Spatial branching processes in random environment

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Dekan
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To my parents

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A little learning is a dangerous thing;
Drink deep, or taste not the Pierian spring:
There shallow draughts intoxicate the brain,
And drinking largely sobers us again.
Fired at first sight with what the Muse imparts,
In fearless youth we tempt the heights of Arts;
While from the bounded level of our mind
Short views we take, nor see the lengths behind,
But, more advanced, behold with strange surprise
New distant scenes of endless science rise!
So pleased at first the towering Alps we try,
Mount o'er the vales, and seem to tread the sky;
The eternal snows appear already past,
And the first clouds and mountains seem the last;
But those attained, we tremble to survey
The growing labours of the lengthened way;
The increasing prospect tires our wandering eyes,
Hills peep o'er hills, and Alps on Alps arise!

– Alexander Pope

Contents

Abstract

The aim of this thesis is the study of two different models of spatially extended branching systems.

First, we consider a one-dimensional branching Brownian motion that evolves in a spatially random environment. We argue that the quenched fluctuations of the maximally displaced particle re-centred at its median remain bounded in time. For the standard branching Brownian motion in a homogeneous environment, an analogous result already follows by the fact that the distribution function of the re-centred maximally displaced particle corresponds to the critical travelling wave solution of the related Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation. This argument, however, cannot be extended to the inhomogeneous setting. In order to achieve our result, we employ certain *tilted* path-measures in order to get fine control on the Feynman-Kac representations to solutions of the F-KPP equation, which we combine with an analytic result on the evolution of the number "zero-crossings" of solutions to parabolic equations, known as a *Sturmian principle*.

The second model we consider is a discrete-time model of branching annihilating random walk on $\mathbb{Z}^d$. In this model, at the end of each generation, all particles produce a mean μ number of offspring that disperse uniformly within a fixed distance R from their parent. Whenever two (or more) child particles try to occupy the same site they get annihilated. This local interaction of particles in the branching system has the interesting but also challenging consequence that high local density of particles leads to more annihilation, making the system *non-monotone*. We investigate and determine regimes of the model parameters for which the system either dies out almost surely or survives with positive probability. Moreover, we exhibit regimes where there is a unique non-trivial ergodic equilibrium distribution that has exponential decay of correlations. Lastly, by keeping track of genealogical information (i.e. parent-child relations), we examine the ancestral lineages of single particles drawn from an equilibrium population by interpreting them as random walks evolving in the dynamic random environment generated by the branching process. We exhibit a law of large numbers and a central limit theorem for this case.

Preface

This thesis is divided into two parts, each of which is dedicated to the study of a particular model of spatially extended branching process evolving in a random environment. These are stochastic processes, describing the evolution of populations of "abstract" objects that grow (and decay) by multiplying and replacing one another and for which the evolution depends additionally on some interaction with an inhomogeneous environment. Each part starts with an introduction aimed at giving a general overview of results related to the respective process and setting the stage for the subsequent chapters which are reproductions of research papers written during my doctoral programme. Since the reproduced papers are largely self-contained, it is inevitable that there is some overlap between the respective introductions and subsequent chapters.

In Part I we consider a one-dimensional branching Brownian motion evolving in a spatially random (branching) environment, i.e. a branching Brownian motion for which locally branching rates are given by an external environmental process. Chapter 2 is a reprint of the following article:

[ČDO22] Jiří Černý, Alexander Drewitz, and Pascal Oswald, *On the tightness of the maximum of branching brownian motion in random environment*, arXiv preprint arXiv.2212.12390, 2022.

In Part II we introduce a branching annihilating random walk (BARW), which evolves in discrete time-steps on $\mathbb{Z}^d$. This process does not interact with an external environment, but rather locally interacts with itself (through annihilation) and in this sense creates its own (random) environment with which it interacts. Chapters 4 and 5 are reproductions of the following two articles:

[BCČ+23] Matthias Birkner, Alice Callegaro, Jiří Černý, Nina Gantert, and Pascal Oswald, *Survival and complete convergence for a branching annihilating random walk*, arXiv preprint arXiv.2304.09127, 2023.

[Osw24] Pascal Oswald, *Ancestral lineages for a branching annihilating random walk*, in preparation, 2024.

Part I

Branching Brownian motion in random environment

1. Introduction

Branching processes form a class of stochastic processes with a long and rich history, dating back at least to the works of Bienyamé [Bie45] as well as Galton and Watson [WG75], whose original interest in such processes came from trying to understand the extinction of aristocratic surnames over the course of time. An interesting account of the early history of branching processes is found in [Ken75]. Since then, (spatial) branching processes have been used in order to describe more varied real-world phenomena, such as, e.g. the spread of diseases, genetic evolution, cosmic rays, particle cascades in nuclear and high-energy physics, to name a few, cf. [McK14, BP19, EEY19, Saw76, Tch17, ALM23, MM18, HKV20].

Next to their far-reaching applications in the natural sciences, branching processes are fascinating mathematical objects in their own right, which find widespread use in many areas of probability theory, such as in the study of random maps [ADS22], in various toy models of (mathematical) statistical physics, e.g. spin glasses [Bov17] or random polymers [Com17], and many more. There is also a deep connection to reaction-diffusion equations, [INW68a, McK75, Bra83, AHR22, ČD20]. In fact, we build heavily upon this connection in this part of the thesis.

In this part of the thesis, we consider a particular variant of the by now classical one-dimensional branching Brownian motion (BBM), cf. [Shi15, Bov17] for comprehensive introductions, which we let evolve with spatially dependent inhomogeneous branching rates. We focus on the particular case, where the environment is given by a stochastic process and is in particular itself random. The resulting branching process is known as the *branching Brownian motion in random environment* (BBMRE). Heuristically, we can describe this process as follows. At time $t = 0$ a single particle is placed at the origin, which, as time starts, evolves as a Brownian motion. To this particle there is attached a random clock (which rings at a rate that depends on the environment the particle has explored thus far). After the clock rings, the initial particle gets replaced by k particles, according to some offspring distribution $(p_k)_{k\in\mathbb{N}}$, each of which carries its own random clock and evolves independently of one another according to the same stochastic dynamics as its parent. We will mainly be interested in the behaviour of the maximally displaced, i.e. right-most, particle in such a process.

After giving details on how to construct branching processes such as the BBMRE

in Section 1.1, we give a brief overview of results relating to the right-most particle in a homogeneous BBM in Section 1.2.1. This helps us put the differences that arise due to the random environment into better perspective and sets the stage for Section 1.2.2, where we consider the right-most particle in a BBMRE.

1.1 Construction of spatial branching processes

We start by giving a formal description of how to construct spatial branching processes evolving in an external landscape. The setup we work with is somewhat abstract but allows us to construct many different models of spatial branching processes both in homogeneous and inhomogeneous environments.

Any minimal description of a spatial branching process should record the location of all particles alive at all times. A natural way to record this information is by representing each particle alive at a given time by a Dirac mass at the particle's position. This approach comes, however, with the drawback that the genealogical structure between particles is lost. Therefore, we choose a slightly different approach that lets us retain genealogical information by encoding it directly into the construction.

The genealogical information is encoded into a *locally finite rooted plane tree* using the Ulam–Harris–Neveu notation. I.e. we associate each node in a locally finite rooted plane tree with a unique label from the set of finite sequences

$$\mathcal{V} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

with the convention that $\mathbb{N}^0 = \{\emptyset\}$. The labelling is carried out as follows. The root of the tree, which represents the initial ancestor, is labelled by $\emptyset$ (the sequence of length 0). The k children of the initial particle are then labelled by length 1 sequences $(1), \dots, (k)$ according to their ordering from left to right. The rest of the tree is labelled recursively, such that $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{V}$ represents the ν_n-th child of the ν_{n-1}-th child of the $\dots$ of the ν_2-nd child of the ν_1-st child of the root, where all children are ordered again from left to right. For $\nu = (\nu_1, \dots, \nu_n)$, we write $|\nu| = n$ for the length of the finite sequence, representing the generation of the node labelled by ν. Further, we introduce the *parent-map* $p : \mathcal{V} \setminus \{\emptyset\} \to \mathcal{V}$ which maps a node to it's parent node and is defined by $p(\nu_1, \dots, \nu_n) = (\nu_1, \dots, \nu_{n-1})$ for $n \geq 1$.

A *locally finite rooted plane tree* can then be seen as a subset $\tau \subseteq \mathcal{V}$ such that

- $\emptyset \in \tau$
- for all $\nu \in \tau \setminus \{\emptyset\}$ it holds that $p(\nu) \in \tau$
- if $(\nu_1, \dots, \nu_n) \in \tau$ and $(\nu_1, \dots, \nu_n, k) \in \tau$ for some $k > 1$, then $(\nu_1, \dots, \nu_n, j) \in \tau$ for all $1 \leq j \leq k$.

We write $\mathfrak{T}$ for the collection of all *locally finite rooted plane trees*. The elements of $\mathfrak{T}$ represent the possible genealogies of a spatial branching process, where for any $\tau \in \mathfrak{T}$, the n-th generation of τ is given by all particles with labels in the set $\{\nu \in \tau : |\nu| = n\}$. Moreover, we say that ν' is an *ancestor* of ν and write $\nu' \prec \nu$ if there exists $m \in \mathbb{N}$ such that $\nu' = p^{(m)}(\nu)$, i.e. iteratively applying the parental map m-times to ν gives ν'.

In the following, we are interested in the case where any individual can have a random number of offspring, leading to random (genealogical) trees. To encode a random offspring distribution $(p_k)_{k\in\mathbb{N}}$ into a tree, we introduce a collection of independent $\mathbb{N}_0$-valued random variables $(O_\nu)_{\nu\in\mathcal{V}}$ on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}})$, such that O_ν represents the number of children of $\nu \in \mathcal{V}$ and is distributed as $(p_k)_{k\in\mathbb{N}_0}$. The tree with offspring distribution $(p_k)_{k\in\mathbb{N}_0}$ is then the $\mathfrak{T}$-valued random variable

$$\{\nu = (\nu_1, \dots, \nu_n) \in \mathcal{V} : \nu_j \leq O_{(\nu_1,\dots,\nu_{j-1})}, 1 \leq j \leq n\}. \tag{1.1.1}$$

These trees are the well-known *Bienyamé-Galton-Watson* trees.

In order to add spatial structure, we enrich the trees $\tau \in \mathfrak{T}$ with additional information. To each $\nu \in \tau$, we associate two *marks*: a life-time $\sigma_\nu \geq 0$ and a map $Z_\nu : [0, \infty) \to \mathbb{R}$, which describes a particle's trajectory throughout its life-time. Formally, such *marked trees* are triplets

$$(\tau, \sigma, Z) = (\tau, (\sigma_\nu)_{\nu\in\tau}, (Z_\nu)_{\nu\in\tau}). \tag{1.1.2}$$

Note that the life-times σ_ν also let us define corresponding *birth-times* b_ν and *death-times* (resp. *branching times*) d_ν by summing up the life-times of ancestors, i.e. for $\nu \in \tau$, we set $b_\nu := \sum_{\nu'\prec\nu} \sigma_{\nu'}$ and $d_\nu := b_\nu + \sigma_\nu$. We set $N(t) = \{\nu \in \tau : b_\nu \leq t < d_\nu\} \subseteq \mathcal{V}$ to be the set of particles that are alive at time $t \geq 0$. For each $\nu \in N(t)$ we moreover inductively define the position of particle ν at time t to be

$$X^\nu(t) := X^\nu_t := Z_\nu(t - b_\nu) + X^{p(\nu)}(d_{p(\nu)}). \tag{1.1.3}$$

We extend this notation to include the ancestors of ν as well. For $\nu' \in N(s)$ for some $s < t$ and such that $\nu' \prec \nu$ we set $X^\nu(s) := X^{\nu'}(s)$.

This construction lets us now define various spatial branching processes by considering different distributions on the space of all marked trees. For any specific choice of $(p_k)_{k\in\mathbb{N}}$, $(\sigma_\nu)_{\nu\in\mathcal{V}}$ and $(Z_\nu)_{\nu\in\mathcal{V}}$, we denote by $\mathcal{X}(t) := (X^\nu_t, \nu \in N(t))$ the resulting *spatial branching process* and by $\mathtt{P}_x$ its distribution when started in $x \in \mathbb{R}$, i.e. such that $\mathtt{P}_x$-a.s. it holds that $\mathcal{X}(0) = x$.

We obtain the standard branching Brownian motion (BBM), with offspring distribution $(p_k)_{k\in\mathbb{N}_0}$ by choosing life-time marks σ_ν to be i.i.d. exp(1) random variables and trajectory marks Z_ν to be independent Brownian motions, where we assume with out loss of generality that the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is rich enough so that all these random objects can be defined on it.

Let us extend this to BBM in an inhomogeneous *branching* environment ξ, where $\xi : \mathbb{R} \to \mathbb{R}$ is given by a non-negative bounded continuous function. Again, we take the trajectory marks Z_ν to be independent Brownian motions. The dependence on the environment is encoded into the life-time marks. To this end, we consider yet another collection of random variables $(S_\nu)_\nu$, defined on $(\widehat{\Omega}, \widehat{\mathcal{F}})$, which are i.i.d. exponentially distributed with mean 1 and set the life-times for particles evolving in the inhomogeneous environment ξ to be defined, using (1.1.3) recursively, by

$$\begin{aligned} \sigma_\nu^\xi &:= \inf\left\{ t > 0 : \int_0^t \xi\big(Z_\nu(t - b_\nu) + X^{p(\nu)}(d_{p(\nu)})\big)\,\mathrm{d}s \geq S_\nu \right\} \\ &= \inf\left\{ t > 0 : \int_0^t \xi(X_s^\nu)\,\mathrm{d}s \geq S_\nu \right\}, \quad \nu \in \mathcal{V}. \end{aligned} \tag{1.1.4}$$

Note that for any $t \geq 0$, the probability that the initial particle $\emptyset$ lives longer than t is given by $\mathtt{P}_x(\sigma_\emptyset^\xi > t) = \mathtt{E}_x[e^{-\int_0^t \xi(X_s^\emptyset)\,\mathrm{d}s}]$. Similar formulas can be derived for all other $\nu \in \mathcal{V}$. Choosing the life-time marks as in (1.1.4) and Brownian trajectory marks then defines a BBM in the inhomogeneous environment ξ. We reflect the environmental dependence of this process by writing $\mathtt{P}_x^\xi$ and $\mathtt{E}_x^\xi$ for its law and expectation when started in $x \in \mathbb{R}$.

The next step is to take ξ to be random, i.e. to be a stochastic process, which is almost surely a non-negative bounded continuous function. To do this, we use the canonical construction of ξ, i.e. we consider a second probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the class of non-negative bounded continuous functions, and let $\xi : \Omega \to \mathbb{R}$ be defined by $\xi(x)(\omega) := \omega(x)$ for $x \in \mathbb{R}$ and $\omega \in \Omega$. Conditioned on such ξ, we would like to proceed as for deterministic environments and define $(\mathcal{X}(t))_{t\geq 0}$ with "law $\mathtt{P}_x^\xi$". This calls, however, for some care, as a priori, it is not clear whether the resulting (random) measures $\mathtt{P}_x^\xi$ exist (i.e. are in fact probability measures). Fortunately, it was shown in [Sav72], using the general framework of [INW68a, INW68b, INW69], that a regular conditional probability of $(\mathcal{X}(t))_{t\geq 0}$ given ξ exists. We denote this regular conditional expectation again by $\mathtt{P}_x^\xi$ in order to keep in line with the (deterministic) inhomogeneous setting. The corresponding conditional expectation is denoted by $\mathtt{E}_x^\xi$.

The random environment

Let us now comment on the random environments that we want to allow in more detail and give two prototypical examples. Throughout Chapters 1 and 2 we make, in particular, the following assumptions. Firstly, we assume that the environment ξ is a stationary ergodic process. This is in effect so that there is "enough randomness" in the environment for certain averaging effects to come into play. In fact, in Section 2.2, we replace ergodicity with a mixing condition, which, however, together with stationary implies ergodicity. Secondly, we also assume that the process is regular enough for

certain PDEs related to the process to have classical solutions, cf. Section 1.2.2 below. It suffices for our purposes to assume that ξ has $\mathbb{P}$-a.s. locally Hölder continuous sample paths, cf. (2.2.3) below for details.

Lastly, we assume that the environment is $\mathbb{P}$-a.s. uniformly bounded away from 0 and infinity. We do this to avoid the non-existence of asymptotic speed. Indeed, it follows from [LT98, Remark 5.4] that if the environment is not almost surely finite, then the (asymptotic) speed with which the process spreads throughout space is unbounded. For convenience, we introduce the two constants

$$0 < \mathtt{ei} := \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \mathtt{es} < \infty. \tag{1.1.5}$$

Note that due to the stationarity of ξ, the constants $\mathtt{ei}$ and $\mathtt{es}$ uniformly bound $\xi(x)$ over all $x \in \mathbb{R}$ for $\mathbb{P}$-a.e. realisation of ξ.

These conditions can also be found in the articles [ČDS23, DS22, HRS23] which are related to this thesis.

Example ([DS22, Example 1.1]). It is informative to have examples of stochastic processes ξ satisfying these conditions in mind.

(i) Let $(Y_x)_{x\in\mathbb{R}}$ be an Ornstein-Uhlenbeck process. It is well known that these are the only stationary Markov processes that are also Gaussian processes (i.e. such that all finite-dimensional distributions are multivariate Gaussians). For such Y we consider the environment

$$\xi(x) = (\mathtt{ei} \vee Y_x) \wedge \mathtt{es}, \quad x \in \mathbb{R}. \tag{1.1.6}$$

This environment inherits the stationarity from Y and also clearly satisfies (1.1.5). Moreover, for any $\gamma \in (0, 1/2)$, the local γ-Hölder continuity of $(Y_x)_{x\in\mathbb{R}}$ follows from the fact that we can write $Y_x = e^{-x} B_{e^{2x}}$, for $x \in \mathbb{R}$, where $(B_t)_{t\geq 0}$ is a Brownian motion. Establishing the ergodicity of (1.1.6) is a bit more involved, and the details are omitted. The idea is to use that $(Y_x)_{x\in\mathbb{R}}$ can be characterised as the solution of the SDE

$$\mathrm{d}Y_x = -Y_x \,\mathrm{d}x + \sqrt{2}\,\mathrm{d}B_x.$$

One can then use the fact that, when started from a single point, the Ornstein-Uhlenbeck process converges to its stationary distribution *sufficiently* fast and the Markov property to conclude.

(ii) Let $\chi : [0, \infty) \to [0, 1]$ be any Hölder-continuous non-increasing function with $\chi(x) = 1$ for $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$. Further, let $\omega = (\omega^i)_{i\in\mathbb{Z}}$ be a homogeneous Poisson point process on $\mathbb{R}$ with intensity one. We then set

$$\xi(x) = \mathtt{ei} + (\mathtt{es} - \mathtt{ei}) \cdot \sup\{\chi(|x - \omega^i|) : i \in \mathbb{Z}\}. \tag{1.1.7}$$

By construction, this environment is $\mathbb{P}$-a.s. locally Hölder-continuous and ξ is bounded from below by `ei` and from above by `es`. Moreover, for fixed $x \in \mathbb{R}$, it holds that $\xi(x) = \texttt{ei}$ whenever $|x - \omega^i| > 2$ for all $i \in \mathbb{Z}$ and $\xi(x) = \texttt{es}$ if there exists $i \in \mathbb{Z}$ such that $|x - \omega^i| \leq 1$. Stationary and ergodicity for this environment follow by properties of the Poisson point process.

1.2 Extreme values in spatial branching processes

A main objective of this part of the thesis is to study the maximally displaced particle in a one-dimensional BBMRE. We denote the maximal displacement at time $t > 0$ by

$$M(t) := \sup_{\nu \in N(t)} X^\nu(t), \tag{1.2.1}$$

where $\mathcal{X}(t) = (X^\nu(t))_{\nu \in N(t)}$ is a BBMRE, as constructed in Section 1.1. A direct consequence of the construction is that for any $t > 0$ such that there are at least two particles alive, the collection $(X_t^\nu)_{\nu \in N(t)}$ comprises a collection of correlated random variables for which the covariance structure is given by the *genealogical distance* between the particles. More precisely, for any two particles $\nu_1, \nu_2 \in N(t)$ one has

$$\mathrm{E}_x^\xi\big[X^{\nu_1}(t)X^{\nu_2}(t)\big] = \inf\{s \geq 0 : X^{\nu_1}(s) \neq X^{\nu_2}(s)\}. \tag{1.2.2}$$

The right-hand side of (1.2.2) corresponds to the span of time that the genealogy of ν_1 and ν_2 spent together before the death of their most recent common ancestor.

Moreover, it follows immediately from (1.2.2) that both for homogeneous and non-homogeneous environments, questions about the extrema of $\{X_t^\nu : \nu \in N(t)\}$ are not covered by the classical theory of extreme value statistics and that these questions need to be handled with care.

Let us now ask a few natural questions that arise about the maximal displacement.

(Q1) What is the first (and second) leading order of the maximally displaced particle at time $t \gg 1$?

(Q2) How strong are the fluctuations of the maximally displaced particle?

(Q3) Is there a meaningful way to describe the limiting distribution of the maximal particle after subtracting its leading orders?

In the rest of this chapter, we discuss these questions and see that they are connected to questions on wave-front propagation for certain non-linear parabolic PDEs. We start off by first considering the case of a homogeneous rate-one environment (i.e. $\xi \equiv 1$). This does not only allow us to introduce many important objects but also serves to highlight differences in the approaches needed in the presence of (random) inhomogeneities.

1.2.1 Maximal particle in homogeneous BBM

The study of homogeneous BBM has a long history that dates back at least to 1937, when question (Q1) was addressed independently in [KPP37] and [Fis37] by analytic means. The connection of these results to BBM was established in the 1960's and 1970's by [Sko64, INW68a, INW68b, INW69, McK75]. All of the questions (Q1)–(Q3) have since been studied extensively in this case and can be answered in quite some detail.

A useful duality for homogeneous BBM

The Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation is a semi-linear parabolic equation of the following form

$$\partial_t w = \frac{1}{2}\partial_x^2 w + F(w), \qquad t > 0, x \in \mathbb{R}, \tag{1.2.3}$$

where the non-linearity $F : [0,1] \to [0,1]$ is a continuously differentiable function that satisfies the *KPP-conditions*

(KPP1) $F(0) = F(1) = 0$

(KPP2) $F(w) > 0$ for $0 < w < 1$

(KPP3) $F'(w) < F'(0)$ for $0 < w \leq 1$.

Equations with such non-linearities were considered by Kolmogorov, Petrovskii and Piskunov [KPP37] in order to study the invasion of a stable state into an unstable state. Independently, Fisher [Fis37], who was interested in the spread of genetic mutations within biological populations living in a one-dimensional habitat, argued that in the infinite density limit, the local proportion of the population carrying a mutation at position x at time t solves a differential equation of type (1.2.3). With this initial motivation in mind, the meaning of the KPP-conditions becomes clear. Condition (KPP1) tells us that both $w \equiv 1$ and $w \equiv 0$ are solutions to the equation and (KPP2)–(KPP3) tell us that the state $w = 1$ is stable and $w = 0$ is unstable.

Remark 1.2.1. Note that both in [Fis37] and [KPP37] the solutions to (1.2.3) represent certain densities and in both contexts it is natural to assume that the stable state is represented by $w = 1$. By replacing the non-linearity F in (1.2.3) with $\widetilde{F}(w) := -F(1-w)$, one gets an analogous equation representing the case where $w = 0$ is stable and invades the unstable state $w = 1$. Indeed, while $\widetilde{F}$ satisfies (KPP1), conditions (KPP2) and (KPP3) are clearly not met, instead, it holds that $\widetilde{F}(w) < 0$ for $0 < w < 1$ and $\widetilde{F}'(w) < \widetilde{F}'(1)$ for $0 < w \leq 1$. Moreover, it is also clear that $1 - w$ solves the analogue of (1.2.3) with the non-linearity $\widetilde{F}$.

Let us now describe how (1.2.3) is connected to BBM. To this end, let $(X_t^\nu, \nu \in N(t))_{t\geq 0}$ be a BBM with homogeneous environment $\xi \equiv 1$, as constructed in Section 1.1, and whose initial particle is situated at the origin. Moreover, assume that the offspring distribution $(p_k)_{k\in\mathbb{N}_0}$ of the BBM is super-critical, has finite variance and $p_0 = 0$ in order to exclude immediate extinction. For measurable functions $g : \mathbb{R} \to [0,1]$, we then examine the following functionals,

$$v(t,x) := \mathtt{E}_0\Big[\prod_{\nu\in N(t)} g(x + X_t^\nu)\Big], \quad (t,x) \in [0,\infty)\times\mathbb{R}. \tag{1.2.4}$$

We can use the branching structure of $(X_t^\nu)_{\nu\in N(t)}$ to obtain a *renewal* equation for v. Conditioning on the lifetime $\sigma_\emptyset > 0$ of the original ancestor, we arrive immediately at the equation,

$$v(t,x) = \mathtt{E}_0\big[g\big(x+X_t^\emptyset\big)\big]\mathtt{P}_0(\sigma_\emptyset > t) + \sum_{k=1}^\infty p_k \int_0^t \mathtt{E}_0\big[v\big(t-s, x+X_s^\emptyset\big)^k\big]\mathtt{P}_0(\sigma_\emptyset \in \mathrm{d}s), \tag{1.2.5}$$

where we recall that $X_t^\emptyset$ is the position of the original particle at time t. By taking derivatives and comparing terms in (1.2.5), one obtains that

$$w(t,x) := 1 - v(t,x), \tag{1.2.6}$$

solves (1.2.3) with initial condition $w(0,x) = 1 - g(x)$ and non-linearity

$$F(w) := (1-w) - \sum_{k=1}^\infty p_k(1-w)^k, \tag{1.2.7}$$

cf. for example [Bov17, Lemma 5.5]. One immediately checks that F in (1.2.7) satisfies (KPP1)–(KPP3). By the same derivation or by Remark 1.2.1 it also follows $v(t,x)$ solves (1.2.3) with non-linearity $-F(1-v) = \sum_{k=1}^\infty p_k v^k - v$. For convenience, we write $f(x) := 1-g(x)$, so that we consider the F-KPP equation (1.2.3) with initial condition $f(x)$.

A particularly useful choice of function f in order to study the extremal particles in a BBM are Heaviside-type indicator functions (i.e. indicator functions on half-lines). Taking $f(x) = \mathbb{1}_{(-\infty,0]}(x)$ (i.e. $g(x) = \mathbb{1}_{[0,\infty)}(x)$) yields immediately by (1.2.4),(1.2.6) and the reflection symmetry of Brownian motion that

$$w(t,x) = \mathtt{P}_0\big(\min_{\nu\in N(t)} X_t^\nu \leq -x\big) = \mathtt{P}_0\big(M(t) \geq x\big). \tag{1.2.8}$$

Similarly, taking instead, $f(x) = \mathbb{1}_{[0,\infty)}(x)$ (i.e. $g(x) = \mathbb{1}_{(-\infty,0]}(x)$) gives

$$w(t,x) = \mathtt{P}_0\big(M(t) \geq -x\big) = \mathtt{P}_0\big(\min_{\nu\in N(t)} X_t^\nu \leq x\big). \tag{1.2.9}$$

The upshot of (1.2.8) and (1.2.9) is that switching the "orientation" of the Heaviside-like initial conditions f, in the F-KPP equation (1.2.3), corresponds to switching between the maximally or minimally displaced particle in the probabilistic representation of its solutions. Note also that by Remark 1.2.1 the counter-probabilities of the right-hand sides of (1.2.8) and (1.2.9) solve the F-KPP equation with non-linearity $-F(1-v)$ and initial condition $v(0,x) = g(x)$.

Travelling wave solutions to the F-KPP equation

An important consequence of the duality (1.2.8) is that we can use analytical tools and the F-KPP equation with Heaviside-type initial conditions in order to study statistical properties of the maximal (or equivalently minimal) displacement of BBM.

The first major result on the analytic side is due to Kolmogorov, Petrovskii and Piskunov. In [KPP37] it is shown that for any solution w of (1.2.3) with Heaviside initial conditions (as in (1.2.8) and (1.2.9)) there exists a map $m : [0,\infty) \to [0,\infty)$ satisfying $\lim_{t\to\infty} m(t)/t = \sqrt{2F'(0)}$ such that the solution of (1.2.3) re-centred at $m(t)$ converges uniformly on compact sets to some limiting shape, as $t \to \infty$. More precisely, there exists some limiting function $g : \mathbb{R} \to [0,1]$ such that

$$w(t, x \pm m(t)) \to g(x), \quad \text{uniformly in } x \in \mathbb{R}, \text{ as } t \to \infty, \tag{1.2.10}$$

where the "$\pm$" takes into account the "orientation" of the initial condition, i.e. whether we are in the situation of (1.2.8) or (1.2.9). In either case, we interpret g as the limiting shape of the *invasion front* of the stable state into the unstable state. It has the same "orientation" as the initial condition, is unique up to translations and is characterised as the solution of the equation

$$\frac{1}{2}g'' \pm \sqrt{2F'(0)}g' + F(g) = 0, \quad \begin{cases} +, & \text{if } \lim_{x\to-\infty} g(x) = 1, \lim_{x\to+\infty} g(x) = 0, \\ -, & \text{if } \lim_{x\to-\infty} g(x) = 0, \lim_{x\to+\infty} g(x) = 0. \end{cases} \tag{1.2.11}$$

Solutions of (1.2.11) are called *travelling wave solutions* of the F-KPP equation, with speed $\sqrt{2F'(0)}$, since the function $g(x \mp \sqrt{2F'(0)}t)$ solves (1.2.3), as is verified by direct calculation.

It is natural to ask whether $\sqrt{2F'(0)}$ is the only possible speed for travelling waves. To this end, in analogy to (1.2.11), we let $\lambda > 0$ and consider solutions g_λ of

$$\frac{1}{2}g_\lambda'' \pm \lambda g_\lambda' + F(g_\lambda) = 0. \tag{1.2.12}$$

Direct computation yields that $g_\lambda(x \mp \lambda t)$ solves (1.2.3) for any $\lambda > 0$, but a priori it is not clear whether the shape of g_λ is wave-like for all $\lambda > 0$. In fact, g_λ only has a wave-shape for $\lambda \geq \sqrt{2F'(0)}$. Indeed, consider the phase variables $q := g_\lambda$ and $p := q'$.

With these, solutions of (1.2.12) can be represented in the phase plane by the system of ODEs

$$\begin{aligned} q' &= p, \\ p' &= -2F(q) \mp 2\lambda p. \end{aligned} \tag{1.2.13}$$

The system (1.2.13) has two fixed points, one at $(0,0)$ and another at $(1,0)$. One of these fixed points is a saddle point, while the other is a stable node. Which is which depends on the sign in (1.2.13), i.e. on the "orientation" of the initial condition of the F-KPP equation. To discover the behaviour of (q,p) near the stable node, we compute the eigenvalues for the fixed points. These are given by $\mp\lambda + \sqrt{\lambda^2 - 2F'(0)}$ and $\mp\lambda - \sqrt{\lambda^2 - 2F'(0)}$, which are imaginary for $\lambda \in (0, \sqrt{2F'(0)})$ and real for $\lambda \geq \sqrt{2F'(0)}$. In the first case, the solution spirals around the stable fixed point, but not in the latter. Consequently, the functions $g_\lambda(x \mp \lambda t)$ are only travelling wave solutions of (1.2.3) for $\lambda \geq \sqrt{2F'(0)}$.

Remark 1.2.2. (i) If instead of $\xi \equiv 1$ we set $\xi \equiv b > 0$, i.e. took a BBM with an arbitrary finite constant branching rate, we would have to replace $F'(0)$ with $bF'(0)$ everywhere in the above discussion. This is elaborated on in Remark 2.3.2 of Chapter 2 below.

(ii) From the analytic perspective, it is natural to consider more general initial conditions (other than Heaviside) and ask whether one still has convergence, as in (1.2.10). This remained unanswered for quite some time after [KPP37], being partially solved in [Kam76, Kan60, Uch78] and fully worked out by Bramson in [Bra83, Theorem A] (literally only for binary BBM, but the generalisations carry over directly) where exact growth conditions of $w(0,x)$ are given that characterise, for any $\lambda \geq \sqrt{2F'(0)}$ the existence of a function $m_\lambda(t)$ such that $w(t, x + m_\lambda(t)) \to g_\lambda(x)$ uniformly in x as $t \to \infty$ (including conditions for convergence towards the critical wave). The upshot of Bramson's characterisation is that the *heavier* the tail of the initial condition, the *faster* the asymptotic propagation speed of the corresponding travelling wave is.

(iii) The original arguments used in the proof of (1.2.10) in [KPP37] are purely analytical, but the convergence result was famously reproved by McKean in [McK75] using the duality (1.2.8) and probabilistic arguments. This interplay between analytic and probabilistic arguments foreshadows a general trend that has since established itself in the study of front propagation as well as BBM.

A better understanding of the leading orders of the re-centering term $m(t)$, for Heaviside-type initial conditions, is also due to Bramson, who showed in [Bra78, Bra83] that

$$m(t) = \sqrt{2F'(0)}t \mp \frac{3}{2\sqrt{2F'(0)}} \log t + O(1), \quad \text{as } t \to \infty. \tag{1.2.14}$$

Interestingly, Bramson's analysis of $m(t)$ is based largely on the application of probabilistic arguments, using representations of solutions to (1.2.3) and solutions of its linearisation (i.e. the equation where $F(w)$ is replaced by $F'(0)w$), near the unstable state, as averages of weighted Brownian paths. These representations are known as *Feynman-Kac* representations and are also heavily featured in Chapter 2. Bramson's result has since been refined and there exist more precise descriptions of the re-centering term $m(t)$, see e.g. [Gra19] for corrections up to the sixth order.

Remark 1.2.3. While the 3/2 in the logarithmic *Bramson correction*, (1.2.14), might seem insignificant, it is conjectured to be universal for many fields with log-correlated structure, such as the two-dimensional discrete Gaussian free field, the logarithm of the characteristic polynomial of random unitary matrices, or the Riemann zeta function in a random interval of length one on the critical line, e.g. [BK22, Zei16, CMN18, ABB$^+$19]. Moreover, the BBM and the 3/2 logarithmic correction correspond to the critical case in the *continuous random energy model* (CREM), which is concerned with the maximum of a collection of random variables with a correlation structure given by a continuous function of the genealogical distance cf. also (1.2.2). In the setting of CREM, the BBM divides a weak correlation regime (with logarithmic second order correction with prefactor 1/2) from a strongly correlated regime (with $O(t^{1/3})$ second order corrections), see e.g. [BK04, Bov17, Har16, MZ16].

Maximally displaced particle in BBM

Let us now come back to the extreme values of BBM and the questions (Q1)–(Q3) we asked at the beginning of this section. A number of results for the extremal particles can be directly read off of the convergence result (1.2.10). We concentrate here only on the behaviour of the maximally displaced particle $M(t)$, corresponding to (1.2.8) and initial condition $f(x) = \mathbb{1}_{(-\infty,0]}(x)$. Analogous results (with the occasional change of sign) hold for the minimally displaced particle. Firstly, we note that the duality (1.2.8) together with the convergence (1.2.10) directly imply

- For any $\varepsilon \in (0,1)$ the quantiles $m_\varepsilon(t) := \inf\{x : \mathtt{P}_0(M(t) \geq x) \leq \varepsilon\}$ can be used as the re-centering term $m(t)$
- The re-centred maximally displaced particle $M(t) - m(t)$ is tight for $t \geq 0$ under $\mathtt{P}_0$.

Combining these two items with Bramson's description of the re-centering term, cf. (1.2.14), then directly yields that, in $\mathtt{P}_0$-probability, the maximally displaced particle $M(t)$ has the expansion

$$M(t) = \sqrt{2F'(0)}t - \frac{3}{2\sqrt{2F'(0)}} \log t + O(1), \quad \text{as } t \to \infty. \tag{1.2.15}$$

The expansion (1.2.15) was shown to hold $\mathtt{P}_0$-a.s. in [HS09]. This in turn implies a strong law of large numbers for $M(t)$, with asymptotic speed $\sqrt{2F'(0)}$, answering questions (Q1)–(Q2).

Question (Q3) is related to the wave-shape g. While Bramson [Bra78, Bra83] analyses the behaviour of the limiting distribution function g, he was not able to give an explicit description of the function in probabilistic terms. Such a description was provided by Lalley and Selke in [LS87], who, based on ideas from [McK75], show that the limiting shape g from (1.2.10) has a representation as

$$g(x) = 1 - \mathtt{E}_0\big[\exp\big\{-cDe^{-\sqrt{2F'(0)}x}\big\}\big], \tag{1.2.16}$$

where $c > 0$ is some constant and $D > 0$ is a random variable whose distribution depends on details of the branching mechanism (it is the a.s. positive limit of the so-called *derivative martingale*). We can interpret the representation (1.2.16) as meaning that, due to the fluctuations of the initial few particles, $M(t) - m(t)$ builds up a delay of $\sqrt{2F'(0)}\log(cD)$, such that after enough time has passed, $M(t)$ fluctuates around $m(t) + \sqrt{2F'(0)}\log(cD)$ with Gumbel fluctuations.

Let us mention briefly that there have been other powerful techniques that have been developed from a purely probabilistic point of view and used in order to study BBM and its extremal particles without the aid of the F-KPP equation. The most prominent of these are *spine-techniques* and *many-to-few lemmas* which reduce questions on the k-th moments of BBM to questions on k dependent Brownian motions, by choosing k special particles (*spines*) in the branching process and using this "additional structure" to construct and understand changes of measures on the branching system, cf. [HH06, HR17, RS20] and references therein. These techniques do not make direct use of the F-KPP equation, however, by virtue of the duality (1.2.6), the many-to-one lemma corresponds to the *Feynman-Kac* representation of the solution to the F-KPP equation.

Building both on probabilistic and analytic literature, questions a lot more refined than (Q1)–(Q3) have been considered and answered for BBM.

(Q4) Does there exist a description of the extremal process, i.e. of all particles close to the extremal particle?

(Q5) Can one get quantitative estimates on level sets?

We do not go into detail about these questions but only provide some references. A full understanding of the statistics of the largest particles in terms of the extremal process was derived independently from each other in [ABBS13] and [ABK11, ABK13], where it was shown, by studying the genealogies of extremal particles, that the extremal point process of BBM converges in distribution to a *randomly shifted decorated Poisson point process* (SDPPP). See also [SZ15] for an overview. For results relating to (Q5), see e.g. [CHL19, CHL21].

Remark 1.2.4. We only consider the one-dimensional case, however, many of the above results have higher-dimensional analogues. The front location for the d-dimensional F-KPP equation was first considered in [AW78, Gär82]. The influence of the dimension on the *Bramson correction*, cf. (1.2.14), is investigated in [Mal15b], where it is shown that there is an additional dimension-dependent geometric term to the logarithmic correction. Recently, in [KLZ23], the "tidal wave" picture of Lalley and Selke was carried over to higher dimensions. Moreover, the full description of the extremal process as a SDPPP is derived in [BKL$^+$22]. See also [KZ24] for a recent result on the shape of the front of multidimensional BBM.

1.2.2 Maximal particle of BBM in random environment

Let us return to the case that we are mainly interested in in this part of the thesis and consider how the presence of an external (branching) environment influences the behaviour of extremal particles.

There is a large literature on spatial branching processes and their extrema in *non-homogeneous* environments, which focuses mostly on the cases of evolution in a deterministic environment, e.g. [LS88, LS89, BBH$^+$15, BN22], or a random time-inhomogeneous environment, e.g. [FZ12a, FZ12b, MZ16, Mal15a, BH14, BH15, BN22].

Recall that we are interested in the case of spatially random branching rates, where the environment is given by a random stationary ergodic process that is locally Hölder continuous and bounded away from zero and infinity, cf. (1.1.6) and (1.1.7) for two prototypical examples. The case of spatially random branching environments has received less attention until fairly recently, cf. [CP07, ČD20, Kri21, Kri22, DS22, HRS23, ČDS23].

A useful duality revisited

A natural approach to addressing the questions (Q1)–(Q3) for inhomogeneous environments is to try to adapt the strategy that was successful for the homogeneous case. To this end, we re-examine the duality relationship of (1.2.4) and (1.2.6). Using a similar *renewal* description as in (1.2.5) we find that for suitable measurable functions $f : \mathbb{R} \to \mathbb{R}$ the functionals

$$w(t,x) = 1 - \mathrm{E}_x^{\xi}\Big[\prod_{\nu\in N(t)} \big(1 - f(X_t^\nu)\big)\Big], \tag{1.2.17}$$

solve the (randomised) F-KPP equation

$$\begin{aligned} \partial_t w(t,x) &= \frac{1}{2}\partial_x^2 w(t,x) + \xi(x)F(w(t,x)), && t>0, x\in\mathbb{R}, \\ w(0,x) &= f(x), && x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

where the non-linearity $F : [0,1] \to [0,1]$ is as in (1.2.7). We refer to [DS22, Proposition 2.1] for a detailed derivation. For convenience, we always refer to the *random* equation (F-KPP) simply as *the* F-KPP equation. As in Remark 1.2.1, it holds again that $v = 1 - w$ solves (F-KPP) with $F(w)$ replaced by $\widetilde{F}(v) = -F(1-v)$. The $\mathbb{P}$-almost sure existence of continuous solutions (for $t > 0$) to (F-KPP) is guaranteed by [Fre85, Theorem 7.4.1] for measurable, non-negative bounded initial conditions.

Due to the inhomogeneity in the environment, the expectations in (1.2.17) are for BBMRE started at x. Since the trajectories X_t^ν are not translationally invariant, we can therefore not relate these expectations directly to expectations for a BBMRE starting at the origin (unlike in the homogeneous case, cf. (1.2.4)). Consequently, while we can again relate solutions to (F-KPP), with Heaviside-like initial conditions f, to the distribution of the extremal particles, the location of the discontinuity in f now plays the role of the argument in the distribution function. That is, taking $f(x) = \mathbb{1}_{[y,\infty)}(x)$ for some $y \in \mathbb{R}$ and denoting the resulting solution to (F-KPP) by w^y, (1.2.17) yields that

$$w^y(t,x) = \mathtt{P}_x^\xi(M(t) \geq y). \tag{1.2.18}$$

Analogously, taking $f(x) = \mathbb{1}_{(-\infty,y]}(x)$ gives us a representation of the corresponding solution of (F-KPP) as in terms of the distribution function of the minimally displaced particle

$$w(t,x) = \mathtt{P}_x^\xi(\min_{\nu \in N(t)} X_t^\nu \geq y). \tag{1.2.19}$$

Consequently, compared to the homogeneous setting, cf. (1.2.8) and (1.2.9), the situation in the inhomogeneous setting is more subtle, as the solutions to (F-KPP) are not directly the distribution functions, but the argument in the F-KPP solution plays the role of the position of the initial particle in the BBMRE. Nonetheless, the behaviour of solutions to (F-KPP) still gives valuable information about statistical properties of the maximally displaced particle $M(t)$.

F-KPP front propagation in random environment

From an analytic perspective, there has been a lot of interest in *heterogeneous* reaction-diffusion equations, such as the F-KPP equation (F-KPP). The main results concern the existence of travelling wave solutions and *spreading properties*, i.e. conditions under which one observes invasion of a steady state ($w = 1$) into an unsteady state ($w = 0$).

It turns out that in general, travelling waves do not exist for space-heterogeneous equations, cf. [NRRZ12, Zla12]. This is even true for generalised notions of travelling waves, as introduced in [Mat03, BH07, BH12]. Accordingly, the question of travelling waves is often the wrong one in the space-inhomogeneous setting. Instead, one often concentrates on identifying appropriate spreading properties, such as the existence of an asymptotic *spreading speed* or obtaining refined information on the *transition front*, i.e. the location where the solution drops from 1 to 0.

A common way to specify fronts for solutions to (F-KPP) with initial condition $f(x) = \mathbb{1}_{[y,\infty)}$ is the following

$$m^\varepsilon(t) := \inf\left\{x \in \mathbb{R} : w^y(t,x) \geq \varepsilon\right\}, \quad \text{for } \varepsilon \in (0,1). \tag{1.2.20}$$

In the following subsections, we review spreading properties for (F-KPP) in the random environments of our setting. We consider leading order terms for the fronts $m^\varepsilon(t)$ as $t \to \infty$ as well as the width of the transition front.

Spreading speed of the F-KPP front

The work on front propagation for the one-dimensional F-KPP equation in stationary ergodic environments was pioneered by Gärtner and Freidlin, cf. [GF79, Fre85], who proved for $\mathbb{P}$-a.e. realisation of the environment, the existence of an exact asymptotic spreading speed $v_0 > 0$. In particular, it follows from Theorem 7.6.1 of [Fre85] that the solutions to (F-KPP) with initial condition $f(x) = \mathbb{1}_{[y,\infty)}$, for any $y \in \mathbb{R}$, satisfy

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x \leq y - vt} w^y(t,x) = 0 \quad \mathbb{P}\text{-a.s.} \quad \text{for any } v > v_0 \\ &\lim_{t\to\infty} \inf_{x \geq y - vt} w^y(t,x) = 1 \quad \mathbb{P}\text{-a.s.} \quad \text{for any } v \in (0, v_0). \end{aligned} \tag{1.2.21}$$

Note that an immediate consequence of (1.2.21) is that the leading order of the F-KPP front is given by v_0, i.e. for any $\varepsilon \in (0,1)$, it holds $\mathbb{P}$-a.s. that $\lim_{t\to\infty} m^\varepsilon(t)/t = v_0$. A key insight of their work was that the speed of the front of (F-KPP), in a bounded stationary ergodic environment, is determined by properties of its linearised equation near the unstable state $w = 0$, which is known as the *parabolic Anderson model* (PAM) and is given by

$$\begin{aligned} \partial_t u(t,x) &= \frac{1}{2}\Delta u(t,x) + \xi(x)F'(0)u(t,x), && t > 0, x \in \mathbb{R}, \\ u(0,x) &= u_0(x), && x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

Due to the prominent role of the linearised equation near the unstable point, one often speaks of *pulled fronts* in this context, cf. [vS03].

Moreover, Gärtner and Freidlin, cf. [GF79, Fre85] show that the asymptotic spreading speed v_0 can be characterised in terms of a family of *Lyapunov exponents* associated with (PAM). We can write this family as a single real-valued function, which we call *the* Lyapunov exponent and which is defined by

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln u(x, vt), \quad v \in \mathbb{R}, \tag{1.2.22}$$

where u is the solution to (PAM), with Heaviside-initial condition. It is shown in [DS22, Proposition A.3, Corollary 3.10] that λ exists for $\mathbb{P}$-a.a. environments and is

a non-random, non-increasing, symmetric, concave function for which there exists a *critical value* $v_c \geq 0$ such that λ is linear on $[0, v_c]$ and strictly concave on $[v_c, \infty)$. Moreover, (1.2.22) does not depend on the choice of initial condition of the solution u to (PAM). Further, the only influence the environment has on (1.2.22) is through $\lambda(0) = \mathtt{es}$. The spreading speed v_0 is given as the unique positive root of the Lyapunov exponent λ.

Remark 1.2.5. The critical value $v_c \geq 0$ plays an important role in the study of the maximal particle of BBMRE as well as in questions of front propagation in random environments. It is linked to large deviation effects of certain (random) path measures, which have been employed in e.g. [Fre85, ČD20, ČDS23, DS22, ČDO22] to obtain detailed information on solutions to the F-KPP equation and the PAM. These path measures are introduced in detail in Section 2.4 below.

There is another useful way of characterising the spreading speed, which is not based on Lyapunov exponents related to (PAM) but has its origins in homogenisation theory. Links between homogenisation theory and spreading properties in heterogeneous equations were first discovered for periodic heterogeneities, cf. [Xin00] for a heuristic summary. In fact, in [GF79, Fre85] Freidlin and Gärtner also consider the case of periodic environments and show that in this case the spreading speed can be characterised in terms of principle eigenvalues of some "effective" linear operator, often called *corrector* in the homogenisation literature, cf. [Fre85, Section 7.3 and Remark 7.6.3 on p.524]. In terms of adapting such an approach to random ergodic environments, one immediate difficulty is that, in general, eigenvalues for the *corrector* (effective linear operator) do not exist. The link between the homogenisation approach for F-KPP type equations in ergodic random environments and spreading properties was clarified by Berestycki, Hamel and Nadin [BHN08] and Berestycki and Nadin [BN12], see also the recent and comprehensive monograph [BN22]. In particular, Section 5.1 of [BN22] is dedicated to exploring this link in detail. The key observation in these references is that in order to characterise the spreading speed, it suffices to consider *approximate correctors*, i.e. to consider the *generalised eigenvalue problem* of the "effective" linear operator on a specific class of admissible functions. More precisely, they consider the linear operator

$$\mathcal{L}_p^\xi \phi := \frac{1}{2}\Delta\phi - p\partial_x\phi + \Big(\frac{p^2}{2} + F'(0)\xi\Big)\phi, \tag{1.2.23}$$

on the very specific class of admissible test functions

$$\mathcal{A}_{-\infty} := \Big\{\phi \in C^2(\mathbb{R}) : \partial_x\phi/\phi \in L^\infty(\mathbb{R}), \phi > 0 \text{ in } \mathbb{R}, \lim_{|x|\to\infty} \frac{1}{x}\log\phi(x) = 0\Big\}.$$

and define the *generalised principle eigenvalues*

$$\begin{aligned} \underline{\lambda}_1(\mathcal{L}_p^\xi) &:= \sup\big\{\lambda : \exists\phi \in \mathcal{A}_{-\infty} \text{ s.t. } \mathcal{L}_p^\xi\phi \geq \lambda\phi \text{ in } \mathbb{R}\big\} \\ \overline{\lambda}_1(\mathcal{L}_p^\xi) &:= \inf\big\{\lambda : \exists\phi \in \mathcal{A}_{-\infty} \text{ s.t. } \mathcal{L}_p^\xi\phi \leq \lambda\phi \text{ in } \mathbb{R}\big\}, \end{aligned} \tag{1.2.24}$$

By Proposition 2.4 of [BN12] it follows that $\mathbb{P}$-a.s. the two principle eigenvalues agree, for bounded stationary ergodic environments, and that $\mathbb{P}$-a.s. the spreading speed v_0, cf. (1.2.21), has the alternative characterisation

$$v_0 = \min_{p>0} \frac{\overline{\lambda}_1(\mathcal{L}^\xi_{-p})}{p} = \min_{p>0} \frac{\underline{\lambda}_1(\mathcal{L}^\xi_{-p})}{p}. \tag{1.2.25}$$

Remark 1.2.6. This approach to characterising the spreading speed in terms of generalised principle eigenvalues works for more general random reaction-diffusion equations than (F-KPP). Moreover, the particular form of the set $\mathcal{A}_{-\infty}$ of test functions is crucial for this ansatz to work. Different choices of test-functions might yield corresponding generalised eigenvalues that disagree. We refer to [BN12, BN22] for more details and an in-depth discussion. Note, moreover, that the operator in (1.2.23) is obtained through $\mathcal{L}^\xi_p \phi = e^{-px}\mathcal{L}^\xi_{\mathrm{PAM}}(e^{px}\phi)$, for $\phi \in C^2(\mathbb{R})$, where $\mathcal{L}^\xi_{\mathrm{PAM}} := 1/2\Delta + F'(0)\xi$ is the *Anderson* operator. In this sense, we can interpret $\mathcal{L}^\xi_p$ as a generalised Doob h-transform of the Markov process with generator $\mathcal{L}^\xi_{\mathrm{PAM}}$, cf. [CT15] for details on generalised Doob h-transforms.

Note that both the characterisation of the spreading speed v_0 with the Lyapunov exponent and the spectral characterisation are based on considerations for the linearised equation, i.e. (PAM). It is therefore reasonable to also investigate front properties of (PAM). Similar to (1.2.29), we introduce the *fronts* of (PAM) by setting for any $M > 0$

$$\overline{m}^M(t) := \inf\big\{x \in \mathbb{R} : u(t,x) \geq M\big\}. \tag{1.2.26}$$

Similarly to the case of the F-KPP fronts, the work of Freidlin and Gärtner, cf. [GF79, Fre85], implies that $\mathbb{P}$-a.s. it holds that $\lim_{t\to\infty} \overline{m}^\varepsilon(t)/t = v_0$, i.e. the first order of the F-KPP and PAM front coincide.

Fluctuations of the F-KPP front

Let us now come to the issue of higher-order fluctuations of the F-KPP front. Unfortunately, sharp asymptotics of $m^\varepsilon(t)$ are not known for non-trivial heterogeneous environments. However, since the environments we consider are random, it is reasonable to consider the *average effect* the environment has on the front.

The first result in this direction is by Nolen, who in [Nol11], examines the F-KPP equation in a bounded, stationary ergodic environment with non-linearity $F(v) = v(1-v)$, corresponding to (1.2.7) with binary branching. By analytic means, he shows that the position of the front satisfies a central limit theorem. In order to achieve this result, he requires very specific initial conditions $w_0(x,\xi)$, which depend on the randomness of the environment. Moreover, he requires $\lim_{x\to-\infty} w_0(x,\xi) = 1$ (note the orientation) as well as

$$c(\xi)Z(x,\xi,\gamma) \leq w_0(x,\xi) \leq C(\xi)Z(x,\xi,\gamma), \quad \text{for } x > 0,$$

where $Z = Z(x, \xi, \gamma)$ is a non-negative solution of the ordinary differential equation $\frac{1}{2}\Delta Z + (\gamma - \xi)Z = 0$ such that $Z(0, \xi, \gamma) = 1$, and which decays to 0 as $x \to \infty$. (This ordinary differential equation should be compared with (2.4.10) in Chapter 2 below.) Due to technical reasons, Nolen makes certain assumptions on the size of γ, which translate to $w_0(x, \xi)$ decaying to 0 with a slow exponential rate as $x \to \infty$. This decay is "slow enough" to make his fronts *supercritical*, i.e. their velocity $v(\gamma)$ satisfies $v(\gamma) > v_0$.

