Birational Transformations of Threefolds

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CHAPTER I

Introduction

In its fundamental form, Algebraic Geometry deals with the study of algebraic varieties, that is zero loci of polynomials in $\mathbf{k}[x_0, \ldots, x_n]$, over a field $\mathbf{k}$. As with any field in Mathematics, the ultimate goal of the theory is the classification of the objects of interest. While this goal is way beyond our reach, a major step towards this is the classification of varieties up to birational equivalence.

The modern framework for birational classification is known as the *Minimal Model Program* (MMP for short). This theory was initiated by the groundbreaking work of Mori [Mor88], for which he was awarded the Fields medal, and settled in most cases of interest by Birkar, Cascini, Hacon and McKernan [BCHM10]. In a nutshell, it is a non-deterministic algorithm that associates to any variety a "simpler variety", birational to the starting one. The outputs of the MMP fall into the following two classes: *minimal models* and ***Mori fiber spaces*** (MFS for short). When the starting variety is rational, or more generally is covered by rational curves, any output of the MMP is a Mori fiber space.

A fundamental, though heuristic, distinction between these two classes is that while minimal models posses a more complicated geometry, the birational relations among them are simpler compared to those among Mori Fiber Spaces. The main tool for the study of birational relations among Mori fiber spaces is the ***Sarkisov Program***. This is, again, an algorithm, in the framework of MMP, that decomposes any birational map between Mori Fiber Spaces into *"simpler"* ones, called ***Sarkisov links***. Ideas for the Sarkisov program stem from the works of Sarkisov and Reid, ultimately being proven by Corti [Cor95] in dimension 3 and later Hacon and McKernan [HM13] in any dimension.

The fundamental example that highlights the aforementioned distinction between minimal models and Mori fiber spaces, is the case of the projective n-space $\mathbb{P}^n$: while its geometry is as simple as possible, its group of birational transformations, the ***Cremona group*** $\mathrm{Cr}_n(\mathbf{k})$, is an immensely complicated object. The Cremona groups have been an object of interest dating back to the 19th century, and apart from the lower dimensional cases, their structure remains a mystery, with many fundamental questions still open or only settled very recently.

As mentioned earlier, the main tool for the study of groups of birational transformations of Mori fiber spaces is the Sarkisov program. Until very recently the Sarkisov program has been utilized only in the study of Mori fiber spaces that admit few, if any, Sarkisov links to other Mori Fiber Spaces (see [CPR00, AZ16, AK16]). However, the theory of **rank r fibrations** developed by Blanc, Lamy and Zimmermann [BLZ21], building upon ideas of Kaloghiros [Kal13], set up a framework to deal with cases when there is an abundance of Sarkisov links, one such being the projective space $\mathbb{P}^n$.

In a nutshell, rank r fibrations encode the relations among Sarkisov links. The first non-trivial case is that of rank 3 fibrations, with the resulting relations being called ***elementary relations***. Similarly to Sarkisov links being the building blocks of all birational maps between Mori fiber spaces, elementary relations are the building blocks of all relations among Mori fiber spaces. Essentially, we have the presentation

$$\mathrm{BirMori}(X) = \left\langle \begin{array}{c} \text{Sarkisov links between} \\ \text{MFS's birational to } X \end{array} \middle| \begin{array}{c} \text{elementary} \\ \text{relations} \end{array} \right\rangle,$$

where $\mathbf{BirMori}(X)$ denotes the groupoid of all birational maps between Mori fiber spaces birational to X.

In this thesis we explore various topics in higher dimensional birational geometry (dimension 3 or more) concerning Mori fiber spaces and their groups of birational transformations. In Chapter II we begin with some preliminaries which shall be used throughout the rest of the text. This includes notions from *Intersection Theory*, some standard results on the various cones of divisors, as well as some machinery on algebraic groups. We also give an overview of the *Minimal Model Program* and discuss some themes in the *Sarkisov Program*.

In Chapter III we study Sarkisov links starting from the projective space $\mathbb{P}^3$. This comes as natural continuation of the results of [BL12]. There the authors classified all smooth curves in $\mathbb{P}^3$ whose blowup $X \to \mathbb{P}^3$ is weak Fano and fits into a Sarkisov link. Along the way, they implicitly proved that any curve that lies in a plane or a quadric surface induces a link if and only if X is Fano (and in particular weak Fano). In Chapter III we work in a similar setting, dropping the restriction that X is weak-Fano but imposing that the curve C is contained in a cubic surface S. After obtaining a list of 6 families of such curves, we explicitly construct the respective Sarkisov links. We finally study some properties of the targets of the links such as their Mori fiber space structure, their singularities and their anti-canonical degree. This article has been accepted for publication in Publicacions Matemàtiques and can be found as a preprint [Zik20].

Chapter IV is concerned with the study of the Cremona group $\mathrm{Cr}_3(\mathbb{C})$ as well as the group of birational transformations $\mathrm{Bir}(X)$ of a smooth cubic 3-fold $X \subset \mathbb{P}^4$. More specifically, we obtain homomorphisms from the aforementioned groups to an uncountably indexed free product of $\mathbb{Z}/2\mathbb{Z}$s. Such homomorphisms were previously constructed in [BLZ21], however our techniques are more explicit in nature, thus reproving the non-simplicity of $\mathrm{Cr}_3(\mathbb{C})$ in an effective way. We furthermore obtain a more precise free product structure, which in turn allows us to prove that the automorphism groups of $\mathrm{Cr}_3(\mathbb{C})$ and $\mathrm{Bir}(X)$ are not generated by inner and field automorphisms. To obtain these results, we utilize the theory of rank r fibrations to study the relations involving certain families of birational involutions. The standalone article for this chapter is [Zik21]

In Chapter V we study algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^n)$, where C is a smooth curve of positive genus and $n \geq 2$. Namely, we show that there exists connected algebraic subgroups which are not contained in a maximal one. This result falls on the other end of the spectrum, as the classification of maximal subgroups of $\mathrm{Cr}_2(\mathbb{C}) = \mathrm{Bir}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)$ and $\mathrm{Cr}_3(\mathbb{C}) = \mathrm{Bir}\left(\mathbb{P}^1 \times \mathbb{P}^2\right)$ (see [Enr93], [Ume80, Ume82a, Ume82b, Ume85]) imply that every connected algebraic subgroup of these groups is contained in a maximal one. The proof uses standard tools from the theory of algebraic groups together with an equivariant version of the Sarkisov program (see [Flo20]). More specifically, we classify all equivariant Sarkisov links from our starting variety and use them as building blocks to embed any given algebraic subgroup in a larger one. This is joint work with Pascal Fong and the preprint is [FZ22].

CHAPTER II

Preliminaries

The main topic of the thesis revolves around the study of birational transformations of 3-folds (i.e. 3 dimensional projective varieties). 3-folds fall under the umbrella of higher dimensional algebraic geometry. Given that, most of the tools of this chapter will be developed in full generality regarding the dimension. We will only sporadically focus on 3-folds. All varieties and morphisms considered are projective and defined over the field of complex numbers $\mathbb{C}$.

1. Some intersection theory

In this section X will denote a smooth variety. Given two subvarieties A and B of X, we generically expect their set theoretic intersection to be a subvariety of codimension $\operatorname{codim}(A) + \operatorname{codim}(B)$. However we would like to attach to the irreducible components of $A \cap B$ certain multiplicities to encode possible tangencies. Intersection theory is the tool to make this precise. The main references for this section are [EH16, Chapter 1] and [Deb01, Chapter 1].

Definition 1.1 (Chow group)**.**

1. *The* ***group of cycles*** $Z(X)$ *of* X *is the free abelian group generated by the set of irreducible subvarieties of* X*. It is naturally graded by dimension, so that* $Z(X) = \bigoplus_k Z_k(X)$*. A* k***-cycle*** *is an element of* $Z_k(X)$*.*
2. *Two* k*-cycles* A_0*,* A_1 *are called* ***rationally equivalent*** *if there is a cycle on* $\mathbb{P}^1 \times X$ *whose restriction to the fibers* $\{t_0\} \times X$ *and* $\{t_1\} \times X$ *is* A_0 *and* A_1 *respectively, where* $t_0, t_1 \in \mathbb{P}^1$*.*
3. *The* ***Chow group*** $A(X)$ *of* X *is the group of cycles modulo rational equivalence. Similarly*

to $Z(X)$, $A(X)$ is naturally graded by dimension, or equivalently by codimension so that $A(X) = \bigoplus_c A^c(X)$.

Remark 1.2 ([EH16, Proposition 1.10])**.** *More simply, two k-cycles A_0, A_1 are rationally equivalent if there exists a $(k+1)$-dimensional subvariety Y of X, so that A_0 and A_1 are linearly equivalent as divisors in Y.*

Two subvarieties $A, B \subset X$ are said to **intersect transversely at** p if they are smooth at that point and we have

$$\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim}(T_pA) + \operatorname{codim}(T_pB)$$

when seen as subspaces of T_pX.

We say that they intersect **generically transversely** if they intersect transversely at an open dense subset of points of their intersection. Two cycles $A = \sum n_i A_i, B = \sum m_j B_j$ intersect **generically transversely** if each A_i intersects each B_j generically transversely.

Theorem 1.3. *There is a unique product structure on $A(X)$ satisfying the condition that if two subvarieties A, B are generically transverse, then*

$$[A][B] = [A \cap B].$$

This structure makes

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

*into an associative, commutative ring, graded by codimesion called the **Chow ring** of X.*

Moreover, there exists a unique map

$$\deg\colon A(X) \to \mathbb{Z}$$

that maps the class $[p]$ of any $p \in X$ to 1 and vanishes on any cycle of positive pure dimension.

When two cycles A and B have complementary dimensions, by abuse of notation, we will write $A \cdot B = n \in \mathbb{Z}$ to mean that $\deg([A][B]) = n$.

1.1. The Chow-ring of some blowups

Proposition 1.4 ([IP99, Lemma 2.2.14])**.** *Let X be a smooth 3-fold and $z \subset X$ be a smooth subvariety. Denote by $\pi\colon X' \to X$ the blowup along z with exceptional divisor E. Let f be the class of a line in E if $E \cong \mathbb{P}^2$ and the class of a fiber if E is a ruled surface. Then:*

1. *$A(X') = \pi^* A(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$ as an additive group; moreover $\pi_* E = \pi_* f = 0$, and $\pi_* \pi^* A(X) = A(X)$.*
2. *The multiplicative structure of $A(X')$ is determined by the following:*

(a) if z is a point, then

$$E^2 = -f, \quad E^3 = E^2 \cdot f = -1, \quad E \cdot \pi^* Z = \pi^* Z \cdot f = 0$$

for any cycle $Z \in A(X)$;

(b) if $z = C$ is a curve, then

$$\begin{aligned} &E^2 = -\pi^*(C) + \deg(N_{C/X}) f, \;\; E^3 = -\deg(N_{C/X}), \;\; E \cdot f = -1, \\ &E \cdot \pi^* D = (D \cdot C) f, \quad \pi^* D \cdot f = 0, \quad \forall D \in A^1(X), \\ &E \cdot \pi^* Z = \pi^* Z \cdot f = 0, \quad \forall Z \in A^2(X), \end{aligned}$$

where $N_{C/X}$ denotes the normal bundle of C in X. In addition, we have the relation:

$$\deg(N_{C/X}) = 2g(C) - 2 - K_X \cdot C,$$

where $g(C)$ denotes the genus of C.

1.2. Singular case

When X is singular intersection theory fails in its full generality. However we can still define the intersection of $\mathbb{Q}$-Cartier divisors with curves [1]. We first recall the definition of a $\mathbb{Q}$-Cartier divisor.

Definition 1.5. *A divisor D is $\mathbb{Q}$-**Cartier** if some multiple of it is Cartier. X is called $\mathbb{Q}$-**factorial** (resp. $\mathbb{Q}$-**Gorenstein**) if all Weil divisors are (resp. the canonical divisor is) $\mathbb{Q}$-Cartier.*

Example 1.6. *Consider the quadric cone $Q = \{x_0x_2 - x_1x_3 = 0\} \in \mathbb{P}^4$ and the planes $P_1 = \{x_0 = x_1 = 0\}$ and $P_3 = \{x_0 = x_3 = 0\}$. One may check, that P_1, when viewed as a Weil divisor on Q, is not a Cartier divisor: there is no neighbourhood of the point $p = (0:0:0:0:1)$ such that P_1 is given by 1 equation. However, $P_1 \sim P_3$ and $2P_1 \sim P_1 + P_3$ is given globally by $x_0 = 0$, and thus it is $\mathbb{Q}$-Cartier. We conclude that X is not factorial, it is however $\mathbb{Q}$-factorial.*

Let $C \subset X$ be a curve and D be a $\mathbb{Q}$-Cartier divisor so that nD is Cartier for some $n \in \mathbb{N}$. Denote by $i\colon C \to X$ the inclusion morphism and by $\nu\colon \hat{C} \to C$ the normalization of C. Then

$$D \cdot C = \frac{1}{n} \deg\big(\nu^* i^*(\mathcal{O}(nD))\big).$$

It can happen that the intersection number between a $\mathbb{Q}$-Cartier divisor and a curve is not an integer.

Example 1.7. *In the setting of Example 1.6 consider the line l parameterized by*

$$\begin{array}{rcl} i\colon \mathbb{P}^1 & \to & Q \\ (u_0 : u_1) & \mapsto & (u_0 : 0 : 0 : 0 : u_1). \end{array}$$

Then $\deg(i^*\mathcal{O}(2P_1)) = \deg(\mathcal{O}_{\mathbb{P}^1}(1)) = 1$ and thus $P_1 \cdot l = \frac{1}{2}$.

[1] or more generally, the intersection of d $\mathbb{Q}$-Cartier divisors with $d = \dim(X)$, see [Deb01, Section 1.2]

2. Cones and morphisms

The main references for this section are [Deb01, Chapter 1] and [Mat02, Exercise 3-5-1].

Definition 2.1. *Two Cartier divisors D, D' on X are said to be* ***numerically equivalent*** *(denoted by $D \equiv D'$) if for any $C \in A_1(X)$ we have $D \cdot C = D' \cdot C$. The group of Cartier divisors up to numerical equivalence is denoted by $N^1(X)_{\mathbb{Z}}$. We set*

$$N^1(X)_{\mathbb{Q}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{Q} \;\; \textit{and} \;\; N^1(X) = N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}.$$

Dually, two curves C, C' are said to be ***numerically equivalent*** *(denoted by $C \equiv C'$) if for any $D \in A^1(X)$ we have $D \cdot C = D \cdot C'$. The group $A_1(X)$ up to numerical equivalence is denoted by $N_1(X)_{\mathbb{Z}}$. We set*

$$N_1(X)_{\mathbb{Q}} := N_1(X)_{\mathbb{Z}} \otimes \mathbb{Q} \;\; \textit{and} \;\; N_1(X) = N_1(X)_{\mathbb{R}} := N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}.$$

We finally define the ***Picard rank*** *$\rho(X)$ of X to be the dimension of the vector space $N^1(X)$ (or equivalently its dual $N_1(X)$).*

Inside these vector spaces lie a number of interesting cones. Here we single out the cone of curves and its closure, the Mori cone.

Definition 2.2. *The* ***cone of curves*** *$\mathrm{NE}(X) \subset N_1(X)$ is the cone spanned by effective 1-cycles, i.e. cycles $C = \sum n_i C_i$ with $n_i \geq 0$. Its closure $\overline{\mathrm{NE}}(X)$ with respect to the Euclidean topology on $N_1(X)$ is known as the* ***Mori cone****, or the cone of pseudo-effective curves.*

We will also make use of the relative versions of these objects. Before we introduce them, we will need the so called **projection formula**.

