\mathbb{P}^2 \setminus \{z = 0\}$ that sends $[x : y : z]$ to $[xz : x^2 - yz : z^2]$ (which is an involution). This example also shows that it is easy to construct reducible counterexamples to Conjecture 2.1.1.

Definition 2.2.3. Let $m \in \mathbb{Z}$. A birational morphism $\pi\colon X \to \mathbb{P}^2$ is called a *m-tower resolution* of a curve $C \subset \mathbb{P}^2$ if

(i) there exists a decomposition

$$\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$$

where π_i is the blow-up of a point p_i, for $i = 1, \dots, n$, such that $\pi_i(p_{i+1}) = p_i$, for $i = 1, \dots, n-1$;

(ii) the strict transform of C by π in X is isomorphic to $\mathbb{P}^1$ and has self-intersection m.

We use the following notational conventions throughout this article. Given a m-tower resolution of a curve $C \subset \mathbb{P}^2$ as above and $i \in \{1, \dots, n\}$, we denote by C_i the strict transform of C by $\pi_1 \circ \dots \circ \pi_i$ in X_i. We usually denote by E_i the exceptional curve of π_i, i.e. $\pi_i^{-1}(p_i) = E_i \subset X_i$. By abuse of notation, we also denote its strict transforms in $X_{i+1}, \dots, X_n$ by E_i.

We will frequently use the following fundamental lemma.

Lemma 2.2.4 ([Bla09])**.** *Let $C \subset \mathbb{P}^2$ be an irreducible curve and $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism, where $D \subset \mathbb{P}^2$ is some curve. Then either φ extends to an automorphism of $\mathbb{P}^2$ or the induced birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has a minimal resolution*

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

where π and η are (-1)-tower resolutions of C and D respectively.

Given a resolution as in Lemma 2.2.4, where π has a decomposition

$$\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$$

with base-points $p_1, \dots, p_n$ and exceptional curves $E_1, \dots, E_n$, we make the following observations that are used throughout this article.

(i) For any $i \in \{1, \dots, n\}$, the curve $E_1 \cup \dots \cup E_i \subset X_i$ has simple normal crossings (SNC) and has a tree structure, i.e. for any two curves from $E_1, \dots, E_i$ there exists a unique chain of curves from $E_1, \dots, E_i$ connecting them.

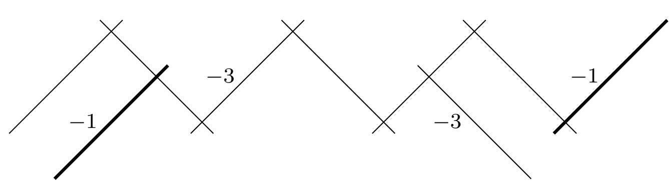
(ii) For any $i \in \{1, \dots, n\}$, the curves $E_1, \dots, E_{i-1} \subset X_i$ have self-intersection ≤ -2 and $E_i \subset X_i$ has self-intersection -1.

(iii) The contracted locus of η is $E_1 \cup \dots E_{n-1} \cup C_n \subset X$ and is also a SNC-curve that has a tree structure. Moreover, E_n is the strict transform of D by η.

Remark 2.2.5. We take the notations of Lemma 2.2.4 and suppose that φ does not extend to an automorphism of $\mathbb{P}^2$. We then have a (-1)-tower resolution $\pi = \pi_1 \circ \dots \circ \pi_n$ of C with exceptional curves $E_1, \dots, E_n$ and a (-1)-tower resolution $\eta = \eta_1 \circ \dots \circ \eta_n$ of D with exceptional curves $F_1, \dots, F_n$. We then have $\{E_1, \dots, E_{n-1}\} = \{F_1, \dots, F_{n-1}\}$ and E_n is the strict transform of D by η and F_n is the strict transform of C by π. One may ask if such a resolution is always symmetric in the sense that

$$E_i \cdot E_j = F_i \cdot F_j \quad \text{and} \quad E_i \cdot F_n = F_i \cdot E_n$$

for all $i, j = 1, \dots, n$. This is in general not the case. For instance, there exists a non-symmetric resolution of an automorphism of the complement of a line with the following configuration of curves, where the unlabeled curves are (-2)-curves.



Starting with either of the (-1)-curves in this configuration, one can successively contract all curves except the other (-1)-curve, whose image is a line in $\mathbb{P}^2$.

Similarly, one can find non-symmetric resolutions of automorphisms of the complement of a conic. However, no example of a non-symmetric resolution of an isomorphism between complements of irreducible singular curves is known to the author.

Proposition 2.2.6. *Let $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ be an open embedding, where C is an irreducible curve and $D = \mathbb{P}^2 \setminus \mathrm{im}(\varphi)$. If φ does not extend to an automorphism of $\mathbb{P}^2$, then one of the following holds.*

(*i*) *C and D both are lines.*

(*ii*) *C and D both are conics.*

(*iii*) *C and D each have a unique proper singular point p and q respectively, such that $C \setminus \{p\}$ and $D \setminus \{q\}$ each are isomorphic to open subsets of $\mathbb{P}^1$, with the same number of complement points.*

Proof. By Lemma 2.2.4 the birational map φ has a minimal resolution

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

where π and η are (-1)-tower resolutions of C and D respectively. Since C and D have the same degree the cases (i) and (ii) are clear and we assume that C (and thus also D) has degree ≥ 3. The curves C and D are both rational since they have a (-1)-tower resolution and hence they have a singular point p and q respectively, by the genus-degree formula for plane curves. Denote by $\hat{C}$ the strict transform of C by π, by $\hat{D}$ the strict transform of D by η, and by E be the union of irreducible curves in X contracted by both π and η. Then $\hat{C} \cup E$ is the exceptional locus of η whose irreducible components form a tree, since η is a (-1)-tower resolution. Likewise, $\hat{D} \cup E$ is the exceptional locus of π and is a tree of irreducible curves. We thus have isomorphisms $C \setminus \{p\} \simeq \hat{C} \setminus (E \cup \hat{D})$ and $D \setminus \{q\} \simeq \hat{D} \setminus (E \cup \hat{C})$ induced by π and η respectively. Since $\hat{C}$ and $\hat{D}$ are both isomorphic to $\mathbb{P}^1$ and they both intersect E transversally it follows that $C \setminus \{p\}$ and $D \setminus \{q\}$ are isomorphic to open subsets of $\mathbb{P}^1$. The number of intersection points between $\hat{C}$ and $E \cup \hat{D}$ is given by

$$\#(\hat{C} \cap E) + \#(\hat{C} \cap \hat{D}) - \#(\hat{C} \cap E \cap \hat{D}).$$

For $\hat{D}$ the same formula holds with $\hat{C}$ and $\hat{D}$ exchanged. It thus suffices to show that $\#(\hat{C} \cap E) = \#(\hat{D} \cap E)$. Since the graphs of curves of $\hat{C} \cup E$ and $\hat{D} \cup E$ define a tree, it follows that $\#(\hat{C} \cap E)$ and $\#(\hat{D} \cap E)$ respectively is the number of connected components of E. $\square$

As a direct consequence, we get the following observation, which we can already find in [Yos84] and [Bla09].

Corollary 2.2.7. *Let $C, D \subset \mathbb{P}^2$ be irreducible closed curves and $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism. If C is not rational or has more than one proper singular point, then φ extends to an automorphism of $\mathbb{P}^2$.*

Proposition 2.2.8. *Let $C \subset \mathbb{P}^2$ be an irreducible curve and $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ an open embedding that does not extend to an automorphism of $\mathbb{P}^2$. Let $p \in C$ be a point such that $C \setminus \{p\}$ is isomorphic to $\mathbb{P}^1 \setminus \{p_1, \ldots, p_k\}$, where $p_1, \ldots, p_k \in \mathbb{P}^1$ are distinct points. If $k \geq 3$, then φ is uniquely determined up to a left-composition with an automorphism of $\mathbb{P}^2$.*

Proof. By Lemma 2.2.4 there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{P}^2$ with exceptional curves $E_1, \dots, E_n$ and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. We denote by $E = E_1 \cup \ldots \cup E_{n-1}$ the union of irreducible curves in X that are contracted by both π and η. Moreover, we denote by $\hat{C} = C_n$ the strict transform of C by π in X, and by $\hat{D} = E_n$ the strict transform of D by η in X. Since π and η are (-1)-tower resolutions, we know that $E \cup \hat{C}$ and $E \cup \hat{D}$ have a tree structure such that $\hat{C}$ and $\hat{D}$ each intersect E in 1 or 2 points. It also follows that $k = \#\hat{C} \cap (E \cup \hat{D})$.

Let us assume first that $k \geq 4$. Then it follows that $\hat{C}$ and $\hat{D}$ intersect in at least two points. This implies that the image of $\hat{C}$ after contracting the (-1)-curve $\hat{D}$ is singular. Hence π is the minimal resolution of singularities of C, i.e. the blow-up of all the singular points of C. By the same argument η is the minimal resolution of singularities of D. Thus the base-points of π and η are completely determined by C and D respectively. But this means that for any other birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ that restricts to an isomorphism $\mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ the composition $\psi \circ \varphi^{-1}$ is an automorphism of $\mathbb{P}^2$. Thus the claim follows in this case.

We now assume that $k = 3$. Then $\hat{C}$ and $\hat{D}$ intersect in $1, 2$, or 3 points. Assume first that $\hat{C}$ and $\hat{D}$ intersect in 2 or 3 points. Then the image of $\hat{C}$ after contracting $\hat{D}$ is singular, so π is the minimal resolution of singularities of C, and analogously η is the minimal resolution of singularities of D. Then for the same reason as before, any other isomorphism $\mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ is just φ composed with an automorphism of $\mathbb{P}^2$.

Finally, we assume that $k = 3$ and that $\hat{C}$ and $\hat{D}$ intersect in only one point. We can assume that this intersection is transversal, otherwise, if they were tangent, π and η would again be the minimal resolutions of the singularities of C and D respectively and we could argue as before. The curve $\hat{D}$ intersects E in two distinct components, say E_i and E_j. If we contract the (-1)-curve $\hat{D}$, there is a triple intersection between the images of $\hat{C}$, E_i and E_j. But this means that π is the minimal resolution of C such that the pull-back $\pi^*(C)$ is a SNC-divisor on X. Hence the base-points of π are again completely determined by the curve C. Likewise, the base-points of η are determined by D. We then argue as before that any isomorphism $\mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ is the composition of φ with an automorphism of $\mathbb{P}^2$. □

Corollary 2.2.9. *Let $C \subset \mathbb{P}^2$ be an irreducible curve such that there exists no point $p \in C$ such that $C \setminus \{p\}$ is isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. Then there exists at most one curve $D \subset \mathbb{P}^2$, up to projective equivalence, such that $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus D$ are isomorphic and such that D is not projectively equivalent to C.*

Proof. This is a direct consequence of Proposition 2.2.8. □

Remark 2.2.10. P. Costa's example ([Cos12]) shows that Corollary 2.2.9 does in general not hold when $C \setminus \{p\} \simeq \mathbb{A}^1$. On the other hand, there is no known example of pairwise projectively non-equivalent curves $C, D, E \subset \mathbb{P}^2$ such that all 3 curves have isomorphic complements and there exists a point $p \in C$ such that $C \setminus \{p\} \simeq \mathbb{A}^1 \setminus \{0\}$.

2.3 Unicuspidal curves with a very tangent line

2.3.1 Very tangent lines

Let $C \subset \mathbb{P}^2$ be an irreducible curve. A singular point $p \in C$ is called a *cusp* if the preimage of p under the normalization $\hat{C} \to C$ consists of only one point. A curve is called *unicuspidal* if it has one cusp and is smooth at all other points. We call a line $L \subset \mathbb{P}^2$ *very tangent* to C if there exists a point q such that $(C \cdot L)_q = \deg(C)$. By Bézout's theorem this means that L intersects C in only one point. A line that is very tangent to C is also tangent in the usual sense, except in the special case where C is a line and the intersection is transversal.

Lemma 2.3.1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve and $L \subset \mathbb{P}^2$ a line. Then $C \setminus L \simeq \mathbb{A}^1$ if and only if L is very tangent to C and one of the following holds:*

(i) C is a line.

(ii) C is a conic.

(iii) C is rational and unicuspidal and L passes through the singular point of C.

Proof. Assume that L is very tangent to C. If C is a line or a conic, then C is isomorphic to $\mathbb{P}^1$ and thus $C \setminus L \simeq \mathbb{A}^1$. We thus assume that C is rational and unicuspidal with singular point p, where L passes through p. It follows that C has a normalization $\eta\colon \mathbb{P}^1 \to C$ such that $\eta^{-1}(p)$ consists of only one point and thus $C \setminus \{p\} \simeq \mathbb{P}^1 \setminus \eta^{-1}(p) \simeq \mathbb{A}^1$. Since L is very tangent to C, the intersection $C \cap L$ consists only of the point p. It follows that $C \setminus L \simeq C \setminus \{p\} \simeq \mathbb{A}^1$.

To prove the converse, assume that $C \setminus L \simeq \mathbb{A}^1$. It follows that C is rational and $\mathrm{Sing}(C) \subset C \cap L$. We consider the normalization $\eta\colon \mathbb{P}^1 \to C$ and obtain $C \setminus L \subset C \setminus \mathrm{Sing}(C) \simeq \mathbb{P}^1 \setminus \eta^{-1}(\mathrm{Sing}(C))$. Since $C \setminus L \simeq \mathbb{A}^1$, it follows that $\eta^{-1}(\mathrm{Sing}(C))$ consists of at most one point. If $\eta^{-1}(\mathrm{Sing}(C))$ is empty, then $C \simeq \mathbb{P}^1$ is smooth and thus either a line or a conic, by the genus-degree formula. Since $C \setminus L \simeq \mathbb{A}^1$, it follows that L intersects C in only one point and is thus very tangent to C. If $\eta^{-1}(\mathrm{Sing}(C))$ is not empty, then it contains exactly one point and thus C is unicuspidal and $C \setminus L = C \setminus \mathrm{Sing}(C)$. Since $C \cap L = \mathrm{Sing}(C)$ consists of only one point, the line L is very tangent to C. □

If C is unicuspidal and rational and has a very tangent line L through the singular point, then $C \setminus L \simeq \mathbb{A}^1$. In other words, C is equivalent to the closure of the image of a closed embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \simeq \mathbb{P}^2 \setminus L$. Note that not all rational unicuspidal curves admit a very tangent line through the singular point. For instance, there exists such a unicuspidal quintic curve that is studied in detail in Section 2.4.2.

We call $C \setminus L \subset \mathbb{P}^2 \setminus L \simeq \mathbb{A}^2$ *rectifiable* if there exists an automorphism $\theta \in \mathrm{Aut}(\mathbb{P}^2 \setminus L)$ such that $\theta(C) = L' \setminus L$ for some line $L' \subset \mathbb{P}^2$ that is distinct from L. Suppose that there exists an open embedding $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$, then the induced birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ contracts the curve C to a point. It turns out that $C \setminus L \subset \mathbb{P}^2 \setminus L$ is then rectifiable. This is a consequence of the following proposition, proven in [BFH16, Proposition 3.16]. It also follows from the work of [KM83] and [Gan85] (see [BFH16, Remark 2.30]).

Proposition 2.3.2. *Let $C \subset \mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$ be a closed curve, isomorphic to $\mathbb{A}^1$, and denote by $\overline{C}$ the closure of C in $\mathbb{P}^2$. Then the following are equivalent:*

(*i*) *There exists an automorphism of $\mathbb{A}^2$ that sends C to a line.*

(*ii*) *There exists a birational transformation of $\mathbb{P}^2$ that sends $\overline{C}$ to a point.*

We call a curve satisfying condition (ii) of Proposition 2.3.2 *Cremona-contractible.* Note that condition (i) is always satisfied if the characteristic of k is 0 by the Abhyankar-Moh-Suzuki theorem ([AM75], [Suz74]), but in general not in positive characteristic. It follows from Proposition 2.3.2 that Theorem 1 holds if $C \setminus L \subset \mathbb{P}^2 \setminus L$ is not rectifiable.

2.3.2 Automorphisms of $\mathbb{A}^2$ and de Jonquières maps

Definition 2.3.3. Let $L \subset \mathbb{P}^2$ be a line and $p \in L$. We denote by $\mathrm{Jon}(\mathbb{P}^2, L, p)$ the group of automorphisms of $\mathbb{P}^2 \setminus L$ that preserve the pencil of lines through p. We call an element in $\mathrm{Jon}(\mathbb{P}^2, L, p)$ a *de Jonquières map with respect to L and p.*

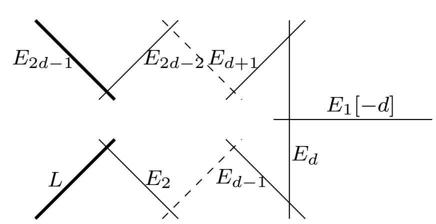
We recall the following standard terminology, for instance as used in [Alb02].

Definition 2.3.4. Let X be a surface and let $p \in X$ be a point. Let E be the exceptional curve of the blow-up of p. We then say that a point $q \in E$ lies in the *first neighborhood of p*. For $k > 1$, we say that a point lies in the *k-th neighborhood of p* if it lies in the first neighborhood of some point in the $(k-1)$-th neighborhood of p. We say that a point is *infinitely near to p* if it lies in the k-th neighborhood of p, for some $k \geq 1$. We call a point q *proximate to p* (denoted $q \succ p$) if q lies on the strict transform of the exceptional curve of the blow-up of p. We sometimes call the points of X *proper* to distinguish them from infinitely near points.

Throughout this section, we fix a line $L \subset \mathbb{P}^2$ and a point $p \in L$. Moreover, we fix projective coordinates $[x : y : z]$ on $\mathbb{P}^2$ and denote the lines

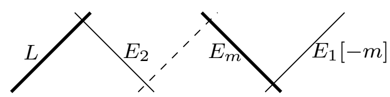
$$L_x \colon x = 0 \qquad L_y \colon y = 0 \qquad L_z \colon z = 0.$$

Lemma 2.3.5. *Let $j \in \mathrm{Jon}(\mathbb{P}^2, L, p) \setminus \mathrm{Aut}(\mathbb{P}^2)$ be of degree d. Then the minimal resolution of j has $2d-1$ base-points with exceptional curves $E_1, \ldots, E_{2d-1}$ as in the following configuration*



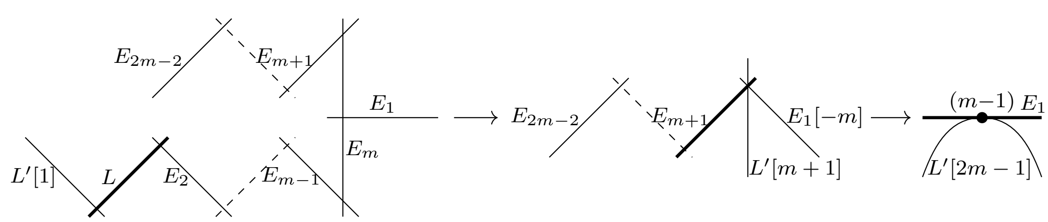
where the self-intersection numbers are -1 for thick lines, -2 for thin lines, or otherwise are indicated in square brackets.

Proof. The map j is an automorphism of $\mathbb{P}^2 \setminus L$ that does not extend to an automorphism of $\mathbb{P}^2$, thus by Lemma 2.2.4 there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of L with exceptional curves $E_1, \dots, E_n$ and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of L such that $j \circ \pi = \eta$. The unique proper base-point of j is p, which is thus the base-point of the first blow-up with exceptional curve E_1. Since π is a (-1)-tower resolution of L, the next base-point is the intersection point between E_1 and the strict transform of L. After this blow-up, the strict transform of L has self-intersection -1 and thus there is no more base-point on this curve. We observe that E_1 is the last curve contracted by η, since j preserves the pencil of lines through p. The next base-point is thus either the intersection point q between E_1 and E_2 or a point on $E_2 \setminus (E_1 \cup L)$. Let $m \geq 0$ be the number of base-points proximate to q. After blowing up those m points we have the following resolution.



The next base-point then lies on $E_m \setminus E_1$. It cannot be the intersection point with E_{m-1}, because then E_{m-1} would have self-intersection < -2 in X. But η first contracts L and then the curves $E_2, \dots, E_{m-2}$. After those contractions the self-intersection of the image of E_{m-1} must be -1. Hence the next base-point lies on $E_m \setminus (E_1 \cup E_{m-1})$. We observe moreover that after η contracts $L, E_2, \dots, E_m$ the image of E_1 has self-intersection $-m+1$. Thus there is a chain of (-2)-curves of length $m-1$ attached to E_m, which are obtained by successively blowing up points that lie on the last exceptional curve but not on the intersection with another one. Since E_1 is the last curve contracted by η, it follows that E_{2m-1} is the last exceptional curve of π.

Let us now determine the degree of j. For this we look at the degree of the image of a line L' that does not pass through the base-points of j. The strict transform of L' is drawn in the diagram on the left below.



After the curves $L, E_2, \dots, E_m$ are contracted the image of L' has self-intersection $m+1$ and L' intersects E_{m+1} and E_1, as shown in the diagram in the middle. Next, the curves $E_{m+1}, \dots, E_{2m-2}$ are contracted and the image of L has self-intersection $2m-1$ and L intersects E_1 with multiplicity $(m-1)$. Thus after E_1 is contracted the self-intersection of the image of L is $2m-1+(m-1)^2 = m^2$ and hence the degree d of j is equal to m. $\square$

We often identify $\mathbb{P}^2 \setminus L_z$ with the affine plane $\mathbb{A}^2$ with coordinates x, y, via the open embedding $(x, y) \mapsto [x : y : 1]$. We call $j \in \operatorname{Aut}(\mathbb{A}^2)$ an *affine de Jonquières map* if it is the restriction of a de Jonquières map with respect to L_z and $[0:1:0]$. Affine de Jonquières maps then preserve the fibration $(x,y) \mapsto x$.

Lemma 2.3.6. *Let $j \in \mathrm{Aut}(\mathbb{A}^2)$ be an affine de Jonquières map. Then j is of the form*

$$(x,y) \mapsto (ax+b, cy+f(x))$$

where $a,c \in \mathrm{k}^*$, $b \in \mathrm{k}$, and $f \in \mathrm{k}[x]$.

Proof. The map j sends (x,y) to $(a(x,y), b(x,y))$, where $a,b \in \mathrm{k}[x,y]$. Since j is an automorphism of $\mathbb{A}^2$, the polynomials a and b are irreducible. Moreover, j preserves the fibration $(x,y) \mapsto x$, thus a is a scalar multiple of some element $x-\lambda$ with $\lambda \in \mathrm{k}$. We can then apply an affine coordinate change and may assume that $a = x$. But then j induces a $\mathrm{k}[x]$-automorphism of the polynomial ring $\mathrm{k}[x][y]$, and thus b is of degree 1 in the variable y. Moreover, the coefficient of y is an element in $\mathrm{k}[x]^* = \mathrm{k}^*$ und thus the claim follows. □

We will use the well known structure theorem of Jung and van der Kulk in the sequel. We denote by $\mathrm{Aff}(\mathbb{P}^2, L)$ the *affine group with respect to* L, which consists of the automorphisms of $\mathbb{P}^2$ that preserve L. Moreover, we denote by $B(\mathbb{P}^2, L, p)$ the intersection $\mathrm{Aff}(\mathbb{P}^2, L) \cap \mathrm{Jon}(\mathbb{P}^2, L, p)$.

Theorem 2.3.7 ([Jun42], [vdK53])**.** *The group $\mathrm{Aut}(\mathbb{P}^2 \backslash L)$ is generated by the subgroups $\mathrm{Aff}(\mathbb{P}^2, L)$ and $\mathrm{Jon}(\mathbb{P}^2, L, p)$. Moreover, $\mathrm{Aut}(\mathbb{P}^2 \setminus L)$ is a free product*

$$\mathrm{Aff}(\mathbb{P}^2, L) *_{B(\mathbb{P}^2,L,p)} \mathrm{Jon}(\mathbb{P}^2, L, p),$$

amalgamated over the intersection of those two subgroups.

Remark 2.3.8. There exist many proofs of Theorem 2.3.7. The proof in [Lam02] uses blow-ups and contractions of the line $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$, in the spirit of the methods used in this article. For more proofs with a similar strategy see [BD11] and [BS15].

Lemma 2.3.9. *Let $\theta \in \mathrm{Aut}(\mathbb{P}^2 \setminus L)$ with*

$$\theta = a \circ j_n \circ a_n \circ \ldots \circ j_1 \circ a_1,$$

where $a_1, a \in (\mathrm{Aff}(\mathbb{P}^2, L) \setminus \mathrm{Jon}(\mathbb{P}^2, L, p)) \cup \{\mathrm{id}\}$, $a_i \in \mathrm{Aff}(\mathbb{P}^2, L) \setminus \mathrm{Jon}(\mathbb{P}^2, L, p)$ for $i = 2, \ldots, n$ and $j_i \in \mathrm{Jon}(\mathbb{P}^2, L, p) \setminus \mathrm{Aff}(\mathbb{P}^2, L)$ for $i = 1, \ldots, n$. Then θ has unique proper base-point $a_1^{-1}(p)$. Moreover, the degree of θ is $\prod_{i=1}^n \deg(j_i)$.

Proof. The map j_1 has unique proper base-point p, and thus $j_1 \circ a_1$ has unique proper base-point $a_1^{-1}(p)$ and $(j_1 \circ a_1)^{-1}$ has unique proper base-point p. We proceed by induction and assume that $j_{n-1} \circ a_{n-1} \circ \ldots \circ j_1 \circ a_1$ has unique proper base-point $a_1^{-1}(p)$ and its inverse has unique proper base-point p. Moreover, the unique proper base-point of $(j_n \circ a_n)$ is $a_n^{-1}(p)$, which is different from p since $a_n \notin \mathrm{Jon}(\mathbb{P}^2, L, p)$. It then follows that the composition $j_n \circ a_n \circ \ldots \circ j_1 \circ a_1$ again has $a_1^{-1}(p)$ as its unique proper base-point. This remains true after a left-composition with $a \in \mathrm{Aff}(\mathbb{P}^2, L)$.

To compute the degree of θ, we observe that $\deg(j_i \circ a_i) = \deg(j_i)$ for all i, since the maps a_i are affine and hence have degree 1. We use again that $(j_{n-1} \circ a_{n-1} \circ \ldots \circ j_1 \circ a_1)^{-1}$ and $j_n \circ a_n$ have no common base-point and obtain the result by induction by using [Alb02, Proposition 4.2.1]. □

Definition 2.3.10. Let X be a surface and let $C \subset X$ be a curve. For a point $p \in C$, let $\mathcal{O}_{X,p}$ be the local ring at p, with unique maximal ideal $\mathfrak{m}_p$. Let moreover $f \in \mathcal{O}_{X,p}$ be a local equation of C at p. We then define the *multiplicity* $m_p(C)$ *of* C *at* p to be the largest integer m such that $f \in \mathfrak{m}_p^m$.

Let Λ be a linear system of curves on $\mathbb{P}^2$ and let p be a proper or infinitely near point of $\mathbb{P}^2$. We then define the *multiplicity of* Λ *at* p to be the smallest multiplicity $m_p(C)$ among all curves C in Λ.

For a birational map $\theta\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, we denote by Λ_θ the linear system of curves on $\mathbb{P}^2$, given by the preimage of θ of the linear system of lines on $\mathbb{P}^2$. For a proper or infinitely near point p of $\mathbb{P}^2$, we define the *multiplicity* $m_p(\theta)$ *of* θ *at* p to be the multiplicity of the linear system Λ_θ at p.

For a more detailed account of these notions, we refer to [Alb02].

We will use the following well known formula in the sequel.

Lemma 2.3.11. *Let $\theta\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and $C \subset \mathbb{P}^2$ a curve that is not contracted by θ. Then the following formula holds:*

$$\deg \theta(C) = \deg(\theta)\deg(C) - \sum_p m_p(\theta) m_p(C)$$

where the sum ranges over all proper and infinitely near points of $\mathbb{P}^2$, but only finitey many summands are different from 0.

Proof. We consider a minimal resolution

$$\begin{array}{ccc} & X & \\ \swarrow^{\sigma_1} & & \searrow^{\sigma_2} \\ \mathbb{P}^2 & \overset{\theta}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where σ_1 and σ_2 are compositions of blow-ups. We denote by $p_1, \ldots, p_n$ the base-points of σ_1 and by $\overline{E}_1, \ldots, \overline{E}_n$ the total transforms of their exceptional divisors in X. Let moreover $L \subset \mathbb{P}^2$ be a line that does not pass through the base-points of θ and θ^{-1}. We then have

$$\mathrm{Pic}(X) \simeq \mathbb{Z}\sigma_1^*(L) \oplus \mathbb{Z}\overline{E}_1 \oplus \ldots \oplus \mathbb{Z}\overline{E}_n$$

with the intersection-numbers $\overline{E}_i \cdot \overline{E}_j = -\delta_{ij}$ and $\overline{E}_i \cdot \sigma_1^*(L) = 0$ for $i, j = 1, \ldots, n$ and $\sigma_1^*(L)^2 = 1$. We find for the strict transform $\hat{C}$ of C by σ_1 and the total transform of L

by σ_2 the following divisor formulas:

$$\hat{C} = \deg(C)\sigma_1^*(L) - \sum_{i=1}^{n} m_{p_i}(C)\overline{E}_i,$$
$$\sigma_2^*(L) = \deg(\theta)\sigma_1^*(L) - \sum_{i=1}^{n} m_{p_i}(\theta)\overline{E}_i.$$

The degree of $\theta(C)$ is equal to the intersection number $\theta(C) \cdot L$. Using the projection formula, we then obtain

$$\deg(\theta(C)) = \theta(C) \cdot L = \hat{C} \cdot \sigma_2^*(L) = \deg(C)\deg(\theta) - \sum_{i=1}^{n} m_{p_i}(C) m_{p_i}(\theta).$$

□

Lemma 2.3.12. *Let $\theta \in \operatorname{Aut}(\mathbb{P}^2 \setminus L_x) \setminus \operatorname{Aut}(\mathbb{P}^2)$ and let $C \subset \mathbb{P}^2$ be a curve different from L_x. Then the following holds.*

(i) θ has a unique proper base-point and contracts L_x to a point $p \in L_x$.

(ii) $\deg(\theta(C)) \leq \deg(\theta)\deg(C)$, and equality holds if and only if $p \notin C$.

(iii) If L is a line and $\theta \in \operatorname{Jon}(\mathbb{P}^2, L_x, [0:1:0])$, then $\theta^{-1}(L)$ is a line if and only if $[0:1:0] \in L$.

Proof. To prove (i), consider the induced birational map $\theta\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Since θ does not extend to an automorphism of $\mathbb{P}^2$, it follows from Lemma 2.2.4 that θ has a minimal resolution

$$\begin{tikzcd} & X \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\theta"] & & \mathbb{P}^2 \end{tikzcd}$$

where σ_1 and σ_2 are (-1)-tower resolutions of L_x. In particular, θ has a unique proper base-point. The strict transform of L_x in X by σ_1 is the exceptional curve of the last blow-up in the tower of σ_2. This means that θ contracts L_x to a point of L_x, which is moreover the unique proper base-point of θ^{-1}. The statements (ii) and (iii) follow directly from the formula

$$\deg\theta(C) = \deg(\theta)\deg(C) - \sum_q m_q(\theta) m_q(C)$$

from Lemma 2.3.11, since θ has a unique proper base-point (which is $[0:1:0]$ if $\theta \in \operatorname{Jon}(\mathbb{P}^2, L_x, [0:1:0])$). □

2.3.3 Isomorphisms between complements of unicuspidal curves

Lemma 2.3.13. *Let $C \subset \mathbb{P}^2$ be a unicuspidal curve such that*

$$\Theta = \{\theta \in \mathrm{Aut}(\mathbb{P}^2 \setminus L_x) \mid \theta(C) = L_z\}$$

is non-empty. Then for any $\theta \in \Theta$ and any minimal resolution

$$\begin{tikzcd} & X \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\theta"] & & \mathbb{P}^2 \end{tikzcd}$$

the following are equivalent.

(*i*) $\deg\theta \leq \deg\theta'$ *for all* $\theta' \in \Theta$.

(*ii*) *The unique proper base-point of θ^{-1} is different from* $[0:1:0]$.

(*iii*) $\deg(\theta) = \deg(C)$.

(*iv*) *The strict transform of C by σ_1 intersects the strict transform of L_x by σ_2 in X.*

(*v*) *The strict transform of C by σ_1 in X has self-intersection* 1.

Proof. Let $\theta \in \Theta$. We first prove $(i) \Rightarrow (ii)$ and thus assume that θ has minimal degree in Θ. We use Theorem 2.3.7 to write

$$\theta^{-1} = a_{n+1} \circ j_n \circ a_n \circ \ldots \circ j_1 \circ a_1,$$

where $a_1, a_{n+1} \in (\mathrm{Aff}(\mathbb{P}^2, L_x) \setminus \mathrm{Jon}(\mathbb{P}^2, L_x, [0:1:0])) \cup \{\mathrm{id}\}$, $a_i \in \mathrm{Aff}(\mathbb{P}^2, L_x) \setminus \mathrm{Jon}(\mathbb{P}^2, L_x, [0:1:0])$ for $i = 2, \ldots, n$, and $j_i \in \mathrm{Jon}(\mathbb{P}^2, L_x, [0:1:0]) \setminus \mathrm{Aff}(\mathbb{P}^2, L_x)$ for $i = 1, \ldots, n$. If $(j_1 \circ a_1)(L_z)$ is a line, we can find $a_1' \in \mathrm{Aff}(\mathbb{P}^2, L_x)$ such that $a_1'(L_z) = (j_1 \circ a_1)(L_z)$. But then $\theta' \coloneqq (a_{n+1} \circ j_n \circ a_n \circ \ldots \circ j_2 \circ a_2 \circ a_1')^{-1}$ lies in Θ and $\deg(\theta') < \deg(\theta)$ by Lemma 2.3.9, which contradicts the minimality of the degree of θ in Θ. It follows moreover from Lemma 2.3.12 that $(j_1 \circ a_1)(L_z)$ is a line if and only if $[0:1:0] \in a_1(L_z)$, i.e. $a_1^{-1}([0:1:0]) \in L_z$. Thus by the minimality of the degree of θ, we have that $a_1^{-1}([0:1:0]) \notin L_z$. Since $a_1^{-1}([0:1:0])$ is the unique proper base-point of θ^{-1}, it follows that it is different from $[0:1:0]$ and hence (ii) is proved.

Assume now that the unique proper base-point of θ^{-1} is different from $[0:1:0]$. From Lemma 2.3.11 we obtain the formula

$$\deg(\theta) = \deg(\theta^{-1}) = \deg(C) + \sum_p m_p(\theta^{-1}) m_p(L_z).$$

Since the unique proper base-point of θ^{-1} lies on L_x and is different from $[0:1:0]$, we have $\deg(\theta) = \deg(C)$. This shows $(ii) \Rightarrow (iii)$. Moreover, if we assume that

$\deg(\theta) = \deg(C)$, then θ has minimal degree in Θ. Thus the implication $(iii) \Rightarrow (i)$ is also proved.

Finally, we show that (iv) and (v) are both equivalent to (ii). We consider a minimal resolution of the induced birational map by θ:

$$\begin{tikzcd} & X \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\theta"] & & \mathbb{P}^2. \end{tikzcd}$$

Since $\theta \in \mathrm{Aut}(\mathbb{P}^2 \setminus L_x) \setminus \mathrm{Aut}(\mathbb{P}^2)$ both σ_1 and σ_2 are (-1)-tower resolutions of L_x. We denote by $\hat{L}_x$ the strict transform of L_x by σ_2 in X and by $\hat{C}$ the strict transform of C by σ_1 (which is also the strict transform $\hat{L}_z$ of L_z by σ_2). Suppose that the unique proper base-point of θ^{-1} is different from $[0:1:0]$. Then $\hat{L}_x$ intersects $\hat{L}_z = \hat{C}$ and $\hat{C}$ has self-intersection 1. This shows that (ii) implies (iv) and (v). On the other hand, if we blow up the point $[0:1:0]$, then the strict transforms of L_x and L_z do not intersect and have self-intersection < 1. Thus the implications $(iv) \Rightarrow (ii)$ and $(v) \Rightarrow (ii)$ also follow. □

Proposition 2.3.14. *Let $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ be an isomorphism, where $C, D \subset \mathbb{P}^2$ are curves such that C is rational and unicuspidal with singular point $[0:1:0]$ and has very tangent line L_x. Let θ_C be an automorphism of $\mathbb{P}^2 \setminus L_x$ such that $\theta_C(C) = L_z$ and suppose that θ_C is of minimal degree with this property.*

Then D is also rational and unicuspidal and, after a suitable change of coordinates, has singular point $[0:1:0]$ and very tangent line L_x. Moreover, there exists an automorphism θ_D of $\mathbb{P}^2 \setminus L_x$ such that $\theta_D(D) = L_z$ and $\psi \in \mathrm{Aut}(\mathbb{P}^2 \setminus L_z)$ that preserves the line L_x such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\theta_C"'] & \mathbb{P}^2 \arrow[d, dashed, "\theta_D"] \\ \mathbb{P}^2 \arrow[r, dashed, "\psi"] & \mathbb{P}^2. \end{tikzcd}$$

Furthermore, θ_D can be chosen such that in the chart $z = 1$, the map ψ has the form

$$(x, y) \mapsto \big(x, y + x^2 f(x)\big)$$

for some polynomial $f \in \mathrm{k}[x]$.

Proof. The map θ_C induces a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. It does not extend to an automorphism of $\mathbb{P}^2$ since C is singular but its image by θ_C is a line. Thus θ_C contracts L_x and no other curves. We consider a minimal resolution of θ_C:

$$\begin{tikzcd} & X \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\theta_C"] & & \mathbb{P}^2. \end{tikzcd}$$

By Lemma 2.2.4, the morphisms σ_1 and σ_2 are (-1)-tower resolutions of L_x. In particular, θ_C has a unique proper base-point. Since the image of C is a line, the unique proper base-point of θ_C is the singular point $[0:1:0]$ and the strict transform of C by σ_1 in X is smooth. Hence σ_1 factors through the minimal SNC-resolution of C. Moreover, by the minimality of the degree of θ_C, it follows from Lemma 2.3.13 that the strict transform of C by σ_1 intersects the strict transform of L_x by σ_2 in X, i.e. the last exceptional curve of σ_1. It follows that the strict transform of C by σ_1 in X has self-intersection 1 by Lemma 2.3.13. In fact, σ_1 is the minimal 1-tower resolution of C that factors through the SNC-resolution of C.

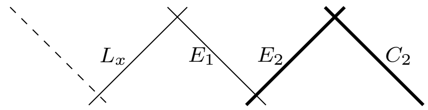
We now consider the induced birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. We assume that φ does not extend to an automorphism of $\mathbb{P}^2$, otherwise the proof is finished. Thus by Lemma 2.2.4 the map φ has a minimal resolution

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

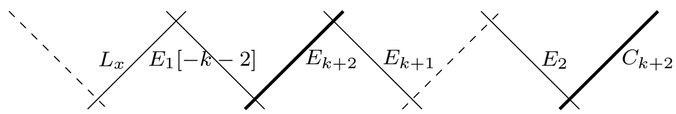
where π and η are (-1)-tower resolutions of C and D respectively. Hence φ has a unique proper base-point, which is the singular point $[0:1:0]$ of C. Since C is unicuspidal, it follows that after each blow-up in the resolution π, the strict transform of C and the exceptional curve intersect in a unique point. Since σ_1 is the minimal 1-tower resolution of C that factors through the SNC-resoltion, it follows that π factors through σ_1. We then get the following commutative diagram:

$$\begin{tikzcd} & & Y \arrow[dl] \arrow[ddrr, "\eta"] & & \\ & X \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & & & \\ \mathbb{P}^2 & & \mathbb{P}^2 \arrow[ll, dashed, "\theta_C"'] \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2. \end{tikzcd}$$

The morphism $Y \to X$ is given by a tower of blow-ups. For $i \in \{0, \dots, n\}$, we denote the intermediate surfaces by X_i, where $X_0 = X$ and $X_n = Y$ and X_i is obtained after the i-th blow-up in this tower. The corresponding exceptional curves, as well as their strict transforms, are denoted by E_i. Moreover, we denote by C_i the strict transform of C in X_i. In the surface $X = X_0$, the curves L_x and C_0 intersect transversally in a unique point and have self-intersections -1 and 1 respectively. Since π is a (-1)-tower resolution of C, the base-point in X_0 lies on the previous exceptional curve, which is the strict transform of L_x by σ_2. Moreover, since the self-intersection of C_0 is 1, the base-point in X_0 also lies on C_0, otherwise C_n would have self-interscetion 1 in Y. Thus the base-point of π in X_0 is the intersection point between C_0 and L_x. We argue similarly that the base-point in X_1 is the intersection point between C_1 and E_1. In X_2 we then have the minimal (-1)-resolution of C and thus have the following configuration of curves, where the dashed line represents the remaining exceptional curves, the unlabeled curves have self-intersection -2, and the thick lines represent (-1)-curves:

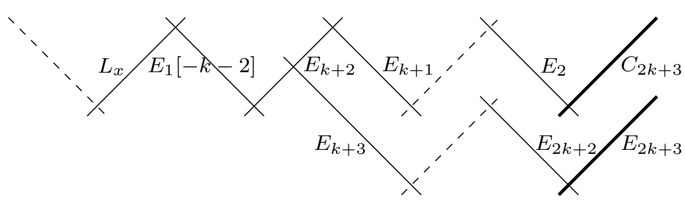


Since C_2 has self-intersection -1, none of the subsequent base-points of π lie on C_2, respectively its strict transforms, otherwise C_n would have self-intersection < -1. Since the curves E_1 and C_2 are not connected in X_2 via the other exceptional curves (except E_2), it follows that π has another base-point in X_2, which must lie on E_2. This base-point is either the intersection point p between E_1 and E_2 or lies on $E_2 \setminus (E_1 \cup C_2)$. Let $k \geq 0$ denote the number of base-points proximate to p. After blowing up those points, we obtain the following configuration in X_{k+2}:



Again, we see that E_1 is not connected to $E_{k+1} \cup \ldots \cup E_2 \cup C_{k+2}$ and thus π has a base-point on E_{k+2}, which now lies on $E_{k+2} \setminus E_1$. This base-point is not the intersection point between E_{k+2} and E_{k+1} since the morphism η first contracts C_n and then the chain of curves $E_2, \ldots, E_k$. This implies that E_{k+1} is a (-2)-curve in X. Thus the next base-point lies on $E_{k+2} \setminus (E_1 \cup E_{k+1})$.

We observe that η first contracts the chain of curves $C_n, E_2, \ldots, E_{k+2}$. After contracting this chain, the image of E_1 has self-intersection $-(k+1)$. This implies that there is a chain of k (-2)-curves attached to E_{k+2}, which then are contracted by η, so the image of E_1 has self-intersection -1 after this chain is contracted. It follows that we have the following configuration in X_{2k+3}:



We now argue that this resolution is in fact π itself. Suppose it were not, then there would be another base-point on $E_{2k+3} \setminus E_{2k+2}$, and thus E_{2k+3} is also contracted by η. We observe that η first contracts C_n, followed by $E_2, \ldots, E_{k+2}$, and then $E_{k+3}, \ldots, E_{2k+2}$. After these contractions, the image of E_1 has self-intersection -1 and is contracted next. After that, L_x and all the exceptional curves of σ_1 are contracted. The next contracted curve must then be the image of E_{2k+3}. But we observe that the image of E_{2k+3} after those contractions is singular. This follows from the fact that C is singular and from the symmetry of the configuration in X_{2k+3}. But then E_{2k+3} cannot be contrated by η

and we have a contradiction. It follows that E_{2k+3} is the last exceptional curve in the (-1)-tower resolution π.

We observe moreover, also by the symmetry of the configuration, that $\eta(L_x)$ is a line in $\mathbb{P}^2$ that is very tangent to $D = \eta(E_{2k+3})$ at the singular point. In fact, using the symmetry of the resolution, we obtain a diagram

$$\begin{tikzcd}
 & & & Y \arrow[dll, "\pi'"'] \arrow[drr, "\eta'"] & & & \\
 & X \arrow[dl, "\sigma_2"'] \arrow[dr, "\sigma_1"] & & & & X' \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\
\mathbb{P}^2 & & \mathbb{P}^2 \arrow[ll, dashed, "\theta_C"'] \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \arrow[rr, dashed, "\theta_D"] & & \mathbb{P}^2
\end{tikzcd}$$

such that $\eta = \tau_1 \circ \eta'$ where τ_1 is the minimal 1-tower resolution of D, η' is the contraction of the curves $C, E_1, \ldots, E_{2k+3}$, and θ_D is an automorphism of $\mathbb{P}^2 \setminus L_x$ that sends D to L_z.

We now consider the birational map $\psi = \theta_D \circ \varphi \circ (\theta_C)^{-1}$, which is an automorphism of $\mathbb{P}^2 \setminus L_z$. With the resolution above, we see that ψ preserves L_x. Hence, in the affine chart $z = 1$, the map ψ has the form $(x, y) \mapsto (ax, by + cx + x^2 f(x))$, where $a, b \in \mathrm{k}^*, c \in \mathrm{k}$ and $f \in \mathrm{k}[x]$. Let α be the map $[x : y : z] \mapsto [a^{-1}x : b^{-1}(y - cx) : z]$, which is an automorphism of $\mathbb{P}^2 \setminus (L_x \cup L_z)$. We define $\psi' := \alpha \circ \psi$ and $\theta'_D := \alpha \circ \theta_D$. Then ψ' has the form $(x, y) \mapsto (x, y + x^2 f(x))$, as claimed.

□

Definition 2.3.15. Let X be an irreducible surface, $C \subset X$ an irreducible curve, and $p \in C$ a point. Let $\mathfrak{a}$ be the kernel of the restriction homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{C,p}$, $f \mapsto f|_C$. Then we denote by $\mathrm{Loc}(X, C, p)$ the group of birational maps $\varphi \colon X \dashrightarrow X$ fixing p, such that φ^* induces

(*i*) an automorphism of $\mathcal{O}_{X,p}$,

(*ii*) a bijection $\mathfrak{a} \to \mathfrak{a}$,

(*iii*) the identity on $\mathcal{O}_{X,p}/\mathfrak{a}^2$,

(*iv*) the identity on $\mathfrak{a}/\mathfrak{a}^3$.

Remark 2.3.16. If $\varphi \in \mathrm{Loc}(X, C, p)$, then φ induces a local isomorphism in a neighborhood of p in X and C. Thus for a birational map $\theta \colon X \dashrightarrow Y$ that is a local isomorphism in a neighborhood of $p \in X$, the conjugation $\psi \mapsto \theta^{-1} \circ \psi \circ \theta$ induces an isomorphism $\mathrm{Loc}(Y, \theta(C), \theta(p)) \to \mathrm{Loc}(X, C, p)$.