Fluctuations in fronts with *critical* velocity v_0 were addressed almost a decade later by Černý and Drewitz in [ČD20] for a discrete-space version of (F-KPP) on $\mathbb{Z}$ (and its linearisation), with positive bounded i.i.d. random environments $\xi : \mathbb{Z} \to (0, \infty)$ and initial conditions $u_0 : \mathbb{Z} \to \mathbb{N}_0$ satisfying

$$C\mathbb{1}_{-\mathbb{N}_0} \geq u_0 \geq \mathbb{1}_{\{0\}}, \quad \text{for some } C \in [1, \infty).$$

By interpreting the discrete-space analogue of (1.2.22) (for which one has an analogous unique positive root v_0 and critical velocity v_c) as a law of large numbers, they inquire first about the fluctuations of the logarithm of solutions to the linear equation, i.e. $\log u(t, \lfloor vt \rfloor)$, for which they attain a *functional* central limit theorem, cf. [ČD20, Theorem 2.8]. This functional central limit theorem is then used to deduce another *functional* central limit theorem, for the front $\overline{m}(t)$ of the linearised equation, defined analogously to (1.2.26), under the additional technical condition that $v_0 > v_c$, cf. [ČD20, Theorem 2.6]. Using a duality analogous to (1.2.18), between the discrete F-KPP equation and the (continuous-time) *branching random walk in random environment* (BRWRE) it is shown moreover in [ČD20, Theorem 2.9] that, again under the technical assumption $v_0 > v_c$, the front of the discrete F-KPP equation satisfies a *non-functional* central limit theorem.

Remark 1.2.7. The BRWRE can be constructed, as in Section 1.1, by taking the trajectory marks $(Z_\nu)_{\nu \in \mathcal{V}}$ to be continuous-time random walks and life-times $(\sigma_\nu)_{\nu \in \mathcal{V}}$ as in (1.1.4).

These results from [ČD20] on the fluctuations of the fronts to the PAM and F-KPP equation were later carried over to the continuous setting in [DS22], and extended to a more general class of initial conditions, satisfying tail-conditions which guarantee *criticality* of the front, cf. [DS22, (KPP-INI), p.881] for details. These results complement the result of Nolen [Nol11].

In particular, Theorem 1.4 of [DS22] gives a *functional* central limit theorem for the PAM front $\overline{m}^\varepsilon(t)$. Moreover, Drewitz and Schmitz are interested in how far the F-KPP front $m^\varepsilon(t)$ lags behind its linearised counterpart $\overline{m}^\varepsilon(t)$. In Theorem 1.5 of [DS22] they show that, under $v_0 > v_c$, there exists a constant $C \in (0, \infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}$ such that for all $t \geq \mathcal{T}$ it holds that

$$\overline{m}^\varepsilon(t) - m^\varepsilon(t) \leq C \log t. \tag{1.2.27}$$

In particular, this lets one deduce a *functional* central limit theorem for the front of the F-KPP equation from the corresponding functional central limit theorem for the front of the PAM, cf. Theorem 1.4 and Corollary 1.6 of [DS22]. In particular, this means that there exists $\sigma > 0$ such that

$$[0, \infty) \ni t \mapsto \frac{m^{\varepsilon}(nt) - v_0 nt}{\sigma\sqrt{n}}, \quad \text{for } n \in \mathbb{N}, \tag{1.2.28}$$

converges, as $n \to \infty$, in $\mathbb{P}$-distribution to a standard Brownian motion.

Note that (1.2.28) implies that in the random setting, the front of the F-KPP equation $\mathbb{P}$-a.s. does *not* have logarithmic (Bramson-)corrections (1.2.14), unlike its homogeneous counterpart.

Shape of the F-KPP front

The last aspect of non-homogeneous F-KPP fronts we want to discuss is their shape. With the convergence of their homogeneous counterparts, cf. (1.2.10), in mind and the existence of an exact spreading speed for the fronts $m^{\varepsilon}(t)$, cf. (1.2.21), it seems intuitive to expect the width of the front to fluctuate around some fixed value. It turns out, however, that this intuition is wrong. In fact, it is shown in [ČDS23, Theorem 2.3] that, for environments as in (1.1.7) that have the additional constraint that $\texttt{es}/\texttt{ei} > 2$, it holds that

$$\limsup_{t\to\infty} \operatorname{diam}\left(\{x \in \mathbb{R} : w^y(t,x) \in [\varepsilon, 1-\varepsilon]\}\right) = \infty. \tag{1.2.29}$$

Moreover, this is purely an effect of the non-linear equation. Despite the logarithmic backlog, cf. (1.2.27), the corresponding transition fronts for (PAM), stay uniformly bounded in time. By this, we mean that for any $0 < \varepsilon < M$ solutions u of (PAM) (say with Heaviside initial condition) satisfy for $\mathbb{P}$-a.e. environment ξ

$$\limsup_{t\to\infty} \operatorname{diam}\left(\{x \in \mathbb{R} : u(t,x) \in [\varepsilon, M]\}\right) < \infty. \tag{1.2.30}$$

This is shown in [ČDS23, Theorem 2.2], under the technical assumption $v_0 > v_c$. (Note that (1.2.29) does not require $v_0 > v_c$.) This striking difference between the width of the linear and non-linear front, is not present in the homogeneous setting, where the widths of both fronts are always uniformly bounded for the F-KPP equation, due to the convergence towards a travelling wave, cf. (1.2.10).

Note, moreover, that the question of the diameter of F-KPP transition fronts and when to expect them to be (un-)bounded has also been raised and investigated in the analytic literature, see e.g. [NR09, Zla12, HN22, BN22] and references therein.

Maximally displaced particle in BBMRE

Coming back to questions (Q1)–(Q3), now for the BBMRE, we need to be more cautious when trying to read off results for the extremal particles from the behaviour of (F-KPP) as compared to in the homogeneous setting. On the one hand, this is because the duality (1.2.18) is more subtle. On the other hand, we can not expect to obtain sharp second order leading terms of $M(t)$ for any fixed realisation of the environment, i.e. under the measure $\mathtt{P}_x^\xi$. Rather, we focus again on the *averaged* effect the environment has on the asymptotic behaviour of $M(t)$, i.e. leading to statements under $\mathbb{P}\times\mathtt{P}_x^\xi$. Typically, the case of a fixed environment is referred to as the *quenched* case and the case of averaging over the environment as the *annealed* case.

Note that many of the results we discussed on the F-KPP (and PAM) fronts require the technical condition $v_0 > v_c$. Similarly, this condition plays an important role in results about $M(t)$. In fact, all the results we refer to in this section rely on this technical condition and we therefore assume it to be fulfilled for the rest of this section.

A first result on the behaviour of the maximally displaced particle $M(t)$ of BBMRE can already be drawn from the existence of an exact spreading speed v_0 for the solutions of the F-KPP equation with initial condition $f(x) = \mathbb{1}_{[0,\infty)}(x)$, cf. (1.2.21). Indeed, a quenched weak law of large numbers can be deduced for $M(t)$ by combining (1.2.21) with the fact that, due to the stationarity of ξ,

$$\mathtt{P}_0^\xi\big(M(t)\geq x\big)\overset{(d)}{=}\mathtt{P}_{-x}^\xi\big(M(t)\geq 0\big),\quad \text{under } \mathbb{P}, \text{ for } (t,x)\in[0,\infty)\times\mathbb{R}. \tag{1.2.31}$$

Thus, for $\mathbb{P}$-a.a. environments ξ, it holds that

$$\frac{M(t)}{t}\xrightarrow{\mathtt{P}_0^\xi} v_0. \tag{1.2.32}$$

This result was strengthened in [HRS23] to a quenched strong law of large numbers for $\mathbb{P}$-a.a. realisations of the environment. It is interesting to note that their method of proof is reminiscent of a classical (and purely probabilistic) proof of the strong law of large numbers for the maximal particle in a homogeneous BBM via convergence properties of an *additive martingale*, cf. e.g. [Ber14, Theorem 58]. More precisely, Hou, Ren and Song introduce

$$W_t^\xi(p) := e^{-\underline{\lambda}_1(\mathcal{L}_{-p})t}\sum_{\nu\in N(t)} e^{-pX_t^\nu}\phi_p(X^\nu(t)),\quad t\geq 0, \tag{1.2.33}$$

where ϕ_p is the eigenfunction of the operator $\mathcal{L}_{-p}^\xi$, cf. (1.2.23), associated to the generalised principle eigenvalue $\underline{\lambda}_1(\mathcal{L}_{-p}^\xi)$, cf. (1.2.24), and $(X_t^\nu)_{\nu\in N(t)}$ is a BBMRE started from x. By [HRS23, Lemma 1.3] this quantity is $\mathbb{P}$-a.s. a positive martingale under the quenched measure $\mathtt{P}_x^\xi$. By studying the quenched limit of $W_t^\xi(p)$ as $t\to\infty$,

it is shown in [HRS23, Corollary 1.5] that for any offspring law $(p_k)_{k\in\mathbb{N}}$ satisfying $\sum_{k=1}^{\infty}(k\log k)p_k < \infty$ it holds for $\mathbb{P}$-a.e. realisation of the environment that

$$\frac{M(t)}{t} \to v_0, \quad \mathtt{P}_x^{\xi}\text{-a.s.} \quad \text{as } t \to \infty.$$

In terms of the questions (Q1)–(Q3), asked in Section 1.2, this gives a partial answer to (Q1) by giving the first leading order of the maximal particle.

Let us now turn to higher-order corrections to the asymptotics of $M(t)$. As mentioned above, we shift the focus here to annealed behaviour and consider the averaged behaviour of $M(t)$ over bounded stationary ergodic environments.

In order to answer what behaviour can be expected, we note that as a direct consequence of (1.2.31), the median $\mathrm{med}(t) := \sup\{x \in \mathbb{R} : \mathtt{P}_0^{\xi}(M(t) \geq x) \geq 1/2\}$ has the same distribution, under $\mathbb{P}$, as the front $m^{1/2}(t)$ of the F-KPP equation, with initial condition $f(x) = \mathbb{1}_{[0,\infty)}(x)$. It is moreover reasonable to expect that the maximal displacement $M(t)$ stays comparable to its median $\mathrm{med}(t)$. Thus, by (1.2.28), we expect $M(t)$ to satisfy an annealed invariance principle.

This strategy was pursued for the maximal displacement $\widetilde{M}(t)$ of a BRWRE in [ČD20], where it is shown in Theorem 2.1 that $\widetilde{M}(t)$ satisfies an invariance principle under $\mathbb{P} \times \mathtt{P}_0^{\xi}$.

Remark 1.2.8. Since the (non-functional) central limit theorem for the F-KPP front in [ČD20] is derived from an invariance principle *for* the median (and not the other way round), the strategy of Černý and Drewitz is literally to first check that $\widetilde{M}(t)$ is comparable to the median, cf. [ČD20, Proposition 2.3], and then to show that for $\mathbb{P}$-a.a. environments, the median lags at most logarithmically behind the front $\overline{m}(t)$ of the (discrete) PAM, for which they prove an invariance principle, cf. [ČD20, Corollary 2.7 and Theorem 2.1].

It is reasonable to expect that one can adapt the argument from [ČD20], on the comparability of $\widetilde{M}(t)$ to its median, to the setting of BBMRE, and then use the invariance principle for the fronts $\overline{m}^{\varepsilon}(t)$ of the PAM, cf. [DS22, Theorem 1.4], in order to obtain an invariance principle for $M(t)$.

A different approach was chosen in [HRS23], where $M(t)$ is directly compared to the front of (PAM), but using an alternative description of the front, which is based on the spectral approach around (1.2.23)–(1.2.24). More precisely, by combining that for $\mathbb{P}$-a.e. environment, $M(t)$ stays within logarithmic distance of their characterisation of the front, cf. [HRS23, Theorem 1.6], with an invariance principle for that front, cf. [HRS23, Lemma 3.7], they then also derive an invariance principle for $M(t)$, cf. [HRS23, Theorem 1.7]. Consequently, there exists $\sigma > 0$ such that under $\mathbb{P} \times \mathtt{P}_x^{\xi}$, the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{M(nt) - v_0 nt}{\sigma\sqrt{n}}, \quad \text{for } n \in \mathbb{N},$$

converges weakly as $n \to \infty$ to a standard Brownian motion. In particular, this implies that $\mathbb{P}$-a.s.

$$M(t) \neq v_0 t + O(\log t), \quad \text{as } t \to \infty \text{ under } \mathtt{P}_0^{\xi},$$

which, as the leading orders of the F-KPP front $m^{\varepsilon}(t)$, stands in direct contrast to (1.2.15) and the logarithmic second order (Bramson-)correction for the maximal displacement in *homogeneous* BBM. Thus, the answer to (Q1) changes significantly in the presence of a random environment.

With regards to (Q2) and quenched fluctuations of $M(t)$ around its leading orders, an interesting issue is raised by the unbounded transition fronts, cf. (1.2.29). We saw that in the homogeneous setting, transition fronts are always bounded due to (1.2.10), which, together with the duality (1.2.8), directly implies that the maximally displaced particle re-centred at its median is *tight*.

In the random heterogeneous setting, however, it is not clear what the ramifications of the unbounded F-KPP transition fronts, cf. (1.2.29), are for the quenched fluctuations of $M(t)$ around its median, if in fact there are any at all. In Chapter 2, we investigate precisely this issue and show that, as in the homogeneous case, the fluctuations around the median remain bounded. With regard to (Q3) this *tightness* result in turn doesn't imply (quenched) convergence in distribution, and in fact, it seems unreasonable that the re-centred law of $M(t)$ should converge in distribution. At the moment, we lack, however, a rigorous argument that disproves convergence in law and (Q3) remains open.

The following chapter is a reproduction of joint work with Jiří Černý and Alexander Drewitz.

2. On the tightness of the maximum of branching Brownian motion in random environment

Jiří Černý, Alexander Drewitz, Pascal Oswald

Abstract. We consider one-dimensional branching Brownian motion in spatially random branching environment (BBMRE) and show that for almost every realisation of the environment, the distribution of the maximal particle of the BBMRE re-centred around its median is tight. This result is in stark contrast to the fact that the transition fronts in the solution to the randomised F-KPP equation are, in general, not bounded uniformly in time. In particular, this highlights that—when compared to the setting of homogeneous branching—the introduction of a random environment leads to a much more intricate behaviour.

2.1 Introduction

The behaviour of the position of the maximally—or, equivalently, minimally—displaced particle in various variants of branching random walk (BRW) and branching Brownian motion (BBM) has been the subject of intensive research over the last couple of decades [Bra78, Bra83, BZ07, ABR09, HS09, Aïd13]. While initially most of the work focused on branching systems with homogeneous branching rates, there has recently been an increased activity in the investigation of branching random walks with non-homogeneous branching rates that depend on either time or space mostly in special *deterministic* ways, see [LS88, LS89, FZ12a, FZ12b, BBH$^+$15, MZ16, Mal15a, BH14, BH15, ČD20, Kri21, HRS23, Kri22].

In this article we continue the study of the maximally displaced particle in the model of branching Brownian motion with spatially random branching environment (BBMRE) which was initiated in [DS22], building on the previous work [ČD20] on

a discrete-space analogue, the branching random walk in i.i.d. random environment (BRWRE). The techniques developed in [ČD20, DS22] also lent themselves to obtain refined information on the front of the solution of the randomised Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation [ČDS23]. Subsequently, the techniques and results of [ČD20] have been extended to the continuum space setting of BBMRE in [HRS23].

We complement the above body of research by addressing a seemingly simple, but subtle problem that arises naturally, and which has also been formulated as an open question in [ČD20]. More precisely, we show that the distributions of the position of the maximally displaced particle of the BBMRE, when re-centred around its median, form a tight family of distributions as time evolves. While establishing tightness might a priori not look like an overly intricate problem, we take the opportunity to emphasise that such a preconception is erroneous, see also [BZ07, BZ09]. Our result is particularly interesting as it sharply contrasts the result established in [ČDS23], where it is shown that the transition fronts of the solution to the randomised F-KPP equation are, in general, unbounded in time. In the homogeneous setting, such a dichotomy cannot be observed since, a fortiori, there is a duality between these two objects in that tightness of the re-centred maximum of BBM is equivalent to the uniform boundedness in time of the transition fronts of the solution to the F-KPP equation.

2.1.1 Homogeneous BBM and F-KPP equation

To explain this duality more in detail, we start with recalling the model in the homogeneous situation, which will also serve as a point of reference throughout the article. For a (binary) branching Brownian motion with homogeneous branching rate equal to one, started from a single particle located at the origin at time 0, we denote its maximal displacement at time $t \geq 0$ by $M(t)$, and write

$$w(t,x) = P(M(t) \geq x), \tag{2.1.1}$$

for the probability that this displacement exceeds $x \in \mathbb{R}$. Then, the function $w(t,x)$ solves a non-linear PDE, known as the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation,

$$\partial_t w(t,x) = \frac{1}{2}\partial_x^2 w(t,x) + w(t,x)(1-w(t,x)), \qquad t>0,\, x \in \mathbb{R}, \tag{2.1.2}$$

with the initial datum $w(0,\cdot) = \mathbf{1}_{(-\infty,0]}$ of Heaviside type, see [INW68a, McK75]. Moreover, it is well known that as $t \to \infty$, the solution to (2.1.2) approaches a *travelling wave* g in the following sense: for an appropriate function $m : [0,\infty) \to [0,\infty)$ one has that

$$w(t, m(t)+\cdot) \to g \quad \text{uniformly as } t \to \infty \tag{2.1.3}$$

for a decreasing function g satisfying $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 1$. A critical ingredient in the proof of this convergence is that, again for $m(t)$ being chosen appropriately, one has

$$\begin{aligned} &w(t, x + m(t)) \text{ is decreasing in } t \text{ for } x < 0, \text{ and} \\ &w(t, x + m(t)) \text{ is increasing in } t \text{ for } x > 0. \end{aligned} \tag{2.1.4}$$

Property (2.1.3) immediately yields for every $\varepsilon > 0$ the existence of some $r_\varepsilon \in (0, \infty)$ such that

$$w(t, m(t) + r_\varepsilon) - w(t, m(t) - r_\varepsilon) > 1 - \varepsilon \qquad \text{for all } t \geq 0. \tag{2.1.5}$$

Put differently, the family $(M(t) - m(t))_{t\geq 0}$ is tight. Another, essentially trivial, consequence of (2.1.3) is the uniform boundedness of the width of the transition front of the solution to (2.1.2); that is, that for every $\varepsilon \in (0, 1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x \in \mathbb{R} : w(t, x) \in [\varepsilon, 1 - \varepsilon]\}\big) < \infty. \tag{2.1.6}$$

In this context, it is worth pointing out that the above line of reasoning implicitly uses the reflection symmetry of Brownian motion and the homogeneity of the branching environment. As a consequence, this proof technique breaks down in the presence of an inhomogeneous environment, and the relationship between the solutions of the F-KPP equation and the maximum of BBMRE becomes more intricate than that given in (2.1.1) and (2.1.2), cf. Section 2.3.1.

2.1.2 Randomised F-KPP equation

In the inhomogeneous setting of a random potential, as considered in the current paper, the respective randomised F-KPP equation has been investigated in [ČDS23]. In that source it has been established that for a canonical choice of random potentials ξ, the transition front of the solution to the inhomogeneous F-KPP equation (which is discussed in more detail in Section 2.3.1)

$$\partial_t w^\xi(t, x) = \frac{1}{2}\partial_x^2 w^\xi(t, x) + \xi(x) w^\xi(t, x)(1 - w^\xi(t, x)), \qquad t > 0, x \in \mathbb{R}, \tag{2.1.7}$$

with the initial condition $w^\xi(0, \cdot) = \mathbf{1}_{(-\infty,0]}$ does not need to be uniformly bounded in time, in the sense that the width of their transition fronts can be unbounded. More precisely, in contrast to (2.1.6), it follows from [ČDS23, Theorem 2.3] that there are random potentials ξ within the class of inhomogeneities considered in the current paper, such that $\mathbb{P}$-a.s., for all $\varepsilon \in (0, 1/2)$,

$$\limsup_{t\to\infty} \operatorname{diam}\big(\{x \in \mathbb{R} : w^\xi(t, x) \in [\varepsilon, 1 - \varepsilon]\}\big) = +\infty. \tag{2.1.8}$$

It might hence be surprising and is non-trivial to prove that for BBMRE in the random potential ξ we obtain tightness for the re-centred family of maxima, and a novel approach is required in order to address this situation adequately.

It is worthwhile to note that the PDE results of [ČDS23] have been obtained by taking advantage of almost exclusively probabilistic techniques. In the current article, however, the probabilistic main result will be proven via a symbiosis of analytic and probabilistic techniques.

2.2 Definition of the model and the main result

We work with a model of *branching Brownian motion in random branching environment* (BBMRE) introduced in [ČDS23, DS22] as a continuous space version of the branching random walk in random environment model studied in [ČD20]. The random environment is given by a stochastic process $\xi = (\xi(x))_{x\in\mathbb{R}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which fulfils the following assumptions.

Assumption 1.

- ξ is stationary, that is, for every $h \in \mathbb{R}$ one has

$$(\xi(x))_{x\in\mathbb{R}} \overset{(d)}{=} (\xi(x+h))_{x\in\mathbb{R}}. \tag{2.2.1}$$

- ξ fulfils a *ψ-mixing* condition: There exists a continuous non-increasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\sum_{k=1}^{\infty} \psi(k) < \infty$ such that (using the notation $\mathcal{F}_A = \sigma(\xi(x) : x \in A)$ for $A \subset \mathbb{R}$) for all $Y \in L^1(\Omega, \mathcal{F}_{(-\infty,j]}, \mathbb{P})$, and all $Z \in L^1(\Omega, \mathcal{F}_{[k,\infty)}, \mathbb{P})$ we have

$$\begin{aligned} \big|\mathbb{E}\big[Y - \mathbb{E}[Y] \mid \mathcal{F}_{[k,\infty)}\big]\big| &\le \mathbb{E}[|Y|]\,\psi(k-j), \\ \big|\mathbb{E}\big[Z - \mathbb{E}[Z] \mid \mathcal{F}_{(-\infty,j]}\big]\big| &\le \mathbb{E}[|Z|]\,\psi(k-j). \end{aligned} \tag{2.2.2}$$

  (Note that this conditions implies the ergodicity of ξ with respect to the usual shift operator.)

- The sample paths of ξ are $\mathbb{P}$-a.s. locally Hölder continuous, that is, for almost every ξ there exists $\alpha = \alpha(\xi) \in (0,1)$ and for every compact $K \subseteq \mathbb{R}$ a constant $C = C(K,\xi) > 0$ such that

$$|\xi(x) - \xi(y)| \le C|x-y|^{\alpha}, \qquad \text{for all } x, y \in K. \tag{2.2.3}$$

- ξ is uniformly elliptic in the sense that the essenital infimum and supremum of the random variable $\xi(0)$ (and thus also of $\xi(x)$, $x \in \mathbb{R}$, by (2.2.1)) satisfy

$$0 < \mathtt{ei} := \operatorname{ess\,inf} \xi(0) < \operatorname{ess\,sup} \xi(0) =: \mathtt{es} < \infty. \tag{2.2.4}$$

In the current article we do not explicitly make use of the mixing condition. However, in particular in Section 2.5, we will employ some of the results developed in [ČDS23, DS22] which depend on this mixing assumption.

The dynamics of BBMRE started at a position $x \in \mathbb{R}$ is as follows. Given a realisation of the environment ξ, we place one particle at x at time $t = 0$. As time evolves, the particle follows the trajectory of a standard Brownian motion $(X_t)_{t\geq 0}$. Additionally and independently of everything else, while at position y, the particle is killed with rate $\xi(y)$. Immediately after its death, the particle is replaced by k independent copies at the site of death, according to some fixed offspring distribution $(p_k)_{k\geq 1}$. All k descendants evolve independently of each other according to the same stochastic diffusion-branching dynamics.

We denote by $\mathtt{P}_x^\xi$ the quenched law of a BBMRE, started at x and write $\mathtt{E}_x^\xi$ for the corresponding expectation. Moreover, we denote by $N(t)$ the set of particles alive at time t. For any particle $\nu \in N(t)$ we denote by $(X_s^\nu)_{s\in[0,t]}$ the spatial trajectory of the genealogy of ancestral particles of ν (unique at any given time) up to time t. Our main focus of interest lies in the maximally displaced particle of the BBMRE at time t,

$$M(t) := \sup\{X_t^\nu : \nu \in N(t)\}.$$

Throughout this article we deal with supercritical branching such that the offspring distribution has second moments and particles always have at least one offspring.

Assumption 2. The offspring distribution $(p_k)_{k\geq 1}$ satisfies

$$\sum_{k=1}^{\infty} kp_k =: \mu > 1, \qquad \text{and} \qquad \sum_{k=1}^{\infty} k^2 p_k =: \mu_2 < \infty. \tag{2.2.5}$$

Under these assumptions the maximally displaced particle $M(t)$ satisfies a law or large numbers for some non-random asymptotic velocity $v_0 \in (0, \infty)$; that is, for $\mathbb{P}$-a.e. ξ one has

$$\lim_{t\to\infty} \frac{M(t)}{t} = v_0, \quad \mathtt{P}_0^\xi\text{-a.s.} \tag{2.2.6}$$

see [HRS23, Corollary 1.5]. (Note also that convergence in probability follows from classical results of Freidlin, [Fre85, Theorem 7.6.1].) The asymptotic velocity can be characterised as the unique positive root of the *Lyapunov exponent* λ, which is a deterministic function $\lambda : \mathbb{R} \to \mathbb{R}$ that admits the representation

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \ln \mathtt{E}_0^\xi\big[\big|\{\nu \in N(t) : X_t^\nu \geq vt\}\big|\big], \quad \mathbb{P}\text{-a.s.} \tag{2.2.7}$$

Under Assumptions 1 and 2, the function λ is non-increasing on $[0, \infty)$, concave, and there exists a critical value $v_c \geq 0$ such that λ is linear on $[0, v_c]$ and strictly concave on $[v_c, \infty)$, see e.g. [DS22, Corollary 3.10]. As in [ČD20, ČDS23, DS22] we make the following technical assumption.

Assumption 3. We only consider BBMREs whose asymptotic speed satisfies

$$v_0 > v_c. \tag{2.2.8}$$

Essentially, this condition allows for the introduction of a *tilted* probability measure in the *ballistic phase*, under which a Brownian particle $(X_t)_{t\geq 0}$ moves *on average* with speed v_0 up to time t, cf. Section 2.4. By the same argument as in [ČD20, Lemma A.4] one can show that (2.2.8) is satisfied by a rich class of environments. We refer also to [DS22, Section 4.4] for a more in-depth discussion on the condition (2.2.8) and in particular to [DS22, Proposition 4.10] where environments are constructed which satisfy Assumption 1, but for which (2.2.8) fails. Due to the length of the construction we do not replicate it here.

Finally, we also define for $\varepsilon \in (0,1)$ the quenched quantiles for the distribution of $M(t)$ where the process is started at the origin,

$$m^\xi_\varepsilon(t) := \inf\left\{y \in \mathbb{R} : \mathtt{P}^\xi_0(M(t) \leq y) \geq \varepsilon\right\}. \tag{2.2.9}$$

For notational convenience, we omit the subscript when $\varepsilon = 1/2$ and write $m^\xi(t)$ for the median of the distribution.

With the above notation at our disposal, we can state our main result.

Theorem 2.2.1. *Under Assumptions 1–3, for almost every realisation of the environment ξ, the family $\big(M(t) - m^\xi(t)\big)_{t\geq 0}$ is tight under $\mathtt{P}^\xi_0$.*

Remark 2.2.2. Note that Theorem 2.2.1 also remains valid if for any $\varepsilon \in (0,1)$, the quantity $m^\xi(t)$ is replaced by $m^\xi_\varepsilon(t)$.

This result should be contrasted with the behaviour (2.1.8) of transition fronts of solutions to the inhomogeneous F-KPP equation (2.1.7) discussed in the introduction. More precisely, in [ČDS23, Theorem 2.3, Theorem 2.4] environments ξ satisfying Assumptions 1–3 of the present paper were constructed for which the width of the transition front grows logarithmically in time, along a sub-sequence. That is, for small enough $\varepsilon > 0$, there exist times and positions $(t_n)_n, (x_n)_n \in \Theta(n)$, and a function $\varphi \in \Theta(\ln n)$ such that

$$w^\xi(t_n, x_n) \geq w^\xi(t_n, x_n + \varphi(n)) + \varepsilon. \tag{2.2.10}$$

(Note, that this not only implies (2.1.8), but also the spatial non-monotonicity of the functions $w^\xi(t,\cdot)$.) The existence of environments for which Theorem 2.2.1, and (2.1.8) or (2.2.10) hold simultaneously seems unintuitive, as it sharply contrasts with the homogeneous case. Indeed, in the latter, as indicated by (2.1.4) and (2.1.5), the standard reasoning for deducing the tightness of BBM is by the uniform boundedness in time of transition fronts for the corresponding homogeneous F-KPP solutions. We will explain the reason for this apparent discrepancy later in the paper (see Section 2.2.1).

Questions of tightness also arise naturally and have been addressed in many other classes of models. In [BZ09], analytic tools have been developed in order to establish tightness for a class of discrete time models whose distribution functions satisfy certain recursive equations, analogous to the F-KPP equation in the case of BBM. These tools are powerful and were applied and adapted to establish tightness for several models, e.g. [ABR09, BDZ11, DRZ21, FZ12a, HS09, NZ21] to name a few. For BBM in a periodic environment, [LTZ22] used an analytic result on the F-KPP front in periodic environment [HNRR16] which directly implies tightness.

In the context of the discrete space model of [ČD20], sub-sequential tightness along a deterministic sequence is shown for the quenched and annealed law of the maximally displaced particle in [Kri21] using a Dekking-Host type argument. Our method relies crucially on analytic properties of solutions to the F-KPP equation, and differs from the approaches in the above mentioned articles.

The tightness result of Theorem 2.2.1 naturally suggests the question whether the random variables $M(t) - m^{\xi}(t)$ converge in distribution as $t \to \infty$. Supported by the numerical simulations presented in Figure 2.1, we conjecture that the answer to this question is negative.

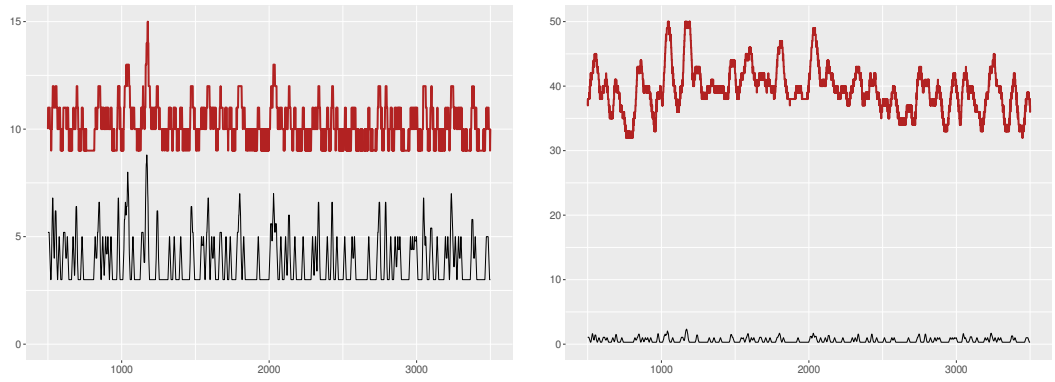

Figure 2.1: Numerical simulations suggesting that the distributions of $M(t) - m^{\xi}(t)$ do not converge as $t \to \infty$. The red line shows the dependence of the "spread" of this distribution, that is of $m^{\xi}_{0.99}(t) - m^{\xi}_{0.01}(t)$, on the median $m^{\xi}(t)$. The black line shows the corresponding potential $\xi(x)$ as function of x. The simulations were performed for a discrete-space model, for realisations of ξ from two different distributions (left and right panel). In both cases, note the similarity of the red and black line, in the sense that at times t when the median $m^{\xi}(t)$ reaches an area where ξ is large, the spread of $M(t)$ tends to be large as well.

Remark 2.2.3. As observed above, in [HRS23] the authors prove an annealed functional central limit theorem for the position of the maximally displaced particle $M(t)$ of

BBMRE in the setting described here. In our notation, this means that for some $\sigma^2 \in (0,\infty)$, the sequence of processes

$$[0,\infty) \ni t \mapsto \frac{M(nt) - v_0 nt}{\sqrt{\sigma^2 n}}, \quad n \in \mathbb{N},$$

under $\mathbb{P} \times \mathtt{P}_x^\xi$ converges weakly in $C([0,\infty))$ to standard Brownian motion.

Using McKean's representation (see Proposition 2.3.1 below), reflecting the potential around the origin by defining $\widetilde{\xi}(-y) := \xi(y)$ for all $y \in \mathbb{R}$, as well as using its stationarity, we obtain that for $x \in \mathbb{R}$, the solution $w^{\widetilde{\xi}}(t,x)$ to (2.1.7) with initial condition $\mathbf{1}_{(-\infty,0]}$ has the same $\mathbb{P}$-distribution as $\mathtt{P}_0^\xi(M(t) \geq x)$. As a consequence, the (functional) central limit theorem [DS22, Corollary 1.6] for the front of this solution $w^{\widetilde{\xi}}(t,x)$ at level $\varepsilon \in (0,1)$ (note that $\widetilde{\xi}$ still satisfies the conditions imposed on ξ for [DS22, Corollary 1.6] to hold) entails a (non-functional) central limit theorem for $m_\varepsilon^\xi(t)$ as defined in (2.2.9) as well; that is, the sequence of random variables $(m^\xi(nt) - v_0 nt)/\sqrt{\sigma^2 n}$, $n \in \mathbb{N}$, converges weakly to a $\mathcal{N}(0,\sigma^2)$-distributed random variable. In combination with Theorem 2.2.1, we recover a non-functional form of the above central limit theorem for $M(t)$ as well. That is, the sequence of random variables $(M(nt) - v_0 nt)/\sqrt{\sigma^2 n}$, $n \in \mathbb{N}$, converges weakly to a $\mathcal{N}(0,\sigma^2)$-distributed random variable under $\mathbb{P} \times \mathtt{P}_x^\xi$.

2.2.1 Strategy of the proof

We now explain the main ideas behind the proof of Theorem 2.2.1, and on the way also comment on the seeming discrepancy between this theorem and properties (2.1.8), (2.2.10).

The first ingredient of the proof is the well-known duality between the distribution of $M(t)$ and the solutions to the randomised F-KPP equation (2.1.7). In the spatially non-homogeneous case this duality states (see Section 2.3.1 below)

$$w^y(t,x) = \mathtt{P}_x^\xi(M(t) \geq y), \tag{2.2.11}$$

where w^y is the solution to (2.1.7) with the initial condition $w^y(0,\cdot) = \mathbf{1}_{[y,\infty)}$. Hence, in order to prove tightness, we need to check that for every $\varepsilon > 0$ there is $\Delta = \Delta(\varepsilon) < \infty$ so that for all $t > 0$ and $x_t = x_t(\varepsilon) \in \mathbb{R}$ characterized via

$$\varepsilon = \mathtt{P}_0^\xi(M(t) \geq x_t) = w^{x_t}(t,0) \tag{2.2.12}$$

it holds that

$$1 - \varepsilon < \mathtt{P}_0^\xi\big(M(t) \geq x_t - \Delta\big) = w^{x_t - \Delta}(t,0). \tag{2.2.13}$$

We note here in passing that this already provides an indication that the above mentioned discrepancy is only apparent: While properties (2.1.8) and (2.2.10) are

linked to the dependency of $w^y(t,x)$ on the spatial variable x, the tightness of $M(t)$ is linked to its dependency on the initial condition $\mathbf{1}_{[y,\infty)}$.

To show that (2.2.13) holds true, we first exploit the fact that solutions to (2.1.7) increase quickly to 1, once they move away from 0. In connection with (2.2.12) this fact implies (cf. Corollary 2.3.6 below) that for some $T = T(\varepsilon) < \infty$, uniformly in ξ and t large, we have

$$1 - \varepsilon < w^{x_t}(t+T,0) =: w^{f_t}(t,0), \tag{2.2.14}$$

where w^{f_t} is the solution of (2.1.7) with the initial condition $f_t := w^{x_t}(T,\cdot)$. In view of this, (2.2.13) follows, if we can show that for some Δ sufficiently large, we have uniformly in t large that

$$w^{x_t-\Delta}(t,0) > w^{f_t}(t,0). \tag{2.2.15}$$

Proving inequality (2.2.15) directly at the spatial coordinate $x = 0$ seems to be difficult, as it requires comparing two solutions to (2.1.7) in the regime where they are away from 0 and 1, and where various approximations to them, e.g. by linearisation, are not precise enough. To work around this difficulty, we take advantage of the *Sturmian principle* for solutions of parabolic PDEs, which we recall in Section 2.3.2. As we will see in the proof of Theorem 2.2.1, this principle implies that (2.2.15) follows if, for some $v > 0$, we can show a modified inequality

$$w^{x_t-\Delta}(t,-vt) > w^{f_t}(t,-vt). \tag{2.2.16}$$

The advantage of inequality (2.2.16) is that if v is sufficiently large, then both its sides are very close to 0, and thus can be controlled using linearisation techniques or the first order Feynman-Kac formulas. The proof of (2.2.16) is still rather technical and is provided in the key Lemma 2.6.1 below.

Finally, we return to the discrepancy between the divergence of the width of the transition front (2.1.8) and the tightness proved in Theorem 2.2.1. The proof of (2.1.8) in [ČDS23] explores the fact that it is easy to construct potentials where, for n large, in any interval $[0,n]$ there is a subinterval of length $\Theta(\log n)$ where the potential is close to `ei`, closely followed by a subinterval of the same length where the potential almost equals `es`. If `es`/`ei` > 2, then the existence of such subintervals forces a creation of "bumps" in the solutions to the randomised F-KPP equation, as illustrated in Figure 2.2. The creation of such bumps directly leads to (2.1.8). It turns out that the existence of those subintervals does not make inequality (2.2.16) invalid, even, e.g., if x_t is located in a subinterval where the potential is close to `es` and $x_t - \Delta$ in a subinterval where ξ is almost `ei`. Via the duality (2.2.11), this is related to the established intuition that the behaviour of the maximum of branching processes is more easily influenced by randomness "occurring" close to their starting point.

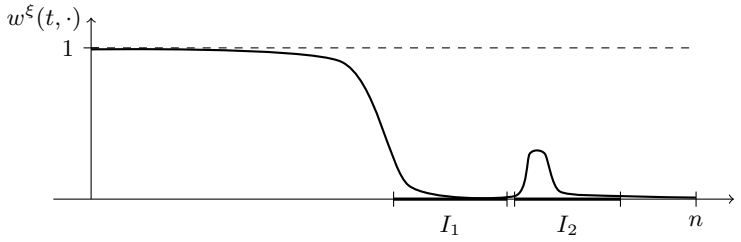

Figure 2.2: A bump in the solution to (2.1.7) which develops shortly after the front of the solution (moving to the right) reaches the interval I_1 where ξ is close to `ei`, and which is followed by the interval I_2 with ξ close to `es`, if `es`/`ei` > 2. Both intervals are of length $\Theta(\log n)$.

Organisation of the article

The rest of the paper is organised as follows. In Section 2.3 we first recall some well-established facts, such as the duality (2.2.11), the Feynman-Kac formulas for the solution of the randomised F-KPP equation and of its linearisation, as well as the parabolic Anderson model. We then discuss a first application of the Sturmian principle to our setting. Section 2.4 reviews tilted measures, which, on a technical level, will play the role of a suitable "gauging-measure" under which we can compare the terms in (2.2.16). In Section 2.5 we explain how to extend the arguments in [ČDS23, DS22] in order obtain a spatial and temporal perturbation result for solutions of the parabolic Anderson model. This perturbation result is then applied in Section 2.6 where we prove the key technical lemma which is related to inequality (2.2.16). Finally, Section 2.7 contains the proof of the main theorem.

Notational conventions

We often use positive finite constants c_1, c_2, etc. in the proofs. This numbering is consistent within every proof and is reset at its end. We use c, C, c' etc. to denote positive finite constants whose value may change during computations.

2.3 Preliminaries

This section recalls two important and well known probabilistic tools which will feature heavily in the proof of our main theorem. Furthermore, we make precise the Sturmian principle alluded to above.

2.3.1 The randomised F-KPP equation and its linearisation

As already mentioned in the introduction, there is a fundamental link between branching Brownian motion and solutions to the homogeneous F-KPP equation. It is often attributed to McKean [McK75], but can already be found in Skorohod [Sko64] and Ikeda, Nagasawa and Watanabe [INW68a]. Such a connection can also be extended to the setting of random branching rates, as we now describe. For this purpose, assume we are given an offspring distribution $(p_k)_{k\geq 1}$ as in (2.2.5). We then consider the random semilinear heat equation

$$\begin{aligned} \partial_t w(t,x) &= \frac{1}{2}\partial_x^2 w(t,x) + \xi(x)F(w(t,x)), && t>0, x\in\mathbb{R}, \\ w(0,x) &= w_0(x), && x\in\mathbb{R}, \end{aligned} \tag{F-KPP}$$

where the non-linearity $F:[0,1]\to[0,1]$ is given by

$$F(w) = (1-w) - \sum_{k=1}^{\infty} p_k (1-w)^k, \qquad w\in[0,1]. \tag{2.3.1}$$

Then the adaptation of McKean's representation of solutions to (F-KPP) takes the following form.

Proposition 2.3.1. *For any function $w_0:\mathbb{R}\to[0,1]$ which is the pointwise limit of an increasing sequence of continuous functions, and for any bounded, locally Hölder continuous function $\xi:\mathbb{R}\to(0,\infty)$, there exists a solution to* (F-KPP) *which is continuous on $(0,\infty)\times\mathbb{R}$ and which, for $t\in[0,\infty)$ and $x\in\mathbb{R}$, can be represented as*

$$w(t,x) = 1 - \mathtt{E}_x^{\xi}\Big[\prod_{\nu\in N(t)} \big(1-w_0(X_t^\nu)\big)\Big]. \tag{2.3.2}$$

A proof of this proposition can be found e.g. in [DS22, Proposition 2.1]; the formulation in that source is under slightly more restrictive conditions, but it transfers verbatim to the assumptions we impose above.

A crucial consequence of Proposition 2.3.1 is that the solution w^y of (F-KPP) with Heaviside-like initial condition $w_0^y = \mathbf{1}_{[y,\infty)}$, for $y\in\mathbb{R}$, is linked to the distribution function of $M(t)$ via the identity

$$w^y(t,x) = \mathtt{P}_x^{\xi}(M(t)\geq y). \tag{2.3.3}$$

Remark 2.3.2. It is common practice in the F-KPP literature to normalise the non-linearity F in such a way that its derivative at the origin is one. Using (2.2.5) it is easy to check that in our case, $F'(0)=\mu-1$. In other words, the standard normalisation of equation (F-KPP) corresponds to a branching processes for which the offspring distribution has mean $\mu=2$, as is also assumed in [DS22]. In (2.2.5), we assume only that

$\mu > 1$ and do not a priori work under the usual F-KPP normalisation. Nevertheless, given any such offspring distribution $(p_k)_{k\in\mathbb{N}}$ with mean $\mu \neq 2$ and a corresponding BBMRE in environment ξ, one can always transform it into another BBMRE in a rescaled environment, so that the transformed process is in the usual normalisation and has the same distribution as the original process. Indeed, the transformation defined by

$$\xi \to (\mu-1)\xi, \quad p_1 \to \frac{\mu + p_1 - 2}{\mu - 1}, \quad \text{and} \quad p_k \to \frac{p_k}{\mu - 1} \text{ for } k \geq 2,$$

yields a new offspring distribution with mean two. Moreover, rescaling the environment guarantees that (F-KPP), and the law $\mathtt{P}_x^\xi$ are invariant under the transformation. After rescaling, it holds that $F'(0) = 1$ and $\mu_2 > 2$; hence, in light of this reasoning, we will from now on always assume that

$$\mu = 2, \qquad F'(0) = 1, \quad \text{and} \quad \mu_2 > 2. \tag{2.3.4}$$

Observe also, that by (2.3.1) this implies that

$$F'(w) \leq 1, \quad \text{and} \quad F''(w) \geq -\mu_2 + 2 \qquad \text{for all } w \in [0,1]. \tag{2.3.5}$$

Another PDE related to BBMRE, which we make use of later on, is the linearisation of (F-KPP), known as the *parabolic Anderson model* (PAM),

$$\begin{aligned} \partial_t u(t,x) &= \frac{1}{2}\partial_x^2 u(t,x) + \xi(x)u(t,x), && t > 0, x \in \mathbb{R} \\ u(0,x) &= u_0(x), && x \in \mathbb{R}. \end{aligned} \tag{PAM}$$

The PAM has been the subject of intense investigation in its own right, see e.g. [Kön16] and reference therein for a comprehensive overview; our main interest, however, lies in space and time perturbation results that have been developed for its solution in [ČDS23, DS22]. These will be considered in more detail in Section 2.5.

An important strategy for probabilistically investigating the solutions to the equations (F-KPP) and (PAM) is via analysing their Feynman-Kac representations. In what comes below we denote, for arbitrary $x \in \mathbb{R}$, by P_x the probability measure under which the process denoted by $(X_t)_{t\geq 0}$ is a standard Brownian motion started at x. The corresponding expectation operator is denoted by E_x. We also make repeated use of the abbreviation $E_x[f; A]$ for $E_x[f\mathbf{1}_A]$.

Proposition 2.3.3. *Under Assumptions 1 and 2, the unique non-negative solution u of* (PAM) *is given by*

$$u(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(X_r)\,\mathrm{d}r\Big\}u_0(X_t)\Big], \qquad t \geq 0, x \in \mathbb{R}, \tag{2.3.6}$$

and the unique non-negative solution w of (F-KPP) *fulfils*

$$w(t,x) = E_x\Big[\exp\Big\{\int_0^t \xi(X_r)\widetilde{F}(w(t-r,X_r))\,\mathrm{d}r\Big\}w_0(X_t)\Big], \qquad t\geq 0, x\in\mathbb{R}, \quad (2.3.7)$$

where $\widetilde{F}(w) = F(w)/w$ for $w\in(0,1]$, which can be continuously extended to $\widetilde{F}(0) = \lim_{w\to 0+}\widetilde{F}(w) = \sup_{w\in(0,1]}\widetilde{F}(w) = 1$.

See e.g. [Bra83, (1.32), (1.33)] for references to the former. Note that the Feynman-Kac representation (2.3.6) is explicit, while (2.3.7) is not (in the sense that the expressions on both sides of the latter equation involve w).

Taking advantage of the above, a (direct) link between the PAM and BBMRE can be derived by combining the Feynman-Kac representation (2.3.6) of the solution to (PAM) with a *many-to-one formula*, see e.g. [DS22, Proposition 2.3], in order to arrive at the representation

$$u(t,x) = \mathtt{E}_x^\xi\Big[\sum_{\nu\in N(t)} u_0(X_t^\nu)\Big]$$

of solutions to (PAM).

2.3.2 Sturmian principle

In this section we present the analytic ingredient of our proof of Theorem 2.2.1. As explained in the introduction (see around (2.2.15)), we are interested in differences of the type $W(\cdot,\cdot) = w^{y_1}(\cdot,\cdot) - w^{y_2}(\cdot+T,\cdot)$ for some $T>0$, and $y_2>y_1$, where we recall that for any $y\in\mathbb{R}$, we denote by w^y the solution of (F-KPP) with initial condition $w_0 = \mathbf{1}_{[y,\infty)}$. It is immediate that the function W satisfies the linear parabolic equation

$$\begin{aligned} \partial_t W(t,x) &= \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(x,t), && t>0, x\in\mathbb{R},\\ W(0,x) &= \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(T,x), && x\in\mathbb{R}, \end{aligned} \quad (2.3.8)$$

where G is the bounded measurable function defined by (using the convention $F'(0) = 1$, cf. Remark 2.3.2)

$$G(t,x) = \begin{cases} \xi(x)\,\frac{F(w^{y_1}(t,x)) - F(w^{y_2}(t+T,x))}{w^{y_1}(t,x) - w^{y_2}(t+T,x)}, & \text{if } w^{y_1}(t,x)\neq w^{y_2}(t+T,x),\\ \xi(x), & \text{if } w^{y_1}(t,x) = w^{y_2}(t+T,x). \end{cases} \quad (2.3.9)$$

Let us state the following simple observation, which will be used at various stages in the following: By Proposition 2.3.1 it follows that

$$0 < w^{y_2}(T,x) < 1 \quad \text{for all } x\in\mathbb{R}. \quad (2.3.10)$$

As a consequence, the initial condition of (2.3.8) has exactly one zero-crossing, and it is located at y_1.

In the analysis literature, it has been known for a long time that the cardinality of the set of zero-crossings of solutions to linear parabolic equations is monotonically non-increasing in time, with the earliest reference dating back to at least an article by Charles Sturm in 1836, cf. [Stu36]. Nevertheless, despite this result having been known for almost two centuries by now, it was not until the eighties of the last century that Sturm's ideas really revived in the theory of linear and non-linear parabolic equations, see, e.g., [Ang88, Ang91, DGM14, EW99, Nad15] for a non-exhaustive list. In this list, the ideas in [EW99] stand out, as they involve a simple and purely probabilistic proof, by interpreting the linear parabolic partial differential equations as generators of Markov processes and reducing the study of the zero-crossings to the study of Markovian transition operators acting on signed measure spaces. A more complete history and a detailed discussion of the Sturmian principle and its applications can be found in [Gal04]. In this context, it is interesting to note that already in their seminal article on the F-KPP equation, Kolmogorov, Petrovskii and Piskunov also make use of a Sturmian principle for equations of the form (2.3.8), see [KPP37, Theorem 11], which is proved using a parabolic maximum principle.

We include a version of such results which is formulated to fit our purpose; a more general version of this result can be found in [Nad15]. Note that the assumptions in particular fit the setting of a single zero-crossing in the initial datum.

Lemma 2.3.4 ([Nad15, Proposition 7.1])**.** *For any $t_0 \in \mathbb{R}$, let $G \in L^\infty((t_0,\infty)\times\mathbb{R})$ and assume $W \in C((t_0,\infty)\times\mathbb{R}) \cap L^\infty((t_0,\infty)\times\mathbb{R})$ to be a weak solution of*

$$\begin{aligned} \partial_t W(t,x) &= \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(x,t), && t > t_0, x \in \mathbb{R}, \\ W(t_0,x) &= W_{t_0}(x), && x \in \mathbb{R}, \end{aligned}$$

where $W_{t_0} \not\equiv 0$ is piecewise continuous and bounded in $\mathbb{R}$, such that for some $z_{t_0} \in \mathbb{R}$ one has

$$W_{t_0}(x) \le 0, \text{ if } x < z_{t_0}, \qquad \text{and} \qquad W_{t_0}(x) \ge 0, \text{ if } x > z_{t_0}.$$

Then, for all $t > t_0$ there exists a unique point $z(t) \in [-\infty,\infty]$ such that

$$W(t,x) < 0, \text{ if } x < z(t), \qquad \text{and} \qquad W(t,x) > 0, \text{ if } x > z(t).$$

As a first application of Lemma 2.3.4, let us consider the effect on the solution of (F-KPP) when the discontinuity of the Heaviside-type initial condition tends to infinity. For this purpose, in order to obtain a non-trivial limit, we perform an appropriate temporal shift. More precisely, we introduce for a given realisation of the environment ξ, any $y \in \mathbb{R}$ and any $\varepsilon > 0$ the "temporal quantile at the origin" as

$$\tau_y^\varepsilon := \inf\{t \ge 0 : w^y(t,0) \ge \varepsilon\}. \tag{2.3.11}$$

Since $\mathbb{P}$-a.s. we have $\lim_{t\to\infty} w^y(t,0) = 1$ (due to, e.g., [Fre85, Theorem 7.6.1]), τ_y^ε is finite. By the continuity of w^y on $(0,\infty)\times\mathbb{R}$, cf. Proposition 2.3.1, the quantity τ_y^ε satisfies $w^y(\tau_y^\varepsilon, 0) = \varepsilon$. From (2.3.3) it follows that $y \mapsto w^y(t,0)$ is decreasing, and thus $y \mapsto \tau_y^\varepsilon$ is increasing. By the law of large numbers for the maximal displacement (cf. (2.2.6)), it follows readily that $\lim\limits_{y\to\infty} \tau_y^\varepsilon = \infty$.

The shift by τ_y^ε allows to establish the following result, which follows already from [Nad15, Lemma 7.3]. Nevertheless, we provide its short proof here for the sake of completeness and as an illustration of how Lemma 2.3.4 can be used in our context.

