ORDINARY DIFFERENTIAL EQUATIONS AND SINGULAR INTEGRALS

Gianluca Crippa

Departement Mathematik und Informatik, Universität Basel
Mathematisches Institut, Rheinsprung 21, CH-4051 Basel, Switzerland

Abstract. We present an informal review of some concepts and results from the theory of ordinary differential equations in the non-smooth context, following the approach based on quantitative a priori estimates introduced in [9] and [7].

1. **Introduction.** In this note we give an informal overview on some results from [9] (collaboration with Camillo De Lellis) and [7] (collaboration with François Bouchut) regarding an approach to non-smooth ordinary differential equations based on quantitative a priori estimates.

Given the *velocity field*

$$b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \tag{1}$$

we consider the *ordinary differential equation*

$$\begin{cases} \dot{X}(t,x) = b(t, X(t,x)) \\ X(0,x) = x\,, \end{cases} \tag{2}$$

where we denote with the "dot" the differentiation with respect to the time variable t. The solution $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is called the *flow* of the velocity field b. We are thus looking for characteristic (or integral) curves of the given velocity field b, i.e., curves with the property that at each point the tangent vector coincides with the value of the given vector field at such point.

The classical Cauchy-Lipschitz theory deals with the case in which the velocity field b is regular enough (Lipschitz with respect to the space variable uniformly with respect to time, see (3)). After a brief review of this smooth theory, in this note we motivate the extension to non-smooth contexts, and we consider first of all the case of $W^{1,p}$ (with $p > 1$) velocity fields, then the case of $W^{1,1}$ velocity fields, and and finally the case of velocity fields whose derivative can be represented as a singular integral operator of an L^1 function. This stratified presentation has the advantage to present the main conceptual and technical differences between these different cases.

The presentation will be *very* informal and only the key points of the proofs will be indicated, with the aim to catch the interest of the reader for the general context and to motivate him or her to further readings on this topic. Emphasis will be put on the ideas, rather than on the details.

2000 *Mathematics Subject Classification.* Primary: 34A12; Secondary: 42B37.

Key words and phrases. Ordinary Differential Equations, Sobolev and Bounded Variation Functions, Singular Integrals.

For this reason, the only references given will be those strictly related to our line of presentation. For a wider presentation of the subject and a detailed bibliography the reader is referred for instance to [5] or [2]. Moreover, the "partial differential equations side" of this problem (well posedness of the transport and the continuity equations in the non-smooth context) will not be addressed here. The interested reader is referred to the two most important papers in this area, namely [10] for the Sobolev case and [1] for the bounded variation case, and again to the bibliographical references in [5] or [2].

2. **The Lipschitz case.** As a warm up, let us start by considering the case of a vector field which is Lipschitz with respect to the space variable uniformly with respect to the time. This means that we assume the existence of a constant L such that

$$|b(t,x) - b(t,y)| \leq L|x-y| \tag{3}$$

for every x, $y \in \mathbb{R}^d$ and every $t \in [0,T]$. Under this assumption, it is known from the classical Cauchy-Lipschitz theorem that a unique solution to (2) exists for every initial point $x \in \mathbb{R}^d$, and moreover the flow $X(t,x)$ inherits the Lipschitz regularity with respect to x.

Uniqueness can be easily proven with the following argument. Consider two (possibly distinct) flows X_1 and X_2. Then for every given $x \in \mathbb{R}^d$ one may compute

$$\begin{aligned}\frac{d}{dt}|X_1(t,x) - X_2(t,x)| \leq& |b(t, X_1(t,x)) - b(t, X_2(t,x))| \\ \leq& L|X_1(t,x) - X_2(t,x)|\,,\end{aligned}$$

where in the last inequality we have used (3). Using Gronwall Lemma (and recalling that $X_1(0,x) = X_2(0,x)$) we deduce immediately that $X_1(t,x) = X_2(t,x)$ for every $t \in [0,T]$, i.e., the desired uniqueness.

The proof of the Lipschitz regularity of the flow $X(t,x)$ with respect to x goes along the same line. Fix two points x, $y \in \mathbb{R}^d$ and compute

$$\begin{aligned}\frac{d}{dt}|X(t,x) - X(t,y)| \leq& |b(t, X(t,x)) - b(t, X(t,y))| \\ \leq& L|X(t,x) - X(t,y)|\,.\end{aligned}$$

Applying again Gronwall Lemma and observing that $|X(0,x) - X(0,y)| = |x-y|$ we obtain

$$|X(t,x) - X(t,y)| \leq e^{Lt}|x-y|\,, \tag{4}$$

i.e., $X(t,x)$ is Lipschitz with respect to x, and the Lipschitz constant depends exponentially on the Lipschitz constant of the given velocity field b.