Let $\pi \colon X \to Y$ be a morphism and C be a curve on X. We define the 1-cycle $\pi_* C$ as

$$\pi_* C := \begin{cases} 0, & \text{if } C \text{ is contracted by } \pi \\ d\pi(C), & \text{otherwise,} \end{cases}$$

where $d = \deg(\pi|_C)$.

We then have the following:

Proposition 2.3. *Let $\pi \colon X \to Y$ be a morphism, C a curve in X and D a Cartier divisor on Y. Then*

$$\pi^* D \cdot C = D \cdot \pi_* C.$$

In particular, if C is a curve contracted by π and D is any Cartier divisor on Y, $\pi^* D \cdot C = 0$.

We can now define a new equivalence relation on $N^1(X)$ and $N_1(X)$:

Definition 2.4. *Let $\pi \colon X \to Y$ be a morphism. Denote by $N_1(X/Y)$ the subspace of $N_1(X)$ spanned by classes of contracted curves, i.e. classes $\sum n_i C_i$ with $\pi_* C_i = 0$. Two Cartier divisors D, D' are said to be **numerically equivalent over** Y (denoted by $D \equiv_Y D'$) if $D \cdot C = D' \cdot C$, for all $C \in N_1(X/Y)$. We denote by $N^1(X/Y)$ the quotient of $N^1(X)$ with respect to this pairing.*

*Similarly, we define the **relative cone of curves** $\mathrm{NE}(X/Y)$ (or more commonly denoted by $\mathrm{NE}(\pi)$) to be the cone in $N_1(X)$ spanned by the classes of contracted curves (which can also be seen as a cone in $N_1(X/Y)$).*

Note that $\mathrm{NE}(\pi) = \mathrm{NE}(X) \cap \ker(\pi_*)$ and so $\mathrm{NE}(\pi)$ is closed in $\mathrm{NE}(X)$. The reason why the notation $\mathrm{NE}(\pi)$ is to be preferred will become evident in the next proposition:

Proposition 2.5 ([Deb01, Proposition 1.14])**.** *Let $\pi \colon X \to Y$ be a morphism.*

1. *The subcone $\mathrm{NE}(\pi)$ of $\mathrm{NE}(X)$ is extremal.*
2. *Assume that $\pi_* \mathcal{O}_X \cong \mathcal{O}_Y$[2] and let $\pi' \colon X \to Y$ be another morphism.*
   - *If $\mathrm{NE}(\pi)$ is contained in $\mathrm{NE}(\pi')$, there is a unique morphism $f \colon Y \to Y'$, such that $\pi' = f \circ \pi$.*
   - *The morphism π is uniquely determined by $\mathrm{NE}(\pi)$ up to isomorphism.*

Definition 2.6. *A **contraction** is a surjective morphism with connected fibers.*

Remark 2.7. *By Proposition 2.5 a contraction is completely determined by $\mathrm{NE}(\pi)$, or equivalently by the curves it contracts. We will sometimes refer to π as **the contraction of the extremal subcone** $\mathrm{NE}(\pi)$. When $\mathrm{NE}(\pi)$ is an extremal ray, we will call π an **extremal contraction**.*

3. Positivity of divisors

Recall that a divisor D on X is called **very ample** if its global sections $H^0(X, D)$ define an embedding. It is called **ample** if some multiple of it is very ample. We have the following two numerical characterizations of ampleness:

Theorem 3.1 (Nakai-Moischezon criterion)**.** *A $\mathbb{Q}$-Cartier divisor D on X is ample if and only if, for every subvariety $Y \subset X$ we have*

$$D^{\dim(Y)} \cdot Y > 0.$$

Theorem 3.2 (Kleiman's criterion)**.** *A $\mathbb{Q}$-Cartier divisor D on X is ample if and only if $D \cdot z > 0$ for all $z \in \overline{\mathrm{NE}}(X)$.*

In light of the two previous criteria, it seems natural to define a weaker notion to ampleness.

Definition 3.3. *A $\mathbb{Q}$-Cartier divisor D on X is called **nef** if $D \cdot C \geq 0$ for all $C \in \mathrm{NE}(X)$.*

[2]since we are working over a field of characteristic zero, this condition is equivalent to π having connected fibers.

We also have the following notions which behave well up to taking multiples of a divisor.

Definition 3.4. *A divisor D is on X is called:*

- ***big*** *if its global section grow asymptotically maximally, i.e. there exists a constant $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq C \cdot m^{\dim X}$;*
- ***semiample*** *if there exists some multiple nD of it such that nD is base point free (abr. bpf).*

We also define the ***stable base locus*** *of a divisor D to be the intersection of the base loci of all multiples of D. Then a divisor is semiample if and only if its stable base locus is empty.*

The following example illustrates the differences between semiample, ample and very ample.

Example 3.5. *Let C be an elliptic curve and P be a point on it, viewed as a divisor. Then $\dim H^0(C, P) = 1$; $\dim H^0(C, 2P) = 2$ and $H^0(C, 2P)$ defines a 2-to-1 covering map to $\mathbb{P}^1$; $\dim H^0(C, 3P) = 3$ and $H^0(C, 3P)$ defines an embedding of C to $\mathbb{P}^2$ as a cubic curve.*

Thus P is semiample even though its base locus is not empty. Moreover it is ample but not very ample.

Proposition 3.6. *For a semiample divisor D and for $n \gg 0$, the morphism $\phi_{nD} \colon X \to Y$ given by nD has connected fibers and Y is normal.*

Moreover,

$$\phi_{nD}(X) \cong \operatorname{Proj}(R(X, D)),$$

where $R(X, D)$ denotes the section ring $\bigoplus\limits_k H^0(X, kD)$.

Proof. Since we are anyways working asymptotically, up to taking a multiple of D we may assume that it is bpf. Let

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\phi_D"'] & X' \arrow[d, "p"] \\ & Y \end{tikzcd}$$

be the Stein factorization for ϕ_D (see [Har77, Chapter III, Corollary 11.5]), so that ϕ has connected fibers, Y is normal and p is a finite morphism. We will show that ϕ is independent of taking multiples and that for a large enough multiple, p is an isomorphism.

For any $n \in \mathbb{Z}$ we have

$$\begin{aligned} H^0(X, \mathcal{O}(nD)) &= H^0(X, \phi^*\mathcal{O}(n)) = H^0(X, \phi^*\mathcal{O}(n) \otimes \mathcal{O}_X) = H^0(Y, \phi_{D*}({\phi_D}^*\mathcal{O}(n) \otimes \mathcal{O}_X)) \\ &\overset{(\star)}{=} H^0(Y, \mathcal{O}(n) \otimes \phi_{D*}\mathcal{O}_X) = H^0(Y, \mathcal{O}(n) \otimes p_*\phi_*\mathcal{O}_X) \overset{(\star)}{=} H^0(Y, p_*(p^*\mathcal{O}(n) \otimes \phi_*\mathcal{O}_X)) \\ &= H^0(X, p^*\mathcal{O}(n) \otimes \phi_*\mathcal{O}_X) = H^0(X, p^*\mathcal{O}(n) \otimes \mathcal{O}'_X) = H^0(X, p^*\mathcal{O}(n)), \end{aligned}$$

where the equalities labeled with ($\star$) come from applying the projection formula. The calculation above yields that ϕ_{nD} factors through ϕ in following way

$$\begin{tikzcd} & & Y_n \\ X \arrow[r, "\phi"] \arrow[urr, bend left, "\phi_{nD}"] \arrow[drr, bend right, "\phi_D"'] & X' \arrow[ur, "p_n"] \arrow[dr, "p"'] & \\ & & Y \arrow[u, "v_n"'] \end{tikzcd}$$

where v_n denotes the restriction of the Veronese embedding to Y. But then $p_n^*\mathcal{O}(1) = p^*v_n^*(\mathcal{O}(1)) = p^*(\mathcal{O}(n)) = np^*\mathcal{O}(1)$ and p being finite implies that $p^*\mathcal{O}(1)$ is again ample. Thus for n large enough, p_n is an isomorphism and then ϕ_{nD} has connected fibers, as it coincides with its Stein factorization.

For the second part we have

$$\begin{aligned}\operatorname{Proj}(R(X,nD)) &= \operatorname{Proj}\left(\bigoplus_k H^0(X,knD)\right) = \operatorname{Proj}\left(\bigoplus_k H^0(X,\phi_{nD}^*\mathcal{O}(k)\otimes\mathcal{O}_X)\right) \\ &= \operatorname{Proj}\left(\bigoplus_k H^0(Y,\phi_{nD*}(\phi_{nD}^*\mathcal{O}(k)\otimes\mathcal{O}_X))\right) = \operatorname{Proj}\left(\bigoplus_k H^0(Y,\mathcal{O}(k))\right) = Y.\end{aligned}$$

Moreover $\operatorname{Proj}(R(X,nD))$ is isomorphic to $\operatorname{Proj}(R(X,D))$ via the Veronese embedding and so we conclude. $\square$

A different, more geometric, proof of the first part of the previous Proposition can be found in [Mat02, Proposition 1-2-16]

By Remark 2.7, the contraction of an extremal subcone $F \subset \mathrm{NE}(X)$ is unique up to isomorphism. However, its existence is not always guaranteed. Using Proposition 3.6 we see that, if there exists a semiample divisor D such that

$$F = \mathrm{NE}(X) \cap \{C \in \mathrm{NE}(X) \mid D \cdot C = 0\},$$

then the contraction of F coincides with the morphism $X \to \operatorname{Proj}(R(X,D))$.

The converse is also true: given the contraction $\pi\colon X \to Y$ of an extremal face, the pullback of any ample divisor A on Y is semiample and zero precisely on F.

It is a celebrated theorem that if an extremal ray of $\mathrm{NE}(X)$ is generated by a curve C such that $K_X \cdot C < 0$, then the contraction of the ray exists (see [Mat02, Theorem 8-1-3]). In its simplest form, this is the well known Castelnuovo's Contractibility Criterion for surfaces ([Mat02, Theorem 1-1-6]).

On a slightly different direction, Artin [Art62] proved that any for any surface S and any connected curve $C = \cup C_i \subseteq S$ with connected components C_i we have the following equivalence:

$$C \text{ is contractible } \Leftrightarrow \begin{cases} \text{1. the intersection matrix } |C_i \cdot C_j| \text{ is negative definite,} \\ \text{2. for every cycle } Z \geq 0 \text{ supported on C }, p(Z) \leq 0, \end{cases}$$

where $p(Z)$ denotes the arithmetic genus of Z. Moreover, if S is projective and $S \to \check{S}$ denotes the contraction morphism, then $\check{S}$ is also projective. As Artin puts it, this can be interpreted as: C is contractible if and only if it "sufficiently negative and rational". Below, we prove the simplest version of this, namely when C is irreducible.

Proposition 3.7. *Let S be a smooth surface and C be a rational curve such that $C^2 < 0$. Then there exists a semiample divisor D with the property*

$$D \cdot z = 0 \iff z \in \mathbb{R}E, \forall z \in \mathrm{NE}(S).$$

In particular the extremal ray $\mathbb{R}C$ is contractible.

Proof. Let A be an ample divisor on S so that $H^1(S, \mathcal{O}_S(A)) = 0$. Write $a = A \cdot C$ and $b = -C^2$ and define the divisor

$$N := bA + aC.$$

Then N is a nef divisor that is zero precisely on C. We will show that N is semiample.

Notice that, since N is positive on every curve except C, any stable base locus will have to lie on C. Consider the short exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

After tensoring with $\mathcal{O}_S(N)$ and taking the associated long exact sequence, we get

$$H^0(S, \mathcal{O}_S(N)) \to H^0(C, \mathcal{O}_C(N)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C} \to H^1(S, \mathcal{O}_S(N - C)).$$

Suppose for now that $H^1(S, \mathcal{O}_S(N-C)) = 0$. Then the first map in the sequence above is surjective, and thus we can lift a nowhere vanishing section of $\mathcal{O}_C(N)$, to a section of $\mathcal{O}_S(N)$ that doesn't vanish on C and we are done.

Thus we are left with showing that $H^1(S, \mathcal{O}_S(N - C)) = 0$. We will consider the divisors $D_i := bA + iC$, for $i = 0, \dots, a-1$ and prove that $H^1(S, \mathcal{O}_S(D_i)) = 0$ for all i. Notice that $N - C = D_{a-1}$. For $i = 0$, $D_0 = A$ and so, by assumption, $H^1(S, \mathcal{O}_S(D_0)) = 0$. Suppose that $H^1(S, \mathcal{O}_S(D_k)) = 0$ for some k and consider the exact sequence

$$H^1(S, \mathcal{O}_S(D_k)) \to H^1(S, \mathcal{O}_S(D_{k+1})) \to H^1(C, \mathcal{O}_C(D_{k+1})) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b(a-k+1)))$$

the last term being isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 - b(a-k+1)))$ by Serre duality. By the inductive hypothesis, for k less or equal to $a-2$, the terms of the left and right of the last exact sequence are 0, thus so is the middle term. For $k = a-2$ we get that $H^1(S, \mathcal{O}_S(D_{k+1})) = H^1(S, \mathcal{O}_S(N-C)) = 0$ completing the proof. □

4. A crash course on the MMP

The Minimal Model Program (MMP for short) is the main tool towards the classification of higher dimensional algebraic varieties. It is a (non-deterministic) algorithm, that birationally transforms a mildly singular projective variety to one satisfying certain positivity conditions. We will not go into any details except only what will be needed for the next chapters. However a good, yet slightly outdated, reference for this theory is [Mat02].

The classical case is the one of surfaces: there Castelnuovo's Contractibility Critrerion says that any (-1)-curve E can be contracted with the trarget being again a smooth surface. Repeating this process as many times as possible, one can reach a smooth surface with no (-1)-curves.

For these notions to make sense in higher dimensions, we need a slight change of perspective. Notice that a (-1)-curve is a curve which has negative intersection with the canonical divisor K_S. Thus a natural way to generalize this theory in higher dimensions, is to try to contract as many curves that have negative intersection with the canonical divisor.

However we encounter the following problem:

Lemma 4.1. *Let $f\colon X \to Y$ be a birational morphism contracting curves that have non-trivial intersection against a $\mathbb{Q}$-Cartier divisor D. Moreover assume that the exceptional locus has codimension at least 2. Then $D' := f(D)$ is not a $\mathbb{Q}$-Cartier divisor.*

Proof. Suppose that it is and write the ramification formula $D = f^*(D') + E$. Now E is a divisor supported on the exceptional locus of f, and is thus 0. Let C be a curve contracted by f. We have

$$0 \neq D \cdot C = f^*(D') \cdot C = D' \cdot f_*C = 0,$$

which is a contradiction. □

Such a situation with $D = K_X$ poses a fundamental problem since the whole strategy was to contract K_X-negative curves, which is a notion that only makes sense when K_X is a $\mathbb{Q}$-Cartier divisor. The ingenious idea of Mori to circumvent this problem is to introduce the notion of a flip.

Definition 4.2. *Let X be a $\mathbb{Q}$-factorial terminal variety and $f\colon X \to Y$ be an extremal contraction which is small (the exceptional locus has codimension at least 2). Suppose moreover that all curve contracted by f are negative against a $\mathbb{Q}$-Cartier divisor D. A **D-flip** of f is a morphism $f^+\colon X^+ \to Y$ that fits into a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\chi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

such that

1. *X^+ is again $\mathbb{Q}$-factorial and terminal;*

2. *f^+ is a birational, small, extremal contraction;*

3. *all curves contracted by f^+ are D^+-positive, where D^+ is the strict transform of D under χ.*

By abuse of notation, we will call χ the flip of f, or sometimes the flip of the ray $\mathrm{NE}(f)$.