Proposition 2.3.5. *For every $\varepsilon \in (0,1)$ and for $\mathbb{P}$-a.a. ξ, the limit*

$$w_\varepsilon^\infty(t,x) := \lim_{y\to\infty} w^y(\tau_y^\varepsilon + t, x) \tag{2.3.12}$$

exists locally uniformly in $(t,x) \in \mathbb{R}^2$, and is a global-in-time (that is, for all $t \in \mathbb{R}$) solution to (F-KPP).

The limiting function w_ε^∞ plays a role comparable to that of a travelling wave solution of the homogeneous F-KPP equation, cf. (2.1.3). However, unlike in the homogeneous situation outlined in the introduction, w_ε^∞ does not directly provide an argument for tightness because we lack a suitable quantitative control of the random variables τ_y^ε as y varies. Nonetheless, the result of Proposition 2.3.5 plays a vital role in our proof of tightness. We restrict ourselves to providing a proof of the convergence for $t > 0$ only, as this is sufficient for our purposes in what follows.

Proof of Proposition 2.3.5. Fix $y_1 < y_2$ and for $t \geq -\tau_{y_1}^\varepsilon = -\tau_{y_1}^\varepsilon \vee -\tau_{y_2}^\varepsilon$ (recall that the latter identity follows from the monotonicity of $y \mapsto \tau_y^\varepsilon$ observed below (2.3.11)) define the function $W(t,x) := w^{y_1}(t+\tau_{y_1}^\varepsilon, x) - w^{y_2}(t+\tau_{y_2}^\varepsilon, x)$. Then, similarly as for (2.3.8) and (2.3.9), it follows that

$$\partial_t W(t,x) = \frac{1}{2}\partial_x^2 W(t,x) + G(t,x)W(t,x), \qquad t > -\tau_{y_1}^\varepsilon, x \in \mathbb{R}, \tag{2.3.13}$$

where G is given by

$$G(t,x) = \begin{cases} \xi(x)\,\frac{F(w^{y_1}(t+\tau_{y_1}^\varepsilon,x)) - F(w^{y_2}(t+\tau_{y_2}^\varepsilon,x))}{w^{y_1}(t+\tau_{y_1}^\varepsilon,x) - w^{y_2}(t+\tau_{y_2}^\varepsilon,x)}, & \text{if } w^{y_1}(t+\tau_{y_1}^\varepsilon,x) \neq w^{y_2}(t+\tau_{y_2}^\varepsilon,x), \\ \xi(x), & \text{if } w^{y_1}(t+\tau_{y_1}^\varepsilon,x) = w^{y_2}(t+\tau_{y_2}^\varepsilon,x). \end{cases}$$

From the assumptions, it follows directly that G is a bounded measurable function. Due to [Fre85, Theorem 7.4.1], there exists for $\mathbb{P}$-a.a. ξ a unique classical solution to (2.3.13). Moreover, since $w^{y_1}(0,x) = \mathbf{1}_{[y_1,\infty)}(x)$, it holds that

$$W(-\tau_{y_1}^\varepsilon, x) = w^{y_1}(0,x) - w^{y_2}(\tau_{y_2}^\varepsilon - \tau_{y_1}^\varepsilon, x) = \mathbf{1}_{[y_1,\infty)}(x) - w^{y_2}(\tau_{y_2}^\varepsilon - \tau_{y_1}^\varepsilon, x). \tag{2.3.14}$$

Together with the fact that $0 < w^{y_i}(t,x) < 1$ for $i = 1, 2$ and for all $t > 0$ and $x \in \mathbb{R}$ (cf. (2.3.10)), display (2.3.14) implies that $W(-\tau^\varepsilon_{y_1}, x) < 0$ if $x < y_1$ and $W(-\tau^\varepsilon_{y_1}, x) > 0$ if $x > y_1$. By Lemma 2.3.4, for all $t > -\tau^\varepsilon_{y_1}$, the sets $\{x \in \mathbb{R} : W(t,x) > 0\}$ and $\{x \in \mathbb{R} : W(t,x) < 0\}$ are intervals. But due to the continuity of w^{y_1} and w^{y_2}, we also know that $W(0,0) = w^{y_1}(\tau^\varepsilon_{y_1}, 0) - w^{y_2}(\tau^\varepsilon_{y_2}, 0) = \varepsilon - \varepsilon = 0$. Therefore, the above reasoning supplies us with

$$\begin{aligned} w^{y_1}(\tau^\varepsilon_{y_1}, x) &\le w^{y_2}(\tau^\varepsilon_{y_2}, x), && \text{if } x < 0, \\ w^{y_1}(\tau^\varepsilon_{y_1}, x) &\ge w^{y_2}(\tau^\varepsilon_{y_2}, x), && \text{if } x > 0. \end{aligned} \tag{2.3.15}$$

That is, the function $y \mapsto w^y(\tau^\varepsilon_y, x)$ is non-decreasing if $x < 0$ and non-increasing on $x > 0$. As a consequence, the limit $w^\infty_\varepsilon(0,x) := \lim_{y\to\infty} w^y(\tau^\varepsilon_y, x)$ exists pointwise, and thus locally uniformly, for all $x \in \mathbb{R}$, and also implies $0 \le w^\infty_\varepsilon(0,\cdot) \le 1$. As a consequence, the right-hand side of (2.3.12) converges locally uniformly for $t = 0$. (This should be compared to (2.1.4) in the introduction, which describes the "spatial stretching" of re-centred solutions to the homogeneous F-KPP equation.)

To prove that the local uniform convergence postulated in (2.3.12) holds true for $t > 0$ also, one uses standard estimates on solutions of quasilinear parabolic equations (see, e.g., [LSU68], Chapter V). As a consequence of these estimates, the solutions $w^y(t,x)$ together with their derivatives are bounded locally uniformly in (t,x), uniformly for all y sufficiently large. Hence the set $\{w^y : y \ge 0\}$ is pre-compact in $C^{1,2}_{loc}(\mathbb{R}_+ \times \mathbb{R})$. It therefore contains converging sub-sequences, and every limit point of such a sub-sequence is a solution to (F-KPP) with initial condition $w^\infty(0,\cdot)$. As the solution to (F-KPP) with that given initial condition is unique, this implies that all subsequential limits must agree and thus (2.3.12) holds for all $t > 0$, as well as that w^∞ solves (F-KPP) for $t \ge 0$. We omit here the proof for $t < 0$, as it will not be needed later on. □

The next corollary is a direct consequence of Proposition 2.3.5. It formalises the idea that when a solution to (F-KPP) moves away from 0, it increases quickly to 1. This is going to be relevant later on (cf. (2.2.14) in the introduction).

Corollary 2.3.6. *For every $\varepsilon \in (0, 1/2)$ there exists a $\mathbb{P}$-a.s. finite random variable $T = T(\xi)$ such that for all $y \in \mathbb{R}$ large enough, and any t for which $w^y(t,0) = \varepsilon$, it holds that*

$$w^y(t + t', 0) \ge 1 - \varepsilon \quad \textit{for all } t' \in [T, T+1].$$

Proof. Let $y \in \mathbb{R}$ and $t \ge 0$ be such that $w^y(t,0) = \varepsilon$. By (2.3.11) and the finiteness of τ^ε_y deduced below that display, there exists some $s_0 = s_0(y) \ge 0$ such that $t = \tau^\varepsilon_y + s_0$.

Consider w^∞_ε from Proposition 2.3.5 and let

$$s_1 = \inf\{s > s_0 : w^\infty_\varepsilon(s', 0) \ge 1 - \varepsilon/2 \text{ for all } s' > s\};$$

note that as w_ε^∞ solves (F-KPP), it follows by [Fre85, Theorem 7.6.1] that for $\mathbb{P}$-a.a. realisations of the environment, $\lim_{s\to\infty} w_\varepsilon^\infty(s,x) = 1$, and hence s_1 is $\mathbb{P}$-a.s. finite. Next, taking advantage of the fact that the convergence in Proposition 2.3.5 is locally uniform in t, due to the continuity of the functions involved and using the compactness of $[s_1, s_1+1]$, it holds for large enough $y \in \mathbb{R}$ that

$$\sup_{s'\in[s_1,s_1+1]} |w^y(\tau_y^\varepsilon + s', 0) - w_\varepsilon^\infty(s', 0)| < \varepsilon/2.$$

Setting $T = s_1 - s_0$, we thus obtain for all y large enough and for all $t' \in [T, T+1]$ (with $s' = s_0 + t' \in [s_1, s_1+1]$) that

$$w^y(t+t', 0) = w^y(\tau_y^\varepsilon + s', 0) \geq w_\varepsilon^\infty(s', 0) - \varepsilon/2 \geq 1 - \varepsilon.$$

This completes the proof. □

This result concludes our analytic preparations on how the set of zero-crossings of solutions to linear parabolic equations evolves, and of how it can be applied to the difference of temporally shifted solutions of (F-KPP).

2.4 Tilting and exponential change of measure

The next tool that we introduce is a change of measure for Brownian paths in the Feynman-Kac representation, which makes certain large deviation events typical. These measures have been featured heavily in [ČD20, ČDS23, DS22] already, including in the proof of their respective versions of Proposition 2.5.1. In the aforementioned articles this change of measure has been employed so as to make solutions to (PAM) amenable to the investigation by more standard probabilistic tools. Here we go a step further and consider the stochastic processes driving the tilted path measures. This in turn gives us even more precise control on the tilted measures and allows for comparisons with Brownian motion with constant drift, see Proposition 2.4.3 below.

To define the tilted measures we set

$$\zeta := \xi - \mathtt{es}. \tag{2.4.1}$$

Due to the uniform ellipticity (2.2.4) it follows that $\mathbb{P}$-a.s. for all $x \in \mathbb{R}$,

$$\zeta(x) \in [\mathtt{ei} - \mathtt{es}, 0], \tag{2.4.2}$$

and ζ is $\mathbb{P}$-a.s. locally Hölder continuous with the same exponent as ξ. Moreover, ζ also inherits the stationarity as well as the mixing property from ξ.

For the Brownian motion $(X_t)_{t\geq 0}$ under the measure P_x, as used in the Feynman-Kac representations of Proposition 2.3.3, we introduce first hitting times as

$$H_y := \inf\{t \geq 0 : X_t = y\} \quad \text{for } y \in \mathbb{R}.$$

Analogously to [ČD20, ČDS23, DS22], we define for $x, y \in \mathbb{R}$ with $y \geq x$, as well as $\eta < 0$, the *tilted* path measures characterised through events $A \in \sigma(X_{t\wedge H_y}, t \geq 0)$ via

$$P^{\zeta,\eta}_{x,y}(A) := \frac{1}{Z^{\zeta,\eta}_{x,y}} E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}; A\Big], \tag{2.4.3}$$

with normalising constant

$$Z^{\zeta,\eta}_{x,y} := E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \in (0,1]. \tag{2.4.4}$$

By the strong Markov property, it follows easily that the measures are consistent in the sense that $P^{\zeta,\eta}_{x,y'}(A) = P^{\zeta,\eta}_{x,y}(A)$ for $x \leq y \leq y'$ and $A \in \sigma(X_{t\wedge H_y}, t \geq 0)$. Hence, for any $x \in \mathbb{R}$, we can extend $P^{\zeta,\eta}_{x,y}$ to a probability measure $P^{\zeta,\eta}_x$ on $\sigma(X_t, t \geq 0)$ with the help of Kolmogorov's extension theorem. We write $E^{\zeta,\eta}_x$ for the expectation with respect to the probability measure $P^{\zeta,\eta}_x$.

Finally, as in [DS22, (2.8)], we introduce the annealed logarithmic moment generating function

$$L(\eta) := \mathbb{E}[\ln Z^{\zeta,\eta}_{0,1}], \tag{2.4.5}$$

and denote by $\overline{\eta}(v) < 0$ the unique solution of the equation $L'(\overline{\eta}(v)) = \frac{1}{v}$ for any $v > v_c$; observe that the former is well-defined as by [DS22, Lemma 2.4],

$$\begin{minipage}{0.85\textwidth}$\overline{\eta}(v)$ exists for every $v > v_c$; $v \mapsto \overline{\eta}(v)$ is a continuous decreasing function and $\lim_{v\to\infty} \overline{\eta}(v) = -\infty$.\end{minipage} \tag{2.4.6}$$

The strong Markov property furthermore entails that, for a fixed realization ζ and any $\eta < 0$, the normalising constants (2.4.4) are multiplicative in the sense that for any $x < y < z$ in $\mathbb{R}$,

$$Z^{\zeta,\eta}_{x,z} = Z^{\zeta,\eta}_{x,y} Z^{\zeta,\eta}_{y,z}. \tag{2.4.7}$$

Defining, for some arbitrary but fixed $x_0 \in \mathbb{R}$, the function

$$Z^{\zeta,\eta}(x) := \begin{cases} (Z^{\zeta,\eta}_{x_0,x})^{-1}, & \text{if } x \geq x_0, \\ Z^{\zeta,\eta}_{x,x_0}, & \text{if } x < x_0, \end{cases} \tag{2.4.8}$$

the identity (2.4.7) thus implies that for all $x < y$ we have

$$Z^{\zeta,\eta}_{x,y} = \frac{Z^{\zeta,\eta}(x)}{Z^{\zeta,\eta}(y)}. \tag{2.4.9}$$

The following lemma states some useful properties of the function $Z^{\zeta,\eta}$.

Lemma 2.4.1. *For every bounded Hölder continuous function $\zeta : \mathbb{R} \to (-\infty, 0]$ and $\eta < 0$, the function $Z^{\zeta,\eta}$ is non-decreasing, strictly positive, twice continuously differentiable and satisfies*

$$\frac{1}{2}\Delta Z^{\zeta,\eta}(x) + (\zeta(x) + \eta)Z^{\zeta,\eta}(x) = 0, \qquad x \in \mathbb{R}. \tag{2.4.10}$$

Furthermore, we have

$$b^{\zeta,\eta}(x) := \frac{\mathrm{d}}{\mathrm{d}x} \ln Z^{\zeta,\eta}(x) \in \big[\underline{v}(\eta), \overline{v}(\eta)\big], \tag{2.4.11}$$

where $\underline{v}(\eta) := \sqrt{2|\eta|}$ and $\overline{v}(\eta) := \sqrt{2(\mathtt{es} - \mathtt{ei} + |\eta|)}$.

Remark 2.4.2. Let us note here that the notation $\underline{v}(\eta)$ and $\overline{v}(\eta)$ introduced in the above lemma is suggestive of velocities. This will be made precise in Lemma 2.4.4 below.

Proof of Lemma 2.4.1. The monotonicity and the strict positivity of $Z^{\zeta,\eta}$ follow directly from its definition (2.4.8), using also (2.4.4).

To show (2.4.10), we observe that, for any interval $[x_1, x_2]$, the equation $\frac{1}{2}\Delta u(x) + (\zeta(x) + \eta)u(x) = 0$, $x \in [x_1, x_2]$, with boundary conditions $u(x_i) = Z^{\zeta,\eta}(x_i)$, $i = 1, 2$, has a unique classical solution (see, e.g., [GT01, Corollary 6.9]). Denoting by T the exit time of X from $[x_1, x_2]$, this solution can be represented as (see [Bas98, Theorem II(4.1), p.48])

$$u(x) = E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \tag{2.4.12}$$

On the other hand, for $x \in [x_1, x_2]$, taking $y = x_2$ in (2.4.9), using (2.4.4), and the strong Markov property at time T,

$$\begin{aligned} Z^{\zeta,\eta}(x) &= Z^{\zeta,\eta}(y)Z^{\zeta,\eta}_{x,y} \\ &= Z^{\zeta,\eta}(y)E_x\big[e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}Z^{\zeta,\eta}_{X_T,y}\big] \\ &= E_x\big[Z^{\zeta,\eta}(X_T)e^{\int_0^T(\zeta(X_s)+\eta)\,\mathrm{d}s}\big]. \end{aligned} \tag{2.4.13}$$

Therefore, $Z^{\zeta,\eta}$ satisfies (2.4.10) on $[x_1, x_2]$. Since the interval $[x_1, x_2]$ is arbitrary, (2.4.10) holds for every $x \in \mathbb{R}$.

To show (2.4.11), note first that $b^{\zeta,\eta}$ is well-defined since $Z^{\zeta,\eta}$ is strictly positive and differentiable, by (2.4.10). Therefore, with $y \geq x$, by (2.4.9) and the strong Markov property again,

$$\begin{aligned} b^{\zeta,\eta}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \ln Z^{\zeta,\eta}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln Z^{\zeta,\eta}_{x,y} \\ &= \lim_{\varepsilon\to 0^+} \varepsilon^{-1}\Big(\ln E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] - \ln E_{x-\varepsilon}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big]\Big) \\ &= -\lim_{\varepsilon\to 0^+} \varepsilon^{-1} \ln E_{x-\varepsilon}\Big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big]. \end{aligned} \tag{2.4.14}$$

It is a known fact that for $\alpha > 0$ and $z_1, z_2 \in \mathbb{R}$, it holds that

$$\ln E_{z_1}[e^{-\alpha H_{z_2}}] = -\sqrt{2\alpha}|z_1 - z_2| \tag{2.4.15}$$

(cf. [BS02, (2.0.1), p. 204]). In combination with the bounds (2.4.2), the expectation on the right-hand side of (2.4.14) thus satisfies

$$\begin{aligned} -\varepsilon\sqrt{2|\eta|} &= \ln E_{x-\varepsilon}\big[e^{H_x\eta}\big] \geq \ln E_{x-\varepsilon}\Big[e^{\int_0^{H_x}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \\ &\geq \ln E_{x-\varepsilon}\big[e^{H_x(\mathtt{ei}-\mathtt{es}+\eta)}\big] = -\varepsilon\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta|)}, \end{aligned} \tag{2.4.16}$$

which together with (2.4.14) implies (2.4.11). □

The function $b^{\zeta,\eta}(x)$ introduced in (2.4.11) is useful in describing the law of X under the tilted measure, as it allows an interpretation of the tilted process as a Brownian motion with an inhomogeneous drift, by constructing an appropriate SDE as follows.

Proposition 2.4.3. *Let $x_0 \in \mathbb{R}$, $\eta < 0$ and let $\zeta : \mathbb{R} \to (-\infty, 0]$ be a locally Hölder continuous function that is uniformly bounded from below. Furthermore, denote by B a standard Brownian motion. Then the distribution of the solution to the SDE*

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t + b^{\zeta,\eta}(X_t)\,\mathrm{d}t, \qquad t > 0, \\ X_0 &= x_0, \end{aligned} \tag{2.4.17}$$

agrees with $P^{\zeta,\eta}_{x_0}$.

Proof. The proof is based on an exponential change of measure for diffusion processes. For the sake of simplicity we write b for $b^{\zeta,\eta}$ and Z for $Z^{\zeta,\eta}$ whenever there is no risk of confusion. By (2.4.11) we obtain that

$$b' = (\ln Z)'' = \Big(\frac{Z'}{Z}\Big)' = \frac{\Delta Z}{Z} - \Big(\frac{Z'}{Z}\Big)^2 = -2(\zeta + \eta) - b^2. \tag{2.4.18}$$

Therefore, the bounds (2.4.11) and (2.4.2) imply that b is a bounded Lipschitz function and thus there is a strong solution to (2.4.17), whose distribution we denote by $Q_{x_0} = Q^{\zeta,\eta}_{x_0}$. Let further, as previously, P_{x_0} be the distribution of Brownian motion started from x_0, and let $Q^t_{x_0}$ and $P^t_{x_0}$ be the restrictions of those distributions to the time interval $[0, t]$, $t > 0$. As a consequence of the Cameron-Martin-Girsanov theorem (see, e.g., [RW00, Theorem V.27.1] for a suitable formulation), it is well known that

$$\frac{\mathrm{d}Q^t_{x_0}}{\mathrm{d}P^t_{x_0}} = \exp\Big\{\int_0^t b(X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t b^2(X_s)\,\mathrm{d}s\Big\} =: M_t, \tag{2.4.19}$$

for a P_{x_0}-martingale M. (The fact that M_t is a martingale follows, e.g., from [RW00, Theorem IV.37.8], since b is a bounded function.)

With the aim of arriving at a comparison with (2.4.3), we claim that

$$M_t = \frac{Z(X_t)}{Z(X_0)} e^{\int_0^t (\zeta(X_s)+\eta)\,\mathrm{d}s}. \tag{2.4.20}$$

To see this, note first that applying Itô's formula to $\ln Z(x) = \int_{x_0}^x b(t)\,\mathrm{d}t$ yields

$$\frac{Z(X_t)}{Z(X_0)} = \exp\Big\{ \ln Z(X_t) - \ln Z(X_0)\Big\} = \exp\Big\{ \int_0^t b(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t b'(X_s)\,\mathrm{d}s\Big\}. \tag{2.4.21}$$

Comparing this with (2.4.19) shows that

$$M_t = \frac{Z(X_t)}{Z(X_0)} \exp\Big\{ -\frac{1}{2}\int_0^t \big(b'(X_s) + b^2(X_s)\big)\,\mathrm{d}s\Big\}, \tag{2.4.22}$$

which together with (2.4.18) implies (2.4.20).

We can now complete the proof of the proposition. For the sake of clarity we sometimes write expectations with respect to a probability measure Q as E^Q. For $y \geq x_0$, let $Q_{x_0,y}$ be the measure Q_{x_0} restricted to the σ-algebra $\mathcal{H}_y = \sigma(X_{s\wedge H_y} : s \geq 0)$. To show that $Q_{x_0} = P^{\zeta,\eta}_{x_0}$, it is sufficient to show that $Q_{x_0,y} = P^{\zeta,\eta}_{x_0,y}$ for all $y > x_0$ (see (2.4.3)). For this purpose, we observe that by Lemma 2.4.1, Z is a bounded function on $(-\infty, y]$ and thus the stopped martingale $M_t^{H_y} = M_{t\wedge H_y}$ is uniformly bounded from above. Therefore, by the optional stopping theorem, for any $A \in \mathcal{H}_y$, using (2.4.19) for the second equality,

$$\begin{aligned}
Q_{x_0,y}(A) &= \lim_{t\to\infty} E^{Q_{x_0,y}}[\mathbf{1}_{A\cap\{H_y\leq t\}}] = \lim_{t\to\infty} E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\leq t\}}] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[E^{P_{x_0}}[M_t \mathbf{1}_{A\cap\{H_y\leq t\}} \mid \mathcal{H}_y]\big] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[\mathbf{1}_{A\cap\{H_y\leq t\}} E^{P_{x_0}}[M_t \mid \mathcal{H}_y]\big] \\
&= \lim_{t\to\infty} E^{P_{x_0}}\big[\mathbf{1}_{A\cap\{H_y\leq t\}} M_{H_y}\big] = E^{P_{x_0}}[M_{H_y}\mathbf{1}_A].
\end{aligned}$$

By (2.4.20), $M_{H_y} = \frac{Z(y)}{Z(x_0)} e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}$, and thus, also by (2.4.9),

$$Q_{x_0,y}(A) = (Z^{\zeta,\eta}_{x_0,y})^{-1} E_{x_0}[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\mathbf{1}_A] = P^{\zeta,\eta}_{x_0,y}(A)$$

as required. This completes the proof. □

We are now ready to reap the fruits of the above considerations. Proposition 2.4.3 together with the uniform bounds (2.4.11) on $b^{\zeta,\eta}$ allows for a comparison between the tilted measures (2.4.3) and Brownian motion with constant drift. The next lemma provides this desired control and makes it precise. For a given drift $\alpha \in \mathbb{R}$, we write P^α_x for the law of Brownian motion with constant drift α started at x and E^α_x for the corresponding expectation.

Lemma 2.4.4. *Let $\zeta : \mathbb{R} \to [-(\mathtt{es} - \mathtt{ei}), 0]$ be locally Hölder continuous and let $\eta < 0$. Then, for any starting point $x \in \mathbb{R}$ and any bounded non-decreasing function $g : \mathbb{R} \to \mathbb{R}$,*

$$E_x^{\underline{v}(\eta)}[g(X_t)] \leq E_x^{\zeta,\eta}[g(X_t)] \leq E_x^{\overline{v}(\eta)}[g(X_t)],$$

where $\underline{v}(\eta)$ and $\overline{v}(\eta)$ have been introduced in Lemma 2.4.1.

Proof. By Proposition 2.4.3, the process X_t driven by the tilted measure $P_{x_0}^{\zeta,\eta}$ has generator $L^{\zeta,\eta} = \frac{1}{2}\Delta + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$. Let further $L^v = \frac{1}{2}\Delta + v\frac{\mathrm{d}}{\mathrm{d}x}$ be the generator of the Brownian motion with drift v. Then, for any non-decreasing $g \in C_b^2(\mathbb{R})$, if follows from (2.4.11) that

$$L^{\underline{v}(\eta)}g \leq L^{\zeta,\eta}g \leq L^{\overline{v}(\eta)}g.$$

Since, by Kolmogorov's forward equation, $\frac{\mathrm{d}}{\mathrm{d}t}E_x^{\zeta,\eta}[g(X_t)] = E_x^{\zeta,\eta}[(L^{\zeta,\eta}g)(X_t)]$ and analogously for the measures $E_x^{\underline{v}}$ and $E_x^{\overline{v}}$, the statement of the lemma follows for any non-decreasing $g \in C_b^2(\mathbb{R})$. The extension to arbitrary non-decreasing functions g follows by approximating g by a sequence of non-decreasing functions in $C_b^2(\mathbb{R})$ and using the dominated convergence theorem. □

2.5 Perturbations of the Feynman-Kac representation

We provide a result on perturbations of the Feynman-Kac representation (2.3.6) of solutions to the parabolic Anderson model (PAM) with Heaviside-like initial condition, with respect to disruption in time and with respect to disruptions in the discontinuity of the initial condition. A variant of such results was developed in [ČDS23, DS22] (cf. Lemmas 3.11 and 3.13 from [DS22], or Lemma 4.1 of [ČDS23]) for the study of the fronts of (F-KPP) and (PAM). In the current setting the perturbation results will be used together with (2.3.3) and the Feynman-Kac representation, Proposition 2.3.3, in order to get bounds on the solutions to (F-KPP) in the proof of the key Lemma 2.6.1 in Section 2.6.

To avoid the dependence of various constants appearing in these perturbation results on the speed, we assume for the rest of the article that the speeds we allow are contained in some arbitrary but fixed compact interval $V \subset (v_c, \infty)$ which has v_0 in its interior (in particular, we require (2.2.8) to hold). As we can otherwise choose V arbitrarily large, this does not pose any further restrictions for what follows in the subsequent sections.

Proposition 2.5.1. *(a) For every $\delta > 0$ and $A > 0$, there exist a finite constant $c_1 \in (1, \infty)$ and a $\mathbb{P}$-a.s. finite random variable $\mathcal{T}_1$ such that for all $t \geq \mathcal{T}_1$*

uniformly in $0 \le h \le t^{1-\delta}$, *and* $x, y \in [-At, At]$ *with* $x < y$, $\frac{y-x}{t} \in V$ *and* $\frac{y-x}{t+h} \in V$,

$$E_x\Big[e^{\int_0^{t+h} \xi(X_s)\,\mathrm{d}s}; X_{t+h} \ge y\Big] \le c_1 e^{c_1 h} E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y\Big].$$

(b) Let $\delta : (0,\infty) \to (0,\infty)$ *be a function tending to* 0 *as* $t \to \infty$, *and let* $A > 0$. *Then there exists a constant* $c_2 \in (1,\infty)$ *and a* $\mathbb{P}$*-a.s. finite random variable* $\mathcal{T}_2$ *such that for all* $t \ge \mathcal{T}_2$, *uniformly in* $0 \le h \le t\delta(t)$ *and* $x, y \in [-At, At]$ *with* $x < y$, $\frac{y-x}{t} \in V$ *and* $\frac{y+h-x}{t} \in V$,

$$E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y + h\Big] \le c_2 e^{-h/c_2} E_x\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \ge y\Big].$$

The proof of this proposition involves comparing the Feynman-Kac representation (2.3.6) to functionals with respect to the family of tilted probability measures that were presented in Section 2.4. It is a rather straightforward, but lengthy adaptation of the proofs of [DS22, Lemma 3.11(b)] and [ČDS23, Lemma 4.1(b)], as shall be discussed next. There are two key differences in the statement of Proposition 2.5.1 and the two respective statements in [ČDS23, DS22], that need to be addressed:

(A) Proposition 2.5.1 requires that its estimates hold uniformly over the "starting point" x and the "target point" y in an interval growing linearly with time t. In the original statements, the target point is always the origin and the starting point satisfies $x = vt$.

(B) Proposition 2.5.1(b) involves a perturbation by the end point (that is, y changes to $y + h$), while the starting point is perturbed in the original statement.

In addition [ČDS23, DS22] consider always the travelling waves going from left to right, while for our purposes it is more suitable to work with waves going from right to left. This difference is easy to be dealt with by mirroring the environment and we do not discuss it further.

Proving Proposition 2.5.1 thus requires checking that these two differences can be dealt with by the original arguments. We do not reproduce the lengthy argument in completeness here, but describe key locations where the arguments of [ČDS23, DS22] have to be adapted. In particular arguments that use only estimates that are uniform in the environment ξ (resp. ζ, cf. (2.4.1)) carry over to the current setting simply by a change of notation. We do not elaborate on such arguments more than necessary, but give frequent references to the corresponding passages in [ČDS23, DS22].

Recall first the definition of the tilted measure $P_x^{\zeta,\eta}$ from below (2.4.4). Similarly to [DS22, (2.13)], we are interested in the (random) tilting parameter $\eta_{x,y}^{\zeta}(v)$ for which

the mean speed of a particle on its way from x to y, under the tilted measure, is precisely v, that is

$$E_x^{\zeta,\eta^\zeta_{x,y}(v)}[H_y] = \frac{y-x}{v}, \qquad v > 0, x < y. \tag{2.5.1}$$

(If no such parameter exists, we set $\eta^\zeta_{x,y}(v) = 0$.) Recall also the definitions of $\overline{\eta}(v) < 0$ from below (2.4.5), which can be seen as the "typical value" of parameters satisfying (2.5.1) (see also [DS22, (2.10)]), as well as the definition of the compact interval $V \subset (v_c, \infty)$ containing v_0 in its interior from above Proposition 2.5.1. By [DS22, Lemma 2.4], there is a compact interval $\Delta \subset (-\infty, 0)$ which contains $\{\overline{\eta}(v) : v \in V\}$ in its interior. In particular, this implies that

$$\infty < \inf_{v\in V} \overline{\eta}(v) \le \sup_{v\in V} \overline{\eta}(v) < 0. \tag{2.5.2}$$

The next lemma shows that $\eta^\zeta_{x,y}(v)$ exists with high probability and that it is close to $\overline{\eta}$. It is an extension of Lemma 2.5 of [DS22] and the first step on the way to dealing with the difference (A) in the above list.

Lemma 2.5.2. *(a) For every $A > 1$ there exists an a.s. finite random variable $\mathcal{N} = \mathcal{N}(A)$ such that for all $v \in V$ and $x < y \in \mathbb{R}$ such that $y - x \ge \mathcal{N}$ and $|x|, |y| \le A(y-x)$, the solution $\eta^\zeta_{x,y}(v)$ to (2.5.1) exists and satisfies $\eta^\zeta_{x,y}(v) \in \Delta$.*

(b) For each $q \in \mathbb{N}$, and each compact interval $V \subset (v_c, \infty)$, there exists $C = C(V, q) \in (0, \infty)$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\sup_{v\in V} \sup_{x\in[-n,-n+1]} \sup_{y\in[0,1]} |\eta^\zeta_{x,y}(v) - \overline{\eta}(v)| \ge C\sqrt{\frac{\ln n}{n}} \right) \le Cn^{-q}. \tag{2.5.3}$$

Proof. As in [DS22], (a) follows directly from (b), using the Borel-Cantelli lemma and (2.5.2), with the help of the stationarity and an additional union bound to take care over the uniformity in y.

The proof of (b) looks almost the same as the proof of Lemmas 2.5, 2.6 in [DS22], where it was shown, in our notation, that

$$\mathbb{P}\left(\sup_{v\in V} \sup_{x\in[-n,-n+1]} |\eta^\zeta_{x,0}(v) - \overline{\eta}(v)| \ge C\sqrt{\frac{\ln n}{n}} \right) \le Cn^{-q}. \tag{2.5.4}$$

Including the additional supremum over $y \in [0, 1]$ essentially only requires notational changes. □

Given this concentration result for the tilting parameters, we next adapt Lemma 2.7 of [DS22], which is used in the proof of the spatial perturbation result, Lemma 4.1 in

[ČDS23] and provides a perturbation result for the tilting parameters, satisfying (2.5.1). Besides [DS22, Lemma 2.5] which we already adapted to our setting in Lemma 2.5.2, its proof only uses steps that are uniform in the potential ξ, and thus requires only notational changes. For this reason we omit the proof and refer to [DS22] for the complete argument.

Lemma 2.5.3. *There exists a constant $c > 0$ and for every $A > 1$ there exists an a.s. finite random variable $\mathcal{N}' = \mathcal{N}'(A)$ such that for all $x, y \in \mathbb{R}$ with $y - x \geq \mathcal{N}'$ and $|x|, |y| \leq A(y - x)$, $v \in V$, and $h \in [0, y - x]$, we have*

$$\left|\eta^{\zeta}_{x,y}(v) - \eta^{\zeta}_{x,y+h}(v)\right| \leq \frac{ch}{y - x}. \tag{2.5.5}$$

Next, we introduce two auxiliary processes which we later relate to the expressions appearing in Proposition 2.5.1. We consider, for $x \leq y \in \mathbb{R}$ and $v > 0$, the quantities (cf. [DS22, (3.7)])

$$\begin{aligned} Y^{\approx}_v(x, y) &:= E_x\Big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y \in \Big[\frac{y - x}{v} - K, \frac{y - x}{v}\Big]\Big], \\ Y^{>}_v(x, y) &:= E_x\Big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y < \frac{y - x}{v} - K\Big], \end{aligned} \tag{2.5.6}$$

where $K > 0$ is a large constant fixed as in (2.5.15) below. It turns out that $Y^{\approx}_v(x, y)$ and $Y^{<}_v(x, y)$ are comparable uniformly in the admissible choices of x and y.

Lemma 2.5.4. *For $A > 1$, let $\mathcal{N} = \mathcal{N}(A)$ be as in Lemma 2.5.2. Then there exists a constant $C \in (1, \infty)$ such that for all $v \in V$ and all $x < y \in \mathbb{R}$ such that $y - x \geq \mathcal{N}$ as well as $|x|, |y| \leq A(y - x)$, we have*

$$\frac{Y^{\approx}_v(x, y)}{Y^{<}_v(x, y)} \in [C^{-1}, C]. \tag{2.5.7}$$

Proof. The proof of this lemma contains a computation that is essential for a step in the proof of the key Lemma 2.6.1, and is also featured in Section 2.6 below. We assume that x, y satisfy the assumptions of the lemma, and, in order to keep the notation simple, we in addition assume that $x, y \in \mathbb{Z}$ (see [DS22, Section 1.9] for notational conventions that allow to deal with non-integer x and y). We write $\eta := \eta^{\zeta}_{x,y}(v)$ and define

$$\sigma = \sigma^{\zeta}_{x,y}(v) := |\eta| \sqrt{\mathrm{Var}_{P^{\zeta,\eta}_x}(H_y)}, \tag{2.5.8}$$

where the variance is with respect of $P^{\zeta,\eta}_x$. As in [DS22, (3.8)], uniformly in ζ and $v \in V$,

$$c^{-1}\sqrt{y - x} \leq \sigma^{\zeta}_{x,y}(v) \leq c\sqrt{y - x} \tag{2.5.9}$$

for some $c \in (1, \infty)$. Indeed, if we define

$$\begin{aligned} \overline{L}^{\zeta}_{x,y}(\eta) &:= (y-x)^{-1} \sum_{z=x+1}^{y} \ln E_{z-1}\Big[e^{\int_0^{H_z}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big] \\ &= (y-x)^{-1} \ln E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}\Big], \end{aligned} \tag{2.5.10}$$

a brief calculation yields that $\mathrm{Var}_{P^{\zeta,\eta}_x}(H_y) = (y-x)(\overline{L}^{\zeta}_{x,y})''(\eta)$. Moreover it follows by the same argument as in [DS22, Lemma A.1] that for each compact interval $\Delta \subset (-\infty, 0)$ it holds $\mathbb{P}$-a.s. that

$$-\infty < \inf_{|y-x|\geq 1} \inf_{\eta\in\Delta} (\overline{L}^{\zeta}_{x,y})''(\eta) \leq \sup_{|y-x|\geq 1} \sup_{\eta\in\Delta} (\overline{L}^{\zeta}_{x,y})''(\eta) < \infty,$$

from which we deduce (2.5.9).

Let further $\tau_z = H_z - H_{z-1}$, $z \in [x+1, y] \cap \mathbb{Z}$, and let $\widehat{\tau}_z := \tau_z - E^{\zeta,\eta}_x[\tau_z]$. Then, by the definition of η, for x, y satisfying the assumptions of Lemma 2.5.2, we have $E^{\zeta,\eta}_x[H_y] = \frac{y-x}{v}$. With this notation, a straightforward computation as in [DS22, (3.12)] yields

$$\begin{aligned} Y^{\approx}_v(x,y) &= E_x\Big[e^{\int_0^{H_y}(\zeta(B_s)+\eta)\,\mathrm{d}s}\, e^{-\eta\sum_{z=x+1}^{y}\widehat{\tau}_z};\ \sum_{i=x+1}^{y} \widehat{\tau}_z \in [-K, 0]\Big] e^{-(y-x)\eta/v} \\ &= E^{\zeta,\eta}_x\Big[e^{-\sigma\frac{\eta}{\sigma}\sum_{z=x+1}^{y}\widehat{\tau}_z};\ \frac{\eta}{\sigma}\sum_{z=x+1}^{y} \widehat{\tau}_z \in \Big[0, -\frac{K\eta}{\sigma}\Big]\Big] e^{-(y-x)(\frac{\eta}{v} - \overline{L}^{\zeta}_{x,y}(\eta))}. \end{aligned} \tag{2.5.11}$$

Defining $\mu^{\zeta,\eta}_{x,y}$ to be the distribution of $\frac{\eta}{\sigma}\sum_{z=x+1}^{y}\widehat{\tau}_z$ under $P^{\zeta,\eta}_x$, this implies

$$Y^{\approx}_v(x,y) = e^{-(y-x)(\frac{\eta}{v} - \overline{L}^{\zeta}_{x,y}(\eta))} \int_0^{\frac{-K\eta}{\sigma}} e^{-\sigma u} \mu^{\zeta,\eta}_{x,y}(\mathrm{d}u). \tag{2.5.12}$$

Completely analogous computation then shows that

$$Y^{<}_v(x,y) = e^{-(y-x)(\frac{\eta}{v} - \overline{L}^{\zeta}_{x,y}(\eta))} \int_{\frac{-K\eta}{\sigma}}^{\infty} e^{-\sigma u} \mu^{\zeta,\eta}_{x,y}(\mathrm{d}u). \tag{2.5.13}$$

The upshot of these computations is that under $P^{\zeta,\eta}_x$, the random variables $\widehat{\tau}_z$, $z = x+1, \ldots, y$ are centred, independent, have uniform exponential moments, and $\mu^{\zeta,\eta}_{x,y}$ has unit variance. This allows, as in the proof of [DS22, Lemma 3.6], to (uniformly) approximate $\mu^{\zeta,\eta}_{x,y}$ by the standard Gaussian measure Φ, and to show that the integrals appearing on the right-hand side of (2.5.12) and (2.5.13) are both of order $(y-x)^{-1/2}$, a.s. uniformly in the ζ and $v \in V$ under consideration and for all x, y satisfying the

assumptions of Lemma 2.5.2. More in detail, due to the well known estimates on the errors of normal approximations, see e.g. [BR10, Theorem 13.3], it follows that

$$\sup_{\mathcal{C}} \left|\mu_{x,y}^{\zeta,\eta}(\mathcal{C}) - \Phi(\mathcal{C})\right| \leq c(y-x)^{-1/2} \tag{2.5.14}$$

where the supremum is over all intervals in $\mathbb{R}$ and $c > 0$ only depends on the uniform bound of the exponential moments of the $\widehat{\tau}_z$. Now, due to (2.5.9), we can choose $K > 0$ large enough so that for some constants $c_2 > c_1 > c$ (for c as in (2.5.14)), for all x, y with $y - x \geq \mathcal{N}$ and $v \in V$

$$c_1(y-x)^{-1/2} \leq \Phi\big([0, -K\eta/\sigma]\big) \leq c_2(y-x)^{-1/2}, \tag{2.5.15}$$

and thus infer

$$(c_1 - c)(y-x)^{-1/2} \leq \mu_{x,y}^{\zeta,\eta}([0, -K\eta/\sigma]) \leq (c_2 + c)(y-x)^{-1/2},$$

cf. [DS22, (3.17) and (3.18)].

With these estimates we can then show that the integrals in (2.5.12) and (2.5.13) are both of the same order $(y-x)^{-1/2}$, from which the claim of the lemma follows. The proof, which uses only simple analytical arguments, is the same as in [DS22, Lemma 3.6]. □

Lemma 2.5.4 has an important corollary allowing to approximate the Feynman-Kac formula for the PAM (cf. (2.3.6)) by expressions involving $Y_v^{\approx}(x, y)$. It extends Lemma 3.7 of [DS22], and will also be used in Section 2.6 below.

Lemma 2.5.5. *For each $A > 1$, with $\mathcal{N} = \mathcal{N}(A)$ as in Lemma 2.5.2, there exists a constant $C \in (1, \infty)$ such that for all $t \in (0, \infty)$ and all $x < y \in \mathbb{R}$ such that $y - x \geq \mathcal{N}$, $|x|, |y| \leq A(y-x)$ and $\frac{y-x}{t} \in V$,*

$$C^{-1} Y_v^{\approx}(x, y) \leq E_x\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y\big] \leq C Y_v^{\approx}(x, y). \tag{2.5.16}$$

Proof. The proof of the corresponding Lemma 3.7 of [DS22] only uses estimates that are uniform in ζ and the starting/target position, as well as the respective version of Lemma 2.5.4, which states the comparability of $Y_v^{\approx}(x, y)$ and $Y_v^{<}(x, y)$, for admissible choices of x, y, v . It can thus be adapted directly to the current setting. □

With this we have made all the necessary extensions of the results in [ČDS23, DS22] which are needed in order accommodate for the differences outlined in (A) and (B) and are equipped to show Proposition 2.5.1.

Proof of Proposition 2.5.1. The proof of part (a) involving the perturbation in time follows the exact same lines as the proof of the temporal perturbation results, given in

Lemma 3.11(b) in [DS22]: We denote $v := (y-x)/t$, $v' := (y-x)/(t+h)$ and observe that by Lemma 2.5.5, for x, y, t and h as in the statement, by choosing $\mathcal{T}_1$ sufficiently large so that $y - x \geq \mathcal{N}$,

$$\frac{E_x\Big[e^{\int_0^{t+h}\xi(X_s)\,\mathrm{d}s};\, X_{t+h}\geq y\Big]}{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s};\, X_{t}\geq y\Big]} \leq C\frac{Y^{\approx\approx}_{v'}(x,y)}{Y^{\approx\approx}_{v}(x,y)}. \tag{2.5.17}$$

The fraction on the right-hand side can be rewritten with the help of (2.5.12). Using also the fact that the integral appearing in (2.5.12) is of order $(y-x)^{-1/2}$ uniformly in $v \in V$ and $x, y \geq \mathcal{N}$, as explained at the end of the proof of Lemma 2.5.4, we obtain (cf. [DS22, (3.36)])

$$\frac{Y^{\approx\approx}_{v'}(x,y)}{Y^{\approx\approx}_{v}(x,y)} \leq C\frac{\exp\big\{-(y-x)\big(\frac{\eta^\zeta_{x,y}(v')}{v'} - \overline{L}^\zeta_{x,y}(\eta^\zeta_{x,y}(v'))\big)\big\}}{\exp\big\{-(y-x)\big(\frac{\eta^\zeta_{x,y}(v)}{v} - \overline{L}^\zeta_{x,y}(\eta^\zeta_{x,y}(v))\big)\big\}}. \tag{2.5.18}$$

Now—cf. [DS22, (3.4)]—denoting for any $\eta < 0$

$$S^{\zeta,v}_{x,y}(\eta) := (y-x)\Big(\frac{\eta}{v} - \overline{L}^\zeta_{x,y}(\eta)\Big), \tag{2.5.19}$$

the logarithm of the fraction on the right-hand side of (2.5.18) can be written as

$$\big(S^{\zeta,v}_{x,y}(\eta^\zeta_{x,y}(v)) - S^{\zeta,v}_{x,y}(\eta^\zeta_{x,y}(v'))\big) + \big(S^{\zeta,v}_{x,y}(\eta^\zeta_{x,y}(v')) - S^{\zeta,v'}_{x,y}(\eta^\zeta_{x,y}(v'))\big) \tag{2.5.20}$$

Recalling the definitions of v and v', the second summand in (2.5.20) satisfies

$$\big(S^{\zeta,v}_{x,y}(\eta^\zeta_{x,y}(v')) - S^{\zeta,v'}_{x,y}(\eta^\zeta_{x,y}(v'))\big) = -h\eta^\zeta_{x,y}(v') \leq ch, \tag{2.5.21}$$

since $\frac{1}{c'} \leq \eta^\zeta_{x,y}(v') \leq c' < 0$ for the considered x, y, v', due to Lemma 2.5.2(a). Moreover, the absolute value of the first summand in (2.5.20) can be upper bounded by $ch^2/t \ll h$ uniformly for x, y, t and h under consideration, exactly as in the paragraph containing [DS22, (3.39)] (this proof uses again only estimates that are uniform in ζ). This completes the proof of part (a).

The proof of part (b) follows the lines of the proof of the spatial perturbation result, Lemma 4.1 in [ČDS23]: Using the same reasoning as in (2.5.17)–(2.5.20), now choosing $v := (y-x)/t$ and $v' := (y+h-x)/t$, where x, y, t and h are as in the statement and $\mathcal{T}_2$ is assumed to be sufficiently large so that $y - x \geq \mathcal{N} \vee \mathcal{N}'$, we infer that

$$\frac{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s};\, X_{t}\geq y+h\Big]}{E_x\Big[e^{\int_0^{t}\xi(X_s)\,\mathrm{d}s};\, X_{t}\geq y\Big]} \leq C\frac{Y^{\approx\approx}_{v'}(x,y+h)}{Y^{\approx\approx}_{v}(x,y)} \tag{2.5.22}$$

as well as

$$\begin{aligned}\ln \frac{Y^{\approx}_{v'}(x,y+h)}{Y^{\approx}_{v}(x,y)} &\leq \big(S^{\zeta,v}_{x,y}(\eta^{\zeta}_{x,y}(v)) - S^{\zeta,v'}_{x,y+h}(\eta^{\zeta}_{x,y}(v))\big) \\ &\quad + \big(S^{\zeta,v'}_{x,y+h}(\eta^{\zeta}_{x,y}(v)) - S^{\zeta,v'}_{x,y+h}(\eta^{\zeta}_{x,y+h}(v'))\big).\end{aligned} \tag{2.5.23}$$

By (2.5.10) and (2.5.19), the first summand on the right-hand side of (2.5.23) (which differs slightly from the corresponding one in [ČDS23], due to the difference (B)) satisfies

$$\begin{aligned}&\Big|S^{\zeta,v}_{x,y}(\eta^{\zeta}_{x,y}(v)) - S^{\zeta,v'}_{x,y+h}(\eta^{\zeta}_{x,y}(v))\Big| \\ &\quad= \Big|\ln E_x\Big[e^{\int_0^{H_{y+h}}(\zeta(X_s)+\eta^{\zeta}_{x,y}(v))\,\mathrm{d}s}\Big] - \ln E_x\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta^{\zeta}_{x,y}(v))\,\mathrm{d}s}\Big]\Big| \\ &\quad= \Big|\ln E_y\Big[e^{\int_0^{H_{y+h}}(\zeta(X_s)+\eta^{\zeta}_{x,y}(v))\,\mathrm{d}s}\Big]\Big| \\ &\quad\leq h\sqrt{2(\mathtt{es}-\mathtt{ei}+|\eta^{\zeta}_{x,y}(v)|)} \leq ch,\end{aligned} \tag{2.5.24}$$

where in the second equality we applied the strong Markov property at time H_y, and used (2.4.16) for the final inequality.

The second summand on the right-hand side of (2.5.23) is bounded by $ch^2/t \ll h$ and is thus negligible. This can be proved exactly as in [ČDS23, (4.13)–(4.16)]. Besides [DS22, Lemma 2.7], which we already extended in Lemma 2.5.3, this proof again only uses uniform estimates and thus does not require any modification. This completes the proof of the proposition. □

2.6 Dependence of solutions to the F-KPP equation on the initial condition

In this section we prove the key technical lemma, Lemma 2.6.1 below, which formalises inequalities (2.2.15) and (2.2.16) from the introduction, and which provides the right ordering of two solutions to (F-KPP) with different initial conditions. Its proof is based on a careful examination of the Feynman-Kac representations of these solutions, using all tools that were introduced in previous sections.

To state the lemma, we introduce two auxiliary velocities,

$$v_1 := \sqrt{2(\mathtt{es}+1)} \quad \text{and} \tag{2.6.1}$$

$$v_2 := \inf\{v > v_1 + 1 : |\overline{\eta}(v)| \geq 2v_1^2 + 2\}, \tag{2.6.2}$$

where $\overline{\eta}(v)$ was defined above (2.4.6); note that display (2.4.6) also ensures that v_2 is finite. By comparing the BBMRE with the BBM with constant branching rate $\mathtt{es}$, for which the speed of the maximum is $\sqrt{2\mathtt{es}}$, we obtain

$$v_0 < v_1 < v_2.$$

Lemma 2.6.1. *For each $u > 0$ and each $v > v_2$, there exists $\Delta_0 = \Delta_0(u, v) > 0$ as well as a $\mathbb{P}$-a.s. finite random variable $\mathcal{T} = \mathcal{T}(u, v)$, such that $\mathbb{P}$-a.s., for all $\Delta > \Delta_0$, $y \in [0, vt]$ and $t \geq \mathcal{T}$,*

$$w^y(t, y - vt) \geq w^{y+\Delta}(t + u, y - vt). \tag{2.6.3}$$

Proof. We start with upper bounding the right-hand side of (2.6.3). By the Feynman-Kac representation (2.3.7) and the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) = 1$, cf. Proposition 2.3.3, it follows with any $\Delta > 0$ that

$$\begin{aligned} w^{y+\Delta}(t + u, y - vt) &= E_{y-vt}\Big[e^{\int_0^{t+u} \xi(X_s)\widetilde{F}(w^{y+\Delta}(t+u-s,X_s))\,\mathrm{d}s}; X_{t+u} \geq y + \Delta\Big] \\ &\leq E_{y-vt}\Big[e^{\int_0^{t+u} \xi(X_s)\,\mathrm{d}s}; X_{t+u} \geq y + \Delta\Big]. \end{aligned} \tag{2.6.4}$$

To the right-hand side of (2.6.4) we now successively apply both parts of the perturbation Proposition 2.5.1 (with V sufficiently large, as explained before Proposition 2.5.1 and $A = 2v$). In order to apply them, we let $t \geq u \vee \mathcal{T}_1 \vee \mathcal{T}_2 =: \mathcal{T}$, where $\mathcal{T}_1, \mathcal{T}_2$ are the $\mathbb{P}$-a.s. finite random variables occurring in the statement of the perturbation lemma. For such t, we then obtain

$$\begin{aligned} w^{y+\Delta}(t + u, y - vt) &\leq c_1 e^{c_1 u} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y + \Delta\Big] \\ &\leq c_1 c_2 e^{c_1 u - \Delta/c_2} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\Big], \end{aligned} \tag{2.6.5}$$

which is our first intermediate inequality.

Let us now turn our focus to bounding the left-hand side of (2.6.3) from below. By the Feynman-Kac representation (2.3.7),

$$w^y(t, y - vt) = E_{y-vt}\Big[\exp\Big\{\int_0^t \xi(X_r)\widetilde{F}(w^y(t - r, X_r))\,\mathrm{d}r\Big\}; X_t \geq y\Big]. \tag{2.6.6}$$

We now claim that $\widetilde{F}$ satisfies,

$$\widetilde{F}(w) = F(w)/w \geq 1 - \tfrac{1}{2}(\mu_2 - 2)w, \qquad w \in [0, 1]. \tag{2.6.7}$$

Indeed, by (2.3.1) and the normalisation (2.3.4) of Remark 2.3.2, the non-linearity F of (F-KPP) satisfies $F(0) = 0$ and $F'(0) = 1$. In addition, by (2.3.5), $F'' \geq -\mu_2 + 2$ on $[0, 1]$. Therefore, by a first order Taylor approximation with Lagrange remainder,

$$F(w) \geq w + \frac{1}{2} \inf_{w^*\in[0,1]} F''(w^*)w^2 = w - \frac{1}{2}(\mu_2 - 2)w^2,$$

from which (2.6.7) directly follows.

