AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND A CHARACTERIZATION OF AFFINE n-SPACE

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Dedicated to Ernest Vinberg
at the occasion of his 80th birthday.

ABSTRACT. We show that the automorphism group of affine n-space $\mathbb{A}^n$ determines $\mathbb{A}^n$ up to isomorphism: If X is a connected affine variety such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.

We also show that every torus appears as $\mathrm{Aut}(X)$ for a suitable irreducible affine variety X, but that $\mathrm{Aut}(X)$ cannot be isomorphic to a semisimple group. In fact, if $\mathrm{Aut}(X)$ is finite dimensional and if $X \not\simeq \mathbb{A}^1$, then the connected component $\mathrm{Aut}(X)^\circ$ is a torus.

Concerning the structure of $\mathrm{Aut}(\mathbb{A}^n)$ we prove that any homomorphism $\mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups either factors through jac: $\mathrm{Aut}(\mathbb{A}^n) \to \mathbb{k}^*$ where jac is the Jacobian determinant, or it is a closed immersion. For $\mathrm{SAut}(\mathbb{A}^n) := \ker(\mathrm{jac}) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ we show that every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ is a closed immersion.

Finally, we prove that every non-trivial homomorphism $\varphi \colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ is an automorphism, and that φ is given by conjugation with an element from $\mathrm{Aut}(\mathbb{A}^n)$.

1. INTRODUCTION AND MAIN RESULTS

Our base field $\mathbb{k}$ is algebraically closed of characteristic zero. For an affine variety X the automorphism group $\mathrm{Aut}(X)$ has the structure of an *affine ind-group*. We will shortly recall the basic definitions in the following section 2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, the group of automorphisms of affine n-space $\mathbb{A}^n = \mathbb{k}^n$.

A fundamental question is how much information about X can be retrieved from $\mathrm{Aut}(X)$. E.g. JELONEK shows in [Jel15] that if $\mathrm{Aut}(X)$ is infinite, then X is *uniruled*. Our main result shows that $\mathbb{A}^n$ is completely determined by its automorphism group.

Theorem 1.1. *Let X be a connected affine variety. If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.*

It is clear that X has to be connected since the automorphism group does not change if we form the disjoint union of $\mathbb{A}^n$ with a variety Y with trivial automorphism group. Some generalization of this result can be found in [Reg17].

The proof of the theorem will follow from a more general result (Theorem 5.5, see Remark 5.4) where the group $\mathrm{Aut}(\mathbb{A}^n)$ is replaced by the subgroup $\mathcal{U}(\mathbb{A}^n)$ generated by the unipotent elements.

Another important question is which groups appear as automorphism groups of affine varieties. For finite groups we have the following result due to JELONEK.

Theorem 1.2 ([Jel15], Proposition 7.2)**.** *For every finite group G and every $n \geq 1$ there is a n-dimensional smooth connected affine variety X such that $\mathrm{Aut}(X) \simeq G$.*

Moreover, there exist surfaces with infinite discrete automorphism groups (see [FK17, Proposition 12.7.1]). As for algebraic groups, we have $\mathrm{Aut}(\mathbb{A}^1) = \mathrm{Aff}_1$, and we will give examples where $\mathrm{Aut}(X)$ is a torus (Example 7.4). But other groups cannot appear as the next result shows.

Theorem 1.3. *Let X be a connected affine variety. If $\dim \mathrm{Aut}(X) < \infty$, then either $X \simeq \mathbb{A}^1$ or the connected component $\mathrm{Aut}(X)^\circ$ is a torus.*

The last results concern the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of affine n-space. This group has a closed normal subgroup $\mathrm{SAut}(\mathbb{A}^n)$ consisting of those automorphism $\mathbf{f} = (f_1, \ldots, f_n)$ whose Jacobian determinant $\mathrm{jac}(\mathbf{f}) := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{(i,j)}$ is equal to 1:

$$\mathrm{SAut}(\mathbb{A}^n) := \ker(\mathrm{jac} \colon \mathrm{Aut}(\mathbb{A}^n) \to \Bbbk^*).$$

For an ind-group $\mathcal{G}$ the tangent space $T_e\mathcal{G}$ carries a canonical structure of a Lie-algebra which we denote by $\mathrm{Lie}\,\mathcal{G}$. For $\mathrm{SAut}(\mathbb{A}^n)$, the Lie algebra can be identified with $\mathrm{Vec}^0(\mathbb{A}^n)$, the vector fields ξ on $\mathbb{A}^n$ with divergence $\mathrm{div}\,\xi = 0$. This Lie algebra is simple, so one could expect that $\mathrm{SAut}(\mathbb{A}^n)$ is simple as an ind-group. This is claimed in [Sha66, Sha81], but the proofs turned out to be not correct (see [FK17, section 15]). What we can show here is the following.

Theorem 1.4. *Let $n \geq 2$.*

    (1) *Let $\varphi \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. Then either φ factors through $\mathrm{jac} \colon \mathrm{Aut}(\mathbb{A}^n) \to \Bbbk^*$, or φ is a closed immersion, i.e. the image is closed and isomorphic to $\mathrm{Aut}(\mathbb{A}^n)$ under φ.*

    (2) *Every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups is a closed immersion.*

This theorem has the following interesting applications. By definition, a *representation of an ind-group $\mathcal{G}$ on a vector space V of countable dimension* is a homomorphism $\mathcal{G} \to \mathrm{GL}(V)$ such that the corresponding map $\mathcal{G} \times V \to V$ is a morphism of ind-varieties (see section 3). An *action of an ind-group $\mathcal{G}$ on an affine variety X* is a homomorphism $\mathcal{G} \to \mathrm{Aut}(X)$ of ind-groups.

Corollary 1.5. *Assume that $n \geq 2$.*

    (1) *The ind-group $\mathrm{SAut}(\mathbb{A}^n)$ does not have a non-trivial finite dimensional representation.*

    (2) *Assume that $\mathrm{SAut}(\mathbb{A}^n)$ acts nontrivially on a connected affine variety X. Then the action is faithful, and there are no fixed points.*

Proof. (1) Let $\rho \colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{GL}(V)$ be a finite dimensional representation. If ρ is nontrivial, then it is a closed immersion, by Theorem 1.4(2). This is impossible, because $\mathrm{GL}(V)$ is finite dimensional.

(2) We have a nontrivial homomorphism $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{Aut}(X)$ which is a closed immersion, by Theorem 1.4(2). Thus the action is faithful, and the same is true for the induced action of $\mathrm{SL}_n \subseteq \mathrm{SAut}(\mathbb{A}^n)$. Since X is connected, it follows that SL_n acts nontrivially on every irreducible component of X. This implies that for every fixed point $x \in X^{\mathrm{SL}_n}$ the tangent representation of SL_n on $T_x X$ is non-trivial. Hence, the tangent representation of $\mathrm{SAut}(\mathbb{A}^n)$ on every fixed point of $\mathrm{SAut}(\mathbb{A}^n)$ is also nontrivial, contradicting (1). $\qquad\square$

It is shown in [BKY12] that every automorphism of the ind-group $\mathrm{Aut}(\mathbb{A}^n)$ is inner, i.e. given by conjugation with a suitable $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$ (cf. [FK17, Theorem 12.5.2]). This can be generalized in the following way.

Theorem 1.6.
  (1) *Every injective homomorphism* $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}(\mathbb{A}^n)$ *is an isomorphism, and* $\varphi = \mathrm{Int}\,\mathbf{g}$ *for a well-defined* $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$.
  (2) *Every nontrivial homomorphism* $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ *is an isomorphism, and* $\varphi = \mathrm{Int}\,\mathbf{g}$ *for a well-defined* $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$.

Remark 1.7. The analogue of Theorem 1.6 for vector fields, namely that *every injective homomorphism* $\varphi\colon \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ *of Lie algebras is an automorphism*, would imply the Jacobian Conjecture in dimension n, see [KR17, Corollary 4.4].

We finally mention the following example showing that bijective homomorphisms of ind-groups are not necessarily isomorphisms. The details can be found in [FK17, section 8], cf. [BW00, section 11, last paragraph]. Denote by $\Bbbk\langle x, y\rangle$ the free associative $\Bbbk$-algebra in two generators. Then $\mathrm{Aut}(\Bbbk\langle x, y\rangle)$ is an ind-group, and we have a canonical homomorphism $\pi\colon \mathrm{Aut}(\Bbbk\langle x, y\rangle) \to \mathrm{Aut}(\Bbbk[x, y])$.