Lemma 2.3.17. *For any* $\lambda \in \mathrm{k}$, *the group* $\mathrm{Loc}(\mathbb{A}^2, L_x, (0, \lambda))$ *coincides with the group of birational maps* $\varphi \colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ *such that* φ *and* φ^{-1} *each can be written of the form*

$$(x, y) \mapsto \big(x + x^3\alpha(x, y), y + x^2\beta(x, y)\big)$$

for some $\alpha, \beta \in \mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$.

Proof. Let φ be a birational map of $\mathbb{A}^2$ of the proposed form. Then φ is defined at $(0,\lambda)$ and fixes $(0,\lambda)$. The same is true for φ^{-1}, so it is a local isomorphism at $(0,\lambda)$ and thus satisfies (i) of Definition 2.3.15. One then checks points $(ii)-(iv)$ for the ideal $\mathfrak{a}=(x)\subset \mathrm{k}[x,y]_{(x,y-\lambda)}=\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$. It follows that $\varphi\in \mathrm{Loc}(\mathbb{A}^2,L_x,(0,\lambda))$.

To prove the converse, let $\varphi\in \mathrm{Loc}(\mathbb{A}^2,L_x,(0,\lambda))$. Since φ^* induces an automorphism of $\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}=\mathrm{k}[x,y]_{(x,y-\lambda)}$ we can write $\varphi^*(x)=f$ and $\varphi^*(y)=g$ for some $f,g\in\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$. As φ^* preserves the ideal (x) and induces the identity on $\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}/(x^2)$, we can express $f(x,y)=x+x^2\alpha(x,y)$ and $g(x,y)=y+x^2\beta(x,y)$, for some $\alpha,\beta\in\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$. Finally, since φ^* induces the identity on $(x)/(x^3)$, it follows that x divides α and hence φ is of the desired form. Since $\mathrm{Loc}(\mathbb{A}^2,L_x,(0,\lambda))$ is a group, also the inverse of φ can be written in this form. □

Proposition 2.3.18. *Let $L\subset\mathbb{P}^2$ be a line and $q_1,q_2\in L$ with $q_1\neq q_2$. Let $\psi\in\bigcap_{p\in L\setminus\{q_2\}}\mathrm{Loc}(\mathbb{P}^2,L,p)$ and $\theta\in\mathrm{Aut}(\mathbb{P}^2\setminus L)\setminus\mathrm{Aut}(\mathbb{P}^2)$ such that θ^{-1} has base-point q_1 and θ has base-point q_2. Then $\theta^{-1}\circ\psi\circ\theta$ lies in $\bigcap_{p\in L\setminus\{q_2\}}\mathrm{Loc}(\mathbb{P}^2,L,p)$.*

Proof. Since the base-point of θ^{-1} is q_1 and the base-point of θ is not q_1 we can by Theorem 2.3.7 write $\theta=j_n\circ a_n\circ\ldots\circ j_1\circ a_1$ with $j_i\in\mathrm{Jon}(\mathbb{P}^2,L,q_1)\setminus\mathrm{Aff}(\mathbb{P}^2,L)$ and $a_i\in\mathrm{Aff}(\mathbb{P}^2,L)\setminus\mathrm{Jon}(\mathbb{P}^2,L,q_1)$ for $i=1,\ldots,n$. By induction, it suffices to prove the claim for $\theta=j\circ a$ with $j\in\mathrm{Jon}(\mathbb{P}^2,L,q_1)\setminus\mathrm{Aff}(\mathbb{P}^2,L)$ and $a\in\mathrm{Aff}(\mathbb{P}^2,L)\setminus\mathrm{Jon}(\mathbb{P}^2,L,q_1)$.

We then find a minimal resolution

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "j\circ a"] & & \mathbb{P}^2 \end{tikzcd}$$

where π^{-1} has the same base-points as $j^{-1}\in\mathrm{Jon}(\mathbb{P}^2,L,q_1)$. Let $d\geq 2$ be the degree of j^{-1}, so we can write π as a composition of $2d-1$ blow-ups $\pi\colon X=X_{2d-1}\xrightarrow{\pi_{2d-1}}\ldots\xrightarrow{\pi_2}X_1\xrightarrow{\pi_1}X_0=\mathbb{P}^2$, as described in Lemma 2.3.5. We denote the exceptional curve of π_i by E_i for $i=1,\ldots,2d-1$.

We want to lift ψ to a birational transformation of X by conjugation with π. To do this, we choose coordinates on $\mathbb{P}^2$ such that $L=L_x$ and $q_1=[0:0:1]$ and $q_2=[0:1:0]$. By Lemma 2.3.17, we can locally express ψ as

$$(x,y)\mapsto\big(x+x^3\alpha(x,y),y+x^2\beta(x,y)\big)$$

for some $\alpha,\beta\in\cap_{\lambda\in\mathrm{k}}\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$. We proceed by conjugating ψ step-by-step with the blow-ups π_i.

The first blow-up has base-point $(0,0)$ and is locally given by $\pi_1\colon(x,y)\mapsto(xy,y)$. We thus obtain:

$$\begin{aligned}\pi_1^{-1}\psi\pi_1(x,y) &= \left(\frac{xy+x^3y^3\alpha(xy,y)}{y+x^2y^2\beta(xy,y)}, y+x^2y^2\beta(xy,y)\right)\\ &= \left(x+x^3y\frac{(y\alpha(xy,y)-b(xy,y))}{1+x^2y\beta(xy,y)}, y+x^2y^2\beta(xy,y)\right)\\ &=: \left(x+x^3y\alpha_1(x,y), y+x^2y^2\beta_1(x,y)\right) =: \psi_1(x,y)\end{aligned}$$

In local coordinates of $\mathbb{A}^2 \subset X_1$, the exceptional curve E_1 of π_1 is given by $y=0$ and $\alpha_1, \beta_1 \in \cap_{\lambda\in \mathrm{k}}\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$.

The base-point of π_2 is then the point $(0,0) \in E_1$. Indeed, the base-points of $\pi_2,\ldots,\pi_d$ all lie on E_1, such that each of these blow-ups is of the form $(x,y)\mapsto(x,xy)$, in local coordinates. We can thus write $\pi_2\circ\ldots\circ\pi_d\colon (x,y)\mapsto(x,x^{d-1}y)$ and thus conjugation with this map yields:

$$\begin{aligned}\psi_d(x,y) &= \left(x+x^{d+2}y\alpha_1(x,x^{d-1}y), \frac{x^{d-1}y+x^{2d}y^2\beta_1(x,x^{d-1}y)}{(x+x^{d+2}y\alpha_1(x,x^{d-1}y))^{d-1}}\right)\\ &= \left(x+x^{d+2}\alpha_1(x,x^{d-1}y), y+x^{d+1}y^2\frac{x^{d-1}y^2\beta_1(x,x^{d-1}y)+\ldots}{(1+x^{d+1}y\alpha_1(x,x^{d-1}y))^{d-1}}\right)\end{aligned}$$

In local coordinates of $\mathbb{A}^2 \subset X_d$, we can write

$$\psi_d(x,y) = \left(x+x^{d+2}\alpha_d(x,y), y+x^{d+1}\beta_d(x,y)\right)$$

for some $\alpha_d, \beta_d \in \cap_{\lambda\in \mathrm{k}}\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$.

The base-point of the blow-up π_{d+1} is a point on E_d but not E_{d-1}. In local coordinates, this means that π_{d+1} can be expressed as $(x,y)\mapsto(x,xy+\mu)$, for some $\mu\in \mathrm{k}^*$. The conjugated map is then:

$$\begin{aligned}\psi_{d+1}(x,y) &= \left(x+x^{d+2}\alpha_d(x,xy+\mu), \frac{xy+x^{d+1}\beta_d(x,xy+\mu)}{x+x^{d+2}\alpha_d(x,xy+\mu)}\right)\\ &= \left(x+x^{d+2}\alpha_d(x,xy+\mu), y+x^d\frac{\beta_d(x,xy+\mu)-xy\alpha_d(x,xy+\mu)}{1+x^{d+1}\alpha_d(x,xy+\mu)}\right)\end{aligned}$$

and thus we can find $\alpha_{d+1}, \beta_{d+1} \in \cap_{\lambda\in \mathrm{k}}\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$ such that

$$\psi_{d+1}(x,y) = \left(x+x^{d+2}\alpha_{2d-1}(x,y), y+x^d\beta_{2d-1}(x,y)\right).$$

After conjugating with the $d-2$ remaining blow-ups $\pi_{d+2},\ldots,\pi_{2d-1}$, we thus obtain

$$\psi_{2d-1}(x,y) = \left(x+x^{d+2}\alpha_{2d-1}(x,y), y+x^2\beta_{2d-1}(x,y)\right)$$

for some $\alpha_{2d-1}, \beta_{2d-1} \in \cap_{\lambda\in \mathrm{k}}\mathcal{O}_{\mathbb{A}^2,(0,\lambda)}$ and hence it follows that $\psi_{2d-1}\in \mathrm{Loc}(X, E_{2d-1}, (0,\lambda))$ for all $\lambda\in \mathrm{k}$ by Lemma 2.3.17.

We now consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, dashed, "(j\circ a)^{-1}\circ\psi\circ(j\circ a)"] \arrow[dd, dashed, bend right=60, "j\circ a"'] & \mathbb{P}^2 \arrow[dd, dashed, bend left=60, "j\circ a"] \\
X \arrow[u, "\eta"] \arrow[r, dashed, "\psi_{2d-1}"] \arrow[d, "\pi"'] & X \arrow[u, "\eta"'] \arrow[d, "\pi"] \\
\mathbb{P}^2 \arrow[r, dashed, "\psi"] & \mathbb{P}^2
\end{tikzcd}$$

For any $p \in L_x \setminus [0:1:0]$, it follows that η induces a local isomorphism $\eta^{-1}(p) \to p$ and thus $(j \circ a)^{-1} \circ \psi \circ (j \circ a) = \eta \circ \psi_{2d-1} \circ \eta^{-1} \in \operatorname{Loc}(\mathbb{P}^2, L_x, p)$.

□

Proof of Theorem 1. By Lemma 2.2.1 the curves C and D have the same degree. Thus the claim of the theorem is clear for lines and conics and we can assume that C has degree at least 3 and is hence singular, in fact unicuspidal. The isomorphism $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ induces a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. If φ extends to an automorphism of $\mathbb{P}^2$, then C and D are projectively equivalent. We thus assume that φ does not extend to an automorphism of $\mathbb{P}^2$, i.e. C is contracted by φ. Since $C \setminus L \simeq \mathbb{A}^1$, we can apply Proposition 2.3.2 by identifying $\mathbb{P}^2 \setminus L \simeq \mathbb{A}^2$, so there exists an automorphism of $\mathbb{P}^2 \setminus L$ that sends C to a line. We can then use Proposition 2.3.14 and for suitable coordinates obtain the diagram

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\theta_C"'] & \mathbb{P}^2 \arrow[d, dashed, "\theta_D"] \\
\mathbb{P}^2 \arrow[r, dashed, "\psi"] & \mathbb{P}^2
\end{tikzcd}$$

where $\theta_C, \theta_D \in \operatorname{Aut}(\mathbb{P}^2 \setminus L_x)$ with $\theta_C(C) = \theta_D(D) = L_z$ and $\psi \in \operatorname{Aut}(\mathbb{P}^2 \setminus L_z)$ has the form $(x, y) \mapsto (x, y + x^2 f(x))$ and thus lies in $\operatorname{Loc}(\mathbb{P}^2, L_x, [0 : \lambda : 1])$ for all $\lambda \in \mathrm{k}$. The base-point p of θ_C is different from $[0 : 1 : 0]$ and is thus of the form $[0 : \lambda : 1]$ for some $\lambda \in \mathrm{k}$. We then define the map $\rho = (\theta_C)^{-1} \circ \psi \circ \theta_C$, which is an automorphism of $\mathbb{P}^2 \setminus (L_x \cup C)$. It follows from Proposition 2.3.18 that ρ lies in $\operatorname{Loc}(\mathbb{P}^2, L_x, [0 : 0 : 1])$ and in particular preserves the line L_x. Thus ρ is an automorphism of $\mathbb{P}^2 \setminus C$ and consequently $\varphi' := \varphi \circ \rho^{-1}$ is an isomorphism $\mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$. On the other hand, $\varphi' = (\theta_D)^{-1} \circ \theta_C$ is an automorphism of $\mathbb{P}^2 \setminus L_x$ and hence does not contract C. We conclude that φ' contracts no curves and is indeed an automorphism of $\mathbb{P}^2$, making the curves C and D projectively equivalent. □

2.4 Curves of low degree

In this section we study Conjecture 2.1.1 for curves of low degree, i.e. degree ≤ 8. It is a case study on the multiplicity sequences that occur (see Definition 2.4.2).

2.4.1 Cases by multiplicity sequences

Lemma 2.4.1. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 3$ such that there exists an open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$. Then C is a rational curve, where all the proper and infinitely near singular points of C can be ordered from p_1 to p_k, with multiplicities $m_1 \geq \ldots \geq m_k \geq 2$, such that $p_1 \in C$ is a proper point and p_{i+1} lies in the first neighborhood of p_i, for $i = 1, \ldots, k-1$. Moreover, the multiplicities satisfy the following relations:*

$$d^2 - 3d + 2 = \sum_{i=1}^{k} m_i(m_i - 1), \tag{A}$$

$$d^2 + 1 \geq \sum_{i=1}^{k} m_i^2. \tag{B}$$

Proof. Let $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ be an open embedding that does not extend to an automorphism of $\mathbb{P}^2$. Then by Lemma 2.2.4 there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \ldots, p_n$ and exceptional curves $E_1, \ldots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. For $i \in \{1, \ldots, n\}$, we denote by m_i the multiplicity of C_i at p_i, so we have $m_1 \geq \ldots \geq m_n$. The strict transform C_n in X is smooth, thus π factors through the minimal resolution of singularities of C and blows up all its $k \leq n$ singular points, hence the first part of the claim follows.

For equation (A), we observe that C is a rational curve since $C_n \simeq \mathbb{P}^1$ and thus has genus $g(C) = 0$. By the genus-degree formula for plane curves we get

$$0 = g(C) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{k} \frac{m_i(m_i - 1)}{2}$$

and hence identity (A) follows. To see the inequality (B), it is enough to observe that for a blow-up π_i with exceptional curve E_i, we get

$$\pi_i^*(C_i) = C_{i+1} + m_i E_i$$

and hence $(C_{i+1})^2 = (C_i)^2 - m_i^2$, using the identities $(E_i)^2 = -1$ and $C_{i+1} \cdot E_i = m_i$. We then inductively obtain

$$-1 = (C_n)^2 = d^2 - \sum_{i=1}^{n} m_i^2.$$

The claim then follows from the fact that the number k of singular points is $\leq n$. $\square$

The previous lemma motivates the following definition.

Definition 2.4.2. Let $C \subset \mathbb{P}^2$ be a curve. We say that C has *multiplicity sequence* $(m_1, \ldots, m_k)$, where $m_1 \geq \ldots \geq m_k \geq 2$, if C has (proper or infinitely near) singular points $p_1, \ldots, p_k$ with multiplicities $m_1, \ldots, m_k$ such that $p_1 \in C$ is a proper point and p_{i+1} lies in the first neighborhood of p_i for $i \geq 1$, and moreover C is smooth at all other points. For a constant subsequence $(m, \ldots, m)$ of length $l \geq 1$, we also use the short notation $(m_{(l)})$.

Remark 2.4.3. It is not known to the author whether there exist irreducible curves $C, D \subset \mathbb{P}^2$ that have isomorphic complements but have different multiplicity sequences.

Lemma 2.4.4. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 3$ with multiplicity sequence $(m_1, \ldots, m_k)$, where we set $m_2 := 1$ if $k = 1$. If there exists an open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$, then the following inequalities hold:*

$$m_1 + m_2 \leq d < 3m_1.$$

Proof. We use the set-up of the proof of Lemma 2.4.1 and extend the multiplicity sequence $(m_1, \ldots, m_k)$ by $m_{k+1} = \ldots = m_n = 1$ such that both (A) and (B) from Lemma 2.4.1 become equalities. We then subtract (A) from (B) for the extended multiplicity sequence and obtain

$$3d - 1 = \sum_{i=1}^{n} m_i.$$

We then multiply this equation by $\frac{d}{3}$ and subtract (B), so we get

$$-\left(1 + \frac{d}{3}\right) = \sum_{i=1}^{n} m_i \left(\frac{d}{3} - m_i\right).$$

Since the right-hand side of this equation is negative, so is the left-hand side. Thus, at least one of the terms $\frac{d}{3} - m_i$ is negative. The inequality $d < 3m_1$ now follows from the fact that the multiplicity sequence is non-increasing.

The inequality $m_1 + m_2 \leq d$ follows from Bézout's theorem, where we intersect C with a line going through points p_1 and p_2 of multiplicity m_1 and m_2 respectively. □

Corollary 2.4.5. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree ≤ 8 such that there exists an open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$. Then C has one of the multiplicity sequences shown in the following table.*

degree	multiplicity sequences
3	(2)
4	$(3); (2_{(3)})$
5	$(4); (3, 2_{(3)}); (2_{(6)})$
6	$(5); (4, 2_{(4)}); (3_{(3)}, 2); (3_{(2)}, 2_{(4)}); (3, 2_{(7)})$
7	$(6); (5, 2_{(5)}); (4, 3_{(3)}); (4, 3_{(2)}, 2_{(3)}); (4, 3, 2_{(6)}); (3_{(4)}, 2_{(3)})$
8	$(7); (6, 2_{(6)}); (5, 3_{(3)}, 2_{(2)}); (5, 3_{(2)}, 2_{(5)}); (4_{(3)}, 3); (4_{(3)}, 2_{(3)}); (4_{(2)}, 3_{(3)});$ $(4_{(2)}, 3_{(2)}, 2_{(3)}); (4_{(2)}, 3, 2_{(6)}); (4, 3_{(5)}); (4, 3_{(4)}, 2_{(3)}); (3_{(7)})$

Table 2.1: Multiplicity sequences for degree ≤ 8.

Proof. This follows from computations using Lemma 2.4.1 and Lemma 2.4.4, but we need to look at one case more carefully. In degree 7 the multiplicity sequence $(3_{(5)})$ is consistent with the inequalities in Lemma 2.4.1 and Lemma 2.4.4. Suppose that there exists such a curve C and denote by p_1, p_2, p_3 the first 3 singular points, all of multiplicity 3. By Bézout's theorem those points are not collinear. Moreover, p_3 is not proximate to p_1 as the sum of the multiplicities of the strict transform of C at p_2 and p_3 is larger than the multiplicity at p_1. Thus there exists a quadratic transformation q with base-points p_1, p_2, p_3. The degree of $q(C)$ is then $2 \cdot 7 - 3 - 3 - 3 = 5$ by Lemma 2.3.11 and has two singular points of multiplicity 3. But this is not possible by Lemma 2.4.4. Hence no curve of of degree 7 with multiplicity sequence $(3_{(5)})$ exists. □

The case of cubic curves is then straightforward.

Lemma 2.4.6. *Let $C \subset \mathbb{P}^2$ be a cubic curve and let $\varphi \colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ an isomorphism, where $D \subset \mathbb{P}^2$ is some curve. Then C and D are projectively equivalent.*

Proof. If φ extends to an automorphism of $\mathbb{P}^2$, the claim is clear. If not, then C is rational and hence singular with a point of multiplicity 2. It is a well known fact that can be checked by simple computations that there are only two singular cubic curves, up to projective equivalence. One class is represented by the cuspidal cubic curve $x^2z - y^3 = 0$ and the other class by the nodal cubic curve $x^2z - y^3 - y^2z = 0$. It follows from Lemma 2.2.1 that D is again a cubic curve and by Proposition 2.2.6 that the singularity of D is of the same type as the singularity of C, i.e. $D \setminus \mathrm{Sing}(D) \simeq \mathbb{A}^1$ if C is unicuspidal or $D \setminus \mathrm{Sing}(D) \simeq \mathbb{A}^1 \setminus \{0\}$ if C is nodal. Hence C and D are projectively equivalent. □

Remark 2.4.7. The complement of a nodal cubic curve has infinitely many automorphisms, up to composition with automorphisms of $\mathbb{P}^2$. For a description, see for instance [Yos85, Lemma 2.24]. The automorphism group of the complement of a cuspidal cubic is even infinite dimensional, see [Yos85, Theorem A (6)].

We will frequently use the following formula for intersection numbers.

Lemma 2.4.8. *Let $C \subset \mathbb{P}^2$ be a curve and $\pi\colon X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ a (-1)-tower resolution of C with base-points $p_1, \dots, p_n$ and exceptional curves $E_1, \dots, E_n$. For $i \leq k \leq n$, we then have*

$$C_k \cdot E_i = m_{p_i}(C_i) - \sum_{p_j \succ p_i, j \leq k} m_{p_j}(C_j).$$

Proof. Let $i, k \in \mathbb{N}$ with $i \leq k \leq n$. We denote by $\overline{E}_j$ the total transform of E_j in X_k for $j = 1, \dots, k$. By [Alb02, Corollary 1.1.25], we can then write

$$E_i = \overline{E}_i - \sum_{p_j \succ p_i, j \leq k} \overline{E}_j.$$

By [Alb02, Corollary 1.1.27], we have $C_k \cdot \overline{E}_j = m_{p_j}(C_j)$ and the claim follows. □

Lemma 2.4.9. *Let $C \subset \mathbb{P}^2$ be an irreducible curve that has multiplicity sequence $(m_1, \dots, m_k)$. If there exist $r < s \leq k-2$ such that*

$$\begin{aligned} m_{r+1} + m_{r+2} > m_r > m_{r+1}, \\ m_{s+1} + m_{s+2} > m_s > m_{s+1}, \\ m_s + m_{s+1} > m_{s-1}, \end{aligned}$$

then every open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.

Proof. Suppose that there exists an open embedding $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$. Then by Lemma 2.2.4 there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \dots, p_n$ and exceptional curves $E_1, \dots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. For any $i \in \{1, \dots, k\}$, we obtain from Lemma 2.4.8 the equation

$$C_n \cdot E_i = m_i - \sum_{p_j \succ p_i} m_j.$$

The point p_{r+1} is proximate to p_r, but p_{r+2} is not, as $C_n \cdot E_r \geq 0$ and $m_{r+1} + m_{r+2} > m_r$. Hence we have $C_n \cdot E_r = m_r - m_{r+1} > 0$. Analogously we get $C_n \cdot E_s > 0$. The curve $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ in X is the exceptional locus of η and thus has a tree structure. By the same argument as before, the point p_{s+1} is not proximate to p_{s-1}, hence it follows that the curves E_r and E_s are connected in $E_1 \cup \ldots \cup E_{n-1}$ via some chain of curves. Since E_r and E_s are also connected via C_n, this yields a contradiction to the tree structure of $E_1 \cup \ldots \cup E_{n-1} \cup C_n$. □

Corollary 2.4.10. *Let $C \subset \mathbb{P}^2$ be an irreducible rational curve with one of the multiplicity sequences $(4, 3, 2_{(6)})$, $(4, 3_{(2)}, 2_{(3)})$, $(4, 3_{(4)}, 2_{(3)})$, $(4_{(2)}, 3, 2_{(6)})$, $(4_{(2)}, 3_{(2)}, 2_{(3)})$, $(5, 3_{(2)}, 2_{(5)})$, or $(5, 3_{(3)}, 2_{(2)})$. Then any open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.*

Proof. This follows directly from Lemma 2.4.9. □

2.4.2 The unicuspidal case and a special quintic curve

If $C \subset \mathbb{P}^2$ is a unicuspidal curve that admits a very tangent line through the singular point, then Theorem 1 gives an affirmative answer to Conjecture 2.1.1. In low degrees this is often the case, as we will see using the following lemma, which we can already find in [Yos84].

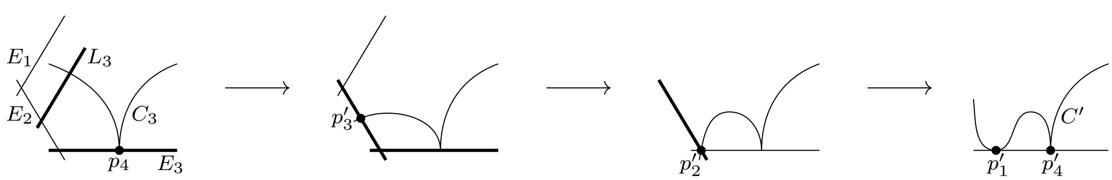
Lemma 2.4.11. *Let $C \subset \mathbb{P}^2$ be a curve with multiplicity sequence $(m_1, \ldots, m_k)$, where we set $m_2 = 1$ if $k = 1$. If $\deg(C) = m_1 + m_2$, then there exists a very tangent line to C through the proper singular point.*

Proof. Let $p_1 \in C$ be the proper singular point of multiplicity m_1 and p_2 a point infinitely near to p_1 with multiplicity m_2. Then there exists a line L through p_1 and p_2. We then get the local intersection $(C \cdot L)_{p_1} \geq m_1 + m_2 = \deg(C)$. By Bézout's theorem L intersects C in no other point and we have equality $(C \cdot L)_{p_1} = \deg(C)$, and thus L is very tangent to C. □

In Table 2.1, we find the multiplicity sequence $(2_{(6)})$ for quintic curves. It follows from Bézout's theorem that such curves do not admit a very tangent line through the singular point and hence Theorem 1 does not apply. We thus have to study this case separately. This seems to be a well known class of curves and was already considered in [Yos84] and [Yos79], but without full proofs. Over the field of complex numbers, unicuspidal quintic curves were classified in [Nam84, Theorem 2.3.10.]. For the sake of completeness, we give a self-contained treatment of the case unicuspidal curves with multiplicity sequence $(2_{(6)})$ below.

Lemma 2.4.12. *Let C and $D \subset \mathbb{P}^2$ be irreducible unicuspidal quintic curves with multplicity sequence $(2_{(6)})$ with singular points $p_1, \ldots, p_6$ and $q_1, \ldots, q_6$ respectively. Then there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\alpha(p_i) = q_i$ for $i = 1, \ldots, 6$.*

Proof. Let $L \subset \mathbb{P}^2$ be the line through p_1 and p_2. The singular points p_1, p_2, p_3 of C all have multiplicity 2, thus they are not collinear by Bézout's theorem. It follows that there exists a quadratic map $\theta_1 \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base-points p_1, p_2, p_3 and exceptional curves E_1, E_2, E_3. The map θ_1 is then given by first blowing up p_1, p_2, p_3 and then contracting L_3, E_2, E_1, as shown below. We denote by p_1', p_2', p_3' the base-points of $(\theta_1)^{-1}$ and by p_4', p_5', p_6' the singular points of $C' := \theta_1(C)$.



By Lemma 2.3.11, the degree of C' is $2 \cdot 5 - 1 \cdot 2 - 1 \cdot 2 - 1 \cdot 2 = 4$ and hence C' is a unicuspidal quartic curve. Likewise, there exists a quadratic map θ_2 that sends D to a unicuspidal quartic curve D', where we analogously denote the points $q_1', \ldots, q_6'$.

We show that there exists an automorphism $\alpha' \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\alpha'(p_i') = q_i'$ for $i = 1, \ldots, 6$, which implies that the map $\alpha = (\theta_2)^{-1} \circ \alpha' \circ \theta_1$ is an automorphism of $\mathbb{P}^2$ that sends p_i to q_i, for $i = 1, \ldots, 6$, since the base-points of $(\theta_1)^{-1}$ are sent to the base-points of $(\theta_2)^{-1}$.

We can assume that, after a linear change of coordinates, we have $p_1' = q_1' = [0:0:1]$ and $p_4' = q_4' = [0:1:0]$. By Bézout's theorem the points p_1', p_4', p_5' are not collinear, thus we can moreover assume that p_5', respectively q_5', corresponds to the tangent direction L_z.

The points p_1', p_2', p_4' are in fact collinear and thus p_2' corresponds to the tangent direction L_x, and the same is the case for q_2'. The linear maps fixing p_1', p_2', p_4', p_5' then correspond to matrices in PGL_3 of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathrm{k}$ and $ac \neq 0$. We now consider the action of those linear maps on the points p_3' and p_6'. We thus blow up the point $p_1' = [0:0:1]$. In local coordinates, this blow-up is given by $(u,v) \mapsto [uv : v : 1]$ and moreover $p_2' = (0,0)$. With a linear map of the above form, we get $[uv : v : 1] \mapsto [auv : buv + cv : 1]$ and the induced map in the blow-up is locally given by $(u,v) \mapsto \left(\frac{au}{bu+c}, (bu+c)v\right)$. The induced map on the exceptional curve is then $[u : v] \mapsto [\frac{a}{c}u : cv] = [\frac{a}{c^2}u : v]$. We observe that p_3' is not proximate to p_1' and that p_3' is not collinear with p_1', p_2' and p_4' by Bézout's theorem. Thus p_3' is neither of the points $[0:1]$ or $[1:0]$ on the exceptional curve and we can assume that $p_3' = q_3' = [1:1]$. From this we obtain the condition $a = c^2$.

For the point p_6', we consider the blow-up of $p_4' = [0:1:0]$, in local coordinates given by $(u,v) \mapsto [u : 1 : uv]$, and $p_5' = (0,0)$. Applying a linear map of the form above, we obtain $[u : 1 : uv] \mapsto [au : bu + c : uv]$ and the induced map on the blow-up is given by $(u,v) \mapsto \left(\frac{au}{bu+c}, \frac{v}{a}\right)$, in local coordinates. The induced map on the exceptional curve is $[u : v] \mapsto [\frac{a}{c}u : \frac{1}{a}v] = [\frac{a^2}{c}u : v] = [c^3 u : v]$. As before, we see that p_6' is not proximate to p_4' and is not collinear with p_4' and p_5'. Hence we can also assume that $p_6' = q_6' = [1:1]$ and get the condition $c = 1$.

We have thus found a linear map that sends p_i' to q_i' for $i = 1, \ldots, 6$ and the claim follows. □

Proposition 2.4.13. *Let $C \subset \mathbb{P}^2$ be an irreducible unicuspidal quintic curve with multiplicity sequence $(2_{(6)})$. Then C is projectively equivalent to the curve*

$$Q\colon (xz + y^2)\left((xz + y^2)z + 2x^2y\right) - x^5 = 0.$$

Proof. We start by constructing a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ that sends the line L_z to the quintic curve Q. To do this we consider first the quadratic map $\theta_1\colon [x:y:z] \mapstochar\dashrightarrow [x^2 : xy : xz+y^2]$. This map is an automorphism of $\mathbb{P}^2 \setminus L_x$ and sends the line L_z to the conic $xz+y^2 = 0$. Next, consider the quadratic map $\theta_2\colon [x:y:z] \mapstochar\dashrightarrow [xz : x^2 - yz : z^2]$, which induces an automorphism of $\mathbb{P}^2 \setminus L_z$. We compute the composition $\psi := (\theta_1)^{-1} \circ \theta_2 \circ \theta_1$ and obtain

$$[x : y : z] \dashrightarrow [x(xz+y^2)^2 : (xz+y^2)\big(x^3-y(xz+y^2)\big) : (xz+y^2)\big(z(xz+y^2)+2x^2y)\big)-x^5].$$

The map ψ is an automorphism of the complement of the conic $xz + y^2 = 0$ in $\mathbb{P}^2$ and is moreover an involution. Hence both ψ and ψ^{-1} contract the conic $xz + y^2 = 0$ and have unique proper base-point $[0 : 0 : 1]$. The image of the line L_z by ψ is exactly the quintic curve Q. The degree of ψ is 5 and the linear system of ψ contains the curve Q whose only proper singular point is $[0 : 0 : 1]$ with multplicity 2, thus by the Noether equations ψ has 6 base-points of multiplicity 2, which then must be the same as the singular points of Q.

Let C be any unicuspidal quintic curve with multiplicity sequence $(2_{(6)})$. We can assume by Lemma 2.4.12 that after a change of coordinates the 6 (proper and infinitely near) singular points of C and Q coincide. Hence by Lemma 2.3.11 the birational map ψ^{-1} sends the curve C to a curve of degree $5\cdot 5-2\cdot 2-2\cdot 2-2\cdot 2-2\cdot 2-2\cdot 2-2\cdot 2 = 1$, i.e. a line. This line is tangent to the conic $xz + y^2 = 0$ since C is unicuspidal and the line does not pass through the base-point $[0 : 0 : 1]$ of ψ. The tangents to the conic $xz + y^2 = 0$ that do not pass through $[0 : 0 : 1]$ are parametrized by the family $L_\alpha\colon \alpha^2x + 2\alpha y - z = 0$, where $\alpha \in \mathrm{k}$. We then compute the equation of the image of L_α under ψ and get

$$Q_\alpha\colon (xz + y^2)\left((xz + y^2)(\alpha^2x - 2\alpha y - z) + 2x^2(\alpha x - y)\right) + x^5 = 0.$$

Thus $C = Q_\alpha$, for some $\alpha \in \mathrm{k}$. A short computation shows that the automorphism of $\mathbb{P}^2$ given by

$$[x : y : z] \mapsto [x : \alpha x + y : -\alpha^2x - 2\alpha y + z]$$

sends the curve Q_α to the curve $Q_0 = Q$. □

Corollary 2.4.14. *Let $Q \subset \mathbb{P}^2$ be an irreducible unicuspidal quintic curve with multiplicity sequence $(2_{(6)})$ and $\varphi\colon \mathbb{P}^2 \setminus Q \to \mathbb{P}^2 \setminus D$ an isomorphism, where $D \subset \mathbb{P}^2$ is some curve. Then D is projectively equivalent to Q.*

Proof. By Lemma 2.2.1 and Proposition 2.2.6, the curve D is also a rational unicuspidal quintic. It thus has one of the multiplicity sequences $(4), (3, 2_{(3)})$, or $(2_{(6)})$ by Corollary 2.4.5. In the first two cases, D admits a very tangent line through the singular point by Lemma 2.4.11, and thus by Theorem 1, this would also hold for the curve Q. Since Q does not admit a very tangent line through the singular point, it follows that D has multiplicity sequence $(2_{(6)})$ and is hence projectively equivalent to Q by Proposition 2.4.13. □

To conclude the case of unicuspidal curves, we need two more observations.

Lemma 2.4.15. *Let $C \subset \mathbb{P}^2$ be a rational irreducible curve with one of the multiplicity sequences $(3_{(4)}, 2_{(3)})$, $(4, 3_{(5)})$, $(4, 3_{(4)}, 2_{(3)})$, or $(5, 2_{(5)})$. Then C is not unicuspidal.*

Proof. Let $\pi\colon X = X_k \xrightarrow{\pi_k} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ be a minimal resolution of singularities of C, where π_i is the blow-up of the singular point $p_i \in X_i$ of multiplicity m_i and has exceptional curve E_i for $i = 1, \dots, k$. It follows that C_k intersects E_k with multiplicity m_k. If there exists some $i \leq k-2$ such that $m_i - m_{i+1} = 1$, it follows from Lemma 2.4.8 that

$$C_k \cdot E_i = m_i - \sum_{p_j \succ p_i} m_j = m_i - m_{i+1} = 1$$

since $C_k \cdot E_i \geq 0$ and $m_{i+2} \geq 2$. If E_i does moreover not intersect E_k, it follows that C is not unicuspidal, as C_k intersects the exceptional locus $E_1 \cup \ldots \cup E_k$ of π in at least two points, one on E_i and one on E_k. We observe that this is the case for the multiplicity sequences $(3, 2_{(7)})$, $(3_{(4)}, 2_{(3)})$, $(4, 3_{(5)})$, and $(4, 3_{(4)}, 2_{(3)})$, since in each case the exceptional curves in their minimal resolution of singularities form a chain where E_i and E_k do not intersect, as one checks with Lemma 2.4.8.

Similarly, we see with Lemma 2.4.8 that for the multiplicity sequence $(5, 2_{(5)})$, either p_3 is proximate to p_1 or not, but in both cases the curve C_7 intersects E_1 and E_7 in distinct points and thus C is again not unicuspidal. □

Lemma 2.4.16. *Let $C \subset \mathbb{P}^2$ be a rational, unicuspidal curve of degree d and multiplicity sequence $(m_1, \dots, m_k)$. There exists an open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$ if and only if exactly one of the following possibilities holds.*

(i) $d^2 - \sum_{i=1}^k m_i^2 = -1$ *and* $m_{k-1} - m_k = 1$.

(ii) $d^2 - \sum_{i=1}^k m_i^2 - m_k = -2$ *and* $m_k = 2$, $m_{k-1} \neq 3$.

(iii) $d^2 - \sum_{i=1}^k m_i^2 - m_k \geq -1$.

Proof. We first prove the direction ($\Rightarrow$), i.e. we suppose that there exists an open embedding $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$ and show that we are in one of the cases $(i), (ii)$, or (iii). It follows by Lemma 2.2.4 that there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \dots, p_n$ and exceptional curves $E_1, \dots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. Then $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is the exceptional locus of η, being the support of an SNC-divisor that has a tree structure. The minimal resolution of singularities of C is $\pi_1 \circ \ldots \circ \pi_k$. The curve C_k intersects E_k and since C is unicuspidal this intersection is in a single point with multiplicity m_k (see Figure 2.1 on the left). Since π is a (-1)-tower resolution of C, the self-intersection of C_k is ≥ -1.

Suppose that $(C_k)^2 = -1$. Then π has no other base-point, as this point would lie on $E_k \setminus C_k$, and this would imply that C_n and E_k do not intersect transversally in X. Moreover, the configuration of the curves $E_1, \ldots, E_{k-1}, C_k$ is connected, i.e. C_k transversally intersects exactly one curve $E \in \{E_1, \ldots, E_{k-1}\}$ in its interesection point with E_k. We observe that C_k intersects $E_1 \cup \ldots \cup E_{k-1}$ only in the curve E, and thus $E_1 \cup \ldots \cup E_{k-1}$ is connected. But this implies that E_k intersects only one curve from $E_1, \ldots, E_{k-1}$, and thus $E = E_{k-1}$. Now it follows from the fact that $E_{k-1} \cdot C_k = 1$ and from Lemma 2.4.8 that $m_{k-1} - 1 = m_k$ and we are thus in case (i).

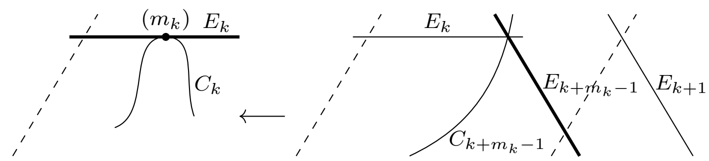


Figure 2.1: Blow-up of the points $p_k, \ldots, p_{k+m_k-2}$.

Suppose now that $(C_k)^2 \neq -1$. Then π has a base-point on $E_k \cap C_k$. Thus $k < n$ and the union of the curves $E_1, \ldots, E_{n-1}, C_n$ is SNC in X. It follows that the base-point p_{i+1} is the intersection point between C_i and E_k for $i = k, \ldots, k + m_k - 2$. The configuration of curves in X_{k+m_k-1} is shown in the diagram on the right in Figure 2.1. The self-intersection of C_{k+m_k-1} is then $d^2 - \sum_{i=1}^{k} m_i^2 - (m_k - 1)$, and this number is ≥ -1, since π is a (-1)-tower resolution of C.

Assume that $d^2 - \sum_{i=1}^{k} m_i^2 - m_k = -2$, i.e. there is no base-point on C_{k+m_k-1}. But this means that there is no more base-point at all, since there is a triple intersection between E_k, E_{k+m_k-1} and C_{k+m_k-1}, which would violate the SNC structure of the exceptional divisor of η if E_{k+m_k-1} was not the last exceptional curve of π. Since the union of $E_1, \ldots, E_{k+m_k-2}, C_{k+m_k-1}$ is connected, it follows that $m_k = 2$ (see Figure 2.1). It also follows that the union of $E_1, \ldots, E_{k+m_k-1}$ is connected and hence C_k does not intersect any other exceptional curve apart from E_k in X_k. It then follows from Lemma 2.4.8 that $m_{k-1} - m_k \neq 1$ and thus $m_{k-1} \neq 3$. We are thus in case (ii).

The last remaining case is when $d^2 - \sum_{i=1}^{k} m_i^2 - m_k \neq -2$, but then this expression is ≥ -1 and we are in case (iii). We observe moreover that the cases (i), (ii), (iii) are mutually exclusive.

We now prove the direction $(\Leftarrow)$. In each case we first blow up the k singular points of C (with exceptional curves $E_1, \ldots, E_k$). In case (i), this yields the resolution in Figure 2.2. By the symmetry of the configuration, there exists a morphism from this surface to $\mathbb{P}^2$ contracting $C_k, E_{k-1}, \ldots, E_1$.

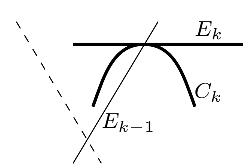


Figure 2.2: Case (i).

In case (ii), we also blow up the the intersection point of C_k and E_k and obtain the diagram in Figure 2.3. Again, by the symmetry of the configuration, there exists a morphism to $\mathbb{P}^2$ that contracts $C_{k+1}, E_k, \dots, E_1$.

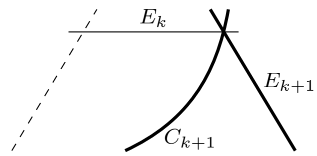


Figure 2.3: Case (ii).

Finally, in case (iii), we blow up m_k points, with exceptional curves $E_{k+1}, \dots, E_{k+m_k}$, all proximate to the intersection point between C_k and E_k. Then C_{k+m_k} intersects E_{k+m_k} transversally and the self-intersection of C_{k+m_k} is ≥ -1. We can thus continue to blow up points until we have a (-1)-tower resolution of C, where C_{n-1} intersects E_{n-1} tranversally. We then blow up any point on E_{n-1} that does not lie on C_{n-1} or any other exceptional curve. We then obtain the configuration in Figure 2.4. By the symmetry of this configuration, there exists a morphism to $\mathbb{P}^2$ by contracting the curves $C_n, E_{n-1}, \dots, E_1$.

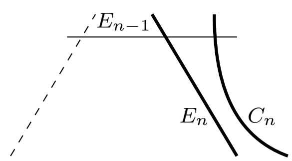


Figure 2.4: Case (iii).

$\square$

Remark 2.4.17. Lemma 2.4.16 allows us to determine for a unicuspidal curve $C \subset \mathbb{P}^2$, whether there exists an open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$, simply by looking at the multiplicity sequence of C.

Corollary 2.4.18. *Let $C \subset \mathbb{P}^2$ be an irreducible unicuspidal curve of degree ≤ 8 and let $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ be an isomorphism, where $D \subset \mathbb{P}^2$ is some curve. Then C and D are projectively equivalent.*

Proof. If φ extends to an automorphism of $\mathbb{P}^2$, the claim is trivial. If not, then C has one of the multiplicity sequences in Table 2.1, by Corollary 2.4.5. In the case of the multiplicity sequence $(2_{(6)})$, the claim follows from Corollary 2.4.14. For the multiplicity sequences $(3, 2_{(7)})$, $(3_{(4)}, 2_{(3)})$, $(4, 3_{(5)})$, $(4, 3_{(4)}, 2_{(3)})$ the claim follows from Lemma 2.4.15 and for $(3_{(7)})$ from Lemma 2.4.16, since $8^2 - 7 \cdot 3^2 - 3 = -2 < -1$. In all other cases, there exists a very tangent line through the proper singular point of C by Lemma 2.4.11. Then the claim follows from Theorem 1. $\square$

2.4.3 Some special multiplicity sequences

In this section we present some extension results for isomorphisms between curves that are not unicuspidal and have a multiplicity sequence of a special form. Together with the previous results this will lead to the proof of Theorem 2.

Proposition 2.4.19. *Let C be an irreducible rational curve of degree $d \geq 4$ and multiplicity sequence $(m_{(k)})$, where $m \geq 2$ and $k \geq 1$, and let $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ be an open embedding that does not extend to an automorphism of $\mathbb{P}^2$. If C is not unicuspidal, then $C \setminus \operatorname{Sing}(C)$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and C has either degree 8 with multiplicity sequence $(3_{(7)})$ or degree 16 with multiplicity sequence $(6_{(7)})$.*

Proof. Suppose that C is not unicuspidal. By Lemma 2.2.4, there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \ldots, p_n$ and exceptional curves $E_1, \ldots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. Then $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is the exceptional locus of η, being the support of an SNC-divisor that has a tree structure. The composition $\pi_k \circ \ldots \circ \pi_1$ is the minimal resolution of singularities of C. By Lemma 2.4.8 we obtain that in the surface X_k, we have the intersection numbers $C_k \cdot E_i = 0$, for $i = 1, \ldots, k-1$, and $C_k \cdot E_k = m$. Since $E_1 \cup \ldots \cup E_{k-1} \cup C_k$ is not connected, we know that $n > k$, hence more points are blown up to obtain the (-1)-tower resolution π. Since we assumed C not to be unicuspidal, the curves C_k and E_k intersect in at least two points in X_k. If C_k and E_k intersect in at least 3 points, then it follows that C_n and E_k intersect in at least two points in X, which is not possible by the tree structure of $E_1 \cup \ldots \cup E_{n-1} \cup C_n$. It thus follows that C_k and E_k intersect in exactly two points and hence $C \setminus \operatorname{Sing}(C) = C \setminus \{p_1\} \simeq \mathbb{A}^1 \setminus \{0\}$. Moreover, it follows (again by the tree structure) that C_n intersects E_k transversally in one point in the surface X, thus C_k intersects E_k in one point transversally and in the point p_{k+1} with intersection multiplicity $m-1$ in X_k. The configuration of curves is illustrated in the diagram on the left in Figure 2.5, where the dashed lines represent chains of (-2)-curves. Again by the fact that C_n and E_k intersect only in one point, the base-points of the blow-ups $\pi_{k+1}, \ldots, \pi_{k+m-1}$ are proximate to p_{k+1} (i.e. all lie on E_k) and we obtain $E_k^2 = -m$ in X_{k+m-1}, as illustrated in the diagram on the right of Figure 2.5. We denote the self-intersection of C_{k+m-1} by δ and thus have $\delta = d^2 - km^2 - (m-1)$. Since π is a (-1)-tower resolution of C we have $\delta \geq -1$.

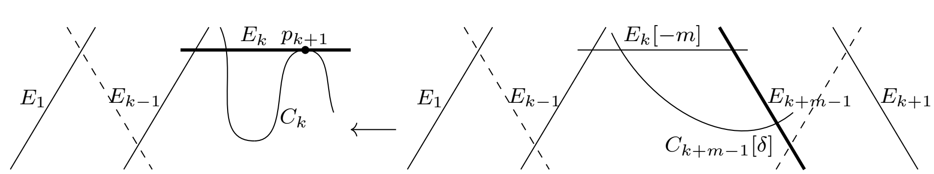


Figure 2.5: Minimal SNC-resolution of C.

To simplify the later cases we first prove the following.