We define a ***flip*** *to be a K_X-flip and an* ***anti-flip*** *to a $(-K_X)$-flip. A* ***flop*** *is a D-flip such that all curves contracted by f (or equivalently by f^+) have trivial intersection with K_X.*

Once a D-flip exists, it is unique and f^+ is precisely the morphism $\mathrm{Proj}(R) \to Y$, where $R = \oplus_{m \geq 0} f_* \mathcal{O}_X(mD)$. Its existence boils down to the finite generation of R, which has been proven for most, relevant to the MMP, cases.

In a nutshell, MMP works in the following way: at each step we contract an extremal ray spanned by K-negative curves. If the contraction is small, we perform a flip instead. This process only ends in two ways: either there are no more K-negative curves, or the last extremal contraction is a contraction to a lower dimensional variety. The outputs of an MMP are called **minimal models** in the former case and **Mori fiber spaces** in the latter.

We will now introduce some notions of singularities that appear naturally in the MMP.

Definition 4.3. *Let D, D' be two $\mathbb{Q}$-divisors (formal sum of prime divisors with rational coefficients) on a normal variety X. We say that D and D' are $\mathbb{Q}$ linearly equivalent, and write $D \sim_{\mathbb{Q}} D'$ if there exists $n \in \mathbb{N}$ so that nD and nD' are divisors (with integer coefficients) and $nD \sim nD'$.*

Example 4.4. *Let E be an elliptic curve and fix a 2-to-1 covering $E \to \mathbb{P}^1$, so that the points P and P' are ramification points. Then $P \not\sim P'$ but $2P \sim 2P'$. Thus P and P' are $\mathbb{Q}$-linearly but not linearly equivalent.*

Definition 4.5. *Let X be a normal, $\mathbb{Q}$-factorial variety and let $f \colon Y \longrightarrow X$ be a resolution of singularities. Write*

$$K_Y \sim_{\mathbb{Q}} f^* K_X + \sum a_i E_i.$$

We say that X is ***terminal*** *if $a_i > 0$ and* ***canonical*** *if $a_i \geq 0$, for every i.*

Let $\Delta = \sum d_j D_j$ be a $\mathbb{Q}$-divisor on X, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and let $g \colon Z \longrightarrow X$ be a log resolution, i.e. a resolution of X such that $\mathrm{Supp}(g^{-1}(D) + \mathrm{Exc}(g))$ *has pure codimension 1 and is simple normal crossings. Write*

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum a_i E_i$$

where $f_*\left(\sum a_i E_i\right) = -\Delta$. We say that the pair (X, Δ) is ***Kawamata log terminal*** *(klt for short) if $a_i > -1$ and* ***log canonical*** *if $a_i \geq -1$, for every i.*

We note that the smallest category in which the MMP works (i.e. all steps of the MMP keeps us in the same category) is the category of $\mathbb{Q}$-factorial terminal varieties.

5. Mori fiber spaces and Sarkisov links

The machinery of the MMP allows us to reduce the study of terminal projective varieties to, roughly, the study of 2 classes of varieties: minimal models and Mori fiber spaces. The contents of this thesis are concerned with the latter, which we now define.

Definition 5.1. *A morphism $\phi\colon X \to Y$ is called a **Mori fiber space** if the following conditions are satisfied:*

1. *X is normal, $\mathbb{Q}$-factorial and terminal;*
2. *ϕ is the contraction of an extremal ray R of $\mathrm{NE}(X)$;*
3. *for all $C \in R$, $K_X \cdot C < 0$;*
4. $\dim Y < \dim X$.

Condition (1) stems from the choice of a category to run the MMP in; conditions (2) and (3) ensures that the morphism is a step of the MMP; finally, condition (4) implies that ϕ is the final step of some MMP.

A fundamental, though heuristic, distinction between minimal models and Mori fiber spaces, is that while the former posses a more complicated geometry, the birational relations among them are simpler compared to those among Mori fiber spaces. The main tool for the study of birational relations among Mori fiber spaces is the Sarkisov Program. This is, again, an algorithm, within the framework of MMP, that decomposes any birational map between Mori fiber spaces into "simpler" ones, called Sarkisov links. Ideas for the Sarkisov program stem from the works of Sarkisov and Reid, ultimately being proven by Corti [Cor95] in dimension 3 and later Hacon and McKernan [HM13] in any dimension.

Definition 5.2. *A **Sarkisov diagram** between two Mori fiber spaces $X_1 \to B_1$ and $X_2 \to B_2$ is a commutative diagram of the form*

$$\begin{tikzcd}
Y_1 \arrow[rr, dotted, "\chi"] \arrow[d, "\alpha_1"'] & & Y_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[dr, "s_1"'] & & B_2 \arrow[dl, "s_2"] \\
& R &
\end{tikzcd}$$

which satisfies the following properties:

1. *all morphisms appearing in the diagram are either isomorphisms or outputs of some MMP on a $\mathbb{Q}$-factorial klt pair (Z, Δ).*

2. *maximal dimensional varieties have $\mathbb{Q}$-factorial and terminal singularities,*

3. *α_1 and α_2 are divisorial contractions or isomorphisms,*

4. *s_1 and s_2 are extremal contractions or isomorphisms,*

5. *χ is an isomorphism or a composition of anti-flips/flop/flips (in that order),*

6. *the relative Picard rank of Y_1 and Y_2 over R is $\rho(Y_1/R) = \rho(Y_2/R) = 2$.*

We call R the base *of the diagram.*

Property 6 implies that α_1 is a divisorial contraction if and only if s_1 is an isomorphism. A similar statement holds for the right hand side of the diagram. Depending whether s_1 or s_2 is an isomorphism, we get four types of Sarkisov diagrams:

Type I

$$\begin{tikzcd} Y_1 \arrow[r, dotted] \arrow[d] & X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[d, "\pi_1"'] & B_2 \arrow[dl] \\ B_1 = R & \end{tikzcd}$$

Type II

$$\begin{tikzcd} Y_1 \arrow[r, dotted] \arrow[d] & Y_2 \arrow[d] \\ X_1 \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ B_1 = R \arrow[r, phantom, "="] & B_2 \end{tikzcd}$$

Type III

$$\begin{tikzcd} X_1 \arrow[r, dotted] \arrow[d, "\pi_1"'] & Y_2 \arrow[d] \\ B_1 \arrow[dr] & X_2 \arrow[d, "\pi_2"] \\ & R = B_2 \end{tikzcd}$$

Type IV

$$\begin{tikzcd} X_1 \arrow[r, dotted] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[dr] & B_2 \arrow[d] \\ & R \end{tikzcd} .$$

The birational map $\psi = \alpha_2 \chi \alpha_1^{-1}$ between X_1 and X_2 is called a Sarkisov link.

Remark 5.3. *Property 2 does not follow directly from the original definition of a Sarkisov diagram of [HM13]. For a proof, see [BLZ21, Proposition 4.25].*

The main reason for introducing Sarkisov links is the following theorem, first proven by Corti in dimension 3 and later by Hacon and McKernan in higher dimensions:

Theorem 5.4 ([Cor95, HM13])**.** *Any birational map between Mori fiber spaces can be decomposed into Sarkisov links and Mori fiber space isomorphisms (i.e. an isomorphism between their sources that induces an isomorphism on the base as well).*

A small note: an isomorphism between the sources of two Mori fiber spaces which is not a Mori fiber space isomorphism is a non-trivial Sarkisov link. The prime example of this is the switch involution $i\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $(x, y) \mapsto (y, x)$. The associated Sarkisov diagram in that case is

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[rr, "i"] \arrow[d, "p_1"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "p_1"] \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& \mathrm{Spec}(\mathbb{C}) &
\end{tikzcd}$$

which is a diagram of Type IV.

We know set out to answer the natural question: why is a Sarkisov link simpler than any other birational map. The fundamental property in the definition of a Sarkisov link is $\star$(6). It allows us to recover the whole diagram from the data of Y_1/R (or equivalently of Y_2/R) by a process known as the **2-ray game**, which we now heuristically explain, leaving the details to Chapter III. $\mathrm{NE}(Y_1/R) \subset N_1(Y_1/R)$ is a cone in a 2-dimensional vector space, and as such it is generated by two extremal rays r_1 and r_2. Notice that one of them, say r_1, corresponds to $\mathrm{NE}(\alpha_1)$, from which we recover the left hand side of the diagram. Thus to recover the right hand side of the diagram, we are forced to "play" with the ray r_2, that is we have to consider its contraction.

Case 1: If the contraction of r_2 is not small then we take χ to be an isomorphism and the contraction of r_2 to be α_2, from which we recover the right hand side of the diagram.

Case 2: If the contraction of r_2 is small then we consider its D-flip (see 4.2)

$$\begin{tikzcd}
Y_1 \arrow[rr, dotted] \arrow[dr, "f"'] & & Y_1^+ \arrow[dl, "f^+"] \\
& W. &
\end{tikzcd}$$

Then again, $\mathrm{NE}(Y_1^+/R)$ is a cone in a 2-dimensional vector space generated by two rays s_1 and s_2. Again, the contraction of one of them, say s_1, corresponds to f^+. Playing with that ray, will get us back to Y_1, thus in order to move forward, we have to play with the ray s_2.

We continue this process until we fall in Case 1, at which point we recover the whole Sarkisov diagram. Thus, not only can the diagram be recovered from the data Y_1/R, but also in a "pseudo-algorithmic" way, which implies that it is also determined by this data.

Vice versa, classifying the Sarkisov links starting from (or ending in) a fixed Mori fiber space $X \to B$, is equivalent to classifying certain extremal contractions either whose target is X, or whose source is B. We will later call them **rank 2 fibrations** dominating $X \to B$ (see IV).

In general, groups of birational transformations seem impossible to describe explicitly. However, classifying all Sarkisov links starting from a given Mori fiber space is way more tangible. In many cases the number of links is limited, and then we can use them as building blocks to describe and understand all birational maps. Examples of this approach include [CPR00, AZ16, AK16], or [FZ22] where an action of a group is also present. On the other hand, even in cases where there is an abundance of Sarkisov links, the theory of rank r fibrations of [BLZ21], still allows us to extract information on the structure of the group of birational transformations.

6. Algebraic subgroups of $\mathrm{Bir}(X)$

In the following $\mathrm{Bir}(X)$ denotes the group of birational transformations of X.

Definition 6.1. *We say that an algebraic group G **acts birationally** on X if:*

1. *there are open dense subsets U, V of $G \times X$ such that the projections to the first factor are surjective onto G and a birational map*

$$\begin{array}{rccc} P_G\colon & G \times X & \dashrightarrow & G \times X \\ & (g,x) & \mapsto & (g, \rho(g,x)) \end{array}$$

   whose restriction between U and V is an isomorphims;

2. *$\rho(e,\cdot) = id_X$ and $\rho(gh,x) = \rho(g,\rho(h,x))$, for all $g, h \in G$ and $x \in X$ such that $\rho(h,x), \rho(gh,x)$ and $\rho(g,\rho(h,x))$ are well defined.*

In that case we get a homomorphism of groups $\Phi_G\colon G \to \mathrm{Bir}(X)$, $g \mapsto \rho(g,\cdot)$.

*We say that G is an **algebraic subgroup** of $\mathrm{Bir}(X)$ if G is an algebraic group acting birationally on X and moreover Φ_G is injective.*

*We say that G **acts regularly** (or more simply acts) on X if the birational map P_G is an isomorphism.*

The reason we define an algebraic subgroup of $\mathrm{Bir}(X)$ in this way is that in general $\mathrm{Bir}(X)$ is not an algebraic group or even an ind-group (see [BF13]).

We have the following fundamental result on rational group actions, know as the Weil regularization theorem originally proven by [Wei55] (see [Zai95, Kra18] for a modern proofs).

Theorem 6.2. *Let G be an algebraic subgroup of $\mathrm{Bir}(X)$. Then there exists a variety Y on which G acts regularly and a G-equivariant birational map $\phi\colon X \to Y$.*

Note that in general the variety Y is neither projective nor normal. However, when G is a connected algebraic group, we can use a G-equivariant completion (see [Bri17, Corollary 3]) to assume that Y is projective. Moreover, we can choose a functorial resolution of singularities, which is G-equivariant (see [Kol07, Proposition 3.9.1]), to furthermore assume Y to be smooth. We thus obtain the following:

Corollary 6.3. *Let G be a connected algebraic subgroup of $\mathrm{Bir}(X)$. Then there exists a smooth projective variety Y on which G acts regularly and a G-equivariant birational map $\phi\colon X \to Y$.*

The following statement is known as Blanchard's lemma and its proof can be found in [BSU13, Proposition 4.2.1].

Proposition 6.4. *Let $f\colon X \to Y$ be a morphism with $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and let G be a connected group acting on X. Then there exists a unique action of G on Y such that f is G-equivariant.*

Using this machinery, we now explain the standard strategy one uses to study algebraic subgroups of $\mathrm{Bir}(X)$:

1. We start with G a connected algebraic subgroup of $\mathrm{Bir}(X)$; using Corollary 6.3, we may assume that G acts regularly on a smooth projective variety Y, and we have a G-equivariant birational map $\phi\colon X \dashrightarrow Y$. Thus we have effectively conjugated G in the automorphism group of Y by
$$\phi G \phi^{-1} \subseteq \mathrm{Aut}(Y) \subseteq \mathrm{Bir}(X).$$
2. Now using Proposition 2.3, for any contraction of an extremal ray (see Remark 2.7) there exists a unique action in the target that makes the contraction G-equivariant. In particular, running an MMP on Y we end up with a minimal model or a Mori fiber space Z on which G acts regularly.

This shows that we may always conjugate a connected algebraic subgroup of $\mathrm{Bir}(X)$ inside the group of automorphisms of a minimal model or a Mori fiber space Z, birational to X. If moreover one starts with a G which is maximal, with respect to inclusion into algebraic subgroups, then G is automatically conjugate to the full automorphism group of Z, which is a huge reduction step.

Remark 6.5 (Finite Groups)**.** *The machinery above applies to the case of finite groups with the only difference being the output Z: given a finite subgroup of* $\mathrm{Bir}(X)$ *we may assume that G acts regularly on a smooth, projective variety Y which is birational to X. (see [PS14,* Lemma-Definition *3.1]); then we may run a G-MMP on Y (see [Pro21, Theorem 3.3.1]) to obtain G-minimal model or a G-Mori fiber space Z on which G acts regularly. In contrast however to the connected case, a G-MMP is not necessarily an MMP. Thus a G-minimal model/Mori fiber space is in general more complicated than an (absolute) minimal model/Mori fiber space.*

CHAPTER III

Sarkisov links with centers space curves on smooth cubic surfaces

Abstract

We construct and study Sarkisov links obtained by blowing up smooth space curves lying on smooth cubic surfaces. We restrict our attention to the case where the blowup is not weak Fano. Together with the results of [BL12] which cover the weak Fano case, we provide a classification of all such curves. This is achieved by computing all curves which satisfy certain necessary criteria on their multisecant curves and then constructing the Sarkisov link step by step.

1. Introduction

The Sarkisov program is a central tool in the study of birational relations between Mori fibre spaces. It was first proved by Corti in [Cor95] for dimension 3 and by Hacon and McKernan in [HM13] in the general case. In recent years there has been an increase in the number of applications of it, mostly to the study of birational automorphisms of certain Mori fibre spaces. [AK16], [Ahm17], [Pro18] and [BLZ21] are but a few papers on the subject. Thus, it is natural to study and try to classify links involving a fixed Mori fibre space.