Proposition 1.8. *The map* $\pi\colon \mathrm{Aut}(\Bbbk\langle x, y\rangle) \to \mathrm{Aut}(\Bbbk[x, y])$ *is a bijective homomorphism of ind-groups, but it is not an isomorphism, because it is not an isomorphism on the Lie algebras.*

Note that $\mathrm{Aut}(\Bbbk\langle x, y\rangle)$ is generated by the closed algebraic subgroups $G \subseteq \mathrm{Aut}(\Bbbk\langle x, y\rangle)$, and that $\pi\colon G \xrightarrow{\sim} \pi(G)$ is an isomorphism for these subgroups.

2. Notation and preliminary results

The notion of an ind-group goes back to Shafarevich who called these objects *infinite dimensional groups*, see [Sha66, Sha81]. We refer to [Kum02] and the notes [FK17] for basic notation in this context.

Definition 2.1. An *ind-variety* $\mathcal{V}$ is a set together with an ascending filtration $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}$ such that the following holds:
  (1) $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$;
  (2) Each $\mathcal{V}_k$ has the structure of an algebraic variety;
  (3) For all $k \in \mathbb{N}$ the inclusion $\mathcal{V}_k \hookrightarrow \mathcal{V}_{k+1}$ is closed immersion.

A *morphism* between ind-varieties $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_m \mathcal{W}_m$ is a map $\varphi\colon \mathcal{V} \to \mathcal{W}$ such that, for every k, there is an m with the properties that $\varphi(\mathcal{V}_k) \subseteq \mathcal{W}_m$ and that the induced map $\mathcal{V}_k \to \mathcal{W}_m$ is a morphism of varieties. *Isomorphisms* of ind-varieties are defined in the usual way.

Two filtrations $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$ and $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}'_k$ are called *equivalent* if, for any k, there is an m such that $\mathcal{V}_k \subseteq \mathcal{V}'_m$ is a closed subvariety as well as $\mathcal{V}'_k \subseteq \mathcal{V}_m$.

Equivalently, the identity map $\mathrm{id}\colon \mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k \to \mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}'_k$ is an isomorphism of ind-varieties.

An ind-variety $\mathcal{V}$ has a natural *topology* where $S \subseteq \mathcal{V}$ is open, resp. closed, if $S_k := S \cap \mathcal{V}_k \subseteq \mathcal{V}_k$ is open, resp. closed, for all k. Obviously, a locally closed subset $S \subseteq \mathcal{V}$ has a natural structure of an ind-variety. It is called an *ind-subvariety*. An ind-variety $\mathcal{V}$ is called *affine* if all $\mathcal{V}_k$ are affine. A subset $X \subseteq \mathcal{V}$ is called *algebraic* if it is locally closed and contained in some $\mathcal{V}_k$. Such an X has a natural structure of an algebraic variety.

Example 2.2.   (1) Any $\Bbbk$-vector space V of countable dimension carries the structure of an (affine) ind-variety by choosing an increasing sequence of finite dimensional subspaces V_k such that $V = \bigcup_k V_k$. Clearly, all these filtrations are equivalent.

(2) If R is a commutative $\Bbbk$-algebra of countable dimension, $\mathfrak{a} \subseteq R$ a subspace, e.g. an ideal, and $S \subseteq \Bbbk[x_1, \ldots, x_n]$ a set of polynomials, then the subset

$$\{(a_1, \ldots, a_n) \in R^n \mid f(a_1, \ldots, a_n) \in \mathfrak{a} \text{ for all } f \in S\} \subseteq R^n$$

is a closed ind-subvariety of R^n.

For any ind-variety $\mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k$ we can define the *tangent space* in $x \in \mathcal{V}$ in the obvious way. We have $x \in \mathcal{V}_k$ for $k \geq k_0$, and $T_x\mathcal{V}_k \subseteq T_x\mathcal{V}_{k+1}$ for $k \geq k_0$, and then define

$$T_x\mathcal{V} := \varinjlim_{k \geq k_0} T_x\mathcal{V}_k$$

which is a vector space of countable dimension. A morphism $\varphi\colon \mathcal{V} \to \mathcal{W}$ induces linear maps $d\varphi_x\colon T_x\mathcal{V} \to T_{\varphi(x)}\mathcal{W}$ for every $x \in X$. Clearly, for a $\Bbbk$-vector space V of countable dimension and for any $v \in V$ we have $T_vV = V$ in a canonical way.

The *product* of two ind-varieties is defined in the obvious way. This allows to define an *ind-group* as an ind-variety $\mathcal{G}$ with a group structure such that multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}\colon (g, h) \mapsto g \cdot h$, and inverse $\mathcal{G} \to \mathcal{G}\colon g \mapsto g^{-1}$, are both morphisms.

Remark 2.3. Let $G \subseteq \mathcal{G}$ be a subgroup. If G is an algebraic subset, i.e. locally closed and contained in $\mathcal{G}_k$ for some k, then G is an algebraic group and is closed in $\mathcal{G}$. We will call such a G an *algebraic subgroup*.

Conversely, if G is an algebraic group and $\varphi\colon G \to \mathcal{G}$ a homomorphism of ind-groups, then $\varphi(G) \subseteq \mathcal{G}$ is a closed subgroup and an algebraic subset. The easy proofs are left to the reader.

If $\mathcal{G}$ is an affine ind-group, then $T_e\mathcal{G}$ has a natural structure of a Lie algebra which will be denoted by $\operatorname{Lie}\mathcal{G}$. The structure is obtained by showing that every $A \in T_e\mathcal{G}$ defines a unique left-invariant vector field δ_A on $\mathcal{G}$, see [Kum02, Proposition 4.2.2, p. 114].

Definition 2.4. An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all k. Clearly, $\mathcal{G}$ is discrete if and only if $\operatorname{Lie}\mathcal{G}$ is trivial.

The next result can be found in [FK17, sections 4.1 and 4.6]. Here $\operatorname{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on X, i.e. $\operatorname{Vec}(X) = \operatorname{Der}(\mathcal{O}(X))$, the Lie algebra of derivations of $\mathcal{O}(X)$.

Proposition 2.5. *Let X be an affine variety. Then $\operatorname{Aut}(X)$ has a natural structure of an affine ind-group, and there is a canonical embedding $\xi\colon \operatorname{Lie}\operatorname{Aut}(X) \hookrightarrow \operatorname{Vec}(X)$ of Lie algebras.*

Remark 2.6. For $X = \mathbb{A}^n$ the embedding ξ identifies $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$ with $\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)$, the vector fields $\delta = \sum_i f_i \frac{\partial}{\partial x_i}$ with constant divergence: $\operatorname{div}\delta := \sum_i \frac{\partial f_i}{\partial x_i} \in \mathbb{k}$, see [FK17, Proposition 4.9.1].

The Jacobian determinant $\operatorname{jac}(\mathbf{f}) := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{(i,j)}$ of an automorphism $\mathbf{f} = (f_1, \ldots, f_n)$ of $\mathbb{A}^n$ defines a homomorphism $\operatorname{jac}\colon \operatorname{Aut}(\mathbb{A}^n) \to \mathbb{k}^*$ of ind-groups. Setting $\operatorname{SAut}(\mathbb{A}^n) := \ker\operatorname{jac}$ one sees that ξ identifies $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n)$ with $\operatorname{Vec}^0(\mathbb{A}^n)$, the vector fields δ with $\operatorname{div}\delta = 0$, see [FK17, Remark 4.9.3].

It is known that for $n \geq 2$ the Lie algebra $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n)$ is simple and that $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n) \subseteq \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ is the only proper ideal, see [Sha81, Lemma 3]. Moreover, both Lie algebras are generated by the subalgebras $\operatorname{Lie}G$ where G is an algebraic subgroup.

Another result which we will need is proved in [FK17, Proposition 2.7.6].

Proposition 2.7. *Let* $\varphi, \psi\colon \mathcal{G} \to \mathcal{H}$ *be two homomorphisms of ind-groups. Assume that* $\mathcal{G}$ *is connected and that* $d\varphi_e = d\psi_e\colon \operatorname{Lie}\mathcal{G} \to \operatorname{Lie}\mathcal{H}$. *Then* $\varphi = \psi$.

A final result which we will use can be found in [KRZ17]. Denote by $\operatorname{Aff}_n \subseteq \operatorname{Aut}(\mathbb{A}^n)$ *the subgroup of affine transformations*, i.e. $\operatorname{Aff}_n = \operatorname{GL}_n(\mathbb{k}) \ltimes (\mathbb{k}^n)^+$. Similarly, the subgroup $\operatorname{SAff}_n \subseteq \operatorname{Aff}_n$ consists of the affine transformations with determinant 1, i.e. $\operatorname{SAff}_n = \operatorname{SL}_n(\mathbb{k}) \ltimes (\mathbb{k}^n)^+$.

Proposition 2.8. *Let* X *be a connected affine variety with a faithful action of* SAff_n. *If* $\dim X \leq n$, *then* X *is* SAff_n-*isomorphic to* $\mathbb{A}^n$.