Claim (1). If $k = 1$, we reach a contradiction.

Proof of Claim (1). Since the degree of C is $d \geq 4$, we obtain $m = d - 1 \geq 3$ by the rationality of C and the genus-degree formula and hence we have $\delta = d + 1 \geq 5$. Since C_n has self-intersection -1, the base-point p_{i+1} is the unique intersection point between C_i and E_i in X_i for $i = m, \ldots, m + 1 + \delta$, as shown in Figure 2.6.

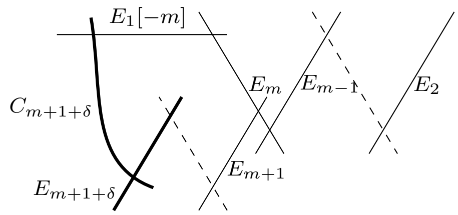


Figure 2.6: Case (m).

If π has another base-point in $X_{m+1+\delta}$, then it lies on $E_{m+1+\delta} \setminus C_{m+1+\delta}$. We know that $\delta \geq 5$ and thus the curves E_m and E_{m+1} have self-intersection -2 in X. Moreover, the curves $E_1, \ldots, E_{n-1}, C_n$ have a tree structure in X, thus C_n and E_m are uniquely connected via E_1 in this tree. The map η successively contracts the curves in this tree, starting with C_n. The chain of curves that connects C_n to E_{m-1}, respectively E_{m+1}, contains E_m, thus η contracts E_1 before E_{m-1} and E_{m+1}. But this is not possible since after contracting E_m, the images of both E_{m-1} and E_{m+1} have self-intersection -1. We thus get a contradiction and conclude that $k \geq 2$.

■

In the sequel, we separately study the cases $\delta \geq 1$, $\delta = 0$, and $\delta = -1$.

Claim (2). If $\delta \geq 1$, we reach a contradiction.

Proof of Claim (2). Since π is a (-1)-tower resolution of C the base-point p_{i+1} is the unique intersection point between C_i and E_i in X_i for $i = k+m-1, \ldots, k+m+\delta$ (see Figure 2.7).

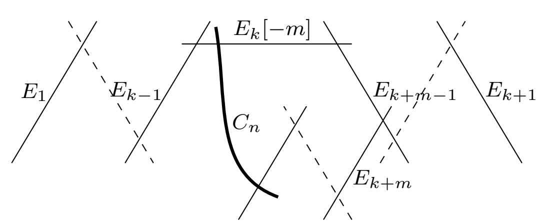


Figure 2.7: Case $\delta \geq 1$.

Since $\delta \geq 1$, it follows that the curve E_{k+m-1} has self-intersection -2 in X. Moreover, we know that $k \geq 2$ (i.e. there is a (-2)-curve E_{k-1} as pictured in Figure 2.7). The map η contracts the curves E_{k-1} and E_{k+m-1} after E_k, since in the tree of curves

$E_1, \dots, E_{n-1}, C_n$ the curves C_n and E_{k-1}, respectively E_{k+m-1}, are connected via E_k. But after contracting E_k, the self-intersections of the images of E_{k-1} and E_{k+m-1} are both -1, which is not possible. We thus conclude that $\delta \geq 1$ is not possible. ■

Claim (3). If $\delta = 0$, we reach a contradiction.

Proof of Claim (3). Since $\delta = 0$, the base-point of the next blow-up π_{k+m} is the unique intersection point between C_{k+m-1} and E_{k+m-1} and we obtain the configuration of curves in the left part of Figure 2.8.

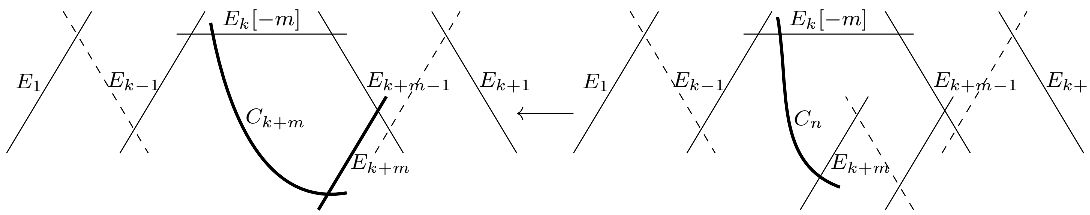


Figure 2.8: Case $\delta = 0$.

In the surface X, the curves $E_{k+m}, \dots, E_n$ all lie in a chain (not necessarily in this order) between C_n and E_{k+m-1}, i.e. the base-points always lie on the intersection points of the chain between C_n and E_{k+m-1}, as otherwise there would be a loop in the configuration of the curves $E_1, \dots, E_{n-1}, C_n$ in X (see the right part of Figure 2.8). Moreover, E_{k+m} intersects C_n in this chain. The map η first contracts C_n and after this contraction the image of E_k has self-intersection $-m+1$. It follows that in the chain of curves between C_n and E_{k+m-1}, after C_n there is a chain of (-2)-curves of length $m-2$, such that the image of E_k is -1, after this chain is contracted. This means that the base-points p_{i+1} for $i = k+m, \dots, k+m+(m-3)$ all lie on E_{k+m-1}. Denote the next curve in the chain after the $m-2$ (-2)-curves by E. After C_n and the chain of $m-2$ (-2)-curves are contracted, the images of E_k and E intersect. Moreover, the self-intersection of E_k is -1 in this surface and thus η then contracts $E_k, \dots, E_1$. Since we assume $k \geq 2$, it follows that the image of E is tangent to E_{k+m-1}. But this means that E is not contracted by η and must in fact be $E_n = E_{k+m+(m-2)}$. Since the base-points $p_{k+m+1}, \dots, p_{k+m+(m-2)}$ all lie on E_{k+m-1}, the self-intersection of E_{k+m-1} in X is $-m$. We observe that after η contracts C_n and the chain $E_k, \dots, E_1$ the image of E_{k+m-1} has self-intersection $-m+k$, which has to be equal to -1, and thus $k = m-1$. From the condition $\delta = 0$ and the genus-degree formula we obtain the equations

$$0 = d^2 - (m-1)m^2 - m + 1,$$
$$0 = d^2 - 3d + 2 - (m-1)m^2.$$

Subtracting the second equation from the first then yields $3d - m^2 - 1 = 0$. We can then substitute $d = \frac{m^2+1}{3}$ in the first equation and obtain

$$0 = \frac{(m^2+1)^2}{9} - (m-1)m^2 - (m-1) = (m^2+1)\left(\frac{m^2+1}{9} - m + 1\right),$$

which has no integer solutions in m. We conclude that $\delta = 0$ is not possible. ■

Claim (4). If $\delta = -1$, then C is of degree 8 or 16 with multiplicity sequence $(3_{(7)})$ or $(6_{(7)})$ respectively.

Proof of Claim (4). We already have a (-1)-tower resolution of C in this case (see Figure 2.9). We observe that blowing up the intersection point between E_k and E_{k+m-1} yields a symmetric diagram and thus there exists a morphism $X \to \mathbb{P}^2$ whose contracted locus is exactly $E_1 \cup \ldots \cup E_{k+m-1} \cup C_{k+m}$.

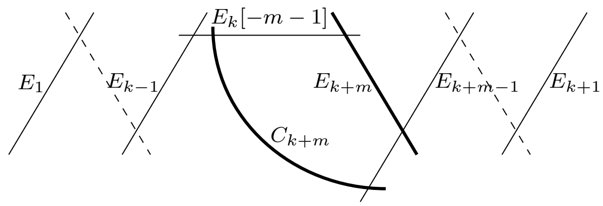


Figure 2.9: Case $\delta = -1$.

The condition $\delta = -1$ and the genus-degree formula give us the following equations for the values of d, m, k:

$$0 = d^2 - km^2 - m + 2,$$
$$0 = d^2 - 3d + 2 - km^2 - km.$$

We see from the first equation that any integer factor of d and m also divides 2. Hence the greatest common divisor of d and m is 1 or 2. Subtracting the equations yields $3d - m - km = 0$, from which we conclude that m divides $3d$. It thus follows that m divides 6. Next, we replace $k = \frac{3d-m}{m}$ in the first equation above and get $d^2 - 3dm - m^2 - m + 2 = 0$. We then check for natural solutions in d for $m \in \{2, 3, 6\}$ and find $(d, m) = (8, 3)$ or $(16, 6)$ (both with $k = 7$) as the only possibilities. ■

This concludes the proof of Proposition 2.4.19. □

Remark 2.4.20. The assumption that $d = \deg(C) \geq 4$ in Proposition 2.4.19 is necessary since the the complement of a nodal cubic has non-extendable automorphisms (see Remark 2.4.7).

Corollary 2.4.21. *Let $C \subset \mathbb{P}^2$ be an irreducible rational curve with one of the multiplicity sequences $(2_{(3)})$, (3), (4), $(2_{(6)})$, (5), (6), or (7). If C is not unicuspidal, then any open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.*

Proof. This is a direct consequence of Proposition 2.4.19. □

Proposition 2.4.22. *Let $C \subset \mathbb{P}^2$ be an irreducible rational curve of degree d and multiplicity sequence $\left(m_{(k)}, (m-1)_{(l)}\right)$, where $m \geq 3$ and $k, l \geq 1$ and let $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ be an open embedding that does not extend to an automorphism of $\mathbb{P}^2$. Then either C is unicuspidal or of degree 6 with multiplicity sequence $(3, 2_{(7)})$ or of degree 13 with multiplicity sequence $(5_{(6)}, 4)$.*

Proof. We suppose that C is not unicuspidal. Since φ does not extend to an automorphism of $\mathbb{P}^2$, it follows by Lemma 2.2.4 that there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \ldots, p_n$ and exceptional curves $E_1, \ldots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. Then $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is the exceptional locus of η, being the support of an SNC-divisor on X that has a tree structure. The composition $\pi_{k+l} \circ \ldots \circ \pi_1$ is the minimal resolution of the singularities of C. By Lemma 2.4.8 we obtain that in the surface X_{k+l}, we have the intersection numbers $C_{k+l} \cdot E_k = 1$ and $C_{k+l} \cdot E_i = 0$ for $i = 1, \ldots, k-1$ and $i = k+1, \ldots, k+l-1$.

Claim (1). If $k \geq 2$ and $l \geq 2$, we reach a contradiction.

Proof of Claim (1). By Lemma 2.4.8 we have $C_{k+l} \cdot E_{k+l} = m - 1$. The configuration is shown in Figure 2.10, where the dashed lines represent chains of (-2)-curves.

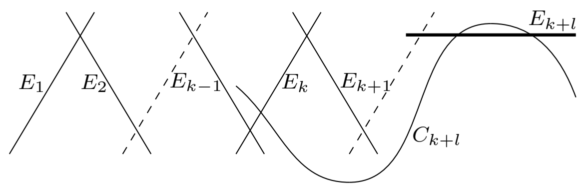


Figure 2.10: Minimal resolution of singularities of C.

If π has a base-point in X_{k+l}, then it lies on the intersection with C_{k+l} and E_{k+l}, otherwise there would be a loop formed by $E_k, \ldots, E_{k+l}$ and C_n in X_n, which is not possible by the tree structure of the curves $E_1, \ldots, E_{n-1}, C_n$. Since E_{k+l} does not intersect the (-2)-curves E_{k-1}, E_k, and E_{k+1}, it follows that their self-intersections in X are also -2. We observe that the map η contracts the curve E_k before E_{k-1} and E_{k+1}, since C_n and E_{k-1}, respectively E_{k+1}, are connected via E_k in the graph of the curves $E_1, \ldots, E_{n-1}, C_n$. But after contracting E_k, the images of E_{k-1} and E_{k+1} both have self-intersection -1, which is a contradiction since η is a (-1)-tower resolution. ∎

In the sequel, we separately look at the more involved cases where $k = 1$ or $l = 1$ (parts (A) and (B) below).

(A) We assume that $k = 1$.

Claim (A.1). If $(C_{l+1})^2 = -1$, then C has degree 6 and multiplicity sequence $(3, 2_{(7)})$.

Proof of Claim (A.1). By Lemma 2.4.8 we have $C_{l+1} \cdot E_{l+1} = m - 1$. If C_{l+1} has self-intersection -1, then by the symmetry of the configuration (see Figure 2.11), there exists a morphism $X \to \mathbb{P}^2$ whose contracted locus is $E_1 \cup \ldots \cup E_l \cup C_{l+1}$.

From $(C_{l+1})^2 = -1$ and the genus-degree formula we obtain the following two identities:

$$0 = d^2 - m^2 - l(m-1)^2 + 1,$$
$$0 = d^2 - 3d + 2 - m(m-1) - l(m-1)(m-2).$$

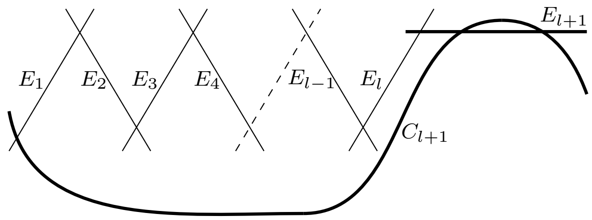


Figure 2.11: Case $k = 1$, $(C_{l+1})^2 = -1$.

Subtracting the second equation from the first yields $3d - 1 - m - l(m-1) = 0$. We then substitute $l(m-1) = 3d - 1 - m$ in the first equation and obtain $d^2 = 3d(m-1)$ and thus $d = 3(m-1)$. Finally, we get

$$0 = 3d - 1 - m - l(m-1) = (9-l)(m-1) - (m+1)$$

and for positive integer values this equation is only satisfied with $m = 2$ and $l = 7$ since $1 < 9 - l = \frac{m+1}{m-1} < 2$, for $m \geq 3$. This leads to the multiplicity sequence $(3, 2_{(7)})$ in degree 6. The corresponding resolution diagram is shown in Figure 2.11, where the dashed line represents one (-2)-curve. ∎

We suppose from now on that we are not in the case of the multiplicity sequence $(3, 2_{(7)})$. We then have $(C_{l+1})^2 > -1$. This implies that π has a base-point in the intersection of C_{l+1} with E_{l+1}. In fact, the curves C_n and E_{l+1} do not intersect in X, otherwise there would be a loop in the graph of the curves $E_1, \ldots, E_{n-1}, C_n$. Thus C_{l+1} and E_{l+1} intersect in a single point in X_{l+1}, and hence the intersection multiplicity is $m-1$. We have thus the configuration of curves shown in the left part of Figure 2.12.

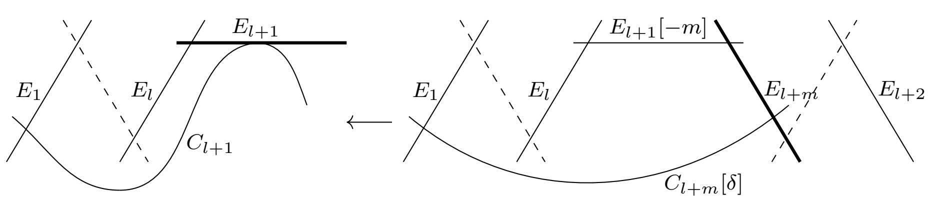


Figure 2.12: Minimal SNC-resolution of C for $k = 1$.

Since C_n and E_{l+1} do not intersect in X, it follows that the base-point p_{i+1} for $i = l+1, \ldots, l+m-1$ is the unique intersection point between C_i and E_i, which also lies on E_{l+1}. The configuration of curves in X_{l+m} is shown in the right part of Figure 2.12. We denote the self-intersection number of C_{l+m} by δ and this number is equal to $d^2 - m^2 - l(m-1)^2 - (m-1)$. Since π is a (-1)-tower resolution we have that $\delta \geq -1$.

Claim (A.2). If $\delta = -1$, we reach a contradiction.

Proof of Claim (A.2). From $\delta = -1$ and the genus-degree formula we obtain

$$0 = d^2 - m^2 - l(m-1)^2 - m + 2,$$
$$0 = d^2 - 3d + 2 - m(m-1) - l(m-1)(m-2).$$

Subtracting the second equation from the first yields $3d - 2m - l(m-1) = 0$. We then replace $l = \frac{3d-2m}{m-1}$ in the first equation and obtain the identity

$$0 = d^2 - m^2 - (3d-2m)(m-1) - m + 2 = d^2 - (m-1)(3d - m + 2).$$

It follows that $m-1$ divides d^2. Let p be a prime number that divides $m-1$. Then p divides d^2 and thus also d. From the equality $l(m-1) = 3d - 2m$ it follows that p divides $2m$. Since $m-1$ and m are coprime, it follows that $p = 2$. We can then write $m - 1 = 2^r$ for some $r \geq 1$. We observe that 2^r divides d^2. Moreover, 2^r divides $3d - 2(2^r + 1)$ and thus also $3d - 2$. But then 2^r divides $d^2 - 3d + 2 = (d-1)(d-2)$. Since d is even, it follows that 2^r divides $(d-2)$. Since 2^r divides $3d - 2 = (d-2) + 2d$, it follows that 2^{r-1} divides d, but also $d - 2$, and thus r must be 1 or 2. Using these values for r, it is easy to check that the equations above have no integral solutions for d. We can thus conclude that $\delta \neq -1$. ■

Claim (A.3). If $\delta = 0$, we reach a contradiction.

Proof of Claim (A.3). The curves C_{l+m} and E_{l+m} have a unique intersection point, hence this is the base-point p_{l+m+1}. After blowing up p_{l+m+1} we obtain a (-1)-tower resolution of C (see the left part of Figure 2.13).

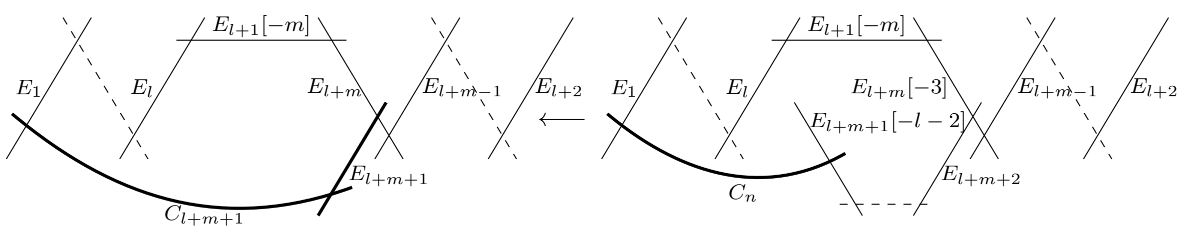


Figure 2.13: Case $k = 1$, $\delta = 0$.

In the surface X, the curves $E_{l+m+1}, \ldots, E_n$ all lie in a chain (not necessarily in this order) between C_n and E_{l+m}, otherwise there would be a loop in the configuration of the curves $E_1, \ldots, E_{n-1}, C_n$. The curve E_{l+m+1} intersects C_n in this chain. The map η contracts first C_n and then the chain $E_1, \ldots, E_l$. The self-intersection of the image of E_{l+m+1} after those contractions increases by $l+1$. Since E_{l+1} is not a (-1)-curve after those contractions (as $m \geq 3$), it follows that E_{l+m+1} is a (-1)-curve in this surface. This implies that in X the curve E_{l+m+1} has self-intersection $-(l+2)$. This means that the base-points $p_{l+m+2}, \ldots, p_{l+m+(l+2)}$ must lie on the strict transform of E_{l+m+1}. Assume first that $l \geq 2$. Then E_{l+m+2} has self-intersection -2 in X. The map η contracts E_{l+m} before the (-2)-curves E_{l+m-1} and E_{l+m+2}, but this is not possible, as the images of both E_{l+m-1} and E_{l+m+2} are (-1)-curves, after contracting E_{l+m}. Hence l must be 1 and the multiplicity sequence of C is then $(m, m-1)$. The condition $\delta = 0$ and the genus-degree formula give

$$\begin{aligned} 0 &= d^2 - m^2 - (m-1)^2 - m + 1, \\ 0 &= d^2 - 3d + 2 - m(m-1) - (m-1)(m-2). \end{aligned}$$

Subtracting those equations yields the identity $3d = 3m$, which is not possible as $m < d$. We conclude that $\delta \neq 0$.

■

Claim (A.4). If $\delta = 1$, we reach a contradiction.

Proof of Claim (A.4). Again, the base-point p_{l+m+1} is the intersection point between E_{l+m} and C_{l+m} and p_{l+m+2} is the intersection point between E_{l+m+1} and C_{l+m+1}. After blowing up p_{l+m+1} and p_{l+m+2} we have a (-1)-tower resolution of C (see the left part of Figure 2.14).

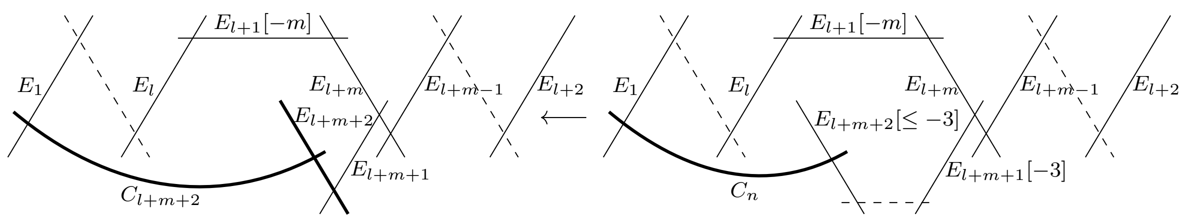


Figure 2.14: Case $k = 1$, $\delta = 1$.

Suppose that this resolution is π. Then η contracts E_{l+m} before the (-2)-curves E_{l+m-1} and E_{l+m+1}, but this is not possible. Hence π has another base-point, which must be the intersection point between E_{l+m+1} and E_{l+m+2}, otherwise there would be loop in the resolution in X. Now in X_{l+m+3}, the curve C_{l+m+3} intersects the (-2)-curves E_1 and E_{l+m+2}. Thus there is another base-point of π, which is the intersection point between E_{l+m+2} and E_{l+m+3}. But this implies that E_{l+m+1} has self-intersection -3 in X (see the right part of Figure 2.14). We know that η contracts E_{l+m} before E_{l+m-1} and E_{l+m+1}. After contracting E_{l+m}, the self-intersections of the images of E_{l+m-1} and E_{l+m+1} are -1 and -2 respectively. But then E_{l+m-1} intersects no other (-2)-curve, so we have $E_{l+m-1} = E_{l+2}$ and hence $m = 3$. The multiplicity sequence of C is thus of the form $(3, 2_{(l)})$. Using $\delta = 1$ and the genus degree formula, we obtain

$$\begin{aligned} 0 &= d^2 - 4l - 10, \\ 0 &= d^2 - 3d - 2l - 4. \end{aligned}$$

Subtracting those equations and rearranging terms, we obtain $l = \frac{3d-6}{2}$, which we can substitute in the first equation and get $d^2 - 6d + 2 = 0$, which has no integer solution in d. Thus $\delta = 1$ is not possible.

■

Claim (A.5). If $\delta \geq 2$, we reach a contradiction.

Proof of Claim (A.5). For $i = l+m, \ldots, l+m+\delta$, the base-point p_{i+1} is then the unique intersection point between C_i and E_i. As $\delta \geq 2$, this means that E_{l+m+1} has self-intersection -2 in X (see Figure 2.15). But this leads to a contradiction, since η contracts E_{l+m} before the (-2)-curves E_{l+m-1} and E_{l+m+1}, whose images both have self-intersection -1, after E_{l+m} is contracted.

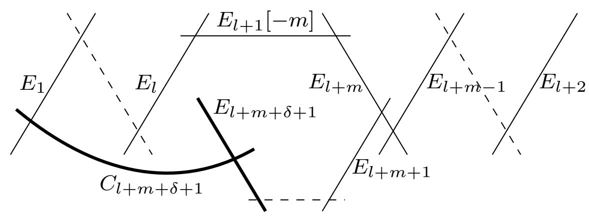


Figure 2.15: Case $k = 1$, $\delta \geq 2$.

■

This conclude the case $k = 1$.

(B) Assume now that $l = 1$, as shown in Figure 2.16. We can also assume that $k \geq 2$, since we have already considered the case $k = 1$. If C_{k+1} has self-intersection -1, then by the symmetry of the configuration, there exists a morphism $X \to \mathbb{P}^2$ whose contracted locus is $E_1 \cup \ldots \cup E_{n-1} \cup C_n$.

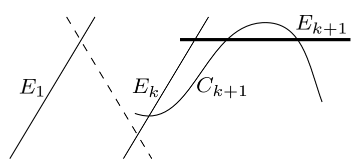


Figure 2.16: Minimal resolution of singularities for $l = 1$.

From $(C_{k+1})^2 = -1$ and the genus-degree formula we get the following two identities

$$
\begin{aligned}
0 &= d^2 - km^2 - (m-1)^2 + 1, \\
0 &= d^2 - 3d + 2 - km(m-1) - (m-1)(m-1).
\end{aligned}
$$

Subtracting the second identity from the first yields $3d - 1 - km - (m-1) = 0$. We then substitute $km = 3d - 1 - (m-1)$ in the first equation and obtain $d^2 = m(3d-2)$. Let p be a prime number that divides $3d-2$ and thus also d. But then $p = 2$ and hence we can write $3d - 2 = 2^r$ for some natural number r. It then follows that $m = \frac{(2^r+2)^2}{9 \cdot 2^r}$, in particular 2^r divides $2^{2r} + 4 \cdot 2^r + 4$ and thus $r = 1$ or $r = 2$. If $r = 1$, then $d = \frac{4}{3}$, which is absurd. If $r = 2$, then $d = 2$ and $m = 1$, which is excluded by hypothesis.

We thus know that $(C_{k+1})^2 > -1$ and hence π has a base-point on E_{k+1} that also lies on C_{k+1}. Since C is not unicuspidal, the curves C_{k+1} and E_{k+1} intersect in at least two points.

There are now two possibilities: either C_{k+1} passes through the intersection point between E_k and E_{k+1}, or it does not. We will look at those cases separately (parts (i) and (ii) below).

(i) We suppose that C_{k+1} passes through the intersection point between E_k and E_{k+1}. Then this point is the next base-point of π, since there can be no triple intersections in the tree of the curves $E_1, \ldots, E_{n-1}, C_n$ in X. Moreover the intersection multiplicity between C_{k+1} and E_{k+1} at p_{k+2} is $m-2$ as C_n and E_{k+1} intersect transversally in X, see the configuration on the left in Figure 2.17.

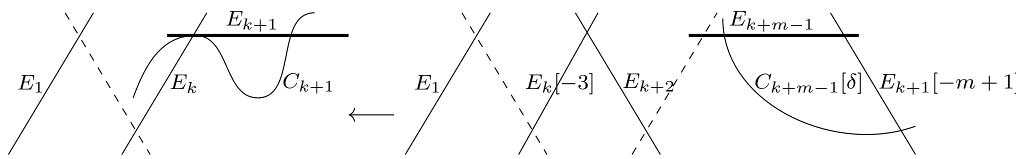


Figure 2.17: Blow-up of $p_{k+2}, \ldots, p_{k+m-1}$.

It follows that the base-point p_{i+1} is the intersection point between E_{k+1} and E_i for $i = k+1, \ldots, k+m-2$. We then denote by δ the self-intersection of C_{k+m-1} in X_{k+m-1}, see the configuration on the right in Figure 2.17. We have $\delta = d^2 - km^2 - (m-1)^2 - (m-2)$ and $\delta \geq -1$, since π is a (-1)-tower resolution.

Claim (B.i.1). If $\delta = -1$, we reach a contradiction.

Proof of Claim (B.i.1). From $\delta = -1$ and the genus-degree formula we obtain

$$0 = d^2 - km^2 - (m-1)^2 - m + 3,$$
$$0 = d^2 - 3d + 2 - km(m-1) - (m-1)(m-2).$$

Subtracting those identities yields $3d - km - 2m + 2 = 0$. Thus the greatest common divisor of d and m divides 2. We then substitute $k = \frac{3d-2m+2}{m}$ in the first equation and obtain $d^2 - 3dm + m^2 - m + 2 = 0$. Let p be any prime number that divides m. Then p divides $3d+2$ and also d^2+2. But then p also divides $d^2 - 3d = d(d-3)$. Assume that p does not divide d, then p divides $d-3$. Then p divides $3d+2-3(d-3) = 11$. On the other hand p also divides $(d^2+2) - (d-3)^2 - 3(d-3) = 2$ and thus we have a contradiction. It follows that p divides d and hence $p = 2$. Dividing the equation above by 2 yields

$$d\frac{d}{2} - 3d\frac{m}{2} + m\frac{m}{2} - \frac{m}{2} + 1 = 0.$$

We conclude that $\frac{m}{2}$ must be odd. Since m is a power of 2 it then follows that $m = 2$. We hence obtain the equation $d^2 - 6d + 4 = 0$, which has no integer solution in d. We conclude that $\delta = -1$ is not possible. ∎

Claim (B.i.2). If $\delta = 0$, then C has degree 13 and multiplicity sequence $(5_{(6)}, 4)$.

Proof of Claim (B.i.2). From $\delta = 0$ and the genus-degree formula we obtain

$$
\begin{aligned}
0 &= d^2 - km^2 - (m-1)^2 - m + 2,\\
0 &= d^2 - 3d + 2 - km(m-1) - (m-1)(m-2).
\end{aligned}
$$

Subtracting those identities yields $3d - km - 2m + 1 = 0$. We thus see that d and m are coprime and that m divides $3d+1$. We substitue $k = \frac{3d-2m+1}{m}$ in the first equation and obtain $d^2 - 3dm + m^2 + 1 = 0$. From this we see that m divides d^2+1. But then m also divides $(d^2+1) - (3d+1) = d(d-3)$. Since d and m are coprime, m divides $d-3$. On the other hand, m also divides $(d^2+1) + (3d+1) = (d+1)(d+2)$. Let p be a prime number that divides m. Then p divides $d-3$ and either $d+1$ or $d+2$, but not both since they are coprime. Thus p must be either 2 or 5. Assume moreover that p^2 divides m. Then p^2 also divides d^2+1 and $3d+1$. Since p divides $d-3$, it follows that p^2 divides $(d-3)^2 = d^2 - 6d + 9 = d^2 + 1 - 2(3d+1) + 10$. But then p^2 divides 10, which is not possible. We conclude that $m \in \{5, 10\}$ (since $m \geq 3$). We then check for integer solutions for d in the equation $d^2 - 3dm + m^2 + 1 = 0$ for those values of m and find $(d, m) = (13, 5)$ as the only possibility. For a diagram of a resolution of such an isomorphism see Remark 2.4.23. We assume from now on that we are not in this case. ■

Claim (B.i.3). If $\delta = 1$, we reach a contradiction.

Proof of Claim (B.i.3). From $\delta = 1$ and the genus-degree formula we get the equations

$$
\begin{aligned}
0 &= d^2 - km^2 - (m-1)^2 - m + 1,\\
0 &= d^2 - 3d + 2 - km(m-1) - (m-1)(m-1).
\end{aligned}
$$

Subtracting those identities yields $3d - km - 2m = 0$. We then substitute $k = \frac{3d-2m}{m}$ in the first equation and obtain $d^2 = m(3d+m+1)$. Let p be any prime number that divides m. But then p divides d^2 and thus also d. It then follows that p divides 1 and we have a contradiction. ■

Claim (B.i.4). If $\delta \geq 2$, we reach a contradiction.

Proof of Claim (B.i.4). Since π is a (-1)-tower resolution of C, the base-point p_{i+1} is the unique intersection point between C_i and E_i, for $i = k+m-1, \ldots, k+m+\delta-1$. The configuration after those blow-ups is shown in Figure 2.18. Since no more base-point of π can lie on E_{k+m}, its strict transform in X has self-intersection -2. If $m > 3$, then E_{k+m-1} intersects the two (-2)-curves E_{k+m-2} and E_{k+m} in X. But η contracts E_{k+m-1} before those two curves and thus this situation is not possible and we have $m = 3$. Since $d < 3m = 9$ by Lemma 2.4.4, the multiplicity sequence of C is in Table 2.1 and can

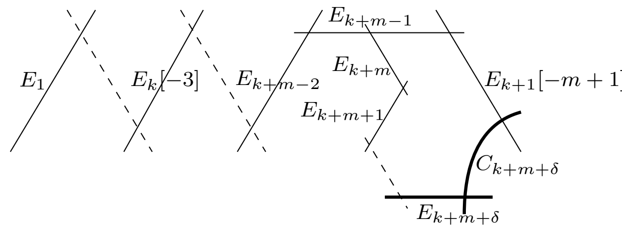


Figure 2.18: Case $l = 1$, $\delta \geq 2$.

only be $(3_{(3)}, 2)$ in degree 6. In this case $\delta = 5$. But this implies that E_{k+m+1} is also a (-2)-curve in X. We hence get a contradiction after η contracts E_{k+m-1}. Then the image of E_{k+m} intersects the (-2)-curves E_k and E_{k+m+1}. ∎

This concludes (i) of part (B).

(ii) Suppose now that C_{k+1} does not pass through the intersection point between E_k and E_{k+1}. Then C_{k+1} intersects E_{k+1} in one point with intersection multiplicity $m-1$, otherwise there would be a loop in the configuration of the curves $E_1, \ldots, E_{n-1}, C_n$. The configuration of curves in X_{k+1} is shown in the left part of Figure 2.19. Since C_n and E_{k+1} do not intersect in X, it follows that the base-point p_{i+1} for $i = k+1, \ldots, k+m-1$ is the unique intersection point between C_i and E_i, which also lies on E_{k+1}. The configuration of curves in X_{k+m} is shown in the right part of Figure 2.19. We denote the self-intersection of C_{k+m} by δ and this number is equal to $d^2 - km^2 - (m-1)^2 - (m-1)$. Since π is a (-1)-tower resolution of C, it follows that $\delta \geq -1$.

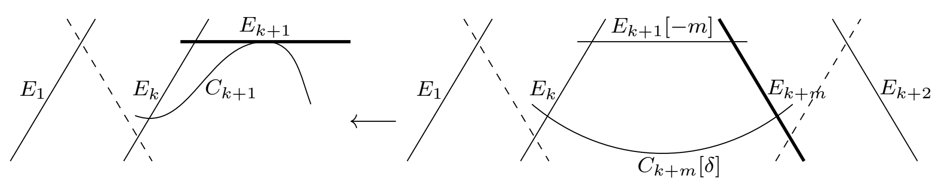


Figure 2.19: Blow-up of $p_{k+2}, \ldots, p_{k+m}$.

In the surface X, let $E \neq E_k$ in $\{E_1, \ldots, E_n\}$ be a curve that intersects C_n. We know that the map η first contracts C_n and then the chain $E_k, \ldots, E_1$. Since $k \geq 2$, it follows that the image of E is tangent to E_{k+1}, after those contractions. This implies that E is not contracted by η and thus $E = E_n$ is the last exceptional curve in the (-1)-tower resolution π. We now look what happens for different values of δ.

Claim (B.ii.1). If $\delta = -1$, we reach a contradiction.

Proof of Claim (B.ii.1). In this case we already have a (-1)-tower resolution of C. This resolution must be π, since there is no more base-point on C_{k+m} and C_n intersects E_n. But we observe that the curves $E_1, \ldots, E_{k+m-1}, C_{k+m}$ are not connected and thus cannot be the contracted locus of η. Hence $\delta = -1$ is not possible. ∎

Claim (B.ii.2). If $\delta = 0$, we reach a contradiction.

Proof of Claim (B.ii.2). The base-point p_{k+m+1} is the unique intersection point between C_{k+m} and E_{k+m}. After this blow-up, we have a (-1)-tower resolution of C, which must be π, for the same reason as in the case $\delta = -1$. The configuration of curves is shown in Figure 2.20.

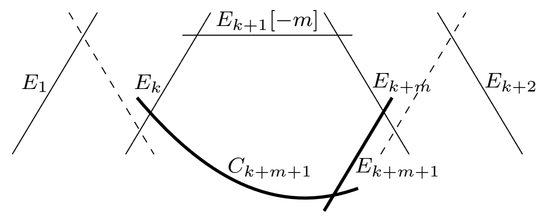


Figure 2.20: Case $l = 1$, $\delta = 0$.

The map η contracts first C_{k+m+1} and then the chain $E_k, \dots, E_1$. After those contractions the self-intersection of the image of E_{k+1} is $-m + k$, but must also be -1 and hence $k = m - 1$. From $\delta = 0$ we then obtain the equation $d^2 = m(m^2 - 1)$. Since m and $m^2 - 1$ are coprime, they are both squares, as $d > 0$. But if $m \geq 2$ is a square, then $m^2 - 1$ is not a square. Hence the only integer solutions to the equation are $(d, m) = (0, -1), (0, 0), (0, 1)$, and thus $\delta = 0$ is also not possible. ■

Claim (B.ii.3). If $\delta \geq 1$, we reach a contradiction.

Proof of Claim (B.ii.3). For $i = l + m, \dots, l + m + \delta$, the base-point p_{i+1} is the unique intersection point between C_i and E_i. After those blow-ups we have a (-1)-tower resolution of C, which has to be π for the same reason as in the previous cases. The configuration of curves is shown in Figure 2.21.

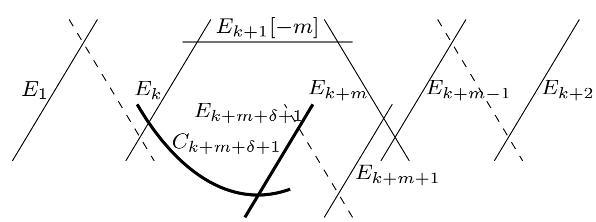


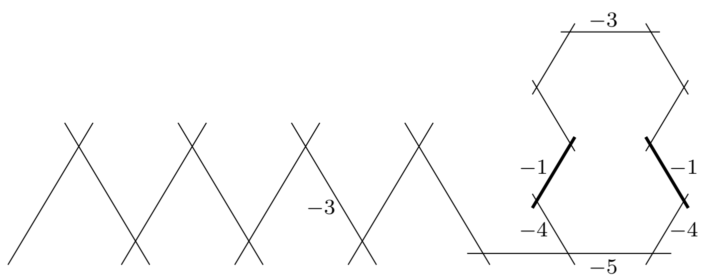
Figure 2.21: Case $l = 1$, $\delta \geq 1$.

Since $\delta \geq 1$, the curve E_{l+m+1} has self-intersection -2. But we know that η contracts E_{k+m} before the (-2)-curves E_{l+m-1} and E_{l+m+1}, which leads to a contradiction. ■

This concludes (ii) of part (B) and hence finishes the proof of Proposition 2.4.22.

□

Remark 2.4.23. Below we see the configuration of exceptional curves of a resolution of a non-extendable isomorphism between two curves of degree 13 with multiplicity sequence $(5_{(6)}, 4)$. All the unlabeled curves have self-intersection -2. Starting with either of the (-1)-curves, one can successively contract all curves in this configuration, except the other (-1)-curve. The image of this curve in $\mathbb{P}^2$, denoted C, then has self-intersection $169 = 13^2$. It remains to be verified whether such curves exist and whether new counterexamples to Conjecture 2.1.1 may arise in this way. We remark that $C \setminus \mathrm{Sing}(C) \simeq \mathbb{A}^1 \setminus \{0\}$ and thus C is different from the unicuspidal examples of degree 13 constructed in [Cos12].



Corollary 2.4.24. *Let $C \subset \mathbb{P}^2$ be an irreducible curve with one of the multiplicity sequences $(3, 2_{(3)})$, $(3_{(2)}, 2_{(4)})$, $(3_{(3)}, 2)$, $(3_{(4)}, 2_{(3)})$, $(4, 3_{(3)})$, $(4, 3_{(5)})$, $(4_{(2)}, 3_{(3)})$, or $(4_{(3)}, 3)$. Then either C is unicuspidal or any open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.*

Proof. This is a direct consequence of Proposition 2.4.22. □

Remark 2.4.25. Note that in Corollary 2.4.24, only curves with the multiplicity sequences $(3_{(3)}, 2)$ and $(4_{(3)}, 3)$ can be unicuspidal.

Proposition 2.4.26. *Let $C \subset \mathbb{P}^2$ be a rational curve of degree d and multiplicity sequence $(m_1, \ldots, m_k)$ such that all multiplicities are even and there exists $l < k$ such that $m_{l+1} = \ldots = m_k = 2$ and $m_j < m_{j+1} + \ldots + m_k$ for all $j \leq l$. Let $\varphi \colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ be an open emedding that does not extend to an automorphism of $\mathbb{P}^2$. Then C is unicuspidal.*

Proof. Suppose that C is not unicuspidal. By Proposition 2.4.19, we can assume that the multiplicity sequence of C is non-constant. By Lemma 2.2.4, there exists a (-1)-tower resolution $\pi \colon X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \ldots, p_n$ and exceptional curves $E_1, \ldots, E_n$, and a (-1)-tower resolution $\eta \colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. Then $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is the exceptional locus of η, being the support of an SNC-divisor that has a tree structure. The composition $\pi_1 \circ \ldots \circ \pi_k$ is the minimal resolution of singularities of C. For $i = 1, \ldots, k$, we obtain the following intersection numbers, by Lemma 2.4.8:

$$C_k \cdot E_i = m_i - \sum_{p_j \succ p_i} m_j.$$

In particular, $C_k \cdot E_k = 2$. Since $m_j < m_{j+1} + \ldots + m_k$ for all $j \leq l$, it follows that, for $i = 1, \ldots, l$, the curves E_i and E_k do not intersect in X_k and hence also not in X. Since all m_i are even, it follows that the intersection numbers $C_k \cdot E_i$ are even. It follows moreover that the intersection numbers $C_n \cdot E_i$ are also even for $i = 1, \ldots, l$, since E_k and E_i do not intersect in X_k. The curve $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is SNC and therefore E_i and C_n do not intersect at all, for $i = 1, \ldots, l$. Since the multiplicities $m_{l+1}, \ldots, m_k$ are all equal to 2, it follows that C_k does not intersect any of the curves $E_1, \ldots, E_{k-1}$, but only E_k. Since C is not unicuspidal, the curves C_k and E_k intersect in two distinct points. We denote by δ the self-intersection of C_k, which is given by $\delta = d^2 - \sum_{i=1}^{k} m_i^2$. Since C has a (-1)-tower resolution, we have $\delta \geq -1$.

Claim (1). If $\delta = -1$, we reach a contradiction.

Proof of Claim (1). We already have a (-1)-tower resolution of C (see Figure 2.22). Since C_k and E_k intersect in two points and there is no more base-point on C_k, there is no more base-point at all. But we observe that C_k and $E_1 \cup \ldots \cup E_{k-1}$ are not connected. This is not possible and hence δ must be ≥ 0.

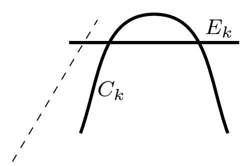


Figure 2.22: Case $\delta = -1$.

■

Claim (2). If $\delta = 0$, we reach a contradiction.

Proof of Claim (2). The genus-degree formula yields

$$d^2 - 3d + 2 = \sum_{i=1}^{k} m_i(m_i - 1).$$

Using $\delta = 0$, we get $3d - 2 = \sum_{i=1}^{k} m_i$. This identity implies that d is even. We can thus find the equations

$$\left(\frac{d}{2}\right)^2 = \sum_{i=1}^{k} \left(\frac{m_i}{2}\right)^2,$$

$$3\left(\frac{d}{2}\right) + 1 = \sum_{i=1}^{k} \frac{m_i}{2}.$$

Adding those identities yields

$$\frac{d}{2}\left(\frac{d}{2} + 3\right) + 1 = \sum_{i=1}^{k} \frac{m_i}{2}\left(\frac{m_i}{2} + 1\right).$$

The left-hand side of this equation is odd, whereas the right-hand side is even. This is a contradiction and thus $\delta = 0$ is not possible. ■

Claim (3). If $\delta = 1$, we reach a contradiction.

Proof of Claim (3). The base-point p_{k+1} is one of the intersection points between C_k and E_k. The curve C_{k+1} has then self-intersection 0 in X_{k+1} and thus the base-point p_{k+2} is the unique intersection point between C_{k+1} and E_{k+1}. The configuration of curves in X_{k+2} is shown in Figure 2.23. In the surface X, the curve E_k has self-intersection -2. This implies that η first contracts C_n and then $E_k, \dots, E_1$, in this order. By assumption, the multiplicity sequence of C is non-constant. This implies that there exists a curve E_j with $j < k$ that intersects 3 other exceptional curves. But this implies that the image of E_{k+1}, after contracting $C_n, E_k, \dots, E_1$, is singular and hence cannot be contracted. We thus reach a contradiction and conclude that $\delta \neq 1$.

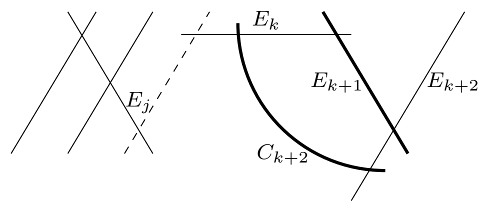


Figure 2.23: Case $\delta = 1$.

■

Claim (4). If $\delta \geq 2$, we reach a contradiction.

Proof of Claim (4). Again, the base-point p_{k+1} is one of the intersection points between C_k and E_k. Since π is a (-1)-tower resolution of C, it follows that for $i = k+1, \dots, k+\delta$, the base-point p_{i+1} is the unique intersection point between C_k and E_k (see Figure 2.24). This implies that in X, the curve E_{k+1} has self-intersection -2. We observe that E_k also intersects the (-2)-curve E_{k-1} in X. Since η contracts E_k before E_{k-1} and E_{k+1}, this leads to a contradiction.

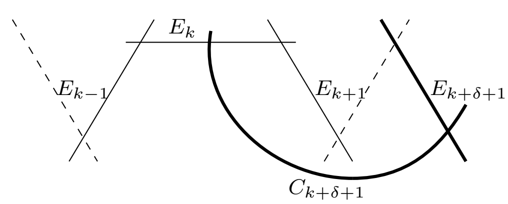


Figure 2.24: Case $\delta \geq 2$.