In this paper we will study links where one of the two Mori fibre spaces is $\mathbb{P}^3 \to pt$. Notice that since the Picard rank of $\mathbb{P}^3$ is 1, the first step has to be a divisorial contraction $f\colon X \to \mathbb{P}^3$. A divisorial contraction to $\mathbb{P}^3$ (or more generally to a smooth Fano 3-fold Y) either be the blowup of a smooth curve or a weighted blowup of a point. The main case of interest for us is the former, i.e.

when f is the blowup of a smooth curve. We note that if one drops the smoothness assumption on Y there could be more than these two classes of divisorial contractions (see [Tzi03]).

In [JPR05], [JPR11] and [CM13] the two sets of authors embark on a classification of weak Fano threefolds obtained by blowups of Fano threefolds of Picard rank 1 under different assumptions. In all three papers, the focus was the classification, while the existence of links was more of a byproduct. Moreover, the classification in the last mentioned paper was numerical in nature leaving the actual existence of some cases open (see [ACM17, Fuk17, Tak22]) for the existence of some cases).

Our approach is more closely related to the one of [BL12]. There, the authors give a complete list of pairs (g, d) such that the blowup of a general space curve C of genus g and degree d produces a weak Fano threefold. This was done via geometric ways, proving also the existence of all the listed cases. Moreover, focus was also given to the links produced by the second K_X-negative contraction. Thus, our focus here will be the case when the blowup produces threefolds which are not weak Fano but nonetheless still fit in some Sarkisov link. In that case we will say that C induces a Sarkisov link.

It is easy to show (and will also be evident in the course of this paper) that not all curves induce a Sarkisov link. In a sense, we need a way to control the K_X-non-negative curves. A way to do so is to reduce to the case when C is contained in a surface of degree less or equal to 4. In this case, all K_X-non-negative curves will be contained in the surface of low degree whose geometry we can exploit to better control their behaviour. This was the main viewpoint in [BL12]. In this paper we treat the case when the curve lies in a smooth cubic.

Smooth cubics in $\mathbb{P}^3$ are isomorphic to $\mathbb{P}^2$ blown up at 6 points. Any curve on the blowup $\pi\colon S \to \mathbb{P}^2$ of $\mathbb{P}^2$ along 6 points is linearly equivalent to $kL - \sum m_i F_i$, where L is the pullback of a general line under π and the F_i's are the π-exceptional curves. In that case, we will say that the curve is of type $(k; m_1, \dots, m_6)$. Identifying the cubic with S, by classifying the curves that induce Sarkisov links we mean classifying their type. We may also reduce to the case where $m_i \geq m_{i+1}$ and $k \geq m_1 + m_2 + m_3$. With that in mind the main theorem of this paper is the following:

Theorem A. *Let $C \subset S \subset \mathbb{P}^3$ be a curve lying on a smooth cubic surface in $\mathbb{P}^3$. Suppose that the blowup of $\mathbb{P}^3$ along C is not weak Fano. Then C induces a Sarkisov link if and only if its type (up to the assumptions $m_i \geq m_{i+1}$ and $k \geq m_1 + m_2 + m_3$) belongs to one of the following two sets:*

$$\begin{aligned} \mathcal{T}^{(II)} &= \{(3;\, 1,1,0,0,0,0),\, (3;\, 2,0,0,0,0,0),\, (4;\, 2,1,1,1,0,0),\, (5;\, 2,1,1,1,1,1)\}\,; \\ \mathcal{T}^{(I)} &= \{(3;\, 2,1,0,0,0,0),\, (5;\, 3,1,1,1,1,1)\}\,. \end{aligned}$$

Moreover, the curves in $\mathcal{T}^{(II)}$ produce Sarkisov links of Type II to terminal Fano 3-folds of Picard rank 1 while the curves in $\mathcal{T}^{(I)}$ produce Sarkisov links of Type I to terminal del-Pezzo fibrations of degree 5 and 4 respectively.

This is proven by constructing the steps of the so-called 2-ray game on X (see section 3.2, for the definition of the 2-ray game).

The outline of the paper is as follows: In Section 3, we introduce some key elements in the Sarkisov program such as the notion of Sarkisov links and that of the central model. We then define Mori dream spaces and explain their connection to the Sarkisov program via the 2-ray game. We also prove that any curve lying on a plane or quadric that induces a Sarkisov link already appears in the literature.

In Section 4 we recall the basic theory of cubic surfaces. We then establish some bounds that the type of a curve must satisfy in order for it to induce a Sarkisov link and compute all possible curves that satisfy these bounds. At this point we obtain the combined list of curves appearing in Theorem A.

In Section 5 we prove that all curves in the aforementioned list actually produce Sarkisov links. This is achieved by constructing the first few steps of the 2-ray game until we hit the weak Fano variety lying over the central model at which point the general theory assures the existence of the links. Finally, in Section 6 we explore all steps of the links as well as compute some invariants of the target variety.

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2. Notation & Conventions

In this paper all varieties are assumed to be normal, projective and defined over $\mathbb{C}$. A 3-fold is a 3-dimensional projective variety. For a variety X we also define:

$$\begin{aligned} \mathrm{WDiv}(X) &:= \text{the group of Weil divisors modulo linear equivalence;}\\ \mathrm{CDiv}(X) &:= \text{the group of Cartier divisors modulo linear equivalence;}\\ N^1(X/Z) &:= \{\mathbb{Q}\text{-Cartier divisors modulo numerical equivalence over } Z\} \otimes \mathbb{R};\\ N_1(X/Z) &:= \{\text{1-cycles modulo numerical equivalence over } Z\} \otimes \mathbb{R}. \end{aligned}$$

For a $\mathbb{Z}$-module M we define $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. If z_1, z_2 are 1-cycles in X we write $z_1 \sim_S z_2$ if there exists a surface S such that z_1 and z_2 are linearly equivalent in S.

Finally, we denote by $\mathrm{NE}(X/Z)$ the cone in $N_1(X/Z)$ spanned by effective 1-cycles and by $\mathrm{Nef}(X)$, $\mathrm{Mov}(X)$, $\mathrm{Eff}(X)$ and $\overline{\mathrm{Eff}}(X)$ the cones in $N^1(X) := N^1(X/pt)$ spanned by nef, movable, effective and pseudoeffective divisors respectively.

3. Preliminaries

3.1. The Sarkisov program

Definition 3.1. *Let X be a normal variety with K_X $\mathbb{Q}$-Cartier and let $f\colon Y \to X$ be a resolution of singularities. Write*

$$K_Y \sim_{\mathbb{Q}} f^*K_X + \sum a_i E_i.$$

*We say that X is **terminal** if $a_i > 0$ and **canonical** if $a_i \geq 0$.*

Let $\Delta = \sum d_j D_j$ be a $\mathbb{Q}$-divisor on X, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and let $g\colon Z \to X$ be a log resolution, i.e. a resolution of X such that $\operatorname{Supp}(g^{-1}(D) + \operatorname{Exc}(g))$ has pure codimension 1 and is simple normal crossings. Write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum a_i E_i$$

*where $f_*\left(\sum a_i E_i\right) = -\Delta$. We say that the pair (X, Δ) is **Kawamata log terminal** (klt for short) if $a_i > -1$ for every i.*

Remark 3.2. *Suppose that X is a smooth threefold and Δ is a prime divisor. If the support of Δ is smooth, then for any $q < 1$, the pair $(X, q\Delta)$ is klt.*

Indeed, in such case we may choose the identity as a log resolution of the pair and compare with the definition above.

First we recall the definition of a Sarkisov link.

Definition 3.3 (Sarkisov link)**.** *A **Sarkisov diagram** is a commutative diagram of the form*

$$\begin{tikzcd}
X' \arrow[rr, dotted, "\chi"] \arrow[d, "p"'] & & Y' \arrow[d, "q"] \\
X \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
S \arrow[dr, "s"'] & & T \arrow[dl, "t"] \\
& R &
\end{tikzcd}$$

which satisfies the following properties

1. *ϕ and ψ are Mori fibre spaces,*
2. *p and q are divisorial contractions or isomorphisms,*
3. *s and t are extremal contractions or isomorphisms,*
4. *χ is a pseudo-isomorphism (i.e. an isomorphism when restricted to a subset whose complement has codimension greater than 1),*

5. *all varieties of maximal dimension are $\mathbb{Q}$-factorial and terminal,*

6. *the relative Picard rank $\rho(Z/R)$ of any variety Z in the diagram is at most 2.*

Property (6) implies that p is a divisorial contraction if and only if s is an isomorphism. A similar statement holds for the right hand side of the diagram. Depending whether s or t is an isomorphism, we get four types of Sarkisov diagrams

Type I

$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d] & Y \arrow[d, "\psi"] \\
X \arrow[d, "\phi"'] & T \arrow[dl] \\
S &
\end{tikzcd}$$

Type II

$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\
X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\
S \arrow[r, "\sim"] & T
\end{tikzcd}$$

Type III

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "\phi"'] & Y' \arrow[d] \\
S \arrow[dr] & Y \arrow[d, "\psi"] \\
& T
\end{tikzcd}$$

Type IV

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
S \arrow[dr] & & T \arrow[dl] \\
& R &
\end{tikzcd} .$$

The induced birational map $X \dashrightarrow Y$ is called a ***Sarkisov link****. A diagram of the form above that satisfies all but condition* (5) *will be called a* ***Sarkisov-like diagram***

Theorem 3.4. *([HM13, Theorem 1.1]) Let $\phi\colon X \to S$, $\psi\colon Y \to T$ be two Mori fibre spaces where X, Y are $\mathbb{Q}$-factorial and terminal. Then any birational map between X and Y can be decomposed as a sequence of Sarkisov links and isomorphisms of Mori fibre spaces.*

Notice that if the starting Mori fibre space is $\phi\colon \mathbb{P}^3 \to pt$, then we can only get links of Type I and II. More specifically, the birational map $\mathbb{P}^3 \dashrightarrow Y$ can be factored as the inverse of a divisorial contraction followed by either a pseudo-isomorphism (Type I link) or a pseudo-isomorphism and a divisorial contraction (Type II link). In the following we study the case where the first divisorial contraction is the inverse of blowing up a smooth curve. We say that such a curve $C \subset \mathbb{P}^3$ **induces a Sarkisov link** if $X := \mathrm{Bl}_C \mathbb{P}^3 \to \mathbb{P}^3$ fits into a Sarkisov diagram.

As explained in [BLZ21, Remark 3.10], if $X_m \dashrightarrow Y_n$ is the pseudo-isomorphism part of a Sarkisov diagram, then its decomposition into anti-flips, flops and flips takes the following form:

$$\begin{tikzcd}
X_m \arrow[d] \arrow[drrr, two heads] & \dots \arrow[l, dashed] \arrow[drr, two heads] & X_0 \arrow[l, dashed] \arrow[r, dashed, leftrightarrow] \arrow[d] & Y_0 \arrow[r, dashed] \arrow[dl] \arrow[ddl, two heads] & \dots \arrow[r, dashed] \arrow[ddll, two heads] & Y_n \arrow[d] \arrow[dddlll, two heads] \\
{} \arrow[drrr, two heads] & & Z \arrow[d] & & & {} \arrow[dlll, two heads] \\
& & R & & &
\end{tikzcd} \qquad (\star)$$

where $X_0 \dashleftarrow\dashrightarrow Y_0$ is either a flop over Z or $X_0 \cong Z \cong Y_0$. In the first case X_0 and Y_0 are $\mathbb{Q}$-factorial and terminal weak Fanos over R, while in the second case Z is $\mathbb{Q}$-factorial and Fano over R. In both cases Z/R is called the **central model** of the Sarkisov link/diagram.

3.2. Mori dream spaces and 2-rays games

Definition 3.5. *Let X and Y be normal, projective varieties.*

A ***small $\mathbb{Q}$-factorial modification*** *(SQM for short) of X is a pseudo-isomorphism $f\colon X \dashrightarrow X'$, where X' is again normal, projective and $\mathbb{Q}$-factorial.*

A ***birational contraction*** *$f\colon X \dashrightarrow Y$ is a birational map such that if $(p,q)\colon W \to X \times Y$ is a resolution of f, then every p-exceptional divisor is also q-exceptional.*

Definition 3.6 (Mori Dream Space)**.** *A normal projective variety X is called a* ***Mori Dream Space*** *(MDS for short) if it satisfies the following:*

1. *X is $\mathbb{Q}$-factorial and $\operatorname{Pic}(X)_{\mathbb{Q}} = \mathcal{N}^1(X)_{\mathbb{Q}}$,*
2. *$\operatorname{Nef}(X)$ is generated by finitely many semi-ample divisors and*
3. *there are finitely many SQMs $f_i\colon X \dashrightarrow X_i$ such that each X_i satisfies (1) and (2) and $\operatorname{Mov}(X)$ is the union of $f_i^* \operatorname{Nef}(X_i)$.*

Proposition 3.7 ([HK00, Proposition 1.11])**.** *Let X be an MDS. Then the following hold.*

1. *The MMP can be carried out for any divisor on X. That is, the necessary contractions and flips exist, any sequence terminates, and if at some point the divisor becomes nef then at that point it becomes semi-ample.*
2. *The f_i's in property (3) of Definition 3.6 are the only SQMs of X. Moreover, there are finitely many birational contractions $g_i\colon X \dashrightarrow Y_i$, such that*

$$\overline{\operatorname{Eff}}(X) = \bigcup_i \mathcal{C}_i,$$

where $\operatorname{Eff}(X)$ denotes the cone of effective divisors in $\mathcal{N}^1(X)$ and

$$\mathcal{C}_i = g_i^* \operatorname{Nef}(Y_i) + \mathbb{R}_{\geq 0}\{E_1, \ldots, E_k\},$$

with $E_1, \ldots, E_k$ being the prime divisors contracted by g_i. The $\mathcal{C}_i$'s are called the ***Mori chambers*** *of X.*

3. *Adjacent Mori chambers are related by a D-flip for some $D \in \operatorname{Div}(X)$.*

Definition 3.8. *A normal, projective and $\mathbb{Q}$-factorial variety X is called*

- ***weak Fano*** *if the anti-canonical divisor $-K_X$ is nef and big;*
- ***of Fano type*** *if there is an effective $\mathbb{Q}$-divisor Δ such that the pair (X, Δ) is klt and $-(K_X + \Delta)$ is ample.*

Lemma 3.9. *If X is terminal and weak Fano, then X is of Fano type.*

Proof. Since X is weak Fano, by definition $-K_X$ is big. Thus is can be written as

$$-K_X \sim A + E$$

where A and E are ample and effective $\mathbb{Q}$-divisors respectively (see [Laz04, Corollary 2.2.7]). For any $k > 1$ we write

$$k(-K_X) = (k-1)(-K_X) + (-K_X) \sim (k-1)(-K_X) + A + E.$$

Since $-K_X$ is nef, $A' := (k-1)(-K_X) + A$ is ample and so $-(K_X + \frac{1}{k}E)$ is ample. Moreover we can choose k sufficiently large such that the pair $(X, \frac{1}{k}E)$ is klt. $\square$

Proposition 3.10 ([BCHM10, Corollary 1.3.2])**.** *If X is of Fano type then X is a Mori Dream Space.*

Property 3.7 (1) is a fundamental property of MDSs and allows us to play 2-ray games and possibly construct Sarkisov links.

Proposition 3.11. *Let $C \subset \mathbb{P}^3$ be a smooth curve and $X := \mathrm{Bl}_C \mathbb{P}^3 \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ along C. Then C induces a Sarkisov link if and only if X is an MDS and for every birational contraction $f\colon X \dashrightarrow Y$ (see Definition 3.5) such that Y is $\mathbb{Q}$-factorial, Y is also terminal.*

Proof. Suppose that C induces a Sarkisov link and let X_0, Y_0 be the varieties over the central model Z/pt of the Sarkisov link

$$\begin{tikzcd} X_0 \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & Y_0 \arrow[dl] \\ & Z \arrow[d] & \\ & pt & \end{tikzcd}$$

so that $X_0 \dashrightarrow Y_0$ is a flop or an isomorphism. Then X_0 is a terminal weak Fano threefold, hence by Lemma 3.9 of Fano type and in turn an MDS. By Proposition 3.7 (2) as well as property (3) of Definition 3.6, X is also an MDS.