Remark 2.9. It is shown in [KRZ17] that the same holds if we replace SAff_n by Aff_n. Using Theorem 1.6 we see that we can replace SAff_n by $\operatorname{Aut}(\mathbb{A}^n)$ or $\operatorname{SAut}(\mathbb{A}^n)$ as well.

3. THE ADJOINT REPRESENTATION

Following [Kum02, section 4.2] we define a *representation of an ind-group* $\mathcal{G}$ on a vector space V of countable dimension to be a homomorphism $\rho\colon \mathcal{G} \to \operatorname{GL}(V)$ of groups such that the induced map $\mathcal{G} \times V \to V$ is a morphism of ind-varieties. Note that $\operatorname{GL}(V)$ does not have the structure of an ind-variety if $\dim V = \infty$. However, if L is a finitely generated Lie algebra, then $\operatorname{Aut}_{\mathrm{Lie}}(L)$ has a natural structure of an ind-group which is defined in the following way (see [FK17, section 7] where we define an ind-group structure on $\operatorname{Aut}(R)$ for any finitely generated general algebra R, i.e. a $\mathbb{k}$-vector space R endowed with a bilinear map $R \times R \to R$).

Choose a finite-dimensional subspace $L_0 \subseteq L$ which generates L as a Lie algebra. Then the restriction map $\operatorname{End}_{\mathrm{Lie}}(L) \to \operatorname{Hom}(L_0, L)$ is injective and the image is a closed affine ind-subvariety. (To see this write L as the quotient of the free Lie algebra $F(L_0)$ over L_0 modulo an ideal I.) Choosing a filtration $L = \bigcup_{k \geq 0} L_k$ by finite-dimensional subspaces, we set $\operatorname{End}_{\mathrm{Lie}}(L)_k := \{\alpha \in \operatorname{End}_{\mathrm{Lie}}(L) \mid \alpha(L_0) \subseteq L_k\}$ which is a closed subvariety of $\operatorname{Hom}(L_0, L_k)$ (see Example 2.2). Then we define the ind-structure on $\operatorname{Aut}_{\mathrm{Lie}}(L)$ by identifying $\operatorname{Aut}_{\mathrm{Lie}}(L)$ with the closed subset

$$\{(\alpha, \beta) \in \operatorname{End}_{\mathrm{Lie}}(L) \times \operatorname{End}_{\mathrm{Lie}}(L) \mid \alpha \circ \beta = \beta \circ \alpha = \operatorname{id}_L\} \subseteq \operatorname{End}_{\mathrm{Lie}}(L) \times \operatorname{End}_{\mathrm{Lie}}(L),$$

i.e.

$$\operatorname{Aut}_{\mathrm{Lie}}(L)_k := \{\alpha \in \operatorname{Aut}_{\mathrm{Lie}}(L) \mid \alpha, \alpha^{-1} \in \operatorname{End}_{\mathrm{Lie}}(L)_k\}.$$

It follows that $\mathrm{Aut}_{\mathrm{Lie}}(L)$ is an affine ind-group with the usual functorial properties. In particular, we have the following result.

Lemma 3.1. *Let $\mathcal{G}$ be an ind-group, and let $\rho\colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(L)$ be an abstract homomorphism where L is a finitely generated Lie algebra. Then ρ is a homomorphism of ind-groups if and only if ρ is a representation, i.e. the map $\rho\colon \mathcal{G} \times L \to L$ is a morphism of ind-varieties.*

Proof. Assume that L is generated by the finite dimensional subspace $L_0 \subseteq L$. If $\mathcal{G} = \bigcup_j \mathcal{G}_j$ and if $\rho\colon \mathcal{G} \times L \to L$ is a morphism, then, for any j, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$ and $\rho(\mathcal{G}_j^{-1} \times L_0) \subseteq L_k$. Hence, $\rho(\mathcal{G}_j) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(L)_k$, and the map $\mathcal{G}_j \to \mathrm{Hom}(L_0, L_k)$ is clearly a morphism.

Now assume that $\mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(L)$ is a homomorphism of ind-groups. Then, for any j, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(L)_k \hookrightarrow \mathrm{Hom}(L_0, L_k)$. Hence, $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$, and $\mathcal{G}_j \times L_0 \to L_k$ is a morphism. $\qquad\square$

The *adjoint representation* $\mathrm{Ad}\colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G})$ of an ind-group $\mathcal{G}$ is defined in the usual way: $\mathrm{Ad}\,g := (d\,\mathrm{Int}\,g)_e\colon \mathrm{Lie}\,\mathcal{G} \xrightarrow{\sim} \mathrm{Lie}\,\mathcal{G}$ where $\mathrm{Int}\,g$ is the inner automorphism $h \mapsto ghg^{-1}$.

Proposition 3.2. *For any ind-group $\mathcal{G}$ the canonical map $\mathrm{Ad}\colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G})$ is a homomorphism of ind-groups.*

Proof. Let $\gamma\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the morphism $(g, h) \mapsto ghg^{-1}$. For any $g \in \mathcal{G}$, the map $\gamma_g\colon \mathcal{G} \to \mathcal{G}$, $h \mapsto ghg^{-1}$, is an isomorphism of ind-groups, and its differential $\mathrm{Ad}(g) = (d\gamma_g)_e\colon \mathrm{Lie}\,\mathcal{G} \to \mathrm{Lie}\,\mathcal{G}$ is an isomorphism of Lie algebras. If $\mathcal{G} = \bigcup_k \mathcal{G}_k$, then for any $p, q \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $\gamma\colon \mathcal{G}_p \times \mathcal{G}_p \to \mathcal{G}_m$. Clearly, for $g \in \mathcal{G}_p$, $\mathrm{Ad}\,g$ is given by $(d\gamma_g)_e\colon T_e\mathcal{G}_q \to T_e\mathcal{G}_m$, and the map $\mathcal{G}_k \to \mathrm{Hom}(T_e\mathcal{G}_q, T_e\mathcal{G}_m)$ is a morphism, by the following lemma. Now the claim follows from Lemma 3.1 above. $\qquad\square$

Lemma 3.3. *Let $\Phi\colon X \times Y \to Z$ be a morphism of affine varieties and set $\Phi_x(y) := \Phi(x, y)$. Assume that there exist $y_0 \in Y$ and $z_0 \in Z$ such that $\Phi_x(y_0) = z_0$ for all $x \in X$. Then the induced map $X \to \mathrm{Hom}(T_{y_0}Y, T_{z_0}Z)$, $x \mapsto d_{y_0}\Phi_x$, is a morphism.*

Proof. We can assume that Y, Z are vector spaces, $Y = W$ and $Z = V$. Choose bases $(w_1, \ldots, w_m)$ of W and $(v_1, \ldots, v_n)$ of V. Then Φ is given by an element of the form

$$\sum_{i=1}^{n} \sum_j f_{ij} \otimes h_{ij} \otimes v_i, \text{ where } f_{ij} \in \mathcal{O}(X) \text{ and } h_{ij} \in \mathcal{O}(Y) = \Bbbk[y_1, \ldots, y_m],$$

and so the differential $(d\Phi_x)_{y_0}\colon W \to V$ is given by the matrix

$$\left(\sum_j f_{ij}(x) \frac{\partial h_{ij}}{\partial y_k}(y_0) \right)_{(i,k)}$$

whose entries are regular functions on x. The claim follows. $\qquad\square$

We have shown in [KR17] that the adjoint representation

$$\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$$

and the induced homomorphism $\rho\colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ are both bijective. They are also homomorphisms of ind-groups: For $\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ this

is Proposition 3.2 above, and for ρ it is obvious. But this does not necessarily imply that the maps are isomorphisms of ind-groups, see Proposition 1.8. However, for $\mathrm{Aut}(\mathbb{A}^n)$ it is true, and we will need this for the proof of Theorem 1.4 in the following section.

Proposition 3.4. *The adjoint representation*

$$\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)} \colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$$

is an isomorphism of ind-groups.

Proof. We will use here the identification of $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ with $\mathrm{Vec}^c(\mathbb{A}^n)$, see Remark 2.6. Put $\partial_{x_i} := \frac{\partial}{\partial x_i}$.

Let $\mathbf{f} = (f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{A}^n)$ and set $\theta := \mathrm{Ad}(\mathbf{f}^{-1}) \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n))$. Then the matrix $\left(\theta(\partial_{x_k})x_j\right)_{(j,k)}$ is invertible, and

$$(*) \qquad \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1} = \mathrm{Jac}(\mathbf{f}) = \left(\frac{\partial f_j}{\partial x_i}\right)_{(i,j)},$$

see [KR17, Remark 4.2]. We now claim that the map

$$\theta \mapsto \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1} \colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \to \mathrm{M}_n(\Bbbk[x_1, \ldots, x_n])$$

is a well-defined morphism of ind-varieties. In fact, $\theta \mapsto \theta(\partial_{x_k})x_j$ is the composition of the orbit map $\theta \mapsto \theta(\partial_{x_k}) \colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \to \mathrm{Vec}^c(\mathbb{A}^n)$ and the evaluation map $\delta \mapsto \delta(x_j) \colon \mathrm{Vec}^c(\mathbb{A}^n) \to \Bbbk[x_1, \ldots, x_n]$, hence $\theta \mapsto \Theta := \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}$ is a morphism. Since $\mathrm{jac}(\Theta) \in \Bbbk^*$ the claim follows.