■

This concludes the proof of Proposition 2.4.26. □

Corollary 2.4.27. *Let $C \subset \mathbb{P}^2$ be an irreducible curve with one of the multiplcity sequences $(4, 2_{(4)})$, $(4_{(3)}, 2_{(3)})$, or $(6, 2_{(6)})$. If C is not unicuspidal, then any open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.*

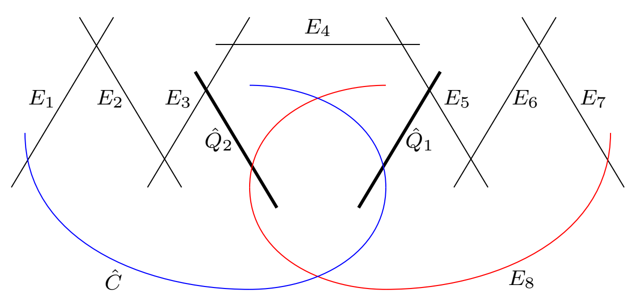
Proof. This is a direct consequence of Proposition 2.4.26. □

2.4.4 A special sextic curve and the proof of Theorem 2

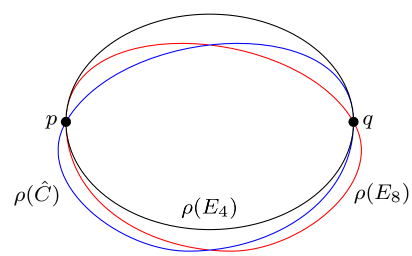
Proposition 2.4.28. *Let $C \subset \mathbb{P}^2$ be a curve of degree 6 and multplicity sequence $(3, 2_{(7)})$ and let $\varphi\colon \mathbb{P}^2 \setminus C \to \mathbb{P}^2 \setminus D$ be an isomorphism, where $D \subset \mathbb{P}^2$ is a curve. Then C and D are projectively equivalent.*

Proof. If φ extends to an automorphism of $\mathbb{P}^2$ the claim is trivial, so we assume this is not the case. Then by Lemma 2.2.4, there exists a (-1)-tower resolution $\pi\colon X \to \mathbb{P}^2$ of C and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of D such that $\eta = \varphi \circ \pi$. The curve C has 8 singular points $p_1, \ldots, p_8$, where p_{i+1} lies in the first neighborhood of p_i for $i = 1, \ldots, 7$. The map π is a (-1)-tower resolution of C and thus blows up the points $p_1, \ldots, p_8$. We denote by E_i the exceptional curve of the blow-up of p_i, for $i = 1, \ldots, 8$. After blowing up those 8 points, the strict transform $\hat{C}$ of C has self-intersection $6^2 - 3^2 - 7 \cdot 2^2 = -1$. We observe that $\hat{C}$ and E_8 intersect with multiplicity 2. Since no other base-point of π lies on $\hat{C}$, it follows that also the strict transforms of $\hat{C}$ and E_8 intersect with multiplicity 2 in X. But this means that E_8 is not contracted by η. It follows that E_8 is the last exceptional curve of π and $\eta(E_8) = D$.

By Bézout's theorem the points p_1, p_2, p_3 are not collinear and hence there exists a conic $Q_1 \subset \mathbb{P}^2$ that passes through $p_1, \ldots, p_5$. Again by Bézout's theorem, it follows that C and Q_1 intersect transversally in some proper point of $\mathbb{P}^2$ that is different from p_1. It then follows that the strict transform $\hat{Q}_1$ of Q_1 in X transversally intersects E_5 and $\hat{C}$. By symmetry there also exists a conic $Q_2 \subset \mathbb{P}^2$ whose strict transform $\hat{Q}_2$ by η intersects E_3 and $\hat{D}$ transversally. The configuration of curves in X is shown below.



To see that $\hat{Q}_1$ and $\hat{Q}_2$ do not intersect in X, we observe that π sends $\hat{Q}_2$ to a rational quartic curve with multiplicity sequence $(2_{(3)})$ and singular points p_1, p_2, p_3. It then follows that $\hat{Q}_1 \cdot \hat{Q}_2 = Q_2 \cdot \pi(\hat{Q}_2) - 2 - 2 - 2 - 1 - 1 = 0$. Moreover, the curves $\hat{Q}_1$ and $\hat{Q}_2$ both have self-intersection -1 in X. We can thus construct a morphism ρ by contracting the curves $\hat{Q}_2, E_3, E_2, E_1$ and $\hat{Q}_1, E_5, E_6, E_7$. The rank of the Picard group of X is 9, and hence the rank of the Picard group of the image of ρ is 1. It thus follows that ρ is a morphism $X \to \mathbb{P}^2$. The images of $\hat{C}, E_4$ and E_8 all have self-intersection 4 and are thus smooth conics in $\mathbb{P}^2$. The curves $\rho(E_4)$ and $\rho(\hat{C})$ intersect in two distinct points $p, q \in \mathbb{P}^2$, with multiplicity 1 in p and multiplicity 3 in q. The curves $\rho(E_4)$ and $\rho(E_8)$ also intersect in p and q, but with multiplicity 3 in p and multiplicity 1 in q. The configuration of the 3 conics is shown below.



Up to a linear change of coordinates, we can assume that the smooth conic $\rho(E_4)$ has equation $xz + y^2 = 0$ and the points p and q are $[1 : 0 : 0]$ and $[0 : 0 : 1]$ respectively. Conics that pass through the points $[1 : 0 : 0]$ and $[0 : 0 : 1]$ are of the form

$$ay^2 + bxy + cxz + dyz = 0$$

where $a, b, c, d \in \mathrm{k}$. A smooth conic with this equation intersects $xz + y^2 = 0$ with multiplicity 3 in $[1 : 0 : 0]$ if and only if $a = c \neq 0$, $b = 0$ and $d \neq 0$. Thus there exists some $\lambda \in \mathrm{k}^*$ such that $\rho(\hat{C})$ has equation $xz + y^2 + \lambda yz = 0$. Analogously, there exists $\mu \in \mathrm{k}^*$ such that $\rho(E_8)$ has equation $xz + y^2 + \mu yz = 0$.

We then find $\theta \in \mathrm{Aut}(\mathbb{P}^2)$ that sends a point $[x : y : z]$ to $[\frac{\lambda}{\mu} z : y : \frac{\mu}{\lambda} x]$. Thus θ preserves the conic $xz + y^2 = 0$ and exchanges $\rho(\hat{C})$ and $\rho(E_8)$. It follows that $\hat{\theta} \coloneqq \rho^{-1} \circ \theta \circ \rho$ is an automorphism of X that exchanges $\hat{C}$ and E_8 and sends E_i to E_{8-i} for $i = 2, \ldots, 7$. But then $\eta \circ \hat{\theta} \circ \pi^{-1}$ is an automorphism of $\mathbb{P}^2$ that sends C to D, and hence C and D are projectively equivalent. □

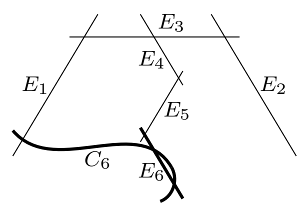
Before we are able to prove Theorem 2, we need to look at one more special case.

Lemma 2.4.29. *Let $C \subset \mathbb{P}^2$ be a curve of degree 7 and multiplicity sequence $(5, 2_{(5)})$. Then every open embedding $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ extends to an automorphism of $\mathbb{P}^2$.*

Proof. Suppose that there exists an open embedding $\varphi\colon \mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$. Then by Lemma 2.2.4, there exists a (-1)-tower resolution $\pi\colon X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$ of C with base-points $p_1, \ldots, p_n$ and exceptional curves $E_1, \ldots, E_n$, and a (-1)-tower resolution $\eta\colon X \to \mathbb{P}^2$ of some curve $D \subset \mathbb{P}^2$ such that $\varphi \circ \pi = \eta$. Then $E_1 \cup \ldots \cup E_{n-1} \cup C_n$ is the exceptional locus of η, being the support of an SNC-divisor that has a tree structure. By Lemma 2.4.8, we obtain the intersection number

$$C_n \cdot E_1 = m_i - \sum_{p_j \succ p_1} m_j.$$

Thus either $C_n \cdot E_1 = 3$ or $C_n \cdot E_1 = 1$. Since C_n can intersect E_1 only transversally in at most one point, we conclude that $C_n \cdot E_1 = 1$ and that p_3 is proximate to p_1. For the first 6 blow-ups of π, we then obtain the configuration of curves illustrated below.



The curves E_2 and E_4 have self-intersection -2 in X since the resolution π is obtained by blowing up more points on E_6. Moreover, the map η contracts E_3 before E_2 and E_4, but this leads to a contradiction. □

We are now ready to give the proof of the second main result.

Proof of Theorem 2. We assume that C is not a line, conic, or a nodal cubic. We can also assume that C is rational and has a unique proper singular point with one of the multiplicity sequences in Table 2.1, by Corollary 2.4.5. Otherwise, φ extends to an automorphism of $\mathbb{P}^2$. If C is unicuspidal, then C and D are projectively equivalent by Corollary 2.4.18. If C is not unicuspidal, then φ extends to an automorphism of $\mathbb{P}^2$ by Corollary 2.4.10, Corollary 2.4.21, Corollary 2.4.24, Corollary 2.4.27, and Lemma 2.4.29, except when C is of degree 6 with multiplicity sequence $(3, 2_{(7)})$ or C is of degree 8 with multiplicity sequence $(3_{(7)})$. If C has multiplicity sequence $(3, 2_{(7)})$, the claim follows from Proposition 2.4.28. If C has multiplicity sequence $(3_{(7)})$, then $C \setminus \operatorname{Sing}(C)$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, by Proposition 2.4.19. □

Remark 2.4.30. For all known examples of irreducible curves $C \subset \mathbb{P}^2$ that have non-extendable open embeddings $\mathbb{P}^2 \setminus C \hookrightarrow \mathbb{P}^2$, we have that $C \setminus \operatorname{Sing}(C) \simeq \mathbb{P}^1 \setminus \{p_1, \ldots, p_k\}$, where $k \in \{1, 2, 3, 9\}$. There are only very few known non-unicuspidal examples. Do there exist examples for any $k \in \mathbb{N}$?

2.4.5 A counterexample of degree 8

It follows from Theorem 2 that if two irreducible curves $C, D \subset \mathbb{P}^2$ of degree ≤ 8 are counterexamples to Conjecture 2.1.1, then C and D are of degree 8 and have multiplicity sequence $(3_{(7)})$. In this section, we show that such counterexamples do indeed exist. First we need the following auxiliary construction.

Lemma 2.4.31. *We denote the conic*

$$\Lambda\colon xy + xz + yz = 0$$

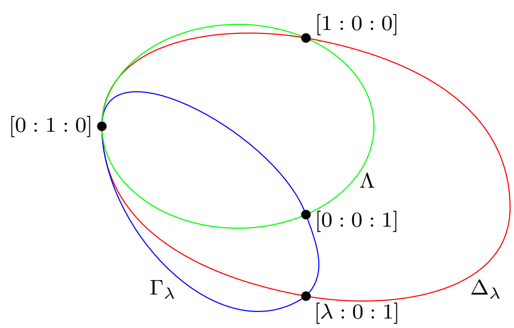
and for $\lambda \in \mathrm{k} \setminus \{0, -1\}$ the conics

$$\Gamma_\lambda\colon x^2 - (1+\lambda)xy - \lambda xz - (1+\lambda)yz = 0,$$
$$\Delta_\lambda\colon z^2 - \left(1 + \frac{1}{\lambda}\right)xy - \frac{1}{\lambda}xz - \left(1 + \frac{1}{\lambda}\right)yz = 0.$$

Then the curves Λ, Γ_λ and Δ_λ intersect in $[0:1:0]$ with multiplicity 3 for each pair. Moreover, the curves

- *Λ and Γ_λ intersect in $[0:0:1]$,*
- *Λ and Δ_λ intersect in $[1:0:0]$,*
- *Γ_λ and Δ_λ intersect in $[\lambda:0:1]$,*

and in no other point apart from $[0:1:0]$. The configuration of these conics is shown below.

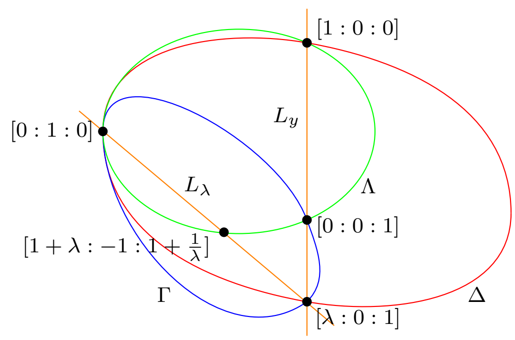


Furthermore, there exists an automorphsim of $\mathbb{P}^2$ that preserves Λ and exchanges Γ_λ and Δ_λ if and only if $\lambda = 1$.

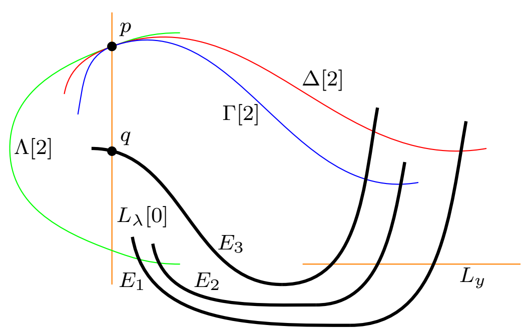
Proof. The curves Λ, Γ_λ and Δ_λ are given by explicit equations and it is a straightforward computation to determine the intersection points and multiplicities.

To prove the last claim, suppose that $\theta \in \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathrm{k})$ preserves Λ and exchanges Γ_λ and Δ_λ. Then θ fixes $[0:1:0]$ and exchanges $[1:0:0]$ and $[0:0:1]$. Those conditions imply that θ is of the form $[x:y:z] \mapsto [\alpha z : y : \beta x]$, for some $\alpha, \beta \in \mathrm{k}^*$. The image of Λ under θ then has equation $\beta xy + \alpha\beta xz + \alpha yz = 0$. Since Λ is preserved, it follows that $\alpha = \beta = \alpha\beta$ and hence $\alpha = \beta = 1$. The map θ also fixes the intersection point $[\lambda : 0 : 1]$ between Γ_λ and Δ_λ. Since $\theta([\lambda:0:1]) = [1:0:\lambda]$, it follows that $\lambda = 1$. For the converse, suppose that $\lambda = 1$. Then the automorphism $[x:y:z] \mapsto [z:y:x]$ preserves Λ and exchanges Γ_1 and Δ_1. $\square$

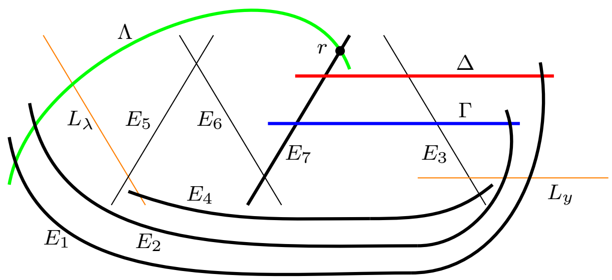
Proof of Theorem 3. With the same notations as in Lemma 2.4.31, we choose some $\lambda \in \mathrm{k} \setminus \{0, \pm 1\}$ and conics Λ, $\Gamma = \Gamma_\lambda$, $\Delta = \Delta_\lambda$. We denote moreover by L_y the line $y = 0$ and by L_λ the line through $[0:1:0]$ and $[\lambda:0:1]$. The line L_λ has equation $x - \lambda z = 0$ and intersects Λ in the points $[0:1:0]$ and $[1+\lambda : -1 : 1+\frac{1}{\lambda}]$. The configuration of those curves in shown below.



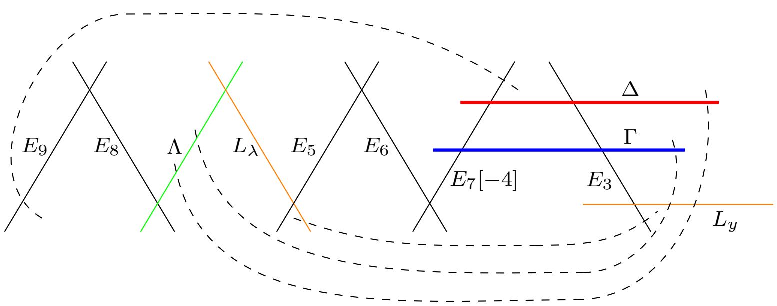
We then blow up the points $[1:0:0]$, $[0:0:1]$ and $[\lambda:0:1]$, with exceptional curves E_1, E_2, and E_3 respectively. The configuration after these blow-ups is shown below. By abuse of notation, we use the same names for the strict transforms of all curves. Curves with self-intersection -1 are drawn with thick lines and all other self-intersection numbers are indicated, except if they are -2.



Next, we blow up the intersection point q between L_λ and E_3, with exceptional curve E_4. The curves Γ, Δ and Λ each intersect with multiplicity 3 in the point p. We then blow up p and two points proximate to p (with exceptional curves E_5, E_6, E_7) so that the strict transforms of Γ, Δ and Λ are disjoint. We thus obtain the following configuration of curves.



Finally, we blow up the intersection point r between Λ and E_7 and two points proximate to r, with exceptional curves E_8, E_9, E_{10}, and obtain the configuration shown below. We denote the surface obtained after these blow-ups by X and denote the composition of all 10 blow-ups by $\rho\colon X \to \mathbb{P}^2$. The curves E_1, E_2, E_4, E_{10} are dashed and unlabeled because they will not be used for what follows.



The rank of the Picard group of X is 11, since this surface is obtained from $\mathbb{P}^2$ by 10 blow-ups. We can now find a morphism $\pi\colon X \to \mathbb{P}^2$, by contracting the 10 curves Δ, E_3, L_y, E_7, E_6, E_5, L_λ, Λ, E_8, E_9, in this order. The image $C := \pi(\Gamma)$ is then a curve of degree 8 in $\mathbb{P}^2$ with multiplicity sequence $(3_{(7)})$. Likewise, we find a morphism $\eta\colon X \to \mathbb{P}^2$, where we first contract Γ instead of Δ. The image $D := \eta(\Delta)$ is then also a curve of degree 8 with multiplicity sequence $(3_{(7)})$. The complements $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus D$ are both isomorphic to the complement of the union of the curves Γ, Δ, E_3, L_y, E_7, E_6, E_5, L_λ, Λ, E_8, E_9 in X.

Suppose now that C and D are projectively equivalent, i.e. there exists $\theta \in \mathrm{PGL}_3(\mathrm{k})$ with $\theta(C) = D$. We observe that the base-points of π are completely determined by C, since π is the minmal SNC-resolution of C followed by the blow-up of the unique intersection point between E_3 and E_7. Likewise, the base-points of η are determined by D. It follows that $\hat{\theta} := \eta^{-1} \circ \theta \circ \pi$ defines an automorphism of X that exchanges Γ and Δ and preserves the other exceptional curves. But then $\hat{\theta}$ induces an automorphism of $\mathbb{P}^2$ (via ρ) that exchanges the conics $\Gamma, \Delta \subset \mathbb{P}^2$ and preserves Λ, L_y and L_λ. But this is not possible by Lemma 2.4.31, since we have chosen $\lambda \neq 1$. We thus reach a contradiction and conclude that C and D are not projectively equivalent. □

Remark 2.4.32. The construction in the proof of Theorem 3 also works if the base-field k is not algebraically closed, except if the fieldk has only 2 or 3 elements. In those cases we cannot choose $\lambda \in \mathrm{k} \setminus \{0, \pm 1\} = \varnothing$.

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Chapter 3

Exceptional isomorphisms between complements of affine plane curves

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(arXiv:1609.06682v3)

ABSTRACT. This article describes the geometry of isomorphisms between complements of geometrically irreducible closed curves in the affine plane $\mathbb{A}^2$, over an arbitrary field, which do not extend to an automorphism of $\mathbb{A}^2$.

We show that such isomorphisms are quite exceptional. In particular, they occur only when both curves are isomorphic to open subsets of the affine line $\mathbb{A}^1$, with the same number of complement points, over any field extension of the ground field. Moreover, the isomorphism is uniquely determined by one of the curves, up to left composition with an automorphism of $\mathbb{A}^2$, except in the case where the curve is isomorphic to the affine line $\mathbb{A}^1$ or to the punctured line $\mathbb{A}^1 \setminus \{0\}$. If one curve is isomorphic to $\mathbb{A}^1$, then both curves are equivalent to lines. In addition, for any positive integer n, we construct a sequence of n pairwise non-equivalent closed embeddings of $\mathbb{A}^1 \setminus \{0\}$ with isomorphic complements. In characteristic 0 we even construct infinite sequences with this property.

Finally, we give a geometric construction that produces a large family of examples of non-isomorphic geometrically irreducible closed curves in $\mathbb{A}^2$ that have isomorphic complements, answering negatively the Complement Problem posed by Hanspeter Kraft [Kra96]. This also gives a negative answer to the holomorphic version of this problem in any dimension $n \geq 2$. The question had been raised by Pierre-Marie Poloni in [Pol16].

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Contents

3.1 Introduction

In the Bourbaki Seminar *Challenging problems on affine n-space* [Kra96], Hanspeter Kraft gives a list of eight basic problems related to the affine n-spaces. The sixth one is the following:

> Complement Problem. Given two irreducible hypersurfaces $E, F \subset \mathbb{A}^n$ and an isomorphism of their complements, does it follow that E and F are isomorphic?

Recently, Pierre-Marie Poloni gave a negative answer to the problem for any $n \geq 3$ [Pol16]. The construction is given by explicit formulas. There are examples where both E and F are smooth, and examples where E is singular, but F is smooth. This article deals with the case of dimension $n = 2$. The situation is much more rigid than in dimension $n \geq 3$, as we discuss in Theorem 4.

We will work over a fixed arbitrary field k and we will only consider curves, surfaces, morphisms, and rational maps defined over k, unless we explicitly state so (and will then talk about $\overline{\mathrm{k}}$-curves, $\overline{\mathrm{k}}$-surfaces, $\overline{\mathrm{k}}$-morphisms, and $\overline{\mathrm{k}}$-rational maps, where $\overline{\mathrm{k}}$ denotes the algebraic closure of k.) We recall that two closed curves $C, D \subset \mathbb{A}^2$ are *equivalent* if there is an automorphism of $\mathbb{A}^2$ that sends one curve onto the other. Note that equivalent curves are isomorphic. A variety (defined over k) is called *geometrically irreducible* if it is irreducible over $\overline{\mathrm{k}}$. A *line* in $\mathbb{A}^2$ is a closed curve of degree 1.

Theorem 4. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, the complement $D \subset \mathbb{A}^2$ of the image of φ is also a geometrically irreducible closed curve. Assuming that φ does not extend to an automorphism of $\mathbb{A}^2$, the following holds:*

(1) *Both C and D are isomorphic to open subsets of $\mathbb{A}^1$, with the same number of complement points. This means that there exist square-free polynomials $P, Q \in \mathrm{k}[t]$ with the same number of roots in* k *and such that*

$$C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}]) \quad \textit{and} \quad D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}]).$$

Moreover, the same result holds for every field extension k'/k.

(2) *If C is isomorphic to $\mathbb{A}^1$, then both C and D are equivalent to lines.*

(3) *If C is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$, then φ is uniquely determined up to a left composition with an automorphism of $\mathbb{A}^2$.*

Corollary 3.1.1. *If $C \subset \mathbb{A}^2$ is a geometrically irreducible closed curve not isomorphic to $\mathbb{A}^1 \setminus \{0\}$, then there are at most two equivalence classes of closed curves whose complements are isomorphic to $\mathbb{A}^2 \setminus C$.*

Corollary 3.1.2. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. Then there exists at most one closed curve $D \subset \mathbb{A}^2$, up to equivalence, such that C and D are non-isomorphic, but have isomorphic complements.*

Corollary 3.1.3. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve, not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. Then, the group $\mathrm{Aut}(\mathbb{A}^2, C) = \{g \in \mathrm{Aut}(\mathbb{A}^2) \mid g(C) = C\}$, which can be naturally identified with a subgroup of $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$, has index 1 or 2 in this group.*

Corollary 3.1.4. *If $C \subset \mathbb{A}^2$ is a singular, geometrically irreducible closed curve and $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ is an isomorphism, for some closed curve D, then φ extends to an automorphism of $\mathbb{A}^2$.*

Corollary 3.1.4 shows in particular that the Complement Problem for $n = 2$ has a positive answer if one of the curves is singular, contrary to the case where $n \geq 3$, as pointed out before. This is also different from the case of $\mathbb{P}^2$, where there exist non-isomorphic geometrically irreducible closed curves with isomorphic complements [Bla09, Theorem 1], but where all these curves are necessarily singular (see Proposition 3.7.1 below).

Theorem 4 moreover shows that the Complement Problem for $n = 2$ has a positive answer if one of the curves is not rational (this was already stated in [Kra96, Proposition 3] and does not need all tools of Theorem 4 to be proven, see for instance Corollary 3.2.7 below). More generally, the answer is positive when one of the curves is not isomorphic to an open subset of $\mathbb{A}^1$. The circle of equation $x^2 + y^2 = 1$ over $\mathbb{R}$ is an example of a smooth rational affine curve which is not isomorphic to an open subset of $\mathbb{A}^1$. Note that [Kra96, Proposition 3] says in addition that the Complement Problem for $n = 2$ and $\mathrm{k} = \mathbb{C}$ has a positive answer if one of the curves has Euler characteristic one; this is also provided by Theorem 4.

Corollary 3.1.1 describes a situation quite different from the case of dimension $n \geq 3$, where there are infinitely many hypersurfaces $E \subset \mathbb{A}^n$, up to equivalence, that have isomorphic complements [Pol16, Lemma 3.1]. It is also in contrast with the case of $\mathbb{P}^2$, where we can find algebraic families of closed curves in $\mathbb{P}^2$, non-equivalent under automorphisms of $\mathbb{P}^2$, that have isomorphic complements (and thus infinitely many if k is infinite). This follows from a construction in [Cos12], see Proposition 3.7.3 below.

All tools necessary to obtain the rigidity result (Theorem 4) are developped in Section 3.3, using some basic results given in Section 3.2. The proof is carried out at the end of Section 3.3. It uses embeddings into various smooth projective surfaces and a detailed study of the configuration of the curves at infinity. We study in particular embeddings into Hirzebruch surfaces that have mild singularities on the boundary and then study blow-ups of these, and completions by unions of trees.

Our second theorem is an existence result which demonstrates the optimality of Theorem 4.

Theorem 5.

(1) *There exists a closed curve $C \subset \mathbb{A}^2$, isomorphic to $\mathbb{A}^1 \setminus \{0\}$, whose complement $\mathbb{A}^2 \setminus C$ admits infinitely many equivalence classes of open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ into the affine plane. Moreover, the set of equivalence classes of curves with this property is infinite.*

(2) *For every integer $n \geq 1$, there exist pairwise non-equivalent closed curves $C_1, \ldots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, such that the surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are all isomorphic. Moreover, if* $\mathrm{char}(\mathrm{k}) = 0$, *we can find an infinite sequence of pairwise non-equivalent closed curves $C_i \subset \mathbb{A}^2$, $i \in \mathbb{N}$, such that the surfaces $\mathbb{A}^2 \setminus C_i$, $i \in \mathbb{N}$, are all isomorphic.*

(3) *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two non-equivalent closed curves $C, D \subset \mathbb{A}^2$, both isomorphic to* $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$*, such that the surfaces $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic. Moreover, the set of equivalence classes of the curves C in such pairs (C, D) is infinite.*

A constructive proof of Theorem 5 is given in Section 3.4. We use explicit equations and work with birational maps which either preserve one projection $\mathbb{A}^2 \to \mathbb{A}^1$ or are compositions of a small number of them.

We then give counterexamples to the Complement Problem in dimension 2:

Theorem 6. *There exist two geometrically irreducible closed curves $C, D \subset \mathbb{A}^2$ which are not isomorphic, but whose complements $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic. Furthermore, these two curves can be chosen of degree 7 if the field admits more than 2 elements and of degree 13 if the field has 2 elements.*

The proof is given in Section 3.5. We first establish Proposition 3.5.1 (mainly via blow-ups of points on singular curves in $\mathbb{P}^2$) which asserts that, for each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 1$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, there exist two closed curves $C, D \subset \mathbb{A}^2$ of degree $d^2 - d + 1$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that the following isomorphisms hold:

$$C \simeq \mathrm{Spec}\Big(\mathrm{k}[t, \frac{1}{P}]\Big) \text{ and } D \simeq \mathrm{Spec}\Big(\mathrm{k}[t, \frac{1}{Q}]\Big), \text{ where } Q(t) = P\Big(\lambda + \frac{1}{t}\Big) \cdot t^{\deg(P)}.$$

Then, the proof of Theorem 6 follows by providing an appropriate pair (P, λ) for every field. The case of infinite fields is quite easy. Indeed, if k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, then $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic, for a general element $\lambda \in \mathrm{k}$ (Lemma 3.5.4). This shows that the isomorphism type of counterexamples to the Complement Problem is as large as possible (indeed, by Theorem 4(1), any curves $C, D \subset \mathbb{A}^2$ providing a counterexample to the Complement Problem are necessarily isomorphic to open subsets of $\mathbb{A}^1$ with at least three complement $\overline{\mathrm{k}}$-points).

We finish this introduction by presenting some easy consequences of Theorem 6 that are further elaborated in Section 3.6:

(i) The negative answer to the Complement Problem for $n = 2$ directly gives a negative answer for any $n \geq 3$ (Proposition 3.6.1): Our construction produces, for each $n \geq 3$, two geometrically irreducible smooth closed hypersurfaces $E, F \subset \mathbb{A}^n$ which are not isomorphic, but whose complements $\mathbb{A}^n \setminus E$ and $\mathbb{A}^n \setminus F$ are isomorphic (Corollary 3.6.2). All the hypersurfaces constructed this way are isomorphic to $\mathbb{A}^{n-2} \times C$ for some open subset $C \subset \mathbb{A}^1$. This does not allow us to give singular examples like those of [Pol16], but provides a different type of example.

(ii) Choosing $\mathrm{k} = \mathbb{C}$, our construction gives families of closed complex curves $C, D \subset \mathbb{C}^2$ whose complements are biholomorphic (because they are isomorphic as algebraic varieties), but which are not themselves biholomorphic (Proposition 3.6.3). From this there directly follows the existence of algebraic hypersurfaces $E, F \subset \mathbb{C}^n$ which are complex manifolds that are not biholomorphic, but have biholomorphic complements, for every $n \geq 2$ (Corollary 3.6.4). This answers a question asked in [Pol16]. Note that in the counterexamples of [Pol16], if both hypersurfaces are smooth, then they are always biholomorphic (even if they are not isomorphic as algebraic varieties).

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3.2 Preliminaries

In the sequel, k is an arbitrary field and $\overline{\mathrm{k}}$ its algebraic closure. Unless otherwise specified, all varieties of dimension at least one are k-varieties, i.e. algebraic varieties defined over k, or equivalently $\overline{\mathrm{k}}$-varieties with a k-structure. When we say for example *rational*, resp. *isomorphic*, we mean k-rational, resp. k-isomorphic (which means that the maps are defined over k). Nevertheless, we will often have to consider $\overline{\mathrm{k}}$-varieties, but we will then always state so explicitly. A variety is called *geometrically rational,* resp. *geometrically irreducible,* if it is rational, resp. irreducible, after the extension to $\overline{\mathrm{k}}$. When dealing with "points" (but also with "base-points" or "complement points") we will always specify k-points or $\overline{\mathrm{k}}$-points. Finally, let us recall that a $\overline{\mathrm{k}}$-base-point of a $\overline{\mathrm{k}}$-birational map $f\colon X \dashrightarrow Y$, where X and Y are smooth projective $\overline{\mathrm{k}}$-surfaces, is either *proper*, when it belongs to X, or *infinitely near*, when it does not belong to X, but to a surface obtained from X via a finite number of blow-ups. If we assume furthermore that f, X, Y are defined over k, then a k-base-point of f is defined in the following obvious way: it is either a proper $\overline{\mathrm{k}}$-base-point defined over k, or it is an infinitely near $\overline{\mathrm{k}}$-base-point of f which is a k-point of a surface obtained from X via a finite number of blow-ups of k-points. Of course, there is no reason for a birational map $f\colon X \dashrightarrow Y$ to admit a k-base-point. For example, when $\mathrm{k} = \mathbb{F}_2$ the birational involution of $\mathbb{P}^2$ given by $[x : y : z] \mapsto [x^2 + y^2 + yz : xz + y^2 + z^2 : x^2 + xy + z^2]$ admits no k-base-point (but has three base-points over $\mathbb{F}_8 = \mathbb{F}_3[u]/(u^3+u+1)$, namely $[1 : u : u^2+u+1]$, $[u : u^2+u+1 : 1]$ and $[u^2+u+1 : 1 : u]$). Similar examples of degree 5 for $\mathrm{k} = \mathbb{R}$ are classical and can be found in [BM15, Example 3.1]. Also, a closed curve in $\mathbb{A}^2$ does not necessarily admit a k-point. For example, the geometrically irreducible closed curve of equation $x^2 + y^2 + 1 = 0$ admits no $\mathbb{R}$-point.

Working over an algebraically closed field, every birational map $\varphi\colon X \dashrightarrow Y$ between two smooth projective irreducible surfaces X and Y admits a resolution, which consists of two birational morphisms $\eta\colon Z \to X$ and $\pi\colon Z \to Y$, where Z is a smooth projective irreducible surface, such that the following diagram is commutative.

$$\begin{tikzcd} & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd}$$

Let us also recall that a birational morphism between two smooth projective irreducible surfaces is a composition of finitely many blow-downs. We can moreover choose this resolution to be *minimal*, which corresponds to asking that no irreducible curve of Z of self-intersection (-1) be contracted by both η and π. The morphism η is obtained by blowing up all base-points in X of φ. Analogously π is obtained by blowing up all base-points in Y of φ^{-1}. In Lemma 3.2.5(2), we will prove that under some additional hypotheses (satisfied by all birational maps that we will consider), such a miminal resolution also exists over an arbitrary field k, and that moreover the morphisms η and π are obtained by sequences of blow-ups of k-points (which may be proper or infinitely near).

3.2.1 Basic properties

In order to study isomorphisms between affine surfaces, it is often interesting to see the affine surfaces as open subsets of projective surfaces and then to see the isomorphisms as birational maps between the projective surfaces. Recall that a rational map $\varphi\colon X \dashrightarrow Y$ between smooth projective irreducible surfaces is defined on an open subset $U \subset X$ such that $F = X \setminus U$ is finite. If C is an irreducible curve of the surface X, its image is defined by $\varphi(C) := \overline{\varphi(C \setminus F)}$. We then say that C is *contracted by* φ if $\varphi(C)$ is a point. The aim of this section is to establish Proposition 3.2.6, that we often use in the sequel. Its proof relies on some easy results that we begin by recalling: Proposition 3.2.3, Corollary 3.2.4 and Lemma 3.2.5.

We begin with the following definition, that we will frequently use, in particular to extend birational maps of $\mathbb{A}^2$ to birational maps of $\mathbb{P}^2$:

Definition 3.2.1. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{P}^2 \\ (x,y) & \mapsto & [x:y:1] \end{array}$$

is called the *standard embedding*. It induces an isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{P}^2 \setminus L_\infty$, where $L_\infty \subset \mathbb{P}^2$ denotes the *line at infinity* given by $z = 0$.

With this embedding every line in $\mathbb{A}^2$, given by an equation $ax+by=c$ where a,b,c are elements of k and a,b are not both zero, is the restriction of a line of $\mathbb{P}^2$, given by the equation $ax+by=cz$ and distinct from L_∞.

Definition 3.2.2. For each birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, we define $J_\varphi \subset \mathbb{P}^2$ to be the reduced curve given by the union of all irreducible $\overline{\mathrm{k}}$-curves contracted by φ.

Proposition 3.2.3. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map.*

(1) *The curve J_φ is defined over* k*, i.e. is the zero locus of a homogeneous polynomial $f \in \mathrm{k}[x,y,z]$.*

(2) *The restriction of φ induces an isomorphism $\mathbb{P}^2 \setminus J_\varphi \to \mathbb{P}^2 \setminus J_{\varphi^{-1}}$. Moreover, the number of irreducible components of J_φ and $J_{\varphi^{-1}}$ over $\overline{\mathrm{k}}$ are equal.*

Proof. (1). The maps φ and φ^{-1} may be written in the form

$$\begin{array}{rll} \varphi\colon & [x:y:z] \mapsto & [s_0(x,y,z):s_1(x,y,z):s_2(x,y,z)] \quad \text{and} \\ \varphi^{-1}\colon & [x:y:z] \mapsto & [q_0(x,y,z):q_1(x,y,z):q_2(x,y,z)], \end{array}$$

where $s_0, s_1, s_2 \in \mathrm{k}[x,y,z]$ (as well as q_0, q_1, q_2) are homogeneous polynomials of the same degree and with no common factor. Since $\varphi^{-1} \circ \varphi = \mathrm{id}$, there exists a homogeneous polynomial $f \in \mathrm{k}[x,y,z]$ such that $q_0(s_0,s_1,s_2) = xf$, $q_1(s_0,s_1,s_2) = yf$, $q_2(s_0,s_1,s_2) = zf$. We now observe that J_φ is the zero locus of f. Indeed, the polynomial f is zero along an irreducible $\overline{\mathrm{k}}$-curve if and only if this curve is sent by φ to a base-point of φ^{-1}. In characteristic zero, note that J_φ is also the zero locus of the Jacobian determinant associated to φ.

(2) By extending the scalars, we may assume that $\mathrm{k} = \overline{\mathrm{k}}$ is algebraically closed. We take a minimal resolution of φ, with the commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

where η and π are birational morphisms. The morphism η, resp. π, is the sequence of blow-ups of the base-points of φ, resp. φ^{-1}.

By computing the Picard rank of X, we see that η and π contract the same number of irreducible curves of X. Let n be this number. We then denote by $E \subset X$, resp. $F \subset X$, the union of the n irreducible curves contracted by η, resp. π. The map φ then restricts to an isomorphism

$$\mathbb{P}^2 \setminus \eta(E \cup F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E \cup F).$$

We now show that $\eta(E \cup F) = \eta(F)$. Since $\eta(E)$ consists of finitely many points, it suffices to see that these are contained in the curves of $\eta(F)$. Each point p of $\eta(E)$ corresponds to a connected component of E, which contains at least one (-1)-curve $\mathcal{E} \subset E$. The curve $\mathcal{E}$ is not contracted by π, by minimality, and hence is sent by π onto a curve $\pi(\mathcal{E}) \subset \mathbb{P}^2$ of self-intersection ≥ 1. This implies that $\mathcal{E}$ intersects F and thus $p \in \eta(F)$. We similarly get that $\pi(E \cup F) = \pi(E)$, and obtain that φ restricts to an isomorphism

$$\mathbb{P}^2 \setminus \eta(F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E).$$

Since $\eta(F)$ is a closed curve in $\mathbb{P}^2$ whose irreducible components are contracted by φ, we have $\eta(F) = J_\varphi$. Similarly, we get $\pi(E) = J_{\varphi^{-1}}$. Moreover, the number of $\overline{\mathrm{k}}$-irreducible components of $\eta(F)$ is equal to the number of irreducible components of $\overline{F \setminus E}$, which is equal to the number of irreducible components of $\overline{E \setminus F}$. This completes the proof. $\square$

Corollary 3.2.4. *Let $\Gamma \subset \mathbb{P}^2$ be a closed curve and $\varphi\colon \mathbb{P}^2 \setminus \Gamma \hookrightarrow \mathbb{P}^2$ an open embedding. Then the complement of $\varphi(\mathbb{P}^2 \setminus \Gamma)$ is a closed curve $\Delta \subset \mathbb{P}^2$ with the same number of irreducible components over $\overline{\mathrm{k}}$ as Γ.*

Proof. Let $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map induced by φ. Proposition 3.2.3 implies that $J_{\hat{\varphi}} \subset \Gamma$, that $J_{\hat{\varphi}}$ and $J_{\hat{\varphi}^{-1}}$ have the same number of irreducible components over $\overline{\mathrm{k}}$, and that $\hat{\varphi}$ induces an isomorphism $\mathbb{P}^2 \setminus J_{\hat{\varphi}} \xrightarrow{\simeq} \mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$.

If $J_{\hat{\varphi}} = \Gamma$, the proof is finished. Otherwise, $\Gamma' = \Gamma \setminus J_{\hat{\varphi}}$ is a closed curve of $\mathbb{P}^2 \setminus J_{\hat{\varphi}}$, which has the same number of irreducible components over $\overline{\mathrm{k}}$ as the closed curve $\Delta' = \hat{\varphi}(\Gamma')$ of $\mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$. The result follows with $\Delta = \Delta' \cup J_{\hat{\varphi}^{-1}}$. □

Lemma 3.2.5. *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between two smooth projective surfaces that restricts to an isomorphism $U = X \setminus C \xrightarrow{\simeq} Y \setminus D = V$, where C, resp. D, is the union of geometrically irreducible closed curves $C_1, \ldots, C_r$ in X, resp. $D_1, \ldots, D_s$ in Y. Then, the following holds.*

(1) *All $\overline{\mathrm{k}}$-base-points of φ, resp. φ^{-1}, are k-rational and belong to C, resp. D.*

(2) *The map φ admits a minimal resolution which is given by birational morphisms $\eta\colon Z \to X$ and $\pi\colon Z \to Y$, which are blow-ups of the base-points of φ and φ^{-1} respectively, as shown in the following diagram:*

$$\begin{tikzcd}
 & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\
X \arrow[rr, dashed, "\varphi"] & & Y \\
U \arrow[u, hook] \arrow[rr, "\simeq"] & & V. \arrow[u, hook]
\end{tikzcd}$$

(3) *In the above resolution, we have $\eta^{-1}(U) = \pi^{-1}(V)$.*

(4) *For each $i \in \{1, \ldots, r\}$, there exists $j \in \{1, \ldots, s\}$ such that either φ restricts to a birational map $C_i \dashrightarrow D_j$ or $\varphi(C_i)$ is a k-point of D_j. In this latter case, the curve C_i is rational (over k).*

Proof. We argue by induction on the total number of $\overline{\mathrm{k}}$-base-points of φ and φ^{-1}. If there is no such base-point, then φ is an isomorphism and everything follows.

Suppose now that $q \in Y$ is a proper $\overline{\mathrm{k}}$-base-point of φ^{-1}. As φ induces an isomorphism $U \xrightarrow{\simeq} V$, we have $q \in D_j(\overline{\mathrm{k}})$ for some $j \in \{1, \ldots, s\}$. There is moreover an irreducible $\overline{\mathrm{k}}$-curve of Y contracted by φ onto q, which is then equal to C_i for some $i \in \{1, \ldots, r\}$. Since C_i is defined over k, so is its image (the generic point of C_i is defined over k and is sent onto the k-point q), i.e. q is k-rational. Let $\varepsilon\colon \hat{Y} \to Y$ be the blow-up of q and let $E \subset \hat{Y}$ be the exceptional divisor (which is isomorphic to $\mathbb{P}^1$). The birational map $\hat{\varphi} = \varepsilon^{-1} \circ \varphi\colon X \dashrightarrow \hat{Y}$ induces an isomorphism $U \xrightarrow{\simeq} \hat{V}$, where $\hat{V} = \varepsilon^{-1}(V) = \hat{Y} \setminus (\tilde{D}_1 \cup \cdots \cup \tilde{D}_s \cup E)$, and where $\tilde{D}_i \subset \hat{Y}$ is the strict transform of D_i for $i = 1, \ldots, s$. The $\overline{\mathrm{k}}$-base-points of $\hat{\varphi}^{-1}$ correspond to the $\overline{\mathrm{k}}$-base-points of φ^{-1} from which the point q is removed and the $\overline{\mathrm{k}}$-base-points of $\hat{\varphi}$ coincide with the $\overline{\mathrm{k}}$-base-points of φ.

We may thus apply the induction hypothesis and obtain assertions (1)–(4) for $\hat{\varphi}$. Denoting by $\hat{\eta}\colon Z \to X$ and $\hat{\pi}\colon Z \to \hat{Y}$ the blow-ups of the base-points of $\hat{\varphi}$ and $\hat{\varphi}^{-1}$

respectively (which give the resolution of $\hat{\varphi}$ as in (2)), we obtain (1)–(2) for φ with $\eta = \hat{\eta}$, $\pi = \varepsilon\hat{\pi}$. Assertion (3) is given by $\eta^{-1}(U) = \hat{\eta}^{-1}(U) \overset{(3) \text{ for } \hat{\varphi}}{=} \hat{\pi}^{-1}(\hat{V}) = \hat{\pi}^{-1}(\epsilon^{-1}(V)) = \pi^{-1}(V)$. Assertion (4) follows from the assertion for $\hat{\varphi}$ and from the fact that ε restricts to a birational morphism $\tilde{D}_i \to D_i$ for each i, and sends $E \simeq \mathbb{P}^1$ onto a k-point of D_j.

In the case where φ^{-1} admits no $\overline{\mathrm{k}}$-base-point, a symmetric argument can be applied to φ^{-1} by starting with a proper $\overline{\mathrm{k}}$-base-point of φ. □

In the sequel, we will frequently use the following result.

Proposition 3.2.6. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, there exists a geometrically irreducible closed curve $D \subset \mathbb{A}^2$ such that $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$. Denote by $\overline{C}$ and $\overline{D}$ the closures of C and D in $\mathbb{P}^2$, using the standard embedding of Definition 3.2.1. Denote also by $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$ the line at infinity and by $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map induced by φ. Then, one of the following three possibilities holds:*

(1) *We have $\hat{\varphi}(\overline{C}) = \overline{D}$. Then, the map φ extends to an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$ that sends C onto D.*

(2) *We have $\hat{\varphi}(\overline{C}) = L_{\mathbb{P}^2}$. Then, the curve D is a line in $\mathbb{A}^2$, i.e. $\overline{D}$ is a line in $\mathbb{P}^2$ and φ extends to an isomorphism $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty \xrightarrow{\simeq} \mathbb{P}^2 \setminus \overline{D}$ that sends C onto $L_\infty \setminus \overline{D}$. In particular, C is equivalent to a line.*

(3) *The map $\hat{\varphi}$ contracts the curve $\overline{C}$ to a* k*-point of $\mathbb{P}^2$. Then, the curve $\overline{C}$ (and therefore, also the curve C) is a rational curve (i.e. is* k*-birational to $\mathbb{P}^1$).*

Proof. The restriction of $\hat{\varphi}$ to $\mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) = \mathbb{A}^2 \setminus C$ gives the open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. By Corollary 3.2.4, we obtain an isomorphism $\mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \Delta$, for some curve $\Delta \subset \mathbb{P}^2$, which is the union of two $\overline{\mathrm{k}}$-irreducible closed curves of $\mathbb{P}^2$. Since L_∞ is included in Δ, there exists an irreducible closed $\overline{\mathrm{k}}$-curve D of $\mathbb{A}^2$ such that $\Delta = L_\infty \cup \overline{D}$. As a conclusion, the restriction of $\hat{\varphi}$ at the source and the target induces an isomorphism

$$\mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (L_\infty \cup \overline{D}).$$

It follows that $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$. The equality $D = \mathbb{A}^2 \setminus \varphi(\mathbb{A}^2 \setminus C)$ proves that the curve D is defined over k and is therefore geometrically irreducible. By Lemma 3.2.5(4), one of the following three possibilities holds:

(1) We have $\hat{\varphi}(\overline{C}) = \overline{D}$. Hence, the restriction of $\hat{\varphi}$ at the source and the target provides an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$ (Proposition 3.2.3).

(2) We have $\hat{\varphi}(\overline{C}) = L_\infty$. Then, the restriction of $\hat{\varphi}$ at the source and the target provides an isomorphism $\mathbb{P}^2 \setminus L_\infty \xrightarrow{\simeq} \mathbb{P}^2 \setminus \overline{D}$ (again by Proposition 3.2.3). Since the Picard group of $\mathbb{P}^2 \setminus \Gamma$ is isomorphic to $\mathbb{Z}/\deg(\Gamma)\mathbb{Z}$, for each irreducible curve Γ, the curve $\overline{D}$ must be a line in $\mathbb{P}^2$.