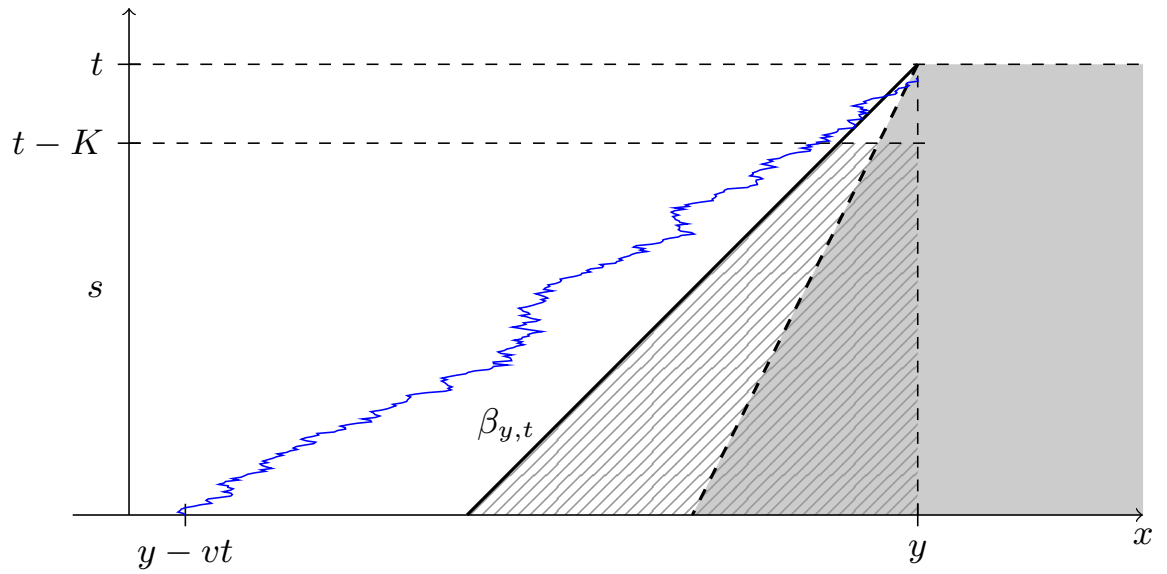

Figure 2.3: Sketch of a trajectory of the Brownian motion $(X_s)_{s\geq 0}$, started at $y - vt$, up until the hitting time H_y of y, which realises the good event $\mathcal{G}$. This trajectory does not hit the moving barrier $\beta_{y,t}(s)$ (thick solid line) in the time interval $[0, t-K]$ and thus avoids the dashed region. The function $w^y(t-s,\cdot)$ is close to 1 in the grey region, close to 0 in its complement, and changes its value from 0 to 1 in the vicinity of the thick dashed line whose slope is v_0.

Plugging (2.6.7) into (2.6.6) and using the uniform ellipticity (2.2.4) from Assumption 1, we arrive at

$$w^y(t, y-vt) \geq E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}\; e^{-\frac{\mathrm{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s}; X_t \geq y\Big]. \tag{2.6.8}$$

In order to obtain a suitable control of the second exponential factor in (2.6.8), we construct an event restricted to which the second exponential is bounded from below in a suitable way. For this purpose, recall the definition of v_1 from (2.6.1), and introduce for given t, y the *moving boundary*

$$\beta_{y,t}(s) := y - v_1(t-s), \qquad s \in [0,t]. \tag{2.6.9}$$

By $\mathcal{T}_{y,t} := \inf\{s \geq 0 : X_s = \beta_{y,t}(s)\}$ we denote the first hitting time of $\beta_{y,t}$ by a Brownian motion started at $y - vt$.

We claim that for $K > 1 \vee v_1^{-2}$ to be fixed later, on the good event $\mathcal{G} := \{\mathcal{T}_{y,t} \in [t-K, t]\}$, it holds that

$$\int_0^{t-K} w^y(t-s, X_s)\,\mathrm{d}s \leq 1, \tag{2.6.10}$$

see Figure 2.3 for an illustration. Indeed, note that using again the Feynman-Kac representation (2.3.7) as well as the uniform ellipticity (2.2.4) of Assumption 1, in combination with the fact that $\sup_{w\in[0,1]} \widetilde{F}(w) = 1$ once more, it holds that

$$\begin{aligned} w^y(t-s, X_s) &\leq E_{X_s}\Big[e^{\int_0^{t-s}\xi(\widetilde{X}_r)\,\mathrm{d}r}; \widetilde{X}_{t-s} \geq y\Big] \\ &\leq e^{\mathtt{es}(t-s)} P_{X_s}\big(\widetilde{X}_{t-s} \geq y\big), \end{aligned}$$

where we write $\widetilde{X}$ for an independent Brownian motion started at X_s in order to avoid confusion of the two processes. On $\mathcal{G}$ one has that $X_s \leq y - v_1(t-s)$ for $s \in [0, t-K]$. Hence, by a straightforward coupling argument, on $\mathcal{G}$ we have

$$P_{X_s}\big(\widetilde{X}_{t-s} \geq y\big) \leq P_0\big(\widetilde{X}_{t-s} \geq v_1(t-s)\big) = P(Z \geq v_1\sqrt{t-s}),$$

where Z is a standard Gaussian random variable. Using this in combination with a standard Gaussian bound (see e.g. [AT07, (1.2.2)]) and taking advantage of the fact that by assumption $v_1\sqrt{(t-s)} \geq v_1\sqrt{K} \geq 1$, it follows that on $\mathcal{G}$ we can upper bound

$$\begin{aligned} \int_0^{t-K} w^y(t-s, X_s)\,\mathrm{d}s &\leq \int_0^{t-K} e^{\mathtt{es}(t-s)} P\big(Z \geq v_1\sqrt{t-s}\big)\,\mathrm{d}s \\ &\leq \frac{1}{\sqrt{2\pi}} \int_0^{t-K} e^{-(v_1^2/2-\mathtt{es})(t-s)}\,\mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_K^t e^{-(v_1^2/2-\mathtt{es})z}\,\mathrm{d}z \\ &\leq \frac{1}{\sqrt{2\pi}(v_1^2/2-\mathtt{es})} e^{-K(v_1^2/2-\mathtt{es})} \leq 1, \end{aligned} \tag{2.6.11}$$

where in the last inequality we used $v_1^2/2 - \mathtt{es} = 1$, which holds by (2.6.1). This proves (2.6.10).

Coming back to the task of finding a lower bound for the right-hand side of (2.6.8), we infer by the above discussion that on $\mathcal{G}$ we can use (2.6.10) to bound the second exponential factor on the right-hand side of (2.6.8) by

$$\begin{aligned} e^{-\frac{\mathtt{es}}{2}(\mu_2-2)\int_0^t w^y(t-s,X_s)\,\mathrm{d}s} &\geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)\left(1+\int_{t-K}^t w^y(t-s,X_s)\,\mathrm{d}s\right)} \\ &\geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)}, \end{aligned} \tag{2.6.12}$$

where in the last inequality we used that $0 \leq w^y(s,y) \leq 1$ uniformly for all $(s,y) \in [0,\infty) \times \mathbb{R}$. Consequently, by restricting the expectation on the right-hand side of (2.6.8) to $\mathcal{G}$, it follows by (2.6.12) that whenever $v > v_1$, then

$$w^y(t, y-vt) \geq e^{-\frac{\mathtt{es}}{2}(\mu_2-2)(1+K)} E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y,\ \mathcal{G}\Big]. \tag{2.6.13}$$

This is our second intermediate inequality.

In order to finish the proof of (2.6.3), we need to compare the expectations on the right-hand side of (2.6.5) and on the right-hand side of (2.6.13). This is the purpose of the following lemma.

Lemma 2.6.2. *Let v_2 be as in (2.6.2). Then for every $v > v_2$ there exists constants $K = K(v)$, $\widetilde{C} = \widetilde{C}(v) \in (0, \infty)$ such that for $\mathbb{P}$-a.a. ξ, for all t large enough and all $y \in [0, vt]$, one has*

$$E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y\big] \leq \widetilde{C} E_{y-vt}\big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y,\, \mathcal{G}\big]. \tag{2.6.14}$$

We postpone the proof of Lemma 2.6.2 and complete the proof of Lemma 2.6.1 first. By combining the lower bound (2.6.13), the upper bound (2.6.5) and Lemma 2.6.2, we obtain.

$$\begin{aligned} &w^y(t, y - vt) - w^{y+\Delta}(t + u, y - vt) \\ &\geq \big(e^{-\frac{\mathtt{es}}{2}(\mu_2 - 2)(K+1)} - \widetilde{C} c_1 c_2 e^{c_1 u - \Delta/c_2}\big) E_{y-vt}\Big[e^{\int_0^t \xi(X_s)\,\mathrm{d}s}; X_t \geq y, \mathcal{G}\Big]. \end{aligned}$$

For every Δ satisfying

$$\Delta \geq \Delta_0 := c_2\Big(c_1 u + \frac{\mathtt{es}}{2}(\mu_2 - 2)(K+1) + \ln(\widetilde{C} c_1 c_2)\Big),$$

the right-hand side is positive, which proves (2.6.3) and thus the lemma. □

Proof of Lemma 2.6.2. To prove the lemma, we use the machinery of tilted measures as introduced in Section 2.4. We recall the notation $\zeta = \xi - \mathtt{es}$ from (2.4.1) and observe that, by multiplying both sides of (2.6.14) by $e^{-\mathtt{es}\,t}$, it is sufficient to show (2.6.14) with ζ in place of ξ.

We start by proving an upper bound for the left-hand side of (2.6.14) in terms of tilted measures. By Lemma 2.5.5 there exist constants $C, L < \infty$ such that for any $\eta < 0$, for t large enough uniformly in $y \in [0, vt]$ it holds that

$$\begin{aligned} E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y\big] &\leq C E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}\,; H_y \in [t-L, t]\big] \\ &\leq C e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-L, t]\big). \end{aligned} \tag{2.6.15}$$

In the next step, we bound the expression appearing on the right-hand side of (2.6.14) from below. To this end, let $p^{\zeta,\eta}_y(t) := P^{\zeta,\eta}_y(X_t \geq y)$. Using the strong Markov property we obtain

$$\begin{aligned} &E_{y-vt}\big[e^{\int_0^t \zeta(X_s)\,\mathrm{d}s}; X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &\geq e^{-(\mathtt{es}-\mathtt{ei})K} E_{y-vt}\big[e^{\int_0^{H_y} \zeta(X_s)\,\mathrm{d}s}; H_y \in [t-K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &\geq e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} E_{y-vt}\big[e^{\int_0^{H_y} (\zeta(X_s)+\eta)\,\mathrm{d}s}; H_y \in [t-K, t], X_t \geq y, \mathcal{T}_{y,t} \geq t - K\big] \\ &= e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} E^{\zeta,\eta}_{y-vt}\big[p^{\zeta,\eta}_y(t - H_y), H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t - K\big] \\ &\geq \frac{1}{2} e^{-(\mathtt{es}-\mathtt{ei}-\eta)K} e^{-\eta t} Z^{\zeta,\eta}_{y-vt,y} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t - K\big), \end{aligned} \tag{2.6.16}$$

where in the last inequality we used Lemma 2.4.4 to infer that for any $\eta < 0$ and $s \geq 0$ one has $p_y^{\zeta,\eta}(s) \geq P_0^{\sqrt{2|\eta|}}(X_s \geq 0) \geq 1/2$.

In view of (2.6.15) and (2.6.16), in order to complete the proof of Lemma 2.6.2, it is sufficient to show that

$$P_{y-vt}^{\zeta,\eta}\big(H_y \in [t-L,t]\big) \leq C P_{y-vt}^{\zeta,\eta}\big(H_y \in [t-K,t], \mathcal{T}_{y,t} \geq t-K\big), \tag{2.6.17}$$

for some suitably chosen parameter η and constants C, K, L, $\mathbb{P}$-a.s. for all t large, uniformly in $y \in [0, vt]$.

To this end we will need two further auxiliary lemmas. The first one will be used to upper bound the probability appearing on the right-hand side of (2.6.17), and also specifies the range of suitable η's.

Lemma 2.6.3. *Let $\eta < 0$ be such that $\sqrt{2|\eta|} > v_1(1 + \frac{2L}{K})$, and let $0 < L < K$ be such that $L/K \leq 1/3$. Then, $\mathbb{P}$-a.s. for every $y \in \mathbb{R}$ and $v > v_1$,*

$$P_{y-vt}^{\zeta,\eta}\big(H_y \leq t, \mathcal{T}_{y,t} \leq t-K\big) \leq 2P_{y-vt}^{\zeta,\eta}(H_y < t-L). \tag{2.6.18}$$

The second auxiliary lemma is a quantitative extension of a part of Proposition 3.5 of [DS22]. It states that under the tilted measure, if the tilting is not too strong, the probability to cross a large interval in time t is comparable to the probability of crossing the same interval in time $t - L$.

Lemma 2.6.4. *For every $v > v_c$ there is $c = c(v) < \infty$ such that for all L large enough and $\eta \in (\overline{\eta}(v) + \frac{c}{L}, 0)$, $\mathbb{P}$-a.s. for all t large enough and $|y| \leq 2vt$,*

$$P_{y-vt}^{\zeta,\eta}(H_y \leq t-L) \leq \frac{1}{4} P_{y-vt}^{\zeta,\eta}(H_y \leq t),$$

and as a consequence,

$$P_{y-vt}^{\zeta,\eta}(H_y \leq t-L) \leq \frac{1}{3} P_{y-vt}^{\zeta,\eta}\big(H_y \in (t-L,t]\big).$$

In order not to hinder the flow or reading, we postpone the proofs of these two lemmas to the end of the current section. We now come back to the proof of Lemma 2.6.2 and complete it by showing (2.6.17). To this end we choose the parameters η, K, and L in such a way that the previous two lemmas can be used simultaneously. More precisely, for a given $v \geq v_2$ we fix arbitrary η so that

$$|\overline{\eta}(v)| - 1 > |\eta| > 2v_1^2, \tag{2.6.19}$$

which is possible by the definition of v_2 in (2.6.2). Then we fix L as large as required in Lemma 2.6.4. Consequently, due to (2.6.19), the required assumptions on η are satisfied in our setting. Finally, we fix $K \geq 3L$ and observe that, in combination

with (2.6.19), $\sqrt{2|\eta|} > 2v_1 \geq v_1(1 + \frac{2L}{K})$, so that the assumptions of Lemma 2.6.3 are satisfied as well.

With this choice of constants, noting that $\{H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t-K\} = \{H_y \leq t, \mathcal{T}_{y,t} \geq t-K\}$ (cf. Figure 2.3 also), the right-hand side of (2.6.17) satisfies

$$\begin{aligned} &P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-K, t], \mathcal{T}_{y,t} \geq t-K\big) \\ &\quad = P^{\zeta,\eta}_{y-vt}\big(H_y \leq t\big) - P^{\zeta,\eta}_{y-vt}\big(H_y \leq t, \mathcal{T}_{y,t} < t-K\big) \\ &\quad \geq P^{\zeta,\eta}_{y-vt}\big(H_y \leq t\big) - 2P^{\zeta,\eta}_{y-vt}\big(H_y \leq t-L\big), \end{aligned} \tag{2.6.20}$$

where the last inequality follows from Lemma 2.6.3. This can be written as

$$P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-L, t]\big) - P^{\zeta,\eta}_{y-vt}\big(H_y \leq t-L\big) \geq \frac{2}{3} P^{\zeta,\eta}_{y-vt}\big(H_y \in [t-L, t]\big), \tag{2.6.21}$$

where the last inequality is a direct consequence of Lemma 2.6.4. Now combining (2.6.20) and (2.6.21) we obtain (2.6.17), which completes the proof. □

It remains to provide the proofs of Lemmas 2.6.3 and 2.6.4.

Proof of Lemma 2.6.3. Using the tower property for conditional expectations we obtain

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y < t-L) &\geq P^{\zeta,\eta}_{y-vt}(H_y < t-L, \mathcal{T}_{y,t} \leq t-K) \\ &= E^{\zeta,\eta}_{y-vt}\big[\mathbf{1}_{\{\mathcal{T}_{y,t} \leq t-K\}} P^{\zeta,\eta}_{y-vt}(H_y < t-L \mid \mathcal{F}_{\mathcal{T}_{y,t}})\big], \end{aligned} \tag{2.6.22}$$

where $\mathcal{F}_{\mathcal{T}_{y,t}}$ is the canonical stopped σ-algebra associated to $\mathcal{T}_{y,t}$. It follows from Lemma 2.4.4 that the drift of X under the tilted measure $P^{\zeta,\eta}_{y-vt}$ is always larger than $\sqrt{2|\eta|}$. On the event $\{0 \leq \mathcal{T}_{y,t} \leq t-K\}$, by the strong Markov property at time $\mathcal{T}_{y,t}$ and using that $X_{\mathcal{T}_{y,t}} = \beta_{y,t}(\mathcal{T}_{y,t})$, it holds that

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}\big(H_y < t-L \mid \mathcal{F}_{\mathcal{T}_{y,t}}\big) &= P^{\zeta,\eta}_{X_{\mathcal{T}_{y,t}}}\big(H_y < t-L-\mathcal{T}_{y,t}\big) \\ &\geq \inf_{0 \leq u \leq t-K} P^{\zeta,\eta}_{\beta_{y,t}(u)}(H_y \leq t-u-L) \\ &\geq \inf_{0 \leq u \leq t-K} P^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}(H_y \leq t-u-L). \end{aligned} \tag{2.6.23}$$

Recalling the assumptions of the lemma, for $u \in [0, t-K]$ we have that

$$\begin{aligned} E^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}(X_{t-u-L}) &= \beta_{y,t}(u) + \sqrt{2|\eta|}(t-u-L) \\ &\geq y - v_1(t-u) + v_1\big(1 + \tfrac{2L}{K}\big)(t-u-L) \\ &\geq y - v_1 L + v_1 \tfrac{2L}{K}(K-L) \geq y + \tfrac{1}{3} v_1 L \geq y, \end{aligned} \tag{2.6.24}$$

where for the penultimate inequality we used $K-L \geq \frac{2}{3}K$, which holds by assumption. In combination with the fact that X is a Brownian motion with drift under $P^{\sqrt{2|\eta|}}_{\beta_{y,t}(u)}$, it follows that the probability on the right-hand side of (2.6.23) is at least $1/2$. Plugging this back into (2.6.22) we arrive at

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y < t-L) &\geq \frac{1}{2} P^{\zeta,\eta}_{y-vt}\big(\mathcal{T}_{y,t} \leq t-K\big) \\ &\geq \frac{1}{2} P^{\zeta,\eta}_{y-vt}\big(\mathcal{T}_{y,t} \leq t-K, H_y \leq t\big), \end{aligned}$$

as claimed. □

Next we give the proof of Lemma 2.6.4.

Proof of Lemma 2.6.4. The first part of the proof of this lemma follows the same steps as the proof of Proposition 3.5 of [DS22] (see also the proof of Lemma 2.5.4). By Lemma 2.5.2(a), $\mathbb{P}$-a.s. for all t large enough, and all $|y| \leq 2vt$, there exist constants $\eta^{\zeta}_{y-vt,y}(v)$ so that

$$E^{\zeta,\eta^{\zeta}_{y-vt,y}(v)}_{y-vt}[H_y] = t. \tag{2.6.25}$$

To simplify the notation we write $\widetilde{\eta} := \eta^{\zeta}_{y-vt,y}(v)$. Using Lemma 2.5.2(b), we can assume that $\widetilde{\eta} < \overline{\eta}(v) + \frac{c}{2L}$, and thus, by the hypothesis of the lemma,

$$\eta - \widetilde{\eta} > \frac{c}{2L}. \tag{2.6.26}$$

By definition of tilted measures (2.4.3),

$$\begin{aligned} P^{\zeta,\eta}_{y-vt}(H_y \leq t-L) &= \frac{1}{Z^{\zeta,\eta}_{y-vt,y}} E_{y-vt}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\eta)\,\mathrm{d}s}; H_y \leq t-L\Big] \\ &= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} \frac{1}{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}} E_{y-vt}\Big[e^{\int_0^{H_y}(\zeta(X_s)+\widetilde{\eta})\,\mathrm{d}s} e^{-H_y(\widetilde{\eta}-\eta)}; H_y \leq t-L\Big]. \\ &= \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} E^{\zeta,\widetilde{\eta}}_{y-vt}\Big[e^{-H_y(\widetilde{\eta}-\eta)}; H_y \leq t-L\Big]. \end{aligned} \tag{2.6.27}$$

Define random variables $\tau_i = H_{y-vt+i} - H_{y-vt+i-1}$, $i = 1, \ldots, \lfloor vt \rfloor$, and $\tau_{vt} = H_y - H_{y-vt+\lfloor vt \rfloor}$, so that $\sum_{i=1}^{\lfloor vt \rfloor} \tau_i + \tau_{vt} = H_y$, and their re-centred versions $\widehat{\tau}_i = \tau_i - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_i]$ for $i = 1, \ldots, \lfloor vt \rfloor$, and $\widehat{\tau}_{vt} = \tau_{vt} - E^{\zeta,\widetilde{\eta}}_{y-vt}[\tau_{vt}]$. Further, let

$$Y^{\zeta}_{y-vt,y} := \tfrac{(\widetilde{\eta}-\eta)}{\widetilde{\sigma}}\big(\sum_{i=1}^{\lfloor vt \rfloor} \widehat{\tau}_i + \widehat{\tau}_{vt}\big), \tag{2.6.28}$$

where

$$\widetilde{\sigma} = \widetilde{\sigma}^{\zeta}_{y-vt,y}(v) = |\widetilde{\eta} - \eta| \sqrt{\mathrm{Var}_{P^{\zeta,\widetilde{\eta}}_{y-vt}}(H_y)}. \tag{2.6.29}$$

is chosen so that the variance of $Y^{\zeta}_{y-vt,y}$ is one. Denoting by $\mu^{\zeta}_{y-vt,y}$ the distribution of $Y^{\zeta}_{y-vt,y}$ under $P^{\zeta,\widetilde{\eta}}_{y-vt,y}$, using also the fact that $E^{\zeta,\widetilde{\eta}}_{y-vt,y}[H_y] = t$, by the definition of $\widetilde{\eta}$, (2.6.27) can be rewritten as

$$\begin{aligned} &P^{\zeta,\widetilde{\eta}}_{y-vt}(H_y \leq t - L) \\ &\quad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} E^{\zeta,\widetilde{\eta}}_{y-vt}\Big[e^{-\widetilde{\sigma} Y^{\zeta}_{y-vt,y}}; Y^{\zeta}_{y-vt,y} \in \Big[\frac{L(\eta-\widetilde{\eta})}{\widetilde{\sigma}}, \infty\Big)\Big] \\ &\quad = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} \int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u). \end{aligned} \tag{2.6.30}$$

Setting $L = 0$ in the above formula we further obtain

$$P^{\zeta,\eta}_{y-vt}(H_y \leq t) = \frac{Z^{\zeta,\widetilde{\eta}}_{y-vt,y}}{Z^{\zeta,\eta}_{y-vt,y}} e^{(\eta-\widetilde{\eta})t} \int_{0}^{\infty} e^{-\widetilde{\sigma} u} \mu^{\zeta}_{y-vt,y}(\mathrm{d}u), \tag{2.6.31}$$

Hence, to finish the proof of the lemma, it suffices to show that the integral on the right-hand side of (2.6.30) is at most 1/4 of the integral on the right-hand side of (2.6.31).

To see this we proceed as in the proof of Lemma 3.6 of [DS22]. By the strong Markov property the random variables $\widehat{\tau}_i$, $i = 1, \ldots, \lfloor vt \rfloor$, and $\widehat{\tau}_{vt}$ are independent under $P^{\zeta,\widetilde{\eta}}_{y-vt}$. Further, it is a straightforward consequence of the definitions of the logarithmic moment generating functions in [DS22, (2.7)] and their being well defined for $\eta < 0$ that these random variables have uniform exponential moments. Moreover, recall that $\widetilde{\sigma}$ was chosen such that the variance of $\mu^{\zeta}_{y-vt,y}$ is one. This allows the application of a local central limit theorem for independent normalised sequences [BR10, Theorem 13.3], which implies that

$$\sup_{B} |\mu^{\zeta}_{y-vt,y}(B) - \Phi(B)| \leq c_1 (\lceil vt \rceil)^{-1/2}, \tag{2.6.32}$$

where the supremum is taken over all intervals B in $\mathbb{R}$ and Φ denotes the standard Gaussian measure. Note that the constant c_1 in the last display depends only on the uniform bound of the exponential moments of the $\widehat{\tau}_i$'s. Without loss of generality, we can assume that $c_1 > 4$. We also note that by [DS22, (3.8)] (see also (2.5.9)) the variance $\widetilde{\sigma}^2$ defined in (2.6.29) satisfies for $\mathbb{P}$-a.a. ζ and t large enough

$$c_2^{-1} \sqrt{\lceil vt \rceil} \leq \widetilde{\sigma} \leq c_2 \sqrt{\lceil vt \rceil}. \tag{2.6.33}$$

We now have all ingredients to finish the proof. To this end, we assume that the constant c from the statement of the lemma satisfies the inequality

$$\ell := \frac{L(\eta - \widetilde{\eta})}{\widetilde{\sigma}} \geq \frac{c}{2c_2\sqrt{vt}} \geq \frac{20 c_1}{\sqrt{vt}}. \tag{2.6.34}$$

To bound the integral in (2.6.30) from above, we observe that for any interval (a, b) of length ℓ we have $\Phi((a, b)) \le \ell/\sqrt{2\pi}$ and thus $\mu^{\zeta}_{y-vt,y}((a, b)) \le (\ell + c_1/\sqrt{vt}) \le 2\ell$, by (2.6.34). Therefore, using (2.6.33) in the last step,

$$\begin{aligned}\int_{L(\eta-\widetilde{\eta})/\widetilde{\sigma}}^{\infty} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u) &\le \sum_{i=1}^{\infty} e^{-\widetilde{\sigma}i\ell}\mu^{\zeta}_{y-vt,y}((i\ell,(i+1)\ell)) \\ &\le \frac{2\ell e^{-\widetilde{\sigma}\ell}}{1-e^{-\widetilde{\sigma}\ell}} \le \frac{2\widetilde{\sigma}\ell e^{-\widetilde{\sigma}\ell}}{1-e^{-\widetilde{\sigma}\ell}} \cdot \frac{c_2}{\sqrt{vt}}.\end{aligned} \tag{2.6.35}$$

On the other hand, using the rough bound $\Phi((0, x)) \ge x/5$ which holds for small enough x, and (2.6.34),

$$\begin{aligned}\int_0^{\infty} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u) &\ge \int_0^{L(\eta-\widetilde{\eta})/2\widetilde{\sigma}} e^{-\widetilde{\sigma}u}\mu^{\zeta}_{y-vt,y}(\mathrm{d}u) \\ &\ge e^{-\widetilde{\sigma}\ell/2}\mu^{\zeta}_{y-vt,y}((0,\ell/2)) \ge e^{-\widetilde{\sigma}\ell/2}\Big(\Phi((0,\ell/2)) - \frac{c_1}{\sqrt{vt}}\Big) \\ &\ge e^{-\widetilde{\sigma}\ell/2}\frac{c_1}{\sqrt{vt}}.\end{aligned} \tag{2.6.36}$$

By increasing the value of the constant c and thus of $\widetilde{\sigma}\ell \ge c/2$, the right-hand side of (2.6.35) can be made at most $1/4$ as large as the right-hand side of (2.6.36). This completes the proof of the lemma. □

2.7 Proof of the tightness of the maximum of the BBMRE

We are now ready to prove the main theorem of this paper.

Proof of Theorem 2.2.1. For $\varepsilon \in (0, 1/2)$ let $x_t = x_t(\varepsilon) \in \mathbb{R}$ the unique location where

$$w^{x_t}(t, 0) = \mathtt{P}^{\xi}_0(M(t) \ge x_t) = \varepsilon. \tag{2.7.1}$$

As already explained in Section 2.2.1, to show the tightness of the re-centered maximum $M(t)$ we need to show that there exists $\Delta = \Delta(\varepsilon) < \infty$ such that for all $t > 0$ it holds that

$$w^{x_t-\Delta}(t, 0) = \mathtt{P}^{\xi}_0(M(t) \ge x_t - \Delta) > 1 - \varepsilon. \tag{2.7.2}$$

Note that (2.7.1) and the law of large numbers for $M(t)$ (that is $\lim_{t\to\infty} M(t)/t = v_0$, cf. (2.2.6)) imply that

$$\lim_{t\to\infty} \frac{x_t}{t} = v_0, \quad \mathbb{P}\text{-a.s.} \tag{2.7.3}$$

A direct consequence of (2.7.3) is that for t large enough, we can guarantee that x_t is large enough in order to apply Corollary 2.3.6 to $w^{x_t}(t,0)$. Therefore, for large enough t, we can infer the existence of a $\mathbb{P}$-a.s. finite time $T < \infty$ such that

$$w^{x_t}(t+t',0) \geq 1-\varepsilon \quad \text{for all } t' \in [T, T+1]. \tag{2.7.4}$$

For any $u \in \mathbb{N}$ define the subset $\Omega_u = \{T \in [u-1, u)\}$ of the probability space on which ξ is defined. We now consider $\xi \in \Omega_u$. Observe that (2.7.2) would follow from (2.7.4) on Ω_u, if for a suitably large $\Delta < \infty$ we had

$$w^{x_t-\Delta}(t,0) \geq w^{x_t}(t+u,0). \tag{2.7.5}$$

Instead of comparing these two function directly at $x = 0$ we use the Sturmian principle, to relate the inequality (2.7.5) at the origin to an inequality at some point on the negative half-line. More precisely, recall from Section 2.3.2 that for any $t > 0$, $u > 0$ and $\widetilde{\Delta} < \infty$ the difference

$$W^{u,\widetilde{\Delta}}(t,x) := w^{x_t-\widetilde{\Delta}}(t,x) - w^{x_t}(t+u,x)$$

solves a linear parabolic equation of the form (2.3.8), with initial condition

$$W^{u,\widetilde{\Delta}}(0,x) = \mathbf{1}_{[x_t-\widetilde{\Delta},\infty)}(x) - w^{x_t}(u,x).$$

Since $0 < w^{x_t}(u,x) < 1$ for all $u > 0$ and $x \in \mathbb{R}$, cf. (2.3.10), it follows moreover, that $W^{u,\widetilde{\Delta}}(0,x) > 0$ for $x > x_t - \widetilde{\Delta}$ and $W^{u,\widetilde{\Delta}}(0,x) < 0$ for $x < x_t - \widetilde{\Delta}$. Therefore it holds by Lemma 2.3.4 that for all $t > 0$ the sets

$$\{x \in \mathbb{R} : w^{x_t-\widetilde{\Delta}}(t,x) > w^{x_t}(t+u,x)\} = \{x \in \mathbb{R} : W^{u,\widetilde{\Delta}}(t,x) > 0\}$$

are open intervals unbounded to the right. Thus, in order to prove (2.7.5) it suffices to find some $x^* = x^*(t) < 0$ and $\Delta < \infty$ such that $W^{u,\Delta}(t,x^*) > 0$, as this implies $0 \in \{x \in \mathbb{R} : W^{u,\Delta}(t,x) > 0\}$, which in turn implies (2.7.5); for an illustration of this argument see Figure 2.4.

To find such $x^*(t)$ take any $v > v_2$, where v_2 is defined in (2.6.2). Since $v_2 > v_0$ and $(x_t - \Delta)/t \to v_0$, by (2.7.3), it follows that $x_t - \Delta \in [0, vt]$ for all t that are sufficiently large. Consequently we can apply Lemma 2.6.1 and infer the existence of an a.s. finite random variable $\mathcal{T}(u,v)$ and some $\Delta_0(u,v) > 0$ such that if we require, additional to the previous conditions on the size of t, that $t > \mathcal{T}(u,v)$ and that $\Delta > \Delta_0(u,v)$, then

$$w^{x_t-\Delta}(t,x^*) \geq w^{x_t}(t+u,x^*),$$

with $x^* = x_t - \Delta - vt < 0$. By the previous discussion this implies

$$w^{x_t-\Delta}(t,0) \geq w^{x_t}(t+u,0) \geq 1-\varepsilon,$$

and hence tightness of the family $(M(t) - m^{\xi}(t))_{t\geq 0}$ for $\mathbb{P}$-a.e. $\xi \in \Omega_u$. As $\Omega = \cup_{u\geq 1}\Omega_u$, this completes the proof. □

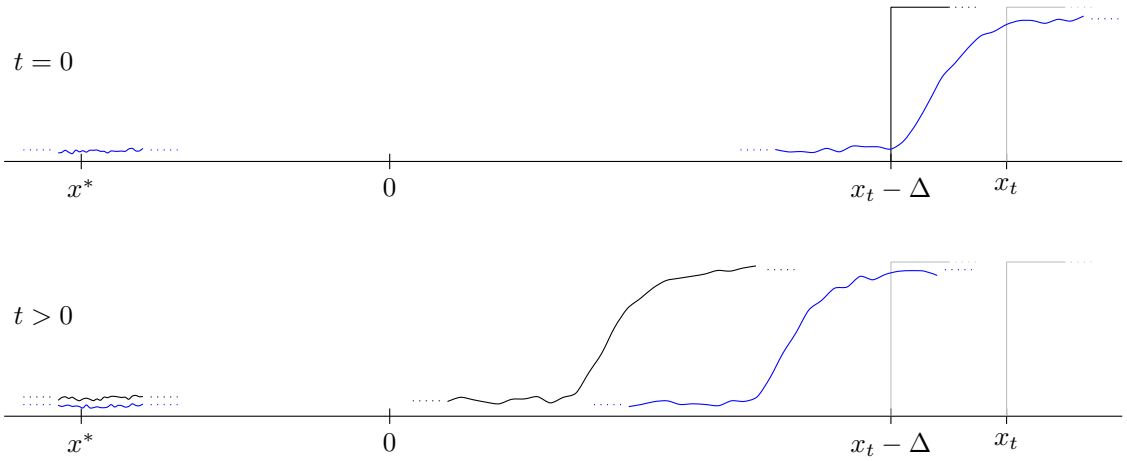

Figure 2.4: The top figure shows the graph of the function $w^{x_t-\Delta}(0,\cdot) = \mathbf{1}_{[x_t-\Delta,\infty)}(\cdot)$ in black and the function $w^{x_t}(T,\cdot)$ in blue. The lower figure shows the graph of the same functions at some positive time $t > 0$. By the Sturmian principle, for any $t > 0$, the region where $w^{x_t-\Delta}(t,\cdot)$ dominates $w^{x_t}(t+T,\cdot)$ is an interval that contains $[x^*,\infty)$.

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Part II

A branching annihilating random walk

3. Introduction

The second part of this thesis is concerned with an entirely different model of spatial branching system, a *branching annihilating random walk* (BARW), which evolves in discrete time on the d-dimensional integer lattice $\mathbb{Z}^d$. Contrary to BBMRE from Part I, the process we are concerned with in here does not evolve in an external random environment, but rather it locally interacts with itself and in this sense, generates its own environment with which it then interacts.

Let us start by giving an informal definition of the process. Consider an initial distribution of particles on the grid $\mathbb{Z}^d$. All particles have a unit life span and die simultaneously after creating the next generation. At the end of each generation, before its death, each particle gives birth to a Poisson number of offspring with mean $\mu > 0$. All the particle's children then jump independently to a uniformly chosen site within a fixed distance $R \in \mathbb{N}$ from their parent. Whenever two or more particles from the offspring generation try to occupy the same site, all particles on that site get *annihilated*. This defines a two-parameter stochastic process depending on the mean number of offspring μ and the maximal length of displacement R of any given particle from its parent particle. All particles are of the same type and for each generation, we are interested in whether a given site $x \in \mathbb{Z}^d$ is occupied or not. To this end, we attach to each site $x \in \mathbb{Z}^d$ the local state space $\{0, 1\}$. The synchronous updating of all local states lets us interpret the BARW $\eta = (\eta_n)_{n\geq 0}$ as a discrete-time Markov process where the full state space is given by its *configuration space* $\{0, 1\}^{\mathbb{Z}^d}$.

Note that the synchronous updating of local states lets us define the BARW directly on the (countably infinite) lattice $\mathbb{Z}^d$. Indeed, we can define the stochastic updating rule of the BARW with the help of a Markov kernel κ from $\{0, 1\}^{\mathbb{Z}^d}$ to $\{0, 1\}^{\mathbb{Z}^d}$, together with the discrete σ-algebras. We use the convention that κ is a probability measure in its *first* argument and a measurable function in its *second*. Now for any generation $n \geq 0$, the particle distribution $\eta_{n+1}(x)$ at a given site $x \in \mathbb{Z}^d$ is a function of the configuration η_n in a neighbourhood of x. More precisely, $\eta_{n+1}(x)$ is a function of $(\eta_n(y))_{y\in B_R(x)}$, where $B_R(x) = \{z \in \mathbb{Z}^d : \|x - z\| \leq R\}$, and of additional randomness. This is true independently for all $x \in \mathbb{Z}^d$. Hence, the conditional probability of finding η_{n+1} in some specific configuration $\zeta \in \{0, 1\}^{\mathbb{Z}^d}$, given the configuration of

$\eta_n \in \{0,1\}^{\mathbb{Z}^d}$, can be written as

$$\kappa(\mathrm{d}\zeta, \eta_n) = \prod_{x\in\mathbb{Z}^d} p_x(\mathrm{d}\zeta(x), \eta_n|_{B_R(x)}), \tag{3.0.1}$$

where $(p_x)_{x\in\mathbb{Z}^d}$ is a family of probability kernels from $\{0,1\}^{B_R(x)}$ to $\{0,1\}$ respectively, giving the local updating-rule at site x. I.e. $p_x(\{1\}, \eta_n|_{B_R(x)})$ is the probability that, given the particle configuration $\eta_n|_{B_R(x)} \in \{0,1\}^{B_R(x)}$, the dynamics of the BARW produces a particle at x at time $n+1$. In Section 4.1 of Chapter 4 below, we derive that for any configuration $\zeta \in \{0,1\}^{\mathbb{Z}^d}$, the measures $p_x(\cdot, \zeta|_{B_R(x)})$ are Bernoulli random measures, with μ-dependent parameter that is unimodal in the density of 1's in $\zeta|_{B_R(x)}$. The unimodality with respect to the density of the Bernoulli-parameters for the local updating rules reflects the intuitive understanding that having many particles near each other leads to more interaction, i.e. annihilation, between those particles.

By the description of the BARW via the kernel (3.0.1), we can interpret the BARW as an *interacting family of Markov processes*, better known as *probabilistic cellular automata* (PCA), cf. [FLN18]. This interpretation of BARW as a PCA is used heavily in Chapters 4 and 5 below. We therefore give a brief overview of some general aspects of PCA and how they can be related to the BARW.

PCA are the discrete-time Markov processes that have a stochastic updating rule that can be described by a product similar to (3.0.1), with any finite (or infinite) local neighbourhoods and any local kernels. Their name is derived from the more classical *cellular automata*, which are the analogous discrete-time (locally interacting) dynamical systems with *deterministic* updating-rule, cf. [Kar05, Kar12]. To better place these PCA in the context of dynamical systems, we collect part of the typical classification of spatially extended dynamical systems in Table 3.1.

Model Class	**Space**	**Time**	**State space**
Probabilistic Cellular Automaton (PCA)	discrete	discrete	discrete
Coupled Map Lattice (CML)	discrete	discrete	continuous
Interacting Particle System (IPS)	discrete	continuous	discrete

Table 3.1: Classification of spatially extended dynamical systems

In the classification of Table 3.1, interacting particle systems (IPS) stand out as not having a deterministic counterpart. Indeed, typically, IPS are defined such that *interaction events* (i.e. motion, branching, etc.) occur at specific random *rates*. We refer to [Swa17] for a comprehensive overview and [Lig85, Lig99] for an in-depth analysis of interacting particle systems.

All three systems in Table 3.1 are related to one-another and we make use of all three of these systems in Chapter 4 below. Importantly, IPS offer an alternative continuous-time approach to studying (discrete space) phenomena on discrete state spaces that are addressed by PCA. For this reason, PCA and IPS are sometimes grouped together under the umbrella term of *spin-systems*, the term coming from the theory of spin-glasses.

Remark 3.0.1. Despite their many similarities, one should distinguish between PCA and IPS carefully. Both are Markovian processes defined by families of interacting stochastic processes but have the essential difference that PCA update in parallel, i.e. synchronously, while IPS update sequentially, i.e. asynchronously, where all interaction events occur at specific rates independently for each particle. Consequently, interactions in IPS are not only due to a shared past but also due to constraints imposed on these arrival rates. This often leads to a more subtle construction for IPS on infinite lattices, by e.g. taking finite-region limits, cf. [Lig85, Section VII.3].

The main questions of interest for these spin-systems have to do with their long-time behaviour. The most basic of these questions being about the (non-)triviality of the system.

(Q1) Can one identify parameter-regimes of a specific model for which there is almost sure extinction? And parameter-regimes for which the process survives with positive probability?

If a non-trivial limiting behaviour can be established, the next natural question is that of describing the *invariant measures* of the model, as these are the possible limits of the distribution of the system as $t \to \infty$. Moreover, one wants to understand the domains of attraction of each of these measures, i.e. determining the class of initial distributions such that, conditioned on survival, the distribution of the process converges to that measure as $t \to \infty$.

(Q2) Do there exist invariant measures for the model? If so, how many?

(Q3) Conditioned on survival, what are the limiting distributions?

These three questions have been addressed for many different examples of spin-systems, both in discrete and continuous time, see e.g. [FLN18, Swa17] and references therein for PCA and IPS respectively. In Chapter 4 below, we add to this list by addressing and partially answering (Q1)–(Q3) for the BARW.

Monotonicity

Many tools and techniques for addressing (Q1)–(Q3) are based on coupling different spin-systems to each other. The most successful of these couplings make use of monotonicity properties that a system might possess.

To discuss (non-)monotonicity of spin-systems, we need to introduce a notion of monotonicity on probability kernels, cf. PCA are defined via kernels of the form (3.0.1). To this end, consider first the local state space $\{0,1\}$, which is endowed with the natural order $0 \leq 1$. This order can be extended to a partial order on the entire configuration space by setting for any $\zeta, \widetilde{\zeta} \in \{0,1\}^{\mathbb{Z}^d}$

$$\zeta \leq \widetilde{\zeta} \quad \text{iff} \quad \zeta(x) \leq \widetilde{\zeta}(x), \quad \text{for all } x \in \mathbb{Z}^d. \tag{3.0.2}$$

This can be further extended to all continuous functions $f \in C(\{0,1\}^{\mathbb{Z}^d})$ by saying that f is *increasing* if

$$\zeta \leq \widetilde{\zeta} \quad \text{implies} \quad f(\zeta) \leq f(\widetilde{\zeta}).$$

Moreover, we can use increasing functions to define the *stochastic order* on probability measures. Let μ_1, μ_2 be probability measures on $\{0,1\}^{\mathbb{Z}^d}$, then we set

$$\mu_1 \leq \mu_2 \quad \text{iff} \int_{\{0,1\}^{\mathbb{Z}^d}} f \, \mathrm{d}\mu_1 \leq \int_{\{0,1\}^{\mathbb{Z}^d}} f \, \mathrm{d}\mu_2, \quad \text{for all increasing } f \in C(\{0,1\}^{\mathbb{Z}^d}).$$

The stochastic order lets us talk about monotonicity properties of spin-systems. We say that a spin-system is *monotone* (or *attractive*) if the Markov kernel κ describing the dynamics, cf. (3.0.1), yields *increasing* probability measures, i.e. if either of the two following equivalent conditions are satisfied

(i) $\kappa(\cdot, \zeta)$ stochastically dominates $\kappa(\cdot, \widetilde{\zeta})$ for all $\zeta, \widetilde{\zeta} \in \{0,1\}^{\mathbb{Z}^d}$ such that $\zeta \geq \widetilde{\zeta}$

(ii) $\int_{\{0,1\}^{\mathbb{Z}^d}} \kappa(\mathrm{d}\zeta, \cdot) f(\zeta)$ is a monotone function whenever $f \in C(\{0,1\}^{\mathbb{Z}^d})$ is monotone.

It follows from e.g. [Lig85, Theorem II.2.4] that domination in the stochastic order is characterised by the existence of a *monotone coupling*, i.e. $\mu_1 \leq \mu_2$ if and only if there is a coupling $(\zeta, \widetilde{\zeta})$ such that $\zeta \sim \mu_1$, $\widetilde{\zeta} \sim \mu_2$ and $\zeta \leq \widetilde{\zeta}$.

This *monotone coupling* is a common (and very powerful) tool for working with spin-systems. It is typically applied by either comparing two distinct spin-systems with the same initial configuration or by comparing two copies of the same spin-system with two distinct initial configurations that are increasing, cf. (3.0.2).

For the specific model of the BARW, it is intuitively clear that the system is non-monotone due to the annihilation. More formally, we can see this by comparing two copies of BARW with different and ordered initial configurations. Let $\eta_0, \widetilde{\eta}_0 \in \{0,1\}^{\mathbb{Z}^d}$ be two initial configurations such that $\eta_0 \leq \widetilde{\eta}_0$ and write $\eta = (\eta_n)_{n \geq 0}$ and $\widetilde{\eta} = (\widetilde{\eta}_n)_{n \geq 0}$ for the processes corresponding to these initial configurations. As noted under (3.0.1), given any configuration $\zeta \in \{0,1\}^{\mathbb{Z}^d}$, the local updating measures $p_x(\cdot, \zeta|_{B_R(x)})$, for $x \in \mathbb{Z}^d$, are Bernoulli measures with (μ-dependent) parameter that is unimodal in the density of 1's in $B_R(x)$. Therefore, an increase in the local density of 1's invariably

leads to Bernoulli measures with *smaller* parameters. This can be used, e.g. to build a counterexample to (i) by showing that no monotone coupling exists uniformly in all initial configurations (i.e. take $\widetilde{\eta}_0$ to be the *full* initial configuration $\widetilde{\eta}_0 \equiv 1$ and choose η_0 so that for some $x \in \mathbb{Z}^d$ the Bernoulli measure $p_x(\cdot, \eta|_{B_R(x)})$ has a larger parameter than the corresponding parameter of the measure $p_x(\cdot, \widetilde{\eta}|_{B_R(x)})$).

It is important to note, however, that the non-monotonicity of the BARW $(\eta_n)_{n\geq 0}$ does *not* imply that it cannot be coupled monotonically to *any* other system $(\zeta_n)_{n\geq 0}$ for *any* initial configuration η_0 and *any* parameters of the model. In fact, in Chapter 4 we construct a coupling between the BARW (defined with model parameters in some specified regime) and a discrete-time particle system (i.e. a PCA), which *is* monotone in the sense introduced above. Note, however, that the coupling depends on the specific regime of model parameters, outside of which the specific coupling we construct does not exist. This comparison with a monotone PCA for suitable parameter-regimes is, in fact, precisely how we address question (Q1) on survival of the BARW in Chapter 4.

Invariant measures and ergodicity

Let us now turn to questions (Q2)–(Q3) about the equilibria of spin-systems. These equilibria are characterised by probability measures on the configuration space, which are left invariant by the dynamics of the system.

Let ν be a probability measure on the configuration space $\{0,1\}^{\mathbb{Z}^d}$ and let the initial configuration η_0 of the spin-system $\eta = (\eta_n)_{n\geq 0}$ be distributed as ν. (We write time as discrete, but the following results are also true for continuous-time spin-systems.) Then ν is an *invariant law* if for all $n > 0$ the distribution of η_n is also given by ν, i.e.

$$\nu(\eta_0 \in A) = \nu(\eta_t \in A), \quad \text{for all } t \in \mathbb{N}, \text{ and } A \subseteq \{0,1\}^{\mathbb{Z}^d}. \tag{3.0.3}$$

We denote by $\mathcal{I}$ the set of all invariant measures for the corresponding spin-system. Moreover, we endow the set of probability measures on $\{0,1\}^{\mathbb{Z}^d}$ with the topology of weak convergence. By the compactness of $\{0,1\}^{\mathbb{Z}^d}$, the set of probability measures on $\{0,1\}^{\mathbb{Z}^d}$ is compact with respect to the weak topology. A standard result based on this compactness is that $\mathcal{I}$ is a non-empty convex and compact set, which is the closed convex hull of its extreme points, cf. [TVS$^+$90] for PCA and [Lig85] for IPS. Consequently, any (two-state) spin-system on $\mathbb{Z}^d$ either has several invariant measures (all or none of which might be attractive for certain classes of initial configurations), a unique non-attractive invariant measure, or a unique attractive invariant measure. Let us comment briefly on these three cases.

The property of a spin-system having a unique attractive invariant measure is typically referred to as the system being *ergodic*. This is slightly different from the usual notion of ergodicity for dynamical systems (triviality of the σ-algebra of events that are invariant under translations in time). However, the notion of ergodicity used for spin-systems implies the usual notion for the corresponding *stationary* process

(i.e. the spin-system that is distributed according to the unique attractive invariant measure). The converse implication cannot be made in general, which calls for some care when discussing ergodic measures in the context of spin-systems.

Next, in order to see that unique non-attractive invariant measures can exist, we note first that it is well-known, both for PCA and IPS, cf. [MM14b, Lig85], that if $\mathcal{I} = \{\nu\}$, then ν is attractive for *Cesàro means*. By this, we mean that if we write μ_k for the distribution of the spin-system at time $k \geq 0$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_k = \nu, \tag{3.0.4}$$

for weak convergence. This does not, however, imply ergodicity. For some initial distribution μ_0, the system might converge to a *periodic orbit* $(\nu^{(1)}, \dots, \nu^{(T)})$, for some $T > 0$. In this case, the Cesàro means converge to $\frac{1}{T}\sum_{k=1}^{T} \nu^{(k)}$, which by (3.0.4) is the unique invariant measure, but the distribution of the spin-system itself does not converge and hence is not ergodic. See also [CM11, JK14, Swa17] for other examples of non-ergodic spin-systems with a unique invariant measure. Note, however, that generally, not much is rigorously known about spin-systems with periodic orbits.

Lastly, in the case where there are several invariant measures, the main objectives are to distinguish between which measures are attractive and which are non-attractive, as well as to describe the various domains of attraction for the attractive invariant measures.

Remark 3.0.2. Many spin-systems exhibit trivial invariant measures. These are Dirac-measures on *monochromatic* configurations. E.g. two-state spin-systems that have an individual based description for which particles cannot be *spontaneously* created (such as the BARW), the Dirac-measure on the empty configuration $\mathbf{0} \in \{0,1\}^{\mathbb{Z}^d}$ is always an invariant measure. In this situation, one is interested in $\mathcal{I} \setminus \{\delta_{\mathbf{0}}\}$ instead of $\mathcal{I}$. Moreover, if $\mathcal{I} \setminus \{\delta_{\mathbf{0}}\}$ contains one attractive measure ν, one typically calls it the *non-trivial ergodic measure*.

Below Table 3.1 we noted that PCA and IPS can be used as alternative models for the same type of phenomena, one in discrete-time, the other in continuous-time. It is, however, important to note that when considering invariant measures, the seemingly subtle differences in the definitions of PCA and IPS can have non-trivial consequences that lead to different behaviour in the two cases. In particular, in the case of one-dimensional finite-range spin-systems, Mountford constructed in [Mou93, Mou95] a refined coupling for IPS, which he uses in order to show that, in this case, any converging subsequence (of distributions) converges towards an invariant measure. This implies, in particular, that *all* one-dimensional finite-range IPS are ergodic. (This inadvertently also precludes the possibility of periodic laws for one-dimensional finite-range IPS, as the argument below (3.0.4) shows.) The same is, however, not true for

one-dimensional finite-range PCA. A long open problem for PCA was the so-called *positive rates conjecture*, which states that any one-dimensional PCA that has *positive rates* (i.e. for which the updated content of any cell has a positive probability of being any of the local states, regardless of the current local configuration of the system) is ergodic. In [Gác01] (see also [Gra01]), a very complicated counter-example to this conjecture is presented.

This difference in the possible behaviour of one-dimensional finite-range IPS and PCA is an indication that the question of ergodicity for PCA can be quite subtle and, in general, quite a hard one. Further evidence of this comes from a computational point of view. It can be shown that the question of ergodicity for PCA is algorithmically undecidable in the sense that there exists no algorithm (i.e. finite set of instructions) that takes any PCA as input and can decide in finitely many steps whether it is ergodic or not, cf. [TVS$^+$90, MM14a, Kar92] for details.

Another aspect showing the complexity of the question of ergodicity for PCA, is that often a lack of ergodicity can be translated into the existence of phase-transitions in particular models of statistical mechanics on the space-time histories of the PCA. This connection goes back at least to [Ver70, Ver76] and was worked out more carefully in [KV80, GJH85, GKLM89, Der89]. See also [Lou02, MM14a, FLN18] for an overview and some applications of this connection.

The upshot of this discussion is that invariant measures for spin-systems offer a very rich range of behaviour and that answers to questions (Q2)–(Q3) can depend on fine details of the particular model one is considering.

In the case of the BARW, we address (Q2)–(Q3) in Chapter 4 below, where we identify a specific regime of model parameters, for which we can guarantee both the existence of a non-trivial invariant measure and the convergence towards it for all (non-trivial) initial configurations, conditioned on non-extinction. Outside of the parameter regime for which there is this non-trivial ergodic measure, not much is known, and simulations suggest the existence of periodic orbits (at least locally), cf. Figure 3.1.