3. **Towards non-Lipschitz velocity fields: The regular Lagrangian flow.** After some reflections on the very simple theory presented in the previous section, a natural question arises: how much of such a theory survives when the velocity field b is less regular than Lipschitz?

We immediately realize that, if we stick to "classical" statements (for instance, if we look for uniqueness of the flow for *every* initial point), then the answer is negative. A possible example is very well known: consider in $\mathbb{R}$ the (Hölder but not Lipschitz) vector field $b(x) = \sqrt{|x|}$. Then it is readily checked that $X_1(t,0) \equiv 0$ and $X_2(t,0) = \frac{1}{4}t^2$ are two *distinct* solutions of (2), with the same value ($x = 0$) at the initial time. Indeed, it is easy to construct an infinite family of distinct solutions.

One may be discouraged by having a "counterexample" in a still fairly simple situation (an Hölder time-independent vector field in one dimension!). However, non-regular transport phenomena do appear in an ubiquitous fashion in physical models: fluid dynamics, conservation laws, kinetic equations... The reader is referred again to [5] and to [2] for a list of references.

The hope is to find some "milder" issues (some "weakened" version of the pointwise uniqueness for (2), i.e., uniqueness of the flow for *every* point $x \in \mathbb{R}^d$, or of the regularity of the flow with respect to the initial position), together with some "reasonable" context in which such new question may allow a positive answer. The two elements of the new theory will be the following:

(1) The velocity field b may be non-Lipschitz, but it must have "a first-order derivative" in some suitable weak sense. The "bad" velocity field $b(x) = \sqrt{|x|}$ is merely 1/2-Hölder, hence it possesses only "half a derivative" at the origin.
(2) We content ourselves with showing uniqueness of (almost) measure preserving flow solutions of (2). That is, we drop the pointwise framework, and we just consider as "admissible" solutions to (2) those flows $X(t, x)$ for which, at every time $t \in [0, T]$, the map $X(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ does not squeeze or expand sets in a crazy fashion. The non-unique trajectories produced by $b(x) = \sqrt{|x|}$ do indeed "compress" long segments into one point, the origin of $\mathbb{R}$. (The non-uniqueness is dynamically due to the stopping of the trajectories at the origin). The reader will notice that the origin is precisely the point at which the regularity of b is degenerating.

We now specify what we mean with "measure preserving flow solution":

Definition 3.1 (Regular Lagrangian flow)**.** We say that a map $X : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a *regular Lagrangian flow* associated to the vector field b if

(i) For $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t, x)$ is a distributional solution to the ordinary differential equation $\dot\gamma(t) = b(t, \gamma(t))$, with $\gamma(0) = x$;
(ii) There exists some constant $M > 0$ such that the *compressibility condition*

$$X(t, \cdot)_\# \mathcal{L}^d \le M \mathcal{L}^d \qquad \text{for every } t \in [0, T] \tag{5}$$

holds.

The condition in (5) involves the push-forward of the d-dimensional Lebesgue measure $\mathcal{L}^d$ and can be equivalently reformulated as follows: there exists some constant $M > 0$ such that for every $t \in [0, T]$ and every $\varphi \in C_c(\mathbb{R}^d)$ with $\varphi \ge 0$ there holds

$$\int_{\mathbb{R}^d} \varphi(X(t, x))\, dx \le M \int_{\mathbb{R}^d} \varphi(x)\, dx\,.$$

This means that we require *a priori*, i.e., as a sort of "selection condition" for our notion of solution, a quantitative control on how much the flow compresses d-dimensional sets. For simplicity, in the following presentation, we shall restrict our attention to those regular Lagrangian flow which *exactly* preserve the Lebesgue measure (in the smooth context, this corresponds to the condition of b having zero divergence, thanks to Liouville's Theorem). We can formulate it by saying that "changes of variable along the flow are performed for free", that is, for every $\varphi \in C_c(\mathbb{R}^d)$ we can compute

$$\int_{\mathbb{R}^d} \varphi(X(t, x))\, dx = \int_{\mathbb{R}^d} \varphi(x)\, dx\,. \tag{6}$$

4. **A stable formal estimate, and a new integral quantity.** In order to make our computations typographically more clear, in the rest of this note we shall only consider time-independent vector fields. The passage to the time-dependent case does not give rise to any complication in the argument.