(3) The map $\hat{\varphi}$ contracts the curve $\overline{C}$ to a $\overline{\mathrm{k}}$-point of $\mathbb{P}^2$. Then, by Lemma 3.2.5(4) this point is necessarily a k-point and the curve $\overline{C}$ is k-rational. ☐

Corollary 3.2.7. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. If C is not rational (i.e. not* k*-birational to $\mathbb{P}^1$), then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. This follows from Proposition 3.2.6 and the fact that cases (2)-(3) occur only when C is rational. ☐

Remark 3.2.8. It follows from Corollary 3.2.7 that the automorphism group $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$, where C is a non-rational geometrically irreducible closed curve, may be identified with the group $\mathrm{Aut}(\mathbb{A}^2, C)$ of automorphisms of $\mathbb{A}^2$ preserving C. By [BS15, Theorem 2], this group is finite (and in particular conjugate to a subgroup of $\mathrm{GL}_2(\mathrm{k})$ if $\mathrm{char}(\mathrm{k}) = 0$, as one can deduce from [DaGi75, Theorem 5], [Serr77, §6.2, Proposition 21] or from [Kam79, Theorem 4.3]). For a general discussion on the group $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$, where C is a geometrically irreducible closed curve, see Section 3.3.5 below.

We find it interesting to prove that case (3) of Proposition 3.2.6 occurs only when $\overline{C}$ intersects L_∞ in at most two $\overline{\mathrm{k}}$-points, even if this will not be used in the sequel.

Corollary 3.2.9. *If $C \subset \mathbb{A}^2$ is a geometrically irreducible closed curve such that $\overline{C}$ intersects $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$ in at least three $\overline{\mathrm{k}}$-points, then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. We may assume that $\mathrm{k} = \overline{\mathrm{k}}$. Assume by contradiction that the extension $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ does not restrict to an automorphism of $\mathbb{A}^2$. By Proposition 3.2.6, the curve $\overline{C}$ is contracted by $\hat{\varphi}$ (because C is not equivalent to a line, so (2) is impossible). We recall that $\hat{\varphi}$ restricts to an isomorphism $\mathbb{A}^2 \setminus C = \mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) \xrightarrow{\simeq} \mathbb{A}^2 \setminus D = \mathbb{P}^2 \setminus (L_\infty \cup \overline{D})$ (Proposition 3.2.6) and that $\overline{C} \subset J_{\hat{\varphi}} \subset L_\infty \cup \overline{C}$, $J_{\hat{\varphi}^{-1}} \subset L_\infty \cup \overline{D}$, where $J_{\hat{\varphi}}$, $J_{\hat{\varphi}^{-1}}$ have the same number of irreducible components (Proposition 3.2.3). We take a minimal resolution of $\hat{\varphi}$ which yields a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\hat{\varphi}"] & & \mathbb{P}^2. \end{tikzcd}$$

We first observe that the strict transforms $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ of $L_\infty, \overline{C}$ by η intersect in at most one point. Indeed, otherwise the curve $\tilde{L}_{\mathbb{P}^2}$ would not be contracted by π, because π contracts $\tilde{C}$, and is sent onto a singular curve, which then has to be $\overline{D}$. We get $J_{\hat{\varphi}} = \overline{C}$, $J_{\hat{\varphi}^{-1}} = L_\infty$ and get an isomorphism $\mathbb{P}^2 \setminus \overline{C} \to \mathbb{P}^2 \setminus L_\infty$, which is impossible, because $\overline{C}$ has degree at least 3.

Secondly, the fact that $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ intersect in at most one point implies that η blows up all points of $\overline{C} \cap L_\infty$, except at most one. Since $J_{\hat{\varphi}^{-1}} \subset D \cup L_\infty$, there are at most two (-1)-curves contracted by η. But L_∞ and $\overline{C}$ intersect in at least three points, so we obtain exactly two proper base-points of $\hat{\varphi}$, corresponding to exactly two

(-1)-curves $E_1, E_2 \subset X$ contracted to two points $p_1, p_2 \in \overline{C} \cap L_\infty$ by η. Moreover, the identity $J_{\hat{\varphi}^{-1}} = D \cup L_\infty$ implies that $J_{\hat{\varphi}} = C \cup L_\infty$ (Proposition 3.2.3). We write $E_i' = \overline{\eta^{-1}(p_i) \setminus E_i}$ and find that π contracts $F = E_1' \cup E_2' \cup \tilde{C} \cup \tilde{L}_{\mathbb{P}^2}$.

We now show that $E_i \cdot F \geq 2$, for $i = 1, 2$, which will imply that $\pi(E_i)$ is a singular curve for $i = 1, 2$, and lead to a contradiction since E_1, E_2 are sent onto L_∞ and $\overline{D}$ by π. As $E_i \cup E_i' = \eta^{-1}(p_i)$, it is a tree of rational curves, which intersects both $\tilde{C}$ and $\tilde{L}_{\mathbb{P}^2}$ since $p_i \in \overline{C} \cap L_\infty$. If E_i' is empty, then $E_i \cdot \tilde{C} \geq 1$ and $E_i \cdot \tilde{L}_{\mathbb{P}^2} \geq 1$, whence $E_i \cdot F \geq 2$ as we claimed. If E_i' is not empty, then $E_i \cdot E_i' \geq 1$. The only possibility to get $E_i \cdot F \leq 1$ would thus be that $E_i \cdot E_i' = 1$, $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$. The equality $E_i \cdot E_i' = 1$ implies that E_i' is connected, and $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$ implies that $\tilde{C} \cdot E_i' \geq 1$ and $\tilde{L}_{\mathbb{P}^2} \cdot E_i' \geq 1$. Since $\tilde{L}_{\mathbb{P}^2}$ and $\tilde{C}$ intersect in a point not contained in E_i', it follows that F contains a loop and thus cannot be contracted. □

Remark 3.2.10. In case (3) of Proposition 3.2.6, it is possible that $\overline{C}$ intersects the line L_∞ in two $\overline{\mathrm{k}}$-points. This is the case in most of our examples (see for example Lemma 3.4.2 or Lemma 3.4.9). The case of one point is of course also possible (see for instance Lemma 3.2.12(1)).

We will also need the following basic algebraic result.

Lemma 3.2.11. *Let $f \in \mathrm{k}[x, y]$ be a polynomial, irreducible over $\overline{\mathrm{k}}$, and let $C \subset \mathbb{A}^2$ be the curve given by $f = 0$. Then, the ring of functions on $\mathbb{A}^2 \setminus C$ and its subset of invertible elements are equal to*

$$\mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x, y, f^{-1}] \subset \mathrm{k}(x, y),\ \mathcal{O}(\mathbb{A}^2 \setminus C)^* = \{\lambda f^n \mid \lambda \in \mathrm{k}^*, n \in \mathbb{Z}\}.$$

In particular, every automorphism of $\mathbb{A}^2 \setminus C$ permutes the fibres of the morphism

$$\mathbb{A}^2 \setminus C \to \mathbb{A}^1 \setminus \{0\}$$

given by f.

Proof. The field of rational functions of $\mathbb{A}^2 \setminus C$ is equal to $\mathrm{k}(x, y)$. We may write any element of this field as u/v, where $u, v \in \mathrm{k}[x, y]$ are coprime polynomials, $v \neq 0$. The rational function is regular on $\mathbb{A}^2 \setminus C$ if and only if v does not vanish on any $\overline{\mathrm{k}}$-point of $\mathbb{A}^2 \setminus C$. This means that $v = \lambda f^n$, for some $\lambda \in \mathrm{k}^*$, $n \geq 0$. This provides the description of $\mathcal{O}(\mathbb{A}^2 \setminus C)$ and $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$. The last remark follows from the fact that the group $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$ is generated by k^* and one single element g, if and only if this element g is equal to $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$: Therefore, every automorphism of $\mathbb{A}^2 \setminus C$ induces an automorphism of $\mathcal{O}(\mathbb{A}^2 \setminus C)$ which sends f onto $\lambda f^{\pm 1}$. □

3.2.2 The case of lines

Proposition 3.2.6 shows that we need to study isomorphisms $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ which extend to birational maps of $\mathbb{P}^2$ that contract the curve C to a point. One can ask whether this point might be a point of $\mathbb{A}^2$ (and would thus be contained in D) or belongs to the boundary line $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$. As we will show (Corollary 3.3.6), the first possibility only occurs in a very special case, namely when C is equivalent to a line. The case of lines is special for this reason, and is treated separately here.

Lemma 3.2.12. *Let $C \subset \mathbb{A}^2$ be the line given by $x = 0$.*

(1) *The group of automorphisms of $\mathbb{A}^2 \setminus C$ is given by:*

$$\operatorname{Aut}(\mathbb{A}^2 \setminus C) = \{(x, y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x, x^{-1})) \mid \lambda, \mu \in \mathrm{k}^*, n \in \mathbb{Z}, s \in \mathrm{k}[x, x^{-1}]\}.$$

(2) *Every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ is equal to $\psi\alpha$, where $\alpha \in \operatorname{Aut}(\mathbb{A}^2 \setminus C)$ and $\psi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$. In particular, the complement of its image, i.e. the complement of $\psi\alpha(\mathbb{A}^2 \setminus C) = \psi(\mathbb{A}^2 \setminus C)$, is a curve equivalent to a line.*

Proof. To prove (1), we first observe that each transformation $(x, y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x, x^{-1}))$ actually yields an automorphism of $\mathbb{A}^2 \setminus C$. Then we only need to show that all automorphisms of $\mathbb{A}^2 \setminus C$ are of this form. An automorphism of $\mathbb{A}^2 \setminus C$ corresponds to an automorphism of $\mathrm{k}[x, y, x^{-1}]$ which sends x to $\lambda x^{\pm 1}$, where $\lambda \in \mathrm{k}^*$ (Lemma 3.2.11). Applying the inverse of $(x, y) \mapsto (\lambda x^{\pm 1}, y)$, we may assume that x is fixed. We are left with an R-automorphism of $R[y]$, where R is the ring $\mathrm{k}[x, x^{-1}]$. Such an automorphism is of the form $y \mapsto ay + b$, where $a \in R^*$, $b \in R$. Indeed, if the maps $y \mapsto p(y)$ and $y \mapsto q(y)$ are inverses of each other, the equality $y = p(q(y))$ implies that $\deg p = \deg q = 1$. This actually proves that p has the desired form, i.e. $p = ay + b$, where $a \in R^*$, $b \in R$.

To prove (2), we use Proposition 3.2.6 and write φ as an isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ where D is a geometrically irreducible closed curve, and only need to see that D is equivalent to a line. We write $\psi = \varphi^{-1}$, choose an equation $f = 0$ for D (where $f \in \mathrm{k}[x, y]$ is an irreducible polynomial over $\overline{\mathrm{k}}$), and get an isomorphism $\psi^*\colon \mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x, y, x^{-1}] \to \mathcal{O}(\mathbb{A}^2 \setminus D) = \mathrm{k}[x, y, f^{-1}]$ that sends x to $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (since the group $\mathcal{O}(\mathbb{A}^2 \setminus D)^*$ is generated by k^* and the single element $\psi^*(x)$, this forces $\psi^*(x) = \lambda f^{\pm 1}$). We can thus write ψ as $(x, y) \mapsto (\lambda f(x, y)^{\pm 1}, g(x, y) f(x, y)^n)$, where $n \in \mathbb{Z}$ and $g \in \mathrm{k}[x, y]$. Replacing ψ by its composition with the automorphism $(x, y) \mapsto ((\lambda^{-1}x)^{\pm 1}, y((\lambda^{-1}x)^{\pm 1})^{-n})$ of $\mathbb{A}^2 \setminus C$, we may assume that ψ is of the form $(x, y) \mapsto (f(x, y), g(x, y))$. If g is equal to a constant $\nu \in \mathrm{k}$ modulo f, we apply the automorphism $(x, y) \mapsto (x, (y - \nu)x^{-1})$ and decrease the degree of g. After finitely many steps we obtain an isomorphism $\mathbb{A}^2 \setminus D \xrightarrow{\simeq} \mathbb{A}^2 \setminus C$ of the form $\psi_0\colon (x, y) \mapsto (f(x, y), g(x, y))$ where g is not a constant modulo f. The image of D by ψ_0 is then dense in C, which implies that ψ_0 extends to an automorphism of $\mathbb{A}^2$ that sends D onto C (Proposition 3.2.6). $\square$

3.3 Geometric description of open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$

3.3.1 Embeddings into Hirzebruch surfaces

We will need not only embeddings of $\mathbb{A}^2$ into $\mathbb{P}^2$, but also embeddings of $\mathbb{A}^2$ into other smooth projective surfaces, and in particular into Hirzebruch surfaces. These surfaces play a natural role in the study of automorphisms of $\mathbb{A}^2$ (and of images of curves by these automorphisms), as we can decompose every automorphism of $\mathbb{A}^2$ into elementary links between such surfaces and then study how the singularities at infinity of the curves behave under these elementary links (see for instance [BS15]).

Example 3.3.1. For $n \geq 1$, the n-th Hirzebruch surface $\mathbb{F}_n$ is

$$\mathbb{F}_n = \{([a:b:c],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \ \mid \ bv^n = cu^n\}$$

and the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ yields a $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$.

Let $S_n, F_n \subset \mathbb{F}_n$ be the curves given by $[1:0:0] \times \mathbb{P}^1$ and $v = 0$, respectively. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (x,y) & \mapsto & ([x:y^n:1],[y:1]) \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{F}_n \backslash (S_n \cup F_n)$.

We recall the following easy classical result:

Lemma 3.3.2. *For each $n \geq 1$, the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ is the unique $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$, up to automorphisms of the target $\mathbb{P}^1$. The curve S_n is the unique irreducible $\overline{\mathrm{k}}$-curve in $\mathbb{F}_n$ of self-intersection $-n$, and we have $(F_n)^2 = 0$.*

Proof. Since $\mathbb{F}_n \backslash (S_n \cup F_n)$ is isomorphic to $\mathbb{A}^2$, whose Picard group is trivial, we have $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n + \mathbb{Z}S_n$ (where the class of a divisor D is again denoted by D). Moreover, F_n is a fibre of π_n and S_n is a section, so $(F_n)^2 = 0$ and $F_n \cdot S_n = 1$. We denote by $S'_n \subset \mathbb{F}_n$ the section given by $a = 0$, and find that S'_n is equivalent to $S_n + nF_n$, by computing the divisor of $\frac{a}{c}$.

Since S_n and S'_n are disjoint, this yields $0 = S_n \cdot (S_n + nF_n) = (S_n)^2 + n$, so $(S_n)^2 = -n$.

To get the result, it suffices to show that an irreducible $\overline{\mathrm{k}}$-curve $C \subset \mathbb{F}_n$ not equal to S_n or to a fibre of π_n has self-intersection at least equal to n. This will show in particular that a general fibre F of any morphism $\mathbb{F}_n \to \mathbb{P}^1$ is equal to a fibre of π_n, since F has self-intersection 0. We write $C = kS_n + lF_n$ for some $k, l \in \mathbb{Z}$. Since $C \neq S_n$ we have $0 \leq C \cdot S_n = l - nk$. Since C is not a fibre, it intersects every fibre, so $0 < F_n \cdot C = k$. This yields $l \geq nk > 0$ and $C^2 = -nk^2 + 2kl = kl + k(l - nk) \geq kl \geq nk^2 \geq n$. $\square$

Lemma 3.3.3. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. Then, there exists an integer $n \geq 1$ and an isomorphism $\iota \colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the following two possibilities:*

(1) $\Gamma \cdot F_n = 1$ *and* $\Gamma \cap F_n \cap S_n = \emptyset$.

(2) $\Gamma \cdot F_n \geq 2$ *and the following assertions hold:*

  (a) *If* $n = 1$*, then* $2m_p(\Gamma) \leq \Gamma \cdot F_1$ *for* $\{p\} = S_1 \cap F_1$*, and* $m_r(\Gamma) \leq \Gamma \cdot S_1$ *for each* $r \in F_1(\mathrm{k})$.

  (b) *If* $n \geq 2$*, then* $2m_r(\Gamma) \leq \Gamma \cdot F_n$ *for each* $r \in F_n(\mathrm{k})$.

Furthermore, in case (1)*, the curve* C *is equivalent to a curve given by an equation of the form*

$$a(y)x + b(y) = 0,$$

where $a, b \in \mathrm{k}[y]$ *are coprime polynomials such that* $a \neq 0$ *and* $\deg b < \deg a$*. Moreover, the following assertions are equivalent:*

(i) *The polynomial* a *is constant;*

(ii) *The curve* C *is equivalent to a line;*

(iii) *The curve* C *is isomorphic to* $\mathbb{A}^1$*;*

(iv) $\Gamma \cdot S_n = 0$.

Proof. Let us take any fixed isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$ for some $n \geq 1$, and denote by Γ the closure of $\iota(C)$.

We first assume that $\Gamma \cdot F_n = 1$. This is equivalent to saying that Γ is a section of π_n. We may furthermore assume that the k-point q_n defined by $\{q_n\} = F_n \cap S_n$ does not belong to Γ, as otherwise we could blow up the point q_n, contract the curve F_n, change the embedding to $\mathbb{F}_{n+1}$ and decrease by one unit the intersection number of Γ with S_n at the point q_n. After finitely many steps we get $q_n \notin \Gamma$, i.e. we are in case (1).

If $\Gamma \cdot F_n = 0$, then Γ is a fibre of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$. Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 3.3.1. Then, the curve C is equivalent to the curve $\psi^{-1}(C)$, which has equation $y = \lambda$, for some $\lambda \in \mathrm{k}$. This proves that C is equivalent to the line $y = \lambda$, and thus to the line $x = \lambda$, sent by the standard embedding onto a curve satisfying conditions (1).

It remains to consider the case where $\Gamma \cdot F_n \geq 2$. If Γ satisfies (2), we are done. Otherwise, we have a k-point $p \in F_n$ satisfying one of the following two possibilities:

(a) $n = 1$, $m_p(\Gamma) > \Gamma \cdot S_1$, and $p \in F_1$.

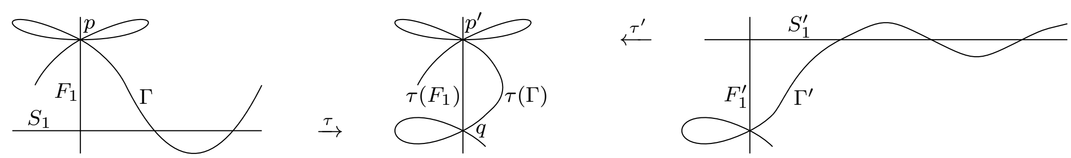
(b) $2m_p(\Gamma) > \Gamma \cdot F_n$ and either $n \geq 2$ or $n = 1$ and $p \in S_1 \cap F_1$.

We will replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$ by another, where the singularities of the curve Γ either decrease (all multiplicities are unchanged, except one which has decreased) or stay the same (as usual, the multiplicities taken into account concern not only the proper points of $\mathbb{F}_n$, but also the infinitely near points). Moreover, the case where the multiplicities stay the same is only in (a), which cannot appear two consecutive times. Note that in all that process the intersection $\Gamma \cdot F_n$ remains unchanged. Then, after finitely many steps, the new curve Γ satisfies the conditions (2).

In case (a), we observe that the inequality $m_p(\Gamma) > \Gamma \cdot S_1$ combined with the inequality $\Gamma \cdot S_1 \geq (\Gamma \cdot S_1)_p \geq m_p(\Gamma) \cdot m_p(S_1)$ implies that $p \notin S_1$. We may then choose p to be a k-point of $F_1 \setminus S_1$ of maximal multiplicity and denote by $\tau\colon \mathbb{F}_1 \to \mathbb{P}^2$ the birational morphism contracting S_1 to a k-point $q \in \mathbb{P}^2$, observe that $\tau(F_1)$ is a line through q, that $\tau(\Gamma)$ is a curve of multiplicity $\Gamma \cdot S_1$ at q and of multiplicity $m_p(\Gamma) > \Gamma \cdot S_1$ at $p' = \tau(p) \in \tau(F_1)$. Moreover, p' is a k-point of $\tau(F_1)$ of maximal multiplicity on that line. Denote by $\tau'\colon \mathbb{F}_1' \to \mathbb{P}^2$ the birational morphism which is the blow-up at p'. Let S_1' be the exceptional fibre of τ', F_1' the strict transform of $\tau(F_1)$ and Γ' the strict transform of $\tau(\Gamma)$. We then replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1 \setminus (S_1 \cup F_1)$ with the analogous isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1' \setminus (S_1' \cup F_1')$ and get

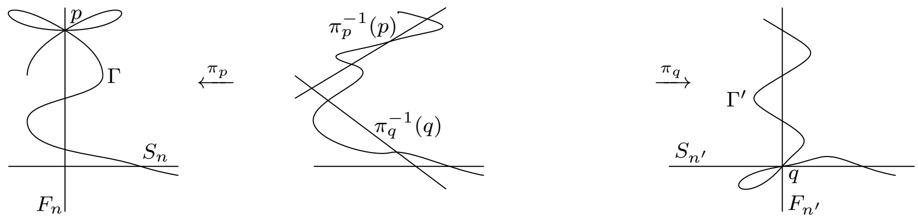
$$\forall\, r \in F_1',\ m_r(\Gamma') \leq \Gamma' \cdot S_1' = m_p(\Gamma).$$

Hence, (a) is no longer possible. Moreover, the singularities of the new curve Γ' have either decreased or stayed the same: Indeed, the multiplicities of the singular points of $\tau(\Gamma)$ are the same as those of Γ, plus one point of multiplicity $\Gamma \cdot S_1$. Similarly, the multiplicities of the singular points of $\tau(\Gamma)$ are the same as those of Γ', plus one point of multiplicity $m_p(\Gamma)$. Of course, we do not really get a singular point if the multiplicity is 1. Therefore, the singularities of the new curve remain the same if and only if $m_p(\Gamma) = 1$ and $\Gamma \cdot S_1 = 0$. The situation is illustrated below in a simple example (which satisfies $m_p(\Gamma) = 3 > \Gamma \cdot S_1 = 2$).



In case (b), we denote by $\kappa\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ the birational map that blows up the point p and contracts the strict transform of F_n. Call q the point to which the strict transform of F_n is contracted. We have $\kappa = \pi_q \circ (\pi_p)^{-1}$, where π_p, resp. π_q, are blow-ups of the point p of $\mathbb{F}_n$, resp. the point q of $\mathbb{F}_{n'}$. The drawing below illustrates the situation in a case where $n' = n - 1$. The composition of ι with κ provides a new isomorphism $\mathbb{A}^2 \to \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$, where $S_{n'}$ is the image of S_n and $F_{n'}$ is the curve corresponding to the exceptional divisor of p. Note that $F_{n'}$ is a fibre of the $\mathbb{P}^1$-bundle $\pi'\colon \mathbb{F}_{n'} \to \mathbb{P}^1$ corresponding to $\pi' = \pi_n \circ \kappa^{-1}$, and that $S_{n'}$ is a section, of self-intersection $-n'$, where $n' = n + 1$ if $p \in S_n$ and $n' = n - 1$ if $p \notin S_n$. Hence, since $n \geq 2$ or $n = 1$ and $\{p\} = S_n \cap F_n$, we get that $(S_{n'})^2 = -n' < 0$, and obtain a new isomorphism $\iota'\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$. The singularity of the new curve Γ' at the point q is equal to $\Gamma \cdot F_n - m_p(\Gamma)$, which is strictly smaller than $m_p(\Gamma)$ by assumption. Moreover

$2m_p(\Gamma) > \Gamma \cdot F_n \geq 2$, which implies that p was indeed a singular point of Γ.



Finally, we must now prove the last statement of our lemma, which concerns case (1). Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 3.3.1. Then, by replacing ι by $\iota \circ \psi$ and C by the equivalent curve $\psi^{-1}(C)$, we may assume that $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$ is the standard embedding. This being done, the restriction of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$ to $\mathbb{A}^2$ is $(x,y) \to [y:1]$. The fibres of π_n, equivalent to F_n being given by $y = \text{cst}$, the degree in x of the equation of C is equal to $\Gamma \cdot F_n$ (this can be done for instance by extending the scalars to $\overline{\mathrm{k}}$ and taking a general fibre). Since $\Gamma \cdot F_n = 1$, the equation is of the form $xa(y) + b(y)$ for some polynomials $a, b \in \mathrm{k}[y]$, $a \neq 0$. Since C is geometrically irreducible, the polynomials a and b are coprime. There exist (unique) polynomials $q, \tilde{b} \in \mathrm{k}[x]$ such that $b = aq + \tilde{b}$ with $\deg \tilde{b} < \deg a$. Then, changing the coordinates by applying $(x,y) \mapsto (x + q(y), y)$, we may furthermore assume that $\deg b < \deg a$.

Let us prove that points (*i*)-(*iv*) are equivalent. The implications (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*) are obvious. We then prove (*iii*) $\Rightarrow$ (*iv*) $\Rightarrow$ (*i*).

(*iii*) $\Rightarrow$ (*iv*): We recall that Γ is a section of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$, so that we have isomorphisms $\Gamma \simeq \mathbb{P}^1$ and $\Gamma \setminus F_n \simeq \mathbb{A}^1$. The fact that $C = \Gamma \setminus (F_n \cup S_n) \simeq \mathbb{A}^1$ implies that $C \cap (S_n \setminus F_n)$ is empty. Since $\Gamma \cap F_n \cap S_n = \emptyset$ by assumption, we get $\Gamma \cdot S_n = 0$.

(*iv*) $\Rightarrow$ (*i*): We use the open embedding

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (u,v) & \mapsto & ([1 : uv^n : u], [v:1]). \end{array}$$

The preimages of Γ and S_n by this embedding are the curves of equations $a(v) + b(v)u = 0$ and $u = 0$. Hence $\Gamma \cdot S_n = 0$ implies that a has no $\overline{\mathrm{k}}$-root and thus is a constant. $\square$

3.3.2 Extension to regular morphisms on $\mathbb{A}^2$

The following proposition is the principal tool in the proof of Proposition 3.3.10, Corollary 3.3.11 and Proposition 3.3.13, which themselves give the main part of Theorem 4.

Proposition 3.3.4. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve, not equivalent to a line, and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, there exists an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$, and such that $\iota(\mathbb{A}^2) = \mathbb{F}_n \setminus (S_n \cup F_n)$ (where S_n and F_n are as in Example 3.3.1).*

Proof. By Proposition 3.2.6, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible closed curve D. If φ extends to an automorphism of $\mathbb{A}^2$ sending C onto D, the result is obvious, by taking any isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n)$, so we may assume that φ does not extend to an automorphism of $\mathbb{A}^2$. Lemma 3.2.12 implies, since C is not equivalent to a line, that the same holds for D. Moreover, Proposition 3.2.6 implies that the extension of φ^{-1} to a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, via the standard embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, contracts the curve $\overline{D}$ to a k-point of $\mathbb{P}^2$. In particular, it does not send $\overline{D}$ birationally onto $\overline{C}$ or onto L_∞.

We choose an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$ given by Lemma 3.3.3, which comes from an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$, such that the closure of $\iota(D)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two possibilities (1)-(2) of Lemma 3.3.3.

We want to show that the open embedding $\iota \circ \varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{F}_n$ extends to a regular morphism on $\mathbb{A}^2$. Using the standard embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$ (Definition 3.2.1), we get a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ and need to show that all $\overline{\mathrm{k}}$-base-points of this map are contained in L_∞. Note that ψ restricts to an isomorphism $\mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n \cup \Gamma)$. This implies that all $\overline{\mathrm{k}}$-base-points of ψ, ψ^{-1} are defined over k (Lemma 3.2.5(1)) and gives the following commutative diagram

$$\begin{tikzcd}
 & & X \arrow[ld, "\eta"'] \arrow[rd, "\pi"] & & \\
\mathbb{A}^2 \arrow[r, hook, "\mathrm{std}"] & \mathbb{P}^2 \arrow[rr, dashed, "\psi"] & & \mathbb{F}_n & \mathbb{A}^2 \arrow[l, hook', "\iota"'] \\
 & \mathbb{A}^2 \setminus C \arrow[lu, hook'] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D, \arrow[ru, hook] &
\end{tikzcd}$$

where η, π are blow-ups of the base-points of ψ and ψ^{-1} respectively, and where $\eta^{-1}(L_\infty \cup \overline{C}) = \pi^{-1}(F_n \cup S_n \cup \Gamma)$ (Lemma 3.2.5(2)-(3)).

We assume by contradiction that ψ has a base-point q in $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_\infty$, which means that one (-1)-curve $E_q \subset X$ is contracted by η to q. This curve is the exceptional divisor of a base-point infinitely near to q, but not necessarily of q. The minimality of the resolution implies that π does not contract E_q, so $\pi(E_q)$ is a curve of $\mathbb{F}_n$ contracted by ψ^{-1} to q, which belongs to $\{\Gamma, F_n, S_n\}$.

We first study the case where ψ has no base-point in L_∞. The strict transform of L_∞ has then self-intersection 1 on X. Hence, it is not contracted by π, and thus sent onto a curve of self-intersection ≥ 1, which belongs to $\{\Gamma, F_n, S_n\}$ by Lemma 3.2.5(4). As $(F_n)^2 = 0$ and $(S_n)^2 = -n \leq -1$, L_∞ is sent onto Γ by ψ. This contradicts the fact that Γ is not sent birationally onto L_∞ by ψ^{-1}.

We can now reduce to the case where ψ also has a base-point p in L_∞. There is thus a (-1)-curve $E_p \subset X$ contracted by η to p and not contracted by π. As above, this curve is the exceptional divisor of a base-point infinitely near to p, but not necessarily of p. Again, $\pi(E_p)$ belongs to $\{\Gamma, F_n, S_n\}$.

We thus have at least two of the curves Γ, F_n, S_n that correspond to (-1)-curves of X contracted by η.

We suppose first that S_n corresponds to a (-1)-curve of X contracted by η. The fact that $(S_n)^2 = -n \leq -1$ implies that $n = 1$ and that π does not blow up any point of S_n. As there is another (-1)-curve of X contracted by η, the two curves are disjoint on X, and thus also disjoint on $\mathbb{F}_1$, since π does not blow up any point of S_1. The other curve is then Γ (since $F_1 \cdot S_1 = 1$), and $\Gamma \cdot S_1 = 0$. If moreover $\Gamma \cdot F_1 = 1$ (condition (1) of Lemma 3.3.3), then the contraction $\mathbb{F}_1 \to \mathbb{P}^2$ of S_1 sends Γ onto a line of $\mathbb{P}^2$, which contradicts the fact that $D \subset \mathbb{A}^2$ is not equivalent to a line. If $\Gamma \cdot F_1 \geq 2$, then condition (2) of Lemma 3.3.3 implies that $m_r(\Gamma) \leq \Gamma \cdot S_1 = 0$ for each $r \in F_1(\mathrm{k})$. Hence, the intersection of Γ with F_1 (which is not empty since $\Gamma \cdot F_1 \geq 2$) consists only of points not defined over k, which are therefore not blown up by π. The strict transforms $\tilde{\Gamma}$ and $\tilde{F_1}$ on X then satisfy $\tilde{\Gamma} \cdot \tilde{F_1} = \Gamma \cdot F_1 \geq 2$. As $\tilde{\Gamma}$ is contracted by η, the image $\eta(\tilde{F_1})$ is a singular curve and is then equal to $\overline{C}$. This contradicts the fact that ψ contracts $\overline{C}$ to a point.

There remains the case is when S_n does not correspond to a (-1)-curve of X contracted by η, which implies that $\{\pi(E_p), \pi(E_q)\} = \{F_n, \Gamma\}$, or equivalently that $\{E_p, E_q\} = \{\tilde{F_n}, \tilde{\Gamma}\}$, where $\tilde{F_n}$ and $\tilde{\Gamma}$ denote the strict transforms of F_n and Γ on X. Since $(F_n)^2 = 0$ and $(\tilde{F_n})^2 = -1$, there exists exactly one $\overline{\mathrm{k}}$-point $r \in F_n$ (and no infinitely near points) blown up by π, which is then a k-point (as all base-points of π are defined over k). We obtain

$$m_r(\Gamma) = \Gamma \cdot F_n \geq 1 \text{ and } \Gamma \cap F_n = \{r\},$$

since $\tilde{F_n}$ and $\tilde{\Gamma}$ are disjoint on X (and because $\Gamma \cdot F_n \geq 1$, as Γ satisfies one of the two conditions (1)-(2) of Lemma 3.3.3).

We now prove that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F_n}$ and $\tilde{\Gamma}$, i.e. the two curves E_p and E_q. For this, it suffices to prove that $r \notin S_n$ and that $S_n \cdot \Gamma \geq 1$. Suppose first that $\Gamma \cdot F_n = 1$ (condition (1) of Lemma 3.3.3). Since $\Gamma \cap F_n \cap S_n = \emptyset$, we get $r \in F_n \setminus S_n$. The inequality $\Gamma \cdot S_n > 0$ is provided by the fact that D is not equivalent to a line (see again condition (1) of Lemma 3.3.3 and the equivalence between (*ii*) and (*iv*) given in that case). Suppose now that $\Gamma \cdot F_n \geq 2$. As $m_r(\Gamma) = \Gamma \cdot F_n \geq 2$, we have $2m_r(\Gamma) > \Gamma \cdot F_n$, which implies that $n = 1$, $r \in F_n \setminus S_n$ and $2 \leq m_r(\Gamma) \leq \Gamma \cdot S_n$ (see again possibility (2) of Lemma 3.3.3).

We conclude by observing that, since $\eta(E_q) = q \in \mathbb{P}^2 \setminus L_\infty$ and $\eta(E_p) = p \in L_\infty$, any connected set of curves of $\eta^{-1}(L_\infty \cup \overline{C})$ which intersects the two curves E_q and E_p must contain the strict transform $\tilde{C}$ of $\overline{C}$. Since $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are included in $\pi^{-1}(F_n \cup S_n \cup \Gamma) = \eta^{-1}(L_\infty \cup \overline{C})$, this contradicts the fact that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F_n}$ and $\tilde{\Gamma}$. $\square$

A direct consequence of Proposition 3.3.4 is the following corollary, which shows that only smooth curves $C \subset \mathbb{A}^2$ are interesting to study. This also follows from Proposition 3.3.10 below. Since the proof of Proposition 3.3.10 is more involved, we prefer first to explain the simpler argument that shows how the smoothness follows from Proposition 3.3.4.

Corollary 3.3.5. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. If C is not smooth, then every open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. By Proposition 3.2.6, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible closed curve D. We apply Proposition 3.3.4 and obtain an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. Embedding $\mathbb{A}^2$ into $\mathbb{P}^2$, we get a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ which is regular on $\mathbb{A}^2$. In particular, the singular $\overline{\mathrm{k}}$-points of C are not blown up in the minimal resolution of ψ. Hence, the curve $\overline{C}$ is not contracted by ψ and is thus sent onto a singular curve $\psi(\overline{C}) \subset \mathbb{F}_n$. Since ψ restricts to an isomorphism $\mathbb{P}^2 \setminus (L_\infty \cup \overline{C}) \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n \cup \overline{D})$, Lemma 3.2.5(4) shows that the singular curve $\psi(\overline{C})$ must be F_n, S_n or $\overline{D}$. As F_n and S_n are smooth, we find that $\psi(\overline{C}) = \overline{D}$. Proposition 3.2.6 then shows that φ extends to an automorphism of $\mathbb{A}^2$. □

Another direct consequence of Proposition 3.3.4 is the following result, which shows that in case (3) of Proposition 3.2.6, the point to which $\overline{C}$ is contracted lies in $\mathbb{A}^2$ only in a very special situation:

Corollary 3.3.6. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. If the extension of φ to $\mathbb{P}^2$ contracts the curve C (or equivalently its closure) to a point of $\mathbb{A}^2$, then there exist automorphisms α, β of $\mathbb{A}^2$ and an endomorphism $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ of the form $(x, y) \mapsto (x, x^n y)$, where $n \geq 1$ is an integer, such that $\varphi = \alpha\psi\beta$. In particular, $C \subset \mathbb{A}^2$ is equivalent to a line, via β.*

Proof. By Proposition 3.2.6, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible closed curve D. Denote by $\varphi^{-1}\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ the birational transformation which is the inverse of φ. Since C is contracted by φ to a point of $\mathbb{A}^2$, it is not possible to find an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the birational map $\iota \circ \varphi^{-1}$ actually defines a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. By Proposition 3.3.4, this implies that D is equivalent to a line. Hence, the same holds for C, by Lemma 3.2.12. Applying automorphisms of $\mathbb{A}^2$ at the source and the target, we may then assume that C and D are equal to the line $x = 0$. By Lemma 3.2.12(1), the map φ is of the form $(x, y) \mapsto (\lambda x, \mu x^n y + s(x))$, where $\lambda, \mu \in \mathrm{k}^*$, $n \geq 1$ and $s \in \mathrm{k}[x]$ is a polynomial. We then observe that $\varphi = \alpha\psi$, where α is the automorphism of $\mathbb{A}^2$ given by $(x, y) \mapsto (\lambda x, \mu y + s(x))$ and ψ is the endomorphism of $\mathbb{A}^2$ given by $(x, y) \mapsto (x, x^n y)$. □

Corollary 3.3.6 also gives a simple proof of the following characterisation of birational endomorphisms of $\mathbb{A}^2$ that contract only one geometrically irreducible closed curve. This result has already been obtained by Daniel Daigle in [Dai91, Theorem 4.11].

Corollary 3.3.7. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let φ be a birational endomorphism of $\mathbb{A}^2$ which restricts to an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$. Then, the following assertions are equivalent:*

(i) *The endomorphism φ contracts the curve C.*

(ii) *The endomorphism φ is not an automorphism.*

(iii) *There exist automorphisms α, β of $\mathbb{A}^2$ and an endomorphism $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ of the form $(x, y) \mapsto (x, x^n y)$, where $n \geq 1$ is an integer, such that $\varphi = \alpha\psi\beta$.*

Proof. $(iii) \Rightarrow (ii)$: This follows from the fact that, for each $n \geq 1$, the map $\psi\colon (x, y) \mapsto (x, x^n y)$ is a birational endomorphism of $\mathbb{A}^2$ which is not an automorphism, as its inverse $\psi^{-1}\colon (x, y) \mapsto (x, x^{-n} y)$ is not regular.

$(ii) \Rightarrow (i)$: Denote by $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map induced by φ. Since φ is an endomorphism of $\mathbb{A}^2$ which is not an automorphism, cases (1)-(2) of Proposition 3.2.6 are not possible. Hence, we are in case (3): C is contracted by $\hat{\varphi}$ to a point of $\mathbb{P}^2$, which is necessarily in $\mathbb{A}^2$ since $\varphi(\mathbb{A}^2) \subset \mathbb{A}^2$.

$(i) \Rightarrow (iii)$: This follows from Corollary 3.3.6. □

3.3.3 Completion with two curves and a boundary

The following technical Proposition 3.3.10 is used to prove Corollary 3.3.11 and Proposition 3.3.13, which yield almost all statements of Theorem 4.

Definition 3.3.8. Let X be a smooth projective surface. A reduced closed curve $C \subset X$ is a k-*forest of* X if C is a finite union of closed curves $C_1, \ldots, C_n$, all isomorphic (over k) to $\mathbb{P}^1$ and if each singular $\overline{\mathrm{k}}$-point of C is a k-point lying on exactly two components C_i, C_j intersecting transversally. We moreover ask that C does not contain any loop. If C is connected, we say that C is a k-*tree*.

Remark 3.3.9. If $\eta\colon X \to Y$ is a birational morphism between smooth projective surfaces such that all $\overline{\mathrm{k}}$-base-points of η^{-1} are defined over k, then the exceptional curve of η (the union of the contracted curves) is a k-forest $E \subset X$. Moreover, the strict transform and the preimage of any k-forest of Y is a k-forest of X. The preimage of a k-tree is a k-tree.

Proposition 3.3.10. *Let $C, D \subset \mathbb{A}^2$ be geometrically irreducible closed curves, not equivalent to lines, and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism which does not extend to an automorphism of $\mathbb{A}^2$. Then there is a smooth projective surface X and two open embeddings $\rho_1, \rho_2\colon \mathbb{A}^2 \hookrightarrow X$ which make the following diagram commutative*

$$\begin{tikzcd} & X & \\ \mathbb{A}^2 \arrow[ur, hook, "\rho_1"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_2"'] \\ \mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D \arrow[u, hook] \end{tikzcd}$$

and such that the following holds:

(*i*) *The curves* $\Gamma = \overline{\rho_1(C)} \subset X$, $\Delta = \overline{\rho_2(D)} \subset X$ *are isomorphic to* $\mathbb{P}^1$.

(*ii*) *For* $i = 1, 2$, *we have* $\rho_i(\mathbb{A}^2) = X \setminus B_i$ *for some* k*-tree* B_i.

(*iii*) *Writing* $B = B_1 \cap B_2$, *we have* $B_1 = B \cup \Delta$ *and* $B_2 = B \cup \Gamma$.

(*iv*) *There is no birational morphism* $X \to Y$, *where* Y *is a smooth projective surface, which contracts one connected component of* B, *and no other* $\overline{\mathrm{k}}$*-curve.*

(*v*) *The number of connected components of* B *is equal to the number of* $\overline{\mathrm{k}}$*-points of* $B \cap \Gamma$ *and to the number of* $\overline{\mathrm{k}}$*-points of* $B \cap \Delta$, *and is at most* 2.

Proof. By Proposition 3.3.4, there exist integers $m, n \geq 1$, and isomorphisms

$$\iota_1 \colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_m \setminus (S_m \cup F_m),\ \ \iota_2 \colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$$

such that both open embeddings $\iota_1\varphi^{-1} \colon \mathbb{A}^2 \setminus D \to \mathbb{F}_m$ and $\iota_2\varphi \colon \mathbb{A}^2 \setminus C \to \mathbb{F}_n$ extend to regular morphisms $u_1 \colon \mathbb{A}^2 \to \mathbb{F}_m$ and $u_2 \colon \mathbb{A}^2 \to \mathbb{F}_n$. Denoting by $\psi \colon \mathbb{F}_m \dashrightarrow \mathbb{F}_n$ the corresponding birational map, equal to $\iota_2(u_1)^{-1} = u_2(\iota_1)^{-1}$, the restriction of ψ gives an isomorphism $\mathbb{F}_m \setminus (S_m \cup F_m \cup \iota_1(C)) \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n \cup \iota_2(D))$ (which corresponds to φ). We then have the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\
\mathbb{F}_m \arrow[rr, dashed, "\psi"] & & \mathbb{F}_n \\
\mathbb{A}^2 \arrow[u, hook, "\iota_1"] \arrow[urr, "u_2"'] & & \mathbb{A}^2 \arrow[u, hook', "\iota_2"'] \arrow[ull, "u_1"] \\
\mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[rr, "\varphi", "\simeq"'] & & \mathbb{A}^2 \setminus D \arrow[u, hook']
\end{tikzcd}$$

where η and π are birational morphisms, which are sequences of blow-ups of k-points, being the base-points of ψ and ψ^{-1} respectively (Lemma 3.2.5).

Since u_1, u_2 are regular on $\mathbb{A}^2$, the $\overline{\mathrm{k}}$-base-points of ψ (which are k-points), resp. ψ^{-1}, are infinitely near to k-points of $F_m \cup S_m \subset \mathbb{F}_m$, resp. $F_n \cup S_n \subset \mathbb{F}_n$. In particular, we get two open embeddings

$$\rho_1 = \eta^{-1}\iota_1 \colon \mathbb{A}^2 \hookrightarrow X,\ \ \rho_2 = \pi^{-1}\iota_2 \colon \mathbb{A}^2 \hookrightarrow X$$

such that $\rho_2\varphi = \rho_1$ (or more precisely $\rho_2\varphi = \rho_1|_{\mathbb{A}^2\setminus C}$). We have $\rho_1(\mathbb{A}^2) = X \setminus B_1$ and $\rho_2(\mathbb{A}^2) = X \setminus B_2$, where $B_1 := \eta^{-1}(S_m \cup F_m)$ and $B_2 := \pi^{-1}(S_n \cup F_n)$ are k-trees (see Remark 3.3.9).

By Lemma 3.2.5, the following equality holds:

$$\eta^{-1}(S_m \cup F_m \cup \iota_1(C)) = \pi^{-1}(S_n \cup F_n \cup \iota_2(D)).$$

The left-hand side is equal to $B_1 \cup \Gamma$, where $\Gamma = \overline{\rho_1(C)} \subset X$ is the strict transform of $\overline{\iota_1(C)} \subset \mathbb{F}_m$ by η and the right-hand side is equal to $B_2 \cup \Delta$, where $\Delta = \overline{\rho_2(D)} \subset X$

is the strict transform of $\overline{\iota_2(D)} \subset \mathbb{F}_n$ by π. The fact that φ does not extend to an automorphism of $\mathbb{A}^2$ implies that $B_1 \neq B_2$, whence $\Delta \neq \Gamma$. Writing $B := B_1 \cap B_2$, the equality $B_1 \cup \Gamma = B_2 \cup \Delta$ yields:

$$B_2 = B \cup \Gamma \text{ and } B_1 = B \cup \Delta \text{ (with } \Gamma = \overline{\rho_1(C)}, \Delta = \overline{\rho_2(D)} \subset X).$$

In particular, since B_1, B_2 are two k-trees, Γ and Δ are isomorphic to $\mathbb{P}^1$ (over k) and intersect transversally B in a finite number of k-points. We have now found the surface X together with the embeddings ρ_1, ρ_2, satisfying conditions (i)–(ii)–(iii). We will then modify X if needed, in order to get also (iv)–(v).

The number of connected components of B is equal to the number of $\overline{\mathrm{k}}$-points of $B \cap \Gamma$, and of $B \cap \Delta$: This follows from the fact that $B \cup \Gamma$ and $B \cup \Delta$ are k-trees. Remember also that each $\overline{\mathrm{k}}$-point of $B \cap \Gamma$, or of $B \cap \Delta$, is a k-point, as mentioned earlier.

Suppose that the number of connected components of B is $r \geq 3$, and let us show that at least $r-2$ connected components of B are contractible (in the sense that there is a birational morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts one component of B and no other $\overline{\mathrm{k}}$-curve). To show this, we first observe that Γ intersects r distinct curves of B. Since Γ is one of the irreducible components of $B_2 = \pi^{-1}(S_n \cup F_n)$, we can decompose π as $\pi_2 \circ \pi_1$ where $\pi_1(\Gamma)$ is an irreducible component of $(\pi_2)^{-1}(S_n \cup F_n)$ intersecting exactly two other irreducible components R_1, R_2, and such that all $\overline{\mathrm{k}}$-points blown up by π_1 are infinitely near points of $\pi_1(\Gamma) \setminus (R_1 \cup R_2)$. This proves that we can contract at least $r-2$ connected components of B.