Let g_1, g_2 denote the birational contractions with targets the Mori fibre spaces $\mathbb{P}^3/pt$ and Y/T. We claim that $\mathcal{F}_1 := g_1^* \operatorname{Nef}(\mathbb{P}^3)$ and $\mathcal{F}_2 := g_2^* \operatorname{Nef}(Y)$ are extremal in the movable cone. Indeed, let D be a divisor in $\mathcal{F}_i$ and denote by E_i the exceptional divisor of g_i. Then E_i is covered by infinitely many g_i-exceptional curves which are E_i-negative. For any $\kappa > 0$, $\kappa D + E_i$ is negative against all those curves and thus must contain them all and by extension E_i. Thus $\kappa D + E_i$ is not movable, proving the claim. Moreover, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are 1-dimensional and $\rho(X) = 2$ they generate the movable cone of X. Thus, by (2) and (3) of Proposition 3.7, the Sarkisov link factors through all the SQMs $f_i\colon X \dashrightarrow X_i$ [1] thus all of them appear in the Sarkisov diagram making all X_i terminal.

[1] uniquely in fact, since $N^1(X)$ is 2-dimensional and so any path in the between $\mathcal{F}_1$ and $\mathcal{F}_2$ hits all other chambers in a unique order

Moreover, the only birational contractions of X with $\mathbb{Q}$-factorial targets are $X \to \mathbb{P}^3$ and $X \dashrightarrow Y$ both of which are terminal.

Conversely, if X is an MDS then we can run the so-called 2-ray game. This is explained in detail in [BLZ21, Section 2F]; here we give a rough idea. Choose an ample divisor A on X and run the $(-A)$-MMP. Any such MMP must terminate with a Mori fibre space. Since $\rho(X) = 2$ on the first step of the $(-A)$-MMP we have a choice between two $(-A)$-negative rays to contract. One MMP outputs $\mathbb{P}^3/pt$, while the other outputs another Mori fibre space Y/T. Now using the fact that the targets of all SQMs are terminal one can check that the diagram produced by this process satisfies all the properties of Definition 3.1. □

3.3. Notation & Setup

Notation 3.12. *Throughout the rest of the paper and unless otherwise stated, C will be a smooth space curve and $\pi\colon X \to \mathbb{P}^3$ will be the blow up of $\mathbb{P}^3$ along C. We will denote by f the numerical class of a fibre of π over a point of C and by l the numerical class of the pullback of a general line in $\mathbb{P}^3$.*

Dually in $\mathcal{N}^1(X)$, we will denote by E the class of the π-exceptional divisor and by H the class of the pullback a general hyperplane in $\mathbb{P}^3$. These classes generate their respective vector spaces and the intersection matrix is determined by the relations: $H \cdot l = 1$, $H \cdot f = E \cdot l = 0$ and $E \cdot f = -1$. Note that in this notation we have the relation: $K_X \sim -4H + E$.

The Mori cone $\overline{\mathrm{NE}}(X)$ of X is a two dimensional cone, with one extremal ray generated by f which is K_X-negative. Thus, whether or not X is a weak Fano is determined by the sign of the second generating ray relative to the canonical divisor.

By abuse of notation, we will also denote by H and l the classes of a hyperplane and a line in $\mathcal{N}^1(\mathbb{P}^3)$ and $\mathcal{N}_1(\mathbb{P}^3)$ respectively.

Lemma 3.13. *The Mori cone $\overline{\mathrm{NE}}(X)$ is spanned by two extremal rays; the first generated by f and the second by the class $l - rf$ with $r \in \mathbb{R}$ maximal among the pseudo-effective classes.*

Dually, $\overline{\mathrm{Eff}}(X)$ is spanned by the two extremal rays; the first generated by E and the second by the class $H - rE$ with $r \in \mathbb{R}$ maximal among the pseudo-effective classes.

Proof. It is clear that f generates an extremal ray since it is the fibre of a contraction. Let $l - sf$ be a pseudo-effective class, with $s \leq r$, then clearly

$$l - sf = l - rf + (r - s)f.$$

The proof is similar for the dual statement. □

Remark 3.14. *The effective representatives of an effective class $dH - mE$, with $d, m \in \mathbb{N}$, are strict transforms of surfaces of degree d having multiplicity m along C.*

Similarly, the effective representatives of an effective class $dl - mf$ which do not lie on E are strict transforms of curves of degree d meeting C at m points counted with multiplicities, that is, the scheme theoretic intersection of C with such a representative is a 0-dimensional scheme of length m.

*We will call such a curve an **m-secant curve** to C and when $d = 1$, an **m-secant line**, when $d = 2$ an **m-secant conic** and so on.*

We note that if C induces a Sarkisov link then by Proposition 3.11 X must be an MDS and thus its cone of curves is closed and rationally generated. In that case the second extremal ray is generated, over $\mathbb{Q}$, by a class of the form $dl - mf$ with $\frac{m}{d}$ maximal among all effective classes.

3.4. Curves in planes or quadrics

We now show that if C lies on a plane or quadric, then it induces a Sarkisov link if and only if the blowup $\mathrm{Bl}_C \mathbb{P}^3$ is Fano. All such curves have been classified in [BL12] and so any open cases cannot lie on surfaces of degree 1 or 2. But first we prove a result which is essential for the rest of the paper.

Proposition 3.15. *Let $\chi\colon Y \dashrightarrow Y'$ be an anti-flip between $\mathbb{Q}$-factorial terminal 3-folds, z be a curve in Y which is not in the indeterminacy locus of χ and z' be its strict transform under χ. Then*

$$K_Y \cdot z \leq K_{Y'} \cdot z'.$$

In particular, if X is the blowup of a Fano threefold of Picard rank 1 such that it admits an infinite number of K_X-non-negative curves then X does not fit into a Sarkisov link.

Proof. Let

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[rr, dashed, "\chi"] & & Y' \end{tikzcd}$$

be a resolution of indeterminacies of χ. Writing the ramification formulas for p and q and comparing them we get

$$q^* K_{Y'} = p^* K_{X_Y} + (E_p - E_q), \tag{III.1}$$

where E_p and E_q are sums of p and q-exceptional divisors, respectively. Since χ is a pseudo-isomorphism, $\operatorname{Supp} E_p = \operatorname{Supp} E_q$. Moreover, the discrepancies decrease under an anti-flip (see [Mat02, Lemma 9.1.3]) and so $E_p - E_q \geq 0$. Let $z_W \subset W$ be the curve dominating both z and z'. Then

$$K_{Y'} \cdot z' = q^* K_{Y'} \cdot z_W = p^* K_Y \cdot z_W + (E_p - E_q) \cdot z_W \geq p^* K_Y \cdot z_W = K_Y \cdot z.$$

Suppose now that X is as described in the statement. Assuming X fits into a Sarkisov diagram we will derive a contradiction. $\overline{\mathrm{NE}}(X)$ has two extremal rays, one of which, which we will denote by R, is K_X-non-negative and the other corresponding to the blowup morphism is K_X-negative. Since X fits into a Sarkisov diagram, R must also be contractible. We distinguish two cases.

If $-K_X$ is nef then all the K_X-non-negative curves are actually K_X-trivial curves. In that case R has to contain all of the infinitely many K_X-trivial curves thus the contraction $f\colon X \to Y$ of R is not small. If it is not divisorial then the induced diagram cannot be a Sarkisov link. Finally, if f is divisorial, writing the ramification formula for f and intersecting both sides with a contracted curve we get that X has canonical but not terminal singularities, giving us a contradiction to the existence of the Sarkisov diagram.

On the other hand if $-K_X$ is not nef, playing the 2-ray game on X and after a finite number of K_X-positive steps, i.e. anti-flips, we must arrive to the variety X_0 lying over the central model of the link which is either Fano or weak-Fano. By the first part of the proposition, X_0 must still contain an infinite number of K_{X_0}-non-negative curves. If X_0 is Fano this is already a contradiction. If X_0 is weak-Fano then the contraction of the K_{X_0}-trivial ray of $\overline{\mathrm{NE}}(X_0)$ must be small, i.e. contain a finite number of curves, again a contradiction. □

The following statement follows implicitly from [BL12].

Proposition 3.16. *Let C be a smooth curve contained in a plane or quadric. If C induces a Sarkisov link, then X is weak Fano.*

Proof. If C is contained in a plane or quadric, in [BL12, Proposition 3.1] the authors prove that if the blowup of X is not weak Fano, then C admits an infinite number of m-secants lines, with $m \geq 5$. By Remark 3.14, their class in $N_1(X)$ is $l - mf$ and thus correspond to K_X-positive curves. By Proposition 3.15, we conclude that C does not induce a Sarkisov link. □

Remark 3.17. *We note that, by Proposition 3.16, the lists of [BL12] are exhaustive in the cases where the curve is contained in a plane or quadric. That is, every curve that lies on plane or quadric whose blowup induces a Sarkisov link is studied and appear in their lists.*

4. Curves on smooth cubic surfaces

The goal of this section is to give necessary conditions for curves lying on smooth cubic surfaces to induce a Sarkisov link.

If S is the blowup of $\mathbb{P}^2$ along 6 points $p_1, \ldots, p_6$ in general position (i.e. no 3 on a line and no 6 on a conic), then $-K_S$ is ample and the anti-canonical system gives an embedding in $\mathbb{P}^3$ where the image is a smooth cubic surface. Conversely, any smooth cubic surface in $\mathbb{P}^3$ is obtained like that (see [BL12, Section 4]). In the following we will denote the image of S under the anti-canonical system again by S and when there is no confusion we will not distinguish between the two.

We recall some basic facts about curves on S. Denote by L the numerical class of the pullback of a general line under $S \to \mathbb{P}^2$ and by F_i the class of the fibre over p_i. These classes generate $\mathcal{N}^1(S)$. The intersection form is given by the diagonal $(1, -1, \ldots, -1)$. We will say that a 1-cycle $z \in \mathcal{N}_1(S)$

is of **type** $(k;\, m_1, m_2, m_3, m_4, m_5, m_6)$ if its class is numerically equivalent to $kL - \sum m_1 F_1$. The canonical class on S is then of type $(-3;\, 1, \dots, 1)$. Moreover the cone of curves $\mathrm{NE}(S)$ is closed and generated by the classes of the (-1)-curves. There are 27 such classes corresponding to the 6 exceptional curves, the strict transforms of the 15 lines through two of the points or the 6 conics through five of the points. Their classes are respectively F_i, $L - F_i - F_j$ and $2L - F_1 - \cdots - F_6 + F_i$. Their images are lines in $\mathbb{P}^3$ which we will denote by e_i, $l_{i,j}$ and c_i respectively.

We stick to the notation introduced in Notation 3.12. For the following statements we will also always assume that C lies on a smooth cubic surface S.

Proposition 4.1. *Let γ be a curve in X such that $\gamma \sim dl - mf$ with $\frac{m}{d} > 3$. Then C admits an n-secant line with $n \geq \frac{m}{d}$. In particular, X is weak Fano if and only if C admits no m-secant line with $m \geq 5$.*

Proof. We write T for the strict transform of S. Then T and S are isomorphic and

$$T \cdot \gamma = (3H - E)(dl - mf) = 3d - m < 0$$

thus γ is contained in T. The cone of curves of T is generated by the strict transforms of the 27 lines l_i in $S \subset \mathbb{P}^3$ and so we can write

$$\gamma \sim_T l_1 + \cdots + l_k \implies \gamma \equiv_X l_1 + \cdots + l_k.$$

Intersecting with the restriction of a general hyperplane of $\mathbb{P}^3$ we get that $k = d$. Intersecting with E we get

$$E(l_1 + \cdots + l_d) = m.$$

Therefore, at least one of the lines l_i intersects E at n points counted with multiplicity, with $n \geq \frac{m}{d}$. □

By the previous lemma we see that the property of X not being weak Fano is equivalent to the existence of at least one m-secant line with $m > 4$. On the other hand, the following, we prove that C must not admit "too many" in a sense such lines.

Lemma 4.2. *Let l_1, l_2 be two distinct, intersecting lines on a smooth cubic surface S. Then there exists a pencil of conics $\mathcal{C}$ on S such that each element $c \in \mathcal{C}$ is linearly equivalent to $l_1 + l_2$.*

Proof. Denote by P_1 the unique plane containing both l_1 and l_2, and let l_3 be the residual line of the intersection of P_1 with S. Then the pencil of planes containing l_3 gives us a residual pencil of conics on S which are linearly equivalent to $l_1 + l_2$. □

Lemma 4.3. *Suppose that C admits two distinct, m_1 and m_2-secant lines l_1 and l_2 which intersect each other. If $m_1 + m_2 \geq 8$, then C does not induce a Sarkisov link.*

Proof. By Lemma 4.2 there is a pencil of conics such that each element intersects C at $m_1 + m_2$ points counted with multiplicity. Their strict transforms on X give us an infinite family of K_X-non-negative curves. We conclude by Corollary 3.15. □

We now set-up an algorithm to compute all the curves on smooth cubics, candidates to induce a Sarkisov link. We note that the conditions we will impose at this stage are necessary but not sufficient to guarantee that the curve induces a Sarkisov link.

Remark 4.4. *Suppose that C is of type $(k; m_1, m_2, m_3, m_4, m_5, m_6)$. Up to reordering, we may assume*

$$m_1 \geq m_2 \geq \cdots \geq m_6.$$

Moreover, as explained in [BL12, Set-up 4.1], by performing a number of quadratic transformations

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

based on 3 of the 6 points and changing the left hand side morphism to the right hand side one we may assume that

$$k \geq m_1 + m_2 + m_3.$$

In what follows, we will assume that the type of C satisfies these conditions.

Lemma 4.5. *Let $C \subset S$ be a curve of type $(k; m_1, m_2, m_3, m_4, m_5, m_6)$. Then we have the following inequalities for the intersection of C with the various lines on S:*

$$C \cdot c_1 \geq C \cdot L, \qquad C \cdot l_{5,6} \geq C \cdot l_{i,j}, \qquad C \cdot l_{k,6} \geq C \cdot l_{k,j}, \qquad C \cdot e_1 \geq C \cdot e_i,$$

where L is any line on S.

Proof. We have

$$C \cdot c_1 = 2k - (m_2 + \cdots + m_6) \geq 2k - (m_1 + \cdots + \hat{m}_i + \cdots + m_6) = C \cdot c_i,$$

where the hat notation means that the corresponding term does not appear in the expression. In a similar manner we get the rest of the inequalities. □

Using these inequalities as well as Lemma 4.3 we will bound the quantities $k, m_1, \ldots, m_6$.

Proposition 4.6. *Let $C \subset S$ be a curve of type $(k; m_1, m_2, m_3, m_4, m_5, m_6)$ and suppose that C induces a Sarkisov link and that $X := \mathrm{Bl}_C\, \mathbb{P}^3$ is not weak Fano. Then*

$$k \leq 9 \qquad m_1 \leq 8 \quad \text{and} \quad m_2, \ldots, m_6 \leq 2.$$

Proof. Since we assume that X is not weak Fano and by Proposition 4.1, C admits an m-secant line with $m \geq 5$. Thus, by Lemma 4.5 we have $C \cdot c_1 \geq 5$.