BARW as a model of mathematical ecology

In the field of mathematical ecology, non-monotone models have generated special interest as models of population dynamics subject to some type of local (self-) regulation, cf. [Eth04, BD07, BEM07]. One goal of mathematical ecology is to study plausible (individual based) stochastic models that *might* describe how spatially heterogeneous populations of animals or plants evolve over time, see e.g. [Eth11]. Throughout their lifetime, individuals in the population live, move (seed-dispersal for plants), reproduce and die. Reproduction of individuals in the population naturally leads to models of spatial branching processes. It is however also clear that models such as branching random walk (BRW), branching Brownian motion (BBM) and their relatives are only of limited use in this context, since by the independence of particles alive at

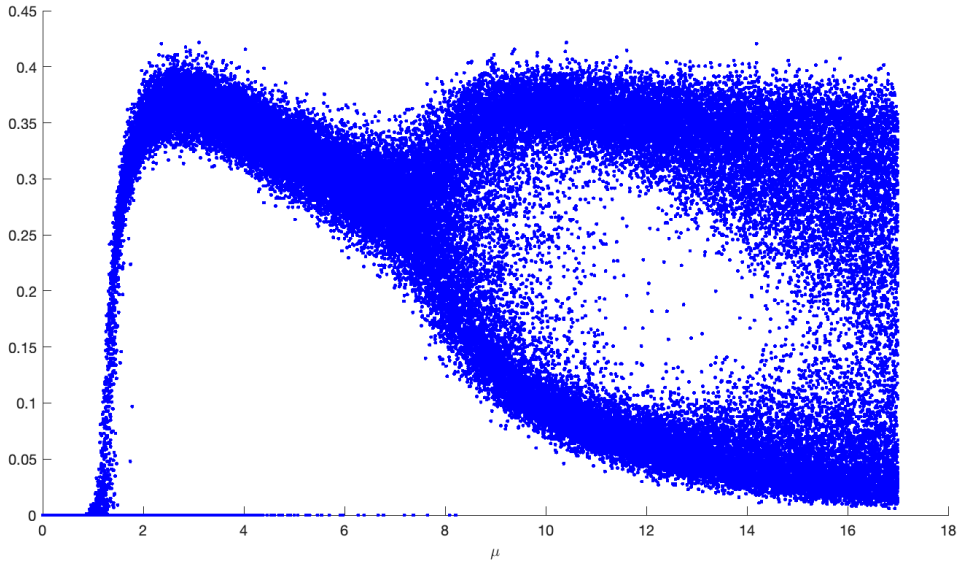

Figure 3.1: Simulation of the "long-term" density in $B_R(0)$ for a one-dimensional BARW on $\mathbb{Z}/10000\mathbb{Z}$ with initial condition $\eta_0 = \delta_0$ and with $R = 500$ and varying μ, showing 20 simulations of the local density after $n = 25$ and $n = 26$ generations. For values of μ between 1 and e^2, the density concentrates a single value and for $\mu > e^2$ the local densities concentrate around multiple values, periodically flipping the value around which they concentrate in each generation.

any given time, a population evolving as a BRW or BBM will either explode or die out completely (there is an interesting discussion on the relevance of branching models in the physical world by Peter Jagers at the end of Anton Wakolbinger's article [Wak95]). In order to describe a stable population, one therefore needs to introduce some type of interaction between individuals. This has been done in many different ways and has led to various different models that have been studied rigorously, e.g. [WK65, DL94, Dur99, NP99, BEM07, Eth04, FM04, BD07, MP22] for an incomplete list. We can interpret the BARW as such a model of population dynamics that allows for arbitrary population sizes and densities, with the very drastic mode of self-regulating interaction given by local annihilation of individuals that try to share a common site. From an ecological standpoint the BARW can be seen as a model of intraspecific competition commonly referred to as *scramble competition*, in which available resources are shared equally among all competitors (at least locally), cf. [Nic54]. Annihilation in the BARW can be seen as modelling scarcity of resources at a given site where the allocated resources per individual are insufficient for survival. In this sense, in sparsely populated regions, there are enough resources for the population to

grow over the course of the next generation, whereas in densely populated regions, the resources are too scarce to support the current population and it decreases over the course of the next generation.

In this context, questions (Q1)–(Q3) give us information on the long-time behaviour and stability of the population. Furthermore, the context of an evolving population also leads to new questions, which are not as natural from the perspective of spin-systems. If we assume for now that the population evolves in a stable way, i.e. is distributed according to some invariant measure $\nu \in \mathcal{I}$, then it is natural to ask the following.

(Q4) What does the space-time history of a *sample* of individuals look like?

If we were to allow for individuals of multiple types, then information on the spatial embedding of *genealogies* could be used to gain information on the spatial distribution of the types. These kinds of questions are mathematically quite challenging. A common approach to making the analysis of space-time histories of a sample of individuals (in a stationary population) more tractable is to locally fix an *effective size* of the population. That is, one divides space into a discrete grid of *demes* and assumes that the local population sizes per deme are constant over time. This results in so-called *stepping-stone* models, [Saw76, WK65]. In stepping-stone models, the ancestral lineages of a sample of individuals perform a coalescing random walk. This allows the application of powerful mathematical tools in order to gain insight into the quantities of ecological interest.

The situation for locally self-regulating populations, such as the BARW, is more complicated than a stepping-stone model. In fact, we can interpret the spatial embedding of the ancestral lineage of a single individual (i.e. of a sample of size one) as a random walk in a *dynamic random environment*, which is generated by the backwards in time history of the entire population. Similarly, the ancestral lineages of samples of size $k \geq 2$ correspond to a collection of random walks in the same dynamic environment, which coalesce upon meeting. Questions about the behaviour of such ancestral lineages were considered in [BČDG13, BGS19, BG21, BBDS23] for the discrete-time contact process and for the logistic branching random walk, first introduced in [BD07]. Moreover, in [BČD16] the question of the behaviour of single ancestral lineages was considered for an entire class of locally regulated models satisfying some abstract conditions.

In Chapter 5 we address the question of the motion of ancestral lineages of a single particle for the BARW (drawn from a population distributed according to the non-trivial ergodic distribution), by showing that it falls into the class of models studied in [BČD16]. We will see that when viewed over large enough space-time scales, the ancestral lineages of individuals in the BARW behave similar to ordinary random walks and thus, on the right scales, the behaviour is in fact close to that predicted by the stepping stone model. Moreover, the fluctuations in the local population sizes ("on the

demes"), due to different realisations of the BARW, only play into the *variance* of this "random walk".

Remark 3.0.3. From an ecological point of view, the rigorous results we derive for the BARW (as well as results in much of the above mentioned literature) should be taken with a grain of salt and be interpreted as being only conceptual in nature. By this, we mean that for these models, results on survival, coexistence of several types, or similarity of ancestral lineages to random walks (etc.) should be read as indicating *possible* real-world behaviour if interaction terms within the population are weak enough. But these results don't give any realistic information about *true* critical values for which such phenomena might hold. This is due to the available mathematical toolbox to address these issues from a rigorous point of view. One common tool, which we shall also use in the subsequent chapters, is e.g. comparison with finite-range oriented percolation. To apply such a comparison, one needs to fine-tune the model parameters away from the critical values. Hence, the information that we can infer from these results on how *true* populations are distributed in space when resources are scarce is quite limited.

The following two chapters address questions (Q1)–(Q3) and (Q4) respectively. Chapter 4 is joint work with Matthias Birkner, Alice Callegaro, Jiří Černý and Nina Gantert.

4. Survival and complete convergence for a branching annihilating random walk

Matthias Birkner, Alice Callegaro, Jiří Černý, Nina Gantert, Pascal Oswald

Abstract. We study a discrete-time branching annihilating random walk (BARW) on the d-dimensional lattice. Each particle produces a Poissonian number of offspring with mean μ which independently move to a uniformly chosen site within a fixed distance R from their parent's position. Whenever a site is occupied by at least two particles, all the particles at that site are annihilated. We prove that for any $\mu > 1$ the process survives when R is sufficiently large. For fixed R we show that the process dies out if μ is too small or too large. Furthermore, we exhibit an interval of μ-values for which the process survives and possesses a unique non-trivial ergodic equilibrium for R sufficiently large. We also prove complete convergence for that case.

4.1 Introduction

As a model for a population evolving in space, one may consider branching random walks. These are systems of particles where the particles reproduce and move randomly in space, independently for different families. For instance, the children may take i.i.d. displacements from their mother particle or, in a more general model, the parent particle may generate a configuration of children according to some point process. Branching random walks are a very active research topic, we refer to [Shi15] for an introduction.

Our goal is to model a population which competes for resources, hence a particle system in which particles reproduce, move randomly in space, and compete with each

other locally. We chose here a rather radical form of interaction: whenever two or more particles are on the same site, they annihilate. The annihilation makes the system non-attractive in the sense of interacting particle systems, i.e. adding more particles initially can stochastically decrease the law of the configuration at later times.

A first question about branching random walks is if the system has a strictly positive survival probability. In the classical case, that is without annihilation, the answer is well-known since the number of particles at time n forms a Galton–Watson process. However, taking into account annihilation, the question is much more difficult and there are relatively few mathematical papers addressing it, see the discussion of related literature in Section 4.1.3 below.

Assuming that the parameters of the model are such that the survival probability is indeed strictly positive, the next question is about invariant measures and the convergence towards the invariant measure in the case of survival. As for the classical branching random walk or the contact process, it is clear that the Dirac measure on the empty configuration is invariant. We can show for our model that in a certain range of parameters there is complete convergence, i.e. there is exactly one non-trivial ergodic invariant measure and the law of the process, conditioned on survival, approaches this invariant measure.

Our model allows for a representation as a probabilistic cellular automaton. Questions about ergodicity and complete convergence are notoriously difficult for such systems, we refer to [MM14a] for an introduction. If we consider the iteration of the expected number of particles at the sites of the lattice, we have a deterministic system, a coupled map lattice, see Section 4.1.4 below. This system is of independent interest and we expect that it admits, in a certain range of parameters, travelling wave solutions. Hence our model can be interpreted as a stochastic perturbation of the coupled map lattice, and this interpretation raises several interesting questions.

Let us now give a more precise definition of the model and describe our results. We study a process $\eta = (\eta_n(x) : x \in \mathbb{Z}^d, n \geq 0)$ evolving in discrete time on $\mathbb{Z}^d$, where $\eta_n(z)$ denotes the state of site z at time n. We write $\eta_n(z) = 1$ if site z is occupied by exactly one particle at time n and $\eta_n(z) = 0$ otherwise. We denote by $\|\cdot\|$ the sup-norm on $\mathbb{Z}^d$ and define $B_R(z) = \{x \in \mathbb{Z}^d : \|z - x\| \leq R\}$ to be the d-dimensional ball (box) of radius $R \in \mathbb{N}$ centred at $z \in \mathbb{Z}^d$. We set $V_R = 2R+1$ to be its side length, so that its volume is V_R^d.

For fixed $R \in \mathbb{N}$, $\mu > 0$, and an initial particle configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, the configurations at later times are obtained recursively as follows. Given η_n, $n \geq 0$, for every $z \in \mathbb{Z}^d$ with $\eta_n(z) = 1$ the particle at z dies and gives birth to a Poisson number of children with mean μ. Each child moves independently to a uniformly chosen site in $B_R(z)$. Whenever there is more than one particle at a given site, then all the particles at that site are killed. This means that if two (or more) children of the same parent jump to the same site they will disappear, but also children coming from different

parents who jump to the same site will annihilate. The particles remaining after the annihilation make up the configuration η_{n+1}.

The thinning and superposition properties of the Poisson distribution give the following equivalent description of the model, which is particularly convenient to carry out calculations. For a configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $z \in \mathbb{Z}^d$, define first the (local) density of particles at z by

$$\delta_R(z;\eta) = V_R^{-d} \sum_{x \in B_R(z)} \eta(x). \tag{4.1.1}$$

Fix η_n and denote by $N_{n+1}(z)$ the number of newborn particles at z in the next generation before the annihilation occurs. This is given by the superposition of the offspring of all particles that can move to z, that is of all $x \in B_R(z)$ with $\eta_n(x) = 1$. Thus $N_{n+1}(z)$ is a Poisson random variable with mean $\mu\delta_R(z;\eta_n)$. Taking the annihilation into account, it then holds that

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{if } N_{n+1}(z) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

Let

$$\varphi_\mu(w) = \mu w\, e^{-\mu w}, \quad w \in [0,\infty) \tag{4.1.3}$$

denote the probability that a Poisson random variable with mean μw equals 1. By construction, the random variables in the family $(\eta_{n+1}(z) : z \in \mathbb{Z}^d)$ are conditionally independent given η_n and by (4.1.2), (4.1.3) we can represent our system as

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{with probability } \varphi_\mu(\delta_R(z;\eta_n)), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.4}$$

This gives a representation of η as a particular example of a probabilistic cellular automata. We point out that this representation is only possible because we choose a Poisson offspring distribution. For more detailed discussion of the assumptions of the model, see the discussion in Section 4.1.2 below.

4.1.1 Main results

We can now state the main results of this paper. For the intuition behind them, we find it useful to first point out a few properties of the function φ_μ introduced in (4.1.3) which governs the behaviour of the process:

(a) For $\mu \leq 1$, φ_μ has a unique fixpoint at 0, which is attractive.

(b) For $\mu > 1$, φ_μ has two fixpoints, 0 and $\theta_\mu = \mu^{-1} \ln \mu$. In this case 0 is always repulsive.

(c) For $\mu \in (1, e^2)$, θ_μ is an attractive fixpoint.

(d) For $\mu > e^2$, there are no attractive fixpoints

In the case (d), the one point iteration $x \mapsto \varphi_\mu(x)$ has rich behaviour. Depending on the value of μ, there can be attractive periodic orbits or chaotic behaviour.

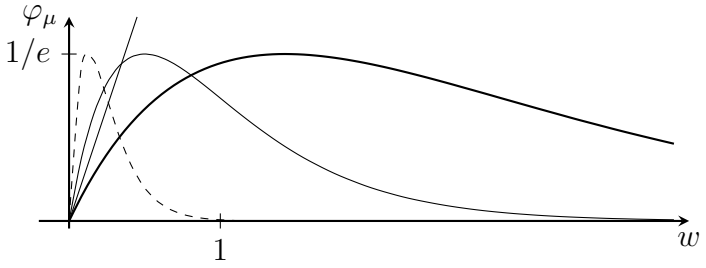

Figure 4.1: Graphs of φ_μ for $\mu = 0.7$ (thick), 2 and 8 (dashed), together with the identity function.

Extinction Our first result identifies a range of parameters (μ, R) where the process dies out a.s. Here, we say that η goes extinct *locally* if $\lim_{n\to\infty} \eta_n(x) = 0$ for every $x \in \mathbb{Z}^d$, and that η goes extinct *globally*, if $\eta_n \equiv 0$ for all n large enough.

Theorem 4.1.1. *For $R \in \mathbb{N}$, let $\mu_1(R), \mu_2(R)$ be the two real solutions of*

$$V_R^d \, \varphi_\mu\big(V_R^{-d}\big) = 1$$

with $1 < \mu_1(R) < \mu_2(R) < \infty$. If $\mu < \mu_1(R)$ or $\mu > \mu_2(R)$, then, for all initial conditions η goes locally extinct a.s., and for all initial conditions containing only a finite number of particles η goes extinct globally a.s. Furthermore, $\mu_1(R) \to 1$ and $\mu_2(R) \to +\infty$ as $R \to \infty$.

The result of the proposition is not optimal, we expect (based on simulations, see Figure 4.7 in Section 4.7 below) that the process goes extinct for many values (μ, R) outside of the specified range. On the other hand, its proof is relatively simple. It is given in Section 4.5 below, and is based on the observation that for (μ, R) in the specified range the killing by annihilation among siblings is already strong enough to make the expected number of "surviving" offspring of a single particle strictly smaller than one, and thus the branching effectively subcritical, even though $\mu > 1$.

Remark 4.1.2. The two values $\mu_1(R)$ and $\mu_2(R)$ can be given explicitly as

$$\mu_1(R) = -V_R^d \, W_0(-V_R^{-d}), \quad \mu_2(R) = -V_R^d \, W_{-1}(-V_R^{-d}),$$

where W_0 and W_{-1} are the two real branches of the Lambert W function. This also describes their asymptotic behaviour as $R \to \infty$, see (4.5.2) and (4.5.3) in the proof of Theorem 4.1.1 below.

Survival The second result identifies a range of parameters where it is possible that the process survives locally, by which we mean that for every $x \in \mathbb{Z}^d$ the set of times n when $\eta_n(x) = 1$ is unbounded. Similarly as in Theorem 4.1.1, the identified range is not optimal.

Theorem 4.1.3. *For every $\mu > 1$ there exists $R_\mu \in \mathbb{N}$ such that η survives with positive probability from any non-trivial initial condition when $R \geq R_\mu$.*

Remark 4.1.4. Inspection of the proof of Theorem 4.1.3 shows that there is $R_0 \in \mathbb{N}$ such that for every $R \geq R_0$ there exist two values $1 < \underline{\mu}_R < \overline{\mu}_R$ such that η survives with positive probability from any non-trivial initial condition when $\mu \in (\underline{\mu}_R, \overline{\mu}_R)$. Furthermore, on the event of survival, it holds that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \eta_j(x) > 0 \quad \text{a.s. for any } x \in \mathbb{Z}^d. \tag{4.1.5}$$

Ergodicity and complete convergence The final set of results discusses the invariant measures of the process, in the case when the system survives. For this we equip the state space $\{0,1\}^{\mathbb{Z}^d}$ with the product topology and the corresponding Borel σ-algebra. In these results we restrict ourselves to $\mu \in (1, e^2)$, where the non-trivial fixpoint of φ_μ is attractive, as pointed out above.

Theorem 4.1.5. *For every $\mu \in (1, e^2)$ there is $R'_\mu < \infty$ such that for every $R \geq R'_\mu$ the process η has two extremal invariant distributions: the first one is trivial and is concentrated on the empty configuration $\eta \equiv 0$, and the second one, $\nu_{\mu,R}$, is non-trivial, translation invariant, ergodic, and has exponential decay of correlations.*

Furthermore, starting from any non-trivial initial condition the process η, conditioned on non-extinction, converges in distribution in the weak topology to the non-trivial extremal invariant distribution $\nu_{\mu,R}$.

The driving result behind Theorem 4.1.5 is the following strong coupling property of the system η, which is of independent interest.

Theorem 4.1.6. *Assume that $\mu \in (1, e^2)$ and $R \geq R'_\mu$ satisfy the assumptions of Theorem 4.1.5. Then there exists a speed $a = a(R, \mu, d) > 0$ such that for every pair of (possibly random) initial conditions $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$ there exists a coupling of the processes $(\eta_n^{(i)})_{n\in\mathbb{N}_0}$, $i = 1, 2$, with the following property. For each $x \in \mathbb{Z}^d$ there is an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable T_x^{coupl} (whose exact law will in general depend on the initial conditions and on x) such that $\{\eta_n^{(i)} \not\equiv 0 \text{ for all } n \in \mathbb{N}, i = 1, 2\} \subseteq \{T_x^{\text{coupl}} < \infty\}$ a.s. and*

$$\eta_n^{(1)}(y) = \eta_n^{(2)}(y) \quad \textit{for all } n > T_x^{\text{coupl}} \textit{ and } \|y - x\| \leq a(n - T_x^{\text{coupl}}).$$

Remark 4.1.7. It follows from the proof of Theorem 4.1.6 that when starting from a finite (or a half-space) initial condition, the system η, given that it survives, will expand into the "empty territory" at least at some (small) linear speed. Furthermore, simple comparison arguments with supercritical branching random walks show that this expansion cannot occur faster than linearly. However, identifying an actual linear speed or even an asymptotic profile of the expanding population near its tip remains a topic for future research.

Remark 4.1.8. Denote by $\overline{\theta}_{\mu,R} = \mathbb{E}_{\nu_{\mu,R}}[\eta_0(0)] \in (0,1)$ the particle density of the non-trivial invariant measure $\nu_{\mu,R}$ from Theorem 4.1.5, where $\mu \in (1, e^2)$ and $R \geq R'_\mu$. By ergodicity, we have almost surely when $\eta_0 \sim \nu_{\mu,R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \eta_n(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_R(x; \eta_n) = \overline{\theta}_{\mu,R} \quad \text{for every } x \in \mathbb{Z}^d. \tag{4.1.6}$$

By the coupling property from Theorem 4.1.6, (4.1.6) holds in fact a.s. for any initial condition given that the system survives.

Furthermore, for $1 < \mu < e^2$, inspection of the proof of Theorem 4.1.6 shows that for every $\varepsilon \in (0,1)$ there exists $R'_{\mu,\varepsilon} < \infty$ such that if $R \geq R'_{\mu,\varepsilon}$ then, conditionally on non-extinction,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{|\delta_R(x;\eta_n) - \theta_\mu| < \varepsilon\}} \geq 1 - \varepsilon \quad \text{almost surely for every } x \in \mathbb{Z}^d, \tag{4.1.7}$$

where we recall that θ_μ is the fixpoint of φ_μ (on "good" blocks, the particle density is close to θ_μ, see Definition 4.4.4 below, and good blocks are shown to occur with high space-time density).

Note that (4.1.6) and (4.1.7) together imply that $|\overline{\theta}_{\mu,R} - \theta_\mu| \leq 2\varepsilon$ for $R \geq R'_{\mu,\varepsilon}$. This corroborates the idea that for large R, the particle system's behaviour is close to that of the corresponding deterministic coupled map lattice, which we discuss in Section 4.1.4 below, and which, as shown in Proposition 4.1.9, converges to the configuration which is constant and equal to θ_μ.

4.1.2 Possible generalisations

The construction of our model might seem very rigid. Therefore, we discuss here the role of the different assumptions in the model and their possible generalisations.

The assumption that the particles jump distribution is uniform over a box of length V_R is non-essential and is made only for convenience of the notation. It can in principle be replaced by an arbitrary (centred) *finite range* transition kernel, and the proofs can be adapted by suitably replacing the particle density (4.1.1) by the convolution of this kernel with η.

The assumption that the number of offspring of a single particle has Poisson distribution is more important, as it allows for the essential representation (4.1.4), and also yields the conditional independence of $(\eta_{n+1}(x) : x \in \mathbb{Z}^d)$ given η_n. Replacing the Poisson distribution would thus require non-trivial modifications to our proofs. On the other hand, if we take (4.1.4) as the definition of the model, the particular form of the function φ_μ used there does not play a strong role. Our techniques will continue to work, if we replace φ_μ by another function of a "similar shape". In fact, for survival we only need that $\varphi_\mu : [0,1] \to [0,1]$ is continuously differentiable and strictly positive on $(0,1]$ with 0 as an unstable fixpoint. For the proof of the convergence result, we also need that there is a unique attracting fixpoint $\theta_\mu \neq 0$.

In a different direction, the "hard" annihilation constraint of at most one particle per site could be relaxed by replacing the definition (4.1.2) of η by $\eta_{n+1}(z) = N_{n+1}(z)\mathbb{1}_{\{N_{n+1}(z)\leq k\}}$ for some $k \in \mathbb{N}$. Since this modification retains the conditional Poisson and independence properties of the $N_n(z)$'s and the sums of truncated Poisson random variables have good concentration properties, we are hopeful that our proofs could be adapted to this scenario with some additional work.

4.1.3 Discussion of related results

One of the first models of branching annihilating random walks was introduced and studied by Bramson and Gray [BG85]. They considered a particle system on $\mathbb{Z}$, in which sites can be occupied as the result of the following mechanisms: particles can either *jump* to one of the two neighbouring sites at a certain rate or *branch* into two by giving birth to a new particle on one of the neighbouring sites. On top of this, particles behave independently except when they land on a site which is already occupied, in which case both particles disappear, *annihilate*. The authors show that, starting from any finite number of particles, the system survives with positive probability if the jumping rate is small compared to the branching rate and that the population dies out almost surely if the jumping rate is sufficiently high. This process is an interacting particle system in the sense of [Lig85, Lig99] but it is not attractive. The authors use contour arguments which rely on the one-dimensional model they chose.

Very general interacting particle systems on $\mathbb{Z}$ are considered in [Sud00b], where pairwise interactions among neighbours can produce annihilation, birth, coalescence, and exclusion and single individuals can die. Conditions on the rates which ensure positive probability of survival are given by making use of self-duality (which has been proved by the same author in [Sud00a]) and supermartingale arguments. In [BDD91] instead, processes on $\mathbb{Z}^d$ with nearest-neighbour birth at rate 1, annihilation and spontaneous death at rate δ have been considered. An extinction result for the branching annihilating process started from one particle at the origin is obtained by comparison with the contact process. On the other hand, survival when δ is small is proved through comparison with oriented percolation.

In cases where survival can be established, natural questions concern the existence of stationary distributions and weak convergence. Sudbury [Sud90] considers a version of Bramson and Gray's model in $\mathbb{Z}^d$ in the case of no random walk and shows that the product measure with rate $1/2$ is the only non-empty limiting distribution. In the case of a double branching and annihilating process on $\mathbb{Z}$ (where each particle can place offspring on both of its neighbouring sites), a richer variety of limiting measures is exhibited. In [BDD91], the authors prove that when $\delta = 0$ the product measure with density $1/2$ is stationary and is the limiting measure, thus obtaining independently the same result proved in [Sud90]. Furthermore the authors show that for any δ there are at most two extremal translation invariant stationary distributions, and if δ is small there exists a non-trivial stationary distribution.

Another question of interest is whether branching processes with annihilation satisfy duality relations. Athreya and Swart [AS12] consider processes in continuous time where particles can annihilate, branch, coalesce or die. They show that annihilation does not play a key role in a duality relation: the process with annihilation is dual to a system of interacting Wright-Fisher diffusions, and this result holds also if annihilation is suppressed (but in the case of annihilation the duality function is different and more complicated). It would be highly interesting to find a useful duality relation for our model as well.

Versions of branching annihilating processes in discrete time are generally more difficult to deal with, since continuous time implies that changes in the configuration can only occur one site at a time, sequentially as opposed to in parallel. A discrete-time analogous of [BG85] has been considered in [AIR01] for a model on $\mathbb{Z}$, where particles at each time move with probability $1 - \varepsilon$ or branch with probability ε, with the rule that two particles occupying the same site will annihilate. The authors show that, if the branching probability is small enough, for any finite initial configuration of particles the probability $p(t)$ that at least one site is occupied at time t decays exponentially fast in t.

Perl, Sen and Yadin [PSY15] consider a branching annihilating random walk on the complete graph which evolves in discrete time, where the number of offspring is Poisson distributed with mean μ and each one of them independently moves to one of the neighbouring sites of their parent. This corresponds to our model on the complete graph. Since on a finite graph there is always a positive probability of total annihilation in one step, the system eventually dies out at some finite time. They show that if $\mu > 1$, then the process on the complete graph with N vertices has an exponentially long lifetime in N and that, conditional on extinction, its last excursion from the "equilibrium value" $\theta_\mu N$ before it reaches the zero state is logarithmic in N.

Besides systems where particles can annihilate, recent research directions have also been focusing on spatial branching systems in which the interaction among particles is regulated by a competition kernel which can reduce the average reproductive success

of an individual at a given site. In this case, rather than annihilating particles in areas with high particle density, the existing particles will produce fewer offspring. Spatial models with local competition are for example investigated in [Eth04, BEM07, BD07, FKK09, Mü15, MP22]. The two papers most related to our present work are [BD07, MP22].

Birkner and Depperschmidt [BD07] consider a discrete time branching system with a finite range (and thus *local*) competition kernel. They show that the system survives with positive probability if the competition term is small enough and obtain complete convergence of the system to a non-trivial equilibrium for some choices of the model parameters. The strategy used in [BD07] to prove survival is building a comparison with an oriented percolation model. We will use similar ideas to show survival for our branching annihilating random walk, as well as complete convergence.

In a more recent paper, Maillard and Penington [MP22] work in continuous time and consider non-local competition kernels, where the range of interaction can be arbitrary, even infinite. Using a contour argument, they prove that in the low competition regime the system survives globally. In the same regime, they also provide a shape theorem, showing that the asymptotic spreading speed of the population is the same as in the branching random walk without competition.

Since we work in discrete time, our model is not an interactive particle system in the sense of [Lig85, Lig99] but rather a probabilistic cellular automaton. We refer to [MM14a] for a survey on probabilistic cellular automata. Ergodicity and complete convergence for probabilistic cellular automata is a notoriously difficult topic where a lot of the proof techniques are model-dependent. For attractive systems there are still some general tools as monotonicity and subadditivity, see [Ham74]. We refer to [FLN18] for a collection of recent results.

4.1.4 Auxiliary coupled map lattice

Our work also raises questions about coupled map lattices which are deterministic versions of the probabilistic cellular automata, see (4.1.4), and which, in our examinations of the BARW, serve as an intuitional guide for the proofs of the survival and the complete convergence. This coupled map lattice is a deterministic $[0, e^{-1}]^{\mathbb{Z}^d}$-valued process Ξ_n (note that $\max_{w\geq 0} \varphi_\mu(w) = e^{-1}$) defined, given any initial condition Ξ_0, by the iteration of

$$\Xi_{n+1}(x) = \varphi_\mu(\delta_R(x; \Xi_n)). \tag{4.1.8}$$

At least for R large, locally, the dynamics of this process is a good approximation for the dynamics of the "density profile" $\delta_R(\cdot; \eta_n)$ of η, as can be heuristically seen from (4.1.4) and the law of large numbers.

We will prove and exploit the fact that in the regime when φ_μ has the unique attractive fixpoint θ_μ, that is for $\mu \in (1, e^2)$, when starting from a non-zero initial

condition, the coupled map lattice converges locally to θ_μ, and the region where it is close to this value expands.

Proposition 4.1.9. *Let $\mu \in (1, e^2)$ and assume that $\Xi_0(0) > 0$. Then*

$$\lim_{n\to\infty} \Xi_n(z) = \theta_\mu \qquad \text{for all } z \in \mathbb{Z}^d,$$

and for every $\varepsilon > 0$ there is a speed $a = a(\mu, \varepsilon, \Xi_0(0)) > 0$ such that $\Xi_n(x) \in (\theta_\mu - \varepsilon, \theta_\mu + \varepsilon)$ for all $|x| \le an$.

We believe that for localised or half-space initial conditions, the process Ξ will approach a "travelling wave". While there is a rich literature addressing travelling waves, we were not able to find results which literally apply in our context, in particular since our model has discrete time and space. We thus prove the above (weaker and non-optimal) proposition by rather bare hand arguments, which involve a construction of a "travelling wave sub-solution", see Section 4.2.3 below. Travelling waves in the context of PDEs have been widely studied, also with a view of biological applications. In the context of discrete time, *continuous space* models, the existence of travelling waves has also been considered quite extensively, see e.g. [Wei78, LLW09, Kot92, KS86]. In particular, in situations where φ_μ in (4.1.8) is replaced by an increasing (and hence monotone) function, existence of such travelling waves has been shown [Ham74, Wei78].

The regime $\mu > e^2$ is also very interesting. In this regime the iteration of φ_μ does not converge to a single point but to a stable orbit, which as μ increases beyond e^2 will increase its number of elements. In this case, we are not aware of results in the literature covering the coupled map lattice model. But even given such results, the behaviour of the stochastic system might be different and more difficult to control than in the stable-fixpoint case treated here. We leave these questions for future work.

4.2 Preliminary results and tools

In this section we collect some preliminary results that will be used throughout the paper.

4.2.1 A general coupling construction

We will frequently make use of the following construction allowing to define the process η for all initial conditions simultaneously and also allowing to compare η with other particle systems, in particular with monotone ones.

Let $U(x, n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{N}_0$, be a collection of i.i.d. uniform random variables on $[0, 1]$. Recall the definition of the function φ_μ from (4.1.3), and let $\psi : [0, 1] \to \mathbb{R}_+$ be any non-decreasing function satisfying

$$\psi(w) \le \varphi_\mu(w) \qquad \text{for all } w \in [0, 1] \cap V_R^{-d}\mathbb{Z}, \tag{4.2.1}$$

that is, for all possible values of the density $\delta_R(\cdot;\eta_n)$. Then, for any initial conditions η_0, $\widetilde{\eta}_0$, define, recursively for $n \geq 0$,

$$\eta_{n+1}(x) = \mathbb{1}_{\{U(x,n+1)\leq\varphi_\mu(\delta_R(x;\eta_n))\}}, \tag{4.2.2}$$
$$\widetilde{\eta}_{n+1}(x) = \mathbb{1}_{\{U(x,n+1)\leq\psi(\delta_R(x;\widetilde{\eta}_n))\}}. \tag{4.2.3}$$

The construction (4.2.2) of η is morally the analogue of the common graphical construction of an interacting particle system in our context, and can be viewed as a stochastic flow on the configuration space $\{0,1\}^{\mathbb{Z}^d}$. The next lemma gives its main properties.

Lemma 4.2.1 (General coupling construction)**.** *(a) The process η defined by* (4.2.2) *has the law of the branching-annihilating random walk with parameters μ and R and initial condition η_0.*

(b) If $\widetilde{\eta}_0(x) \leq \eta_0(x)$ for all $x \in \mathbb{Z}^d$, then $\widetilde{\eta}_n(x) \leq \eta_n(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$.

Proof. Part (a) follows immediately from (4.1.4). To see part (b) assume $\widetilde{\eta}_1(x) = 1$ for some $x \in \mathbb{Z}^d$. Then, by construction $U(x,1) \leq \psi(\delta_R(x;\widetilde{\eta}_0))$. Since $\widetilde{\eta}_0 \leq \eta_0$ and ψ is non-decreasing, and φ_μ dominates ψ, this yields $U(x,1) \leq \varphi_\mu(\delta_R(x;\eta_0))$, and so $\eta_1(x) = 1$. It follows that $\widetilde{\eta}_1 \leq \eta_1$, and by iteration, $\widetilde{\eta}_n \leq \eta_n$. □

In what follows we always assume that η is constructed as in (4.2.2) and define the filtration

$$\mathcal{F}_n := \sigma\big(\eta_0(x) : x \in \mathbb{Z}^d\big) \vee \sigma\big(U(x,i) : x \in \mathbb{Z}^d, i \leq n\big) \supseteq \sigma\big(\eta_i(x) : x \in \mathbb{Z}^d, i \leq n\big). \tag{4.2.4}$$

4.2.2 Concentration and comparison with deterministic profiles

As remarked under (4.1.3), the random variables $(\eta_{n+1}(x))_{x\in\mathbb{Z}^d}$ are conditionally independent given η_n. Therefore, the density $\delta_R(x;\eta_{n+1})$ should concentrate, at least for R large. We need estimates providing quantitative control of this concentration. These estimates involve certain sequences of functions $\xi_k^\pm$ on $\mathbb{Z}^d$, which we call *comparison density profiles*, that have the property that if at some time t the local density of η_t is controlled by $\xi_k^\pm$, then, at least locally, the density of η_{t+1} is controlled by $\xi_{k+1}^\pm$. In fact, the sequences ξ_k^- and ξ_k^+ that we use later can be regarded as a travelling wave sub- and super-solution, respectively, of the coupled map lattice iteration (4.1.8).

Definition 4.2.2. For a given $\varepsilon, \delta > 0$, *comparison density profiles* are deterministic functions $\xi_k^-, \xi_k^+ : \mathbb{Z}^d \to [0,\infty)$, $k = 0, 1, \ldots, k_0$, satisfying:

(i) For every $k = 0, \dots, k_0$, $\xi_k^-(\cdot) \le \xi_k^+(\cdot)$.

(ii) For every $k = 0, \dots, k_0$, $\mathrm{Supp}(\xi_k^-) := \{x \in \mathbb{Z}^d : \xi_k^-(x) > 0\}$ is finite, and $\xi_k^-(x) \ge \varepsilon$ for every $x \in \mathrm{Supp}(\xi_k^-)$.

(iii) For every $k = 0, \dots, k_0 - 1$, and $x \in \mathrm{Supp}(\xi_k^-)$ it holds that if $\zeta : B_R(x) \to \mathbb{R}$ satisfies $\zeta(y) \in [\xi_k^-(y), \xi_k^+(y)]$ for all $y \in B_R(x)$, then

$$(1+\delta)\xi_{k+1}^-(x) \le V_R^{-d} \sum_{y \in B_R(x)} \varphi_\mu(\zeta(y)) \le (1-\delta)\xi_{k+1}^+(x). \tag{4.2.5}$$

Note that ξ_k^-, ξ_k^+ will in general depend on R, μ, ε and δ, but we do not make this explicit in the notation (in fact, δ, ε could also depend on R and μ).

Lemma 4.2.3. *(a) For comparison density profiles $\xi_k^\pm$, if for some $x \in \mathbb{Z}^d$ and for $k \in \{0, \dots, k_0 - 1\}$*

$$\delta_R(y; \eta_k) \in \big[\xi_k^-(y), \xi_k^+(y)\big] \quad \textit{for all } y \in B_R(x), \tag{4.2.6}$$

then

$$\mathbb{P}\Big(\xi_{k+1}^-(x) \le \delta_R(x; \eta_{k+1}) \le \xi_{k+1}^+(x) \,\Big|\, \mathcal{F}_k\Big) \ge 1 - 2\exp(-cV_R^d), \tag{4.2.7}$$

where $c = (\delta\varepsilon)/\big(1/(2\delta\varepsilon) + 2/3\big)$.

(b) If, in (4.2.5), φ_μ is replaced by any ψ satisfying (4.2.1), then statement (a) holds for the monotone dynamics $\widetilde{\eta}$ defined in (4.2.3) in place of η.

Remark 4.2.4. If only a lower bound is required, as e.g. in the proof of survival, one can use the "trivial" upper bound for ξ_n^+, namely $\xi_n^+(\cdot) \equiv \max(\varphi_\mu)/(1-\delta) = e^{-1}/(1-\delta)$, and then apply (4.2.7) only for the lower bound.

Proof of Lemma 4.2.3. We only show (a), the proof of (b) is completely analogous. We consider first the lower bound, that is we want to show that the conditional probability of the event $\{\delta_R(x; \eta_{k+1}) < \xi_{k+1}^-(x)\}$ is small. Note that, by (4.2.6) and (4.2.5),

$$\sum_{y \in B_R(x)} \mathbb{E}[\eta_{k+1}(y) \mid \mathcal{F}_k] = \sum_{y \in B_R(x)} \varphi_\mu\big(\delta_R(y; \eta_k)\big) \ge (1+\delta) V_R^d \xi_{k+1}^-(x).$$

Therefore,

$$\begin{aligned} &\mathbb{P}\Big(\delta_R(x; \eta_{k+1}) < \xi_{k+1}^-(x) \,\Big|\, \mathcal{F}_k\Big) \\ &\quad \le \mathbb{P}\Big(\textstyle\sum_{y \in B_R(x)} \big(\eta_{k+1}(y) - \mathbb{E}[\eta_{k+1}(y) \,|\, \mathcal{F}_k]\big) < -\delta V_R^d \xi_{k+1}^-(x) \,\Big|\, \mathcal{F}_k\Big) \end{aligned} \tag{4.2.8}$$

and

$$\mathrm{Var}\big(\delta_R(x;\eta_{k+1})\,\big|\,\mathcal{F}_k\big) = V_R^{-2d}\sum_{y\in B_R(x)}\varphi_\mu\big(\delta_R(y;\eta_k)\big)\big(1-\varphi_\mu\big(\delta_R(y;\eta_k)\big)\big) \le \frac{1}{4}V_R^{-d}.$$

We now apply the Bernstein inequality (which we recall in Lemma A.1 in the Appendix) to the right-hand side of (4.2.8) with $n = V_R^d$, $\sigma_n \le V_R^{-d/2}/2$, $m_n \le 1$ and $w = \delta V_R^d \xi_{k+1}^-(x) \ge \delta\varepsilon V_R^d$ (since, by assumption ((ii)) $\xi_{k+1}^-(x) \ge \varepsilon$ if $\xi_{k+1}^-(x) > 0$, and there is nothing to prove if $\xi_{k+1}^-(x) = 0$). The expression in the exponent of the right-hand side of (A.1) then satisfies

$$\frac{w^2}{2\sigma_n^2 + (2/3)m_n w} = \frac{w}{2\sigma_n^2/w + (2/3)m_n} \ge \frac{w}{V_R^d/(2w) + 2/3} \ge \frac{\delta\varepsilon}{1/(2\delta\varepsilon) + 2/3}V_R^d,$$

which completes the proof of the lower bound in (4.2.7).

The proof of the upper bound, that is showing that the probability (conditional on η_k) of the event $\{\delta_R(x;\eta_{k+1}) > \xi_{k+1}^+(x)\}$ is small, is completely analogous. □

4.2.3 Lower bounds on travelling waves

The goal of this section is to construct explicit comparison density profiles ξ_k^- which can later be used as the lower bounds on $\delta_R(\cdot;\eta)$ in the proofs of survival and complete convergence. As pointed out before, these can be viewed as travelling wave sub-solutions to the iteration (4.1.8).

We start by providing the basic building block for this construction. To this end we concentrate first on the one-dimensional setting. For parameters $a > 1$, $\varepsilon_0 \in (0,1)$, $w > 0, s > 0$ and $R \in \mathbb{N}$ we say that a non-decreasing function $f : \mathbb{Z} \to [0,\infty)$ is a linear travelling wave shape with width $\lceil wR\rceil$, shift $\lceil sR\rceil$, growth factor a and minimal step size ε_0 if it fulfils

$$f(x) = 0 \text{ for } x < 0, \qquad f(0) = \varepsilon_0, \qquad f(x) = 1 \text{ for } x \ge \lceil wR\rceil \tag{4.2.9}$$

and

$$a\delta_R(x;f) \ge f\big(x + \lceil sR\rceil\big) \quad \text{for all } x \in \mathbb{Z}. \tag{4.2.10}$$

In this parametrisation, we think of a "wave profile" which, when subjected to one iteration of the operation $f(\cdot) \mapsto a\delta(\cdot;f)$, moves to the left by at least $\lceil sR\rceil$ in each time step. Note that by construction, one necessarily has that $s \le 1$.

We now show that such a function f exists for any $a > 1$ and R large.

Lemma 4.2.5. *For every $a > 1$, there is $w \ge 2$, $\varepsilon_0 \in (0,1)$, $s \in (0,1)$, and $R_0 \in \mathbb{N}$ such that the function*

$$f(x) = \min\big\{(\varepsilon_0 + x/\lceil wR\rceil)\mathbb{1}_{x\ge 0}, 1\big\}$$

satisfies (4.2.9), (4.2.10) *for all $R \ge R_0$.*

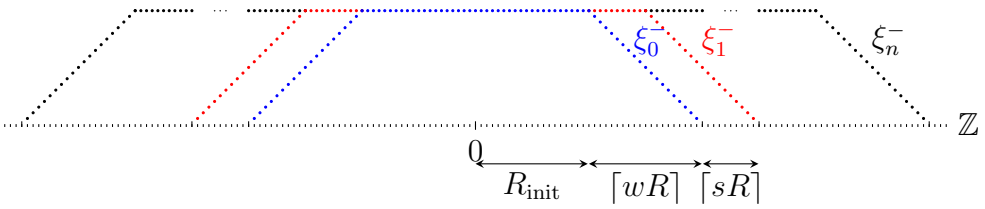

Figure 4.2: The one-dimensional deterministic comparison density profile ξ_n^- built from the linear travelling wave shape f, with fronts of width $\lceil wR \rceil$ that get shifted outward by $\lceil sR \rceil$ in every time step.

The proof of Lemma 4.2.5 is a straightforward, albeit somewhat lengthy computation, and is given in Section 4.6.1. In fact, with even lengthier computations it could be shown that the lemma holds for any $R \geq 1$.

Using this travelling wave shape we can now define the desired comparison density profile ξ_n^-. For this, fix $R_{\text{init}} \in \mathbb{N}$ with $R_{\text{init}} > 2R$ and set, for $x \in \mathbb{Z}$,

$$\widetilde{\xi}_n(x) = f\big(R_{\text{init}} + n\lceil sR \rceil + \lceil wR \rceil - |x|\big) \tag{4.2.11}$$

with f from Lemma 4.2.5, see Figure 4.2 for an illustration. Note that by construction, $\widetilde{\xi}_n(\cdot) \equiv 1$ on $B_{R_{\text{init}}+n\lceil sR \rceil}(0)$ and $\text{Supp}(\widetilde{\xi}_n) = B_{R_{\text{init}}+n\lceil sR \rceil+\lceil wR \rceil}(0)$. Furthermore, using (4.2.10), $a\delta_R(x;\widetilde{\xi}_n) \geq \widetilde{\xi}_{n+1}(x)$ for all $x \in \mathbb{Z}$, and $\widetilde{\xi}_n(x) > 0$ implies $\widetilde{\xi}_n(x) \geq \varepsilon_0$.

Finally, for any $d \geq 1$, write $x = (x_1, \ldots, x_d)$ and set

$$\xi_n^-(x) := b \prod_{i=1}^{d} \widetilde{\xi}_n(x_i), \quad x \in \mathbb{Z}^d, n \in \mathbb{N}_0 \tag{4.2.12}$$

with some $0 < b \leq 1$ that will be suitably tuned later. Note that ξ_n^- implicitly depends on d, R, R_{init}, a and b but our notation does not make this explicit. We summarise the relevant properties of ξ_n^- in the following lemma.

Lemma 4.2.6. *The functions ξ_n^- have the following properties:*

(i) $0 \leq \xi_n^-(x) \leq b$ *for every* $n \in \mathbb{N}_0$ *and* $x \in \mathbb{Z}^d$,

(ii) *for* $n \in \mathbb{N}_0$, $\xi_n^-(\cdot) \equiv b$ *on* $B_{R_{\text{init}}+n\lceil sR \rceil}(0)$ *and* $\text{Supp}(\xi_n^-) = B_{R_{\text{init}}+n\lceil sR \rceil+\lceil wR \rceil}(0)$,

(iii) $a^d\delta_R(x;\xi_n^-) \geq \xi_{n+1}^-(x)$ *for all* $n \in \mathbb{N}_0$, $x \in \mathbb{Z}^d$,

(iv) $\xi_n^-(x) > 0$ *implies* $\xi_n^-(x) \geq b\varepsilon_0^d$.

Proof. The properties (i), (ii) and (iv) follow directly from (4.2.9), (4.2.11) and (4.2.12). Using (4.2.10), (4.2.11), (4.2.12), it follows moreover that

$$\begin{aligned} a^d \delta_R(x;\xi_n^-) &= a^d V_R^{-d} \sum_{y\in B_R(0)} \xi_n^-(y+x) \\ &= b a^d V_R^{-d} \sum_{y_1=-R}^{R} \cdots \sum_{y_d=-R}^{R} \prod_{i=1}^{d} \widetilde{\xi}_n(x_i+y_i) \\ &= b \prod_{i=1}^{d} \Big(a V_R^{-1} \sum_{y=-R}^{R} \widetilde{\xi}_n(x_i+y) \Big) = b \prod_{i=1}^{d} \big(a\delta_R(x_i;\widetilde{\xi}_n) \big) \\ &\geq b \prod_{i=1}^{d} \widetilde{\xi}_{n+1}(x_i) = \xi_{n+1}^-(x), \end{aligned}$$

which shows (iii) and completes the proof. □

4.3 Survival for large R: Proof of Theorem 4.1.3

In this section we prove Theorem 4.1.3, stating that the system survives for any $\mu > 1$, given that R is chosen sufficiently large. The proof is based on the comparison with a monotone system $\widetilde{\eta}$, which in turn is shown to survive using a comparison with finite range oriented percolation. The latter is a by now classical technique for interacting particle systems, we refer to [Czu16], [Lan17] or [Swa17] for recent and reader-friendly introductions.

The monotone system $\widetilde{\eta}$ is constructed as in Section 4.2.1: we first fix parameters $\widetilde{a} \in (1,\mu)$ and $b \in (0,1)$, so that the function ψ defined by

$$\psi(w) := \widetilde{a}(w \wedge b) \tag{4.3.1}$$

satisfies (4.2.1). This is possible since $\mu > 1$. With this ψ, we define $\widetilde{\eta}$ as in (4.2.3) and simultaneously η as in (4.2.2) on the probability space supporting the i.i.d. uniform random variables $(U(x,n))_{x\in\mathbb{Z}^d, n\in\mathbb{N}_0}$.

We then fix $a > 1$ such that $a^d < \widetilde{a}$, and for this choice of a, we fix R_0, w, s and ε_0 according to Lemma 4.2.5. For $R \geq R_0$, we set $R_{\text{init}} := \lceil wR/2 \rceil$ and define ξ_n^- as in (4.2.12). We claim that $\xi_n^-(x)$ (and the trivial ξ_n^+, as explained in Remark 4.2.4) is a comparison density profile in the sense of Definition 4.2.2 with $\delta = (\widetilde{a}/a^d) - 1$ and $\varepsilon = b\varepsilon_0^d$. Moreover the lower bound of (4.2.5) even holds with ψ in place of φ_μ. Indeed, ((i)) is trivially true, ((ii)) follows from Lemma 4.2.6(iv). To show ((iii)), that is (4.2.5) (with ψ in place of φ_μ), let $\zeta = (\zeta(y)) \in [0,1]^{\mathbb{Z}^d}$ be such that $\zeta(\cdot) \geq \xi_n^-(\cdot)$ for

some $n \in \mathbb{N}_0$. Then, using Lemma 4.2.6(iii) for the inequality,

$$\begin{aligned} V_R^{-d} \sum_{y \in B_R(x)} \psi\big(\zeta(y)\big) &= V_R^{-d} \sum_{y \in B_R(x)} \widetilde{a}\big(\zeta(y) \wedge b\big) \\ &\geq \frac{\widetilde{a}}{a^d} \cdot a^d V_R^{-d} \sum_{y \in B_R(x)} \xi_n^-(y) \\ &= \frac{\widetilde{a}}{a^d} \cdot a^d \delta_R(x; \xi_n^-) \geq \frac{\widetilde{a}}{a^d} \cdot \xi_{n+1}^-(x), \end{aligned}$$

as required. As a consequence, we will later be able to apply the concentration result of Lemma 4.2.3(b) to the process $\tilde{\eta}$.

Define $R'_{\text{block}} = \lceil wR/2 \rceil$. To set up the comparison with oriented percolation, we coarse-grain the system by using blocks spaced by $L'_{\text{block}} := 2R'_{\text{block}}$, of side length $L_{\text{block}} := 5L'_{\text{block}}$ and temporal size $T_{\text{block}} := \big\lceil \lceil wR \rceil / \lceil sR \rceil \big\rceil$. Since we often refer to radii rather than block lengths, it is convenient to define $R_{\text{block}} = L_{\text{block}}/2$.

For (z,t) in the sub-lattice $\mathbb{L} := L'_{\text{block}}\mathbb{Z}^d \times T_{\text{block}}\mathbb{N}_0$, we define

$$\texttt{Block}(z,t) = \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{N}_0 : \|x - z\| \leq R_{\text{block}},\, t \leq n \leq t + T_{\text{block}}\big\}.$$

Note that blocks in the same time-layer have non-trivial overlap with their neighbours but the number of overlapping neighbours in $\mathbb{L}$ per block does not grow with R. In the time direction, only the top time slice of a given block coincides with the bottom layer of the next block(s).

Definition 4.3.1. We call $\texttt{Block}(z,t)$ *well-started* if the density of the monotone system $\widetilde{\eta}$ dominates the (suitably shifted) density profile ξ_0^- at the bottom of the block, that is

$$\delta_R\big(x; \widetilde{\eta}_t\big) \geq \xi_0^-(x - z) \quad \text{for } \|x - z\| \leq R_{\text{block}}. \tag{4.3.2}$$

Note that for any $(z,t) \in \mathbb{L}$ the event $\{\texttt{Block}(z,t) \text{ is well-started}\}$ is measurable with respect to the filtration $\mathcal{F}_t$, which was defined in (4.2.4).

Definition 4.3.2. $\texttt{Block}(z,t)$ is called *good* if it is well-started and the random variables $U(x,n)$ are such that the domination property of (4.3.2) propagates over the block. That is, $\texttt{Block}(z,t)$ is *good* if it holds that

$$\delta_R\big(x; \widetilde{\eta}_{t+n}\big) \geq \xi_n^-(x - z) \quad \text{for } \|x - z\| \leq R_{\text{block}},\, n = 0, \ldots, T_{\text{block}}.$$

The properties of the comparison density profiles ξ_n^-, see Lemma 4.2.6, enforce

$$\{\texttt{Block}(z,t) \text{ is good}\} \subseteq \bigcap_{\substack{z' \in L'_{\text{block}}\mathbb{Z}^d : \\ \|z - z'\| \leq L'_{\text{block}}}} \{\texttt{Block}(z', t + T_{\text{block}}) \text{ is well-started}\}. \tag{4.3.3}$$

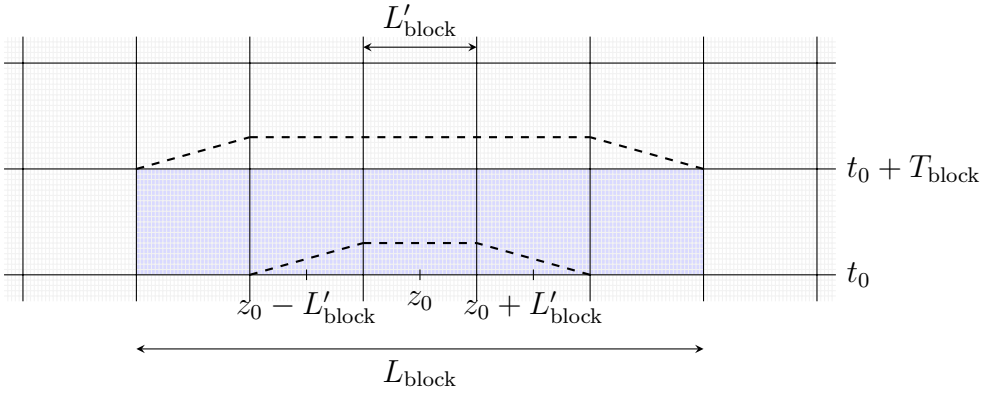

Figure 4.3: Sketch of a $\texttt{Block}(z_0, t_0)$ (light blue), centred at the coarse-grained space-time lattice point (z_0, t_0). The thick dashed lines depict the deterministic comparison density profiles $\xi^-_{t_0}(\cdot)$ and $\xi^-_{t_0+T_{\text{block}}}(\cdot)$ which have to be dominated by the density $\delta_R(\cdot; \widetilde{\eta}_n)$, $t_0 \le n \le t_0 + T_{\text{block}}$ in order for the block to be *good*. Note the picture is not drawn to scale: L_{block} and L'_{block} are both growing linearly in R while T_{block} does not grow with R.