If one connected component of B is contractible, there exists a morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts this component of B, and no other curve. Since the component intersects Δ transversally in one point, and also Γ in one point, we can replace X by Y, ρ_1, ρ_2 by their compositions with the morphism $X \to Y$ and still fulfill conditions (i)–(ii)–(iii). After finitely many steps, condition (iv) is satisfied. By the observation made earlier, the number of connected components of B, after this is done, is at most 2, giving then (v). □

Corollary 3.3.11. *Let $C, D \subset \mathbb{A}^2$ be geometrically irreducible closed curves and let $\varphi \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism which does not extend to an automorphism of $\mathbb{A}^2$.*

Then, the curves C, D are isomorphic to open subsets of $\mathbb{A}^1$: there exist polynomials $P, Q \in \mathrm{k}[t]$ without square factors, such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$. Moreover, the numbers of $\overline{\mathrm{k}}$-roots of P and Q are the same (i.e. extending the scalars to $\overline{\mathrm{k}}$, the curves C and D become isomorphic to $\mathbb{A}^1$ minus some finite number of points, the same number for both curves). The numbers of k-roots of P and Q are also the same.

Remark 3.3.12. When $\mathrm{k} = \mathbb{C}$, this follows from the fact that C and D are isomorphic to open subsets of $\mathbb{A}^1$, since the curves are rational (Corollary 3.2.7) and smooth (Corollary 3.3.5). Indeed, since $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic, they have the same Euler characteristic, so C and D also have the same Euler characteristic.

Proof. If C or D is equivalent to a line, so are both curves (Lemma 3.2.12), and the result holds. Otherwise, we apply Proposition 3.3.10 and get a smooth projective surface X and two open embeddings $\rho_1, \rho_2 \colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$ and satisfying the conditions (*i*)-(*ii*)-(*iii*)-(*iv*)-(*v*). In particular, C is isomorphic to $\Gamma \setminus B_1 = \Gamma \setminus ((\Gamma \cap B) \cup (\Gamma \cup \Delta))$. Since Γ is isomorphic to $\mathbb{P}^1$ and $\Gamma \cap B$ consists of one or two k-points, this shows that Γ is isomorphic to an open subset of $\mathbb{A}^1$. Proceeding similarly for D, we get isomorphisms $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ where $P, Q \in \mathrm{k}[t]$ are polynomials, which we may assume without square factors.

The number of $\overline{\mathrm{k}}$-roots of P is equal to the number of $\overline{\mathrm{k}}$-points of $\Gamma \cap B_1$ minus 1. Similarly, the number of $\overline{\mathrm{k}}$-roots of Q is equal to the number of $\overline{\mathrm{k}}$-points of $\Delta \cap B_2$ minus 1. To see that these numbers are equal, we observe that $\Gamma \cap B_1 = (\Gamma \cap B) \cup (\Gamma \cap \Delta)$, that $\Delta \cap B_2 = (\Delta \cap B) \cup (\Delta \cap \Gamma)$, and that the number of $\overline{\mathrm{k}}$-points of $\Gamma \cap B$ is the same as the number of $\overline{\mathrm{k}}$-points of $\Delta \cap B$ (this follows from (*v*)). As each point of $\Gamma \cap B$ that is contained in $\Gamma \cap \Delta$ is also contained in $\Delta \cap B$, this shows that P and Q have the same number of $\overline{\mathrm{k}}$-roots. As each $\overline{\mathrm{k}}$-point of $\Gamma \cap B_1$ or $\Delta \cap B_2$ which is not a k-point is contained in $\Gamma \cap \Delta$, the polynomials P and Q have the same number of k-roots. □

Proposition 3.3.13. *Let $C, D, D' \subset \mathbb{A}^2$ be geometrically irreducible closed curves, not equivalent to lines, and let $\varphi \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$, $\varphi' \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D'$ be isomorphisms which do not extend to automorphisms of $\mathbb{A}^2$. Then, one of the following holds:*

(*a*) *The map $\varphi'(\varphi)^{-1}$ extends to an automorphism of $\mathbb{A}^2$ (sending D to D');*

(*b*) *The curves C, D, D' are isomorphic to $\mathbb{A}^1$;*

(*c*) *The curves C, D, D' are isomorphic to $\mathbb{A}^1 \setminus \{0\}$.*

Remark 3.3.14. Case (*b*) never occurs, as we will show later. Indeed, since C is not equivalent to a line, the existence of φ, φ' is excluded (Proposition 3.3.16 below).

Proof. If $C \simeq \mathbb{A}^1$ or $C \simeq \mathbb{A}^1 \setminus \{0\}$, then $D \simeq C \simeq D'$ by Corollary 3.3.11. We may thus assume that C is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. We apply Proposition 3.3.10 with φ and φ' and get smooth projective surfaces X, X' and open embeddings $\rho_1, \rho_2, \rho_1', \rho_2' \colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$, $\rho_2'\varphi' = \rho_1'$ and satisfying the conditions (*i*)-(*ii*)-(*iii*)-(*iv*)-(*v*). In particular, we obtain an isomorphism $\kappa \colon X \setminus (B \cup \Gamma \cup \Delta) \xrightarrow{\simeq} X' \setminus (B' \cup \Gamma' \cup \Delta')$ (where $\Gamma = \overline{\rho_1(C)} \subset X$, $\Delta = \overline{\rho_2(D)} \subset X$, $\Gamma' = \overline{\rho_1'(C)} \subset X'$, $\Delta' = \overline{\rho_2'(D')} \subset X'$) and a commutative diagram

$$\begin{tikzcd}
& X \arrow[rr, dashed, "\kappa"] & & X' & \\
\mathbb{A}^2 \arrow[ur, hook, "\rho_2"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_1"'] \arrow[ur, hook, "\rho_1'"] & & \mathbb{A}^2 \arrow[ul, hook', "\rho_2'"'] \\
\mathbb{A}^2 \setminus D \arrow[u, hook] & & \mathbb{A}^2 \setminus C \arrow[u, hook] \arrow[ll, "\varphi", "\simeq"'] \arrow[rr, "\varphi'", "\simeq"'] & & \mathbb{A}^2 \setminus D' \arrow[u, hook]
\end{tikzcd}$$

By construction, κ sends birationally $\Gamma = \overline{\rho_1(C)}$ onto $\Gamma' = \overline{\rho_1'(C)}$. If κ also sends Δ birationally onto Δ', then $\varphi'\varphi^{-1}$ extends to a birational map that sends birationally D onto D' and then extends to an automorphism of $\mathbb{A}^2$ (Proposition 3.2.6). It remains then to show that this is the case.

Using Lemma 3.2.5, we take a minimal resolution of the indeterminacies of κ:

$$\begin{tikzcd} & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\kappa"] & & X' \end{tikzcd}$$

where η and π are the blow-ups of the $\overline{\mathrm{k}}$-base-points of κ and κ^{-1}, all being k-rational. We want to show that the strict transforms $\tilde{\Delta}$ and $\tilde{\Delta}'$ of $\Delta \subset X$, $\Delta' \subset X'$ are equal. We will do this by studying the strict transform $\tilde{\Gamma} = \tilde{\Gamma}'$ of Γ and Γ' and its intersection with $\tilde{\Delta}$ and $\tilde{\Delta}'$ and with the other components of $B_Z = \eta^{-1}(B \cup \Gamma \cup \Delta) = \pi^{-1}(B' \cup \Gamma' \cup \Delta')$.

Recall that $B_1 = B \cup \Delta$, $B_2 = B \cup \Gamma$, $B_1' = B' \cup \Delta'$, $B_2' = B' \cup \Gamma'$ are k-trees and that C is isomorphic to $\Gamma \setminus B_1$ and $\Gamma' \setminus B_1'$ (Proposition 3.3.10).

(i) Suppose first that $\Gamma \cap B_1$ contains some $\overline{\mathrm{k}}$-points which are not defined over k. None of these points is thus a base-point of κ and each of these points belongs to $\Gamma \cap \Delta$, so $\tilde{\Gamma} \cap \tilde{\Delta}$ contains $\overline{\mathrm{k}}$-points not defined over k. Since B_2' is a k-tree, $\pi^{-1}(B_2')$ is a k-tree, so $\tilde{\Gamma} = \tilde{\Gamma}'$ intersects all irreducible components of B_Z into k-points, except maybe $\tilde{\Delta}'$. This yields $\tilde{\Delta} = \tilde{\Delta}'$ as we wanted.

(ii) We can now assume that all $\overline{\mathrm{k}}$-points of $\Gamma \cap B_1$ are defined over k, which implies that all intersections of irreducible components of B_Z are defined over k. We will say that an irreducible component of B_Z is *separating* if the union of all other irreducible components is a k-forest (see Definition 3.3.8).

Since $B_1 = B \cup \Delta$ is a k-tree, its preimage on B_Z is a k-tree. The union of all components of B_Z distinct from $\tilde{\Gamma}$ being equal to the disjoint union of $\eta^{-1}(B_1)$ with some k-forest contracted to points of $\Gamma \setminus B_1$, we find that $\tilde{\Gamma}$ is separating. The same argument shows that $\tilde{\Delta}$ and $\tilde{\Delta}'$ are also separating.

It remains then to show that any irreducible component $E \subset B_Z$ which is not equal to $\tilde{\Delta}$ or $\tilde{\Gamma}$ is not separating. We use for this the fact that $C \simeq \Gamma \setminus B_1$ is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$, so the set $\Gamma \cap B_1$ contains at least 3 points. If $\eta(E)$ is a point q, then the complement of $\eta^{-1}(q)$ in B_Z contains a loop, since Γ intersects the k-tree B_1 into at least two points distinct from q. If $\eta(E)$ is not a point, it is one of the components of B. We denote by F the union of all irreducible components of $B \cup \Gamma \cup \Delta$ not equal to $\eta(E)$, and prove that F is not a k-forest, since it contains a loop. This is true if $\Delta \cap \Gamma$ contains at least 2 points. If $\Delta \cap \Gamma$ contains one or less points, then $\Delta \cap B$ contains at least two points, so contains exactly two points, on the two connected components of B which both intersect Γ and Δ (see Proposition 3.3.10(v)). We again get a loop on the union of Γ, Δ and of the connected component of B not containing $\eta(E)$. The fact that F contains a loop implies that $\eta^{-1}(F)$ contains a loop, and achieves to prove that E is not separating. $\square$

3.3.4 The case of curves isomorphic to $\mathbb{A}^1$ and the proof of Theorem 4

To finish the proof of Theorem 4, we still need to handle the case of curves isomorphic to $\mathbb{A}^1$. The case of lines has already been treated in Lemma 3.2.12. In characteristic zero, this finishes the study by the Abyhankar-Moh-Suzuki theorem, but in positive characteristic, there are many closed curves of $\mathbb{A}^2$ which are isomorphic to $\mathbb{A}^1$, but are not equivalent to lines (these curves are sometimes called "bad lines" in the literature). We will show that an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ always extends to $\mathbb{A}^2$ if C is isomorphic to $\mathbb{A}^1$, but not equivalent to a line.

Lemma 3.3.15. *Let $n \geq 1$ and let $\Gamma \subset \mathbb{F}_n$ be a geometrically irreducible closed curve such that $\Gamma \cdot F_n \geq 2$. If there exists a birational map $\mathbb{F}_n \dashrightarrow \mathbb{P}^2$ that contracts Γ to a point (and perhaps contracts some other curves), then Γ is geometrically rational and singular. Moreover, one of the following occurs:*

(*a*) *There exists a point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$.*

(*b*) *We have $n = 1$ and there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$.*

Proof. We may assume that $\mathrm{k} = \overline{\mathrm{k}}$. Denote by $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ the birational map that contracts C to a point (and maybe some other curves). The minimal resolution of this map yields a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{F}_n \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^2 \end{tikzcd}$$

In $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n \bigoplus \mathbb{Z}S_n$ we write

$$\begin{array}{rcl} \Gamma & = & aS_n + bF_n \\ -K_{\mathbb{F}_n} & = & 2S_n + (2+n)F_n \end{array}$$

for some integers a, b. Note that $a = \Gamma \cdot F_n \geq 2$ and that $b - an = \Gamma \cdot S_n \geq 0$. By hypothesis, the strict transform $\tilde{\Gamma}$ of Γ on X is a smooth curve contracted by π. In particular, Γ is rational and the divisor $2\tilde{\Gamma} + aK_X$ is not effective, since

$$(2\tilde{\Gamma} + aK_X) \cdot \pi^*(L) = aK_X \cdot \pi^*(L) = a\pi^*(K_{\mathbb{P}^2}) \cdot \pi^*(L) = aK_{\mathbb{P}^2} \cdot L = -3a < 0$$

for a general line $L \subset \mathbb{P}^2$.

Denoting by $E_1, \ldots, E_r \in \mathrm{Pic}(X)$ the pull-backs of the exceptional divisors blown up by η (which satisfy $(E_i)^2 = -1$ for each i and $E_i \cdot E_j = 0$ for $i \neq j$) we have

$$\begin{array}{rcccccl} \tilde{\Gamma} & = & a\eta^*(S_n) & + & b\eta^*(F_n) & - & \sum_{i=1}^r m_i E_i \\ -K_X & = & 2\eta^*(S_n) & + & (2+n)\eta^*(F_n) & - & \sum_{i=1}^r E_i \\ 2\tilde{\Gamma} + aK_X & = & & & (2b - a(2+n))\eta^*(F_n) & + & \sum_{i=1}^r (a - 2m_i)E_i \end{array}$$

which implies, since $2\tilde{\Gamma} + aK_X$ is not effective, that either $2b < a(2+n)$ or $2m_i > a$ for some i. If $2m_i > a$ for some i, we get (a), since the m_i are the multiplicities of $\tilde{\Gamma}$ at the points blown up by η.

It remains to study the case where $2m_i \leq a$ for each i, and where $2b < a(2+n)$. Remembering that $b - an = \Gamma \cdot S_n \geq 0$, we find $n \leq \frac{b}{a} < \frac{2+n}{2}$, whence $n = 1$ and thus $2b < 3a$. We then compute

$$3\tilde{\Gamma} + bK_X \quad = \quad (3a-2b)\eta^*(S_n) \quad + \quad \textstyle\sum_{i=1}^{r}(b - 3m_i)E_i$$

which is again not effective, since $(3\tilde{\Gamma} + bK_X) \cdot \pi^*(L) = bK_X \cdot \pi^*(L) = -3b < 0$ for a general line $L \subset \mathbb{P}^2$, because $b \geq an = a \geq 2$. This implies that there exists an integer i such that $3m_i > b$. Since $2m_i \leq a$, we find $m_i > b - a = \Gamma \cdot S_1$, which implies ($b$). $\square$

Proposition 3.3.16. *Let $C \subset \mathbb{A}^2$ be a closed curve, isomorphic to $\mathbb{A}^1$ (over k). The following are equivalent:*

(a) *The curve C is equivalent to a line.*

(b) *There exists an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$.*

(c) *There exists a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ that contracts the curve C (or its closure) to a $\overline{\mathrm{k}}$-point (and perhaps contracts some other curves). In this statement $\mathbb{A}^2$ is identified with an open subset of $\mathbb{P}^2$ via the standard embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) can be observed, for example, by taking the map $(x, y) \mapsto (x, xy)$, which is an open embedding of $\mathbb{A}^2 \setminus \{x = 0\}$ into $\mathbb{A}^2$, which does not extend to an automorphism of $\mathbb{A}^2$, and whose extension to $\mathbb{P}^2$ contracts the line $x = 0$ to a point.

To prove (b) $\Rightarrow$ (c), we take an open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$ and look at the extension to $\mathbb{P}^2$. By Proposition 3.2.6, either this contracts C, or C is equivalent to a line, in which case (c) is true as was shown earlier.

It remains to prove (c) $\Rightarrow$ (a). We apply Lemma 3.3.3, and obtain an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two cases (1)-(2) of Lemma 3.3.3. In case (1), the curve is equivalent to a line as it is isomorphic to $\mathbb{A}^1$ (equivalence (ii) $-$ (iii) of Lemma 3.3.3). It remains to study the case where Γ satisfies conditions (2) of Lemma 3.3.3 (in particular $\Gamma \cdot F_n \geq 2$), and to show that these, together with (c), yield a contradiction. We prove that there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$. Indeed, since $\Gamma \cdot F_n \geq 2$, such a point would be a singular point of Γ, and since $\Gamma \setminus (S_n \cup F_n) = \iota(C) \simeq C$ is isomorphic to $\mathbb{A}^1$, p would be a k-point and the unique $\overline{\mathrm{k}}$-point of $\Gamma \cap (S_n \cup F_n)$. Moreover, as $\Gamma \cdot F_n \geq 2$, we would find that $p \in F_n$. Since $2m_p(\Gamma) > \Gamma \cdot F_n$ and because Γ satisfies conditions (2) of Lemma 3.3.3, the only possibility would be that $n = 1$, $p \in F_1 \setminus S_1$

and $0 < m_p(\Gamma) \leq \Gamma \cdot S_1$. This contradicts the fact that $\Gamma \cap (S_1 \cup F_1)$ contains only one $\overline{\mathrm{k}}$-point.

Denote by $\psi_0 \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map that contracts C (and maybe some other curves) to a $\overline{\mathrm{k}}$-point. Observe that $\psi_0 \circ \iota^{-1}$ yields a birational map $\psi \colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ which contracts Γ to a $\overline{\mathrm{k}}$-point. As there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$, Lemma 3.3.15 implies that $n = 1$ and that there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$. Again, this point is a k-point, since C is isomorphic to $\mathbb{A}^1$. This contradicts the conditions (2) of Lemma 3.3.3. □

Remark 3.3.17. If k is algebraically closed, the equivalence between conditions (*a*) and (*c*) of Proposition 3.3.16 can also be proved using Kodaira dimension. We introduce the following conditions:

(*a*)′ The Kodaira dimension $\kappa(C, \mathbb{A}^2)$ of C is equal to $-\infty$.

(*c*)′ There exists a birational transformation of $\mathbb{P}^2$ that sends C onto a line.

The equivalence between (*a*) and (*a*)′ follows from [Gan85, Theorem 2.4.(1)] and the equivalence between (*a*)′ and (*c*)′ is Coolidge's theorem (see e.g. [KM83, Theorem 2.6]). We now recall how the classical equivalence between (*c*) and (*c*)′ can be proven. Every simple quadratic birational transformation of $\mathbb{P}^2$ contracts three lines. This proves (*c*)′ ⇒ (*c*). To get (*c*) ⇒ (*c*)′, we take a birational transformation φ of $\mathbb{P}^2$ that contracts C to a point and decompose φ as $\varphi = \varphi_r \circ \cdots \circ \varphi_1$, where each φ_i is a simple quadratic transformation (using the Castelnuovo-Noether factorisation theorem). If $i \geq 1$ is the smallest integer such that $(\varphi_i \circ \cdots \circ \varphi_1)(C)$ is a $\overline{\mathrm{k}}$-point, the curve $(\varphi_{i-1} \circ \cdots \circ \varphi_1)(C)$ is contracted by φ_i and is thus a line.

Remark 3.3.18. If the field k is perfect, then every curve that is geometrically isomorphic to $\mathbb{A}^1$ (i.e. over $\overline{\mathrm{k}}$) is also isomorphic to $\mathbb{A}^1$. This can be seen by embedding the curve in $\mathbb{P}^1$ and considering the complement point, necessarily defined over k. For non-perfect fields, there exist closed curves $C \subset \mathbb{A}^2$ geometrically isomorphic to $\mathbb{A}^1$, but not isomorphic to $\mathbb{A}^1$ (see [Rus70]). Corollary 3.3.11 shows that every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$ for all such curves.

We can now conclude this section by proving Theorem 4:

Proof of Theorem 4. We recall the hypotheses of the theorem: we have a geometrically irreducible closed curve $C \subset \mathbb{A}^2$ and an isomorphism $\varphi \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ for some closed curve $C \subset \mathbb{A}^2$. Moreover, φ does not extend to an automorphism of $\mathbb{A}^2$. We consider the following three cases:

If C is isomorphic to $\mathbb{A}^1$, then the implication (*b*) ⇒ (*a*) of Proposition 3.3.16 shows that C is equivalent to a line and Lemma 3.2.12(2) implies that the same holds for D. In particular, the curves C and D are isomorphic. This achieves the proof of the theorem in this case.

If C is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ then so is D by Corollary 3.3.11. This also gives the result in this case.

It remains to assume that C is not isomorphic to $\mathbb{A}^1$ or to $\mathbb{A}^1 \setminus \{0\}$. Proposition 3.3.13 shows that the isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ (not extending to an automorphism of $\mathbb{A}^2$) is uniquely determined by C, up to left composition by an automorphism of $\mathbb{A}^2$. In particular, there are at most two equivalence classes of curves of $\mathbb{A}^2$ that have complements isomorphic to $\mathbb{A}^2 \setminus C$. Corollary 3.3.11 gives the existence of isomorphisms $C \simeq \operatorname{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \operatorname{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some square-free polynomials $P, Q \in \mathrm{k}[t]$ that have the same number of roots in k, and also the same number of roots in the algebraic closure of k. By replacing k with any field k′ containing k we obtain the result. $\square$

Corollaries 3.1.1, 3.1.2 and 3.1.4 are then direct consequences of Theorem 4.

3.3.5 Automorphisms of complements of curves

Another consequence of Theorem 4 is Corollary 3.1.3, which we now prove:

Proof of Corollary 3.1.3. Recall the hypothesis of the corollary: we start with a geometrically irreducible closed curve $C \subset \mathbb{A}^2$ not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. We want to show that $\operatorname{Aut}(\mathbb{A}^2, C)$ has index at most 2 in $\operatorname{Aut}(\mathbb{A}^2 \setminus C)$. If φ_1, φ_2 are automorphisms of $\mathbb{A}^2 \setminus C$ which do not extend to automorphisms of $\mathbb{A}^2$, it is enough to show that $(\varphi_2)^{-1}\varphi_1$ extends to an automorphism of $\mathbb{A}^2$. This follows from Theorem 4(3). $\square$

Remark 3.3.19. With the assumptions of Corollary 3.1.3, the group $\operatorname{Aut}(\mathbb{A}^2 \setminus C)$ is a semidirect product of the form $\operatorname{Aut}(\mathbb{A}^2, C) \rtimes \mathbb{Z}/2\mathbb{Z}$ if and only if there exists an involutive automorphism of $\mathbb{A}^2 \setminus C$ which does not extend to an automorphism of $\mathbb{A}^2$.

Corollary 3.3.20. *If* k *is a perfect field and* $C \subset \mathbb{A}^2$ *is a geometrically irreducible closed curve that is*

(*i*) *not equivalent to a line,*

(*ii*) *not equivalent to a cuspidal curve with equation* $x^m - y^n = 0$*, where* $m, n \geq 2$ *are coprime integers,*

(*iii*) *not geometrically isomorphic to* $\mathbb{A}^1 \setminus \{0\}$*,*

then $\operatorname{Aut}(\mathbb{A}^2 \setminus C)$ *is a zero dimensional algebraic group, hence is finite.*

Proof. Conditions (*i*)-(*ii*)-(*iii*) imply that $\operatorname{Aut}(\mathbb{A}^2, C)$ is a zero dimensional algebraic group [BS15, Theorem 2]. If moreover C is not isomorphic to $\mathbb{A}^1$, then $\operatorname{Aut}(\mathbb{A}^2 \setminus C)$ is also zero dimensional by Corollary 3.1.3. If C is isomorphic to $\mathbb{A}^1$ (but not equivalent to a line by (*i*)), then $\operatorname{Aut}(\mathbb{A}^2 \setminus C) = \operatorname{Aut}(\mathbb{A}^2, C)$ by Proposition 3.3.16. $\square$

Remark 3.3.21. Let us make a few comments on the group $\operatorname{Aut}(\mathbb{A}^2 \setminus C)$ when $C \subset \mathbb{A}^2$ is a geometrically irreducible closed curve not satisfying the conditions of Corollary 3.3.20.

(i) If C is equivalent to a line, we may assume without loss of generality that C is the line $x = 0$. Then, $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$ is described in Lemma 3.2.12.

(ii) If C does not satisfy (ii), we may assume that C has equation $x^m - y^n = 0$, where $m, n \geq 2$ are coprime integers. Since the curve C is singular, we have $\mathrm{Aut}(\mathbb{A}^2 \setminus C) = \mathrm{Aut}(\mathbb{A}^2, C)$ by Corollary 3.3.5. Moreover, we have $\mathrm{Aut}(\mathbb{A}^2, C) = \{(x, y) \mapsto (t^n x, t^m y) \mid t \in \mathrm{k}^*\}$ by [BS15, Theorem 2(ii)].

(iii)(a) If C is geometrically isomorphic to $\mathbb{A}^1 \setminus \{0\}$, but not isomorphic to $\mathbb{A}^1 \setminus \{0\}$, then $\mathrm{Aut}(\mathbb{A}^2, C)$ has index 1 or 2 in $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$ by Corollary 3.1.3. The group $\mathrm{Aut}(\mathbb{A}^2, C)$ is then an algebraic group of dimension ≤ 1 by [BS15, Theorem 2], so the same holds for $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$. An example of dimension 1 is given by the curve of equation $x^2 + y^2 = 1$, in the case where $\mathrm{k} = \mathbb{R}$ (see [BS15, Theorem 2(iv)]).

(iii)(b) If C is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, we do not have a complete description of $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$. The simplest cases where C has equation $x^m y^n - 1$, where $m, n \geq 1$ are coprime, can be completely described. In particular, $\mathrm{Aut}(\mathbb{A}^2 \setminus C)$ contains elements of arbitrarily large degree.

3.4 Families of non-equivalent embeddings

In this section, we study mainly the curves of $\mathbb{A}^2$ given by an equation of the form

$$a(y)x + b(y) = 0$$

where $a, b \in \mathrm{k}[y]$ are coprime polynomials such that $\deg b < \deg a$. This will lead us to the proof of Theorem 5.

These curves already appeared in Lemma 3.3.3, where we proved in particular that they are isomorphic to $\mathbb{A}^1$ if and only if $a(y)$ is a constant (Lemma 3.3.3(*i*)-(*iii*)). Actually, we have the following obvious and stronger result:

Lemma 3.4.1. *Let $C \subset \mathbb{A}^2$ be the irreducible curve given by the equation*

$$a(y)x + b(y) = 0,$$

where $a, b \in \mathrm{k}[y]$ are coprime polynomials and a is nonzero. Then, the algebra of regular functions on C is isomorphic to $\mathrm{k}[y, 1/a(y)]$.

Proof. The algebra of regular functions on C satisfies

$$\mathrm{k}[C] = \mathrm{k}[x, y]/(a(y)x + b(y)) \simeq \mathrm{k}[y, -b(y)/a(y)] = \mathrm{k}[y, 1/a(y)],$$

where the last equality comes from the fact that there exist $c, d \in \mathrm{k}[y]$ with $ad - bc = 1$, which implies that $\frac{1}{a} = \frac{ad-bc}{a} = d - c \cdot \frac{b}{a} \in \mathrm{k}[y, \frac{b}{a}]$. $\square$

3.4.1 A construction using elements of $\mathrm{SL}_2(\mathrm{k}[y])$

Lemma 3.4.2. *For each matrix* $\begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}[y])$, *we have an isomorphism*

$$\begin{array}{rccc} \varphi\colon & \mathbb{A}^2 \setminus C & \xrightarrow{\simeq} & \mathbb{A}^2 \setminus D \\ & (x,y) & \mapsto & \left(\frac{c(y)x+d(y)}{a(y)x+b(y)}, y\right) \end{array}$$

where $C, D \subset \mathbb{A}^2$ *are given by* $a(y)x+b(y)=0$ *and* $a(y)x-c(y)=0$ *respectively.*

Proof. Note first that φ is a birational transformation of $\mathbb{A}^2$, with inverse $\psi\colon (x,y) \mapsto (\frac{-b(y)x+d(y)}{a(y)x-c(y)}, y)$. It remains to prove that the isomorphism $\varphi^*\colon \mathrm{k}(x,y) \to \mathrm{k}(x,y)$, $x \mapsto \frac{cx+d}{ax+b}$, $y \mapsto y$ induces an isomorphism $\mathrm{k}[x,y,\frac{1}{ax-c}] \to \mathrm{k}[x,y,\frac{1}{ax+b}]$. This follows from the equalities:

$$\varphi^*(x) = \tfrac{cx+d}{ax+b}, \quad \varphi^*(y) = y, \quad \varphi^*\big(\tfrac{1}{ax-c}\big) = ax+b \quad \text{and}$$

$$\psi^*(x) = \tfrac{-bx+d}{ax-c}, \quad \psi^*(y) = y, \quad \psi^*\big(\tfrac{1}{ax+b}\big) = ax-c.$$ □

The curves C and D of Lemma 3.4.2 are always isomorphic thanks to Lemma 3.4.1. We now prove that they are in general not equivalent.

Lemma 3.4.3. *Let* $C_1, C_2 \subset \mathbb{A}^2$ *be two geometrically irreducible closed curves given by*

$$a_1(y)x + b_1(y) = 0 \text{ and } a_2(y)x + b_2(y) = 0$$

respectively, for some polynomials $a_1, a_2, b_1, b_2 \in \mathrm{k}[y]$ *such that* $\deg a_1 > \deg b_1 \geq 0$ *and* $\deg a_2 > \deg b_2 \geq 0$. *Then, the curves* C_1 *and* C_2 *are equivalent if and only if there exist constants* $\alpha, \lambda, \mu \in \mathrm{k}^*$ *and* $\beta \in \mathrm{k}$ *such that*

$$a_2(y) = \lambda \cdot a_1(\alpha y + \beta), \quad b_2(y) = \mu \cdot b_1(\alpha y + \beta).$$

Proof. We first observe that if $a_2(y) = \lambda \cdot a_1(\alpha y+\beta)$ and $b_2(y) = \mu \cdot b_1(\alpha y + \beta)$ for some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$, then the automorphism $(x,y) \mapsto (\frac{\lambda}{\mu}x, \alpha y + \beta)$ of $\mathbb{A}^2$ sends C_2 onto C_1.

Conversely, we assume the existence of $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ that sends C_2 onto C_1 and want to find $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ as above. Writing φ as $(x,y) \mapsto (f(x,y), g(x,y))$ for some polynomials $f, g \in \mathrm{k}[x,y]$, we get

$$\mu\big(a_1(g)f + b_1(g)\big) = a_2(y)x + b_2(y) \tag{A}$$

for some $\mu \in \mathrm{k}^*$.

(i) If $g \in \mathrm{k}[y]$, the fact that $\mathrm{k}[f,g] = \mathrm{k}[x,y]$ implies that $g = \alpha y + \beta$, $f = \gamma x + s(y)$ for some $\alpha, \gamma \in \mathrm{k}^*, \beta \in \mathrm{k}$ and $s(y) \in \mathrm{k}[y]$. This yields $a_1(g)f + b_1(g) = a_1(g)(\gamma x + s(y)) + b_1(g)$, so that equation (A) gives:

$$a_2 = \mu\gamma \cdot a_1(g), \quad b_2 = \mu \cdot \big(a_1(g)s(y) + b_1(g)\big).$$

This shows in particular that $\deg a_1 = \deg a_2$, whence $\deg b_2 < \deg a_1(g)$. Since $\deg b_1(g) < \deg a_1(g)$, we find that $s = 0$, and thus that $b_2 = \mu \cdot b_1(g)$, as desired. This concludes the proof, by choosing $\lambda = \mu\gamma$.

(ii) It remains to consider the case where $g \notin \mathrm{k}[y]$, which corresponds to $\deg_x(g) \geq 1$. We have $\deg_x a_1(g) = \deg a_1 \cdot \deg_x(g) > \deg b_1 \cdot \deg_x(g) = \deg_x b_1(g)$, which implies that $\deg_x \big(a_1(g)f + b_1(g)\big) = \deg(a_1) \cdot \deg_x(g) + \deg_x(f)$. Equation (A) shows that this degree is 1, and since $\deg a_1 \geq 1$, we find $\deg a_1 = 1$. Similarly, the automorphism sending C_1 onto C_2 satisfies the same condition, so $\deg a_2 = 1$. This implies that $b_1, b_2 \in \mathrm{k}^*$. There thus exist some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that $a_2(y) = \lambda \cdot a_1(\alpha y + \beta)$ and $b_2(y) = \mu \cdot b_1(\alpha y + \beta)$. □

Proposition 3.4.4. *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two closed curves $C, D \subset \mathbb{A}^2$, both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, that are non-equivalent and have isomorphic complements. Moreover, the set of equivalence classes of the curves C appearing in such pairs (C, D) is infinite.*

Proof. We choose an irreducible polynomial $b \in \mathrm{k}[t]$ which does not divide f. For each $n \geq 1$ such that $\deg(f^n) > 2\deg(b)$, we then choose two polynomials $c, d \in \mathrm{k}[t]$ such that $f^n d - bc = 1$ (this is possible since $\gcd(f^n, b) = 1$). Replacing c, d by $c + \alpha f^n, d + \alpha b$, we may moreover assume that $\deg c < \deg f^n$. The curves $C_n, D_n \subset \mathbb{A}^2$ given by $f(y)^n x + b(y) = 0$ and $f(y)^n x - c(y) = 0$ are both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f^n}]) = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$ by Lemma 3.4.1 and have isomorphic complements by Lemma 3.4.2. Moreover, as $\deg bc = \deg(f^n d - 1) \geq \deg(f^n) > 2\deg(b)$, we find that $\deg c > \deg b$, which implies by Lemma 3.4.3 that C_n and D_n are not equivalent. Moreover, the curves C_n are all non-equivalent, again by Lemma 3.4.3. □

3.4.2 Curves isomorphic to $\mathbb{A}^1 \setminus \{0\}$

We consider now families of curves in $\mathbb{A}^2$ of the form $xy^d + b(y) = 0$, for some $d \geq 1$ and some polynomial $b(y) \in \mathrm{k}[y]$ satisfying $b(0) \neq 0$. Note that all these curves are isomorphic to $\mathrm{Spec}(\mathrm{k}[y, \frac{1}{y^d}]) = \mathrm{Spec}(\mathrm{k}[y, \frac{1}{y}]) \simeq \mathbb{A}^1 \setminus \{0\}$ by Lemma 3.4.1.

Lemma 3.4.5. *Let $d \geq 1$ be an integer and $b(y) \in \mathrm{k}[y]$ be a polynomial satisfying $b(0) \neq 0$. We define $D_b \subset \mathbb{A}^2$ to be the curve given by the equation*

$$xy^d + b(y) = 0$$

and φ_b to be the birational endomorphism of $\mathbb{A}^2$ given by

$$\varphi_b(x, y) = (xy^d + b(y), y).$$

Denote by L_x, resp. L_y, the line in $\mathbb{A}^2$ given by the equation $x = 0$, resp. $y = 0$.

(1) *The transformation φ_b induces an automorphism of $\mathbb{A}^2 \setminus L_y$ and an isomorphism*

$$\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup L_x).$$

(2) *Assume now that b has degree $\leq d-1$ and fix an integer $m \geq 1$. Then, there exists a unique polynomial $c \in \mathrm{k}[y]$ of degree $\leq d-1$ satisfying*

$$b(y) \equiv c(yb(y)^m) \pmod{y^d}. \tag{B}$$

Furthermore, we have $c(0) \neq 0$.

(3) *Define the birational transformations τ and $\psi_{b,m}$ of $\mathbb{A}^2$ by*

$$\tau(x,y) = (x, xy) \text{ and } \psi_{b,m} = (\varphi_c)^{-1}\tau^m\varphi_b.$$

Then, $\psi_{b,m}$ induces an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$ whose expression is

$$\psi_{b,m}(x,y) = \left(\frac{x + \lambda + yf(x,y)}{\left(xy^d + b(y)\right)^{md}}, \, y\left(xy^d + b(y)\right)^m \right),$$

for some constant $\lambda \in \mathrm{k}$ and some polynomial $f \in \mathrm{k}[x,y]$ (depending on b and m).

(4) *Fixing the polynomial b, all open embeddings $\mathbb{A}^2 \setminus D_b \hookrightarrow \mathbb{A}^2$ given by $\psi_{b,m}$, $m \geq 1$, are non-equivalent.*

Proof. (1): The automorphism $(\varphi_b)^*$ of $\mathrm{k}(x,y)$ satisfies

$$(\varphi_b)^*(x) = xy^d + b(y) \text{ and } (\varphi_b)^*(y) = y.$$

The result follows from the following two equalities:

$$\begin{array}{rcl} (\varphi_b)^*(\mathrm{k}[x,y,\frac{1}{y}]) & = & \mathrm{k}[xy^d + b(y), y, \frac{1}{y}] = \mathrm{k}[x,y,\frac{1}{y}] \quad \text{and} \\ (\varphi_b)^*(\mathrm{k}[x,y,\frac{1}{x},\frac{1}{y}]) & = & \mathrm{k}[xy^d + b(y), \frac{1}{xy^d+b(y)}, y, \frac{1}{y}] = \mathrm{k}[x,y,\frac{1}{y},\frac{1}{xy^d+b(y)}]. \end{array}$$

(2): Since $b(0) \neq 0$, the endomorphism of the algebra $\mathrm{k}[y]/(y^d)$ defined by $y \mapsto yb(y)^m$ is an automorphism. If the inverse automorphism is given by $y \mapsto u(y)$, note that (B) is equivalent to $c(y) \equiv b(u(y)) \pmod{y^d}$. This determines uniquely the polynomial c. Finally, replacing x by zero in (B), we get $c(0) = b(0) \neq 0$.

(3): Since τ induces an automorphism of $\mathbb{A}^2 \setminus (L_y \cup L_x)$, assertion (1) implies that ψ induces an isomorphism $\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup D_c)$ (this would be true for any choice of c). It remains to see that the choice of c which we have made implies that ψ extends to an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$ of the desired form.

Since $(\varphi_c)^{-1}(x,y) = \left(\frac{x-c(y)}{y^d}, y\right)$, $\tau^m(x,y) = (x, x^m y)$, and $\psi_{b,m} = (\varphi_c)^{-1}\tau^m\varphi_b$, we get:

$$\begin{array}{rcl} \psi_{b,m}(x,y) & = & (\varphi_c)^{-1}\tau^m(xy^d + b(y), y) \\ & = & \left(\frac{xy^d+b(y)-c(y\Delta)}{y^d\Delta^d}, y\Delta\right), \text{ with } \Delta = (xy^d + b(y))^m. \end{array} \tag{C}$$

To show that $\psi_{b,m}$ has the desired form, we use $b(y) \equiv c(yb(y)^m) \pmod{y^d}$ (equation (B)), which yields $\lambda \in \mathrm{k}$ such that $b(y) \equiv c(yb(y)^m) + \lambda y^d \pmod{y^{d+1}}$. Since $y\Delta \equiv yb(y)^m \pmod{y^{d+1}}$, we get $b(y) \equiv c(y\Delta) + \lambda y^d \pmod{y^{d+1}}$. There is thus $f \in \mathrm{k}[x,y]$ such that

$$xy^d + b(y) - c(y\Delta) = y^d(x + \lambda + yf(x,y)).$$

This yields the desired form for $\psi_{b,m}$ and shows that $\psi_{b,m}$ restricts to the automorphism $x \mapsto x + \lambda$ on L_y and then restricts to an isomorphism $\mathbb{A}^2 \setminus D_b \xrightarrow{\simeq} \mathbb{A}^2 \setminus D_c$.

(4): It suffices to check that for $m > n \geq 1$ the birational transformation $\theta = \psi_{b,n} \circ (\psi_{b,m})^{-1}$ of $\mathbb{A}^2$ does not correspond to an automorphism of $\mathbb{A}^2$. Setting $l = m - n \geq 1$ and denoting by c_m and c_n the elements of $\mathrm{k}[y]$ associated to b and to the integers m and n respectively, we get

$$\theta = \big((\varphi_{c_n})^{-1}\tau^n\varphi_b\big) \circ \Big((\varphi_{c_m})^{-1}\tau^m\varphi_b\Big)^{-1} = (\varphi_{c_n})^{-1}\tau^{-l}\varphi_{c_m}.$$

The second component of $\theta(x,y)$ is thus equal to the second component of $\tau^{-l}\varphi_{c_m}(x,y)$ which is $\frac{y}{(xy^d+c_m(y))^l} \in \mathrm{k}(x,y) \setminus \mathrm{k}[x,y]$. This shows that θ is not an automorphism of $\mathbb{A}^2$ (and not even an endomorphism) and completes the proof. $\square$

Remark 3.4.6. Note that Lemma 3.4.5(1) provides an isomorphism $\mathbb{A}^2 \setminus (L_y \cup D_b) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup L_x)$ where the reducible curves $(L_y \cup D_b)$ and $(L_y \cup L_x)$ are not isomorphic. Indeed, the reducible curve $(L_y \cup D_b)$ has two connected components (since $L_y \cap D_b = \emptyset$), while the reducible curve $(L_y \cup L_x)$ is connected (since $L_y \cap L_x \neq \emptyset$). As noted in [Kra96], this kind of easy example explains why the complement problem in $\mathbb{A}^n$ has only been formulated for irreducible hypersurfaces.

Remark 3.4.7. Geometrically, the construction of Lemma 3.4.5(3) can be interpreted as follows: the birational morphism $\varphi_b\colon (x,y) \mapsto (xy^d + b(y), y)$ contracts the line $y = 0$ to the point $(b(0), 0)$. If $d = 1$ then φ_b just sends the line onto the exceptional divisor of $(b(0), 0)$. If $d \geq 2$, it sends the line onto the exceptional divisor of a point in the $(d-1)$-st neighbourhood of $(b(0), 0)$. The coordinates of these points are determined by the polynomial b. The fact that $\tau^m\colon (x,y) \mapsto (x, x^m y)$ contracts the line $x = 0$ implies that $\psi_{b,m}$ contracts the curve D_b given by $xy^d + b(y) = 0$. Moreover, τ^m fixes the point $(b(0), 0)$ and induces a local isomorphism around it, hence acts on the set of infinitely near points. This action changes the polynomial b and replaces it by another one, which is the polynomial $c = c_{b,m}$ provided by Lemma 3.4.5(2).

Proposition 3.4.8. *There exists an infinite sequence of curves $C_i \subset \mathbb{A}^2$, $i \in \mathbb{N}$, all pairwise non-equivalent, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and such that for each i there are infinitely many open embeddings $\mathbb{A}^2 \setminus C_i \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$.*

Proof. It suffices to choose the curve C_i given by $xy^{i+2} + y + 1$, for each $i \geq 2$. These curves are all isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by Lemma 3.4.1 and are pairwise non-equivalent by Lemma 3.4.3. The existence of infinitely many open embeddings $\mathbb{A}^2 \setminus C_i \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$, is then ensured by Lemma 3.4.5(4). $\square$

One can compute the polynomial $c = c_{b,m}$ provided by Lemma 3.4.5(2), in terms of b and m, and find explicit formulas. We obtain in particular the following result:

Lemma 3.4.9. *For each $\mu \in \mathrm{k}$ define the curve $C_\mu \subset \mathbb{A}^2$ by*

$$xy^3 + \mu y^2 + y + 1 = 0.$$

Then, there exists an isomorphism $\mathbb{A}^2 \setminus C_\mu \xrightarrow{\simeq} \mathbb{A}^2 \setminus C_{\mu-1}$. In particular, if $\mathrm{char}(\mathrm{k}) = 0$, we obtain infinitely many closed curves of $\mathbb{A}^2$, pairwise non-equivalent, which have isomorphic complements.

Proof. The isomorphism between $\mathbb{A}^2 \setminus C_\mu$ and $\mathbb{A}^2 \setminus C_{\mu-1}$ follows from Lemma 3.4.5 with $d = 3$, $m = 1$, $b = \mu y^2 + y + 1$ and $c = (\mu - 1)y^2 + y + 1$.

To get the last statement, we assume that $\mathrm{char}(\mathrm{k}) = 0$ and observe that the affine surfaces $\mathbb{A}^2 \setminus C_n$ are all isomorphic for each $n \in \mathbb{Z}$. To show that the curves C_n, $n \in \mathbb{Z}$ are pairwise non-equivalent, we apply Lemma 3.4.3: for $m, n \in \mathbb{Z}$, the curves C_m and C_n are equivalent only if there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^3 = \lambda \cdot (\alpha y + \beta)^3, \; my^2 + y + 1 = \mu \cdot \big(n(\alpha y + \beta)^2 + (\alpha y + \beta) + 1\big).$$

The first equality gives $\beta = 0$, so that the second one becomes $my^2 + y + 1 = \mu \cdot (n\alpha^2 y^2 + \alpha y + 1)$. We finally obtain $\mu = 1$, $\alpha = 1$ and thus $m = n$, as we wanted. $\square$

If $\mathrm{char}(\mathrm{k}) = p > 0$, Lemma 3.4.9 only gives p non-equivalent curves that have isomorphic complements. We can get more curves by applying Lemma 3.4.3 to polynomials of higher degree:

Lemma 3.4.10. *For each integer $n \geq 1$ there exist curves $C_1, \ldots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, pairwise non-equivalent, such that all surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are isomorphic.*

Proof. The case where $\mathrm{char}(\mathrm{k}) = 0$ is settled by Lemma 3.4.9 so we may assume that $\mathrm{char}(\mathrm{k}) = p \geq 2$. Set $b(y) = 1 + y$ and $d = p^n + 2$. For each integer i with $1 \leq i \leq n$, we apply Lemma 3.4.5(2) with $m = p^i$. Hence, there exists a unique polynomial $c_i \in \mathrm{k}[y]$ of degree $\leq d - 1$ satisfying

$$b(y) \equiv c_i(yb(y)^{p^i}) \pmod{y^d}. \tag{D}$$

Let $C_i \subset \mathbb{A}^2$ be the curve given by the equation

$$xy^d + c_i(y) = 0.$$

By Lemma 3.4.5(3), all surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are isomorphic to $\mathbb{A}^2 \setminus D$, where $D \subset \mathbb{A}^2$ is given by

$$xy^d + b(y) = 0.$$

It remains to check that $C_1, \dots, C_n$ are pairwise non-equivalent. Assume therefore that C_i and C_j are equivalent. By Lemma 3.4.3, there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^d = \lambda \cdot (\alpha y + \beta)^d, \quad c_j(y) = \mu \cdot c_i(\alpha y + \beta).$$

The first equality gives $\beta = 0$, so that we get:

$$c_j(y) = \mu \cdot c_i(\alpha y). \tag{E}$$

However, by equation (D) we have

$$1 + y \equiv c_i(y + y^{p^i+1}) \pmod{y^{p^i+2}}$$

and this equation admits the unique solution

$$c_i = 1 + y - y^{p^i+1} + (\text{terms of higher order}).$$

(Unicity follows for example again from Lemma 3.4.5(2)). Hence, looking at equation (E) modulo y^2, we obtain $1 + y = \mu(1 + \alpha y)$, so that $\alpha = \mu = 1$. Equation (E) finally yields $c_i = c_j$, so that the above (partial) computation of c_i gives us $i = j$. □

The proof of Theorem 5 is now complete:

Proof of Theorem 5. Part (1) corresponds to Proposition 3.4.8. Part (2) is given by Lemma 3.4.9 ($\mathrm{char}(\mathrm{k}) = 0$) and Lemma 3.4.10 ($\mathrm{char}(\mathrm{k}) > 0$). Part (3) corresponds to Proposition 3.4.4. □

3.5 Non-isomorphic curves with isomorphic complements

3.5.1 A geometric construction

We begin with the following fundamental construction:

Proposition 3.5.1. *For each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 3$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, there exist two closed curves $C, D \subset \mathbb{A}^2$ of degree $d^2 - d + 1$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that the following isomorphisms hold:*

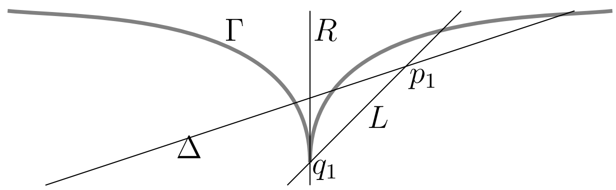
$$C \simeq \mathrm{Spec}\left(\mathrm{k}[t, \frac{1}{P}]\right) \text{ and } D \simeq \mathrm{Spec}\left(\mathrm{k}[t, \frac{1}{Q}]\right), \text{ where } Q(t) = P\left(\lambda + \frac{1}{t}\right) \cdot t^d.$$

Proof. The polynomial $P_d(x, y) := P(\frac{x}{y}) y^d \in \mathrm{k}[x, y]$ is a homogeneous polynomial of degree d such that $P_d(x, 1) = P(x)$. Let then $\Gamma, \Delta, L, R \subset \mathbb{P}^2$ be the curves given by the equations

$$\Gamma : y^{d-1} z = P_d(x, y), \quad \Delta : z = 0, \quad L : x = \lambda y, \quad R : y = 0.$$

By construction, P_d is not divisible by y. Moreover, the two lines L and Δ satisfy $L \cap \Gamma = \{p_1, q_1\}$ where $p_1 = [\lambda : 1 : P(\lambda)]$, $q_1 = [0 : 0 : 1]$ and Δ does not pass through p_1 or q_1.