The natural attempt is to rephrase the strategy of §2 in a way that will be robust when lowering the regularity of the velocity field from Lipschitz to "weakly differentiable". Denoting by ∇ the gradient with respect to the x variable, we can *formally* compute as follows:

$$\begin{aligned}\frac{d}{dt}\log|\nabla X| \leq & \frac{1}{|\nabla X|}\left|\frac{d}{dt}\nabla X\right| \\ = & \frac{1}{|\nabla X|}\left|\nabla\big(b(X)\big)\right| = |\nabla b|(X)\,.\end{aligned} \tag{7}$$

Notice that this computation is effective in the case of a smooth velocity field b possessing a smooth flow X. Anyhow, in the Lipschitz context, it allows us to recover the estimate for the regularity of the flow with respect to the initial position already established in (4). Indeed, if b satisfies (3), then $|\nabla b| \leq L$, and so by integrating (7) we deduce

$$\log|\nabla X| \leq Lt + \log|\nabla \mathrm{Id}| = Lt\,,$$

from which (4).

We apply a similar strategy in order to show uniqueness. For this we fix a small parameter $\delta > 0$. If X_1 and X_2 are flows of b, then we compute

$$\frac{d}{dt}\log\left(1+\frac{|X_1-X_2|}{\delta}\right) \leq \frac{\delta}{\delta+|X_1-X_2|}\;\frac{|b(X_1)-b(X_2)|}{\delta} \leq L\,,$$

where L is the Lipschitz constant of b. Hence

$$\log\left(1+\frac{|X_1-X_2|}{\delta}\right) \leq Lt + \log\left(1+\frac{|X_1(0,\cdot)-X_2(0,\cdot)|}{\delta}\right) = Lt\,,$$

and finally

$$\frac{|X_1-X_2|}{\delta} \leq e^{Lt}\,.$$

Since $\delta > 0$ can be chosen arbitrarily small, we deduce that $X_1 = X_2$.

The remarkable advantage of this argument is that it allows an integral version, which can be used for non-Lipschitz vector fields. In the rest of this note, we focus on the uniqueness issue for the regular Lagrangian flow associated to a given velocity field, as defined in Definition 3.1. Given a velocity field b, two (possibly distinct) associated regular Lagrangian flows X_1 and X_2, and a small parameter $\delta > 0$ we consider

$$\Phi_\delta(t) = \int \log\left(1+\frac{|X_1(t,x)-X_2(t,x)|}{\delta}\right)dx\,. \tag{8}$$

Notice that suitable truncations are necessary in order to make this integral convergent, but for the sake of clarity in this exposition we will ignore this technical issue.

This integral functional has been first considered in a joint paper with De Lellis [9], where we were inspired by some similar computations due to Ambrosio, Lecumberry and Maniglia [3]. Although we are now focussing our presentation on the uniqueness issue, we remark that similar integral quantities are useful to prove regularity, compactness and quantitative stability rates for regular Lagrangian flows.

5. **A condition for uniqueness.** In the unlucky situation of non-uniqueness of the regular Lagrangian flow, that is, when there are two distinct regular Lagrangian flows X_1 and X_2, we easily discover that there is a set $A \subset \mathbb{R}^d$ of measure at least $\alpha > 0$ such that $|X_1(t,x) - X_2(t,x)| \geq \gamma > 0$ for some $t \in [0,T]$ and for all $x \in A$. Hence we can estimate the integral functional $\Phi_\delta(t)$ from below as follows:

$$\Phi_\delta(t) \geq \int_A \log\left(1 + \frac{\gamma}{\delta}\right) dx \geq \alpha \log\left(1 + \frac{\gamma}{\delta}\right).$$

We then discover that a condition guaranteeing uniqueness is:

$$\frac{\Phi_\delta}{\log\left(\frac{1}{\delta}\right)} \to 0 \qquad \text{as } \delta \downarrow 0\,. \tag{9}$$