In particular, the process $\widetilde{\eta}$ survives up to time $t + T_{\text{block}}$ in a good $\texttt{Block}(z, t)$ and the region of the desired density control by the profiles ξ^-_n expands, see Figure 4.3.

By the construction (4.2.3) of $\widetilde{\eta}$, given $\mathcal{F}_t$, if $\texttt{Block}(z, t)$ is well-started, it can be decided whether or not the event $\{\texttt{Block}(z, t) \text{ is good}\}$ occurs for $(z, t) \in \mathbb{L}$ by inspecting (only) the values of

$$\big(U(x, n) \in \mathbb{Z}^d \times \mathbb{N}_0 : \|x - z\| \le R_{\text{block}} + T_{\text{block}} R,\, t < n \le t + T_{\text{block}}\big) \tag{4.3.4}$$

(in fact, strictly speaking it suffices to observe the values of U's at the space-time points $\{(x, n) : \|x - z\| \le R_{\text{block}} + (t + T_{\text{block}} - n)R,\, t < n \le t + T_{\text{block}}\}$). Note that for $(z, t) \in \mathbb{L}$ and $(z', t) \in \mathbb{L}$ with

$$\|z' - z\| > L_{\text{block}} + 2T_{\text{block}} R \approx (5 + 2/s) L'_{\text{block}} \quad (\text{when } R \text{ is large}) \tag{4.3.5}$$

the space-time regions corresponding to (4.3.4) will be disjoint.

Furthermore, by invoking Lemma 4.2.3(b) we can uniformly bound the probability of the density of $\widetilde{\eta}$ dominating the comparison density profile ξ^- for all space-time sites in $\texttt{Block}(z, t)$, which in turn yields

$$\mathbb{P}\big(\texttt{Block}(z, t) \text{ is good} \,\big|\, \mathcal{F}_t\big) \ge \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}} \big(1 - q(T_{\text{block}}, R)\big) \tag{4.3.6}$$

with

$$q(T_{\text{block}}, R) = 2\big|\texttt{Block}(z, t)\big| e^{-cV_R^d}, \tag{4.3.7}$$

which tends to 0 as $R \to \infty$, since $|\mathtt{Block}(z,t)|$ grows only polynomially in R.

In order to make the comparison with oriented percolation, we define random variables

$$Y(z,t) := \mathbb{1}_{\{\mathtt{Block}(z,t) \text{ is good}\}}, \quad (z,t) \in \mathbb{L}, \tag{4.3.8}$$

and say that $(z,t) \in \mathbb{L}$ is *connected to infinity* in Y if there is a path $((z_i, t+iT_{\mathrm{block}}) : i \in \mathbb{N}_0)$ in $\mathbb{L}$ with $z_0 = z$ and $\|z_i - z_{i-1}\| \le L'_{\mathrm{block}}$ for all $i \in \mathbb{N}$, such that $Y(z_i, t+iT_{\mathrm{block}}) = 1$ for all $i \in \mathbb{N}_0$ (such a path is called *open* in Y). By the argument above, it follows that if (z,t) is well-started and connected to infinity in Y, then the process $\widetilde{\eta}$ survives.

In order to show that the latter event occurs with positive probability, we iteratively construct a coupling between the $Y(z,t)$ from (4.3.8) and a family $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ of i.i.d. Bernoulli random variables with parameter $p(R)$ which satisfies $p(R) \to 1$ as $R \to \infty$ such that we have

$$Y(z,t) \ge \mathbb{1}_{\{\mathtt{Block}(z,t) \text{ is well-started}\}} \widetilde{Y}(z,t) \quad \text{for all } (z,t) \in \mathbb{L}. \tag{4.3.9}$$

We construct $\widetilde{Y}(\cdot,t)$ inductively over t and begin with a slightly informal description of this construction: Assume that for some $t' \in T_{\mathrm{block}}\mathbb{N}$, a coupling satisfying (4.3.9) has been achieved for all $(z,t) \in \mathbb{L}$ with $T_{\mathrm{block}}\mathbb{N} \ni t < t'$. We then work conditionally on $\mathcal{F}_{t'}$. The (random) set of nodes

$$W(t') := \{z' \in L'_{\mathrm{block}}\mathbb{Z}^d : \mathtt{Block}(z',t') \text{ is well-started}\},$$

viewed as a graph where z' and z'' are connected by an edge if the space-time regions from (4.3.4) centred at (z',t') and at (z'',t'), respectively, overlap, is a locally finite graph with uniformly bounded degrees. In fact, we see from (4.3.4) that we have irrespective of the realisation of $\widetilde{\eta}_{t'}$ the deterministic bound $(11+4/s)^d$ on the degree of any node (up to rounding, see (4.3.5)). Thus, by (4.3.4)–(4.3.7), using well known stochastic domination arguments for percolation models with finite-range dependencies [LSS97], it follows that the family $(Y(z,t'))_{z\in W(t')}$ stochastically dominates a family $(\widetilde{Y}(z,t'))_{z\in W(t')}$ of i.i.d. Bernoulli random variables with parameter $p(R)$, where $p(R) \to 1$ as $R \to \infty$ and the $(\widetilde{Y}(z,t'))_{z\in W(t')}$ are independent of $\mathcal{F}_{t'}$ given $W(t')$, i.e. (4.3.9) holds for all $z \in W(t')$. In fact, $p(R)$ is a function of the maximal degree $(11+4/s)^d$ of the dependence graph and the minimal guaranteed density $1 - q(T_{\mathrm{block}}, R)$ of good blocks, see Theorem 1.3 in [LSS97]. For $z \notin W(t')$, (4.3.9) imposes no condition at all on $\widetilde{Y}(z,t')$. Thus, we can simply define $\widetilde{Y}(z,t') = \widehat{Y}(z,t')$ for $z \notin W(t')$ where $(\widehat{Y}(z,t))_{(z,t)\in\mathbb{L}}$ is an independent family of i.i.d. Bernoulli$(p(R))$ random variables.

In order to formalise this construction and, in particular, to show that the random variables $\widetilde{Y}(z,t)$ are independent over different time layers, note that by the construction (4.2.2) from Lemma 4.2.1, we can write

$$Y(\cdot,t') = g\big(\eta_{t'}, (U(\cdot,n) : t' < n \le t' + T_{\mathrm{block}})\big)$$

for some deterministic function $g : \{0,1\}^{\mathbb{Z}^d} \times [0,1]^{\mathbb{Z}^d \times \{1,\dots,T_{\text{block}}\}} \to \{0,1\}^{L'_{\text{block}}\mathbb{Z}^d}$, furthermore $W(t') = W(\eta_{t'}) = \{z \in L'_{\text{block}}\mathbb{Z}^d : \texttt{Block}(z,t') \text{ is well started}\}$. For every $\zeta = (\zeta(z))_{z\in\mathbb{Z}^d} \in \{0,1\}^{\mathbb{Z}^d}$, [LSS97, Thm. 1.3] and the discussion above provides a coupling ν_ζ of $\mathcal{L}(Y(\cdot,t') \,|\, \eta_{t'} = \zeta)$ and $\text{Ber}(p(R))^{\otimes\mathbb{Z}^d}$ with the desired properties. We can then disintegrate this joint law with respect to its first marginal and describe the joint law ν_ζ in a two-step procedure. It is convenient to describe this via an auxiliary function $h(\zeta;\cdot,\cdot)$ using additional independent randomness and obtain that given $\eta_{t'} = \zeta$,

$$Y(\cdot,t') = g\big(\zeta, (U(\cdot,n) : t' < n \le t' + T_{\text{block}})\big), \quad \widetilde{Y}(\cdot,t') = h\big(\zeta; Y(\cdot,t'), \widetilde{U}_{t'}\big)$$

where $\widetilde{U}_{t'}$ is independent of everything else and uniformly distributed on $[0,1]$ (see, for example, Theorem 5.10 in [Kal97]). By construction, since $U(\cdot,n), n > t'$ and $\widetilde{U}_{t'}$ are independent of $\mathcal{F}_{t'}$, we have for $A \in \mathcal{F}_{t'}$ and measurable $B \subseteq \{0,1\}^{L'_{\text{block}}\mathbb{Z}^d}$

$$\begin{aligned}\mathbb{P}\big(A \cap \{\widetilde{Y}(\cdot,t') \in B\}\big) &= \mathbb{E}\Big[\mathbb{1}_A \mathbb{P}\big(h(\eta_{t'}; Y(\cdot,t'), \widetilde{U}_{t'}) \in B \,|\, \mathcal{F}_{t'}\big)\Big] \\ &= \mathbb{P}(A)\text{Ber}(p(R))^{\otimes L'_{\text{block}}\mathbb{Z}^d}(B).\end{aligned}$$

This shows the required independence of $\widetilde{Y}$ and completes the induction step.

We see from (4.3.8), (4.3.9) and (4.3.3) that every open path in $\widetilde{Y}(\cdot,\cdot)$ is automatically also an open path in $Y(\cdot,\cdot)$. Furthermore, by well known properties of oriented site percolation, we have

$$\mathbb{P}\big((z,t) \text{ is connected to infinity in } \widetilde{Y}\big) = \mathbb{P}\big((0,0) \text{ is connected to infinity in } \widetilde{Y}\big) > 0$$

if $p(R)$ is sufficiently close to 1, i.e. for all R large enough.

To conclude, let η_0 be any initial configuration containing at least one particle, and let $\widetilde{\eta}_0 = \eta_0$. It is then easy to see (as this involves requiring only finitely many random variables $U(x,n)$ to be sufficiently small), that one can find $(z,t) \in \mathbb{L}$, so that the probability that $\texttt{Block}(z,t)$ is well-started is positive.

Therefore, due to the above properties,

$$\begin{aligned}\mathbb{P}(\eta \text{ survives}) &\ge \mathbb{P}(\widetilde{\eta} \text{ survives}) \\ &\ge \mathbb{E}\big[\mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}} \mathbb{1}_{\{(z,t) \text{ is connected to infinity in } Y\}}\big] \\ &\ge \mathbb{P}\big(\texttt{Block}(z,t) \text{ is well-started}\big)\mathbb{P}\big((z,t) \text{ is connected to infinity in } \widetilde{Y}\big) \\ &> 0 \qquad \text{for all } R \text{ large enough,}\end{aligned}$$

which completes the proof of Theorem 4.1.3.

4.4 Complete convergence

In this section we show our main results in the regime where the particle system survives with a positive probability and is well approximated by the deterministic coupled map lattice introduced in Section 4.1.4. In particular, we assume that $\mu \in (1, e^2)$ and R is large enough. In Section 4.4.1, we start with Theorem 4.1.6 providing the coupling of processes started with different initial conditions. Theorem 4.1.5 is then shown in Section 4.4.3.

4.4.1 Coupling construction: Proof of Theorem 4.1.6

As in Section 4.3, the central ingredient will be a block construction and then a suitable comparison with oriented percolation. The definition of "good blocks" will be more involved than in Section 4.3 and is inspired by the construction in [BD07, Section 5].

In brief, the construction of a good block around z is as follows. We consider a (large) ball B around z and assume that $\eta^{(1)}$ and $\eta^{(2)}$ agree on B and the respective R-densities of the two processes are close to θ_μ. On an even larger ball B' we add milder and milder requirements (as the distance from the centre increases) on the densities of the processes. The contraction property of φ_μ, see Lemma 4.4.1 below, together with the concentration property of the densities of $\eta^{(i)}$ guaranteed by Lemma 4.2.3 then ensure that the area in which the $\eta^{(1)}$ and $\eta^{(2)}$ are coupled expands in time with high probability. In order to guarantee survival of the processes we also require that the respective densities of $\eta^{(1)}, \eta^{(2)}$ dominate the deterministic comparison density profile as defined in (4.2.12) (the latter was also used in Section 4.3).

We now proceed with the formal definitions. Throughout this section, we again use the coupling construction from Section 4.2.1: Given two initial conditions $\eta_0^{(1)}$ and $\eta_0^{(2)}$, we construct both $(\eta_n^{(1)})_n$ and $(\eta_n^{(2)})_n$ using (4.2.2) with the same $U(x,n)$'s, that is, we set

$$\eta_{n+1}^{(i)}(x) = \mathbb{1}_{\{U(x,n+1)\le\varphi_\mu(\delta_R(x;\eta_n^{(i)}))\}}, \quad i \in \{1,2\}, \quad (x,n) \in \mathbb{Z}^d \times \mathbb{N}_0. \tag{4.4.1}$$

Since we are from now on interested in two copies of the branching annihilating process, we redefine the filtration $(\mathcal{F}_n)$ from (4.2.4) by including both initial conditions, i.e.

$$\mathcal{F}_n := \sigma\big(\eta_0^{(i)}(x) : x \in \mathbb{Z}^d, i = 1,2\big) \vee \sigma\big(U(x,j) : x \in \mathbb{Z}^d, j \le n\big).$$

It is clear that this updated filtration is finer than the natural filtration of the two processes, in the sense that for all $n \ge 0$, it holds that $\mathcal{F}_n \supseteq \sigma\big(\eta_j^{(i)}(x) : x \in \mathbb{Z}^d, j \le n, i = 1,2\big)$.

In order to define the comparison density profiles that are used to determine whether a block is good, we need a simple lemma which gives some useful properties of the function φ_μ in the vicinity of its non-trivial fixpoint θ_μ. The result is fairly

standard, we provide a proof for completeness' sake in Section 4.6.2 (cf. also [BD07, Proof of Lemma 12]).

Lemma 4.4.1. *For every $\mu \in (1, e^2)$ there is $\varepsilon > 0$ and $\kappa(\mu, \varepsilon) < 1$ such that φ_μ is a contraction on $[\theta_\mu - \varepsilon, \theta_\mu + \varepsilon]$, that is,*

$$|\varphi_\mu(w_1) - \varphi_\mu(w_2)| \le \kappa(\mu, \varepsilon)|w_1 - w_2| \qquad \text{for } w_1, w_2 \in [\theta_\mu - \varepsilon, \theta_\mu + \varepsilon].$$

Moreover, there exist a strictly increasing sequence $\alpha_m \uparrow \theta_\mu$ and a strictly decreasing sequence $\beta_m \downarrow \theta_\mu$ such that $\varphi_\mu([\alpha_m, \beta_m]) \subseteq (\alpha_{m+1}, \beta_{m+1})$ for all $m \in \mathbb{N}$. Furthermore, it is possible to choose $\alpha_1 > 0$ arbitrarily small and $\beta_1 > 1/e$.

We now take b as in (4.3.1) and fix ε, $\kappa(\mu, \varepsilon) < 1$, as well as sequences $\alpha_m \uparrow \theta_\mu$, $\beta_m \downarrow \theta_\mu$ as in Lemma 4.4.1, with $\alpha_1 = b$ and $\beta_1 > 1/e$. Then we choose m_0 such that $\beta_m - \alpha_m < \varepsilon$ for every $m \ge m_0$. These choices will remain fixed throughout the remainder of this section.

Next define the size of the blocks

$$L'_{\text{block}} = 2\lceil R \log R \rceil, \qquad L_{\text{block}} = c_{\text{space}} L'_{\text{block}} \quad \text{and} \quad T_{\text{block}} = c_{\text{time}} \lceil \log R \rceil, \tag{4.4.2}$$

where $c_{\text{time}} > -(d+1)/\log \kappa(\mu, \varepsilon)$ and $c_{\text{space}} = 4(1 + c_{\text{time}})$ are integer constants. Remark 4.4.5 below explains these choices. As in Section 4.3, we introduce $R'_{\text{block}} = L'_{\text{block}}/2$ and $R_{\text{block}} = L_{\text{block}}/2$ for the radii of the blocks, and, for (z, t) in the sub-lattice $\mathbb{L} := L'_{\text{block}}\mathbb{Z}^d \times T_{\text{block}}\mathbb{N}_0$, we define

$$\texttt{Block}(z, t) = \big\{(x, n) \in \mathbb{Z}^d \times \mathbb{N}_0 : \|x - z\| \le R_{\text{block}},\, t \le n \le t + T_{\text{block}}\big\}.$$

Further, let us specify the radius for which the strongest form of density control, alluded to in the above informal description, holds. More precisely set $c_{\text{dens}} = 1 + 2c_{\text{time}}$ and $R_{\text{dens}} := 2c_{\text{dens}} R'_{\text{block}}$. Again, the discussion on the choice of c_{dens} is postponed to Remark 4.4.5.

Recall the functions $\xi_n^-(x)$ defined in (4.2.12). We use them here with $R_{\text{init}} = R_{\text{dens}} + m_0 R$ in (4.2.11). For $k \in \{0, \dots, T_{\text{block}}\}$ set $R_{\text{dens}}(k) = R_{\text{dens}} + k\lceil sR \rceil$, then let

$$\zeta_k^-(x) := \begin{cases} \alpha_{m_0} & \text{if } \|x\| \le R_{\text{dens}}(k) \\ \alpha_{m_0 - j + 1} & \text{if } R_{\text{dens}}(k) + (j-1)R < \|x\| \le R_{\text{dens}}(k) + jR,\ 1 \le j \le m_0 \\ \xi_k^-(x) & \text{if } \|x\| > R_{\text{dens}}(k) + m_0 R, \end{cases}$$

and

$$\zeta_k^+(x) := \begin{cases} \beta_{m_0} & \text{if } \|x\| \le R_{\text{dens}}(k) \\ \beta_{m_0 - j + 1} & \text{if } R_{\text{dens}}(k) + (j-1)R < \|x\| \le R_{\text{dens}}(k) + jR,\ 1 \le j \le m_0 \\ 1 \vee \beta_1 & \text{if } \|x\| > R_{\text{dens}}(k) + m_0 R. \end{cases}$$

See also Figure 4.4.

The functions $\zeta_k^-(\cdot) < \zeta_k^+(\cdot)$ are comparison density profiles in the sense of Definition 4.2.2, in particular, they satisfy the following analogue of (4.2.5).

Lemma 4.4.2. *There exists $\delta > 0$ with the following property: For $k \in \mathbb{N}_0$ and any $(\zeta(x))_{x\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d}$ satisfying $\zeta_k^- \le \zeta \le \zeta_k^+$ on $\mathrm{Supp}(\zeta_k^-)$, it follows that*

$$(1+\delta)\zeta_{k+1}^-(x) \le V_R^{-d} \sum_{y\in B_R(x)} \varphi_\mu\big(\zeta(y)\big) \le (1-\delta)\zeta_{k+1}^+(x) \quad \text{for all } x \in \mathrm{Supp}(\zeta_{k+1}^-).$$

Proof. For x such that $\zeta_k^-(x)$ agrees with the previously defined profile $\xi_k^-(x)$ the lower bound in the statement follows easily from Lemma 4.2.6.

Let $x \in \mathrm{Supp}(\zeta_{k+1}^-)$ with $\zeta_{k+1}^-(x) = \alpha_j$, for some $2 \le j \le m_0$. Then

$$\zeta_k^-(y) = \alpha_j \text{ if } y \in B_R(x) \cap \Upsilon_R \qquad \text{and} \qquad \zeta_k^-(y) \ge \alpha_{j-1} \text{ if } y \in B_R(x) \cap \Upsilon_R^c,$$

where $\Upsilon_R := \{z : R_{\mathrm{dens}}(k)+(j-1)R \le \|z\| \le R_{\mathrm{dens}}(k)+jR\}$. Note that $|\Upsilon_R \cap B_R(x)| \ge cV_R^d$ for some $c > 0$, uniformly in the x we consider here. The properties of sequences α_m, β_m from Lemma 4.4.1 then imply that there exists $\delta > 0$ (depending on $(\alpha_m)_{m\le m_0}$, $(\beta_m)_{m\le m_0}$ and d) such that

$$V_R^{-d} \sum_{y\in B_R(x)} \varphi_\mu(\zeta(y)) \ge \alpha_{j+1}|B_R(x)\cap\Upsilon_R|V_R^{-d} + \alpha_j|B_R(x)\cap\Upsilon_R^c|V_R^{-d} \ge \alpha_j(1+\delta)$$

and similarly for the upper bound. This completes the claim for the remaining parts of the profile (those in orange in Figure 4.4). □

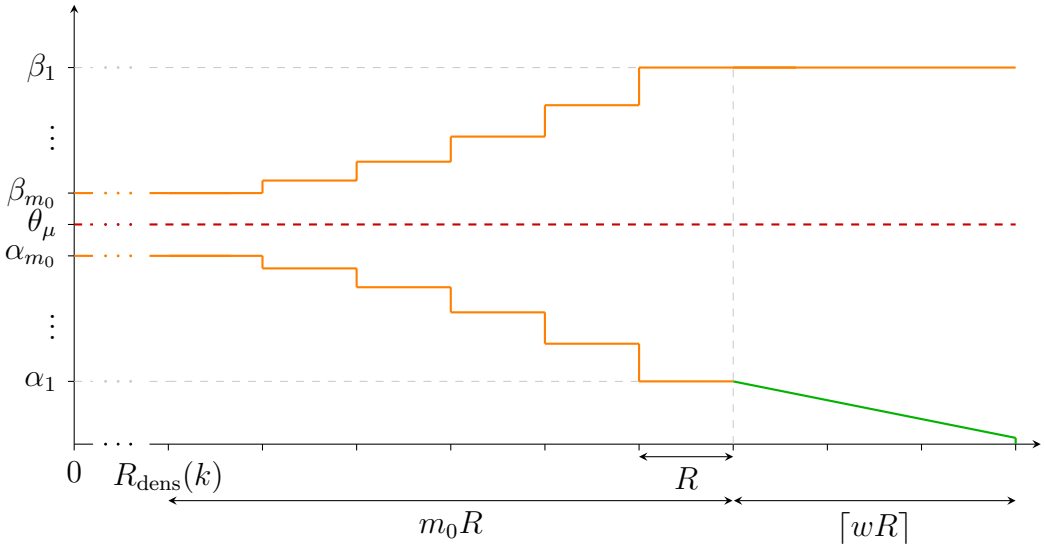

Figure 4.4: The part of the deterministic comparison density profiles ζ_k^+ and ζ_k^- (in orange and green) left from $R_{\mathrm{dens}}(k)$. In a *good* block the densities of both $\eta^{(1)}$ and $\eta^{(2)}$ stay between the union of the orange lines and the green profile, which is glued to the bottom orange profile. The green line is the (suitably recentred and shifted) profile of ξ^-, which we introduced to prove survival.

We proceed in a similar fashion as in Section 4.3 and introduce a new notion of well-started and of good blocks. These updated definitions involve two copies $\eta^{(1)}$, $\eta^{(2)}$ of the system. A well-started block is now determined by the local density of the true system being controlled by the ζ_k^+, ζ_k^- profiles, in addition to which we require agreement of the true processes in the central part of the block.

Definition 4.4.3. A $\texttt{Block}(z,t)$ based at $(z,t) \in \mathbb{L}$ is *well-started* if

$$\delta_R(x; \eta_t^{(i)}) \in \left[\zeta_0^-(x-z), \zeta_0^+(x-z)\right] \quad \text{for all } x \in z + \mathrm{Supp}(\zeta_0^-),\ i = 1,2 \tag{4.4.3}$$

and

$$\eta_t^{(1)}(x) = \eta_t^{(2)}(x) \quad \text{for all } x \in B_{R'_{\mathrm{block}}}(z). \tag{4.4.4}$$

Again as in Section 4.3 we use this as the starting point off of which we base our notion of goodness as the spreading of the control given by well-startedness to neighbouring regions.

Definition 4.4.4. We call a $\texttt{Block}(z,t)$ based at $(z,t) \in \mathbb{L}$ *good* if

(i) $\texttt{Block}(z,t)$ is *well-started*,

(ii) $\eta^{(1)}_{t+T_{\mathrm{block}}}(x) = \eta^{(2)}_{t+T_{\mathrm{block}}}(x)$ for $\|x - z\| \le 3R'_{\mathrm{block}}$,

(iii) $\left(\eta^{(1)}_{t+T_{\mathrm{block}}}, \eta^{(2)}_{t+T_{\mathrm{block}}}\right)$ satisfy (4.4.3) around $z + L'_{\mathrm{block}}e$ for all $e \in B_1(0)$.

Property (iii) implies that if $\texttt{Block}(z,t)$ is good, then $\texttt{Block}(z + L'_{\mathrm{block}}e, t + T_{\mathrm{block}})$ will be well-started for all $\|e\| \le 1$.

Remark 4.4.5. Let us now comment on our choice of the constants c_{space}, c_{dens}, c_{time}. It is instructive to first give c_{space} as a function of c_{dens}, then c_{dens} as a function of c_{time}, and ultimately fixing c_{time} large enough.

(i) Note first that $\zeta_0^{\pm}$ are constant on a box of size R_{dens} (which is of order $R \log R$) and then increase (resp. decrease) on boxes with length of order R. It follows readily that $\mathrm{Supp}(\zeta_0^-) \subseteq B_{2R_{\mathrm{dens}}}(0)$ for large enough R. Therefore $2c_{\mathrm{dens}}$ blocks of size L'_{block} fully cover the spatial region determining whether a block is well-started. Furthermore, we need to to provide additional space for the well-started configurations to spread to in time T_{block}. This warrants the choice $c_{\mathrm{space}} = 2c_{\mathrm{dens}} + 2$. Note in this context that a much smaller L_{block} would suffice, but defining it to be a multiple of L'_{block} gives a more convenient notation.

(ii) In order to have a *well-started* block at (z,t) for which property (4.4.4) spreads to a region of radius $3R'_{\text{block}}$ around z in time T_{block}, the region of space around z for which the densities of $\eta^{(1)}, \eta^{(2)}$ are near θ_μ must be large enough. As will be seen later on (see Section 4.4.2) this is due to the crucial role that the contraction property of Lemma 4.4.1 plays in the expansion of the coupling and translates loosely to R_{dens} being large enough, namely

$$R_{\text{dens}} > R'_{\text{block}} + T_{\text{block}}\lceil sR \rceil + T_{\text{block}}R.$$

This can also be seen as an incentive for taking T_{block} to be of order $\log R$ and R_{dens} to be of order $R \log R$. Further it shows that c_{dens} needs to be chosen suitably large; it suffices to take $c_{\text{dens}} = 1 + 2c_{\text{time}}$.

(iii) Assume that on the event that a block at (z,t) is well started, property (ii) of Definition 4.4.4 does not hold, i.e. there is a site at the top of the block at which $\eta^{(1)}$ and $\eta^{(2)}$ disagree. As will be seen in Section 4.4.2, the probability of the two processes disagreeing at a site (in a well-started block) decays by a factor of $\kappa(\mu,\varepsilon)$ at each time step, when tracing the unsuccessful coupling backwards in time though the block. By a union bound, it follows that the probability that a well-started block at (z,t) does not satisfy (ii), is bounded by $\kappa(\mu,\varepsilon)^{T_{\text{block}}}$ multiplied by the number of sites that are within distance $R'_{\text{block}} + T_{\text{block}}\lceil sR \rceil$ of z. For this probability to decay in R, the constant c_{time} must satisfy $c_{\text{time}} > -(d+1)/\log \kappa(\mu,\varepsilon)$.

In order to set up comparison with oriented percolation, in the same fashion as in Section 4.3, we need to show that the *good* blocks have high density and that the block dependencies have finite range that does not depend on R. To this end, note first that the event $\{\texttt{Block}(z,t) \text{ is good}\}$ depends (only) on $\{\eta_t^{(i)}(x), x \in B_{R_{\text{block}}}(z), i = 1,2\}$ and $\{U(y,t+k) : y \in B_{3R_{\text{block}}}(z), k = 1,2,\dots,T_{\text{block}}\}$.

Lemma 4.4.6. *For $(z,t) \in \mathbb{L}$,*

$$\mathbb{P}\big(\texttt{Block}(z,t) \text{ is good} \,\big|\, \mathcal{F}_t\big) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\big(1 - q(R,\mu)\big)$$

with $q(R,\mu) \to 0$ as $R \to \infty$.

See Section 4.4.2 for the proof.

Armed with Lemma 4.4.6 we can repeat the comparison construction from Section 4.3 and obtain the analogues of (4.3.8) and (4.3.9) in our context. That is we define $Y(z,t) = \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is good}\}}$, and then couple $(\eta^{(1)}, \eta^{(2)})$ with a (high density) i.i.d. Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ such that

$$Y(z,t) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\widetilde{Y}(z,t) \quad \text{for all } (z,t) \in \mathbb{L}$$

and $p(R) = \mathbb{P}(\widetilde{Y}(z,t) = 1) \to 1$ as $R \to \infty$.

This shows that the density of good blocks (and thus also the density of space-time sites where $\eta^{(1)}$ and $\eta^{(2)}$ agree) will be high. In order to conclude that in fact $\eta^{(1)}$ and $\eta^{(2)}$ will agree a.s. from some time on in a growing space-time region, we invoke the fact that "dry" ($\widehat{=}$ "uncoupled") clusters of blocks do not percolate when $p(R)$ is close to 1. More precisely we set

$$C_0 := \left\{ (z,t) \in \mathbb{L} : \begin{array}{l} \text{There exists a path } (z_0,0),(z_1,T_{\text{block}}),\dots,(z_t,t) \text{ in} \\ \mathbb{L} \text{ with } z_0 = 0, z_t = z \text{ such that } \|z_i - z_{i-1}\| \le L'_{\text{block}} \\ \text{and } \widetilde{Y}(z_i, iT_{\text{block}}) = 1 \text{ for } i \in \{1,\dots,t/T_{\text{block}}\} \end{array} \right\}$$

to be the cluster of sites which are connected to the origin by an open path in the Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$. Further we say that a space-time point $(z,t) \in \mathbb{L}$ is C_0-*exposed* if there is an arbitrary path from it to the zero-time slice, which entirely avoids C_0, i.e. if there is a path $(z_0,0),\dots,(z_t,t)$ in $\mathbb{L}$ with $z_t = z$ such that $\|z_k - z_{k-1}\| \le L'_{\text{block}}$ and $(z_k, kT_{\text{block}}) \notin C_0, k = 1,\dots,t/T_{\text{block}}$.

It follows from [Dur92, Section 3] that there is a truncated cone originating from the origin in which there exist no C_0-exposed sites. The exact statement we are interested in is a direct reformulation of [BD07, Lemma 14].

Lemma 4.4.7 ([BD07, Lemma 14])**.** *If $p(R)$ is sufficiently close to 1, then there is a positive constant $c > 0$ and an almost surely finite random time τ, such that conditioned on $\{|C_0| = \infty\}$ there are no C_0-exposed sites in $\{(z,t) \in \mathbb{L} : \|z\| \le ct, t \ge \tau\}$.*

For large enough R the Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ contains an infinite cluster of open sites with probability one. Similarly to Section 4.3, because a *good* block will be created with positive probability from any non-trivial initial condition, we can assume without loss of generality that this cluster contains the origin and that the block at the origin is good.

Lemma 4.4.7 together with Lemma 4.4.6 imply that for sufficiently large R, on $\{|C_0| = \infty\}$ there is a (random) time $\tau > 0$ and a constant $c > 0$ such that no sites in $\{(z,t) \in \mathbb{L} : \|z\| \le ct, t \ge \tau\}$ are C_0-exposed. We show that this implies that $\eta^{(1)}$ agrees with $\eta^{(2)}$ on the space-time cone $A = \{(z,t) \in \mathbb{Z}^d \times \mathbb{N} : \|z\| \le c(t-\tau),\ t \ge \tau\}$ centered at $(0,\tau)$. Indeed, assume to the contrary that there exists $(z,s) \in A$ such that $\eta^{(1)}_s(z) \ne \eta^{(2)}_s(z)$. Then we can find a path $(z,s),(x_{s-1},s-1),\dots,(x_0,0)$ in $\mathbb{Z}^d \times \mathbb{N}_0$ such that $x_u \in B_R(x_{u+1})$ and $\eta^{(1)}_u(x_u) \ne \eta^{(2)}_u(x_u)$ for all $0 \le u \le s-1$. By disregarding all u's which are not a multiple of $T = T_{\text{block}}$, there exists some integer k and a sub-path $(z,s),(x_{kT},kT),\dots,(x_0,0)$ in $\mathbb{Z}^d \times \mathbb{N}_0$ "backwards in time". Assume without loss of generality that s is a multiple of T and associate to the sub-path the nearest neighbour path $\big((Z,k+1),(X_k,k),\dots,(X_0,0)\big) \subseteq \mathbb{L}$ where $Z, X_k \in L'_{\text{Block}}\mathbb{Z}^d$ are the respective closest grid-points to z and x_{kT} in the coarse-grained lattice. In particular $\|X_k - x_{kT}\| \le R'_{\text{Block}}$ for $k = 0,\dots,s$. By definition $\widetilde{Y}(X_k,kT) = 0$ for $k = 0,\dots,s$,

whence (Z, s) is a C_0-exposed site, contradicting Lemma 4.4.7 and yielding that in fact $\eta_s^{(1)}(z) = \eta_s^{(2)}(z)$. As $(z, s) \in A$ was chosen arbitrarily, the claim of Theorem 4.1.6 follows with $T^{\text{coupl}} = \tau$ and $a(R, \mu, d) = c$.

4.4.2 Proof of Lemma 4.4.6

The key step in proving Theorem 4.1.6 is showing that the coupled region in the *well-started* configuration of a *good* block expands to the neighbouring sites with high probability. In order to keep the notation lighter we only show this property for a block centred at the origin at time 0. That is, we show that for some $q = q(R, \mu) \to 0$ as $R \to \infty$,

$$\mathbb{P}\big(\texttt{Block}(0,0) \text{ is good} \,\big|\, \mathcal{F}_0\big) \geq \mathbb{1}_{\{\texttt{Block}(0,0) \text{ is well-started}\}}\big(1 - q(R, \mu)\big) \tag{4.4.5}$$

Shifting the block yields the desired property for blocks centred at arbitrary space-time sites. Note that we still condition on $\mathcal{F}_0$, as we allow for possibly random initial configurations $\eta_0^{(i)}, i = 1, 2$. As was already anticipated in Remark 4.4.5, in order to see the spreading of the coupling after T_{block} steps, we need a large number of sites within distance R_{dens} of the origin for which the densities of both $\eta^{(1)}, \eta^{(2)}$ are close to θ_μ. This is made precise by the following auxiliary events, where the densities have the prescribed behaviour on balls whose radii decrease by R at each time step.

Recall that $T_{\text{block}} = c_{\text{time}}\lceil \log R \rceil$ and write $R'(k) := R'_{\text{block}} + k\lceil sR \rceil$. For $n \in \mathbb{N}$ let

$$\Psi_n = \Big\{ |\delta_R(x; \eta_j^{(i)}) - \theta_\mu| < \varepsilon,\ \|x\| \leq R'(n) + (n - j)R,\ \forall j \in \{1, \dots, n\},\ i \in \{1, 2\} \Big\}.$$

(Recall also that ε was chosen at the beginning of Section 4.4.1, above (4.4.2).)

Note that in a *well-started* configuration around the origin we have $|\delta_R(x; \eta_0^{(i)}) - \theta_\mu| < \varepsilon$ for every x such that $\|x\| \leq R'(T_{\text{block}}) + T_{\text{block}}R$ (in fact, this holds for all x within distance R_{dens} from the origin and $R_{\text{dens}} \geq R'(T_{\text{block}}) + T_{\text{block}}R$). The sites, where the densities of $\eta^{(1)}, \eta^{(2)}$ are close to θ_μ due to the well-startedness, encompass the entire $n = 0$ (bottom) level of the space-time pyramid $\Psi_{T_{\text{block}}}$, see also Figure 4.5. Due to this the the event $\Psi_{T_{\text{block}}}$ holds with high probability. Indeed, by defining the events $A_0 = \emptyset$ and

$$A_j = \big\{ \exists z \in B_{R'(T_{\text{block}}) + (T_{\text{block}} - j)R}(0) : |\delta_R(z; \eta_j^{(1)}) - \theta_\mu| > \varepsilon \big\},$$

we see that on the event $\{\texttt{Block}(0,0) \text{ is well-started}\}$

$$\begin{aligned}
\mathbb{P}\big(\Psi^c_{T_{\text{block}}} \big| \mathcal{F}_0\big) &\leq 2\,\mathbb{P}\left(\bigcup_{j=1}^{T_{\text{block}}} A_j \,\middle|\, \mathcal{F}_0 \right) \\
&\leq 2 \sum_{j=1}^{T_{\text{block}}} \mathbb{P}\big(A_j \cap A_{j-1}^c \big| \mathcal{F}_0\big) \leq 2 \sum_{j=1}^{T_{\text{block}}} \mathbb{P}(A_j \mid A_{j-1}^c, \mathcal{F}_0).
\end{aligned}$$

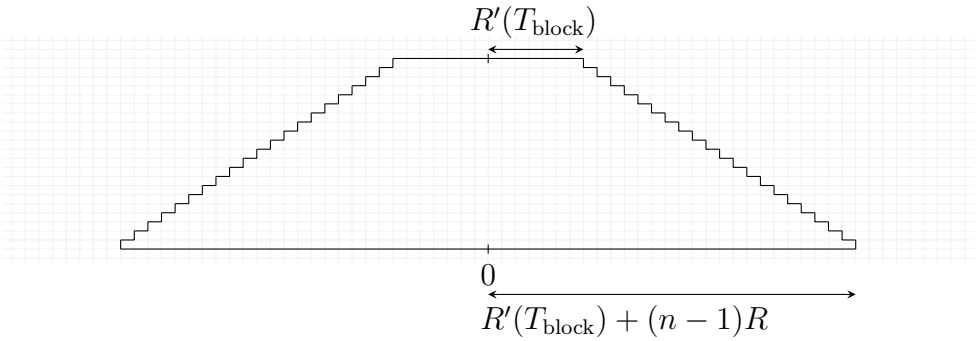

Figure 4.5: The event Ψ_n occurs if the local density of $\eta^{(i)}, i = 1, 2$ is within ε distance of the fixed point θ_μ for all space-time points in the above pyramid. For convenience of presentation the spatial axis in the sketch is scaled by R, while the temporal axis is not scaled.

Together with Lemma 4.2.3 it follows with some constants $c_1, c_2 > 0$ that

$$\mathbb{1}_{\{\texttt{Block}(0,0) \text{ is well-started}\}} \mathbb{P}\big(\Psi^c_{T_{\text{block}}} \big| \mathcal{F}_0\big) \leq c_1 T_{\text{block}} (L_{\text{block}})^d \exp(-c_2\, V_R^d). \tag{4.4.6}$$

In order to utilise the control guaranteed by the pyramids Ψ_n we introduce events that describe properties (ii) and (iii) in Definition 4.4.4:

$$\begin{aligned} \mathcal{C} &= \big\{\eta^{(1)}_{T_{\text{block}}}(x) = \eta^{(2)}_{T_{\text{block}}}(x) \text{ for } \|x\| \leq 3R'_{\text{block}}\big\} \\ \mathcal{D} &= \big\{\big(\eta^{(1)}_{T_{\text{block}}}, \eta^{(2)}_{T_{\text{block}}}\big) \text{ satisfy (4.4.3) around } L'_{\text{block}} e \text{ for all } \|e\| \leq 1\big\}. \end{aligned}$$

We are interested in the conditional probability $\mathbb{P}(\mathcal{C} \cap \mathcal{D} | \mathcal{F}_0)$ on the event that $\texttt{Block}(0,0)$ is well-started. Clearly it holds that

$$\mathbb{P}\big(\mathcal{C}^c \cup \mathcal{D}^c \big| \mathcal{F}_0\big) \leq \mathbb{P}\big(\mathcal{C}^c \cap \Psi_{T_{\text{block}}} \big| \mathcal{F}_0\big) + \mathbb{P}\big(\Psi^c_{T_{\text{block}}} \big| \mathcal{F}_0\big) + \mathbb{P}\big(\mathcal{D}^c \big| \mathcal{F}_0\big). \tag{4.4.7}$$

By (4.4.6) the second term in (4.4.7) decays in R for well-started configurations. To deal with the third term, note that it follows from Lemma 4.4.2 and Lemma 4.2.3 that for some constants $c_3, c_4 > 0$

$$\mathbb{1}_{\{\texttt{Block}(0,0) \text{ is well-started}\}} \mathbb{P}(\mathcal{D}^c | \mathcal{F}_0) \leq c_3 T_{\text{block}} (L_{\text{block}})^d \exp(-c_4\, V_R^d). \tag{4.4.8}$$

It remains to find a bound for $\mathbb{P}\big(\mathcal{C}^c \cap \Psi_{T_{\text{block}}} \big| \mathcal{F}_0\big)$. To this end fix $k \in \{1, \dots, T_{\text{block}}\}$. By a union bound and Markov's inequality

$$\begin{aligned} &\mathbb{P}\big(\{\exists\, |x| \leq R'(k) \text{ such that } \eta^{(1)}_k(x) \neq \eta^{(2)}_k(x)\} \cap \Psi_k \big| \mathcal{F}_0\big) \\ &\quad \leq \sum_{x \in B_{R'(k)}(0)} \mathbb{E}\big[\mathbb{1}_{\Psi_k} |\eta^{(1)}_k(x) - \eta^{(2)}_k(x)| \big| \mathcal{F}_0\big] \\ &\quad = \mathbb{E}\Big[\sum_{x \in B_{R'(k)}(0)} \mathbb{1}_{\Psi_{k-1}} \mathbb{E}\big[|\eta^{(1)}_k(x) - \eta^{(2)}_k(x)| \big| \mathcal{F}_{k-1}\big] \Big| \mathcal{F}_0 \Big]. \end{aligned} \tag{4.4.9}$$

In light of the coupling (4.4.1), we have

$$\begin{aligned}\mathbb{E}\big[|\eta_k^{(1)}(x)-\eta_k^{(2)}(x)|\,\big|\,\mathcal{F}_{k-1}\big] &= \mathbb{P}\Big(U(x,k)\le\big|\varphi_\mu(\delta_R(x;\eta_{k-1}^{(1)}))-\varphi_\mu(\delta_R(x;\eta_{k-1}^{(2)}))\big|\,\Big|\,\mathcal{F}_{k-1}\Big)\\ &= \big|\varphi_\mu(\delta_R(x;\eta_{k-1}^{(1)}))-\varphi_\mu(\delta_R(x;\eta_{k-1}^{(2)}))\big|.\end{aligned}$$

Now $\delta_R(x;\eta_{k-1}^{(i)})\in[\theta_\mu-\varepsilon,\theta_\mu+\varepsilon]$ for $i=1,2$ on the event Ψ_{k-1} and by Lemma 4.4.1, φ_μ is a contraction with Lipschitz constant $\kappa(\mu,\varepsilon)<1$ on this interval. Therefore

$$\begin{aligned}\mathbb{1}_{\Psi_{k-1}}\mathbb{E}\big[|\eta_k^{(1)}(x)-\eta_k^{(2)}(x)|\,\big|\,\mathcal{F}_{k-1}\big] &\le \mathbb{1}_{\Psi_{k-1}}\kappa(\mu,\varepsilon)\big|\delta_R(x;\eta_{k-1}^{(1)})-\delta_R(x;\eta_{k-1}^{(2)})\big|\\ &\le \mathbb{1}_{\Psi_{k-1}}\kappa(\mu,\varepsilon)V_R^{-d}\sum_{y\in B_R(x)}\big|\eta_{k-1}^{(1)}(y)-\eta_{k-1}^{(2)}(y)\big|.\end{aligned}$$

Plugging this back into (4.4.9) yields

$$\begin{aligned}&\mathbb{P}\big(\{\exists\,|x|\le R'(k)\text{ such that }\eta_k^{(1)}(x)\ne\eta_k^{(2)}(x)\}\cap\Psi_k\big|\mathcal{F}_0\big)\\ &\quad\le\kappa(\mu,\varepsilon)V_R^{-d}\sum_{x\in B_{R'(k)}(0)}\sum_{y\in B_R(x)}\mathbb{E}\Big[\mathbb{1}_{\Psi_{k-1}}\big|\eta_{k-1}^{(1)}(y)-\eta_{k-1}^{(2)}(y)\big|\Big|\mathcal{F}_0\Big].\end{aligned}$$

By inductively repeating this step another $k-1$ times, we can upper bound the right hand side of the last display by

$$\kappa(\mu,\varepsilon)^kV_R^{-dk}\,\mathbb{E}\Big[\sum_{x\in B_{R'(k)}(0)}\sum_{y_1\in B_R(x)}\sum_{y_2\in B_R(y_1)}\cdots\sum_{y_k\in B_R(y_{k-1})}\big|\eta_0^{(1)}(y_k)-\eta_0^{(2)}(y_k)\big|\Big|\mathcal{F}_0\Big].$$

Since $\big|\eta_0^{(1)}(y_k)-\eta_0^{(2)}(y_k)\big|\le 1$, with $k=T_{\text{block}}$ we obtain

$$\mathbb{1}_{\{\texttt{Block}(0,0)\text{ is well-started}\}}\mathbb{P}\big(\mathcal{C}^c\cap\Psi_{T_{\text{block}}}\big|\mathcal{F}_0\big)\le\kappa(\mu,\varepsilon)^{T_{\text{block}}}V_{R'(T_{\text{block}})}^d. \tag{4.4.10}$$

The choice $c_{\text{time}}>-(d+1)/\log\kappa$ guarantees that this probability tends to zero as R goes to infinity. Combining (4.4.10) together with (4.4.8) and (4.4.6) gives that, on the event that $\texttt{Block}(0,0)$ is well-started, all the terms on the right-hand side of (4.4.7) tend to zero as R goes to infinity, thus proving (4.4.5).

4.4.3 Proof of Theorem 4.1.5

We now have all required tools to prove complete convergence of the BARW. Given these tools, the proof is relatively standard and thus it is kept brief.

Proof. As the Dirac measure concentrated around $\eta\equiv 0$ is an invariant distribution for η we only need to show existence of a unique non-trivial limiting invariant measure which does not charge the empty configuration. To this end, let ν_0 be the product

measure on $\mathbb{Z}^d$ such that, for all $x \in \mathbb{Z}^d$, $\eta_0(x) = 1$ with probability $p > 0$ and $\eta_0(x) = 0$ otherwise. For any $n \geq 1$, denote by ν_n the distribution of η_n given that η_0 is distributed as ν_0.

Since the set of all probability measures on $\{0,1\}^{\mathbb{Z}^d}$ is compact, there exists a subsequence along which $\frac{1}{N}\sum_{n=0}^{N} \nu_n$ converges to some probability measure ν on $\{0,1\}^{\mathbb{Z}^d}$. From a standard result for interacting particle systems, see e.g. [Lig85, Proposition 1.8], any such subsequential limit ν must be invariant for the process η.

To show that ν is non-trivial (and actually gives zero mass to the empty configuration $\eta \equiv 0$), it suffices to show that η survives almost surely. As we chose ν_0 to be a product measure and since for any fixed R the blocks defined in Section 4.3 depend only on finitely many sites, it follows that at time 0 there are almost surely infinitely many *well-started* blocks and hence by (4.3.6) infinitely many *good* blocks. By the correspondence of the blocks with supercritical oriented site percolation and the fact that supercritical oriented site percolation starting from infinitely many occupied sites does not die out (see e.g. [Lig99, Theorem B24]), we have $\mathbb{P}_{\nu_0}(\exists n \geq 1 : \eta_n \equiv 0) = 0$.

Furthermore, the measure ν is extremal, because any limiting invariant distribution ν' which gives zero mass to $\eta \equiv 0$ must be unique. Indeed, if two stationary distributions existed with this property, then by Theorem 4.1.6 they would coincide on finite subsets of $\mathbb{Z}^d$, and would therefore be equal. Furthermore, under ν, η has exponentially decaying correlations in space and in time, which in particular implies ergodicity w.r.t. spatial shifts. Indeed, using the construction of good blocks from the proof of Theorem 4.1.6 below, this can be deduced from the corresponding property of supercritical oriented percolation in a fairly straightforward way, see for example the analogous construction in [Dep08, Section 3.4] for the related model of a locally regulated population from [BD07].

Finally, in order to verify the complete convergence, consider any (fixed) initial condition $\widetilde{\eta}_0 \in \{0,1\}^{\mathbb{Z}^d}$, a finite box $B \subseteq \mathbb{Z}^d$ centred at the origin and a configuration $\zeta \in \{0,1\}^B$. With $\mathcal{S} := \{\eta_m \not\equiv 0 \text{ for all } m \in \mathbb{N}\}$ we have to check that

$$\lim_{n\to\infty} \mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \mathcal{S}) = \mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S})\nu(\{\eta_0|_B = \zeta\}). \tag{4.4.11}$$

Pick $\varepsilon > 0$. The coupling construction from the proof of Theorem 4.1.6 and standard properties of supercritical oriented percolation show that one can pick $L' \in \mathbb{N}$ and $T' \in \mathbb{N}$ large so that $\left|\mathbb{P}_{\eta'_0}(\{\eta_m|_B = \zeta\}) - \nu(\{\eta_0|_B = \zeta\})\right| \leq \varepsilon$ for all $m \geq T'$ and all starting configurations

$$\eta'_0 \in G' := \left\{ \widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d} : \begin{array}{l} \text{The density of well-started sub-boxes, where the lo-} \\ \text{cal density of } \widetilde{\eta} \text{ satisfies (4.4.3) from Definition 4.4.3,} \\ \text{in a box of radius } L'R_{\text{block}} \text{ is at least } 1/2. \end{array} \right\}.$$

Furthermore, since starting from any non-trivial initial condition there is a positive chance of producing a well-started box in a finite number of steps, a "restart" argument

together with the construction from Theorem 4.1.6 shows that for all large enough n it holds that $\mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S}\Delta\{\eta_n \in G'\}) \le \varepsilon$ for all large enough n. Thus

$$\begin{aligned}
&|\mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \mathcal{S}) - \mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S})\nu(\{\eta_0|_B = \zeta\})| \\
&\quad \le \left|\mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \{\eta_{n/2} \in G'\}) - \mathbb{P}_{\widetilde{\eta}_0}(\{\eta_{n/2} \in G'\})\nu(\{\eta_0|_B = \zeta\})\right| + 2\varepsilon \\
&\quad \le \mathbb{E}_{\widetilde{\eta}_0}\Big[\mathbb{1}_{\{\eta_{n/2}\in G'\}}\left|\mathbb{P}_{\widetilde{\eta}_0}(\eta_n|_B = \zeta \,|\, \mathcal{F}_{n/2}) - \nu(\{\eta_0|_B = \zeta\})\right|\Big] + 2\varepsilon \le 3\varepsilon.
\end{aligned}$$

Taking $n \to \infty$ and then $\varepsilon \downarrow 0$ proves (4.4.11). □

4.5 Extinction results

We provide here a simple proof of Theorem 4.1.1 describing the extinction regime.

Proof of Theorem 4.1.1. Let $R \in \mathbb{N}$ and $\mu > 0$ be such that

$$\widetilde{\mu} := V_R^d\, \varphi_\mu\big(V_R^{-d}\big) = \mu e^{-\mu V_R^{-d}} < 1. \tag{4.5.1}$$

Then $\psi(w) := \widetilde{\mu} w$ fulfils $\varphi_\mu(w) \le \psi(w)$ on $[0,1] \cap V_R^{-1}\mathbb{Z}$ (note that if $w \ge V_R^{-1}$, we have $\varphi_\mu(w) = \mu w \exp(-\mu w) \le \mu w \exp(-\mu V_R^{-d}) = \widetilde{\mu} w$ and $\varphi_\mu(0) = \psi(0)$).

Thus, we can define a process $(\widetilde{\eta}_n)_{n\in\mathbb{N}_0}$ with $\widetilde{\eta}_0 = \eta_0$ using this ψ as in (4.2.3). By the coupling construction from Section 4.2.1 and specifically Lemma 4.2.1(b) we conclude that $\eta_n(x) \le \widetilde{\eta}_n(x)$ holds for all $n \in \mathbb{N}, x \in \mathbb{Z}^d$. Since ψ is a linear function, we have

$$\mathbb{E}[\widetilde{\eta}_n(x)] = \widetilde{\mu} V_R^{-d} \sum_{y \in B_R(x)} \mathbb{E}[\widetilde{\eta}_{n-1}(y)]$$

Iterating this n times shows

$$\mathbb{E}[\widetilde{\eta}_n(x)] = \widetilde{\mu}^n \sum_{z\in\mathbb{Z}^d} p^{(n)}(x,z)\mathbb{E}[\eta_0(z)] \le \widetilde{\mu}^n$$

where $p^{(n)}$ is the n-fold convolution of the uniform transition kernel on $B_R(0)$ with itself. Since $\widetilde{\mu} < 1$ this combined with the coupling shows that $\sum_{n=1}^{\infty} \mathbb{P}(\eta_n(x) > 0) < \infty$ so that indeed for every $x \in \mathbb{Z}^d$

$$\mathbb{P}(\eta_n(x) = 0 \text{ for all } n \text{ large enough}) = 1.$$

Next note that the equation $\mu \exp(-\mu V_R^{-d}) = 1$, i.e. the equality in (4.5.1), has two positive real solutions μ_1, μ_2 such that $1 < \mu_1 < \mu_2 < \infty$ when $R \ge 1$ (when $R = 0$ there is always extinction). The function $\mu \mapsto \mu \exp(-\mu V_R^{-d})$ is unimodal and vanishes at 0 as well as at $+\infty$, so if $\mu < \mu_1$ or $\mu > \mu_2$ there is extinction.