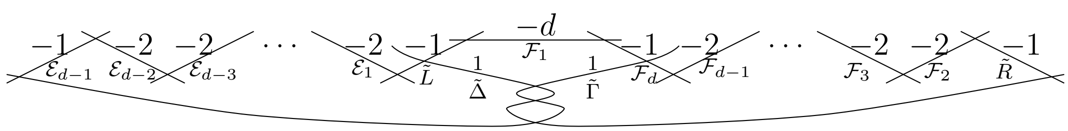
Note that $\Gamma \subset \mathbb{P}^2$ is a cuspidal rational curve, that the point $q_1 = [0 : 0 : 1] \in \mathbb{P}^2(\mathrm{k})$ has multiplicity $d-1$ on Γ, and is therefore the unique singular point of this curve (this follows for example from the genus formula of a plane curve). The situation is then as follows.



Denote by $\pi\colon X \to \mathbb{P}^2$ the birational morphism given by the blow-up of p_1, q_1, followed by the blow-up of the points $p_2,\dots,p_{d-1}$ and $q_2,\dots,q_d$ infinitely near p_1 and q_1 respectively and all belonging to the strict transform of Γ. Denote by $\tilde{\Gamma}$, $\tilde{\Delta}$, $\tilde{L}$, $\tilde{R}$, $\mathcal{E}_1,\dots,\ \mathcal{E}_{d-1}$, $\mathcal{F}_1,\dots,\mathcal{F}_d \subset X$ the strict transforms of Γ, Δ, L, R and of the exceptional divisors above $p_1,\dots,p_{d-1}$, q_1, $\dots,q_d$. Consider the tree (which is in fact a chain)

$$B = \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i.$$

We now prove that the situation on X is as in the symmetric diagram (F),



(F)

where all curves are isomorphic to $\mathbb{P}^1$, all intersections indicated are transversal and consist in exactly one k-point, except for $\tilde{\Gamma} \cap \tilde{\Delta}$, which can be more complicated (the picture shows only the case where we get 3 points with transversal intersection).

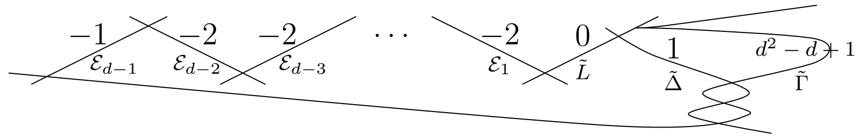
Blowing up once the singular point q_1 of Γ, the strict transform of Γ becomes a smooth rational curve having $(d-1)$-th order contact with the exceptional divisor. The unique point of intersection between the strict transform and the exceptional divisor corresponds to the direction of the tangent line R. Hence, all points q_2, $\dots$, q_d belong to the strict transform of the exceptional divisor of q_1. This gives the self-intersections of $\mathcal{F}_1$, $\dots$, $\mathcal{F}_d$ and their configurations, as shown in diagram (F). As p_1 is a smooth point of Γ, the curves $\mathcal{E}_1$, $\dots$, $\mathcal{E}_{d-1}$ form a chain of curves, as shown in diagram (F). The rest of the diagram is checked by looking at the definitions of the curves Γ, Δ, L, R.

We now show the existence of isomorphisms

$$\psi_1\colon X \setminus (B \cup \tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2\colon X \setminus (B \cup \tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

such that $C = \psi_1(\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}))$ are of degree $d^2 - d + 1$.

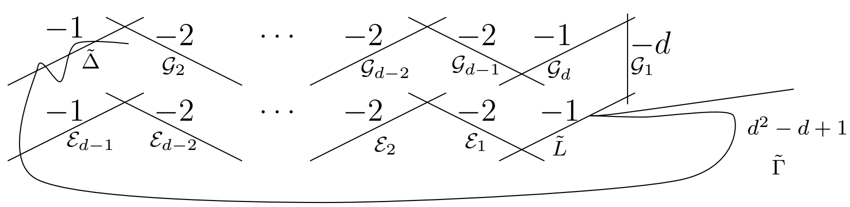
We first show that ψ_1 exists (the case of ψ_2 is similar, as diagram (F) is symmetric). We observe that since π is the blow-up of $2d-1$ points defined over k, the Picard group of X is of rank $2d$, over k and over its algebraic closure $\overline{\mathrm{k}}$. We contract the curves $\mathcal{F}_d$, $\ldots$, $\mathcal{F}_1$ and obtain a smooth projective surface Y of Picard rank d (again over k and $\overline{\mathrm{k}}$). The configuration of the image of the curves $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}, \tilde{L}, \tilde{\Gamma}$ is then depicted in diagram (G) (we omit the curve $\tilde{R}$ as we will not need it):



(G)

In fact, Y is just the blow-up of the points $p_1, \ldots, p_{d-1}$ starting from $\mathbb{P}^2$.

In order to show that $X \setminus (B \cup \tilde{\Delta}) \simeq Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2})$ is isomorphic to $\mathbb{A}^2$, we will construct a birational map $\hat{\psi}_1 \colon Y \dashrightarrow \mathbb{P}^2$ which restricts to an isomorphism $Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \mathcal{L}$ for some line $\mathcal{L}$. Let us now describe this map. Denote by r_1 the unique point of Y such that $\{r_1\} = \tilde{\Delta} \cap \tilde{L}$ in Y. We blow up r_1 and then the point r_2 lying on the intersection of the exceptional curve of r_1 and of the strict transform of $\tilde{\Delta}$. For $i = 3, \ldots, d$, denoting by r_i the point lying on the intersection of the exceptional curve of r_{i-1} and on the strict transform of the exceptional curve of r_1, we successively blow up r_i. We thus obtain a birational morphism $\theta \colon Z \to Y$. The configuration of curves on Z is depicted in diagram (H) (we again use the same name for a curve on Y and its strict transform on Z; we also denote by $\mathcal{G}_i \subset Z$ the strict transform of the exceptional divisor of r_i):



(H)

We can then contract the curves $\tilde{\Delta}, \mathcal{G}_2, \ldots, \mathcal{G}_{d-1}, \tilde{L}, \mathcal{E}_1, \ldots, \mathcal{E}_{d-2}, \mathcal{G}_1$ and obtain a birational morphism $\rho \colon Z \to \mathbb{P}^2$. The image of the target is $\mathbb{P}^2$, because it has Picard rank 1; note also that the image $\mathcal{L}$ of $\mathcal{G}_d$ is actually a line of $\mathbb{P}^2$ since it has self-intersection 1. The birational map $\hat{\psi}_1 \colon Y \dashrightarrow \mathbb{P}^2$ given by $\hat{\psi}_1 = \rho\theta^{-1}$ is the desired birational map. The closure $\overline{C}$ of $C \subset \mathbb{A}^2$ in $\mathbb{P}^2$ is then equal to the image of $\tilde{\Gamma}$ by ρ.

For each contracted curve above, the multiplicity (on $\overline{C}$) at the point where it is contracted, is equal to d for $\tilde{\Delta}, \mathcal{G}_2, \ldots, \mathcal{G}_{d-1}$, to $d-1$ for $\tilde{L}, \mathcal{E}_1, \ldots, \mathcal{E}_{d-2}$, and is equal to $(d-1)^2$ for $\mathcal{G}_1$. Adding the singular point of multiplicity $d-1$ of $\tilde{\Gamma}$, we obtain the two sequences of multiplicities $(\underbrace{d, \ldots, d}_{d-1})$ and $((d-1)^2, \underbrace{d-1, \ldots, d-1}_{d})$. The self-intersection of $\overline{C}$ is then

$$(d^2-d+1) + (d-1) \cdot d^2 + (d-1) \cdot (d-1)^2 + ((d-1)^2)^2 = (d^2-d+1)^2,$$

which implies that the curve has degree $d^2 - d + 1$.

The case of ψ_2 is similar, since the diagram (F) is symmetric.

In particular, this construction provides an isomorphism $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$, where $C, D \subset \mathbb{A}^2$ are closed curves isomorphic to $\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}) \simeq \Gamma \setminus (\Delta \cup \{q_1\})$ and $\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}) \simeq \Delta \setminus (\Gamma \cup L)$ respectively, both of degree $d^2 - d + 1$.

Since $\Gamma \setminus \{q_1\}$ is isomorphic to $\mathbb{A}^1$ via $t \mapsto [t : 1 : P_d(t, 1)] = [t : 1 : P(t)]$, we obtain that $C \simeq \Gamma \setminus (\Delta \cup \{q_1\})$ is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$.

We then take the isomorphism $\mathbb{A}^1 \xrightarrow{\simeq} \Delta \setminus L = \Delta \setminus \{[\lambda : 1 : 0]\}$ given by $t \mapsto [\lambda t + 1 : t : 0]$. The pull-back of $\Delta \cap \Gamma$ corresponds to the zeros of $P_d(\lambda t + 1, t) = t^d P_d(\lambda + \frac{1}{t}, 1) = Q(t)$. Hence, D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ as desired. $\square$

Corollary 3.5.2. *For each $d \geq 0$ and every choice of distinct points $a_1, \ldots, a_d, b_1, b_2 \in \mathbb{P}^1(\mathrm{k})$, there are two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{P}^1 \setminus \{a_1, \ldots, a_d, b_1\}$ and $D \simeq \mathbb{P}^1 \setminus \{a_1, \ldots, a_d, b_2\}$.*

Proof. The case where $d \leq 2$ is obvious: Since $\mathrm{PGL}_2(\mathrm{k})$ acts 3-transitively on $\mathbb{P}^1(\mathrm{k})$, we may take $C = D$ given by the equation $x = 0$, resp. $xy = 1$, resp. $x(x-1)y = 1$, if $d = 0$, resp. $d = 1$, resp. $d = 2$. Let us now assume that $d \geq 3$. Since $\mathrm{PGL}_2(\mathrm{k})$ acts transitively on $\mathbb{P}^1(\mathrm{k})$, we may assume without restriction that b_1 is the point at infinity $[1 : 0]$. Therefore, there exist distinct constants $\mu_1, \ldots, \mu_d, \lambda \in \mathrm{k}$ such that $a_1 = [\mu_1 : 1], \ldots, a_d = [\mu_d : 1]$ and $b_2 = [\lambda : 1]$. We now apply Proposition 3.5.1 with $P = \prod_{i=1}^d (t - \mu_i)$. We get two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}]) \simeq \mathbb{A}^1 \setminus \{\mu_1, \ldots, \mu_d\} \simeq \mathbb{P}^1 \setminus \{a_1, \ldots, a_d, b_1\}$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}]) \simeq \mathbb{A}^1 \setminus \{\frac{1}{\mu_1 - \lambda}, \ldots, \frac{1}{\mu_d - \lambda}\}$, where $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^d$. It remains to observe that D is isomorphic to $\mathbb{P}^1 \setminus \{[\mu_1 : 1], \ldots, [\mu_d : 1], [\lambda : 1]\}$ via $t \mapsto [\lambda t + 1 : t]$. $\square$

Corollary 3.5.3. *If k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, we can find two curves $C, D \subset \mathbb{A}^2$ that have isomorphic complements, such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$, but D is not.*

Proof. By Lemma 3.5.4 below, there exists a constant λ in k such that $P(\lambda) \neq 0$ and such that the curves $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic. The result now follows from Proposition 3.5.1. $\square$

Lemma 3.5.4. *If k is infinite and $P \in \mathrm{k}[t]$ is a polynomial with at least 3 roots in $\overline{\mathrm{k}}$, then for a general $\lambda \in \mathrm{k}$, the polynomial $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^{\deg(P)}$ has the property that the curves $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic.*

Proof. Let $\lambda_1, \ldots, \lambda_d \in \overline{\mathrm{k}}$ be the single roots of P. It suffices to check that for a general λ there is no automorphism of $\mathbb{P}^1$ that sends $\{\lambda_1, \ldots, \lambda_d, \infty\}$ to $\{\frac{1}{\lambda_1 - \lambda}, \ldots, \frac{1}{\lambda_d - \lambda}, \infty\}$, or equivalently that there is no automorphism that sends $\{\lambda_1, \ldots, \lambda_d, \infty\}$ to $\{\lambda_1, \ldots, \lambda_d, \lambda\}$. But if an automorphism sends $\{\lambda_1, \ldots, \lambda_d, \infty\}$ to $\{\lambda_1, \ldots, \lambda_d, \lambda\}$, it necessarily belongs to the set $\mathcal{A}$ of automorphisms φ such that $\varphi^{-1}(\{\lambda_1, \lambda_2, \lambda_3\}) \subset \{\lambda_1, \ldots, \lambda_d, \infty\}$. Since an automorphism of $\mathbb{P}^1$ is determined by the image of 3 points, the set $\mathcal{A}$ has at most $6\binom{d+1}{3} = (d+1)d(d-1)$ elements. In conclusion, if λ is not of the form $\varphi(\mu)$

for some $\varphi \in \mathcal{A}$ and some $\mu \in \{\lambda_1, \dots, \lambda_d, \infty\}$, then no automorphism of $\mathbb{P}^1$ sends $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$. □

Remark 3.5.5. If k is a finite field (with at least 3 elements), then the conclusion of Corollary 3.5.3 is false for the polynomial $P = \prod_{\alpha \in \mathrm{k}} (x-\alpha)$. Indeed, if $C, D \subset \mathbb{A}^2$ are two curves such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathbb{A}^2 \setminus C$ is isomorphic to $\mathbb{A}^2 \setminus D$, then D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some polynomial Q that has no square factors and the same number of roots in k and in $\overline{\mathrm{k}}$ as P (Theorem 4(1)). This implies that Q is equal to μP for some $\mu \in \mathrm{k}^*$ and thus that C and D are isomorphic.

A similar argument holds for $P = \prod_{\alpha \in \mathrm{k}^*} (x-\alpha)$ and $P = \prod_{\alpha \in \mathrm{k} \setminus \{0,1\}} (x-\alpha)$ (when the field has at least 4, respectively 5 elements) since $\mathrm{PGL}_2(\mathrm{k})$ acts 3-transitively on $\mathbb{P}^1(\mathrm{k})$.

Corollary 3.5.6. *For each ground field* k *with more than* 27 *elements, there exist two geometrically irreducible closed curves* $C, D \subset \mathbb{A}^2$ *of degree* 7 *which are not isomorphic, but such that* $\mathbb{A}^2 \setminus C$ *and* $\mathbb{A}^2 \setminus D$ *are isomorphic.*

Proof. We fix some element $\zeta \in \mathrm{k} \setminus \{0,1\}$. For each $\lambda \in \mathrm{k} \setminus \{0,1,\zeta\}$, we apply Corollary 3.5.2 with $d = 3$, $a_1 = [0:1]$, $a_2 = [1:1]$, $a_3 = [\zeta:1]$, $b_1 = [1:0]$, $b_2 = [\lambda:1]$ and get two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{A}^1 \setminus \{0,1,\zeta\} = \mathbb{P}^1 \setminus \{[0:1],[1:1],[\zeta:1],[1:0]\}$ and $D \simeq \mathbb{P}^1 \setminus \{[0:1],[1:1],[\zeta:1],[\lambda:1]\}$. It remains to see that we can find at least one λ such that C and D are not isomorphic. Note that C and D are isomorphic if and only if there is an element of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathrm{k})$ that sends $\{[0:1],[1:1],[\zeta:1],[\lambda:1]\}$ onto $\{[0:1],[1:1],[\zeta:1],[1:0]\}$. The image of this element is determined by the image of $[0:1],[1:1],[\zeta:1]$, so we have at most 24 automorphisms to avoid, hence at most 24 elements of $\mathrm{k} \setminus \{0,1,\zeta\}$ to avoid. Since the field k has at least 28 elements, we find at least one λ with the desired property. □

We can now prove Theorem 6.

Proof of Theorem 6. If the field is infinite (or simply has more than 27 elements), the theorem from Corollary 3.5.6. Let us therefore assume that k is a finite field. We again apply Proposition 3.5.1 (with $\lambda = 0$). Therefore, if $|\mathrm{k}| > 2$ (resp. $|\mathrm{k}| = 2$), it suffices to give a polynomial $P \in \mathrm{k}[t]$ of degree 3 (resp. 4) such that $P(0) \neq 0$ and such that if we set $Q := P(\frac{1}{t}) t^{\deg P}$, then the k-algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are not isomorphic.

We begin with the case where the characteristic of k is odd. Then, the kernel of the morphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ is equal to $\{-1,1\}$, so that this map is not surjective. Let us pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^2$. Let us check that we can take $P = (t-1)((t-1)^2 - \alpha)$. Indeed, up to a multiplicative constant, we have $Q = (t-1)((t-1)^2 - \alpha t^2)$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replaced P and Q by

$$\tilde{P} = P(t+1) = t\big(t^2 - \alpha\big) \text{ and } \tilde{Q} = Q(t+1) = t\big(t^2 - \alpha(t+1)^2\big).$$

This would produce an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t:1]$, which sends the polynomial $uv(u^2 - \alpha v^2)$ onto a multiple of $uv(u^2 - \alpha(u+v)^2)$. This automorphism preserves the set of k-roots: $\{[0:1],[1:0]\}$, and is of the form either $[u:v] \mapsto [\mu u : v]$ or $[u:v] \mapsto [\mu v : u]$ where $\mu \in \mathrm{k}^*$. The polynomial $u^2 - \alpha v^2$ must be sent to a multiple of $u^2 - \alpha(u+v)^2$, which is not possible, because of the term uv.

We now treat the case where k has characteristic 2. We divide it into three cases, depending on whether the cube homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^3$ is surjective or not (which corresponds to asking that $|\mathrm{k}|$ not be a power of 4) and setting aside the field with two elements.

If the cube homomorphism is not surjective, we can pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^3$. We may take the irreducible polynomial $P = t^3 - \alpha \in \mathrm{k}[t]$. Indeed, up to a multiplicative constant, we have $Q = t^3 - \alpha^{-1}$. Assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, there should exist constants $\lambda, \mu, c \in \mathrm{k}$ with $\lambda c \neq 0$ such that

$$c(t^3 - \alpha^{-1}) = (\lambda t + \mu)^3 - \alpha.$$

This gives us $\mu = 0$ and $\lambda^3 = c = \alpha^2$. Since the square homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ is bijective, there is a unique square root for each element of k^*. Taking the square root of the equality $\alpha^2 = \lambda^3$, we obtain $\alpha = (\nu)^3$, where ν is the square root of λ. This is impossible since α was chosen not to be a cube.

If the cube homomorphism is surjective, then 1 is the only root of $t^3 - 1 = (t-1)(t^2 + t + 1)$, so $t^2 + t + 1 \in \mathrm{k}[t]$ is irrreducible. If moreover k has more than 2 elements, we can choose $\alpha \in \mathrm{k} \setminus \{0,1\}$ and take $P = (t - \alpha)(t^2 + t + 1)$. Up to a multiplicative constant, we have $Q = (t - \alpha^{-1})(t^2 + t + 1)$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replaced P and Q by

$$\tilde{P} = P(t+\alpha) = t(t^2 + t + \alpha^2 + \alpha + 1) \text{ and } \tilde{Q} = Q(t + \alpha^{-1}) = t(t^2 + t + \alpha^{-2} + \alpha^{-1} + 1).$$

This would yield an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t:1]$, which sends the polynomial $uv(u^2 + uv + (\alpha^2 + \alpha + 1)v^2)$ onto a multiple of $uv(u^2 + uv + (\alpha^{-2} + \alpha^{-1} + 1)v^2)$. The same argument as before gives $\alpha^2 + \alpha + 1 = \alpha^{-2} + \alpha^{-1} + 1$, i.e. $\alpha^2 + \alpha + 1 = \alpha^{-2}(\alpha^2 + \alpha + 1)$. This is impossible since $\alpha^2 + \alpha + 1 \neq 0$ and $\alpha^2 \neq 1$.

The last case is that in which $\mathrm{k} = \{0,1\}$ is the field with two elements. Here the construction does not work with polynomials of degree 3: the only ones which are not symmetric and do not vanish at 0 are $t^3 + t^2 + 1$ and $t^3 + t + 1$, and they are equivalent via $t \mapsto t + 1$. We then choose for P the irreducible polynomial $P = t^4 + t + 1$ (it has no root and is not equal to $(t^2 + t + 1)^2 = t^4 + t^2 + 1$). This gives $Q = t^4 + t^3 + 1$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, there would exist constants $\lambda, \mu, c \in \mathrm{k}$ such that $\lambda c \neq 0$ and

$$c(t^4 + t^3 + 1) = (\lambda t + \mu)^4 + (\lambda t + \mu) + 1.$$

This is impossible since $(\lambda t + \mu)^4 + (\lambda t + \mu) + 1 = \lambda^4 t^4 + \lambda t + (\mu^4 + \mu + 1)$. $\square$

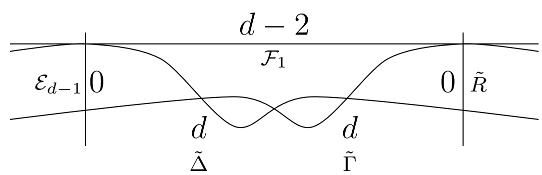
3.5.2 Finding explicit formulas

To obtain the equations of the curves C, D and the isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ given by Proposition 3.5.1, we could follow the construction and explicitly compute the birational maps described: The proposition establishes the existence of isomorphisms

$$\psi_1 \colon X \setminus (B \cup \tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2 \colon X \setminus (B \cup \tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

such that $C = \psi_1(\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}))$ are of degree $d^2 - d + 1$, where $B = \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i$, and ψ_1, ψ_2 are given by blow-ups and blow-downs, so it is possible to compute $\psi_i \pi^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with formulas (looking at the linear systems), and then to get the isomorphism $\psi_2 \pi^{-1} \circ (\psi_1 \pi^{-1})^{-1} \colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$. However, the formulas for $\psi_1 \pi^{-1}, \psi_2 \pi^{-1}$ are complicated.

Another possibility is the following: we choose a birational morphism $X \to W$ that contracts $\tilde{L}, \mathcal{E}_1, \dots, \mathcal{E}_{d-2}$ and $\mathcal{F}_d, \dots, \mathcal{F}_2$ to two smooth points of W, passing through the image of $\mathcal{F}_1$ (this is possible, see diagram (F)). The situation of the image of the curves $\tilde{R}, \mathcal{E}_{d-1}, \mathcal{F}_1, \tilde{\Gamma}, \tilde{\Delta}$ (which we again denote by the same name) in W is as follows:



Computing the dimension of the Picard group, we find that W is a Hirzebruch surface. Hence, the curves $\mathcal{E}_{d-1}, \tilde{R}$ are fibres of a $\mathbb{P}^1$-bundle $W \to \mathbb{P}^1$ and $\mathcal{F}_1, \tilde{\Delta}, \tilde{\Gamma}$ are sections of self-intersection $d-2, d, d$. We can then find many examples in $\mathbb{F}_1$ and $\mathbb{F}_0$ (depending on the parity of d), but also in $\mathbb{F}_m$ for $m \geq 2$ if the polynomial chosen at the outset is special enough.

The case where $d = 3$ corresponds to curves of degree 7 in $\mathbb{A}^2$ (Proposition 3.5.1), which is the first interesting case, as it gives non-isomorphic curves for almost every field (Theorem 6). When $d = 3$, we find that $\mathcal{F}_1$ is a section of self-intersection 1 in $W = \mathbb{F}_1$, so $\mathbb{F}_1 \setminus \mathcal{F}_1$ is isomorphic to the blow-up of $\mathbb{A}^2$ at one point, and $\tilde{\Gamma}, \tilde{\Delta}$ are sections of self-intersection 3 and are thus strict transforms of parabolas passing through the point blown up. This explains how the following result is derived from Proposition 3.5.1. However, the statement and the proof that we give are independent of the latter proposition:

Proposition 3.5.7. *Let us fix some constants $a_0, a_1, a_2, a_3 \in \mathrm{k}$ with $a_0 a_3 \neq 0$ and consider the two irreducible polynomials $P, Q \in \mathrm{k}[x, y]$ of degree 2 given by*

$$P = x^2 - a_2 x - a_3 y \quad \textit{and} \quad Q = y^2 + a_0 x + a_1 y.$$

(1) *Denoting by* $\eta\colon \hat{\mathbb{A}}^2 \to \mathbb{A}^2$ *the blow-up of the origin and by* $\tilde{\Gamma}, \tilde{\Delta} \subset \hat{\mathbb{A}}^2$ *the strict transforms of the curves* $\Gamma, \Delta \subset \mathbb{A}^2$ *given by* $P = 0$ *and* $Q = 0$ *respectively, the rational maps*

$$\begin{array}{rccc} \varphi_P\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(-\dfrac{x}{P(x,y)}, P(x,y)\right) \end{array} \quad \textit{and} \quad \begin{array}{rccc} \varphi_Q\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(\dfrac{y}{Q(x,y)}, Q(x,y)\right) \end{array}$$

are birational maps that induce isomorphisms

$$\psi_P = (\varphi_P\eta)|_{\hat{\mathbb{A}}^2\setminus\tilde{\Gamma}}\colon \quad \hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2 \quad \textit{and} \quad \psi_Q = (\varphi_Q\eta)|_{\hat{\mathbb{A}}^2\setminus\tilde{\Delta}}\colon \quad \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2.$$

(2) *Define the curves* $C, D \subset \mathbb{A}^2$ *by* $C = \psi_Q(\tilde{\Gamma} \setminus \tilde{\Delta})$, $D = \psi_P(\tilde{\Delta} \setminus \tilde{\Gamma})$ *and denote by* $\psi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ *the isomorphism induced by the birational transformation* $\psi_P(\psi_Q)^{-1}\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$. *Then, the curves* $C, D \subset \mathbb{A}^2$ *are given by* $f = 0$ *and* $g = 0$ *respectively, where the polynomials* $f, g \in \mathrm{k}[x,y]$ *are defined by:*

$$\begin{aligned} f &= \Big(1 - x(xy + a_1)\Big)\Big(y\Big(1 - x(xy + a_1)\Big) - a_0a_2\Big) - x(a_0)^2a_3, \\ g &= \Big(1 - x(xy + a_2)\Big)\Big(y\Big(1 - x(xy + a_2)\Big) - a_1a_3\Big) - xa_0(a_3)^2. \end{aligned}$$

The following isomorphisms hold:

$$C \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 a_it^i}]\right) \quad \textit{and} \quad D \simeq \operatorname{Spec}\left(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 a_{3-i}t^i}]\right).$$

Moreover, ψ *and* ψ^{-1} *are given by*

$$\begin{array}{lccc} \psi\colon & (x,y) & \mapsto & \left(\dfrac{a_0\Big(x(xy+a_1)-1\Big)}{f(x,y)}, \dfrac{y\,f(x,y)}{(a_0)^2}\right) \\ & \left(\dfrac{a_3\Big(x(xy+a_2)-1\Big)}{g(x,y)}, \dfrac{y\,g(x,y)}{(a_3)^2}\right) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Proof. (1): Let us first prove that φ_P is birational and that $\varphi_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2$. We observe that $\kappa\colon (x,y) \mapsto (x, x^2 - a_2x - a_3y)$ is an automorphism of $\mathbb{A}^2$ that sends Γ onto the line $L_y \subset \mathbb{A}^2$ of equation $y = 0$. Moreover $\tilde{\varphi}_P = \varphi_P\kappa^{-1}\colon (x,y) \mapsto (-\frac{x}{y}, y)$ is birational, so φ_P is birational. Since κ fixes the origin, $\eta^{-1}\kappa\eta$ is an automorphism of $\hat{\mathbb{A}}^2$ that sends $\tilde{\Gamma}$ onto the strict transform $\tilde{L}_y \subset \hat{\mathbb{A}}^2$ of L_y. The fact that $\tilde{\varphi}_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{L}_y \xrightarrow{\simeq} \mathbb{A}^2$ is straightforward using the classical description of the blow-up $\hat{\mathbb{A}}^2$ in which

$$\hat{\mathbb{A}}^2 = \{((x,y),[u:v]) \mid xv = yu\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

and $\eta\colon \hat{\mathbb{A}}^2 \to \mathbb{A}^2$ is the first projection. Actually, with this description $\tilde{L}_y = L_y \times [1:0]$ is given by the equation $v = 0$ and the following morphisms are inverses of each other:

$$\begin{array}{ll} \hat{\mathbb{A}}^2 \setminus \tilde{L}_y \to \mathbb{A}^2, & ((x,y),[u:v]) \mapsto (-\frac{u}{v}, y) \\ \mathbb{A}^2 \to \hat{\mathbb{A}}^2 \setminus \tilde{L}_y, & (x,y) \mapsto ((-xy,y),[-x:1]). \end{array}$$

It follows that $(\tilde{\varphi}_P\eta)(\eta^{-1}\kappa\eta) = \varphi_P\eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2$. The case of φ_Q and $\varphi_Q\eta$ would be treated similarly, using the automorphism of $\mathbb{A}^2$ given by $(x,y) \mapsto (y^2 + a_0x + a_1y, y)$. This proves (1).

(2): Now that (1) is proven, we get two isomorphisms

$$\psi_P|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus D, \quad \psi_Q|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus C,$$

where $U = \hat{\mathbb{A}}^2 \setminus (\tilde{\Gamma} \cup \tilde{\Delta})$. Remembering that $\Gamma \subset \mathbb{A}^2$ is given by $x(x-a_2) = a_3y$, we have an isomorphism

$$\begin{array}{cccc} \rho\colon & \mathbb{A}^1 & \xrightarrow{\simeq} & \Gamma \\ & t & \mapsto & (ta_3 + a_2, t(ta_3+a_2)) \\ & \frac{1}{a_3}(x-a_2) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Replacing $\rho(t)$ in the polynomial $Q(x,y) = xa_0 + ya_1 + y^2$ used to define Δ, we find

$$Q(ta_3 + a_2, t(ta_3+a_2)) = (ta_3+a_2)(t^3a_3 + t^2a_2 + ta_1 + a_0).$$

The root of $ta_3 + a_2$ is sent by ρ to the origin, which is itself blown up by η. Hence, the map $\eta^{-1}\rho$ induces an isomorphism from $V = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 t^ia_i}]) \subset \mathbb{A}^1$ to $\tilde{\Gamma} \setminus \tilde{\Delta}$. Applying $\psi_Q = (\varphi_Q\eta)|_{\hat{\mathbb{A}}^2\setminus\tilde{\Delta}}$, we get an isomorphism $\theta = (\varphi_Q\rho)|_V\colon V \xrightarrow{\simeq} C$. Since $(\varphi_Q)^{-1}$ is given by

$$(\varphi_Q)^{-1}\colon (x,y) \mapsto \left(\frac{y\Big(1 - x(xy+a_1)\Big)}{a_0}, xy\right),$$

we can explicitly give θ and its inverse:

$$\begin{array}{cccc} \theta\colon & \mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 t^ia_i}]) & \xrightarrow{\simeq} & C \\ & t & \mapsto & \left(\dfrac{t}{\sum_{i=0}^3 t^ia_i}, (ta_3+a_2)(\sum_{i=0}^3 t^ia_i)\right) \\ & \dfrac{1}{a_3}\left(\dfrac{y\Big(1-x(xy-a_1)\Big)}{a_0} - a_2\right) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Computing the extension of θ to a morphism $\mathbb{P}^1 \to \mathbb{P}^2$, we see that the curve $C \subset \mathbb{A}^2$ has degree 7. To find its equation, we can compute $((\varphi_Q)^{-1})^*(P)$: since $(a_0)^2P(x,y) =$

$(a_0x)(a_0x - a_0a_2) - (a_0)^2a_3y$, we get

$$\begin{array}{rcl} (a_0)^2((\varphi_Q)^{-1})^*(P) & = & (a_0)^2 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right) \\ & = & y\,(1-x(xy+a_1))\,(y\,(1-x(xy+a_1)) - a_0a_2) - xy(a_0)^2a_3 \\ & = & yf(x,y), \end{array}$$

where

$$f = (1-x(xy+a_1))\,(y\,(1-x(xy+a_1)) - a_0a_2) - x(a_0)^2a_3 \in \mathrm{k}[x,y]$$

is the equation of C (note that the polynomial $y = 0$ appears here, because it corresponds to the line contracted by $(\psi_Q)^{-1}$, corresponding to the exceptional divisor of $\hat{\mathbb{A}}^2 \to \mathbb{A}^2$ via the isomorphism $\mathbb{A}^2 \to \hat{\mathbb{A}}^2 \setminus \hat{\Delta}$). The linear involution of $\mathbb{A}^2$ given by $(x,y) \mapsto (-y,-x)$ exchanges the polynomials P and Q and the maps φ_P and φ_Q, by replacing a_0, a_1, a_2, a_3 by a_3, a_2, a_1, a_0 respectively. This shows that $D \subset \mathbb{A}^2$ has equation $g = 0$, where g is obtained from f on replacing a_0, a_1, a_2, a_3 by a_3, a_2, a_1, a_0, i.e.

$$g = (1-x(xy+a_2))\,(y\,(1-x(xy+a_2)) - a_1a_3) - xa_0(a_3)^2 \in \mathrm{k}[x,y].$$

Therefore, D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} \alpha_{3-i}t^i}])$. It remains to compute the isomorphism $\psi\colon \mathbb{A}^2 \setminus C \to \mathbb{A}^2 \setminus D$, which is by construction equal to the birational maps $\psi_P(\psi_Q)^{-1} = \varphi_P(\varphi_Q)^{-1}$. Using the equation $(a_0)^2 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right) = yf(x,y)$, we get:

$$\begin{array}{rcl} \psi(x,y) & = & \varphi_P\left(\dfrac{y\,(1-x(xy+a_1))}{a_0}, xy\right) \\ & = & \left(-\dfrac{y\,(1-x(xy+a_1))}{a_0 P\left(\frac{y(1-x(xy+a_1))}{a_0}, xy\right)}, P\left(\dfrac{y\,(1-x(xy+a_1))}{a_0}, xy\right)\right) \\ & = & \left(\dfrac{a_0\,(x(xy+a_1)-1)}{f(x,y)}, \dfrac{y\,f(x,y)}{(a_0)^2}\right). \end{array}$$

By symmetry, the expression of ψ^{-1} is obtained from that of ψ by replacing a_0, a_1, a_2, a_3 by a_3, a_2, a_1, a_0, i.e. it is given by $\psi^{-1}(x,y) = \left(\dfrac{a_3\,(x(xy+a_2)-1)}{g(x,y)}, \dfrac{y\,g(x,y)}{(a_3)^2}\right)$. $\square$

Remark 3.5.8. Proposition 3.5.7 yields an isomorphism $\psi^*\colon \mathrm{k}[x,y,\frac{1}{g}] \xrightarrow{\simeq} \mathrm{k}[x,y,\frac{1}{f}]$ which sends the invertible elements onto the invertible elements and thus sends g onto $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (see Lemma 3.2.11). This corresponds to saying that ψ induces an isomorphism between the two fibrations

$$\mathbb{A}^2 \setminus C \xrightarrow{f} \mathbb{A}^1 \setminus \{0\} \quad \text{and} \quad \mathbb{A}^2 \setminus D \xrightarrow{g} \mathbb{A}^1 \setminus \{0\},$$

possibly exchanging the fibres. To study these fibrations, we use the equalities

$$(\varphi_Q)^*(f) = \frac{(a_0)^2 P}{Q}, \quad (\varphi_P)^*(g) = \frac{(a_3)^2 Q}{P}, \tag{I}$$

which can either be checked directly, or deduced as follows: the first equality follows from $((\varphi_Q)^{-1})^*(P) = \frac{yf(x,y)}{(a_0)^2}$, applying $(\varphi_Q)^*$, and the second is obtained by symmetry.

Note that equation (I) provides $\psi^*(g) = \frac{(a_0 a_3)^2}{f}$, since $\psi = \varphi_P(\varphi_Q)^{-1}$.

For each $\mu \in \mathrm{k}$, the fibre $C_\mu \subset \mathbb{A}^2$ given by $f(x,y) = \mu$ is an algebraic curve isomorphic to its preimage by the isomorphism $\psi_Q = (\varphi_Q \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Delta}} \colon \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2$ of Proposition 3.5.7(1). By construction, $(\psi_Q)^{-1}(C_\mu)$ is equal to $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$, where $\tilde{\Gamma}_\mu \subset \hat{\mathbb{A}}^2$ is the strict transform of the curve $\Gamma_\mu \subset \mathbb{A}^2$ given by $(a_0)^2 P - \mu Q = 0$ (follows from equation (I)). The closure of Γ_μ in $\mathbb{P}^2$ is the conic given by

$$(a_0)^2 x^2 - \mu y^2 - z\Big(a_0(\mu + a_0 a_2)x - (\mu a_1 + (a_0)^2 a_3)y\Big) = 0,$$

which passes through $[0:0:1]$ and is irreducible for a general μ. Projecting from the point $[0:0:1]$ we obtain an isomorphism with $\mathbb{P}^1$ (still for a general μ). The curve $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$ is then isomorphic to $\mathbb{P}^1$ minus three $\overline{\mathrm{k}}$-points of $\tilde{\Delta}$, which are fixed and do not depend on μ, and minus the two points at infinity, which correspond to $(a_0)^2 x^2 - \mu y^2 = 0$.

When the field is algebraically closed, we thus find that the general fibres of f are isomorphic to $\mathbb{P}^1$ minus 5 points, whereas the zero fibre is isomorphic to $\mathbb{P}^1$ minus 4 points (if $\sum_{i=0}^3 a_i t^i$ is chosen to have three distinct roots). Moreover, the two points of intersection with the line at infinity say that this curve is a *horizontal curve of degree* 2, or a *horizontal curve which is not a section* (in the usual notation of polynomials and components on boundary, see [NN02, AC96, CD17]), so the polynomials f and g are rational, but not of simple type (see [NN02, CD17]). When $\mathrm{k} = \mathbb{C}$, this implies that the polynomial has non-trivial monodromy [ACD98, Corollary 2, page 320].

3.6 Related questions

3.6.1 Higher dimensional counterexamples

The negative answer to the Complement Problem for $n = 2$ also furnishes a negative answer for any $n \geq 3$. This relies mainly on the cancellation property for curves, as explained in the following result:

Proposition 3.6.1. *Let $C, D \subset \mathbb{A}^2$ be two closed geometrically irreducible curves that have isomorphic complements. Then for each $m \geq 1$, the varieties $H_C = C \times \mathbb{A}^m$ and $H_D = D \times \mathbb{A}^m$ are closed hypersurfaces of $\mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ that have isomorphic complements. Moreover, C and D are isomorphic if and only if $C \times \mathbb{A}^m$ and $D \times \mathbb{A}^m$ are.*

Proof. Denoting by $f, g \in \mathrm{k}[x, y]$ the geometrically irreducible polynomials that define the curves C, D, the varieties $H_C, H_D \subset \mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ are given by the same polynomials and are thus again geometrically irreducible closed hypersurfaces. The isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ then extends naturally to an isomorphism $\mathbb{A}^{m+2} \setminus H_C \xrightarrow{\simeq} \mathbb{A}^{m+2} \setminus H_D$.

The last equivalence is the well-known cancellation property for curves, proven in [AHE72, Corollary (3.4)]. □

Corollary 3.6.2. *For each ground field* k *and each integer* $n \geq 3$*, there exist two geometrically irreducible smooth closed hypersurfaces* $E, F \subset \mathbb{A}^n$ *which are not isomorphic, but whose complements* $\mathbb{A}^n \setminus E$ *and* $\mathbb{A}^n \setminus F$ *are isomorphic. Furthermore, the hypersurfaces can be given by polynomials* $f, g \in \mathrm{k}[x_1, x_2] \subset \mathrm{k}[x_1, \ldots, x_n]$ *of degree* 7 *if the field admits more than* 2 *elements and of degree* 13 *if the field has* 2 *elements. The hypersurfaces* E, F *are isomorphic to* $C \times \mathbb{A}^{n-2}$ *and* $D \times \mathbb{A}^{n-2}$ *for some smooth closed curves* $C, D \subset \mathbb{A}^2$ *of the same degree.*

Proof. It suffices to choose for f, g the equations of the curves $C, D \subset \mathbb{A}^2$ given by Theorem 6. The result then follows from Proposition 3.6.1. □

3.6.2 The holomorphic case

Proposition 3.6.3. *For every choice of* $d+1$ *distinct points* $a_1, \ldots, a_d, a_{d+1} \in \mathbb{C}$*, with* $d \geq 3$*, there exist two closed algebraic curves* $C, D \subset \mathbb{C}^2$ *of degree* $d^2 - d + 1$ *such that* C *and* D *are algebraically isomorphic to* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_d\}$ *and* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_{d+1}\}$ *respectively, and such that* $\mathbb{C}^2 \setminus C$ *and* $\mathbb{C}^2 \setminus D$ *are algebraically isomorphic.*

In particular, if we choose the points in general position, the curves C *and* D *are not biholomorphic, but their complements are.*

Proof. The existence of C, D follows directly from Proposition 3.5.1. It remains to observe that C and D are not biholomorphic if the points are in general position. If $f \colon C \to D$ is a biholomorphism, then f extends to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$, as it cannot have essential singularities. The same holds for f^{-1}, so f is just an element of $\mathrm{PGL}_2(\mathbb{C})$, hence an algebraic automorphism of the projective complex line. Removing at least 4 points of $\mathbb{CP}^1$ (this is the case since $d \geq 3$) and moving one of them produces infinitely many curves with isomorphic complements, up to biholomorphism. □

Corollary 3.6.4. *For each* $n \geq 2$*, there exist algebraic hypersurfaces* $E, F \subset \mathbb{C}^n$ *which are complex manifolds that are not biholomorphic, but have biholomorphic complements.*

Proof. It suffices to take polynomials $f, g \in \mathbb{C}[x_1, x_2]$ provided by Proposition 3.6.3, whose zero sets are smooth algebraic curves $C, D \subset \mathbb{C}^2$ that are not biholomorphic, but have holomorphic complements. We then use the same polynomials to define $E, F \subset \mathbb{C}^n$, which are smooth complex manifolds that have biholomorphic complements and are biholomorphic to $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ respectively. It remains to observe that $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ are not biholomorphic. Denote by $p_C \colon C \times \mathbb{C}^{n-2} \to C$ and

$p_D\colon D \times \mathbb{C}^{n-2} \to D$ the projections on the first factor. If $\psi\colon \mathbb{C}^{n-2} \times C \to \mathbb{C}^{n-2} \times D$ is a biholomorphism, then $p_D \circ \psi\colon \mathbb{C}^{n-2} \times C \to D$ induces, for each $c \in C$, a holomorphic map $\mathbb{C}^{n-2} \to D$ which must be constant by Picard's theorem (since it avoids at least two values of $\mathbb{C}$). Therefore, the map $p_D \circ \psi$ factors through a holomorphic map $\chi\colon C \to D$: we have $p_D \circ \varphi = \chi \circ p_C$. We analogously get a holomorphic map $\theta\colon D \to C$, which is by construction the inverse of χ, so C and D are biholomorphic, a contradiction. □

3.7 Appendix: The case of $\mathbb{P}^2$

In this appendix, we describe some results on the question of complements of curves in $\mathbb{P}^2$ explained in the introduction. These are not directly related to the rest of the text and serve only as comparison with the affine case.

We recall the following simple argument, known to specialists, for lack of reference:

Proposition 3.7.1. *Let $C, D \subset \mathbb{P}^2$ be two geometrically irreducible closed curves such that $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus D$ are isomorphic. If C and D are not equivalent, up to automorphism of $\mathbb{P}^2$, then C and D are singular rational curves.*

Proof. Denote by $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map which restricts to an isomorphism $\mathbb{P}^2 \setminus C \xrightarrow{\simeq} \mathbb{P}^2 \setminus D$. If φ is an automorphism of $\mathbb{P}^2$, then C and D are equivalent. Otherwise, the same argument as in Proposition 3.2.6 shows that both C and D are rational (this also follows from [Bla09, Lemma 2.2]). If C and D are singular, we are done, so we may assume that one of them is smooth, and then has degree 1 or 2. Since the Picard group of $\mathbb{P}^2 \setminus C$ is $\mathbb{Z}/\deg(C)\mathbb{Z}$, we find that C and D have the same degree. This implies that C and D are equivalent under automorphisms of $\mathbb{P}^2$. The case of lines is obvious. For conics, it is enough to check that a rational conic over any field is necessarily equivalent to the conic of equation $xy + z^2 = 0$. Actually, we may always assume that the rational conic contains the point $[1 : 0 : 0]$, since it contains a rational point. We may furthermore assume that the tangent at this point has equation $y = 0$. This means that the equation of the conic is of the form $xy+u(y,z)$, where u is a homogenous polynomial of degree 2. Using a change of variables of the form $(x, y, z) \mapsto (x + ay + bz, y, z)$, where $a, b \in \mathrm{k}$, we may assume that the equation is of the form $xy + cz^2 = 0$, where $c \in \mathrm{k}^*$. Then, using the change of variables $(x, y, z) \mapsto (cx, y, z)$, we finally get, as announced, the equation $xy + z^2 = 0$. □

In order to get families of (singular) curves in $\mathbb{P}^2$ that have isomorphic complements, we here give explicit equations from the construction of Paolo Costa [Cos12]. We thus obtain unicuspidal curves in $\mathbb{P}^2$ which have isomorphic complements, but which are non-equivalent under the action of $\mathrm{Aut}(\mathbb{P}^2)$. We give the details of the proof for self-containedness, and also because the results below are not explicitly stated in [Cos12].