This means that a good strategy to prove uniqueness is to derive upper bounds for the integral functional $\Phi_\delta(t)$. The natural computation starts with a time differentiation, aimed at making the difference quotients of the velocity field b appear. We calculate

$$\begin{aligned} \Phi'_\delta(t) &\leq \int \frac{\partial_t |X_1 - X_2|}{\delta + |X_1 - X_2|}\, dx \leq \int \frac{|b(X_1) - b(X_2)|}{\delta + |X_1 - X_2|}\, dx \\ &\leq \int \min\left\{ \frac{2\|b\|_{L^\infty}}{\delta} \;;\; \frac{|b(X_1) - b(X_2)|}{|X_1 - X_2|} \right\} dx\,. \end{aligned} \tag{10}$$

For a Lipschitz velocity field, it is sufficient to estimate

$$\frac{|b(X_1) - b(X_2)|}{|X_1 - X_2|} \leq L$$

in (10) to obtain that $\Phi'_\delta(t)$ (and thus $\Phi_\delta(t)$) is bounded by a constant. We recover again uniqueness in the Lipschitz case.

But a milder condition to obtain boundedness of $\Phi_\delta(t)$ would be the difference-quotients estimate

$$\frac{|b(x) - b(y)|}{|x - y|} \leq \psi(x) + \psi(y) \tag{11}$$

for some function $\psi \in L^1_{\mathrm{loc}}$. Indeed, getting back to (10), we estimate

$$\Phi'_\delta(t) \leq \int \big(\psi(X_1) + \psi(X_2)\big)\, dx = 2 \int \psi(x)\, dx\,,$$

where in the last equality we change variables as in (6), and we conclude again that $\Phi_\delta(t)$ is bounded by a constant. Notice that the first term in the minimum in (10) has been simply neglected. A smarter computation allowing for its use will be explained in §7.

6. **Maximal functions, strong and weak estimates, and uniqueness for $W^{1,p}$ velocity fields with $p > 1$.** In the paper [9] with De Lellis we realized that condition (11) is satisfied (and so uniqueness holds) in the case of velocity fields with Sobolev $W^{1,p}$ regularity, for any $p > 1$.

Indeed, in such case, the estimate for the difference quotients

$$\frac{|b(x) - b(y)|}{|x - y|} \leq C_{p,d}\Big(MDb(x) + MDb(y)\Big) \tag{12}$$

holds, where the *maximal function* of a locally summable function f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mathcal{L}^d(B(x,r))} \int_{B(x,r)} |f(y)|\, dy\,. \tag{13}$$

It is classical (see for instance [11]) that the maximal function enjoys the *strong estimate*

$$\|Mf\|_{L^p} \leq C_{d,p}\|f\|_{L^p} \tag{14}$$

for any $1 < p \leq \infty$, but unfortunately this fails for $p = 1$. In that case, only the *weak estimate*

$$\mathcal{L}^d\Big(\Big\{x \ : \ |Mf(x)| > \lambda\Big\}\Big) \leq C_{d,1}\frac{\|f\|_{L^1}}{\lambda} \qquad\qquad \text{for any } \lambda > 0 \tag{15}$$

is available.

We see from (12) that (11) holds if we take

$$\psi = MDb\,, \tag{16}$$

and using the strong estimate (14) we deduce from the assumption $Db \in L^p$ that $\psi \in L^p$, and uniqueness follows. The failure of the strong estimate (14) for $p = 1$ is precisely the reason why the uniqueness theorem in [9] was limited to the case $p > 1$. The cases of $W^{1,1}$ or even of BV velocity fields were missing.

7. **Uniqueness for $W^{1,1}$ velocity fields.** Together with Bouchut, we discovered in [7] how to extend this argument to the case of $W^{1,1}$ velocity fields. The proof uses some more elaborate tools from harmonic analysis (as a general reference the interested reader can consult [11]).

Introducing the quantity

$$|||f|||_{M^1} = \sup\Big\{\lambda\,\mathcal{L}^d\big(\{|f| > \lambda\}\big) \ : \quad \lambda > 0\Big\}\,, \tag{17}$$

we see that (15) can be rewritten as

$$|||Mf|||_{M^1} \leq C_{d,1}\|f\|_{L^1}\,. \tag{18}$$

The space M^1 consisting of all functions for which the quantity in (17) is finite is called weak Lebesgue space (or alternatively Lorentz space or Marcinkiewicz space). It is endowed with the natural pseudo-norm $|||f|||_{M^1}$, which is however not a norm, lacking the subadditivity property. Notice that M^1 is strictly bigger than L^1.