Now recall that the gradient $\Bbbk[x_1, \ldots, x_n] \to \Bbbk[x_1, \ldots, x_n]^n$, $f \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, defines an isomorphism

$$\gamma \colon \Bbbk[x_1, \ldots, x_n]_{\geq 1} \xrightarrow{\sim} \Gamma := \{(h_1, \ldots, h_n) \mid \frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i} \text{ for all } i < j\}.$$

It follows from $(*)$ that the rows of the matrix $(h_{ij})_{(i,j)} := \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1}$ belong to Γ, so that we get a morphism

$$\psi \colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \to \Bbbk[x_1, \ldots, x_n]^n, \ \theta \mapsto (f_1, \ldots, f_n),$$

where $f_i := \gamma^{-1}(h_{i1}, \ldots, h_{in}) \in \Bbbk[x_1, \ldots, x_n]_{\geq 1}$. By construction, we have

$$(**) \qquad \psi(\theta) = \psi(\mathrm{Ad}(\mathbf{f}^{-1})) = \mathbf{f}_0 := (f_1 - f_1(0), \ldots, f_n - f_n(0)) = \mathbf{t}_{-\mathbf{f}(0)} \circ \mathbf{f}$$

where $\mathbf{t}_a$ is the translation $v \mapsto v + a$. Let $S \subseteq \mathrm{Aff}_n$ be the subgroup of translations, and set $\tilde{S} := \mathrm{Ad}(S)$. Then $\tilde{S} \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n))$ is a closed algebraic subgroup and $\mathrm{Ad} \colon S \to \tilde{S}$ is an isomorphism. It follows from $(**)$ that $\mathrm{Ad}(\psi(\theta)) \cdot \theta = \mathrm{Ad}(\mathbf{t}_{-\mathbf{f}(0)}) \in \tilde{S}$, and so

$$\tilde{\psi}(\theta) := \psi(\theta)^{-1} \cdot (\mathrm{Ad}|_S)^{-1}(\mathrm{Ad}(\psi(\theta)) \cdot \theta)$$

is a well-defined morphism $\tilde{\psi} \colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \to \mathrm{Aut}(\mathbb{A}^n)$ with the property that

$$\mathrm{Ad}(\tilde{\psi}(\theta)) = \mathrm{Ad}(\psi(\theta)^{-1}) \cdot \mathrm{Ad}(\psi(\theta)) \cdot \theta = \theta.$$

Thus $\mathrm{Ad} \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$ is an isomorphism, with inverse $\tilde{\psi}$. $\square$

Remark 3.5. Clearly, the restriction $\rho\colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a homomorphism of ind-groups, and it is bijective, see [KR17]. It follows from (1) that the composition $\rho \circ \mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups. Now we use Theorem 1.4(1) to conclude that $\rho \circ \mathrm{Ad}$ is an isomorphism, hence ρ is an isomorphism, too.

4. Proof of the Theorems 1.4 and 1.6

Proof of Theorem 1.4. (1) Let $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups such that $d\varphi$ is injective. We can assume that $\mathcal{G} = \overline{\varphi(\mathrm{Aut}(\mathbb{A}^n))}$, and we will show that φ is an isomorphism. The basic idea is to construct a homomorphism $\psi\colon \mathcal{G} \to \mathrm{Aut}(\mathbb{A}^n)$ such that $\psi \circ \varphi = \mathrm{id}$. By Proposition 4.1 below this implies that φ is a closed immersion, hence an isomorphism.

Denote by $L \subseteq \mathrm{Lie}\,\mathcal{G}$ the image of $d\varphi$. For any $g \in \mathrm{Aut}(\mathbb{A}^n)$ we have $d\varphi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\varphi(g)) \circ d\varphi$. In particular, L is stable under $\mathrm{Ad}(\varphi(g))$, hence stable under $\mathrm{Ad}(\mathcal{G})$, because $\varphi(\mathrm{Aut}(\mathbb{A}^n))$ is dense in $\mathcal{G}$. Thus we get the following commutative diagram of homomorphisms of ind-groups

$$
\begin{array}{ccc}
\mathrm{Aut}(\mathbb{A}^n) & \xrightarrow{\ \varphi\ } & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))) & \xrightarrow{\ \simeq\ } & \mathrm{Aut}_{\mathrm{Lie}}(L)
\end{array}
$$

where the first vertical map is an isomorphism, by Proposition 3.4. Thus, the composition $\mathrm{Ad}_{\mathcal{G}} \circ \varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(L) \simeq \mathrm{Aut}(\mathbb{A}^n)$ is an isomorphism, and so φ is also an isomorphism, by Proposition 4.1 below.

If $d\varphi$ is not injective, then $\ker d\varphi \supseteq \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ (Remark 2.6) and so $d\varphi = f \circ d\,\mathrm{jac}$ where $f\colon \mathbb{k} \to \mathrm{Lie}\,\mathcal{G}$ is a Lie algebra homomorphism. If $\mathbb{k}^* \subseteq \mathrm{GL}_n(\mathbb{k})$ denotes the center, then $\varphi|_{\mathbb{k}^*}\colon \mathbb{k}^* \to \mathcal{G}$ factor through $?^n\colon \mathbb{k}^* \to \mathbb{k}^*$, because $\mathrm{SL}_n(\mathbb{k}) \subseteq \ker\varphi$, i.e. $\varphi(z) = \rho(z^n)$ for any $z \in \mathbb{k}^*$ and a suitable homomorphism $\rho\colon \mathbb{k}^* \to \mathcal{G}$ of ind-groups. By construction, $d\rho_e = f\colon \mathbb{k} \to \mathrm{Lie}\,\mathcal{G}$, and so the two homomorphisms φ and $\rho \circ \mathrm{jac}$ have the same differential. Thus, by Proposition 2.7, we get $\varphi = \rho \circ \mathrm{jac}$, and we are done.

(2) Let $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. If $d\varphi_e$ is not injective, then $d\varphi_e$ is the trivial map (Remark 2.6), hence $d\varphi_e = d\bar\varphi_e$ where $\bar\varphi\colon \mathbf{g} \mapsto e$ is the constant homomorphism. Again by Proposition 2.7 we get $\varphi = \bar\varphi$.

If $d\varphi_e$ is injective, set $L := d\varphi_e(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathrm{Lie}\,\mathcal{G}$. As above we can assume that $\mathcal{G} = \overline{\varphi(\mathrm{SAut}(\mathbb{A}^n))}$. Since L is stable under $\mathrm{Ad}\,\varphi(\mathbf{g})$ for all $\mathbf{g} \in \mathrm{SAut}(\mathbb{A}^n)$ it is also stable under $\mathcal{G}$, and we get, as above, the following commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Aut}(\mathbb{A}^n) & \xleftarrow{\ \supseteq\ } & \mathrm{SAut}(\mathbb{A}^n) & \xrightarrow{\ \varphi\ } & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{Ad}_{\mathrm{SAut}(\mathbb{A}^n)}}\big\downarrow{\scriptstyle \subseteq} & & \big\downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))) & \underset{\mathrm{bijective}}{\xrightarrow{\ \rho\ }} & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))) & \underset{\simeq}{\xrightarrow{\ \Phi\ }} & \mathrm{Aut}_{\mathrm{Lie}}(L)
\end{array}
$$

where $\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ is an isomorphism, by (1). Since ρ is bijective ([KR17]) the composition $\rho \circ \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ is an isomorphism, again by (1). Therefore, the image $\mathfrak{A} := \mathrm{Ad}(\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a closed subgroup isomorphic to $\mathrm{SAut}(\mathbb{A}^n)$, and $\mathfrak{A} \xrightarrow{\sim} \Phi(\mathfrak{A}) = \mathrm{Ad}_{\mathcal{G}}(\varphi(\mathrm{SAut}(\mathbb{A}^n)))$. But $\varphi(\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathcal{G}$ is dense, and so

$\mathrm{Ad}_{\mathcal{G}}(\mathcal{G}) = \Phi(\mathfrak{A})$. Thus, the composition $\mathrm{Ad}_{\mathcal{G}} \circ \varphi \colon \mathrm{SAut}(\mathbb{A}^n) \to \Phi(\mathfrak{A})$ is an isomorphism, hence φ is an isomorphism, by Proposition 4.1 below. $\qquad\square$