For C to induce a Sarkisov link, by Lemma 4.3 we need to make sure that the sum of any two intersecting lines has intersection less than 8 with C. Applying this to the lines intersecting c_1 we get

$$C \cdot (c_1 + l_{1,j}) \leq 7 \quad \text{and} \quad C \cdot (c_1 + e_n) \leq 7$$

for $n \neq 1$. The second inequality above gives $C \cdot e_n \leq 2$, hence

$$m_2, \ldots, m_6 \leq 2.$$

Now the inequality $C \cdot (c_1 + l_{1,j})$ for $j = 6$ together with the bounds on the $m_2, \ldots, m_6$ give

$$3k - m_1 - \cdots - m_5 - 2m_6 \leq 7 \Rightarrow 3k \leq 19 + m_1. \tag{$\star$}$$

For $k = 1$, the curve C cannot have any m-secants with $m \geq 5$. Since we assume that X is not weak Fano, we get $k \neq 1$ and so by Bézout's theorem on $\mathbb{P}^2$ we get $k > m_1$. Combining with ($\star$) we get the last bounds $k \leq 9$ and $m_1 \leq 8$. $\square$

Using the bounds and checks introduced above we can set up an algorithm to compute all the candidate curves. They are presented in the table bellow.

#	Type	$(C \cdot c_1, \ldots, C \cdot c_6)$	$(C \cdot l_{1,2}, \ldots, C \cdot l_{5,6})$	$\deg(C)$	$g(C)$
1	$(3; 1, 1, 0, 0, 0, 0)$	$(5, 5, 4, 4, 4, 4)$	$(1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3)$	7	1
2	$(3; 2, 0, 0, 0, 0, 0)$	$(6, 4, 4, 4, 4, 4)$	$(1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$	7	0
3	$(4; 2, 1, 1, 1, 0, 0)$	$(5, 4, 4, 4, 3, 3)$	$(1, 1, 1, 2, 2, 2, 2, 3, 3, 2, 3, 3, 3, 3, 4)$	7	2
4	$(5; 2, 1, 1, 1, 1, 1)$	$(5, 4, 4, 4, 4, 4)$	$(2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$	8	5
5	$(3; 2, 1, 0, 0, 0, 0)$	$(5, 4, 3, 3, 3, 3)$	$(0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3)$	6	0
6	$(5; 3, 1, 1, 1, 1, 1)$	$(5, 3, 3, 3, 3, 3)$	$(1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$	7	3

Table III.1: List of candidate curves

Remark 4.7. *The curves above are exactly the ones of Theorem A.*

5. Existence of the links

In the previous section we produced a table of types of curves lying on some smooth cubic, which satisfy the necessary criterion set by Proposition 3.15 to induce a Sarkisov link. In this section we will prove that they actually do induce Sarkisov links.

Again, unless otherwise stated, we stick to the notation introduced in Notation 3.12.

5.1. Some properties of the surfaces $\mathbb{F}_n$

For a proof of the facts that follow see [Bea96, Section IV].

We will denote by $\mathbb{F}_n$ the $\mathbb{P}^1$-bundle $p\colon \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$. Then $\mathbb{F}_n$ admits a unique section σ with self intersection $-n$. We will call any section of self-intersection n an n**-section**. If we denote by f a fibre of p then the Mori cone $\overline{\mathrm{NE}}(\mathbb{F}_n)$ is spanned by the classes of σ and f. The intersection matrix is

$$\begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the basis $\{\sigma, f\}$. The canonical class is $K_{\mathbb{F}_n} = -2\sigma - (n+2)f$. Finally, $\mathbb{F}_n$ is smooth and rational, so $h^i(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}) = h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$, for $i \geq 1$. Thus the Riemann-Roch theorem takes the form

$$\chi(\mathcal{O}_{\mathbb{F}_n}(D)) = \frac{D(D - K_{\mathbb{F}_n})}{2} + 1,$$

for any $D \in \mathrm{Div}(\mathbb{F}_n)$. In particular, if $D \sim a\sigma + bf$, we have

$$\chi(D) = (a+1)(b+1) - n\frac{a(a+1)}{2}.$$

We now prove a lemma on the cohomology of divisors on $\mathbb{F}_n$.

Lemma 5.1. *Let D be a divisor on $\mathbb{F}_n$. Then the following hold:*

1. *if $D \cdot f = -1$, then $h^i(\mathbb{F}_n, D) = 0$ for every $i \geq 0$;*
2. *if $D \cdot f \geq 0$ and $D \cdot \sigma \geq -1$, then $h^1(\mathbb{F}_n, D) = 0$.*

Proof. The proof can be found in [CH20], we merely replicate it for the convenience of the reader.

Assume first that $D \cdot f = -1$. Then D is linearly equivalent to $-\sigma + kf$ for some $k \in \mathbb{Z}$. Since $\overline{\mathrm{NE}}(\mathbb{F}_n)$ is spanned by the classes of σ and f we get that $h^0(\mathbb{F}_n, -\sigma + kf) = 0$ and by Serre duality, $h^2(\mathbb{F}_n, -\sigma + kf) = h^0(\mathbb{F}_n, -\sigma - (n+2+k)f) = 0$. Finally using Riemann-Roch one can compute $\chi(\mathbb{F}_n, D) = 0$ and so $h^1(\mathbb{F}_n, -\sigma + kf) = 0$. Hence, we obtain (1).

Now assume that $D \cdot f = k \geq 0$ and $D \cdot \sigma \geq -1$. Define $N := D - (k+1)\sigma$. We will show by induction on i that $h^1(\mathbb{F}_n, N + i\sigma) = h^1(\mathbb{F}_n, D - (k+1-i)\sigma) = 0$ for every $0 \leq i \leq k+1$. For $i = 0$ we have $N \cdot f = -1$ and thus we are done by (1).

For the inductive step we assume that $h^1(\mathbb{F}_n, N + (i-1)\sigma) = 0$ and consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(N + (i-1)\sigma) \to \mathcal{O}_{\mathbb{F}_n}(N + i\sigma) \to \mathcal{O}_\sigma(N + i\sigma) \to 0.$$

Then the long exact sequence in cohomology yields

$$H^1(\mathbb{F}_n, \mathcal{O}(N + (i-1)\sigma)) \to H^1(\mathbb{F}_n, \mathcal{O}(N + i\sigma)) \to H^1(\sigma, \mathcal{O}(N + i\sigma)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\kappa)) = 0$$

where the last equality follows from the fact that $\kappa = D \cdot \sigma + n(k+1-i) \geq -1$. Thus using the inductive hypothesis we get $h^1(\mathbb{F}_n, N - i\sigma) = 0$.

For $i = k+1$ we get $h^1(\mathbb{F}_n, N + (k+1)\sigma) = h^1(\mathbb{F}_n, D) = 0$ This is (2). $\square$

In the following, $\overline{\mathbb{F}}_n$ will denote the surface obtained by contracting the $(-n)$-section of $\mathbb{F}_n$. To conclude the subsection, we prove a lemma about the intersection theory on $\overline{\mathbb{F}}_n$.

Lemma 5.2. *Let $\phi\colon \mathbb{F}_n \to \overline{\mathbb{F}}_n$ be the contraction of the $(-n)$-section on $\mathbb{F}_n$. Then $\mathrm{WDiv}(\overline{\mathbb{F}}_n) = \langle \bar{f}\rangle$ and $\mathrm{CDiv}(\overline{\mathbb{F}}_n) = \langle \bar{\sigma}_+\rangle$ with $\bar{\sigma}_+ = n\bar{f}$, where $\bar{\sigma}$ and $\bar{f}$ denote the images of an n-section and a fibre respectively. We also have $\bar{\sigma}_+^2 = n$ which implies that $\bar{f}^2 = \frac{1}{n}$.*

Proof. The morphism ϕ is given by the linear system $|\sigma + nf|$ (i.e. the linear system of n-sections). The rank of $\mathrm{CDiv}(\overline{\mathbb{F}}_n)$ is 1 and a general hyperplane section is the image of an n-section, thus $\mathrm{CDiv}(\overline{\mathbb{F}}_n) = \langle\bar{\sigma}\rangle$. Moreover $(\sigma + nf)f = 1$ thus the image of a fibre is a line in $\mathbb{P}^N$ and so $\mathrm{WDiv}(\overline{\mathbb{F}}_n) = \langle\bar{f}\rangle$.

The hyperplanes sections passing through the singular point $\phi(\sigma)$ are images of members of $|\sigma + nf|$ which have σ as an irreducible component. Since σ is contracted, hyperplane sections through $\phi(\sigma)$ are equivalent to $n\bar{f}$ which implies that $\bar{\sigma}_+ = n\bar{f}$.

Finally, since an n-section does not meet the exceptional divisor of ϕ we have $\bar{\sigma}_+^2 = (\sigma+nf)^2 = n$ from which we also conclude that $\bar{f}^2 = \left(\frac{1}{n}\bar{\sigma}_+\right)\left(\frac{1}{n}\bar{\sigma}_+\right) = \frac{1}{n}$. $\square$

5.2. X is a Mori Dream Space

Proposition 5.3. *Let $C \subset \mathbb{P}^3$ be a smooth curve lying on a smooth cubic surface $S \subset \mathbb{P}^3$. Then $X := \mathrm{Bl}_C\,\mathbb{P}^3$ is an MDS.*

Proof. If X is weak Fano, then by Lemma 3.9 and Proposition 3.10, it is an MDS.

Suppose X is not weak Fano. By Lemma 3.13, $\overline{\mathrm{NE}}(X)$ is generated by f and $dl - mf$ with $\frac{m}{d} > 4$ and then by Proposition 4.1, $d = 1$ and $m \geq 5$. Moreover, $K_X \sim -4H + E$ and $S \sim 3H - E$, where by abuse of notation S also denotes the strict transform of S in X. The intersections among those classes are given by the table

	f	$l - mf$
S	1	$3 - m$
K_X	-1	$-4 + m$

Then by Remark 3.2, for any rational number $0 < q < 1$ the pair (X, qS) is klt. If moreover $1 - \frac{1}{m-3} < q < 1$, then intersecting $-(K_X + qS)$ with both f and $l - mf$ we get strictly positive numbers. Kleiman's criterion for ampleness (see [Mat02, Theorem 1-2-5] for a statement and [Kle66] for a proof) implies that $-(K_X + qS)$ is ample and so (X, qS) is log Fano. Thus by definition, X is of Fano type and by Proposition 3.10, X is an MDS. $\square$

5.3. Construction of the pseudo-isomorphisms

Proposition 5.4. *Let l be the strict transform of an m-secant line under $X \to \mathbb{P}^3$. If $m \geq 3$, then*

$$N_{l/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3 - m),$$

where $N_{l/X}$ denotes the normal bundle of l in X.

Proof. Consider the short exact sequence

$$0 \to N_{l/S} \to N_{l/X} \to (N_{S/X})|_l \to 0$$

obtained by dualizing the conormal exact sequence (see [FL85, p. 79]). We have

$$\deg(N_{S/X})|_l = S|_S \cdot l = (3H - E)|_S \cdot l = (-3K_S - C) \cdot l = 3 - m.$$

Moreover $\deg(N_{l/S}) = -1$ because l is a (-1)-curve in S. Since l is a rational curve, all line bundles on it are of the form $\mathcal{O}(d)$, where d is the degree of the bundle and so the exact sequence above becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to N_{l/X} \to \mathcal{O}_{\mathbb{P}^1}(3 - m) \to 0. \tag{†}$$

Extensions of lines bundles

$$0 \to L \to M \to N \to 0$$

are classified by $H^1(X, N^{-1} \otimes L)$ (see [Fri98, p. 31]) and in the case of (†) by $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-4))$. However,

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-4)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-m)) = 0$$

since $m \geq 3$. Thus (†) is the unique extension and is thus trivial. We conclude that $N_{l/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3-m)$. □

Lemma 5.5. *Let C be a smooth rational curve lying in the smooth locus of a 3-fold X with normal bundle $N_{C/X} \cong \mathcal{O}_C(\alpha) \oplus \mathcal{O}_C(\beta)$ for some $\alpha \geq \beta \in \mathbb{Z}$. Let $p \colon E \subset X' \to C \subset X$ be the blowup of C with exceptional divisor $E = \mathbb{P}(N_{C/X}) \cong \mathbb{F}_{\alpha-\beta}$ and let C' be the unique negative section of $E \to C$ or a 0-section if $E \cong \mathbb{F}_0$.*

1. *We have $E \cdot C' = \alpha$; in particular $E|_E \sim_E -C' + \beta f$.*
2. *Suppose that $S \subset X$ is a surface containing C which is smooth along C and that $(C^2)_S = \kappa$. If S' is the strict transform of S, then $D := S'|_E \sim_E C' + (\alpha - \kappa)f$.*
3. *If $N_{C'/X'} = \mathcal{O}_C(\alpha') \oplus \mathcal{O}_C(\beta')$ with $\alpha' \geq \beta'$, then $\alpha' + \beta' = \beta$. If furthermore $2\alpha - \beta < 2$ then $\alpha' = \beta - \alpha$ and $\beta' = \alpha$.*

Proof. We first use the adjunction formula on $C' \subset E$ and $C \subset X$ to obtain

$$K_E \cdot C' = 2g - 2 - (C')^2_E \ \text{ and } \ K_X \cdot C = 2g - 2 - \deg N_{C/X} = 2g - 2 - (\alpha + \beta)$$

respectively. We have

$$K_E = (K_{X'} + E)|_E = (p^*K_X + 2E)|_E \implies K_E \cdot C' = p^*K_X \cdot C' + 2E \cdot C' = K_X \cdot C + 2E \cdot C'$$
$$\implies E \cdot C' = \frac{K_E \cdot C' - K_X \cdot C}{2} = \frac{2g - 2 + \alpha - \beta - (2g-2) + \alpha + \beta}{2} = \alpha.$$

We can then write $E|_E \sim_E kC' + lf$ and use the facts that $E \cdot f = -1$ and $E \cdot C' = \alpha$ to deduce that $k = -1$ and $l = \beta$. This is (1).

Since S is smooth along C, $p\colon D \subset S' \to C \subset S$ is an isomorphism. We write $D \sim_E kC' + lf$. By S being smooth (hence of multiplicity 1) along C we get $k = 1$. We also have

$$\kappa = (D)^2_{S'} = D \cdot E = (C' + lf)(-C' + \beta f) = \alpha - \beta + \beta - l \implies l = \alpha - \kappa.$$

This is (2).

Finally, we have

$$\begin{aligned} \deg N_{C'/X'} = 2g - 2 - K_{X'} \cdot C' = 2g - 2 - (p^* K_X + E) \cdot C' = \\ (2g - 2 - K_X \cdot C) - E \cdot C' = \deg N_{C/X} - \alpha \\ \implies \alpha' + \beta' = \alpha + \beta - \alpha = \beta. \end{aligned}$$

Moreover, since $C' \subset E$ and $E \subset X'$ are regular imbeddings, the normal bundle sequence (see [FL85, Proposition 3.4]) yields

$$0 \to N_{C'/E} \to N_{C'/X'} \to N_{E/X'}|_{C'} \to 0$$

which is actually

$$0 \to \mathcal{O}_{C'}(\beta - \alpha) \to \mathcal{O}_{C'}(\alpha') \oplus \mathcal{O}_{C'}(\beta') \to \mathcal{O}_{C'}(\alpha) \to 0.$$

Such extensions are classified by

$$\operatorname{Ext}(\mathcal{O}_{C'}(\beta - \alpha), \mathcal{O}_{C'}(\alpha)) \cong H^1(\mathbb{P}^1, \mathcal{O}(\beta - 2\alpha)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2 + 2\alpha - \beta)).$$

If $2\alpha - \beta < 2$, then $\operatorname{Ext}(\mathcal{O}_{C'}(\beta - \alpha), \mathcal{O}_{C'}(\alpha)) = 0$, thus the extension above is trivial and we deduce (3). □

Lemma 5.6. *Let N be a nef divisor such that $N^\perp = \mathbb{R}_{\geq 0} C$, where C is the numerical class of a curve and let E be any divisor with $E \cdot C < 0$. Then there exists some $r_0 > 0$ such that for each $r \geq r_0$ the divisor $rN - E$ is ample.*

Proof. First we fix a norm $\|\cdot\|$ on $N_1(X)$ such that $\|C\| = 1$. We consider the linear functionals

$$\begin{array}{cccccccc} n\colon N_1(X) & \to & \mathbb{R}, & \qquad & e\colon N_1(X) & \to & \mathbb{R} \\ C & \mapsto & N \cdot C & & C & \mapsto & E \cdot C. \end{array}$$

Since $N_1(X)$ is a finite dimensional vector space, the functionals n, e are continuous.