Going back to (16), and observing that we are now concerned with the case when $Db \in L^1$, we discover that condition (11) now is satisfied for some $\psi \in M^1$. In general ψ does not belong to $L^1_{\rm loc}$: we need some additional considerations in order to conclude uniqueness.

Let us go back to (10). Using (11) and changing variable using (6) we obtain

$$\Phi'_\delta(t) \leq \int_{\mathbb{R}^d} \min\left\{\frac{2\|b\|_{L^\infty}}{\delta} \ \ ; \ \ 2\psi\right\}\,dx\,. \tag{19}$$

None of the two terms inside the minimum suffices by itself to deduce (9). The first term is L^∞, but with a norm which blows up as $\delta \downarrow 0$, while the second term is merely M^1. However, an interpolation inequality between M^1 and L^∞ is at our disposal (see [7] for a proof):

$$\|f\|_{L^1} \leq |||f|||_{M^1}\left[1 + \log\left(C\frac{\|f\|_{L^\infty}}{|||f|||_{M^1}}\right)\right]\,.$$

We apply this interpolation inequality to

$$f = \min\left\{\frac{2\|b\|_{L^\infty}}{\delta} \ \ ; \ \ 2\psi\right\}\,,$$

and we observe that

$$\|f\|_{L^\infty} = \frac{2\|b\|_{L^\infty}}{\delta} \leq \frac{C}{\delta}$$

$$\text{and} \qquad |||f|||_{M^1} = 2|||\psi|||_{M^1} = 2|||MDb|||_{M^1} \leq C\|Db\|_{L^1} \,,$$

by (18). We go back to (19) and employing these estimates we deduce

$$\Phi'_\delta(t) \leq C\|Db\|_{L^1} \left[1 + \log\left(\frac{C}{\delta\|Db\|_{L^1}}\right)\right] . \tag{20}$$

Remember our criterion for uniqueness (9): the bound (20) is exactly the critical growth of the functional $\Phi_\delta(t)$ which is relevant for the uniqueness! In fact, the ratio that criterion (9) requires to be infinitesimal for $\delta \downarrow 0$, is now merely bounded. Still, we cannot conclude uniqueness with this information only.

It is at this point that we exploit the information that Db is an L^1 function, and not just a Radon measure. (Notice that all arguments carried out until now would work verbatim if we substitute $\|\cdot\|_{L^1}$ with the total variation norm $\|\cdot\|_{\mathcal{M}}$, i.e., for b being a BV velocity field). Up to a remainder in L^2, we can assume that Db not only *belongs to* L^1, but also that it is *small in* L^1. (The existence of such a decomposition is due to the equi-integrability of L^1 functions). This smallness allows to fullfill the criterion (9), while the residual part of the functional originated by the L^2 remainder can be treated with the arguments of §6. This allows to conclude uniqueness for $W^{1,1}$ velocity fields, but it is still far from giving any result for BV velocity fields: a measure does *not* allow a decomposition in a small L^1 part plus an L^2 remainder!

8. **Vector fields whose derivative is a singular integral of an L^1 function.** The strategy described in the previous section extends in a (technical but) natural way to the case in which the derivatives of the velocity field b can be expressed as

$$\partial_j b^i = \sum_k S_{ijk} g_{ijk} \,,$$

where $g_{ijk} \in L^1(\mathbb{R}^d)$ and every S_{ijk} is a *singular integral operator*. In more details, we assume that any of these operators can be expressed as a convolution

$$S_{ijk} g_{ijk} = K_{ijk} * g_{ijk} \,,$$

where the singular kernel K_{ijk} is smooth away of the origin of $\mathbb{R}^d$, is homogeneous of degree $-d$ and satisfies the usual cancellation property.

Observe that this class of vector fields includes $W^{1,1}$. However, it does neither include BV, nor it is included in BV. The relevance of this class of vector fields is due to their appearance in some physical problems: for instance, in two dimensional incompressible fluid dynamics, this is the regularity enjoyed by fluid velocities with L^1 vorticity.

It is well known (see again [11]) that singular integrals enjoy the same estimates as maximal functions: namely, strong estimates for $1 < p < \infty$

$$\|Sf\|_{L^p} \leq C_{d,p}\|f\|_{L^p}$$

(the case $p = \infty$ has now to be excluded), and the weak estimates for the case $p = 1$

$$|||Sf|||_{M^1} \leq C_{d,1}\|f\|_{L^1} \,.$$

Also in this case, no strong estimate for $p = 1$ is available.