Proposition 4.1. *Let $\mathcal{H}, \mathcal{G}$ be two ind-groups, and let $\varphi \colon \mathcal{H} \to \mathcal{G}$, $\psi \colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms. If $\psi \circ \varphi = \mathrm{id}_{\mathcal{H}}$, then φ is a closed immersion, i.e. $\varphi(\mathcal{H}) \subseteq \mathcal{G}$ is a closed subgroup and φ induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \varphi(\mathcal{H})$.*

Proof. By base change we can assume that the base field $\Bbbk$ is uncountable. Let $\mathcal{H} = \bigcup_i \mathcal{H}_i$ and $\mathcal{G} = \bigcup_j \mathcal{G}_j$ where we can assume that $\mathcal{H}_i \subseteq \mathcal{G}_i$ for all i. Moreover, for every i there is a $k = k(i)$ such that $\psi(\mathcal{G}_i) \subseteq \mathcal{H}_k$. By assumption, the composition $\psi \circ \varphi \colon \mathcal{H}_i \to \mathcal{G}_i \to \mathcal{H}_k$ is the closed embedding $\mathcal{H}_i \hookrightarrow \mathcal{H}_k$, hence the first map is a closed embedding. Thus $H_i := \varphi(\mathcal{H}_i)$ is a closed subset of $\mathcal{G}_i$ and $H := \varphi(\mathcal{H}) = \bigcup_i H_i$. Now the claim follows from Lemma 4.2 below by setting $S := \ker \psi$. $\qquad\square$

Recall that a subset $S \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V}$ is called *ind-constructible* if $S = \bigcup_i S_i$ where $S_i \subseteq S_{i+1}$ are constructible subsets of $\mathfrak{V}$.

Lemma 4.2. *Let $\mathcal{G}$ be an ind-group, $H \subseteq \mathcal{G}$ a subgroup and $S \subseteq \mathcal{G}$ an ind-constructible subset. Assume that $\Bbbk$ is uncountable and that*

(1) *$H = \bigcup_i H_i$ where $H_i \subseteq H_{i+1} \subseteq \mathcal{G}$ are closed algebraic subsets,*
(2) *the multiplication map $S \times H \to \mathcal{G}$ is bijective.*

Then H is a closed subgroup of $\mathcal{G}$.

Proof. Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$. We have to show that for every k there exists an $i = i(k)$ such that $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$. We can assume that $e \in S = \bigcup_i S_i$. Then, by assumption, $\mathcal{G} = \bigcup_j S_j H_j$. Since $S_j H_j \cap \mathcal{G}_k$ is a constructible subset of $\mathcal{G}_k$ it follows that there exists a $j = j(k)$ such that $\mathcal{G}_k \subseteq S_j H_j$ ([FK17, Lemma 1.6.4]). Setting $\dot{S} := S \setminus \{e\}$ we get $\dot{S}H \cap H = \emptyset$. Thus, $\mathcal{G}_k = (\dot{S}_i H_i \cap \mathcal{G}_k) \cup (H_i \cap \mathcal{G}_k)$ and $H \cap \dot{S}_i H_i = \emptyset$, hence $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$. $\qquad\square$

Finally, we can prove Theorem 1.6.

Proof of Theorem 1.6. (1) We already know from Theorem 1.4 that an injective homomorphism $\varphi \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}(\mathbb{A}^n)$ is a closed immersion. We claim that $d\varphi_e \colon \mathrm{Lie\,Aut}(\mathbb{A}^n) \to \mathrm{Lie\,Aut}(\mathbb{A}^n)$ is an isomorphism. To show this, consider the linear action of $\mathrm{GL}_n(\Bbbk)$ on $\mathrm{Lie\,Aut}(\mathbb{A}^n)$. We then have

$$\mathrm{Lie\,Aut}(\mathbb{A}^n) \subseteq \mathrm{Vec}(\mathbb{A}^n) \simeq \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n] = \bigoplus_d \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n]_d$$

and the latter is multiplicity-free as a $\mathrm{GL}_n(\Bbbk)$-module as well as an $\mathrm{SL}_n(\Bbbk)$-module.

Now $\varphi(\mathrm{GL}_n(\Bbbk)) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is a closed subgroup isomorphic to $\mathrm{GL}_n(\Bbbk)$. Moreover, $d\varphi_e \colon \mathrm{Lie\,Aut}(\mathbb{A}^n) \to \mathrm{Lie\,Aut}(\mathbb{A}^n)$ is an injective linear map which is equivariant with respect to $\varphi \colon \mathrm{GL}_n(\Bbbk) \xrightarrow{\sim} \varphi(\mathrm{GL}_n(\Bbbk))$. Since $\varphi(\mathrm{GL}_n(\Bbbk))$ is conjugate to the standard $\mathrm{GL}_n(\Bbbk) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ and since the representation of $\mathrm{GL}_n(\Bbbk)$ on $\mathrm{Lie\,Aut}(\mathbb{A}^n)$ is multiplicity-free, it follows that $d\varphi_e$ is an isomorphism. Thus $\mathcal{G} := \varphi(\mathrm{Aut}(\mathbb{A}^n)) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is a closed subgroup with the same Lie algebra as $\mathrm{Aut}(\mathbb{A}^n)$, and we get the following commutative diagram (see proof of Theorem 1.4):

$$
\begin{array}{ccccc}
\mathrm{Aut}(\mathbb{A}^n) & \xrightarrow{\ \varphi\ } & \mathcal{G} & \xrightarrow{\ \subseteq\ } & \mathrm{Aut}(\mathbb{A}^n) \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie\,Aut}(\mathbb{A}^n))) & \xrightarrow{\ \simeq\ } & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G}) & =\!=\!= & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie\,Aut}(\mathbb{A}^n))
\end{array}
$$

As a consequence, all maps are isomorphisms, and so $\mathcal{G} = \mathrm{Aut}(\mathbb{A}^n)$ and φ is an isomorphism.

It remains to see that every automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ is inner. Since Ad is bijective (see [KR17]) and $d\varphi_e \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$ we get $d\varphi_e = \mathrm{Ad}(\mathbf{g})$ for some $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$. This means that $d\varphi_e = (d\,\mathrm{Int}\,\mathbf{g})_e$ and so $\varphi = \mathrm{Int}\,\mathbf{g}$, by Proposition 2.7.

(2) The same argument as above shows that every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ is an isomorphism where we use the fact that the action of $\mathrm{SL}_n(\mathbb{k})$ on $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is multiplicity-free.

Moreover, $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups, see [KR17]. Hence, for every $\varphi \in \mathrm{SAut}(\mathbb{A}^n)$ there is a $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$ such that $d\varphi_e = \mathrm{Ad}\,\mathbf{g}$ which implies that $\varphi = \mathrm{Int}\,\mathbf{g}$. □

5. A special subgroup of $\mathrm{Aut}(X)$, proof of Theorem 1.1

Our Theorem 1.1 will follow from a more general result which we will describe now. For any affine variety X consider the normal subgroup $\mathcal{U}(X)$ of $\mathrm{Aut}(X)$ generated by the unipotent elements of $\mathrm{Aut}(X)$, or, equivalently, by the closed algebraic subgroups of $\mathrm{Aut}(X)$ isomorphic to the additive group $\mathbb{k}^+$. This is an instance of a so-called *connected group of automorphisms* defined by Ramanujam in [Ram64]. The group $\mathcal{U}(X)$ defined above was introduced and studied in [AFK+13] where it is called the *group of special automorphisms*[1] *of* X. In particular, they give a very interesting connection between transitivity properties of the group $\mathcal{U}(X)$ and the flexibility of the variety X.

We do not know if $\mathcal{U}(X) \subseteq \mathrm{Aut}(X)$ is closed, but we still have the notion of an *algebraic subgroup* $G \subseteq \mathcal{U}(X)$, namely a subgroup which is algebraic as a subgroup of $\mathrm{Aut}(X)$, see Remark 2.3. We will also need the notion of an "algebraic" homomorphism between these groups.

Definition 5.1. A homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ is *algebraic*, if for any algebraic subgroup $U \subseteq \mathcal{U}(X)$ isomorphic to $\mathbb{k}^+$ the image $\varphi(U) \subseteq \mathcal{U}(Y)$ is an algebraic subgroup and $\varphi|_U\colon U \to \varphi(U)$ is a homomorphism of algebraic groups. We say that $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ are *algebraically isomorphic*, $\mathcal{U}(X) \simeq \mathcal{U}(Y)$, if there exists a bijective homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ such that φ and φ^{-1} are both algebraic.