We can rewrite $\mu \exp(-\mu V_R^{-d}) = 1$ as $ye^y = x$ where $y = -\mu V_R^{-d}$ and $x = -V_R^{-d}$. When $x \in [-1/e, 0)$, this equation has two real solutions $y_1 = W_0(x)$ and $y_2 = W_{-1}(x)$, where W_0 and W_{-1} are two branches of the Lambert W function. Since $\mu = -V_R^d y$, the two solutions of $\mu \exp(-\mu V_R^{-d}) = 1$ are

$$\mu_1 = -V_R^d W_0(-V_R^{-d}), \qquad \mu_2 = -V_R^d W_{-1}(-V_R^{-d}).$$

Since $x \in [-1/e, 0)$, we can express $W_0(x)$ with its Taylor series centred at 0, which has radius of convergence $1/e$, that is

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \cdots$$

This gives

$$\mu_1 = -V_R^d W_0(-V_R^{-d}) = 1 + V_R^{-d} + \frac{3}{2}V_R^{-2d} + \cdots \tag{4.5.2}$$

For the second solution, we use that

$$-1 - \sqrt{2u} - u < W_{-1}(-e^{-u-1}) < -1 - \sqrt{2u} - \frac{2u}{3}$$

for every $u > 0$. Take $u = d \log V_R - 1$. Then the formula above gives

$$\begin{aligned} -\sqrt{2d \log V_R - 2} - d \log V_R < W_{-1}(-V_R^{-d}) \\ < -\frac{1}{3} - \sqrt{2d \log V_R - 2} - \frac{2d}{3} \log V_R, \end{aligned} \tag{4.5.3}$$

which gives the result. □

4.6 Auxiliary results

We prove here the auxiliary technical results that were omitted in the previous sections. Section 4.6.1 deals with Lemma 4.2.5 which was used in the construction of the comparison density profiles ξ_n^- in Section 4.3. In Section 4.6.2, we then show Lemma 4.4.1 used in the proof of the complete convergence in Section 4.4. Finally, in Section 4.6.3 we provide a proof of Proposition 4.1.9.

4.6.1 Proof of Lemma 4.2.5

Recall that $f : \mathbb{Z} \to [0, 1]$ is defined by

$$f(x) = \min\big\{(\varepsilon_0 + x/\lceil wR \rceil)\mathbb{1}_{x \geq 0}, 1\big\}.$$

It is immediate that this function satisfies the properties in (4.2.9). Therefore, it only remains to show that (4.2.10) holds for a suitable choice of parameters.

It is clear that the larger the growth factor a is, the easier it is for (4.2.10) to be satisfied. Setting $\varepsilon_0 = \min\{(a-1)^2, 1/100\}$, it follows that

$$a\delta_R(x;f) \geq (1+\sqrt{\varepsilon_0})\delta_R(x;f) \quad \text{for all } x \in \mathbb{Z},$$

which lets us reduce to the case where $1 < a < 11/10$ and $a = 1+\sqrt{\varepsilon_0}$.

We now set

$$w = 1/\sqrt{\varepsilon_0}, \tag{4.6.1}$$

and define

$$C_0 = \{y \in \mathbb{Z} : y < 0\}, \qquad C_1 = \{y \in \mathbb{Z} : y \geq \lceil wR \rceil\},$$

so that $f(y) = 0$ for every $y \in C_0$ and $f(y) = 1$ for every $y \in C_1$. Since $w > 1$, exactly one of the two sets $B_R(x) \cap C_0$ and $B_R(x) \cap C_1$ can be non-empty. Clearly (4.2.10) holds when $B_R(x) \subseteq C_0$, or $B_R(x) \subseteq C_1$.

When $B_R(x) \cap (C_0 \cup C_1) = \emptyset$, then $f(y) = \varepsilon_0 + y/\lceil wR \rceil$ for every $y \in B_R(x)$ and thus $\delta_R(x;f) = f(x)$, so (4.2.10) holds as well.

The remaining two cases are more delicate. When $B_R(x) \cap C_0 \neq \emptyset$ and $B_R(x) \not\subseteq C_0$, that is when $-R \leq x < R$, then the density of f around x can be written as

$$\begin{aligned}\delta_R(x;f) = V_R^{-1}\sum_{y=0}^{x+R} f(y) &= V_R^{-1}\sum_{y=0}^{x+R}\Big(\varepsilon_0 + \frac{y}{\lceil wR \rceil}\Big) \\ &= V_R^{-1}\Big((x+R+1)\varepsilon_0 + \frac{1}{2\lceil wR \rceil}(x+R)(x+R+1)\Big).\end{aligned}$$

Using this, (4.2.10) is equivalent to

$$\begin{aligned}a(x+R+1)\varepsilon_0 + \frac{a}{2\lceil wR \rceil}(x+R)(x+R+1) &\geq V_R f(x+\lceil sR \rceil) \\ &= V_R\Big(\varepsilon_0 + \frac{x}{\lceil wR \rceil} + \frac{\lceil sR \rceil}{\lceil wR \rceil}\Big).\end{aligned}$$

Rearranging terms, we arrive at a quadratic inequality

$$\alpha x^2 + \beta x + \gamma \geq 0, \tag{4.6.2}$$

where

$$\begin{aligned}\alpha &= \frac{a}{2\lceil wR \rceil}, \\ \beta &= a\varepsilon_0 + \frac{R}{\lceil wR \rceil}\Big(\frac{a}{2} - 1\Big)\Big(2 + \frac{1}{R}\Big), \\ \gamma &= a\varepsilon_0 R\Big(1 + \frac{1}{R}\Big) + \frac{aR^2}{2\lceil wR \rceil}\Big(1 + \frac{1}{R}\Big) - R\Big(2 + \frac{1}{R}\Big)\Big(\varepsilon_0 + \frac{\lceil sR \rceil}{\lceil wR \rceil}\Big).\end{aligned}$$

As $\alpha > 0$ for our choices of parameters, (4.6.2) and hence (4.2.10) follow immediately if the polynomial $\alpha x^2 + \beta x + \gamma$ has no real roots. The discriminant of (4.6.2) is given by

$$\begin{aligned}\beta^2 - 4\alpha\gamma &= \Big(a\varepsilon_0 + \frac{2}{w}\Big(\frac{a}{2} - 1\Big)\Big)^2 - \frac{2a}{w}\Big(a\varepsilon_0 + \frac{a}{2w} - 2\Big(\varepsilon_0 + \frac{s}{w}\Big)\Big) + O(R^{-1}) \\ &= (a\varepsilon_0)^2 + \frac{4}{w^2}(1 - a + as) + O(R^{-1}).\end{aligned} \tag{4.6.3}$$

We now choose

$$s = \frac{\sqrt{\varepsilon_0}}{1 + \sqrt{\varepsilon_0}} - \varepsilon_0,$$

which is clearly positive for $\varepsilon_0 \in (0, 1/100)$. Recalling also (4.6.1) and that $a = 1 + \sqrt{\varepsilon_0}$, the right-hand side of (4.6.3) (without the error term) equals

$$\varepsilon_0^2(\sqrt{\varepsilon_0} - 3)(1 + \sqrt{\varepsilon_0})$$

which is clearly negative. As consequence, the quadratic inequality (4.6.2) holds for all R big enough, depending only on a, and thus (4.2.10) holds also in this case.

For the final case, when $B_R(x) \cap C_1 \neq \emptyset$ that is $\lceil wR \rceil - R \leq x \leq \lceil wR \rceil - R$, we observe that the right-hand side of (4.2.10) is bounded by one and the left-hand side is increasing in x. It is thus sufficient to show that $a\delta_R(\lceil wR \rceil - R - 1) \geq 1$. Using again the fact that f is linear in the R-neighbourhood of $\lceil wR \rceil - R - 1$, this is equivalent to showing $af(\lceil wR \rceil - R - 1) \geq 1$. Recalling the definitions of a, w and s in terms of ε_0, we have

$$\begin{aligned}af(\lceil wR \rceil - R - 1) &= a\Big(\varepsilon_0 + \frac{\lceil wR \rceil - R - 1}{\lceil wR \rceil}\Big) \\ &= a\big(\varepsilon_0 + 1 - w^{-1} + O(R^{-1})\big) \\ &= (1 + \sqrt{\varepsilon_0})\big(\varepsilon_0 + 1 - \sqrt{\varepsilon_0} + O(R^{-1})\big) \\ &= 1 + \varepsilon_0^{3/2} + O(R^{-1}),\end{aligned}$$

and thus the required inequality is satisfied for R large enough. □

4.6.2 Proof of Lemma 4.4.1

We now prove Lemma 4.4.1, exploiting properties of φ_μ in the vicinity of its fixpoint θ_μ.

Proof of Lemma 4.4.1. To prove that φ_μ is a contraction in the vicinity of its critical point $\theta_\mu = \mu^{-1} \log \mu$, it suffices to observe that $|\varphi'_\mu(w)| < 1$ in some neighbourhood of θ_μ. Since $\varphi'_\mu(w) = \mu e^{-\mu w}(1 - \mu w)$, it holds that $|\varphi'_\mu(\theta_\mu)| = |1 - \log \mu| < 1$ if $\mu \in (1, e^2)$. The statement then follows by the continuity of the derivative.

To find the sequences α_m and β_m, note first that φ_μ is increasing on $[0, 1/\mu]$ and decreasing on $[1/\mu, \infty]$. It is convenient to consider three cases (cf. also Figure 4.6):

(1) If $\mu \in (1, e)$, then $\theta_\mu < 1/e < 1/\mu$, and thus φ_μ is a strictly increasing on $[0, 1/e] \ni \theta_\mu$, and $\varphi_\mu(w) > w$ if $w < \theta_\mu$, and $\varphi_\mu(w) < w$ when $w \in (\theta_\mu, 1/e]$. Pick $\alpha_1 < \theta_\mu$ and $\beta_1 > 1/\mu$ satisfying $\varphi_\mu(\beta_1) \geq \varphi_\mu(\alpha_1)$. Put $\alpha_2 = (\alpha_1 + \varphi_\mu(\alpha_1))/2$, $\beta_2 = (e^{-1} + \mu^{-1})/2$, then we have indeed $\varphi_\mu([\alpha_1, \beta_1]) \subseteq (\alpha_2, \beta_2)$. From here on, we can simply iterate by setting

$$\alpha_{m+1} = \frac{\alpha_m + \varphi_\mu(\alpha_m)}{2}, \qquad \beta_{m+1} = \frac{\beta_m + \varphi_\mu(\beta_m)}{2}, \quad m \geq 2. \tag{4.6.4}$$

This defines two sequences converging to θ_μ. Furthermore $\alpha_m < \alpha_{m+1} < \varphi_\mu(\alpha_m)$ and $\varphi_\mu(\beta_m) < \beta_{m+1} < \beta_m$, so $(\alpha_m)_{m\geq 1}$ is strictly increasing and $(\beta_m)_{m\geq 1}$ is strictly decreasing. Since φ_μ is strictly increasing on $[0, 1/e]$ we also have $\varphi_\mu([\alpha_m, \beta_m]) \subseteq (\alpha_{m+1}, \beta_{m+1})$ for every $m \geq 0$, as required.

(2) Consider now the case $\mu = e$, that is when $\theta_\mu = 1/e$ and $\varphi'_\mu(\theta_\mu) = 0$. Pick any $\alpha_1 < 1/e$ and $\beta_1 = 1/e$ such that $\varphi_\mu(\beta_1) \geq \varphi_\mu(\alpha_1)$; then build the sequence $(\alpha_m)_{m\geq 1}$ in the same way as in the case $\mu \in (1, e)$, i.e. as in (4.6.4), using $m \geq 1$ there. By construction, this sequence is strictly increasing and converges to θ_μ. For every $m \geq 2$, let β_m be the largest solution of $\varphi_\mu(x) = \varphi_\mu(\alpha_m)$. Since $w \mapsto \varphi_\mu(w)$ is (strictly) increasing if and only if $w \in [0, 1/e]$, this defines a strictly decreasing sequence $(\beta_m)_{m\geq 1}$ converging to θ_μ and such that

$$\varphi_\mu([\alpha_m, \beta_m]) \subseteq [\varphi_\mu(\alpha_m), 1/e] \subseteq (\alpha_{m+1}, \beta_{m+1}),$$

as required.

(3) Finally, let $\mu \in (e, e^2)$, which implies $1/\mu < \theta_\mu$ and $\varphi'_\mu(\theta_\mu) \in (-1, 0)$. For the initial piece, pick $\alpha_1 < 1/\mu$ and $\lambda > 0$ so small that $\mu^{-1} + \lambda e^{-1} < \theta_\mu$. Define, similarly to (4.6.4),

$$\alpha_{m+1} = \lambda\varphi_\mu(\alpha_m) + (1 - \lambda)\alpha_m, \ m \leq m_0 - 1,$$

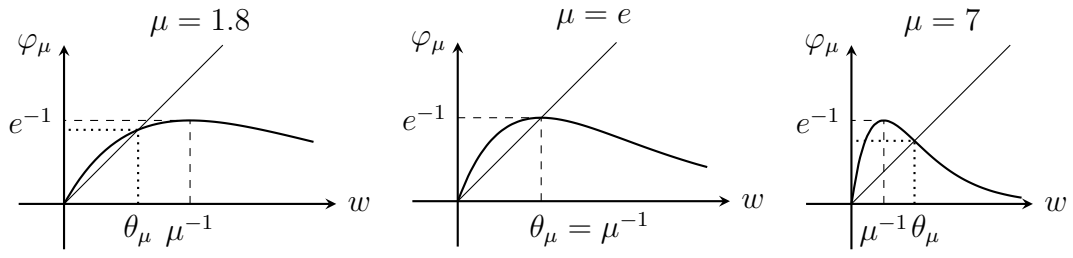

Figure 4.6: The function φ_μ, its fixpoint θ_μ and its maximum in the case $\mu < e$ (left), $\mu = e$ (middle) and $e < \mu < e^2$ (right)

where m_0 is the smallest integer satisfying $\alpha_{m_0} > 1/\mu$. Note that by construction and the choice of λ, since φ_μ is strictly increasing on $[0, 1/\mu]$ and bounded by $1/e$, we have $\alpha_1 < \alpha_2 < \cdots < \alpha_{m_0-1} \leq 1/\mu < \alpha_{m_0} < \theta_\mu$. Choose $\beta_1 > \beta_2 > \cdots > \beta_{m_0} > 1/e$ $(> \theta_\mu)$ so that $\varphi_\mu(\beta_m) > \varphi_\mu(\alpha_m)$ for $m = 1, \ldots, m_0$, then we have $\varphi_\mu([\alpha_m, \beta_m]) \subseteq (\alpha_{m+1}, \beta_{m+1})$ for $m = 1, \ldots, m_0 - 1$.

Since $\alpha_{m_0} > 1/\mu$, the iteration has reached the decreasing part of φ_μ after m_0 steps and we thus must swap the roles of the upper and the lower boundary in each step: Set for $m \geq m_0$

$$\alpha_{m+1} = \frac{\varphi_\mu(\beta_m) + \alpha_m}{2}, \qquad \beta_{m+1} = \frac{\varphi_\mu(\alpha_m) + \beta_m}{2}.$$

We note that if $\varphi_\mu(\alpha_m) < \beta_m$ and $\varphi_\mu(\beta_m) > \alpha_m$ then the same holds for α_{m+1} and β_{m+1}. Indeed $\varphi_\mu(\beta_m) > \alpha_m$ implies that $\alpha_{m+1} > \alpha_m$ and since φ_μ is decreasing then $\varphi_\mu(\alpha_{m+1}) < \varphi_\mu(\alpha_m)$. Similarly $\varphi_\mu(\alpha_m) < \beta_m$ implies that $\beta_{m+1} < \beta_m$ and so $\varphi_\mu(\alpha_m) = 2\beta_{m+1} - \beta_m < \beta_{m+1}$. Combining the two gives $\varphi_\mu(\alpha_{m+1}) < \varphi_\mu(\alpha_m) < \beta_{m+1}$. In the same way we can prove that $\varphi_\mu(\beta_{m+1}) > \alpha_{m+1}$. Hence for $m \geq m_0$

$$\varphi_\mu((\alpha_m, \beta_m)) \subseteq [\varphi_\mu(\beta_m), \varphi_\mu(\alpha_m)] \subseteq (\alpha_{m+1}, \beta_{m+1}).$$

It is clear from the construction that in each one of the three cases α_1 can be chosen arbitrarily small and $\beta_1 > 1/e$ (if a large β_1 is required, this can be achieved by decreasing α_1 appropriately). □

4.6.3 Proof of Proposition 4.1.9

Note again that since $\max_{w \geq 0} \varphi_\mu(w) = 1/e$, for every initial condition $\Xi_0 \in \mathbb{R}_+^{\mathbb{Z}^d}$ of the coupled map lattice defined in (4.1.8) we have $\Xi_1 \in [0, 1/e]^{\mathbb{Z}^d}$. Thus we can assume without loss of generality that $0 \leq \Xi_0(z) \leq 1/e$ for every $z \in \mathbb{Z}^d$. Assume moreover that $\Xi_0(z_0) > 0$ for some $z_0 \in \mathbb{Z}^d$, as it otherwise obviously holds that $\Xi_n \equiv 0$ for all n. The proof follows ideas from Section 4 in [BD07].

Proof of Proposition 4.1.9. Fix $\varepsilon > 0$ and let $a > 1$ and $b > 0$ be such that $\psi(w) = aw \wedge b$ satisfies $\varphi_\mu(w) \geq \psi(w)$ for every $w \in [0, 1]$. Since θ_μ is a stable fixpoint when $\mu \in (1, e^2)$, we can choose sequences $(\alpha_m)_{m \geq 0}$, $(\beta_m)_{m \geq 0}$ as in Lemma 4.4.1 with $\alpha_1 < b/2$ and a suitable $\beta_1 > 1/e$, such that $\varphi_\mu([\alpha_m, \beta_m]) \subseteq (\alpha_{m+1}, \beta_{m+1})$ and $\beta_{m^*} - \alpha_{m^*} < \varepsilon$ for some $m^* \in \mathbb{N}$.

For a fixed $z \in \mathbb{Z}^d$ we show that there exists $n_0 > m^*$ such that $\Xi_n(z) \in [\alpha_{m^*}, \beta_{m^*}]$ for all $n \geq n_0$. We start by showing that

$$\Xi_n(z) \geq \sum_{y \in \mathbb{Z}^d} p^{(n)}(z, y) \Big[\big(a^n \Xi_0(y) \big) \wedge b \Big], \tag{4.6.5}$$

where $p^{(n)}(\cdot,\cdot)$ are the n-step transition probabilities of a random walk whose steps are uniformly distributed in $B_R(0) \cap \mathbb{Z}^d$. We can check (4.6.5) by induction. Using Jensen's inequality, it holds that

$$\begin{aligned}\Xi_{n+1}(z) = \varphi_\mu(\delta_R(z;\Xi_n)) &\geq \psi\Big(V_R^{-d} \sum_{x\in B_R(0)} \Xi_n(z+x)\Big) \\ &\geq V_R^{-d} \sum_{x\in B_R(0)} \psi\big(\Xi_n(z+x)\big).\end{aligned}$$

Using the inductive assumption,

$$\begin{aligned}\psi\big(\Xi_n(z+x)\big) &\geq \Big[a \sum_{y\in\mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^n\Xi_0(y)\big)\wedge b\Big)\Big]\wedge b \\ &= \sum_{y\in\mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big)\wedge ab\Big)\wedge b \\ &\geq \sum_{y\in\mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big)\wedge b\Big)\wedge b \\ &= \sum_{y\in\mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big)\wedge b\Big),\end{aligned}$$

so

$$\Xi_{n+1}(z) \geq V_R^{-d} \sum_{x\in B_R(0)} \sum_{y\in\mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big)\wedge b\Big)$$

and the conclusion follows from the fact that

$$V_R^{-d} \sum_{x\in B_R(0)} p^{(n)}(z+x,y) = \sum_{x\in B_R(0)} p(z,z+x)p^{(n)}(z+x,y) = p^{(n+1)}(z,y).$$

For our fixed choice of z, we show that

$$\Xi_n(x) \in [\alpha_1,\beta_1] \text{ for all } n\geq n_0 \text{ and } \|x-z\| \leq 2Rm^*. \tag{4.6.6}$$

Take $n_1 > \big(4Rm^* + 2\|z-z_0\|\big)^2 \vee \big(\big(\ln(b) - \ln(\Xi_0(z_0))\big)/\ln(a)\big)$ large enough. By a local central limit theorem for symmetric finite range random walks, cf. [LL10, Theorem 2.1.1] there exists $c>0$ such that $p^{(n_1)}(y,z_0) \geq cn_1^{-d/2}$ if $\|y-z_0\| \leq \sqrt{n_1}$. By letting $n_1 > \big(\ln(b) - \ln(\Xi_0(z_0))\big)/\ln(a)$ it holds that $a^{n_1}\Xi_0(z_0)\wedge b = b$ and hence it follows with (4.6.5) that

$$\Xi_{n_1}(y) \geq \sum_{w\in\mathbb{Z}^d} p^{(n_1)}(y,w)\Big[\big(a^{n_1}\Xi_0(w)\big)\wedge b\Big] \geq p^{(n_1)}(y,z_0)\Big[(a^{n_1}\Xi_0(z_0))\wedge b\Big] \geq cn_1^{-d/2}b.$$

Using (4.6.5) again, we deduce that for any $n_2 < \sqrt{n_1}/2$

$$\begin{aligned}\Xi_{n_1+n_2}(x) &\geq \sum_{y\in\mathbb{Z}^d} p^{(n_2)}(x,y)\Big[(a^{n_2}\Xi_{n_1}(y)) \wedge b\Big] \\ &\geq \sum_{y\in B_{\sqrt{n_1}}(z_0)} p^{(n_2)}(x,y)\Big(\big(a^{n_2}cn_1^{-d/2}\big) \wedge 1\Big)b.\end{aligned}$$

Choosing $n_2 = d\log n_1 - 2\log c$ gives that $\big(a^{n_2}cn_1^{-d/2}\big) \wedge 1 = 1$ and, since $B_{n_2}(x) \subseteq B_{\sqrt{n_1}}(z_0)$ when $n_1 > \big(4Rm^* + 2\|z - z_0\|\big)^2$, the above is larger than b.

Since $b > 2\alpha_1$ and trivially $\varphi_\mu(w) \leq 1/e < \beta_1$ for every $w \geq 0$, this shows (4.6.6). It follows that

$$\Xi_{n+1}(x) = \varphi_\mu(\delta_R(x;\Xi_n)) \in [\alpha_2,\beta_2] \text{ for all } n \geq n_0 \text{ and } \|x-z\| \leq (2m^*-1)R$$

and iterating m^* steps shows that

$$\Xi_{n+m^*-1}(x) \in [\alpha_m,\beta_m] \text{ for all } n \geq n_0 \text{ and } \|x-z\| \leq m^*R.$$

Take $x = z$ to conclude that $\Xi_n(z) \in [\alpha_m,\beta_m]$ for $n \geq n_0 + m^*$. □

4.7 Open Questions

We collect here some natural follow-up questions to our results, several of them were already mentioned in the text.

- Is there a sharp transition? That is, for given R, is the survival region a (possible empty) interval of values of μ? See also Figure 4.7.
- Is there always extinction for small values of R? Simulations suggest that in $d = 1$ for $R \leq 2$ the process dies out for all values of μ, see Figure 4.7 again.
- Can one give results for "soft" annihilation, allowing multiple occupancy of the sites? Of course, instead of the strong competition we consider, one could look at truncation, keeping for instance at most N particles per site at the same time and removing the others. Theorem 1.1 in [Mü15] implies for this truncation in our model that there is, for each $\mu > 1$ and all R, a critical value $N_c \in \{2,3,4,\ldots\}$ such that the survival probability is 0 for $N \leq N_c$ and strictly positive for $N > N_c$.
- What is the speed for the stochastic "travelling waves" in our model? Is there a shape theorem?

- The representation (4.1.4) suggests an interesting connection to spread-out oriented site percolation: let each site be open with probability p and closed with probability $1-p$, where $p = \min\{\varphi_\mu((2R+1)^{-d}), \varphi_\mu(1)\}$. Connect the open sites at time $n+1$ to their "parent" (with distance $\leq R$) at time n, provided it is open. Then the "wet" sites at time n are a lower bound for η_n.
  Let $p_c(d, R)$ be the percolation threshold for the event that there is an infinite connected cluster. How does the percolation threshold in directed space-time percolation behave for $R \to \infty$?
  We have the following conjecture, based on the analogy with "spread-out oriented bond percolation", see [vdHS05]:

$$\lim_{R\to\infty} (2R+1)^d p_c(d, R) = 1 \quad \text{for every } d > 4.$$

  It is plausible since the lattice should be more and more tree-like in high dimensions but we could not find a proof in the literature. Since $\varphi'_\mu(0) > 1$, this conjecture would lead to an alternative proof of survival for large R in $d > 4$.

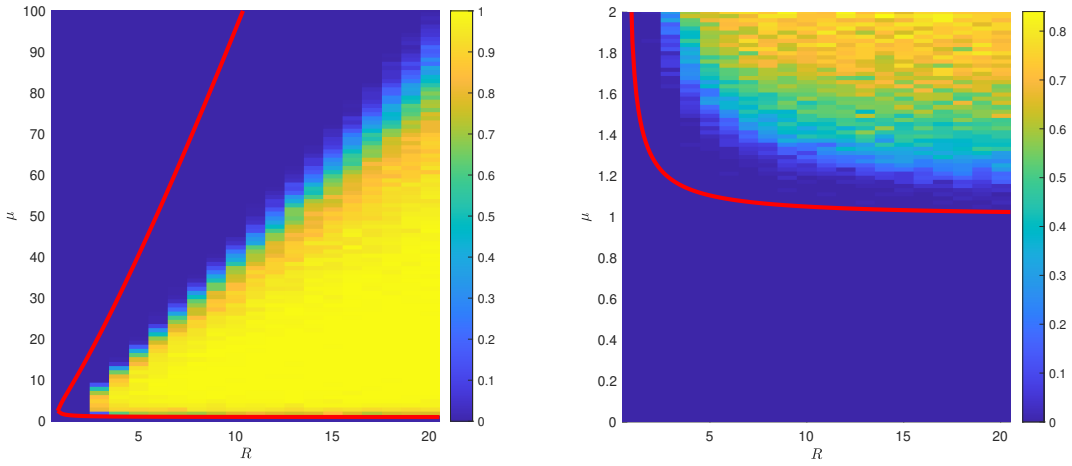

Figure 4.7: Simulations of the "phase diagram" for a one-dimensional BARW on $\mathbb{Z}/1000\mathbb{Z}$ with initial condition $\eta_0 = \delta_0$, showing a Monte Carlo estimate of the survival probability as a function of R and μ. On the left, 200 iterations of this process were run and the proportion of realisations that survived the first 250 generations is shown. Dark blue colour corresponds to no surviving realisations and yellow to only surviving realisations. The right image zooms in the region of small μ's. In both cases the red line is our theoretical bound for extinction from Theorem 4.1.1.

A Appendix

For completeness and ease of reference, we state the following concentration estimate for sums of independent Bernoulli random variables, which is a straightforward consequence of Bernstein's inequality.

Lemma A.1. *Let $(X_i)_{i=1,\dots,n}$ be independent Bernoulli random variables with $p_i = \mathbb{P}(X_i = 1)$, and let $S_n := X_1 + \cdots + X_n$. Then, setting $\mu_n := \mathbb{E}[S_n] = \sum_{i=1}^n p_i$, $\sigma_n^2 := \operatorname{Var} S_n = \sum_{i=1}^n p_i(1-p_i)$, and $m_n := \max_{1\le i\le n} \max\{p_i, 1-p_i\} = \max_{1\le i\le n} \operatorname{ess\,sup}|X_i - \mathbb{E}[X_i]|$ (≤ 1), we have*

$$\mathbb{P}(S_n - \mu_n \ge w) \le \exp\Big(-\frac{w^2}{2\sigma_n^2 + (2/3)m_n w}\Big), \quad w \ge 0, \tag{A.1}$$

and the same bound applies to $\mathbb{P}(S_n - \mu_n \le w)$ for $w \le 0$.

Proof. By Bernstein's inequality (see e.g. [Ben62, Ineq. (8)]), for every $t \ge 0$,

$$\mathbb{P}(S_n \ge \mu_n + t\sigma_n) \le \exp\Big(-\frac{t^2}{2 + 2m_n t/(3\sigma_n)}\Big) = \exp\Big(-\frac{(\sigma_n t)^2}{2\sigma_n^2 + (2/3)m_n t\sigma_n}\Big).$$

Reparametrising $t\sigma_n = w$ (and implicitly assuming $\sigma_n > 0$, otherwise the problem becomes trivial) we can rewrite this as (A.1).

Applying the argument to the $1 - X_i$'s gives the same bound for $\mathbb{P}(S_n - \mu_n \le w)$, for $w \le 0$. □

5. Ancestral lineages for a branching annihilating random walk

Pascal Oswald

Abstract. We study ancestral lineages of individuals of a stationary discrete-time branching annihilating random walk (BARW) on the lattice $\mathbb{Z}^d$. Each individual produces a Poissonian number of offspring with mean μ which then jump independently to a uniformly chosen site with a fixed distance R of their parent. By interpreting the ancestral lineage of such an individual as a random walk in a dynamical random environment, we obtain a law of large numbers and a functional central limit theorem for the ancestral lineage.

1 Model and main result

In this article, we reconsider a model of discrete-time branching annihilating random walk (BARW) on $\mathbb{Z}^d$, $d \geq 1$, that was first examined in [BCČ+23]. There model specific parameter regimes were identified for which the BARW survives with positive probability and for which it exhibits a unique non-trivial ergodic equilibrium. Building upon these results we are interested in the long-term statistical properties of the position of a single individual's ancestors in a population evolving as a stationary BARW. Our main result is that such ancestral lineages satisfy a law of large numbers as well as an annealed central limit theorem, cf. Theorem 1.3 below.

The main tool in proving these results is a renormalisation construction together with a result from [BČD16]. In [BČD16] Birkner, Černý and Depperschmidt develop an abstract program, which allows to study ancestral lineages of spatial population models with local self-interactions lying in the universality class of oriented percolation, via a renewal argument. More precisely the authors work out conditions that imply a law of large numbers and an annealed central limit theorem for the spatial paths of ancestral lineages for individuals drawn from a stationary population. We check that the BARW

satisfies these conditions by exploiting and adapting the renormalisation construction that was used to show the existence of a unique stationary law in [BCČ+23].

Let us now introduce the model. We study a discrete-time Markov process η with state space $\{0,1\}^{\mathbb{Z}^d}$ and denote by $\eta_n(z)$ the state of site $z \in \mathbb{Z}^d$ at time $n \in \mathbb{N}$ (later when dealing with the stationary process we take $n \in \mathbb{Z}$). We interpret $\eta_n(z) = 1$ as the site z being occupied by a single particle at time n and $\eta_n(z) = 0$ as the site being vacant. In order to describe the dynamics of η we introduce the following notational conventions: we write $\|\cdot\|$ for the uniform norm on $\mathbb{Z}^d$ and let $B_R(z) = \{x \in \mathbb{Z}^d : \|z - x\| \leq R\}$ be the d-dimensional ball (box) of radius $R \in \mathbb{N}$ centred at $z \in \mathbb{Z}^d$. Moreover, we set $V_R := 2R + 1$ to be its side length, so that the volume of $B_R(z)$ is V_R^d.

For fixed $R \in \mathbb{N}$, $\mu > 0$, and an initial particle configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, the configurations η_n at times $n \geq 1$ are obtained recursively through a three-step procedure. Let $z \in \mathbb{Z}^d$ be such that $\eta_n(z) = 1$. Then in a first step the particle at site z dies and gives birth to a Poisson number of offspring with mean μ. Secondly, each offspring moves independently to a uniformly chosen site in $B_R(z)$. Lastly, whenever there are two or more particles at a given site, then all the particles at that site are removed, i.e. annihilated. The particles remaining after the annihilation step make up the configuration η_{n+1}.

The thinning and superposition properties of the Poisson distribution give the following equivalent description of the model. For any configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $z \in \mathbb{Z}^d$, define first the (local) density of particles at z by

$$\delta_R(z;\eta) := V_R^{-d} \sum_{x \in B_R(z)} \eta(x). \tag{1.1}$$

Then, in order to get from a configuration η_n to η_{n+1} we fix $\eta_n \in \{0,1\}^{\mathbb{Z}^d}$ and denote by $N_{n+1}(z)$ the number of newborn particles at z in generation $n+1$ after the dispersal step but before annihilation has occurred. This number is given by the superposition of the offspring of all particles that can move to z (that is, of all $x \in B_R(z)$ with $\eta_n(x) = 1$). Thus, using the notation of (1.1), $N_{n+1}(z)$ is a Poisson random variable with mean $\mu\delta_R(z;\eta_n)$. Taking the annihilation into account, it then holds that

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{if } N_{n+1}(z) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Let

$$\varphi_\mu(w) := \mu w\, e^{-\mu w}, \quad w \in [0,\infty) \tag{1.3}$$

denote the probability that a Poisson random variable with mean μw equals 1. By construction, the random variables in the family $(\eta_{n+1}(z) : z \in \mathbb{Z}^d)$ are conditionally

independent given η_n and by (1.2), (1.3) we can represent our system as

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{with probability } \varphi_\mu(\delta_R(z;\eta_n)), \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

This gives a representation of η as a particular example of a probabilistic cellular automaton (PCA), see e.g. [MM14a] for an introduction to PCA. Such PCA can be seen as discrete-time counterpart to interacting particle systems, in which the entire system updates "in parallel", as opposed to "sequentually" as is true for interacting particle systems.

Remark 1.1. (a) The representation (1.4) relies on the offspring distribution being Poissonian and is not possible for a non-Poissonian choice of offspring distributions. Moreover, the constructions used subsequently depend very delicately on properties of φ_μ. We refer to [BCČ$^+$23, Section 1.2] for a detailed discussion of the models assumptions and possible generalisations.

(b) The BARW is non-monotone/non-attractive in the sense of particle systems. That is, adding more particles to a given generation does not guarantee an increase in the number of particles in the succeeding generation as a higher number of particles leads to more annihilation. A consequence of this seemingly simple fact is that many tools of monotone systems (e.g. comparisons using coupling, subadditivity arguments) are not directly applicable.

In [BCČ$^+$23] the existence of (non-trivial) invariant measures for the BARW is examined (note that by (1.4) the *empty configuration* $\mathbf{0} \in \{0,1\}^{\mathbb{Z}^d}$ is always an absorbing state). We summarise the relevant statements for the current objective in the following proposition.

Proposition 1.2 (Survival and complete convergence, [BCČ$^+$23])**.** *Let $\mu \in (1, e^2)$.*

(i) There exists R_μ such that for any $R > R_\mu$ the process η survives with positive probability and has a unique non-trivial invariant extremal distribution $\nu_{\mu,R}$.

(ii) Conditioned on non-extinction, the law of η_n converges to $\nu_{\mu,R}$ in the weak topology.

We are only interested in the parameter regime for which there is a unique non-trivial extremal invariant distribution. Therefore we only consider $\mu \in (1, e^2)$ and $R > R_\mu$ in the rest of the paper. By doing so the existence of $\nu_{\mu,R}$ is always guaranteed by Proposition 1.2.

Let us now introduce the main object of interest, namely the ancestral lineages of single particles in a BARW. We consider the stationary process $\eta = (\eta_n)_{n\in\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$, η_n is distributed as $\nu_{\mu,R}$. It is clear from the informal description of the

model, cf. the paragraph before (1.1), that the model can be enriched with genealogical information, by relating child and parent particles.

For simplicity of notation we condition η on having a particle at the space-time origin and always consider the ancestral lineage of the particle at the space-time origin. Conditioned on $\{\eta_0(0) = 1\}$, the dynamics of the ancestral lineage of the particle at the origin are described by the time-inhomogeneous Markov chain $X = (X_k)_{k\in\mathbb{N}_0}$ given by

$$X_0 = 0, \quad \text{and} \quad \mathbb{P}(X_{k+1} = y | X_k = x, \eta) = p_\eta(k; x, y), \quad k \geq 1 \tag{1.5}$$

where the transition probabilities $p_\eta(k; x, y)$ are given by

$$p_\eta(k; x, y) := \frac{\eta_{-k-1}(y)}{\sum_{z\in B_R(x)} \eta_{-k-1}(z)}. \tag{1.6}$$

Indeed, contingent on the random walk being at site x at time k, for any $y \in B_R(x)$ the number of particles sent from y to x is $\mathrm{Pois}(\mu V_R^{-d})$ distributed, conditional on the total sum over all $y \in B_R(x)$ being equal to one. Now, since a vector of independent Poisson random variables, conditioned on the total size of its sum has a multinomial distribution, it follows readily that a particle "selects" it's predecessor uniformly among all particles alive one generation earlier which are within distance R. That is, the transition kernel in (1.5) can be written as in (1.6).

The random walk X defined in (1.5) is a random walk in the (relatively complicated) random environment η and describes the space-time embedding of ancestral lineages of particles in η. Hence the randomness of X comes solely from the genealogy of the particle. We stress moreover that the forwards in time direction of the random walk, corresponds to the backwards in time direction of the environment η.

Our main result states that X satisfies a law of large numbers and a central limit theorem when averaging over the genealogical randomness (i.e. randomness due to X taking steps) and the randomness in the environment. To this end we write P_η for the conditional law of $\mathbb{P}$, given η, so that $p_\eta(k; x, y) = P_\eta(X_{k+1} = y | X_k = x)$. Moreover we $\mathbb{P}_0(\cdot) := \mathbb{P}(\cdot|\eta_0(0) = 1)$ and write $\mathbb{E}_0$ and E_η for the corresponding expectations.

Theorem 1.3. *For $\mu \in (1, e^2)$ there exists $\widetilde{R}_\mu \geq R_\mu$ such that for $R \geq \widetilde{R}_\mu$ the random walk as defined in* (1.5)–(1.6) *satisfies*

$$P_\eta\big(k^{-1}X_k \overset{k\to\infty}{\longrightarrow} 0\big) = 1 \quad \textit{for} \quad \mathbb{P}_0(\cdot) - a.a.\ \eta. \tag{1.7}$$

Moreover for any $g \in C_b(\mathbb{R}^d)$

$$\mathbb{E}_0\big[g(k^{-1/2}X_k)\big] \overset{k\to\infty}{\longrightarrow} \mathbb{E}[g(Z)], \tag{1.8}$$

where Z is a (non-degenerate) centered isotropic d-dimensional normal random variable, i.e. $Z \sim \mathcal{N}(0, \sigma^2 I)$ for some $\sigma^2 > 0$, where I is the identity matrix. Moreover a functional version of (1.8) *holds as well.*

The proof of Theorem 1.3 amounts to verifying the applicability of [BČD16, Theorem 3.1], which gives abstract conditions for a law of large numbers and a central limit theorem for random walks in dynamic random environments to hold.

Related literature

The questions addressed in [BČD16] for a general class of population models with local self-interactions were motivated by earlier work of the same authors. In [BČDG13] random walks on the backbone of an oriented percolation cluster on $\mathbb{Z}^d, d \geq 1$ (which correspond to ancestral lineages of a discrete-time contact process) are considered. A quenched law of large numbers and an annealed central limit theorem similar to Theorem 1.3 above are shown. Moreover, the natural question of the behaviour of two random walks with transition probabilities as in (1.5), which corresponds to the jointly describing the ancestral lineages of two distinct individuals is considered. By controlling two copies of the random walk, the annealed central limit theorem was strengthened to a quenched one. It follows moreover from [BČDG13] that analogous annealed and quenched central limit theorems hold if one allows the discrete-time contact process to have random i.i.d. carrying capacities (i.e. every space-time site can carry a random number of particles, instead of just one). This was generalised in [Mil17a] to the case where the carrying capacity is not i.i.d. but mixing. It is shown that a quenched law of large numbers and an annealed central limit theorem hold under ϕ-mixing in time (for $\phi_n \in O(n^{-1-\delta})$ resp. $\phi_n \in O(n^{-2-\delta})$ for some $\delta > 0$) and a quenched central limit theorem under an exponential mixing in space and time, see [Mil17a, Mil17b] for details. For unit carrying capacity the quenched central limit theorem was recently extended to a quenched *local* limit theorem for $d \geq 3$ in [BBDS23] (note that the dimensional constraint seems rectifiable as is commented upon in [BBDS23, Outlook and open questions]).

Moreover, the abstract program developed in [BČD16] was also applied to derive an a quenched law of large numbers and an annealed central limit theorem for the logistic branching random walk first studied in [BD07] as a population model with logistic local self-regulation. A comprehensive overview of these and related models can be found in [BG21].

Organisation of the article

The rest of the article is organised as follows. In Section 2 we formalise and make precise the conditions that need to be checked in order to apply the abstract machinery of [BČD16], comment on why they are needed and prove Theorem 1.3. In Section 3 we introduce the main ideas behind the notion of goodness used for the renormalisation construction, which are based on suitable control of local densities of η. Lastly Sections 4 and 5 contain the proofs that the necessary conditions are indeed met.

2 Abstract conditions and proof of Theorem 1.3

Recall that our strategy to prove Theorem 1.3 is to check the abstract conditions that let us apply Theorem 3.1 of [BČD16]. These conditions are rather involved and not straightforward to present in isolation. We present them in the following as propositions which need to be proven. The proofs are given in Sections 4 and 5.

The conditions from [BČD16] can be divided into two parts. Firstly, into conditions on the random environment η, in which the random walk X evolves and secondly into conditions on the random walk itself.

2.1 Conditions on the environment

The first condition on the environment ([BČD16, Assumption 3.2]) is that it is Markovian and admits a "local flow construction" that allows to couple the process with different (and arbitrary) initial conditions.

To this end we make use of an appropriate analogue of the *graphical construction* of interacting particle systems, with which we can view the evolution of η as a stochastic flow on its configuration space $\{0,1\}^{\mathbb{Z}^d}$. This offers the advantage of letting us define the process $\eta = (\eta_n)_{n\geq m}$ for all initial conditions $\eta_m \in \{0,1\}^{\mathbb{Z}^d}$, at any starting time $m \in \mathbb{Z}$, simultaneously. More precisely, we let $U(x,n)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{Z}$, be a collection of i.i.d. uniform random variables on $[0,1]$. Then, for any $m \in \mathbb{Z}$ and any initial condition $\eta_m \in \{0,1\}^{\mathbb{Z}^d}$, we define, recursively for $n \geq 0$,

$$\eta_{m+n+1}(x) := \mathbb{1}_{\{U(x,m+n+1)\leq \varphi_\mu(\delta_R(x;\eta_{m+n}))\}}, \tag{2.1}$$

where $\delta_R(x;\eta_{m+n})$ is as in (1.1). Comparing (2.1) to (1.4) with $m = 0$ it follows immediately that the process defined by (2.1) has the law of the BARW. In this sense the i.i.d. field $(U(x,n))_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}}$ of Unif$[0,1]$ random variables, acts as *driving noise* for the evolution of η. Moreover, the construction is local, as the value of η at any space-time site $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ is fully determined by the value $U(x,n)$ and by the values $\{\eta_{n-1}(y) : y \in B_R(x)\}$, and hence Assumption 3.2 of [BCČ$^+$23] is satisfied.

Expanding on the idea of η as a stochastic flow on the configuration space we introduce for $-\infty < m < n$ the σ-algebras

$$\mathcal{G}_{m,n} := \sigma(U(x,k) : m < k \leq n, x \in \mathbb{Z}^d). \tag{2.2}$$

By iterating (2.1) we can define a (random) family of $\mathcal{G}_{m,n}$-measurable mappings

$$\Phi_{m,n} : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}, \quad -\infty < m < n, \tag{2.3}$$

such that $\eta_n = \Phi_{m,n}(\eta_m)$. More precisely we define for any $\zeta \in \{0,1\}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$

$$\big(\Phi_{m,m+1}(\zeta)\big)(x) := \mathbb{1}_{\{U(x,m+1)\leq \varphi_\mu(\delta_R(x;\zeta))\}}$$

and then set

$$\Phi_{m,n} := \Phi_{n-1,n} \circ \cdots \circ \Phi_{m,m+1}.$$

By using these mappings, the dynamics of $(\eta_n)_{n\geq m}$ defined simultaneously for all initial conditions $\eta_m \in \{0,1\}^{\mathbb{Z}^d}$ and any $m \in \mathbb{Z}$.

The second condition on the environment ([BČD16, Assumption 3.3]) asks that a comparison with supercritical oriented percolation can be made on an appropriately scaled space-time grid, using a very specific notion of "good" blocks. This specific notion of goodness, is introduced in Proposition 2.1 below, where it is also stated that an appropriate coarse-graining exists for the BARW.

To this end, we introduce the following notation. For spatial and temporal scales $L_s, L_t \in \mathbb{N}$ we consider space-time blocks whose "bottom parts" are centered at the points in the coarse-grained grid $\mathbb{L} := L_s\mathbb{Z}^d \times L_t\mathbb{Z}$. Points in $\mathbb{L}$ are labeled by $\mathbb{Z}^d \times \mathbb{Z}$ such that $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ is the label for the point $(L_s x, nL_t) \in \mathbb{L}$. For $m \in \mathbb{N}$ and $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ we consider blocks

$$\texttt{block}_m(x,n) := \big\{(y,k) \in \mathbb{Z}^d \times \mathbb{Z} : \|y - L_s x\| \leq mL_s, nL_t < k \leq (n+1)L_t\big\}. \quad (2.4)$$

Note that these blocks are overlapping in the spatial directions but never in the temporal direction. We further write for any $A \subseteq \mathbb{Z}^d \times \mathbb{Z}$, $U|_A$ for the restriction of the field U of driving noise to the set A. In particular this means that for any $\texttt{block}_m(x,n)$ the restriction of $U|_{\texttt{block}_m(x,n)}$ is an element of $[0,1]^{B_{mL_s}(L_s x)\times\{1,\ldots,L_t\}}$.

With this notation we can present the second condition of [BČD16] on the environment, in form of a proposition, the proof of which is given in Section 4 below.

Proposition 2.1. *For any $\mu \in (1, e^2)$ and $\varepsilon > 0$ there exists $\widetilde{R}_{\mu,\varepsilon} \geq R_\mu$, such that for every $R \geq \widetilde{R}_{\mu,\varepsilon}$ there is a spatial scale L_s, a temporal scale L_t, a set of good (local) configurations $G_{\text{conf}} \subseteq \{0,1\}^{B_{2L_s}(0)}$ and a set of good (local) driving noise realisations $G_U \subseteq [0,1]^{B_{4L_s}(0)\times\{1,\ldots,L_t\}}$ such that*

$$\mathbb{P}(U|_{\texttt{block}_4(0,0)} \in G_U) \geq 1 - \varepsilon \quad (2.5)$$

and such that the following contraction and coupling conditions are satisfied: For any $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ and any configurations $\eta^{(1)}_{nL_t}, \eta^{(2)}_{nL_t} \in \{0,1\}^{\mathbb{Z}^d}$ at time nL_t, if $\eta^{(i)}_{nL_t}|_{B_{2L_s}(L_s x)} \in G_{\text{conf}}$ for $i = 1,2$ and $U|_{\texttt{block}_4(x,n)} \in G_U$, then

(i) $\eta^{(1)}_{(n+1)L_t}(y) = \eta^{(2)}_{(n+1)L_t}(y)$ *for all* $\|y - L_s x\| \leq 3L_s$

(ii) $\eta^{(1)}_{(n+1)L_t}|_{B_{2L_s}(L_s(x+e))} \in G_{\text{conf}}$ *for all* $e \in B_1(0)$.

Moreover, when the $\eta^{(i)}_{nL_t}$'s agree on $B_{2L_s}(L_s x)$ (i.e. at the bottom center of the block), then they agree on all space-time points in $B_{L_s}(L_s x) \times \{1,\ldots,L_t\}$.

For a realisation η of the BARW, we call a coarse-grained block $\mathtt{block}_4(x,n)$ *good*, if $U|_{\mathtt{block}_4(x,n)} \in G_U$ and $\eta_{nL_t}|_{B_{2L_s}(L_s x)} \in G_{\mathrm{conf}}$. More formally, let for any $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$

$$\Gamma(x,n) := \mathbb{1}_{\{U|_{\mathtt{block}_4(x,n)} \in G_U\}} \mathbb{1}_{\{\eta_{nL_t}|_{B_{2L_s}(L_s x)} \in G_{\mathrm{conf}}\}}, \tag{2.6}$$

then $\mathtt{block}_4(x,n)$ is *good*, if $\Gamma(x,n) = 1$. This notion of goodness can be viewed as a type of contractivity property of the local dynamics, in the sense that, on a good block, the flow Φ, cf. (2.3), tends to merge local configurations.

Remark 2.2. The block-construction from [BCČ+23], which we recapitulate in Section 3.2, does not satisfy Proposition 2.1 verbatim. The issue is that next to depending on the driving noise in a box, the notion of "good block" that was used there also depends on the *specific* realisation of η at the bottom of the block. Nonetheless we can reuse the ideas of [BCČ+23, Sectio 4], cf. Section 4 below, and this technicality can be remedied by introducing (next to G_U) the set G_{conf} and by choosing slightly larger scales. The difference in the size of scales is elaborated upon in some more detail in Remark 3.2 after having introduced some more details on the construction of [BCČ+23].

2.2 Conditions on the random walk

Let X be the random walk as defined in (1.5), evolving in the dynamic random environment given by the stationary process η, which is defined as in (2.1) with parameters μ, R for which Proposition 2.1 holds for some $\varepsilon > 0$ and let $L_s, L_t \in \mathbb{N}$ be the scales corresponding to these parameters.

There are again two conditions in [BČD16], on the random walk X, that need to be verified. The first is that if the random walk starts anywhere from the middle half of the top of a good block, i.e. a block $\mathtt{block}_4(x,n)$ such that $\Gamma(x,n) = 1$, for some $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$, then with high probability it doesn't cover long distances within the block, cf. [BČD16, Assumption 3.9]. The precise statement that needs to be checked is summarised in the following proposition. Recall for this that P_η denotes the "quenched" probability measure, i.e. the measure, conditioned on a realisation of η.

Proposition 2.3. *For $\varepsilon, \delta > 0$ there exists $R_{\mu,\delta,\varepsilon} > \widetilde{R}_{\mu,\varepsilon}$ such that for all $R \geq R_{\mu,\delta,\varepsilon}$ and $L_s, L_t, G_U, G_{\mathrm{conf}}$ as in Proposition 2.1, the random walk X satisfies for $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$*

$$\min_{z : \|L_s x - z\| \leq L_s/2} P_\eta\Big(\max_{(n-1)L_t < k \leq nL_t} \|X_k - z\| \leq L_s/4 \Big| X_{(n-1)L_t} = z, \Gamma(x,n) = 1 \Big) \geq 1 - \delta. \tag{2.7}$$

The proof of Proposition 2.3 follows from the fact that in a block with good driving noise the relative fluctuations of the local density of η over R balls and r-balls with $1 \ll r < R$ are small. Thus in each step the increments of X do not deviate much

from the increments of a simple random walk. As the precise notion of goodness of a block depends on the construction in the proof of Proposition 2.1, and the specifics of the fluctuations of the density of η are described in Section 3.3 below, we postpone the details of the proof to Section 5.

The second condition on X is that it behaves symmetrically with respect to spatial point reflections, when η is reflected accordingly, cf. [BČD16, Assumption 3.11]. As for any time $k \in \mathbb{N}$ the random walk in (1.5) chooses uniformly among the possible ancestors of the particle at X_k, this condition holds trivially. Note that this symmetry corroborates that asymptotically the average speed of X is zero.

With the results of Sections 2.1 and 2.2 at hand, the proof of Theorem 1.3 follows directly.

Proof of Theorem 1.3. The assertion of the theorem follows by a combination of Proposition 2.1 and 2.3 as well as Theorem 3.1 of [BČD16] for all η defined with parameters $\mu \in (1, e^2)$ and for R large enough. □

3 Renormalisation construction

In this section we fist collect some results on the function φ_μ that will be essential in the proof of Proposition 2.1 in Section 4. These give insight into the behaviour of local densities $\delta_r(\cdot;\eta)$ for $1 \ll r \leq R$ which in turn lets us to outline the idea of the block construction from [BCČ$^+$23, Section 4], on which our construction in Section 4 is based. Moreover, we discuss how to get quantitative control on the local densities.

3.1 Properties of φ_μ

The following lemma summarises useful properties of the function $\varphi_\mu(w) = \mu w e^{-\mu w}$, which appears in the definition of the dynamics of η, cf. (1.4) and (2.1).

Lemma 3.1. *(a) For $\mu > 1$, φ_μ has two fixpoints, 0 and $\theta_\mu := \mu^{-1} \log \mu$. The fixpoint 0 is always repulsive.*

(b) For $\mu \in (1, e^2)$, θ_μ is an attractive fixpoint and for $\mu > e^2$, there are no attractive fixpoints.