One basic consequence of the cancellation property assumed for the singular kernels under consideration is the weak estimate for the composition of two singular integral operators. Namely, if we consider a composition $S = S_2 \circ S_1$, the associated singular kernel is given by the convolution $K = K_2 * K_1$, and it is again a singular kernel. Thus we still have

$$|||Sf|||_{M^1} = |||S_2 \circ S_1 f|||_{M^1} \leq C\|f\|_{L^1} \,. \tag{21}$$

Note carefully that estimate (21) *cannot* be obtained by composing the two analogue estimates (from L^1 to M^1) which hold for the two singular integral operators S_1 and S_2 separately. At a formal level, (21) requires *cancellations* in the convolutions.

9. **Back to the proof of the uniqueness.** We describe now how to modify the strategy described in §7 in order to prove uniqueness of the regular Lagrangian flow associated to vector fields with the regularity described in §8. This result is contained in [7].

Going back to (16), we realise that in the present context we have

$$\psi = MSg \,,$$

for $g \in L^1$. We thus need a bound of the type

$$|||\psi|||_{M^1} \leq C\|g\|_{L^1} \,, \tag{22}$$

in order to conclude the proof along the lines of §7. In general, however, estimate (22) does not hold if the classical maximal function (13) is considered. Inspired by the cancellation phenomenon which allows (21), we can prove that (22) holds if we consider instead a smooth version of the maximal function, defined as

$$M_\rho f(x) = \sup_{r>0} \left| \int_{\mathbb{R}^d} \rho_r(x-y) f(y)\, dy \right| \,,$$

where ρ is a given smooth convolution kernel. This smooth version of the maximal function is well known in the context of Hardy spaces, under the name of grand maximal function. It is possible to prove that

$$|||\psi|||_{M^1} = |||M_\rho S g|||_{M^1} \leq C\|g\|_{L^1} \,,$$

and this estimate is sufficient to conclude using the strategy in §7, yielding uniqueness of the regular Lagrangian flow for the class of vector fields considered in §8.

Acknowledgments. I thank the scientific committee of the conference HYP2012 for the invitation to give the talk that gave rise to this note. The topics presented here are part of very enjoyable collaborations with Camillo De Lellis and François Bouchut. I thank Anna Bohun and Laura Keller for some comments on a preliminary version of this note. I also acknowledge support by the Swiss National Science Foundation through the grant 140232

REFERENCES

[1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math., **158** (2004), 227–260.

[2] L. Ambrosio and G. Crippa, *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields*, in "Transport equations and multi-D hyperbolic conservation laws", Lect. Notes Unione Mat. Ital., **5** (2008), 1–41.

[3] L. Ambrosio, M. Lecumberry and S. Maniglia, *Lipschitz regularity and approximate differentiability of the DiPerna–Lions flow*, Rend. Sem. Mat. Univ. Padova, **114** (2005), 29–50.

[4] G. Crippa, *The ordinary differential equation with non-Lipschitz vector fields*, Boll. Unione Mat. Ital., **1** (2008), 333–348.

[5] G. Crippa, "The flow associated to weakly differentiable vector fields", Theses of Scuola Normale Superiore di Pisa (New Series), **12**, Edizioni della Normale, Pisa, 2009.

[6] F. Bouchut and G. Crippa, *Équations de transport à coefficient dont le gradient est donné par une intégrale singulière*, Sémin. Équ. Dériv. Partielles, Exp. No. I, École Polytech., Palaiseau, 2009.
[7] F. Bouchut and G. Crippa, *Lagrangian flows for vector fields with gradient given by a singular integral*, J. Hyperbolic Differ. Equ., **10** (2013), 235–282.
[8] G. Crippa and C. De Lellis, *Regularity and compactness for the DiPerna-Lions flow*, Hyperbolic problems: theory, numerics, applications, Springer, Berlin (2008), 423–430.
[9] G. Crippa and C. De Lellis, *Estimates and regularity results for the DiPerna-Lions flow*, J. Reine Angew. Math., **616** (2008), 15–46.
[10] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., **98** (1989), 511–547.
[11] E. Stein, "Singular integrals and differentiability properties of functions", Princeton University Press, 1970.

E-mail address: gianluca.crippa@unibas.ch