Lemma 5.2. *Let* $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ *be an algebraic homomorphism. Then, for any algebraic subgroup* $G \subseteq \mathcal{U}(X)$ *generated by unipotent elements the image* $\varphi(G) \subseteq \mathcal{U}(Y)$ *is an algebraic subgroup and* $\varphi|_G\colon G \to \varphi(G)$ *is a homomorphism of algebraic groups.*

Proof. There exist closed subgroups $U_1, \ldots, U_m \subseteq G$ isomorphic to $\mathbb{k}^+$ such that the multiplication map $\mu\colon U_1 \times U_2 \times \cdots \times U_m \to G$ is surjective. This gives the following commutative diagram

$$
\begin{array}{ccc}
U_1 \times U_2 \times \cdots \times U_m & \xrightarrow{\ \mu\ } & G \\
\Big\downarrow{\scriptstyle \bar{\varphi}:=\varphi|_{U_1}\times\cdots\times\varphi|_{U_m}} & & \Big\downarrow{\scriptstyle \varphi|_G} \\
\varphi(U_1) \times \varphi(U_2) \times \cdots \times \varphi(U_m) & \xrightarrow{\ \bar{\mu}\ } & \varphi(G)
\end{array}
$$

---

[1]They denote this group by $\mathrm{SAut}(X)$ which should not be confused with our definition of $\mathrm{SAut}(\mathbb{A}^n)$ and of $\mathrm{SAut}^{alg}(X)$ below.

where all maps are surjective. It follows that $\overline{\varphi(G)} \subseteq \mathrm{Aut}(Y)$ is a (closed) algebraic subgroup, and thus $\varphi(G) = \overline{\varphi(G)}$, because $\varphi(G)$ is constructible. It remains to show that $\varphi|_G$ is a morphism. This follows from the next lemma, because G is normal, and μ and the composition $\varphi|_G \circ \mu = \bar{\mu} \circ \tilde{\varphi}$ are both morphisms. $\qquad\square$

Lemma 5.3. *Let X, Y, Z be irreducible affine varieties where Y is normal. Let $\mu\colon X \to Y$ be a surjective morphism and $\varphi\colon Y \to Z$ an arbitrary map. If the composition $\varphi \circ \mu$ is a morphism, then φ is a morphism.*

Proof. We have the following commutative diagram of maps

$$
\begin{array}{ccc}
\Gamma_{\varphi\circ\mu} & \xrightarrow{\;\subseteq\;} & X \times Z \\[4pt]
\Big\downarrow{\scriptstyle\bar{\mu}} & & \Big\downarrow{\scriptstyle\mu\times\mathrm{id}} \\[4pt]
\Gamma_{\varphi} & \xrightarrow{\;\subseteq\;} & Y \times Z \\[4pt]
\Big\downarrow{\scriptstyle p} & & {\scriptstyle\mathrm{pr}_Y}\Big\downarrow \\[4pt]
Y & =\!=\!= & Y
\end{array}
$$

where $\Gamma_{\varphi\circ\mu}$ and Γ_{φ} denote the graphs of the corresponding maps. We have to show that $\Gamma_{\varphi} \subseteq Y \times Z$ is closed and that p is an isomorphism. The diagram shows that $\bar{\mu}$ is surjective, hence Γ_{φ} is constructible, and p is bijective. Thus, the induced morphism $\bar{p}\colon \overline{\Gamma_{\varphi}} \to Y$ is birational and surjective, hence an isomorphism since Y is normal (see [Igu73, Lemma 4, page 379]). Since p is bijective, we finally get $\Gamma_{\varphi} = \overline{\Gamma_{\varphi}}$. $\qquad\square$

Remark 5.4. If $\varphi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ is a homomorphism of ind-groups, then the induced homomorphism $\varphi_{\mathcal{U}}\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ is algebraic. If $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic as ind-groups, then $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ are algebraically isomorphic.

The remark shows that the following result generalizes Theorem 1.1. The proof will be given in the next section.

Theorem 5.5. *Let X be a connected affine variety. If $\mathcal{U}(X)$ is algebraically isomorphic to $\mathcal{U}(\mathbb{A}^n)$, then X is isomorphic to $\mathbb{A}^n$.*

Finally, we define the following closed subgroups of $\mathrm{Aut}(X)$:

$$
\mathrm{Aut}^{alg}(X) := \overline{\langle G \mid G \subseteq \mathrm{Aut}(X) \text{ connected algebraic}\rangle},
$$
$$
\mathrm{SAut}^{alg}(X) := \overline{\langle U \mid U \subseteq \mathrm{Aut}(X) \text{ unipotent algebraic}\rangle}.
$$

We have $\mathrm{SAut}^{alg}(X) = \overline{\mathcal{U}(X)} \subseteq \mathrm{Aut}^{alg}(X) \subseteq \mathrm{Aut}(X)$. A similar argument as above gives the next result, again as a consequence of Theorem 5.5 above.

Corollary 5.6. *Let X be a connected affine variety. If $\mathrm{SAut}^{alg}(X)$ is isomorphic to $\mathrm{SAut}^{alg}(\mathbb{A}^n)$ as ind-groups, then X is isomorphic to $\mathbb{A}^n$, and the same holds if we replace SAut^{alg} by Aut^{alg}.*

A special case of Theorem 1.1. Going back to our original Theorem 1.1 there is the following rather short proof in case X is irreducible which was suggested by a referee. We first remark that the subgroup of translations $\mathcal{T} \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is self-centralizing, i.e. $\mathrm{Cent}_{\mathrm{Aut}(\mathbb{A}^n)} \mathcal{T} = \mathcal{T}$. Denote by $\mathcal{T}' \subseteq \mathrm{Aut}(X)$ the image of $\mathcal{T}$. We claim that $\mathcal{T}'$ has a dense orbit. Since $\mathcal{T}'$ is a unipotent group, this implies that X is an orbit, hence isomorphic to $\mathbb{A}^m$ for some $m \leq n$. Since an n-dimensional torus acts faithfully on X, we have $n = m$, and we are done.

It remains to see that $\mathcal{T}'$ has a dense orbit in X, or equivalently, that every $\mathcal{T}'$-invariant function on X is a constant. Assume that this is not the case, and let $f \in \mathcal{O}(X)^{\mathcal{T}'} \setminus \Bbbk$. Then we can "modify" every automorphism $\mathbf{t} \in \mathcal{T}$ by f (see the following section 6) to obtain new unipotent automorphism $f \cdot \mathbf{t}$ in $\mathrm{Aut}(X)$ which do not belong to $\mathcal{T}'$, but commute with $\mathcal{T}'$, contradicting the fact that $\mathcal{T}'$ is self-centralizing. (It is here where we use the irreducibility of X. Otherwise it is not clear why these modified automorphisms do not belong to $\mathcal{T}'$.)

6. Modifications and Root Subgroups, proof of Theorem 5.5

Let X be an affine variety and consider a nontrivial action of $\Bbbk^+$ on X, given by $\lambda \colon \Bbbk^+ \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\Bbbk^+$-invariant, then we define the *modification* $f \cdot \lambda$ of the action λ in the following way (see [AFK+13] where a modified action is called a replica; cf. [FK17, section 12.4]):

$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x \quad \text{for} \quad s \in \Bbbk \text{ and } x \in X.$$

It is easy to see that this is again a $\Bbbk^+$-action. In fact, the action λ corresponds to a locally nilpotent vector field $\delta_\lambda \in \mathrm{Vec}(X)$. Since f is λ-invariant, it follows that $f\delta_\lambda \in \mathrm{Vec}(X)$ is again locally nilpotent, and defines the modified $\Bbbk^+$-action $f \cdot \lambda$. Note that if $U_\lambda \subseteq \mathrm{Aut}(X)$ denotes the image of λ, then $\mathrm{Lie}(U_\lambda) \xrightarrow{\sim} \Bbbk\delta_\lambda$ under the canonical homomorphism $\mathrm{Lie}\,\mathrm{Aut}(X) \hookrightarrow \mathrm{Vec}(X)$.

This modified action $f \cdot \lambda$ is trivial if and only if f vanishes on every irreducible component X_i of X where the action λ is nontrivial. It is clear that the orbits of $f \cdot \lambda$ are contained in the orbits of λ, and that they are equal on the open subset $X_f := \{x \in X \mid f(x) \neq 0\}$ of X. In particular, if X is irreducible and $f \neq 0$, then λ and $f \cdot \lambda$ have the same invariants.

If $U \subseteq \mathrm{Aut}(X)$ is a closed subgroup isomorphic to $\Bbbk^+$ and if $f \in \mathcal{O}(X)^U$ is a U-invariant, then we can define the *modification $f \cdot U$ of U* by choosing an isomorphism $\lambda \colon \Bbbk^+ \xrightarrow{\sim} U$ and setting $f \cdot U := (f \cdot \lambda)(\Bbbk^+)$, the image of the modified action.

Let $\mathcal{G}$ be an ind-group, and let $T \subseteq \mathcal{G}$ be a torus.