(c) For every $\mu \in (1, e^2)$ there is $\varepsilon_{\mathrm{FP}} = \varepsilon_{\mathrm{FP}}(\mu) > 0$ and $\kappa(\mu, \varepsilon_{\mathrm{FP}}) < 1$ such that φ_μ is a contraction on $[\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}]$, that is,

$$|\varphi_\mu(w_1) - \varphi_\mu(w_2)| \leq \kappa(\mu, \varepsilon_{\mathrm{FP}})|w_1 - w_2| \qquad \text{for } w_1, w_2 \in [\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}].$$

(d) There exist a strictly increasing sequence $\alpha_m \uparrow \theta_\mu$ and a strictly decreasing sequence $\beta_m \downarrow \theta_\mu$ such that $\varphi_\mu([\alpha_m, \beta_m]) \subseteq (\alpha_{m+1}, \beta_{m+1})$ for all $m \in \mathbb{N}$. Furthermore, it is possible to choose $\alpha_1 > 0$ arbitrarily small and $\beta_1 > 1/e$.

Proof. Properties (a)–(b) are a direct consequence of the definition of φ_μ and (c)–(d) are the contents of [BCČ+23, Lemma 4.1]. □

Note that by Lemma 3.1(d) for any $\mu \in (1, e^2)$ and any choice of $\alpha_1 > 0$ and $\beta_1 > 1/e$, there is a finite index m_0 such that $\alpha_m, \beta_m \in [\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}]$ for all $m \geq m_0$, i.e.

$$m_0 = m_0(\mu, \alpha_1, \beta_1) := \inf\big\{m \geq 1 : \alpha_m, \beta_m \in [\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}]\big\}. \tag{3.1}$$

This value will later play a role in establishing the properties Proposition 2.1(i)–(ii).

3.2 Local densities and goodness

In [BCČ+23, Section 4] complete convergence of the BARW is shown by a comparison with supercritical oriented percolation with a coarse-graining and a notion of goodness on blocks that is reminiscent, but not identical to that of Proposition 2.1 and (2.6), cf. Remark 3.2. The exact notion of goodness in [BCČ+23] is tailored specifically to showing complete convergence, i.e. convergence (conditioned on survival) of the law of the BARW towards the *unique* non-trivial extremal invariant distribution $\nu_{\mu,R}$. To achieve this, *good* blocks were defined to make two distinct configurations which partially agree at the "bottom" of the block, evolve into configurations that agree on a larger portion of the "top" of the block (this should be compared to Proposition 2.1(i)).

To motivate why it is reasonable to expect distinct local configurations of the BARW, following the same dynamics, to merge, recall that the random variables $(\eta_{n+1}(x))_{x\in\mathbb{Z}^d}$ are conditionally independent and conditionally Bernoulli distributed, given η_n, with respective parameters $\varphi_\mu(\delta_R(x;\eta_n))$ cf. (1.4). Therefore local densities $\delta_r(x;\eta_{n+1})$ (as sums of these conditional Bernoulli random variables) should concentrate around some value, at least for r large. In [BCČ+23] only the case $r = R$ was considered and it was used there that the value, around which the local densities concentrate converges to θ_μ, and thus, by repeated application of Lemma 3.1(c)–(d) and (2.1), the flow Φ tends to merge any two distinct realisations of the BARW over long enough time-spans. Controlling how distinct configurations merge under the dynamics of the BARW thus amounts to gaining control on local densities.

We follow the same idea as in [BCČ+23] to gain quantitative control of the (local) densities by introducing certain families of *reference density profiles* $\zeta_k^{r,-} : \mathbb{Z}^d \to [0,1]$ and $\zeta_k^{r,+} : \mathbb{Z}^d \to [0,1]$, for $k \in \{0, \dots, k_0\}$. We are then interested in the situation where the local r-densities of the true system η_k are wedged in between the two reference density profiles, i.e. when

$$\zeta_k^{r,-}(x) \leq \delta_r(x;\eta_k) \leq \zeta_k^{r,+}(x), \quad \text{for suitable } x \in \mathbb{Z}^d, k \in \mathbb{N}_0 \text{ and } r \text{ large enough.} \tag{3.2}$$

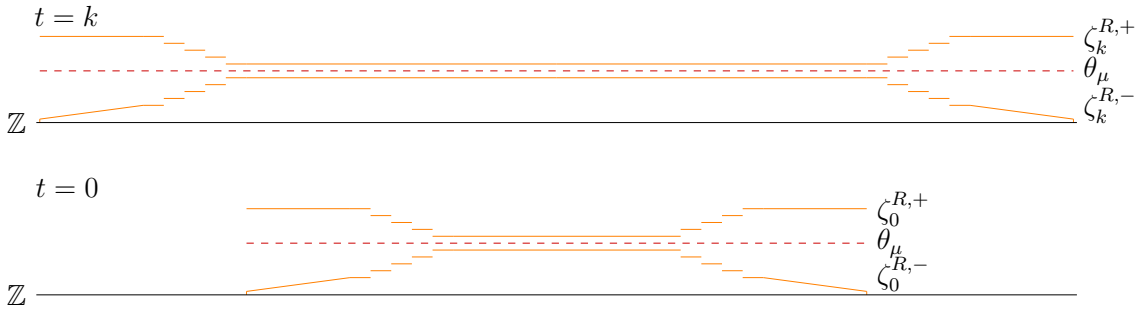

Figure 5.1: Sketch of the density profiles $\zeta_k^{R,\pm}$ (in orange) from [BCČ+23] for dimension $d = 1$. The bottom shows the profiles $\zeta_0^{R,\pm}$, which then expand to $\zeta_k^{R,\pm}$ in the top image. The profiles are chosen such that the distance of the profiles to θ_μ is smaller than $\varepsilon_{\mathrm{FP}}$ in the central constant part of the profiles.

In Section 4.1 of [BCČ+23] a family of reference density profiles is introduced, which satisfies (3.2) with high probability for $r = R$. Moreover the profiles are chosen in such a way that $|\zeta_k^{R,+} - \zeta_k^{R,-}| \in [\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}]$ on part of the support of $\zeta_k^{R,-}$, and such that they have a fixed deterministic "front" that expands by a fixed distance in every time step, see Figure 5.1 for a sketch of the one-dimensional profiles $\zeta_k^{R,\pm}$ of [BCČ+23]. The control (3.2) by the expanding families $\zeta^{R,\pm}$ thus lets one apply Lemma 3.1(c)–(d) in a growing spatial region throughout a block.

Based on this, and a suitable coarse-graining of space-time, the notion of goodness in [BCČ+23] is defined using a two-step procedure for any two realisations $(\eta_n^{(1)})_{n\geq 0}$, $(\eta_n^{(2)})_{n\geq 0}$ of the BARW, with different initial conditions, coupled through the flow construction (2.1).

(I) A block based at some $(z,t) \in \mathbb{Z}^d \times \mathbb{Z}$ is called *well-started* if for $i = 1, 2$

$$\delta_R(x; \eta_t^{(i)}) \in \big[\zeta_0^{R,-}(x-z), \zeta_0^{R,+}(x-z)\big] \quad \text{for all } x \in \{z + y : \zeta_0^{R,-}(y) > 0\}, \tag{3.3}$$

and

$$\eta_t^{(1)}(x) = \eta_t^{(2)}(x) \quad \text{for all } x \text{ in a suitably large ball around } z. \tag{3.4}$$

(II) A block based at some $(z,t) \in \mathbb{Z}^d \times \mathbb{Z}$ is called *good*, if it is well-started and the domination by the $\zeta^{R,\pm}$ profiles and the region where (3.4) holds spread throughout suitably large portions of the block (we refer to [BCČ+23] for details on what suitable means in this context).

The first step is concerned with guaranteeing that the configurations of $(\eta_n^{(i)})_{n\geq 0}$ at the "bottom" of a block are controlled by a suitable shifted version of the reference

profiles $\zeta_0^{R,\pm}$, whereas the second is concerned with the "spreading" of the region where the two configurations agree, as well as the region where the density control by the reference profiles holds.

Remark 3.2. (a) Morally (2.6) is a "uniformisation" of the properties given by (I)–(II) above, by making the notion of goodness independent of any specific configuration at the "bottom" of the block. To accomplish this, special care needs to be taken of (3.4), which is a condition of two configurations partially agreeing in a region at the bottom of the block. To overcome this, we introduce below, in (4.5) the set of reference configurations $\mathrm{C}_{\mathrm{ref}}$, given by all configurations $\widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d}$ such that the local densities are globally in the interval $[\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}]$ to which configurations which are locally in G_{conf} will need to couple successfully.

(b) The size of blocks and thus the scales of the coarse-graining in the construction outlined by (I)–(II) are linked to concrete details of the reference density profiles $\zeta_k^{R,\pm}$ that are used, as the scales need to be chosen such that the desired properties spread to suitable large portions of the blocks.

In contrast to [BCČ+23, Section 4.1], for the current purposes it does not suffice to only have control on the local R-densities of the process η, but in order to prove Proposition 2.3 we also need control on local r_0-densities for some $r_0 < R$ which is specified in Section 5 as a fixed proportion of R. With out loss of generality we assume that r_0 divides R. Importantly, taking R large has the effect that r_0 is also large. To this end we work with two sequences of reference density profiles in Sections 4–5 below, one controlling the local R-densities and one controlling local r_0-densities, cf. (3.2).

3.3 Concentration and comparison of local densities with deterministic profiles

Let us now elaborate upon and formalise the idea of wedging local r-densities of η_k in between two reference functions as in (3.2). To this end we introduce sequences of functions $\zeta_k^{r,\pm}$ on $\mathbb{Z}^d$, which serve as these reference functions and which we call *comparison density profiles* (c.d.p.'s). These are analogous to the functions introduced in [BCČ+23, Definition 2.2], with the difference that we require an averaging property over r-balls instead of R-balls.

Definition 3.3. For given $\varepsilon, \delta > 0$ and $r \in \mathbb{N}$, (ε, δ, r)-*comparison density profiles* ((ε, δ, r)-*c.d.p.*'s) are deterministic functions $\zeta_k^{r,-}, \zeta_k^{r,+} : \mathbb{Z}^d \to [0, \infty)$, $k = 0, 1, \ldots, k_0$, satisfying:

(i) For every $k = 0, \ldots, k_0$, $\zeta_k^{r,-}(\cdot) \leq \zeta_k^{r,+}(\cdot)$.

(ii) For every $k = 0, \ldots, k_0$, $\mathrm{Supp}(\zeta_k^{r,-}) := \{x \in \mathbb{Z}^d : \zeta_k^{r,-}(x) > 0\}$ is finite, and $\zeta_k^{r,-}(x) \geq \varepsilon$ for every $x \in \mathrm{Supp}(\zeta_k^{r,-})$.

(iii) For every $k = 0, \dots, k_0 - 1$, and $x \in \mathrm{Supp}(\zeta_k^{r,-})$ it holds that if $\rho : B_r(x) \to \mathbb{R}$ satisfies $\rho(y) \in [\zeta_k^{r,-}(y), \zeta_k^{r,+}(y)]$ for all $y \in B_r(x)$, then

$$(1+\delta)\zeta_{k+1}^{r,-}(x) \le V_r^{-d} \sum_{y \in B_r(x)} \varphi_\mu(\rho(y)) \le (1-\delta)\zeta_{k+1}^{r,+}(x). \tag{3.5}$$

In Lemma 3.6 we show that such c.d.p.'s in fact exist, by constructing specific examples (in (3.9)–(3.10)). A comment is in order to clarify the role of the model-parameter R in these c.d.p.'s and in particular in (3.5). Recall that by (1.4) the distribution of $\eta_{k+1}(x)$ is conditionally Bernoulli, given η_k, with parameter $\varphi_\mu(\delta_R(x;\eta_k))$. Therefore, in order to make any statement of r-densities at time $k+1$ we must have some knowledge of R-densities at time k. In terms of applying (ε,δ,r)-c.d.p.'s to control local r-densities of η we interpret ρ in (iii) as representing values of local R-densities. This means that in order to make use of the control guaranteed by (3.5) we need R-densities(!) at time k to be controlled by the (ε,δ,r)-c.d.p.'s.

In the following we introduce for any (ε,δ,r)-c.d.p. $\zeta^{r,\pm}$ satisfying Definition 3.3 with $r \in \{1,\dots,R\}$ and any $x \in \mathbb{Z}^d$ the sets of driving noise that provide uniform (in the configurations of η) r-density control at x by the (ε,δ,r)-c.d.p.'s centered at x at time $k+1$ given that R-densities in $B_r(x)$ are controlled by (ε,δ,r)-c.d.p.'s at time k.

$$\mathcal{U}_k^r(\zeta^{r,\pm};x) := \left\{ u \in [0,1]^{B_r(0)} : \begin{array}{l} \text{for all configurations } \tilde\eta \in \{0,1\}^{\mathbb{Z}^d} \\ \text{s.t. } \delta_R(y;\tilde\eta) \in [\zeta_k^{r,-}(y), \zeta_k^{r,+}(y)] \text{ for} \\ \text{all sites } y \in B_r(x), \text{ it holds that} \\ V_r^{-d} \sum_{y \in B_r(x)} \mathbb{1}_{\{u_{y-x} \le \varphi_\mu(\delta_R(y;\tilde\eta))\}} \in \\ [\zeta_{k+1}^{r,-}(x), \zeta_{k+1}^{r,+}(x)] \end{array} \right\}.$$

Moreover, let $\mathcal{U}_k^{r,\pm}(\zeta^{r,\pm};x)$ be the corresponding sets, where only the upper/lower bound holds for the last sum, i.e. $\mathcal{U}_k^r(\zeta^{r,\pm};x) = \mathcal{U}_k^{r,-}(\zeta^{r,\pm};x) \cap \mathcal{U}_k^{r,+}(\zeta^{r,\pm};x)$. To ease the presentation we drop the dependence on $\zeta^\pm$ in the notation, whenever the choice of the c.d.p.'s is clear and write simply $\mathcal{U}_k^r(x)$ and $\mathcal{U}_k^{r,\pm}(x)$ respectively.

Lemma 3.4. *For $\varepsilon,\delta > 0$, $r \in \mathbb{N}$ let $\zeta_k^{r,\pm}$ be a family of (ε,δ,r)-c.d.p.'s. It then holds for $k = 0,1,\dots,k_0-1$ that*

$$\mathbb{P}\left(U|_{B_r(0)\times\{k\}} \in \mathcal{U}_k^r(0)\right) \ge 1 - 2e^{-cV_r^d}, \tag{3.6}$$

where $c = (\delta\varepsilon)/(1/(2\delta\varepsilon) + 2/3)$.

Proof. The proof is an adaptation of the proof of [BCČ+23, Lemma 2.3]. Note that (3.6) follows if one shows that the two probabilities $\mathbb{P}\left(U|_{B_r(0)\times\{k\}} \in \mathcal{U}_k^{r,-}(0)\right)$ and $\mathbb{P}\left(U|_{B_r(0)\times\{k\}} \in \mathcal{U}_k^{r,+}(0)\right)$ are both greater than $1 - e^{-cV_r^d}$. We start by showing the first inequality.

For $k \in \{0, \dots, k_0 - 1\}$ and $y \in B_r(x)$ for $x \in \mathbb{Z}^d$ introduce the quantities

$$\underline{\lambda}_k(y) := \operatorname*{arg\,inf}_{\lambda \in [\zeta_k^{r,-}(y), \zeta_k^{r,+}(y)]} \varphi_\mu(\lambda).$$

Note that by (3.5) it follows from $\underline{\lambda}_k(y) \in [\zeta_k^{r,-}(y), \zeta_k^{r,+}(y)]$ for all $y \in B_r(x)$, that

$$V_r^{-d} \sum_{y \in B_r(x)} \varphi_\mu(\underline{\lambda}_k(y)) \geq (1+\delta)\zeta_{k+1}^{r,-}(x). \tag{3.7}$$

Moreover, for $k \in \{0, \dots, k_0 - 1\}$ and $y \in B_r(x)$ let $Z_y^k := \mathbb{1}_{\{U(y,k) \leq \varphi_\mu(\underline{\lambda}_k(y))\}}$ where U is the i.i.d. field of driving noise. These are independent Bernoulli random variables with parameters $\varphi_\mu(\underline{\lambda}_k(y))$.

Using the Z_y^k and (3.7), we see that

$$\begin{aligned}\mathbb{P}\left(U|_{B_r(0)\times\{k\}} \notin \mathcal{U}_k^{r,-}(0)\right) &\leq \mathbb{P}\Big(V_r^{-d} \sum_{y \in B_r(x)} \mathbb{1}_{\{U(y,k) \leq \varphi_\mu(\underline{\lambda}_k(y))\}} < \zeta_{k+1}^{r,-}(x)\Big) \\ &\leq \mathbb{P}\Big(\sum_{y \in B_r(x)} \big(Z_y^k - \varphi_\mu(\underline{\lambda}_k(y))\big) < -\delta V_r^d \zeta_{k+1}^{r,-}(x)\Big).\end{aligned}$$

Since the right hand side of the above is a sum of independent centered random variables, and since

$$\operatorname{Var}\Big(V_r^{-d} \sum_{y \in B_r(x)} Z_y^k\Big) = V_r^{-2d} \sum_{y \in B_r(x)} \varphi_\mu(\underline{\lambda}_k(y))\big(1 - \varphi_\mu(\underline{\lambda}_k(y))\big) \leq \frac{1}{4} V_r^{-d},$$

a concentration estimate for the sum of independent Bernoulli random variables, see e.g. [BCČ+23, Lemma A.1] for an estimate based on Bernstein's concentration inequality, can be used, exactly as in the proof of [BCČ+23, Lemma 2.3] in order to conclude that

$$\mathbb{P}\left(U|_{B_r(0)\times\{k\}} \notin \mathcal{U}_k^{r,-}(0)\right) \leq \exp(-cV_r^d),$$

where $c = (\delta\varepsilon)/(1/(2\delta\varepsilon) + 2/3)$ is due to details of the concentration estimate.

Proving that the probability of $\{U|_{B_r(0)\times\{k\}} \notin \mathcal{U}_k^{r,+}(0)\}$ is small is completely analogous, using $\overline{\lambda}_k(y) := \arg\sup_{\lambda \in [\zeta_k^{r,-}(y), \zeta_k^{r,+}(y)]} \varphi_\mu(\lambda)$ instead of $\underline{\lambda}_k(y)$ and

$$V_r^{-d} \sum_{y \in B_r(x)} \varphi_\mu(\overline{\lambda}_k(y)) \leq (1-\delta)\zeta_{k+1}^{r,+}(x)$$

instead of (3.7). □

Remark 3.5. Lemma 3.4 is a (slightly) strengthened version of Lemma 2.3, from [BCČ$^+$23], where it was shown that for any given c.d.p.'s $\zeta_k^{R,\pm}$ and any (possibly random) configuration $\widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d}$, such that for some $k \in \{0,\dots,k_0-1\}$ and $x \in \mathbb{Z}^d$

$$\delta_R(y;\widetilde{\eta}) \in \big[\zeta_k^-(y), \zeta_k^+(y)\big], \quad \text{for all } y \in B_R(x),$$

it holds that

$$\mathbb{P}\Big(\zeta_{k+1}^-(x) \le \delta_R\big(x;\Phi_{0,1}(\widetilde{\eta})\big) \le \zeta_{k+1}(x)\Big|\mathcal{F}\Big) \ge 1 - 2e^{-cV_R^d}$$

for some $c > 0$ where $\mathcal{F} = \sigma(\widetilde{\eta}(x) : x \in \mathbb{Z}^d) \vee \sigma(U(x,0) : x \in \mathbb{Z}^d)$.

The uniform (in the configurations of η) concentration result of local r-densities of Lemma 3.4 plays an important role in the following section, where it is used as a building block for the set G_U of good driving noise.

Our proofs of Propositions 2.1 and 2.3 are based on two specific choices $\zeta^{\pm,r_0}$, $\zeta^{R,\pm}$ of c.d.p.'s which are slight modifications of the c.d.p.'s introduced in [BCČ$^+$23]. They have the same general shape as sketched in Figure 5.1 (for $d = 1$), but with scale dependent fronts (i.e. the steps in the "staircase" that make up the fronts have different lengths for $r = r_0$ and for $r = R$). We describe these in more detail now.

The construction of the c.d.p.'s we work with relies on a family of auxiliary functions which are constructed explicitly in Lemma 2.5, Lemma 2.6 of [BCČ$^+$23] and the discussion following those results. To introduce these auxiliary functions let $r \in \mathbb{N}$ and recall the definition of $(\alpha_m)_{m\ge1}$ from Lemma 3.1 and the definition of m_0 from (3.1). With these objects at hand, we consider for any integer $R_{\max} \in \mathbb{N}$ and $\varepsilon_0 \in (0,\alpha_1)$, $s \in (0,1)$ and $w \ge 2$ the family $(\chi_k^r(x))_{k\ge0}$ of functions defined as follows. On $\{\|x\| \le R_{\max} + m_0 r + k\lceil sr\rceil\}$ we let $\chi_k^r(x) \equiv \alpha_1$, and for all x outside of this set we let

$$\chi_k^r(x) = \alpha_1 \prod_{i=1}^d \min\Big\{\Big((\varepsilon_0/\alpha_1)^{1/d} + \frac{\widetilde{R}_k - |x_i|}{\lceil wr\rceil}\Big)\mathbb{1}_{\{\widetilde{R}_k \ge |x_i|\}}, 1\Big\}, \tag{3.8}$$

where $\widetilde{R}_k = R_{\max} + m_0 r + k\lceil sr\rceil + \lceil rR\rceil$, x_i is the i-th coordinate of x. Note that for $d = 1$, the non-constant section of χ_k^r has a width of $\lceil wr\rceil$ and on this section the function decreases linearly to the value ε_0. Moreover, in this case, $\chi_{k+1}^r(|x|) = \chi_k^r(|x| - \lceil sr\rceil)$, i.e. in the one-dimensional case, increasing k by one, shifts the non-constant parts of χ_k^r to the outside by $\lceil sr\rceil$. Note that these properties also hold along the coordinate axis when $d > 1$.

For convenience we set $R_{\max}^r(k) := R_{\max} + k\lceil sr\rceil$ for $k \in \{0,\dots,k_0\}$ where k_0 is any finite value (once the scales are defined properly we will take $k_0 = L_t$). Note that with this notation $\chi_k^r \equiv \alpha_1$ on the ball of radius $R_{\max}^r(k) + m_0 r$. With the help of the

family $(\chi_n^r)_{n\geq 0}$, we introduce the following functions

$$\zeta_k^{r,-}(x) = \begin{cases} \alpha_{m_0} & \text{if } \|x\| \leq R^r_{\max}(k), \\ \alpha_{m_0-j+1} & \text{if } R^r_{\max}(k) + (j-1)r \leq \|x\| \leq R^r_{\max}(k) + jr,\ 1 \leq j \leq m_0, \\ \chi_k^r(x) & \text{if } \|x\| \geq R^r_{\max}(k) + m_0 r, \end{cases} \tag{3.9}$$

and

$$\zeta_k^{r,+}(x) = \begin{cases} \beta_{m_0} & \text{if } \|x\| \leq R^r_{\max}(k), \\ \beta_{m_0-j+1} & \text{if } R^r_{\max}(k) + (j-1)r \leq \|x\| \leq R^r_{\max}(k) + jr,\ 1 \leq j \leq m_0, \\ 1 \vee \beta_1 & \text{if } \|x\| \geq R^r_{\max}(k) + m_0 r. \end{cases} \tag{3.10}$$

For the sake of readability we do not make the dependence of these functions on $R_{\max}$, $(\alpha_m)_{m\geq 0}$, s, w, ε_0 explicit in the notation. Moreover $\zeta_k^{r,-}$ is supported on a ball of radius $R^r_{\max}(k) + m_0 r + \lceil wr \rceil$ and gives the strongest density control on a ball of radius $R^r_{\max}(k)$. The next result shows that these families of functions are c.d.p.'s for large enough choices of r.

Lemma 3.6. *For any $M \geq 1$ there exists $R_{\mu,M}$ such that for all $R \geq R_{\mu,M}$ and $r \geq \lceil R/M \rceil$ there exists $s \in (0,1)$, $w \geq 2$ and $\varepsilon_0, \delta_0 > 0$ such that for $R_{\max} \geq 2R$ the family of functions $\zeta^{r,\pm}$ as defined in (3.8)–(3.10) are $(\varepsilon_0, \delta_0, r)$-c.d.p.'s in the sense of Definition 3.3.*

Proof. It is a direct consequence of Lemma 2.5 in [BCČ+23] that for large enough r there exists $s \in (0,1), w \geq 2$ and $\varepsilon_0 \in (0, \alpha_1)$ of Lemma 4.2 in [BCČ+23] (literally a trivial modification of this result, as $R_{\max}$ may differ from the corresponding quantity there) that there exists $\delta_0 > 0$ such that the functions $\zeta^{r,\pm}$ are $(\varepsilon_0, \delta_0, r)$-c.d.p.'s. For fixed $M \geq 1$ the lemma follows by taking $R_{\mu,M}$ to be the smallest value of R such that Lemma 2.5 and 4.2 of [BCČ+23] are applicable with $\lceil R/M \rceil$. □

Without loss of generality we can take the same value of $\varepsilon_0, \delta_0 > 0$ for the two c.d.p.'s $\zeta^{\pm,r_0}$ and $\zeta^{R,\pm}$ where r_0 is the proportion of R that is fixed in Section 5. By taking R large enough Lemma 3.6 guarantees that both $\zeta^{\pm,r_0}$ and $\zeta^{R,\pm}$ are c.d.p.'s in the sense of Definition 3.3. Note also that for $r = R$ and up to the length $2R_{\max}$ of the constant center section of (3.9)–(3.10) these functions correspond exactly to the profiles used in [BCČ+23, Section 4].

4 Proof of Proposition 2.1

We now have all the tools in order to prove Proposition 2.1. We introduce scales L_s, L_t and sets G_{conf}, G_U, in dependence of R, and show that with increasing R, the properties (i)–(ii) hold with arbitrarily high probability.

We define the spatial and temporal scales L_s, L_t as follows. First we fix $R_{\max} := c_{\text{dens}}\lceil R\log R\rceil$, where $c_{\text{dens}} := 1 + 2c_{\text{time}}$ and $c_{\text{time}} > -(d+1)/\kappa(\mu, \varepsilon_{\text{FP}})$ and then choosing

$$L_s := R_{\max} \quad \text{and} \quad L_t := T^{\text{spread}} + T^{\text{couple}}, \tag{4.1}$$

where

$$T^{\text{spread}} := \left\lceil 3c_{\text{dens}}\lceil R\log R\rceil/\lceil sR\rceil\right\rceil, \quad \text{and} \quad T^{\text{couple}} := c_{\text{time}}\lceil \log R\rceil. \tag{4.2}$$

With this choice of scales it follows immediately that the support of $\zeta_0^{R,-}$ is contained in $B_{2L_s}(0)$, since $((m_0+1)R + \lceil wR\rceil) < L_s$.

We use the c.d.p.'s $\zeta^{\pm,r_0}$ and $\zeta^{R,\pm}$, cf. (3.9)–(3.10), to define the set G_{conf} of good local configurations as follows

$$G_{\text{conf}} := \left\{ \widetilde{\eta} \in \{0,1\}^{B_{2L_s}(0)} : \begin{array}{l} \zeta_0^{R,-}(y) \le \delta_R(y;\widetilde{\eta}) \le \zeta_0^{R,+}(y) \ \text{ for } \ y \in \text{Supp}(\zeta_0^{R,-}), \\ \text{and} \\ \zeta_0^{-,r_0}(y) \le \delta_{r_0}(y;\widetilde{\eta}) \le \zeta_0^{+,r_0}(y) \ \text{ for } \ y \in \text{Supp}\left(\zeta_0^{-,r_0}\right) \end{array} \right\}. \tag{4.3}$$

The set G_{conf} should be seen as the property which replaces the *well-startedness* property of [BCČ+23], cf. (3.3). We note again that the control on the local r_0-densities that we ask for here is a technicality that is needed in Section 5 and has no analogue in the renormalisation construction of [BCČ+23].

For the sake of notational convenience we also introduce the "cylinder set" of G_{conf} defined, by the configurations that are locally in G_{conf},

$$\widehat{G}_{\text{conf}} := \left\{ \widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d} : \widetilde{\eta}|_{B_{2L_s(0)}} \in G_{\text{conf}} \right\}. \tag{4.4}$$

The introduction of the set G_U of good driving noise, is a bit more subtle. In order for properties (i)–(ii) to be satisfied we require that for any $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ such that $\Gamma(x,n) = 1$, cf. (2.6), the driving noise on $\texttt{block}_4(x,n)$ is such that it ensures the following two items:

(A) Whenever $\eta_{nL_t}|_{B_{2L_s}(L_s x)} \in G_{\text{conf}}$ the strongest control of local R-densities by the c.d.p.'s $\zeta_k^{R,\pm}$ from (3.9)–(3.10) spreads throughout the entire spatial extent of $\texttt{block}_4(x,n)$ by time T^{spread}, and the expanding control of the local r_0-densities holds throughout the block. I.e.

$$\begin{aligned} \alpha_{m_0} \le \delta_R(z;\eta_{nL_T+T^{\text{spread}}}) \le \beta_{m_0}, &\quad \text{for all } \ z \in B_{4L_s}(L_s x) \\ \zeta_k^{-,r_0}(z) \le \delta_{r_0}(z;\eta_k) \le \zeta_k^{+,r_0}(z), &\quad \text{for all } \ z \in B_{4L_s}(L_s x), k \in \{0,\dots,L_t\} \end{aligned}$$

(B) Given that the control of R-densities has spread as in the first item, the process couples successfully to any reference configuration on $B_{3L_s}(L_s x)$ in an additional T^{couple} time steps, where the reference configurations are given by

$$\mathrm{C}_{\text{ref}} = \Big\{ \eta^{\text{ref}} \in \{0,1\}^{\mathbb{Z}^d} : |\delta_R(\cdot;\eta^{\text{ref}}) - \theta_\mu| < \varepsilon_{\text{FP}} \Big\} \subseteq \widehat{G}_{\text{conf}}. \tag{4.5}$$

See also Remark 3.2 above.

Let us denote the event of (A) occurring by $A^{\text{spread}}(x,n)$, and the event of (B) occurring by $A^{\text{couple}}(x,n)$. The set G_U can then be defined implicitly as the set of driving noise configurations such that

$$\big\{ U|_{\texttt{block}_4(0,0)} \in G_U \big\} = A^{\text{spread}}(0,0) \cap A^{\text{couple}}(0,0). \tag{4.6}$$

Note that T^{spread} was chosen to be the time that it takes for the c.d.p.'s $\zeta_k^{R,\pm}$ to spread so far that $\zeta^{R,-}_{T^{\text{spread}}}|_{B_{4L_s}(0)} \equiv \alpha_{m_0}$. The probability of $A^{\text{spread}}(x,n)$ can easily be bound with the help of Lemma 3.4.

Lemma 4.1. *It holds that*

$$\mathbb{P}\big(A^{\text{spread}}(x,n)\big) \geq 1 - q^{(1)}(R,\mu), \quad (x,n) \in \mathbb{Z}^d \times \mathbb{Z},$$

where $q^{(1)}(R,\mu) \downarrow 0$ *for* $R \to \infty$.

Proof. Without loss of generality, we consider only the case $(x,n) = (0,0)$. The proof follows directly by applying a union bound and using Lemma 3.4. Indeed, if we denote by $\mathcal{C}_R$ the (spatial) R-fattening of the (truncated) cone $\bigcup_{m=1}^{T^{\text{spread}}} \text{Supp}(\zeta_m^{R,-}) \times \{m\}$ and by $\mathcal{C}_{r_0}$ the (spatial) r_0-fattening of the (truncated) cone $\bigcup_{m=1}^{L_t} \text{Supp}(\zeta_m^{-,r_0}) \times \{m\}$ then Lemma 3.4 (applied once with $r = R$ and once with $r = r_0$) gives

$$\begin{aligned} 1 - \mathbb{P}\big(A^{\text{spread}}\big(0,0)\big) &\leq \mathbb{P}\big(\exists (y,k) \in \mathcal{C}_R : U|_{B_R(y)\times\{k\}} \notin \mathcal{U}_k^R(y)\big) \\ &\quad + \mathbb{P}\big(\exists (y,k) \in \mathcal{C}_{r_0} : U|_{B_{r_0}(y)\times\{k\}} \notin \mathcal{U}_k^{r_0}(y)\big) \\ &\leq cT^{\text{spread}}\text{Volume}\big(\mathcal{C}_R\big)\exp(-c'V_R^d) \\ &\quad + cL_t\text{Volume}\big(\mathcal{C}_{r_0}\big)\exp(-c'V_{r_0}^d), \end{aligned}$$

for some constants $c, c' > 0$. As T^{spread} and L_t are of order $\log R$ and $\text{Volume}(\mathcal{C}_R)$, $\text{Volume}(\mathcal{C}_{r_0})$ are polynomial in R (recall r_0 is a fixed proportion of R), the right hand side of the above display tends to zero as R tends to infinity. □

The next lemma shows that we have a corresponding bound for the probability of $A^{\text{couple}}(x,n)$.

Lemma 4.2. *For large enough R it holds that*

$$\mathbb{P}\big(A^{\text{couple}}(x,n)\big) \geq 1 - q^{(2)}(R,\mu),$$

where $q^{(2)}(R,\mu) \downarrow 0$ for $R \to \infty$.

Proof. With out loss of generality we set again $(x,n) = (0,0)$. The proof is an adaptation of arguments that can already be found in [BCČ$^+$23, Lemma 4.6].

We start with the trivial observation that $A^{\text{couple}}(0,0)$ gives conditions on the behaviour of particle configurations $\widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d}$ which satisfy the local density condition

$$\zeta^{R,-}_{T^{\text{spread}}}(y) < \delta_R(y;\widetilde{\eta}) < \zeta^{R,+}_{T^{\text{spread}}}(y), \quad \text{for all } y \in \text{Supp}(\zeta^{R,-}_{T^{\text{spread}}}). \tag{4.7}$$

For later convenience we denote the set of all such particle configuration by $\widehat{G}^{(2)}_{\text{conf}}$. Moreover, we use the following convention throughout the proof: $\widetilde{\eta}_n := \Phi_{0,n}(\widetilde{\eta})$ for any $\widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d}$. With this notation we can write

$$A^{\text{couple}}(0,0)^c = \Big\{\exists y \in B_{3L_s}(0), \exists \widetilde{\eta} \in \widehat{G}^{(2)}_{\text{conf}}, \exists \eta^{\text{ref}} \in \text{C}_{\text{ref}} : \widetilde{\eta}_{T^{\text{couple}}}(y) \neq \eta^{\text{ref}}_{T^{\text{couple}}}(y)\Big\}$$

The proof of the lemma is based on the following calculation. Recall the the definition of the σ-algebras from (2.2). For any $y \in \mathbb{Z}^d$ and $k > 1$ (cf. (2.2)) it holds by Markov's inequality that

$$\begin{aligned}
&\mathbb{P}\Big(\exists \eta^{(1)}, \eta^{(2)} \in \widehat{G}^{(2)}_{\text{conf}} : \eta^{(1)}_k(y) \neq \eta^{(2)}_k(y)\Big|\mathcal{G}_{0,k}\Big) \leq \mathbb{E}\Big[\sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \widetilde{\eta}_k(y) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \widetilde{\eta}_k(y)\Big|\mathcal{G}_{0,k}\Big] \\
&= \mathbb{E}\Big[\sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \mathbb{1}_{\{U(y,k)\leq\varphi_\mu(\delta_R(y;\widetilde{\eta}_{k-1}))\}} - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \mathbb{1}_{\{U(y,k)\leq\varphi_\mu(\delta_R(y;\widetilde{\eta}_{k-1}))\}}\Big|\mathcal{G}_{0,k}\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{U(y,k)\leq\sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}}\varphi_\mu(\delta_R(y;\widetilde{\eta}_{k-1}))\}} - \mathbb{1}_{\{U(y,k)\leq\inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}}\varphi_\mu(\delta_R(y;\widetilde{\eta}_{k-1}))\}}\Big|\mathcal{G}_{0,k}\Big] \\
&= \mathbb{P}\Big(U(y,k) \leq \Big|\sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big)\Big|\Big|\mathcal{G}_{0,k}\Big) \\
&= \Big|\sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\text{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big)\Big|
\end{aligned} \tag{4.8}$$

The second equality holds, because the supremum and infinmum are really a maximum and minimum over what happens in the R-neighbourhood of y, which only involves finitely many local configurations.

Now, if we had uniformly in the configurations in $\widehat{G}^{(2)}_{\text{conf}}$ that $|\delta_R(y;\Phi_{0,k-1}(\cdot)) - \theta_\mu| < \varepsilon_{\text{FP}}$, i.e. the local density around x, at time $k-1$ were uniformly in the region where

φ_μ is a contraction, cf. Lemma 3.1(c), then we would get

$$\begin{aligned}
&\Big| \sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \varphi_\mu\big(\delta_R(y;\widetilde{\eta}_{k-1})\big)\Big| \\
&\qquad \le \kappa(\mu,\varepsilon_{\mathrm{FP}})\Big| \sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \delta_R(y;\widetilde{\eta}_{k-1}) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \delta_R(y;\widetilde{\eta}_{k-1})\Big| \\
&\qquad \le \kappa(\mu,\varepsilon_{\mathrm{FP}})V_R^{-d} \sum_{y_1\in B_R(y)} \Big| \sup_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}_{k-1}(y_1) - \inf_{\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}_{k-1}(y_1)\Big|,
\end{aligned} \tag{4.9}$$

and we could (contingent on having corresponding density control in a slightly larger region, i.e. an R-fattening of the region, in order to apply Lemma 3.1(c) again) iterate this calculation.

In order to formalise the uniform density control, that lets us apply Lemma 3.1(c) in the above, we introduce for any $r\in\mathbb{N}$ functions $\psi_r : \{0,1\}^{\mathbb{Z}^d}\to\{0,1\}$ with

$$\psi_r(\widetilde{\eta}) = \mathbb{1}_{\{\delta_R(z;\widetilde{\eta})\in[\theta_\mu-\varepsilon_{\mathrm{FP}},\theta_\mu+\varepsilon_{\mathrm{FP}}] \text{ for all } z\in B_r(0)\}}, \quad \widetilde{\eta}\in\{0,1\}^{\mathbb{Z}^d},$$

and the event

$$\Psi_{T^{\mathrm{couple}}} := \Big\{\psi_{3L_s+T^{\mathrm{couple}}\lceil sR\rceil+(T^{\mathrm{couple}}-l)R}(\widetilde{\eta}_{T^{\mathrm{couple}}-l}) = 1 : l = 1,\dots,T^{\mathrm{couple}}, \widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}\Big\}. \tag{4.10}$$

On this event, we have the necessary density control in order to iterate the calculation in (4.8)–(4.9).

It is not hard to see that $\Psi_{T^{\mathrm{couple}}}$ has high probability. Note that for any $\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}$ it holds that $\psi_{4L_s}(\widetilde{\eta}) = 1$ and and by definition

$$T^{\mathrm{couple}}\lceil sR\rceil + T^{\mathrm{couple}}R = c_{\mathrm{time}}\lceil sR\rceil\lceil\log R\rceil + c_{\mathrm{time}}\lceil\log R\rceil R \le 2c_{\mathrm{dens}}\lceil R\log R\rceil = L_s,$$

so that the case $l = T^{\mathrm{couple}}$ in (4.10) is satisfied. Then using the same argument as in the proof of Lemma 4.1, this density control by the $\zeta_k^{R,-}$ profiles spreads by $\lceil sR\rceil$ in every time step, with high probability, such that

$$\mathbb{P}(\Psi_{T^{\mathrm{couple}}}) \ge 1 - \widetilde{q}^{(1)}(R,\mu),$$

for some $\widetilde{q}^{(1)}(R,\mu) > 0$ such that $\lim_{R\to\infty}\widetilde{q}^{(1)}(R,\mu) = 0$.

With this we can now prove the lemma. We consider the probability of the complement of $A^{\mathrm{couple}}(0,0)$ conditioned on $\Psi_{T^{\mathrm{couple}}}$,

$$\begin{aligned}
&\mathbb{P}\Big(A^{\mathrm{couple}}(0,0)^c\Big|\Psi_{T^{\mathrm{couple}}}\Big) \\
&=\mathbb{P}\Big(\exists y\in B_{3L_s}(0), \exists\widetilde{\eta}\in\widehat{G}^{(2)}_{\mathrm{conf}}, \exists\eta^{\mathrm{ref}}\in\mathrm{C}_{\mathrm{ref}} : \widetilde{\eta}_{T^{\mathrm{couple}}}(y)\neq\eta^{\mathrm{ref}}_{T^{\mathrm{couple}}}(y)\Big|\Psi_{T^{\mathrm{couple}}}\Big).
\end{aligned}$$

Moreover, since $\mathrm{C}_{\mathrm{ref}} \subseteq \widehat{G}^{(2)}_{\mathrm{conf}}$, cf. (4.5) and (4.7), and writing $T = T^{\mathrm{couple}}$ in the following, the right hand side of the last display is bounded from above by

$$\mathbb{P}\Big(\max_{y \in B_{3L_s}(0)} \big(\sup_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_T)\widetilde{\eta}_T(y) - \inf_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_T)\widetilde{\eta}_T(y) \big) \geq 1 \Big| \Psi_T \Big). \tag{4.11}$$

This probability can now be dealt with as in the iteration of (4.8)–(4.9). Indeed for any $k \in \{1, \dots, T\}$ one gets with a union bound and Markov's inequality that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{P}\Big(\max_{y \in B_{3L_s}(0)} \big(\sup_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_k)\widetilde{\eta}_k(y) - \inf_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_k)\widetilde{\eta}_k(y) \big) \geq 1 \Big| \mathcal{G}_{0,k} \Big) \Big| \Psi_T \Big] \\
&\leq \Big[\sum_{y \in B_{3L_s}(0)} \mathbb{E}\Big[\sup_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_k)\widetilde{\eta}_k(y) - \inf_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \psi_{L_s}(\widetilde{\eta}_k)\widetilde{\eta}_k(y) \big) \Big| \mathcal{G}_{0,k} \Big] \Big| \Psi_T \Big] \\
&\leq \kappa(\mu, \varepsilon_{\mathrm{FP}}) V_R^{-d} \Big[\sum_{y \in B_{3L_s}(0)} \sum_{y_1 \in B_R(y)} \mathbb{E}\big[\sup_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}_{k-1}(y_1) - \inf_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}_{k-1}(y_1) \big| \mathcal{G}_{0,k} \big] \Big| \Psi_T \Big] \\
&\leq \kappa(\mu, \varepsilon_{\mathrm{FP}})^k V_R^{-dk} \mathbb{E}\Big[\sum_{y \in B_{3L_s}(0)} \sum_{y_1 \in B_R(y)} \sum_{y_2 \in B_R(y_1)} \cdots \\
&\qquad\qquad \sum_{y_k \in B_R(y_{k-1})} \Big(\sup_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}(y_k) - \inf_{\widetilde{\eta} \in \widehat{G}^{(2)}_{\mathrm{conf}}} \widetilde{\eta}(y_k) \Big) \Big| \Psi_T \Big] \\
&\leq \kappa(\mu, \varepsilon_{\mathrm{FP}})^k V_{3L_s}^d.
\end{aligned}$$

By the choice of c_{time}, see above (4.1), the last line tends to zero as R tends to infinity. The claim follows, as

$$\begin{aligned}
\mathbb{P}\big(A^{\mathrm{couple}}(0,0)\big) &= \mathbb{P}\big(A^{\mathrm{couple}}(0,0) \big| \Psi_{T^{\mathrm{couple}}}\big) \times \mathbb{P}\big(\Psi_{T^{\mathrm{couple}}}\big) \\
&\geq \big(1 - \kappa(\mu, R)^{T^{\mathrm{couple}}} V_{3L_s}^d\big)\big(1 - \widetilde{q}^{(1)}(R, \mu)\big),
\end{aligned}$$

the right hand of which can be made to be arbitrarily close to one, by choosing R large. □

By Lemmas 4.1 and 4.2 the proof of Proposition 2.1 follows directly with

$$\widetilde{R}_{\mu,\varepsilon} := \inf\{R > 0 : \max\{q^{(1)}(R, \mu), q^{(2)}(R, \mu)\} < \varepsilon\}.$$

5 Proof of Proposition 2.3

Let $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ be such that $\Gamma(x, n) = 1$. In order to prove Proposition 2.3 we require uniform in $z \in B_{L_s/2}(L_s x)$ control of the conditional probabilities

$$P_\eta\Big(\max_{(n-1)L_t < k \leq nL_t} \|X_k - z\| \geq L_s/4 \Big| X_{(n-1)L_t} = z, \Gamma(x, n) = 1 \Big). \tag{5.1}$$

Without loss of generality we assume that $x = 0$ and $n = -1$, such that $z \in B_{L_s/2}(0)$ and set for readability $P^z_\eta(\cdot) := P_\eta(\cdot|X_0 = z, \Gamma(0,-1) = 1)$. In order to upper bound (5.1) we split the random walk increments into a martingale and non-martingale part, i.e.

$$X_k - X_{k-1} = E^z_\eta\big[X_k - X_{k-1}\big|X_{k-1}\big] + Y_k, \tag{5.2}$$

where Y_k is the martingale part. We can thus write

$$X_k - z = \sum_{i=1}^{k} X_i - X_{i-1} = \sum_{i=1}^{k} E^z_\eta\big[X_i - X_{i-1}\big|X_i\big] + \sum_{i=1}^{k} Y_i. \tag{5.3}$$

Let us introduce the event

$$A_{\text{mart}} := \Big\{ \max_{0<k\le L_t} \Big\| \sum_{i=1}^{k} Y_i \Big\| \ge L_s/8 \Big\}. \tag{5.4}$$

Using (5.3) and (5.4) we can thus bound (5.1) as follows

$$\begin{aligned} &P^z_\eta\big(\max_{0<k\le L_t} \|X_k - z\| \ge L_s/4\big) \\ &\quad\le P^z_\eta\Big(\max_{0<k\le L_t} \Big\| \sum_{i=1}^{k} E^z_\eta[X_i - X_{i-1}|X_i] \Big\| + \max_{0<k\le L_t} \Big\| \sum_{i=1}^{k} Y_i \Big\| \ge L_s/4 \Big) \\ &\quad\le P^z_\eta\Big(\max_{0<k\le L_t} \sum_{i=1}^{k} \|E^z_\eta[X_i - X_{i-1}|X_i]\| \ge L_s/8, A^c_{\text{mart}} \Big) + P^z_\eta(A_{\text{mart}}) \end{aligned} \tag{5.5}$$

To deal with the first summand in the last line of (5.5) we first claim that on A^c_{mart} it holds for all $k \in \{0, \dots, L_t\}$ that $\|X_k - z\| \le L_s/2$. To see this, note first that on A^c_{mart} we can write for any $k \in \{1, \dots, L_t\}$

$$\|X_k - z\| \le L_s/8 + \sum_{i=1}^{k} \|E^z_\eta[X_i - X_{i-1}|X_{i-1}]\|. \tag{5.6}$$

We can bound the sum of expected differences using the following lemma, the proof of which we postpone to the end of this section.

Lemma 5.1. *For $k \in \{1, \dots, L_t\}$ it holds on $\{\|X_{k-1} - z\| \le L_s/2\}$ that*

$$\|E^z_\eta[X_k - X_{k-1}|X_{k-1}]\| < \frac{L_s}{8L_t}. \tag{5.7}$$

Now since $X_0 = z$ we can apply Lemma 5.1 and it follows with (5.6) that $\|X_1 - z\| \le L_s/8 + \frac{L_s}{8L_t} < L_s/8 + L_s/8 = L_s/4$. Thus applying Lemma 5.1 and (5.6) inductively yields

$$\|X_k - z\| \le L_s/8 + \frac{kL_s}{8L_t} \le L_s/4, \quad k \in \{1, \dots, L_t\}.$$

In particular this implies that on A^c_{mart} that

$$\max_{0<k\le L_t}\sum_{i=1}^{k}\|E^z_\eta[X_i - X_{i-1}|X_i]\| < L_s/8,$$

which in turn implies that the first sum in the last line of (5.5) vanishes.

The probability of A_{mart} can be dealt with by applying the Azuma-Hoeffding inequality to the partial sum process of $(Y_k)_{k\ge1}$. More precisely, if $S_k := \sum_{i=1}^k Y_i$ and $S_0 = 0$, then it follows by definition, cf. (5.2), that $\|S_k - S_{k-1}\| = \|Y_k\| \le R$ and thus

$$P^z_\eta\big(\|S_k - S_0\| \ge L_s/8\big) \le \exp\Big(\frac{-(L_s/8)^2}{2L_tR^2}\Big), \quad k \in \{0,\dots,L_t\}.$$

By a union bound, it follow that

$$P^z_\eta(A_{\mathrm{mart}}) \le L_t \exp\Big(\frac{-(L_s/8)^2}{2L_tR^2}\Big).$$

The right hand side of the last display is, by the choice of L_t and L_s in (4.1) of order $O(\log(R)/R)$ which tends to zero for R large. We thus set $R_{\mu,\delta,\varepsilon}$ to be the smallest R such that Proposition 2.1 holds and that the above display is smaller than a given $\delta > 0$ as in the statement of Proposition 2.3, i.e. for given $\varepsilon, \delta > 0$ we set

$$R_{\mu,\delta,\varepsilon} := \inf\big\{R \in \mathbb{N} : R > \widetilde{R}_{\mu,\varepsilon}, L_t \exp\big(-(L_s/8)^2/(2L_tR^2)\big) < \delta\big\}.$$

It still remains to prove Lemma 5.1 in order to complete the proof of Proposition 2.3.

Proof of Lemma 5.1. By the definition of a good block (i.e. $\Gamma(0,-1) = 1$) the driving noise $U|_{\mathtt{block}_4(0,-1)}$ is such that the following local R- and r_0-density conditions are satisfied for $k \in \{0,\dots,L_t\}$,

$$\begin{aligned}&\zeta_k^{R,-}(x) \le \delta_R(x;\eta_{-L_t+k}) \le \zeta_k^{R,+}, \quad \text{for all } x \in \mathrm{Supp}(\zeta_k^{R,-}),\\ &\zeta_k^{-,r_0}(x) \le \delta_{r_0}(x;\eta_{-L_t+k}) \le \zeta_k^{+,r_0}, \quad \text{for all } x \in \mathrm{Supp}(\zeta_k^{-,r_0}),\end{aligned} \tag{5.8}$$

and in particular, by the definition of the $\zeta_k^{R,\pm}$, $\zeta_k^{\pm,r_0}$ profiles, cf. (3.9)–(3.10), it holds that

$$\delta_R(x;\eta_k), \delta_{r_0}(x;\eta_k) \in [\theta_\mu - \varepsilon_{\mathrm{FP}}, \theta_\mu + \varepsilon_{\mathrm{FP}}], \quad (x,k) \in B_{L_s}(0) \times \{-L_t,\dots,0\} \tag{5.9}$$

Recall that we assumed with out loss of generality that r_0 divides R. Let $M > 1$ denote the resulting quotient, i.e. let $M > 1$ be such that $R/r_0 = M$. Then the ball $B_R(0)$

can be divided into M^d sub-balls of radius r_0 centered at points $y_1, \dots, y_{M^d} \in B_R(0)$. Now by (5.9) it follows immediately that for any $m \in \{1, \dots, M^d\}$.

$$\begin{aligned} P_\eta^z(X_k - X_{k-1} \in B_{r_0}(y_m) | X_{k-1}) &= \frac{\delta_{r_0}(y_m + X_{k-1}; \eta_{-k-1}) V_{r_0}^d}{\delta_R(X_{k-1}; \eta_{-k-1}) V_R^d} \\ &\in \Big[(1 - \widetilde{\varepsilon}) \frac{1}{M^d}, (1 + \widetilde{\varepsilon}) \frac{1}{M^d}\Big], \end{aligned}$$

for some $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\varepsilon_{\mathrm{FP}}, R, M, d) > 0$ which decreases to zero for $R \to \infty$. Moreover, denote on $\{X_k - X_{k-1} \in B_{r_0}(y_m)\}$ the relative displacement of $X_k - X_{k-1}$ with respect to y_m by z_m. In particular it holds that $\|z_m\| \leq r_0$. Then we have that

$$\begin{aligned} \big\| E_\eta^z[X_k - X_{k-1} | X_{k-1}] \big\| &= \Big\| \sum_{m=1}^{M^d} E_\eta^z \big[(y_m + z_m) \mathbb{1}_{\{(X_k - X_{k-1} \in B_{r_0}(y_m)\}} | X_{k-1} \big] \Big\| \\ &\leq \Big\| \sum_{m=1}^{M^d} E_\eta^z \big[y_m \mathbb{1}_{\{(X_k - X_{k-1} \in B_{r_0}(y_m)\}} | X_{k-1} \big] \Big\| \\ &\quad + \Big\| \sum_{m=1}^{M^d} z_m P_\eta^z (X_k - X_{k-1} \in B_{r_0}(y_m) | X_{k-1}) \Big\| \\ &\leq \widetilde{\varepsilon} + (1 + \widetilde{\varepsilon}) r_0 \leq 2 r_0. \end{aligned} \tag{5.10}$$

Recall that $L_s = c_{\mathrm{dens}} \lceil R \log R \rceil$. If we write c_1 for the constant such that $L_t = c_1 \log R$ it follows for any fixed $M > 16 c_1 / c_{\mathrm{dens}}$ that $2 r_0 = 2R/M < L_s / 8 L_t$. □

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