Let U be a neighbourhood of C such that for all $x \in U$, $e(x) < 0$. Then for every $x \in U \cap \overline{\mathrm{NE}}(X)$ and $\epsilon \geq 0$ we have $(N - \epsilon E)\, x > 0$.

On the other hand, the set $S = \left(S^1 \setminus U\right) \cap \overline{\mathrm{NE}}(X)$ is a closed subset of the compact set S_1, thus compact. This implies that $n(S)$ is also compact and is contained in $(0, +\infty)$, since $\mathbb{R}_{\geq 0} C \cap S = \emptyset$. Let m be the minimum of $n(S)$. Then for any $x \in S$ and $\epsilon > 0$ we have

$$(N - \epsilon E)\, x = N \cdot y - \epsilon E \cdot y \geq m - \epsilon \|E\| \|y\| = m - \epsilon \|E\|.$$

Then for every $\epsilon \leq \epsilon_0 = \frac{m}{\|E\|}$, $N - \epsilon E$ is strictly positive on $\overline{\mathrm{NE}}(X) \cap S^1$ and thus on the whole $\overline{\mathrm{NE}}(X)$. Finally we set $r_0 = \frac{1}{\epsilon_0}$, $r = \frac{1}{\epsilon}$ and multiply $N - \epsilon E$ by r. Kleiman's criterion for ampleness yields the desired result. □

Next we prove a base-point free type lemma.

Lemma 5.7. *Let C be a smooth rational curve lying on a smooth surface E in the smooth locus of a 3-fold X. We assume that:*

- *C generates an extremal ray of $\overline{\mathrm{NE}}(X)$*
- *$E \cong \mathbb{F}_n$;*
- *$E \cdot C < 0$;*
- *$C \subset E$ is the unique $(-n)$-section of $\mathbb{F}_n$;*
- *C is the unique irreducible curve on X whose numerical class lies on $\mathbb{R}_{\geq 0}[C] \subset \overline{\mathrm{NE}}(X)$.*

Then C is contractible, i.e. there exists a birational morphism $X \to X'$ that contracts C and only C.

Proof. Since C generates an extremal ray of $\overline{\mathrm{NE}}(X)$, there exists a nef divisor N such that $N^{\perp} = \mathbb{R}_{\geq 0}[C]$. By Lemma 5.6, the divisor $A := rN - E$ is ample for $r \gg 0$. Then there exists some $k_0 \in \mathbb{N}$ such that $h^1(X, kA) = 0$ for every $k \geq k_0$. We fix such k and using induction on i we will prove that $h^1(X, kA + iE) = 0$ for $0 \leq i \leq k - 1$.

For the base case $i = 0$ we get $h^1(X, kA) = 0$ by our choice of A and k. For the inductive step we assume that $h^1(X, kA + (i-1)E) = 0$ and consider the exact sequence

$$H^1(X, \mathcal{O}(kA + (i-1)E)) \to H^1(X, \mathcal{O}(kA + iE)) \to H^1(E, \mathcal{O}(kA + iE)).$$

By possibly repicking r even larger, again using Lemma 5.6, we may assume that $kA + iE$ is positive against any fibre of $E \cong \mathbb{F}_n \to \mathbb{P}^1$. Moreover it is also positive against C and thus using Lemma 5.1 we get $h^1(E, kA + iE) = 0$. Consequently, using the inductive hypothesis we get that $h^1(X, kA + iE) = 0$ for $0 \leq i \leq k - 1$. Especially, for $i = k - 1$ we get $h^1(X, krN - E) = 0$.

Notice that $rN = A + E$ where A is ample and E is effective. This implies that among the global sections of rN there are sections of the form $s_i = a_i e_i \in H^0(X, A) \otimes H^0(X, E) \subseteq H^0(X, rN)$. Since A is ample, this immediately implies that rN has no base points away from E.

As for any base points on E we consider the exact sequence

$$H^0(X, \mathcal{O}(krN)) \to H^0(E, \mathcal{O}(krN)) \to H^1(X, \mathcal{O}(krN - E)) = 0.$$

Then $krN|_E$ is a nef divisor on a smooth surface, zero only against a rational curve of negative self-intersection. Consequently, it is semi-ample (see for example the proof of [Mat02, Theorem 1-1-6]). Finally using the exact sequence above we may lift sections of $krN|_E$ to sections of krN, proving that the stable base locus of N does not meet E. Thus N is semiample. □

Proposition 5.8 (The $(-1,-m)$-FLIP)**.** *Let C be smooth rational curve lying in the smooth locus of a 3-fold X. Suppose that $X \to Z$ is a contraction morphism, contracting only C and that the normal bundle $N_{C/X}$ of C in X is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m)$ with $1 \leq m \leq 3$. Then the anti-flip of C exists (i.e. there exists an SQM of X over Z centred at C and the target variety has at worst terminal singularities).*

Proof. The cases $m = 1$ and 2 are the classical cases of the Atiyah flop and the Francia flip respectively. We refer to [Deb01, 6.10, p. 162] for the explicit construction of the resolution of the anti-flip.

Suppose that $m = 3$. Write $Y \to X$ for the blowup of C with exceptional divisor $E \cong \mathbb{F}_{m-1} = \mathbb{F}_2$. Denote by σ the (-2)-section and by f a fibre of $E \to C$. Then the relative cone of curves $\mathrm{NE}(Y/Z)$ is generated by the classes of σ and f and by Lemma 5.7, σ is contractible.

By Lemma 5.5 the normal bundle of σ in Y is $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-2)$ and so the anti-flip $\chi\colon Y \dashrightarrow Y'$ of σ exists. More specifically, χ is the inverse of a Francia flip. If $Y \to \hat{Z}$ is the contraction of σ, we have the diagram

$$\begin{tikzcd} Y \arrow[rr, dashed, "\chi"] \arrow[dr] \arrow[d] & & Y' \arrow[dl] \\ X \arrow[dr] & \hat{Z} \arrow[d] & \\ & Z & \end{tikzcd}$$

where $\hat{Z} \to Z$ is the morphism induced by the inclusion $\overline{\mathrm{NE}}(Y/\hat{Z}) \subset \overline{\mathrm{NE}}(Y/Z)$. Using the explicit resolution of the Francia flip in [Deb01, 6.10, p. 162] in conjunction with Lemma 5.5(2), one can check that the restriction of χ on E is the contraction of the (-2)-curve. The relative cone of curves $\mathrm{NE}(Y'/Z)$ is generated, over $\mathbb{Q}$, by the classes of the anti-flipped curve as well as the class of any curve in the strict transform E' of E. This is because $E' \cong \bar{\mathbb{F}}_2$ and so, by Lemma 5.2, $\rho(E') = 1$, thus the numerical class of any curve on it covers E'. Thus if σ_+ is a section of E disconnected from σ, $\sigma'_+ := \chi(\sigma_+)$ generates an extremal ray of $\mathrm{NE}(Y')$. Furthermore since we chose σ'_+ to be disconnected from the centre of χ we have

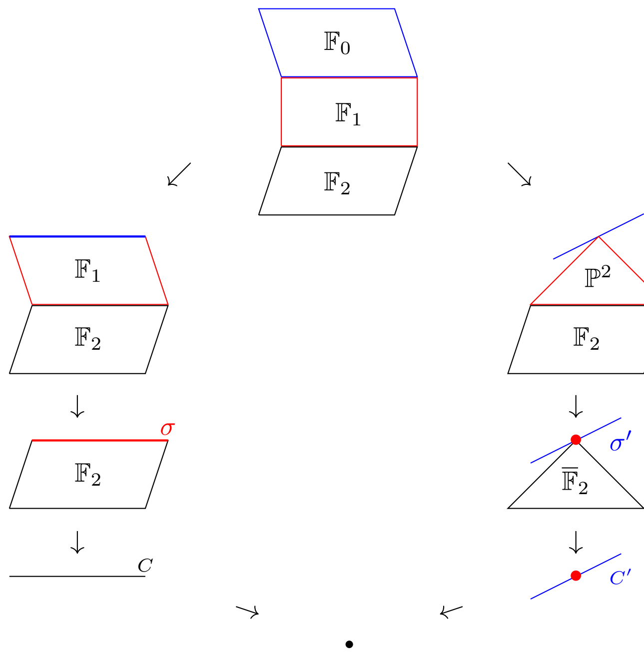
$$K_{Y'} \cdot \sigma'_+ = K_Y \cdot \sigma_+ = K_Y \cdot (\sigma + 2f) = 1 - 2 = -1.$$

Thus σ'_+ is contractible by a divisorial contraction $Y' \to X'$ and since it's $K_{Y'}$-negative and Y' has terminal singularities, then so does X'. The diagram above becomes

$$\begin{tikzcd} Y \arrow[rr, dashed, "\chi"] \arrow[dr] \arrow[d] & & Y' \arrow[dl] \arrow[d] \\ X \arrow[rr, dashed, "\psi"] \arrow[dr] & \hat{Z} & X' \arrow[dl] \\ & Z & \end{tikzcd}$$

where ψ is the birational map induced by the diagram and actually the required pseudo-isomorphism. $\square$

Schematically the resolution described in the proof above looks as follows



where the dot represents the terminal point of X' which is actually a quotient singularity of type $\frac{1}{3}(1,1,2)$.

5.4. Conclusion

Proposition 5.9. *Let $C \subset S$ be one of the curves in Table III.1 so that the strict transforms of the maximal secant lines have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m)$, with $1 \le m \le 3$. Let $\chi \colon X \dashrightarrow X'$ be the anti-flip of those strict transforms as constructed in Proposition 5.8.*

If C admits no 4-secant lines, then X' is Fano. Otherwise, X' is weak-Fano and the strict transforms of 4-secant lines generate (and are the only irreducible curves whose class is contained in) an extremal ray of $\mathrm{NE}(X')$.

Proof. Let $f' \colon X' \dashrightarrow Z$ be the anti-canonical model of X'. Since χ is a pseudo isomorphism, we have

$$Z = \mathrm{Proj}\left(\bigoplus_{m\in\mathbb{N}} H^0(X', -mK_{X'})\right) = \mathrm{Proj}\left(\bigoplus_{m\in\mathbb{N}} H^0(X, -mK_X)\right).$$

Now any curve c' in X' such that $-K_{X'} \cdot c' \le 0$ is either contracted by or in the base locus of f'. If $f := f' \circ \chi$ and c is the strict transform of c' under χ^{-1} then, by the equality above, c is again either contracted by or in the base locus of f. Thus $-K_X \cdot c \leq 0$. Assume for a second that those curves are exactly the 4, 5 or 6-secant lines. Then a look at the table shows that when C admits a 6-secant line then it admits no 5-secant lines. Thus, after the anti-flip, X' is Fano if and only if C admits no 4-secant lines. Moreover, if C admits 4-secant lines, then their strict transforms are the only $K_{X'}$-zero curves.

To complete the proof we have to show that the only curves with $-K_X \cdot c \leq 0$ are the 4, 5 or 6-secant lines. Let c be such a curve, i.e. $c \sim dl - mf$ with $\frac{m}{d} \geq 4$, which is not a line. Then c is contained in the cubic surface S and so we may write

$$c \sim_S l_1 + \cdots + l_d,$$

with $l_i \sim l - m_i f$ being lines in S. We first note that if for some $1 \leq i \leq d$, l_i does not meet any of the other lines in the decomposition then $(c \cdot l_i)_S \leq -1$ and thus c is not irreducible. Thus we may assume that every line intersects another one. If all l_i were 4-secants then, by Lemma 4.3, they wouldn't intersect each other and so, for $\frac{m}{d}$ to be greater than or equal to 4, we may assume that some of the lines in the decomposition of c are 5 or 6-secants. From Table III.1, we see that in all cases there are at most two 5 or 6-secant lines. We rearrange the decomposition of c in the following way:

$$c \sim_S a_1 l_1 + a_2 l_2 + \sum_i^e l_i^{12} + \sum_j^{r_1} l_j^1 + \sum_k^{r_2} l_k^2 + \sum_n^D l_n^0,$$

where $D = d - a_1 - a_2 - e - r_1 - r_2$ and l_1 and l_2 are the two 5 or 6-secants; the lines l_i^{12} are the lines that meet both l_1 and l_2; the lines l_j^1 meet l_1 but not l_2; the lines l_k^2 meet l_2 but not l_1; the lines l_n^0 meet none of the l_1 and l_2. If there is only one 5 or 6-secant, we simply choose $a_2 = 0$ and get empty sums for l_i^{12} and l_k^2. Intersecting with l_1 and l_2 respectively we get $a_1 \leq e + r_1$ and $a_2 \leq e + r_2$. By Lemma 4.3, for any j and k, we have $C \cdot (l_1 + l_j^1), C \cdot \left(l_2 + l_k^2\right) \leq 7$. Similarly, checking Table III.1 we see that for any i, we have $C \cdot (l_1 + l_2 + l_i^{12}) \leq 11$. Finally, we have

$$\begin{aligned}
m = C \cdot c = C \cdot & \left(a_1 l_1 + a_2 l_2 + \sum_i^e l_i^{12} + \sum_j^{r_1} l_j^1 + \sum_k^{r_2} l_k^2 + \sum_n^D l_n^0 \right) \\
&\leq C \cdot \left((e + r_1) l_1 + (e + r_2) l_2 + \sum_i^e l_i^{12} + \sum_j^{r_1} l_j^1 + \sum_k^{r_2} l_k^2 + \sum_n^D l_n^0 \right) \\
&= C \cdot \left(\sum_i^e (l_1 + l_2 + l_i^{12}) + \sum_j^{r_1} (l_1 + l_j^1) + \sum_k^{r_2} (l_2 + l_k^2) + \sum_n^D l_n^0 \right) \\
&\leq 11e + 7(r_1 + r_2) + 4D \leq 4d - e - r_1 - r_2 < 4d,
\end{aligned}$$

which contradicts $\frac{m}{d} \geq 4$. $\square$

Remark 5.10. *In cases* 5 *and* 6 *of Table III.1, a similar argument as the one presented in the proof above, gives us a more precise result. Namely, that any curve with* $-K_X \cdot c < 1$ *is a* 4, 5 *or* 6*-secant line.*

Indeed, we may repeat the calculation above but instead of singling out only the 5*-secant lines, we single out the* 4*-secants as well. We then only need to observe that in those cases, any two intersecting lines we* l_1 *and* l_2 *we have* $C \cdot (l_1 + l_2) \leq 6$ *and for any line* l_3 *joining a* 4*-secant* l_1 *and a* 5*-secant* l_2 *we have* $C \cdot (l_1 + l_2 + l_3) \leq 9$.