Definition 6.1. An algebraic subgroup $U \subseteq \mathcal{G}$ isomorphic to $\Bbbk^+$ and normalized by T is called a *root subgroup* with respect to T. The character of T on $\mathrm{Lie}\,U \simeq \Bbbk$ is called the *weight of U*.

If $U = U_\lambda$ is the image of a nontrivial $\Bbbk^+$-action λ, then U is a root subgroup if and only if $\Bbbk\delta_\lambda \subseteq \mathrm{Vec}(X)$ is stable under T. If α is the weight of U_λ, we have

$$t \cdot \lambda(s) \cdot t^{-1} = \lambda(\alpha(t)s) \text{ for } t \in T, s \in \Bbbk.$$

If a torus T acts on an affine variety X, then we get a locally finite and rational representation of T on the coordinate ring $\mathcal{O}(X)$, and thus a decomposition of $\mathcal{O}(X)$ into weight spaces. A locally finite and rational representation of T is called *multiplicity-free* if the dimensions of the weight spaces are ≤ 1. The following lemma is crucial.

Lemma 6.2. *Let X be an irreducible affine variety, and let $T \subseteq \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subseteq \mathrm{Aut}(X)$ with respect to T such that $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

Proof. The first inequality $\dim T \leq \dim X$ is clear, because T acts faithfully on X. It follows from [DK08, Propositions 2.7 and 2.9] that there exists a T-semi-invariant

$f \in \mathcal{O}(X)^U$ such that the localization $\mathcal{O}(X)_f^U = \mathcal{O}(X_f)^U$ is finitely generated. Clearly, $\mathcal{O}(X)_f^U$ is T-stable and multiplicity-free, and $\mathcal{O}(X)_f^U$ is the coordinate ring of the algebraic quotient $Z := X_f /\!\!/ U$ on which T acts. It follows from [Kra84, II.3.4 Satz 5]) that T has a dense orbit in Z, and so $\dim Z \leq \dim T$. Since $\dim Z = \dim X_f /\!\!/ U = \dim X_f - 1 = \dim X - 1$, we get the second inequality. $\square$

Lemma 6.3. *We have $\mathcal{U}(\mathbb{A}^n) \subseteq \mathrm{SAut}(\mathbb{A}^n)$, and its closure $\overline{\mathcal{U}(\mathbb{A}^n)}$ is connected. Moreover, $\mathrm{Lie}\,\overline{\mathcal{U}(\mathbb{A}^n)} = \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$, hence it is a simple Lie algebra.*

Proof. The first statement is obvious, since every unipotent algebraic subgroup is contained in $\mathrm{SAut}(\mathbb{A}^n)$. The second claim follows from $\mathcal{U}(\mathbb{A}^n) \subseteq \overline{\mathcal{U}(\mathbb{A}^n)}^{\circ}$ (see Lemma 7.3 in the next section). For the last statement we remark that $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is generated by the Lie algebras of the algebraic subgroups of $\mathrm{SAut}(\mathbb{A}^n)$ (Remark 2.6) which are all contained in $\mathcal{U}(\mathbb{A}^n)$ (Lemma 5.2). $\square$

Denote by $T_n \subseteq \mathrm{GL}_n(\Bbbk) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ the diagonal torus and set $T'_n := T_n \cap \mathrm{SL}_n(\Bbbk)$. The next result can be found in [Lie11, Theorem 1].

Lemma 6.4. *Root subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ with respect to T'_n exist, and their weights are all different.*

Now we can give the proof of Theorem 5.5.

Proof of Theorem 5.5. The algebraic subgroups $\mathrm{SL}_n(\Bbbk)$ and $\mathrm{SAff}_n(\Bbbk)$ of $\mathrm{Aut}(\mathbb{A}^n)$ both belong to $\mathcal{U}(\mathbb{A}^n)$ as well as all root subgroups U. Fix an algebraic isomorphism $\varphi \colon \mathcal{U}(\mathbb{A}^n) \xrightarrow{\sim} \mathcal{U}(X)$ and set by $T' := \varphi(T'_n) \subseteq \mathcal{U}(X)$.

Let $X = \bigcup_i X_i$ be the decomposition into irreducible components. Since $\overline{\mathcal{U}(X)}$ is connected by Lemma 6.3, the components X_i are stable under $\overline{\mathcal{U}(X)}$. Denote by $A \subseteq \mathcal{U}(X)$ the image of $\mathrm{Aff}_n(\Bbbk)$ under φ. Since every nontrivial closed normal subgroup of $\mathrm{Aff}_n(\Bbbk)$ contains the translations, one of the restriction maps $\rho_i \colon \mathcal{U}(X) \to \mathcal{U}(X_i)$, say ρ_1, is injective on A.

Let $T_1 := \rho_1(T') \subseteq \mathcal{U}(X_1)$ be the image of T'. We will show that there is a root subgroup $U_1 \subseteq \mathcal{U}(X_1)$ such that $\mathcal{O}(X_1)^{U_1}$ is multiplicity-free. Then Lemma 6.2 implies that $\dim X_1 \leq n$ and so, by Proposition 2.8, X_1 is isomorphic to $\mathbb{A}^n$ with a transitiv action of A. Since X is connected, this implies that $X = X_1 \simeq \mathbb{A}^n$.

In order to construct U_1 we choose a root subgroup $U \subseteq \varphi(\mathrm{SL}_n(\Bbbk)) \subseteq \mathcal{U}(X)$, and set $U_1 := \rho_1(U) \subseteq \mathcal{U}(X_1)$. Since U is a maximal unipotent subgroup of a closed subgroup $S \subseteq \mathcal{U}(X)$ isomorphic to $\mathrm{SL}_2(\Bbbk)$ and since the restriction map $\mathrm{res}\colon \mathcal{O}(X) \to \mathcal{O}(X_1)$ is a surjective homomorphism of S-modules, it follows that $\mathrm{res}\colon \mathcal{O}(X)^U \to \mathcal{O}(X_1)^{U_1}$ is also surjective (see [Kra84, III.3.1, Bemerkung 2]). If α is the weight of U and U_1 and if $f \in \mathcal{O}(X_1)^{U_1}$ is an invariant of weight β, then $f = \tilde{f}|_{X_1}$ for an invariant $\tilde{f} \in \mathcal{O}(X)^U$ of weight β, and so $\tilde{f} \cdot U$ is a root subgroup of weight $\alpha + \beta$ with $\rho_1(\tilde{f} \cdot U) = f \cdot U_1$. Since the root subgroups of $\mathrm{Aut}(X)$ have different weights, it finally follows that $\mathcal{O}(X_1)^{U_1}$ is multiplicity-free. $\square$

7. Finite dimensional automorphism groups

It is well-known that for a smooth affine curve C the automorphism group $\mathrm{Aut}(C)$ is finite except for $C \simeq \Bbbk, \Bbbk^*$. Theorem 1.2 implies that every finite group appears as automorphism group of a smooth affine curve. There also exist examples of smooth affine surfaces with a discrete non-finite automorphism group, see [FK17,

Proposition 12.7.1]. Recall that an ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all k, or equivalently, if $\operatorname{Lie} \mathcal{G} = \{0\}$.

Definition 7.1. An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *finite dimensional*, $\dim \mathcal{G} < \infty$, if $\dim \mathcal{G}_k$ is bounded above. In this case we put $\dim \mathcal{G} := \max_k \dim \mathcal{G}_k$.

Definition 7.2. For an ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ we define
$$\mathcal{G}^\circ := \bigcup_k \mathcal{G}_k^\circ$$
where $\mathcal{G}_k^\circ$ denotes the connected component of $\mathcal{G}_k$ which contains $e \in \mathcal{G}$.

An ind-variety $\mathcal{V}$ is called *curve-connected* if for every $v, w \in \mathcal{V}$ there is an irreducible curve D and a morphism $D \to \mathcal{V}$ whose image contains v and w. This is equivalent to the condition that $\mathcal{V}$ admits a filtration with irreducible varieties (see [FK17, Lemma 1.6.3]). The following result can be found in [FK17, Lemma 2.2.2]).