Lemma 3.7.2. *Let* k *be a field. Let $d \geq 1$ be an integer and $P \in \mathrm{k}[x,y]$ a homogenous polynomial of degree d, not a multiple of y. We define the homogeneous polynomial $f_P \in \mathrm{k}[x,y,z]$ of degree $4d+1$ by the following formula, where $w := xz - y^2$:*

$$f_P = zw^{2d} + 2yw^d P(x^2, w) + xP^2(x^2, w).$$

Denote by $C_P, \mathcal{L}, \mathcal{Q} \subset \mathbb{P}^2$ the curves of equations $f_P = 0$, resp. $z = 0$, resp. $w = 0$, and by $V_P, V_{\mathcal{L}}, V_{\mathcal{Q}} \subset \mathbb{A}^3$ their corresponding cones (given by the same equations). Then:

(1) *The polynomial f_P is geometrically irreducible (i.e. irreducible in $\overline{\mathrm{k}}[x,y,z]$).*

(2) *The rational map $\psi_P \colon \mathbb{A}^3 \dashrightarrow \mathbb{A}^3$ which sends (x,y,z) to*

$$\Big(x, y + xP\left(x^2w^{-1}, 1\right), z + 2yP\left(x^2w^{-1}, 1\right) + xP^2\left(x^2w^{-1}, 1\right)\Big)$$

is a birational map of $\mathbb{A}^3$ that restricts to isomorphisms

$$\mathbb{A}^3 \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\mathcal{Q}},\ V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}} \text{ and } \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\mathcal{L}}).$$

Since ψ_P is homogeneous, the same formula induces a birational map of $\mathbb{P}^2$ that restricts to isomorphisms

$$\mathbb{P}^2 \setminus \mathcal{Q} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \mathcal{Q},\ C_P \setminus \mathcal{Q} \xrightarrow{\simeq} \mathcal{L} \setminus \mathcal{Q} \text{ and } \mathbb{P}^2 \setminus (\mathcal{Q} \cup C_P) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (\mathcal{Q} \cup \mathcal{L}).$$

Since the point $[0:0:1]$ is the unique intersection point between C_P and $\mathcal{Q}$, it is also the unique singular point of C_P.

(3) *Let λ be a nonzero element of* k*. Then, the rational map*

$$\varphi_\lambda \colon (x,y,z) \mapsto \big(x + (\lambda - 1)wz^{-1}, y, z\big) = (\lambda x - (\lambda - 1)y^2z^{-1}, y, z)$$

is a birational map of $\mathbb{A}^3$ that restricts to automorphisms of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ and $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$. The same formula then gives automorphisms of $\mathbb{P}^2 \setminus \mathcal{L}$, $\mathcal{Q} \setminus \mathcal{L}$ and $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$.

(4) *Set $\tilde{P}(x,y) = P(\lambda x, y)$ and $\kappa = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P$. Then, the rational map κ restricts to an isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$. In particular, κ also induces an isomorphism $\mathbb{P}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{P}^2 \setminus C_{\tilde{P}}$.*

(5) *For each homogeneous polynomial $\tilde{P} \in \mathrm{k}[x,y]$ of degree d which is not divisible by y, the curves C_P and $C_{\tilde{P}}$ are equivalent up to automorphisms of $\mathbb{P}^2$, if and only if there exist some constants $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ such that*

$$\tilde{P}(x,y) = \rho P(\rho^2 x, y) + \mu y^d.$$

Proof. (1)-(2): As does each rational map $\mathbb{A}^3 \dashrightarrow \mathbb{A}^3$, the rational map ψ_P supplies a morphism of k-algebras $(\psi_P)^*\colon \mathrm{k}[x,y,z] \to \mathrm{k}(x,y,z)$. This sends x, y, z onto $x, y + xP(x^2w^{-1}, 1), z + 2yP(x^2w^{-1}, 1) + xP^2(x^2w^{-1}, 1)$. Note that $(\psi_P)^*$ fixes x and w. This implies that $(\psi_P)^*$ extends to an endomorphism of $\mathrm{k}[x,y,z,w^{-1}]$, which is moreover an automorphism since $(\psi_P)^* \circ (\psi_{-P})^* = \mathrm{id}$. Extending to the quotient field $\mathrm{k}(x,y,z)$, we get an automorphism of $\mathrm{k}(x,y,z)$, that we again denote by $(\psi_P)^*$, so ψ_P is a birational map of $\mathbb{A}^3$ and induces moreover an isomorphism of $\mathbb{A}^3 \setminus V_{\mathcal{Q}}$, because $(\psi_P)^*(\mathrm{k}[x,y,z,w^{-1}]) = \mathrm{k}[x,y,z,w^{-1}]$. We then observe that $(\psi_P)^*(z) = f_P w^{-2d}$ where f_P and $w = xz - y^2$ are coprime since $f_P(1,0,0) = P^2(1,0) \neq 0$. Let us also notice that $V_P \cap V_{\mathcal{Q}} = \{(x,y,z) \in \mathbb{A}^3 \mid x = y = 0\}$ and that $V_{\mathcal{L}} \cap V_{\mathcal{Q}} = \{(x,y,z) \in \mathbb{A}^3 \mid y = z = 0\}$. Hence ψ_P restricts to an isomorphism of surfaces $V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}}$. This implies that V_P and C_P are rational, and that f_P is geometrically irreducible, which proves (1). This also implies that ψ_P restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\mathcal{L}})$. As ψ_P is homogeneous, we get the analogous results by replacing $\mathbb{A}^3$, V_P, $V_{\mathcal{L}}$, $V_{\mathcal{Q}}$ by $\mathbb{P}^2$, C_P, $\mathcal{L}$, $\mathcal{Q}$ respectively.

(3): We check that $\varphi_\lambda \circ \varphi_{\lambda^{-1}} = \mathrm{id}$, so φ_λ is a birational map of $\mathbb{A}^3$, which restricts to an automorphism of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, since the denominators only involve z. Moreover, $(\varphi_\lambda)^*(w) = \lambda w$ (where $(\varphi_\lambda)^*$ is the automorphism of $\mathrm{k}(x,y,z)$ corresponding to φ_λ), so the surface $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ is preserved, hence φ_λ restricts to automorphisms of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ and $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$. Since φ_λ is homogeneous, the same formula then gives automorphisms of $\mathbb{P}^2 \setminus \mathcal{L}$, $\mathcal{Q} \setminus \mathcal{L}$ and $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$.

(4): By (2)-(3), the transformation $\kappa = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P$ restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\tilde{P}})$. Let us prove that with the special choice of $\tilde{P}$ that we have made, κ then restricts to an isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$. For this, we prove that the restriction of κ is the identity automorphism on $V_{\mathcal{Q}} \setminus V_P = V_{\mathcal{Q}} \setminus V_{\tilde{P}} = V_{\mathcal{Q}} \setminus \{(x,y,z) \in \mathbb{A}^3 \mid x = y = 0\}$. We compute

$$\varphi_\lambda\psi_P(x,y,z) = \left(x + (\lambda - 1)w^{2d+1}f_P^{-1}, y + xP(x^2, w)w^{-d}, f_P w^{-2d}\right)$$

which satisfies $(\varphi_\lambda\psi_P)^*(w) = (\varphi_\lambda)^*(w) = \lambda w$. To simplify the notation, we write $\delta = (\lambda - 1)w^{2d+1}f_P^{-1}$ and get that $\kappa(x,y,z) = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P(x,y,z)$ is equal to

$$\left(x + \delta, y + xP(x^2, w)w^{-d} - (x + \delta)\tilde{P}\left(\lambda^{-1}(x + \delta)^2 w^{-1}, 1\right), z + \zeta\right)$$

for some $\zeta \in \mathrm{k}(x,y,z)$. Since $\tilde{P}(x,y) = P(\lambda x, y)$, the second component is

$$\kappa^*(y) = y + \frac{xP(x^2, w) - P((x+\delta)^2, w)(x + \delta)}{w^d}.$$

As w^{d+1} divides the numerator of δ, we can write $\kappa^*(y)$ as $y + w(f_P)^{-n}R$, for some $R \in \mathrm{k}[x,y,z]$ and $n \geq 0$. Similarly, $\kappa^*(x) = x + wf_P^{-1}S$, where $S \in \mathrm{k}[x,y,z]$. Since $\kappa^*(w) = \lambda w$, we get

$$\lambda w = (x + wf_P^{-1}S)(z + \zeta) - (y + wf_P^{-n}R)^2,$$

which shows that $\zeta(x+wf_P^{-1}S)=wf_P^{-\tilde{m}}\tilde{T}$ for some $\tilde{T}\in \mathrm{k}[x,y,z]$, $\tilde{m}\geq 0$. Hence we can write $\kappa^*(z)=z+\zeta=z+wf_P^{-m}T$ for some $T\in\mathrm{k}[x,y,z]$ and $m\geq 0$. This shows that κ is well defined on $V_{\mathcal{Q}}\setminus V_P=V_{\mathcal{Q}}\setminus V_{\tilde{P}}=V_{\mathcal{Q}}\setminus\{(x,y,z)\in\mathbb{A}^3\mid x=y=0\}$ and restricts to the identity on this surface.

Since κ is homogeneous, the isomorphism $\mathbb{A}^3\setminus V_P\xrightarrow{\simeq}\mathbb{A}^3\setminus V_{\tilde{P}}$ also induces an isomorphism $\mathbb{P}^2\setminus C_P\xrightarrow{\simeq}\mathbb{P}^2\setminus C_{\tilde{P}}$, which fixes pointwise the curve $\mathcal{Q}\setminus C_P=\mathcal{Q}\setminus C_{\tilde{P}}$.

(5): Suppose first that $\tilde{P}(x,y)=\rho P(\rho^2x,y)+\mu y^d$ for some $\rho\in\mathrm{k}^*,\mu\in\mathrm{k}$. Define the transformation $\alpha\in\mathrm{GL}_3(\mathrm{k})$ by

$$\alpha(x,y,z)=(x,\rho y-\mu x,\rho^2z-2\rho\mu y+\mu^2x)$$

and the birational transformation $s\in\mathrm{Bir}(\mathbb{A}^3)$ by $s=\psi_{\tilde{P}}\alpha(\psi_P)^{-1}$. Let us note that $s^*=(\psi_P^*)^{-1}\alpha^*\psi_{\tilde{P}}^*$. We check that $\alpha^*(w)=\rho^2w$, from which we get $s^*(w)=\rho^2w$. The equality

$$\begin{array}{rcl}\alpha^*(\psi_{\tilde{P}}^*(y)) & = & \alpha^*(y+x\tilde{P}(x^2w^{-1},1))=\rho y-\mu x+x\tilde{P}(\rho^{-2}x^2w^{-1},1)\\ & = & \rho y+\rho xP(x^2w^{-1},1)=\rho\,\psi_P^*(y)\end{array}$$

gives us $s^*(y)=\rho y$. The relation $z=x^{-1}(w-y^2)$ combined with the equality $s^*(x)=x$ now proves that $s^*(z)=\rho^2z$. But we have $(\psi_P)^*(z)=f_Pw^{-2d}$ and $(\psi_{\tilde{P}})^*(z)=f_{\tilde{P}}w^{-2d}$, so that we get $\alpha^*(f_{\tilde{P}}w^{-2d})=\rho^2f_Pw^{-2d}$. In turn, this latter equality yields

$$\alpha^*(f_{\tilde{P}})=\rho^{4d+2}f_P.$$

This shows that α induces an automorphism of $\mathbb{P}^2$ sending C_P onto $C_{\tilde{P}}$.

Conversely, suppose that there exists $\tau\in\mathrm{Aut}(\mathbb{P}^2)$ sending C_P onto $C_{\tilde{P}}$.

We begin by proving that τ preserves the conic $\mathcal{Q}$. Since $C_P\setminus\mathcal{Q}\simeq C_{\tilde{P}}\setminus\mathcal{Q}\simeq L\setminus\mathcal{Q}\simeq\mathbb{A}^1$, the irreducible conic $\mathcal{Q}\subset\mathbb{P}^2$ intersects C_P (respectively $C_{\tilde{P}}$) in exactly one $\overline{\mathrm{k}}$-point, the unique singular point $[0:0:1]$ of C_P (resp. $C_{\tilde{P}}$). The irreducible conic $\tau(\mathcal{Q})$ thus also intersects $C_{\tilde{P}}$ in one $\overline{\mathrm{k}}$-point, namely $[0:0:1]$. Observe that this implies that $\tau(\mathcal{Q})=\mathcal{Q}$. We first notice that $C_{\tilde{P}}\setminus\{[0:0:1]\}\simeq\mathbb{A}^1$, so there is one k-point at each step of the resolution of $C_{\tilde{P}}$. We can then write $q_1=[0:0:1]$ and define a sequence of points $(q_i)_{i\geq1}$ such that q_i is the point infinitely near q_{i-1} belonging to the strict transform of $C_{\tilde{P}}$, for each $i\geq 2$. Denote by r the biggest integer such that q_r belongs to the strict transform of $\mathcal{Q}$ and by r' the biggest integer such that $q_{r'}$ belongs to the strict transform of $\tau(\mathcal{Q})$. By Bézout's Theorem (since $\mathcal{Q}$ and $\tau(\mathcal{Q})$ are smooth), we have

$$\sum_{i=1}^{r}m_{q_i}(C_{\tilde{P}})=\deg(\mathcal{Q})\deg(C_{\tilde{P}})=\deg(\tau(\mathcal{Q}))\deg(C_{\tilde{P}})=\sum_{i=1}^{r'}m_{q_i}(C_{\tilde{P}}),$$

which yields $r=r'$. On the blow-up $X\to\mathbb{P}^2$ of $q_1,\dots,q_r$, the strict transform of the curve $C_{\tilde{P}}$ is then disjoint from those of $\mathcal{Q}$ and $\tau(\mathcal{Q})$, which are linearly equivalent. Assume by contradiction that we have $\tau(\mathcal{Q})\neq\mathcal{Q}$. Then, we claim that the strict transform of any irreducible conic $\mathcal{Q}'$ in the pencil generated by $\mathcal{Q}$ and $\tau(\mathcal{Q})$ is also disjoint from the strict transform of $C_{\tilde{P}}$. Indeed, we first note that $C_{\tilde{P}}$ and $\mathcal{Q}'$ have no common irreducible component since $C_{\tilde{P}}$ is an irreducible curve whose degree satisfies

$$\deg C_{\tilde{P}} \geq 5 > 2 = \deg \mathcal{Q}'.$$

Finally, since the (infinitely near) points $q_1, \ldots, q_r$ belong to both $\mathcal{Q}'$ and $C_{\tilde{P}}$ and since $\sum_{i=1}^r m_{q_i}(C_{\tilde{P}}) = \deg(\mathcal{Q}')\deg(C_{\tilde{P}})$, the curves $\mathcal{Q}'$ and $C_{\tilde{P}}$ do not have any other common (infinitely near) point.

Choose now a general point q of $\mathbb{P}^2$ which belongs to $C_{\tilde{P}} \setminus \{q_1\} \simeq \mathbb{A}^1$ and choose the conic $\mathcal{Q}'$ in the pencil generated by $\mathcal{Q}$ and $\tau(\mathcal{Q})$ which passes through q. Then, the strict transforms of $\mathcal{Q}'$ and $C_{\tilde{P}}$ intersect in X (at the point q). This contradiction shows that $\mathcal{Q}$ is preserved by τ.

Since $\tau \in \operatorname{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathrm{k})$ fixes the point $[0:0:1]$ (which is the unique singular point of both C_P and $C_{\tilde{P}}$) and preserves the line $x = 0$ (which is the tangent line of both C_P and $C_{\tilde{P}}$ at the point $[0:0:1]$), it admits a (unique) lift $\alpha \in \mathrm{GL}_3(\mathrm{k})$ which is triangular and satisfies $\alpha^*(x) = x$. This means that α is of the form:

$$\alpha\colon (x, y, z) \mapsto (x, \rho y - \mu x, \gamma z + \delta y + \varepsilon x),$$

for some constants $\rho, \mu, \gamma, \delta, \varepsilon \in \mathrm{k}$ (satisfying $\rho\gamma \neq 0$). Since $\alpha^*(w)$ is proportional to w, we get $\gamma = \rho^2$, $\delta = -2\rho\mu$ and $\varepsilon = \mu^2$, i.e. α is of the form

$$\alpha\colon (x, y, z) \mapsto (x, \rho y - \mu x, \rho^2 z - 2\rho\mu y + \mu^2 x).$$

Set $s := \psi_{\tilde{P}}\alpha(\psi_P)^{-1} \in \operatorname{Bir}(\mathbb{A}^3)$. Since $\alpha^*(w) = \rho^2 w$, we also get $s^*(w) = \rho^2 w$. Since $(\psi_P)^*(z) = f_P w^{-2d}$, $(\psi_{\tilde{P}})^*(z) = f_{\tilde{P}} w^{-2d}$ and since $\alpha^*(f_{\tilde{P}})$ and f_P are proportional, the fractions $s^*(z)$ and z are also proportional. Therefore, there exists a nonzero constant $\xi \in \mathrm{k}$ such that

$$s^*(x) = x, \quad s^*(w) = \rho^2 w, \quad s^*(z) = \xi z. \tag{J}$$

Moreover, s induces a birational map $\hat{s}$ of $\mathbb{P}^2$ which is an automorphism of $\mathbb{P}^2 \setminus \mathcal{Q}$, because the same holds for α, ψ_P and $\psi_{\tilde{P}}$. Let us observe that $\hat{s}$ is in fact an automorphism of $\mathbb{P}^2$. Indeed, otherwise $\hat{s}$ would contract $\mathcal{Q}$ to one point. This is impossible: Since $\hat{s}$ preserves the two pencils of conics given by $[x:y:z] \mapsto [w:x^2]$ and $[x:y:z] \mapsto [w:z^2]$, which have distinct base-points $[0:0:1]$ and $[1:0:0]$, these base-points are fixed by $\hat{s}$. Hence, there exist some constants $\zeta, \eta, \theta \in \mathrm{k}$ such that $s^*(y) = \zeta x + \eta y + \theta z$. Hence (J) gives us $\zeta = \theta = 0$, i.e. $s^*(y) = \eta y$. But the equality $s = \psi_{\tilde{P}}\alpha(\psi_P)^{-1}$ is equivalent to $\psi_{\tilde{P}}\alpha = s\psi_P$ and by taking the second coordinate we get

$$(\rho y - \mu x) + x\tilde{P}\left(\rho^{-2}x^2w^{-1}, 1\right) = (\psi_{\tilde{P}}\alpha)^*(y) = (s\psi_P)^*(y) = \eta\left(y + xP\left(x^2w^{-1}, 1\right)\right)$$

which yields $\rho = \eta$ and $\tilde{P}(\rho^{-2}x^2w^{-1}, 1) = \rho P(x^2w^{-1}, 1) + \mu$. By substituting $\rho^{-2}y + x^{-1}y^2$ for z and by noting that $w(x, y, \rho^{-2}y + x^{-1}y^2) = \rho^{-2}xy$, we obtain $\tilde{P}(xy^{-1}, 1) = \rho P(\rho^2 xy^{-1}, 1) + \mu$, which is equivalent to $\tilde{P}(x, y) = \rho P(\rho^2 x, y) + \mu y^d$, as we required. $\square$

The construction of Lemma 3.7.2 yields, for each $d \geq 1$, families of curves of degree $4d+1$ having isomorphic complements. These are equivalent for $d = 1$, at least when k is algebraically closed (Lemma 3.7.2(5)), but not for $d \geq 2$. We can now easily provide explicit examples:

Proposition 3.7.3. *Let $d \geq 2$ be an integer. Set $P = x^d + x^{d-1}y$ and $w = xz - y^2 \in \mathrm{k}[x, y]$. All curves of $\mathbb{P}^2$ given by*

$$zw^{2d} + 2yw^d P(\lambda x^2, w) + xP^2(\lambda x^2, w) = 0$$

for $\lambda \in \mathrm{k}^*$, have isomorphic complements and are pairwise not equivalent up to automorphisms of $\mathbb{P}^2$.

Proof. The curves correspond to the curves $C_{P(\lambda x,y)}$ of Lemma 3.7.2 and thus have isomorphic complements by Lemma 3.7.2(4). It remains to show that if $C_{P(\lambda x,y)}$ is equivalent to $C_{P(\tilde{\lambda} x,y)}$, then $\lambda = \tilde{\lambda}$. Lemma 3.7.2(4) yields the existence of $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ such that $P(\tilde{\lambda}x, y) = \rho P(\rho^2 \lambda x, y) + \mu y^d$. Since $d \geq 2$, both $P(\tilde{\lambda}x, y)$ and $\rho P(\rho^2 \lambda x, y)$ do not have component with y^d, so $\mu = 0$. We then compare the coefficients of x^d and $x^{d-1}y$ and get

$$\tilde{\lambda}^d = \rho(\rho^2\lambda)^d, \quad \tilde{\lambda}^{d-1} = \rho(\rho^2\lambda)^{d-1},$$

which yields $\tilde{\lambda} = \rho^2 \lambda$, whence $\rho = 1$ and $\tilde{\lambda} = \lambda$ as desired. $\square$

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Chapter 4

Lines in the affine plane in positive characteristic

Abstract. In this chapter, we summarize some results on embeddings of the affine line in the affine plane. It is well known by the theorem of Abhyankar-Moh-Suzuki that any line in the affine plane is rectifiable if the characteristic of the base-field k is 0. This result does not hold in positive characteristic and the classification of lines in the plane is completely unknown. A conjecture related to this problem asks the following: given a polynomial $f \in \mathrm{k}[x, y]$ that defines a line in $\mathbb{A}^2$, does it follow that $f - \lambda$ defines a line for all $\lambda \in \mathrm{k}$? We show that this conjecture holds for all lines of degree at most 11.

Contents

4.1 Introduction

Throughout this section, we fix an algebraically closed field k of characteristic $p \geq 0$. Our aim is to study lines in the affine plane $\mathbb{A}^2$. We call a closed curve $C \subset \mathbb{A}^2$ a *line* if it is isomorphic to $\mathbb{A}^1$. Correspondingly, we call a polynomial $f \in \mathrm{k}[x, y]$ a *line* if $\mathrm{k}[x, y]/(f) \simeq \mathrm{k}[t]$, i.e. the curve defined by f is a line. A line in $\mathbb{A}^2$ can also be described as the image of a closed embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$. Such an embedding is given by $t \mapsto (u(t), v(t))$ such that $u, v \in \mathrm{k}[t]$ with $\mathrm{k}[u, v] = \mathrm{k}[t]$.

We call two closed curves $C, D \subset \mathbb{A}^2$ *equivalent* if there exists an automorphism of $\mathbb{A}^2$ that sends C to D. We say that a line is *rectifiable* if it is equivalent to a coordinate line. Correspondingly, we call a line $f \in \mathrm{k}[x, y]$ a *variable* if there exists a polynomial $g \in \mathrm{k}[x, y]$ such that $\mathrm{k}[f, g] = \mathrm{k}[x, y]$. In the literature non-rectifiable lines have also been called *bad*, *wild*, or *exotic*. The foundational result in the study of lines in $\mathbb{A}^2$ was given S. S. Abhyankar and T. T. Moh.

Theorem 4.1.1 ([AM75]). *Let $f \in \mathrm{k}[x, y]$ be a line. If $p = \mathrm{char}(\mathrm{k})$ does not divide $\deg_x(f)$ or $\deg_y(f)$, then f is a variable. In particular, every line is a variable if $\mathrm{char}(\mathrm{k}) = 0$.*

Remark 4.1.2. Theorem 4.1.1 was proven independently in [Suz74] for the field of complex numbers, with different methods. The complex version of Theorem 4.1.1 is thus usually called the Abhyankar-Moh-Suzuki Theorem.

We will see in Example 4.2.10 that not all lines are variables if $p > 0$. We observe that if $f \in \mathrm{k}[x, y]$ is a variable, then every fiber of f is a line. This naturally leads to the following conjecture which can be found in [Sat76], but according to [Gan11] was already posed by S. S. Abhyankar in 1968.

Conjecture 4.1.3. *Let $f \in \mathrm{k}[x, y]$ be a line. Then $f - \lambda$ is a line for all $\lambda \in \mathrm{k}$.*

Remark 4.1.4. It is shown in [Gan11, Theorem 4.12] that $f - \lambda$ is a line for all $\lambda \in \mathrm{k}$ if and only if $f - \lambda$ is a line for infinitely many $\lambda \in \mathrm{k}$. Moreover, it is shown that if f is a line, then $f - \lambda$ is irreducible, smooth and has one place at infinity for all but finitely many $\lambda \in \mathrm{k}$. To prove Conjecture 4.1.3 it is thus sufficient to show that $f - \lambda$ is rational for infinitely many $\lambda \in \mathrm{k}$.

4.2 Preliminaries

The results and proofs in this section are all well known and can also be found in various sources such as [AM75], [Gan79], [Moh88], or [Dai90].

As usual, we identify $\mathbb{A}^2$ as an open subset of $\mathbb{P}^2$ via the embedding $(x, y) \mapsto [x : y : 1]$ and boundary curve $L_\infty = \mathbb{P}^2 \setminus \mathbb{A}^2$, given by the equation $z = 0$. For a closed curve $C \subset \mathbb{A}^2$ we denote by $\overline{C}$ its closure in $\mathbb{P}^2$. We know from Lemma 2.3.1 that if $C \subset \mathbb{A}^2$ is a line, then $\overline{C} \subset \mathbb{P}^2$ is either a line, a conic, or a unicuspidal curve that has the very tangent line L_∞. If $\overline{C}$ is unicuspidal, its minimal resolution of singularities is a tower resolution. Thus, if $\overline{C}$ is singular, it has a sequence of singular points, called the *multiplicity sequence at infinity*, where the first singular point is proper and any subsequent singular point lies in the first neighborhood of the previous one.

Lemma 4.2.1. *Let $C \subset \mathbb{A}^2$ be a line, defined by a polynomial $f \in \mathrm{k}[x, y]$, and let $u, v \in \mathrm{k}[t]$ be polynomials such that $\mathrm{k}[u, v] = \mathrm{k}[t]$ and $f(u, v) = 0$, where $\deg(u) < \deg(v)$. Then the following hold:*

(*i*) $\deg(f) = \deg(v)$.

(*ii*) $m_{[0:1:0]}(\overline{C}) = \deg(v) - \deg(u)$.

(*iii*) $\deg_x(f) = \deg(v)$ *and* $\deg_y(f) = \deg(u)$.

Proof. To prove (i) it is enough to observe that the closure $\overline{C} \subset \mathbb{P}^2 = \mathbb{A}^2 \cup L_\infty$ intersects the line L_∞ with intersection multiplicity $\deg(v)$. Thus $\deg(f) = \deg(\overline{C}) = \overline{C} \cdot L_\infty = \deg(v)$.

The number $\deg_x(f)$ is the intersection number between C and the affine line $y = 0$ and thus coincides with $\deg(v)$. Analogously, we get $\deg_y(f) = \deg(u)$ and thus we obtain (iii). The intersection number between $\overline{C}$ and the projective line $x = 0$ is $\deg(u) + m_{[0:1:0]}(\overline{C})$, but also $\deg(\overline{C}) = \deg(v)$, and hence we get (ii). □

Corollary 4.2.2. *Let $C \subset \mathbb{A}^2$ be a line such that $\deg(C)$ is a prime number. Then C is rectifiable.*

Proof. Up to a linear change of coordinates we can assume that C is given by a polynomial $f \in \mathrm{k}[x, y]$ such that $\deg_y(f) < \deg_x(f)$. Suppose that C is not rectifiable. Then p divides $\deg(C) = \deg(f) = \deg_x(f)$ by Theorem 4.1.1, and since $\deg(C)$ is a prime number, it follows that $\deg(C) = p$. Moreover, p divides the first multiplicity $m_1 = \deg_x(f) - \deg_y(f)$ at infinity by Theorem 4.1.1. We thus reach a contradiction since $m_1 < \deg(C) = p$. □

Lemma 4.2.3. *Let θ be an automorphism of $\mathbb{A}^2$ and denote $U = \theta(x) \in \mathrm{k}[x, y]$ and $V = \theta(y) \in \mathrm{k}[x, y]$. Then $\deg_x(U)$ divides $\deg_x(V)$ or vice versa.*

Proof. We observe that the claim is true if θ is an affine map. Next, suppose that θ is of the form $j_n \circ a_n \circ \ldots \circ j_1 \circ a_1 \circ j_0$ where $j_i \in \mathrm{Jon}_2 \setminus \mathrm{Aff}_2$ for $i = 0, \ldots, n$ and $a_i \in \mathrm{Aff}_2 \setminus \mathrm{Jon}_2$ for $i = 1, \ldots, n$. We show by induction on n that θ is then of the form

$$(x, y) \mapsto (ax^m + u(x, y), bx^n + v(x, y))$$

where $m < n$ such that m divides n, $\deg(u) < m$, $\deg(v) < n$, and $a, b \in \mathrm{k}^*$. This holds for $n = 0$ by the definition of a de Jonquières map. Suppose by the induction hypothesis that $j_n \circ a_n \circ \ldots j_1 \circ a_1 \circ j_0$ is of the claimed form and let $a_{n+1} \in \mathrm{Aff}_2 \setminus \mathrm{Jon}_2$ and $j_{n+1} \in \mathrm{Jon}_2 \setminus \mathrm{Aff}_2$. Then

$$a_{n+1}(x, y) = (\alpha_1 x + \alpha_2 y + \alpha_3, \beta_1 x + \beta_2 y + \beta_3)$$

for some $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathrm{k}$ with $\alpha_2 \neq 0$, and

$$j_{n+1}(x, y) = (\gamma_1 x + \gamma_2, \delta y + \delta_k x^k + \ldots + \delta_1 x + \delta_0)$$

where $\gamma_1, \gamma_2 \in \mathrm{k}$ with $\gamma_1 \neq 0$ and $\delta, \delta_0, \ldots, \delta_k \in \mathrm{k}$ with $\delta, \delta_k \neq 0$ and $k \geq 2$. It follows that

$$(j_{n+1} \circ a_{n+1} \circ \ldots \circ j_1 \circ a_1 \circ j_0)(x, y) = (\alpha_2 b \gamma_1 x^n + u'(x, y), a_2^k \gamma_1^k \delta_k x^{kn} + v'(x, y))$$

for some $u', v' \in \mathrm{k}[x, y]$ with $\deg(u') < n$ and $\deg(v') < kn$ and so the induction step is complete.

We have proved the claim if θ is of the form $j_n \circ a_n \circ \ldots \circ j_1 \circ a_1 \circ j_0$ and hence the claim also follows if θ is of the form $a_{n+1} \circ j_n \circ a_n \circ \ldots \circ j_1 \circ a_1 \circ j_0 \circ a_0$ for $a_0, a_{n+1} \in (\mathrm{Aff}_2 \setminus \mathrm{Jon}_2) \cup \{\mathrm{id}\}$. This finishes the proof by Theorem 2.3.7. □

Corollary 4.2.4. *Let $f \in \mathrm{k}[x,y]$ be a variable. Then either $\deg_x(f)$ divides $\deg_y(f)$ or $\deg_y(f)$ divides $\deg_x(f)$.*

Proof. Let f be parametrized by $u, v \in \mathrm{k}[t]$, i.e. $f(u,v) = 0$ and $\mathrm{k}[u,v] = \mathrm{k}[t]$. Then $\deg_x(f) = \deg(v)$ and $\deg_y(f) = \deg(u)$ by Lemma 4.2.1. The map $x \mapsto (u(x), v(x))$ defines a closed embedding of $\mathbb{A}^1$ in $\mathbb{A}^2$. Since f is a variable, this embedding extends to an automprhism of $\mathbb{A}^2$, i.e. there exist $U, V \in \mathrm{k}[x,y]$ with $U(x,0) = u(x)$ and $V(x,0) = v(x)$ and $\mathrm{k}[U,V] = \mathrm{k}[x,y]$. We then have $\deg_x(U) = \deg(u)$ and $\deg_x(V) = \deg(v)$ and thus the claim follows from Lemma 4.2.3. $\square$

Remark 4.2.5. We should mention that historically, the main difficulty in the proof of the Abhyankar-Moh Theorem in characteristic 0 consisted in showing that if there are elements $u, v \in \mathrm{k}[t]$ such that $\mathrm{k}[u,v] = \mathrm{k}[t]$, then $\deg(u)$ divides $\deg(v)$ or vice versa. From this fact one can also deduce the theorem of Jung. In this sense, the order of results in this section is somewhat unusual, but in this way, we obtain all the needed results without assumptions on the characteristic. For a more detailed historical account, see for instance [vdE04].

Lemma 4.2.6. *Let $f \in \mathrm{k}[x,y]$ be a line and $u, v \in \mathrm{k}[t]$ such that $f(u,v) = 0$. Then there exists $\alpha \in \mathrm{k}^*$ such that $\partial_x f(u,v) = \alpha \partial_t v$ and $\partial_y f(u,v) = -\alpha \partial_t u$.*

Proof. Applying the derivative in t to the equation $f(u,v) = 0$ yields

$$\partial_x f(u,v)\partial_t u + \partial_y f(u,v)\partial_t v = 0.$$

Since f is a line, we can find $g \in \mathrm{k}[x,y]$ such that $t = g(u,v)$. Taking the derivative in t then yields

$$\partial_x g(u,v)\partial_t u + \partial_y g(u,v)\partial_t v = 1.$$

In particular $\partial_t u$ and $\partial_t v$ are coprime. To prove the claim, it is sufficient to show that $\partial_x f(u,v)$ and $\partial_y f(u,v)$ are coprime. Suppose that $\partial_x f(u,v)$ and $\partial_y f(u,v)$ have a common non-constant divisor d. Let $\alpha \in \mathrm{k}$ be a root of d. Then $(u(\alpha), v(\alpha))$ is a singular point of the curve defined by f, but this is not possible since f is a line and thus smooth. $\square$

Lemma 4.2.7. *Let $f \in \mathrm{k}[x,y]$ be a line and $u, v \in \mathrm{k}[t]$ such that $f(u,v) = 0$. Then $f \in \mathrm{k}[x, y^p]$ if and only if $u \in \mathrm{k}[t^p]$.*

Proof. We observe that $f \in \mathrm{k}[x,y^p] \iff \partial_y f = 0$ and $u \in \mathrm{k}[t^p] \iff \partial_t u = 0$. Moreover, $\partial_x f$ and $\partial_y f$ cannot both be 0, otherwise f lies in $\mathrm{k}[x^p, y^p]$ and cannot be a line. Likewise, $\partial_t u$ and $\partial_t v$ cannot both be 0. The claim then follows from the identity

$$\partial_x f(u,v)\partial_t u + \partial_y f(u,v)\partial_t v = 0,$$

obtained by taking the derivative in t of $f(u,v) = 0$. $\square$

Lemma 4.2.8. *Let $u, v \in \mathrm{k}[t]$. Then $\mathrm{k}[u^p, v] = \mathrm{k}[t]$ if and only if $\mathrm{k}[u,v] = \mathrm{k}[t]$ and $\partial_t v \in \mathrm{k}^*$.*

Proof. Suppose that $\mathrm{k}[u^p, v] = \mathrm{k}[t]$. Since $u \in \mathrm{k}[u^p, v]$, we also have $\mathrm{k}[u, v] = \mathrm{k}[t]$. Moreover, there exists a polynomial $g \in \mathrm{k}[x, y]$ such that $t = g(u^p, v)$. Then the derivative in t yields $1 = \partial_y g(u^p, v)\partial_t v$, and thus $\partial_t v \in \mathrm{k}[t]^* = \mathrm{k}^*$.

For the converse, suppose that $\mathrm{k}[u, v] = \mathrm{k}[t]$ and $\partial_t v \in \mathrm{k}^*$. Then we have $t^p \in \mathrm{k}[u^p, v^p] \subset \mathrm{k}[u^p, v]$. Moreover, we can write $v(t) = at + b(t^p)$, where $a \in \mathrm{k}^*$ and $b \in \mathrm{k}[t]$, and hence $t \in \mathrm{k}[u^p, v]$. □

For a polynomial $f = \sum a_{ij}x^i y^j \in \mathrm{k}[x, y]$ and $n \in \mathbb{N}$ we define

$$f^{(n)} := \sum a_{ij}^n x^i y^j$$

by raising all coefficients to the n-th power. With this notation we obtain the identity $f^p = f^{(p)}(x^p, y^p)$.

Corollary 4.2.9. *Let $f \in \mathrm{k}[x, y]$ be a polynomial. Then $f^{(p)}(x, y^p)$ is a line if and only if f is a line and $\partial_x f \in \mathrm{k}^*$.*

Proof. Suppose that $f^{(p)}(x, y^p)$ is a line. Then by Lemma 4.2.7 there exists $u^p \in \mathrm{k}[t^p]$ and $v \in \mathrm{k}[t]$ such that $f^{(p)}(u^p, v^p) = 0$ and $\mathrm{k}[u^p, v] = \mathrm{k}[t]$. But then $f(u, v) = 0$ and $\mathrm{k}[u, v] = \mathrm{k}[t]$ and thus f is a line. By Lemma 4.2.8 we have $\partial_t v \in \mathrm{k}^*$ and thus also $\partial_x f(u, v) \in \mathrm{k}^*$. Since we have $f(u, v) = 0$, it follows that $\partial_x f \in \mathrm{k}^*$.

For the converse, suppose that f is a line and $\partial_x f \in \mathrm{k}^*$. We have $\mathrm{k}[u, v] = \mathrm{k}[t]$ and $\partial_t v \in \mathrm{k}^*$ by Lemma 4.2.7. It then follows from Lemma 4.2.8 that $\mathrm{k}[u^p, v] = \mathrm{k}[t]$. We also have $0 = f(u, v)^p = f^{(p)}(u^p, v^p)$ and thus $f^{(p)}(x, y^p)$ is a line. □

Example 4.2.10. The best known examples of non-rectifiable lines are the so-called (generalized) *Segre lines*, which first appear in [Seg56] (see also [Gan11]). They can be constructed as follows. We start with a polynomial of the form $f = y - u(x^p) - x$, where $u \in \mathrm{k}[x]$ such that $p \nmid \deg(u) > 1$. Then f is a line and $\partial_x f \in \mathrm{k}^*$. It follows from Corollary 4.2.9 that for any $n \in \mathbb{N}$ the polynomial

$$g = f^{(p^n)}\left(x, y^{p^n}\right) = y^{p^n} - v(x^p) - x$$

is a line, where we denote by $v(x^p) = u^{(p^n)}(x^p)$. We have $\deg_x(g) = p\deg(v) = p\deg(u)$ and $\deg_y(g) = p^n$ and thus by Corollary 4.2.4 it follows that g is not a variable if $n \geq 2$. Additionally, we can find the parametrization $g(t^{p^n}, u(t^p) + t) = 0$. We can also see that Conjecture 4.1.3 holds for Segre lines. To see this, let $\lambda \in \mathrm{k}$. Then we can choose a p^n-th root μ of λ. It follows that $g - \lambda = (y - \mu)^{p^n} - v(x^p) - x = g(x, y - \mu)$ is again a line.

Corollary 4.2.9 allows us to find many examples of non-rectifiable lines. Suppose that $f \in \mathrm{k}[x, y]$ with $\partial_x f \in \mathrm{k}^*$ and $f - \lambda$ is a line for all $\lambda \in \mathrm{k}$. Then for any $\lambda \in \mathrm{k}$ the polynomial $f^{(p)}(x, y^p) - \lambda$ is a line since $f - \lambda$ is a line and $\partial_x(f - \lambda) = \partial_x f \in \mathrm{k}^*$. Thus the construction in Corollary 4.2.9 will not lead us to counterexamples of Conjecture 4.1.3.

To conclude this section we mention two other conjectures related to lines in $\mathbb{A}^2$. The first one can be found in [Moh88] (respectively a slightly stronger version).

Conjecture 4.2.11. *Let $f \in \mathrm{k}[x,y]$ be a line. Then there exists an automorphism $\theta \in \mathrm{Aut}_{\mathrm{k}}(\mathrm{k}[x,y])$ such that $\theta(f) \in \mathrm{k}[x, y^p]$.*

In [Dai90] it is shown that Conjecture 4.2.11 implies Conjecture 4.1.3. Moreover, it is shown that Conjecture 4.2.11 implies that every line in $\mathbb{A}^2$ can be obtained from a coordinate line by iteratively applying automorphisms of $\mathbb{A}^2$ and the construction in Corollary 4.2.9.

The second conjecture is the following.

Conjecture 4.2.12. *Let $f \in \mathrm{k}[x,y]$ be a line. Then there exists some $n \in \mathbb{N}$ such that $\mathrm{k}(t)[x,y]/(f - t^{p^n}) \simeq \mathrm{k}(t)[x]$.*

This conjecture also implies Conjecture 4.1.3 and holds for Segre lines.

4.3 Lines of low degree

Lemma 4.3.1. *Every line of degree ≤ 5 is rectifiable.*

Proof. Let $C \subset \mathbb{A}^2$ be a line of degree ≤ 5. Then $\overline{C} \subset \mathbb{P}^2$ is a rational curve. Moreover, either $\overline{C}$ is a line, a conic or is unicuspidal and has one of the following multiplicity sequences at infinity: $(2), (3), (2_{(3)}), (4), (3, 2_{(3)})$, or $(2_{(6)})$. Using Lemma 2.4.16 we see that in all of these cases there exists an open embedding $\mathbb{P}^2 \setminus \overline{C} \hookrightarrow \mathbb{P}^2$ that does not extend to an automorphism of $\mathbb{P}^2$. In particular, $\overline{C}$ is Cremona-contractible. It then follows from Proposition 3.3.16 (in [BFH16]) that C is rectifiable. $\square$

We have seen in Example 4.2.10 that non-rectifiable lines of degree 6 do exist. Using Lemma 2.4.16 and Proposition 3.3.16, one can check that any non-rectifiable line of degree 6 has multiplicity sequence $(2_{(10)})$ at infinity and any non-rectifiable line of degree 9 has multiplicity sequence $(3_{(9)}, 2)$ at infinity. In fact, the following result from [Gan85, Theorem 2.4] shows that non-rectifiable lines of degree 6 or 9 are all equivalent to Segre lines.

Proposition 4.3.2. *Let $f \in \mathrm{k}[x,y]$ be a non-rectifiable line.*

(*i*) *If $\deg(f) = 6$, then $p = 2$ and f is equivalent to a Segre line of the form*

$$y^4 - x^6 - \lambda x$$

for some $\lambda \in \mathrm{k}^*$.

(*ii*) *If $\deg(f) = 9$, then $p = 3$ and f is equivalent to a Segre line of the form*

$$y^9 - x^6 - \mu x$$

for some $\mu \in \mathrm{k}^*$.

We will moreover use the following result from [Moh88, Corollary of Theorem 2].

Proposition 4.3.3. *Let $p = 2$ and let $f \in \mathrm{k}[x, y]$ be a line such that $\deg_x(f) = 2m$ and $\deg_y(f) = 2n$, where m and n are coprime. Then Conjecture 4.2.11 holds for f.*

Proposition 4.3.4. *Conjecture 4.2.11 holds for all lines of degree ≤ 11.*

Proof. Let $C \subset \mathbb{A}^2$ be a line and $\overline{C}$ its closure in $\mathbb{P}^2$. If $\deg(\overline{C}) \leq 5$, then C is rectifiable by Lemma 4.3.1 and thus Conjecture 4.2.11 holds in this case. If $\deg(\overline{C})$ is 6 or 9, then C is either rectifiable or equivalent to a Segre line by Proposition 4.3.2 and Conjecture 4.2.11 also holds. If $\deg(\overline{C})$ is 7 or 11, then C is rectifiable by Corollary 4.2.2 and thus Conjecture 4.2.11 also holds for those degrees.

The cases of degree 8 and 10 remain to be checked. Assume first that $\deg(C) = 8$. If $p \neq 2$, then C is rectifiable by Theorem 4.1.1. Thus we assume that $p = 2$ and that C is not rectifiable. Then the first multiplicity at infinity is even and is thus $2, 4$ or 6. If this multiplicity is 2 or 6 we can apply Proposition 4.3.3 and Conjecture 4.2.11 holds. We thus assume that the first multiplicity of C at infinity is 4. Using Lemma 2.4.16 and Proposition 3.3.16 and the fact that $\overline{C}$ is unicuspidal, we find that C must have one of the multiplicity sequences $(4, 2_{(15)})$ or $(4_{(2)}, 2_{(9)})$ at infinity.

Assume first that the multiplicity sequence is $(4, 2_{(15)})$. We denote by $p_1, \ldots, p_{16}$ the sequence of (proper and infinitely near) singular points of $\overline{C}$. Since L_∞ is very tangent to $\overline{C}$ it follows from Bézout's theorem that p_1, p_2, p_3, p_4 lie on L_∞ (respectively its strict transforms). On the other hand, $\overline{C}$ is unicuspidal and thus p_3 is proximate to p_1, i.e. lies on the strict transform of the exceptional curve of the blow-up of p_1, since the first multiplicity is the sum of the second and the third. We thus reach a contradiction since p_3 cannot both be proximate to p_1 and lie on the strict transform of L_∞.

We now assume that the multiplicity sequence of C at infinity is $(4_{(2)}, 2_{(9)})$. By Bézout's theorem the first 3 singular points in the sequence of singular points of $\overline{C}$ are not collinear. Thus there exists an affine quadratic map q with those 3 base-points. The map q is an automorphism of $\mathbb{P}^2 \setminus L_\infty$ and $\deg(q(\overline{C})) = 2 \cdot 8 - 4 - 4 - 2 = 6$ by Lemma 2.3.11. It follows that C is equivalent to a Segre line by Proposition 4.3.2 and hence Conjecture 4.2.11 holds in this case.

Assume now that $\deg(\overline{C}) = 10$. If p is different from 2 and 5, then C is rectifiable by Theorem 4.1.1. If $p = 2$ and C is not rectifiable, then the first multiplicity at infinity of C is $2, 4, 6$ or 8. In all of these cases we can apply Proposition 4.3.3 and Conjecture 4.2.11 holds. If $p = 5$ and C is not rectifiable, then the first multiplicity at infinity of C must be 5. Using the fact that $\overline{C}$ is unicuspidal, one checks that C must have multiplicity sequence $(5_{(3)}, 4)$ at infinity. But then $\overline{C}$ is Cremona-contractible by Lemma 2.4.16 and hence C is rectifiable by Proposition 3.3.16. $\square$

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