Lemma 7.3. *Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$ be an ind-group.*
  (1) *$\mathcal{G}^\circ \subseteq \mathcal{G}$ is a curve-connected open (and thus closed) normal subgroup of countable index. In particular, $\operatorname{Lie} \mathcal{G} = \operatorname{Lie} \mathcal{G}^\circ$.*
  (2) *We have $\dim \mathcal{G} < \infty$ if and only if $\mathcal{G}^\circ \subseteq \mathcal{G}$ is an algebraic group.*
  (3) *We have $\dim \mathcal{G} < \infty$ if and only if $\dim \operatorname{Lie} \mathcal{G} < \infty$.*

Example 7.4.   (1) We have $\operatorname{Aut}(\Bbbk^*) \simeq \mathbb{Z}/2 \ltimes \Bbbk^*$, hence $\operatorname{Aut}(\Bbbk^*)^\circ \simeq \Bbbk^*$. Similarly, $\operatorname{Aut}(\Bbbk^{*n}) \simeq \operatorname{GL}_n(\mathbb{Z}) \ltimes \Bbbk^{*n}$, and so $\operatorname{Aut}(\Bbbk^{*n})^\circ \simeq \Bbbk^{*n}$.
  (2) Let $C := V(y^2 - x^3) \subseteq \Bbbk^2$ be NEILE's parabola. Then $\operatorname{Aut}(C) = \Bbbk^*$. In fact, every automorphism of C defines an automorphism of the normalization $\mathbb{A}^1$ of C fixing the origin. From this the claim follows immediately.
  (3) Let C be a smooth curve with trivial automorphism group, and consider the one dimensional variety $Y_C = \mathbb{A}^1 \cup C$ where the two irreducible components meet in $\{0\} \in \mathbb{A}^1$. Then $\operatorname{Aut}(Y_C) \simeq \Bbbk^*$. Moreover, the disjoint union $Y_{C_1} \cup Y_{C_2} \cup \cdots \cup Y_{C_m}$ with pairwise non-isomorphic curves C_i has automorphism group $\Bbbk^{*m}$. We will show in section 8 that for every n there is even an irreducible affine variety X whose automorphism group $\operatorname{Aut}(X)$ is an n-dimensional torus.

Theorem 1.3 claims that if $\dim \operatorname{Aut}(X)$ is finite, then either $X \simeq \mathbb{A}^1$ or $\operatorname{Aut}(X)^\circ$ is a torus. This follows immediately from the next result.

Proposition 7.5. *Let X be a connected affine variety. If X admits a nontrivial action of the additive group $\Bbbk^+$, then either $X \simeq \mathbb{A}^1$ or $\dim \operatorname{Aut}(X) = \infty$.*

Proof. If X contains a one-dimensional irreducible component X_i with a nontrivial action of $\Bbbk^+$, then X_i is an orbit under $\Bbbk^+$, hence $X = X_i \simeq \mathbb{A}^1$. Otherwise, $\Bbbk^+$ acts non-trivially on an irreducible component X_j of dimension ≥ 2. Denote by $U \subseteq \operatorname{Aut}(X)$ the image of $\Bbbk^+$. We claim that the modifications $f \cdot U$ for $f \in \mathcal{O}(X)^U$ form an infinite dimensional subgroup $\mathcal{O}(X)^U \cdot U \subseteq \operatorname{Aut}(X)$. This follows if we show that the image of $\mathcal{O}(X)^U$ in $\mathcal{O}(X_j)$ is infinite dimensional. For that we first remark that there is a nonzero U-invariant f which vanishes on all X_k for $k \neq j$, because the vanishing ideal is U-stable. This implies that $X_f \subseteq X_j$, and so
$$\mathcal{O}(X)_f^U = \mathcal{O}(X_f)^U = \mathcal{O}(X_j)_f^U = (\mathcal{O}(X)^U|_{X_j})_f.$$
Thus the image $\mathcal{O}(X)^U|_{X_j} \subseteq \mathcal{O}(X_j)$ is infinite dimensional. $\square$

The following result—a partial converse of the proposition above—is due to Arzhantsev-Gaĭfullin.

Proposition 7.6 ([AG17])**.** *Let X be an affine variety which does not admit a nontrivial $\Bbbk^+$-action. Then $\mathrm{Aut}(X)$ contains a unique maximal torus T. If the action of T on X is one-fix-pointed, then $\mathrm{Aut}(X)^\circ = T$ and $\mathrm{Aut}(X)/T$ is a finite group.*

(A T-action on X is called *one-fix-pointed* if there is a unique fixed point $x_0 \in X$ and no other closed orbit.) The paper [AG17] contains many examples of such varieties, e.g. cones over projective varieties with a finite automorphism group, or the so-called trinomial hypersurfaces.

8. An example with a torus as automorphism group

In Example 7.4 we have mentioned that Neile's parabola $C := V(y^2 - x^3) \subseteq \Bbbk^2$ has an automorphism group isomorphic to $\Bbbk^*$, and we have given an example of a reducible curve with automorphism group isomorphic to $\Bbbk^{*m}$. We now construct an irreducible variety X of dimension d with $\mathrm{Aut}(X) \xrightarrow{\sim} \Bbbk^{*d}$.

Definition 8.1. A plane curve $C \subseteq \Bbbk^2$ given by an equation of the form $y^m - x^n = 0$ where $n > m \geq 2$ and m, n are relatively prime, is called a *cuspidal curve*. It has an isolated singularity in the origin 0.

For the cuspidal curve $C_{m,n}$ with equation $y^m = x^n$ we have a canonical isomorphism $\Bbbk^* \xrightarrow{\sim} \mathrm{Aut}(C_{m,n})$ given by the action $t(x, y) := (t^m x, t^n y)$. The induced representation on the tangent space $T_0 C_{m,n} = \Bbbk^2$ has weights m, n. In particular, $C_{m,n}$ is isomorphic to $C_{m',n'}$ if and only if $(m, n) = (m', n')$. Moreover, the normalization is given by the bijective morphism $\mu_{C_{m,n}} \colon \mathbb{A}^1 \to C_{m,n}, s \mapsto (s^m, s^n)$.

Proposition 8.2. *Let X be a product of d cuspidal curves which are pairwise non-isomorphic. Then $\mathrm{Aut}(X) \simeq \Bbbk^{*m}$.*

Proof. (a) Let $X = C_1 \times C_2 \times \cdots \times C_d$ be such a product. We have a canonical injective homomorphism $\rho_X \colon \Bbbk^{*d} \hookrightarrow \mathrm{Aut}(X)$. The normalization of X is given by the bijective morphism

$$\eta := \eta_1 \times \cdots \times \eta_d \colon \mathbb{A}^d \to X$$

where $\eta_i \colon \mathbb{A}^1 \to C_i$ is the normalization of C_i. For $j = 1, \ldots, d$ define

$$\tilde{C}_j := \{(0, \ldots, c_j, \ldots, 0) \mid c_j \in C_j\} \subseteq X,$$

i.e. $\tilde{C}_j$ is the image of the jth coordinate line $L_j \subseteq \mathbb{A}^d$ under the normalization η. Then we have

$$A_X := \bigcup_j \tilde{C}_j = \{x \in X \mid \dim T_x X \geq 2d - 1\}.$$

Now let $Y = D_1 \times D_2 \times \cdots \times D_d$ be another product of non-isomorphic cuspidal curves, and define $\tilde{D}_j$ and A_Y as above. It follows from the description of A_X and A_Y that every isomorphism $\mu \colon X \xrightarrow{\sim} Y$ induces an isomorphism $A_X \xrightarrow{\sim} A_Y$. Hence there is a permutation σ of $\{1, \ldots, d\}$ such that $C_i \simeq \tilde{C}_i \simeq \mu(\tilde{C}_i) = \tilde{D}_{\sigma(i)} \simeq D_{\sigma(i)}$.

(b) Now define $X_j := \{c = (c_1, \ldots, c_d) \in X \mid c_j = 0\} \simeq \prod_{i \neq j} C_i$. Clearly, X_j is the image of the jth coordinate hyperplane $H_j \subseteq \mathbb{A}^d$ given by $x_j = 0$ under the normalization $\eta \colon \mathbb{A}^d \to X$. Since the singular points of X are given by $X_{sing} = \bigcup_j X_j$, it follows that every automorphism $\varphi \colon X \xrightarrow{\sim} X$ permutes the irreducible

components X_j of X_{sing}. Now (a) implies that the X_j are pairwise non-isomorphic, hence $\varphi(X_j) = X_j$.

(c) By induction, we can assume that $\rho_{X_j} \colon \Bbbk^{*d-1} \to \operatorname{Aut}(X_j)$ is an isomorphism, and so φ_{X_j} is given by an element $t_j \in \Bbbk^{*d-1}$. Looking at the intersections $X_j \cap X_k$ we see that there is a $t \in \Bbbk^{*d}$ such that $\varphi|_{X_{sing}}$ is given by t. Therefore, the automorphism $\psi := t^{-1} \circ \varphi \in \operatorname{Aut}(X)$ induces the identity on X_{sing}. It follows that the normalization $\tilde{\psi} \colon \mathbb{A}^d \xrightarrow{\sim} \mathbb{A}^d$ fixes the coordinate hyperplanes H_j pointwise which implies that $\tilde{\psi}$ is the identity. In fact, if $\tilde{\psi} = (f_1, \ldots, f_d)$, $f_i \in \Bbbk[x_1, \ldots, x_d]$, then we get $x_i | f_i$, and the claim follows because all f_i are irreducible (see e.g. [Jel91] for a more general result). □

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