Theorem 5.11. *Let* C *be a space curve lying on a smooth cubic surface* S *such that* $X := \mathrm{Bl}_C \mathbb{P}^3$ *is not weak Fano. Then* C *induces a Sarkisov link if and only if its class on* S *appears in Table III.1 (up to the assumptions of Remark 4.4).*

Proof. The first implication is clear since the curves appearing in Table III.1 are exactly those satisfying the necessary conditions of Proposition 4.6.

Conversely, assume that C is one of the curves in the Table. We first note that in all cases Proposition 5.3 implies that the blowup of C is always an MDS. In turn, by Proposition 3.11, we are guaranteed the existence of a Sarkisov link as long as the varieties produced by the 2-ray game are terminal, which we now check: By Propositions 5.4 and 5.8 the anti-flip of the 5 or 6-secant lines produces a terminal 3-fold. Furthermore, by Proposition 5.9 the 3-fold is (weak-)Fano. Thus any further step in the 2-ray game is K-non-positive and so retains the terminal singularities. ☐

6. Study of the links

In this final section we aim to finish the construction of the links produced in the previous section. We also calculate some invariants of the targets of the links such as their singularities and the cube of the anti-canonical divisor.

6.1. Some preliminary calculations

Lemma 6.1 ([BL12, Lemma 2.4])**.** *Let* $C \subset Y$ *be a smooth curve of genus* g *in a smooth* 3*-fold and let* $\pi \colon X \to Y$ *be the blowup of* C*. Then*

$$(-K_X)^3 = (-K_Y)^3 + 2(-K_Y)C - 2 + 2g.$$

In particular, if $Y = \mathbb{P}^3$ *and* C *is a curve of degree* d*, we have* $(-K_X)^3 = 62 - 8d + 2g$.

In the following Lemma we use the notation introduced in Lemma 5.2.

Lemma 6.2. *Let* $p \colon X \to Y$ *be a divisorial contraction to a point, between* $\mathbb{Q}$*-factorial terminal threefolds with exceptional divisor* E*. If* E *lies on the smooth locus of* X *and* $K_X = p^*K_Y + \alpha E$ *is the ramification formula then*

if $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ *with normal bundle* $N_{E/X} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1)$, $\alpha = 1$*;*

if $E \cong \mathbb{P}^2$ with normal bundle $N_{E/X} = \mathcal{O}_{\mathbb{P}^2}(-2)$, $\alpha = \frac{1}{2}$.
Moreover, we have $-K_Y^3 = -K_X^3 + 2$, $-K_Y^3 = -K_X^3 + \frac{1}{2}$ in the two cases respectively.

Similarly, without any extra assumptions on the singularities this time, if $E \cong \overline{\mathbb{F}}_2$ with normal sheaf $N_{E/X} = \mathcal{O}_{\overline{\mathbb{F}}_2}(-3\bar{f})$ and $K_X \cdot \bar{\sigma} = -1$, then $\alpha = \frac{1}{3}$. Moreover, we have $-K_Y^3 = -K_X^3 + \frac{1}{6}$.

Proof. We first compute the discrepancy. Denote by l a line in E, if $E \cong \mathbb{P}^2$ or any ruling of E, if $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. By the adjunction formula for E we have $K_E = (K_Y + E)|_E$. Intersecting both formulas with l and solving for α we get

$$\alpha = \frac{K_E \cdot l - E \cdot l}{E \cdot l}.$$

In the first case we have $K_E \cdot l = -2$ and $E \cdot l = -1$ giving us $\alpha = 1$. In the second case we have $K_E \cdot l = -3$ and $E \cdot l = -2$ and we get $\alpha = \frac{1}{2}$.

In the third case, intersecting the ramification formula for p with $\bar{\sigma}$ we get

$$\alpha = \frac{K_X \cdot \bar{\sigma} - p^*K_Y \cdot \bar{\sigma}}{E \cdot \bar{\sigma}} = \frac{-1 - p^*K_Y \cdot \bar{\sigma}}{(-3\bar{f})\bar{\sigma}},$$

which, since $\bar{\sigma}$ is p-exceptional, equals to $\frac{1}{3}$.

Finally we have

$$K_X^3 = (p^*K_Y + \alpha E)^3 = K_Y^3 + 3\alpha(p^*K_Y)^2 E + 4\alpha^2(p^*K_Y)E^2 + \alpha^3 E^3.$$

In all cases the middle terms vanish. In the first case we have $\alpha = 1$ and $E^3 = 2$, in the second case we have $\alpha = \frac{1}{2}$ and $E^3 = 4$ and in the last case we have $\alpha = \frac{1}{3}$ and $E^3 = \frac{9}{2}$ and so a computation completes the proof. □

Corollary 6.3. *Let $X \dashrightarrow X'$ be a $(1,2)$-flip. Then $-K_{X'}^3 = -K_X^3 + k\frac{1}{2}$ where k is the number of flipped curves.*

Similarly, if $X \dashrightarrow X'$ is a $(1,3)$-flip, then $-K_{X'}^3 = -K_X^3 + k\frac{8}{3}$ where k is the number of flipped curves.

Proof. For simplicity we will assume that the number of flipped curves is 1. The general case is similar.

We first treat the $(1,2)$-flip. Consider the resolution

$$\begin{tikzcd}
& Y_0 \arrow[dl, "r_0"'] \arrow[dr, "s_0"] & \\
Y_1 \arrow[d, "r_1"'] & & Y_1' \arrow[d, "s_1"] \\
X \arrow[rr, dashrightarrow] & & X'
\end{tikzcd}$$

as constructed in [Deb01, 6.10, p. 162]. By [BL12, Lemma 2.4] we have

$$-K_{Y_1}^3 = -K_X^3 + 2K_X \cdot C - 2 + 2g$$

where C is the flipped curve and g is its genus. Since $K_X \cdot C = 1$ and $g = 0$ we get $-K_{Y_1}^3 = -K_X^3$. We also have $-K_{Y_1}{}^3 = -r_0^* K_{Y_1}{}^3 = (-K_{Y_0} + E)^3 = -s_0^* K_{Y_1'}{}^3 = -K_{Y_1'}^3$. Finally, by Lemma 6.2 we have $-K_{X'}^3 = -K_{Y_1'}^3 + \frac{1}{2} = -K_X^3 + \frac{1}{2}$.

For the $(1,3)$-flip we again consider the resolution

$$\begin{tikzcd} Y \arrow[r, dotted, "\chi"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, dotted] & X' \end{tikzcd}$$

constructed in Proposition 5.8, where now χ is a $(1,2)$-flip. Using the formula $-K_Y^3 = -K_X^3 + 2K_X \cdot C - 2 + 2g$ and since $K_X \cdot C = 2$ we get $-K_Y^3 = -K_X^3 + 2$. By the previous statement and by Lemma 6.2, following the diagram counter clockwise we get $-K_{X'}^3 = -K_{Y'}^3 + \frac{1}{6} = -K_Y^3 + \frac{1}{2} + \frac{1}{6} = -K_X^3 + 2 + \frac{1}{2} + \frac{1}{6} = -K_X^3 + \frac{8}{3}$. □

6.2. Some properties of the links

Dimension of linear system of cubics

A careful examination of Table III.1 reveals that the curves 1 through 4, all admit a pencil of 7-secant conics. Indeed, using Lemma 4.2 this amounts to finding two distinct, intersecting, m_1 and m_2-secant lines such that $m_1 + m_2 = 7$. For example, a pair of such lines that works in all cases is c_1 and $l_{1,5}$. This immediately implies that in those cases, C is contained in a unique cubic.

However, this is not true for the last two cases of the table as the following Lemma shows.

Lemma 6.4. *Let C be one of the last two curves of Table III.1. Then C is contained in a pencil of cubics.*

Proof. We consider the exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(3H - E)) \to H^0(X, \mathcal{O}_S(3H - E|_S)) \to H^1(X, \mathcal{O}_X).$$

Since X is projective, rational with rational singularities, we have $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$. Thus

$$h^0(X, \mathcal{O}_X(3H - E)) = h^0(X, \mathcal{O}_S(3H - E|_S)) + 1,$$

where the sections on the left-hand side of this equation correspond to cubics containing C. Then one can check that the divisor $3H - E|_S$ is effective and fixed. We do this calculation for the case 5 of the table.

We first note that $3H - E|_S$ has negative intersection with and thus contains all 4 and 5-secant lines in its base locus with multiplicity 1 and 2 respectively. We then have

$$3H - E|_S - 2c_1 - c_2 = -3K_S - C - 2c_1 - c_2 = 0$$

and so the movable part of $3H - E|_S$ is zero, i.e. $h^0(X, \mathcal{O}_S(3H - E|_S)) = h^0(X, \mathcal{O}_S) = 1$, proving the claim. $\square$

Mori chambers

Let

$$\begin{tikzcd} X \arrow[rr, dotted, "\chi"] \arrow[d] & & Y \arrow[d] \\ \mathbb{P}^3 \arrow[dr] & & Z \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

be a Sarkisov diagram, where $X \to \mathbb{P}^3$ is the blowup of one of the curves of Table III.1 and χ is a pseudo-isomorphism which is a composition of anti-flips, flops and flips ($Y \to Z$ can be either divisorial or of fibre type).

We have already proven that X is an MDS and so by Proposition 3.7 the pseudo-effective cone of X admits a decomposition into chambers of the form

$$\mathcal{C}_i = g_i^* \operatorname{Nef}(Y_i) + \mathbb{R}_{\geq 0}\{E_1, \dots, E_k\},$$

where g_i are birational contractions and E_j are prime divisors contracted by g_i. Moreover, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are neighbouring chambers, Y_i and Y_j are connected by an SQM or an extremal contraction.

Since $\rho(X) = 2$, the pseudo-effective cone $\overline{\operatorname{Eff}}(X)$ is 2-dimensional. In the following Lemma we will compute $\overline{\operatorname{Eff}}(X)$.

Lemma 6.5. *If C is one of the curves of Table III.1, then $\overline{\operatorname{Eff}}(X)$ is spanned by the divisors E and $3H - E$.*

Proof. By Lemma 3.13, $\overline{\operatorname{Eff}}(X)$ is spanned by E and a divisor $dH - mE$ with $\frac{m}{d}$ maximal. We first note that in all 6 cases the pencil of conics $\mathcal{P}$ associated to the lines c_1 and $l_{1,5}$ (see Lemma 4.2) is a pencil of 6 or 7-secant conics (depending on the case), which spans a cubic containing C. If now $D \sim dH - mE$ is an effective divisor with $\frac{m}{d} > \frac{1}{3}$ whose support is irreducible, then using Bézout's theorem, we deduce that all conics in $\mathcal{P}$ are contained in D. Since $\mathcal{P}$ spans a cubic S, we have $S \subseteq D$. Since D is irreducible we get $S = D$. $\square$

We then have the following dichotomy:

- As discussed in the beginning of this subsection, curves in cases 1 through 4 of Table III.1 are contained in a unique cubic surface S , thus the morphism associated to the linear system of cubics is $X \to pt$. This implies that the rightmost chamber $\mathcal{C}$, which is the one associated to Z, contains only big divisors (since they all are a positive combination of an ample and an effective divisor). Thus Z is birational to X and so it is a Fano 3-fold of Picard rank 1. Moreover $Y \to Z$ is the contraction of the strict transform of S. This is a link of **Type II**.

- In cases 5 and 6, Lemma 6.4 yields that the rational map associated to the system of cubics is $X \to \mathbb{P}^1$. This implies that $Z = \mathbb{P}^1$ and in turn that Y is a del-Pezzo fibration with fibres the strict transforms of the cubics containing C. This is a link of **Type I**.

The matrix of the transformation

The pseudo-isomorphism χ induces an isomorphism χ_* between the groups $\mathrm{WDiv}(X)$ and $\mathrm{WDiv}(Y)$. The isomorphism is given by restricting prime divisors in the regular locus of χ and taking the closure of their image in Y_0. We may extend this to an isomorphism between the $\mathbb{Q}$-vector spaces $\mathrm{WDiv}_{\mathbb{Q}}(X)$ and $\mathrm{WDiv}_{\mathbb{Q}}(Y)$.

We fix a cubic S containing C and define $T := \chi_* S$. We also fix the bases $\mathrm{WDiv}_{\mathbb{Q}}(X) = \langle H, E \rangle$ and $\mathrm{WDiv}_{\mathbb{Q}}(Y_0) = \langle K_{Y_0}, T \rangle$. Note that these divisors do not necessarily generate the $\mathbb{Z}$-modules $\mathrm{WDiv}(X)$ and $\mathrm{WDiv}(Y_0)$. Since $K_X = -4H + E \mapsto K_{Y_1}$ and $S = 3H - E \mapsto T$, the matrix of the isomorphism is $\begin{pmatrix} -1 & -3 \\ -1 & -4 \end{pmatrix}$ with inverse $\begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix}$.

6.3. Some invariants of the targets of the links - Cases 1 through 4

As proven earlier, in the cases 1-4 the Sarkisov diagram takes the form

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d] & X_0 \arrow[r, dotted] & Y_0 \arrow[d, "p"] \\ \mathbb{P}^3 \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

where Y is a Fano 3-fold of Picard rank 1.

The contraction $Y_0 \to Y$

The restriction of the pseudo-isomorphism $\chi\colon X \dashrightarrow Y_0$ to the cubic S is the contraction of the anti-fliped/flopped curves. This can be verified using the explicit resolutions of the $(1, m)$-flips of Proposition 5.8 as well as Lemma 5.5.

In cases $1, 2$ and 4 the restriction $\chi|_S\colon S \to T$ is just the contraction of 6 (-1)-curves and so $T \cong \mathbb{P}^2$. More specifically, $\chi|_S$ fits into the diagram

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "\chi|_S"] & \\ \mathbb{P}^2 & & T \end{tikzcd}$$

where p is the blow up of 6-points followed by the contraction of the 6 conics through 5 of the 6 points. We may compute that the pullback of line $l \subset T \cong \mathbb{P}^2$ on S is of type $(5; 2, 2, 2, 2, 2, 2)$. We then have

$$T|_T \cdot l = S|_S \cdot (\chi|_S)^* l,$$

which in all 3 cases can be computed to be -2. Thus $T \cong \mathbb{P}^2$ with $N_{T/Y_0} = \mathcal{O}_{\mathbb{P}^2}(-2)$.

In case 3, the restriction $\chi|_S \colon S \to T$ contracts 5 (-1)-curves, thus T can be isomorphic to either $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$. However, in reference to the morphism $S \to \mathbb{P}^2$, $\chi|_S$ contracts the strict transforms of: 4 conics through 5 of the points; 1 line through 2 of the points. This implies that T is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. More specifically, $\chi|_S \colon S \to T$ factors as

$$\begin{tikzcd}
 & S \arrow[dl] \arrow[dr] \arrow[rr] & & S' \arrow[dl] \arrow[dr] & \\
\mathbb{P}^2 & & T' \cong \mathbb{P}^2 & & T
\end{tikzcd}$$

where starting from left to right the morphism are: the blowup of 6 points, the contraction of the 6 conics though 5 of the 6 points, the blowup of 2 points and finally the contraction of the line through the 2 points.

Pulling back classes of the two rullings of T under the morphism $S \to T$, we find that they are the strict transforms of lines in T' passing through the 2 blown up points. Their classes on S are $(5; 2, 2, 2, 2, 2, 2) - c_5$ and $(5; 2, 2, 2, 2, 2, 2) - c_6$. As above, intersecting $S|_S$ with both those classes we get -1 and so we find that the normal bundle of T in Y_0 is $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.

Singularities of Y

Following the Sarkisov diagram clockwise we may compute the singularities of Y. We will do this case by case.

- #1. We have the $(1, 2)$-flip of 2 curves $X \dashrightarrow X_0$, followed by the $(1, 1)$-flop of 4 more curves $X_0 \dashrightarrow Y_0$ and finally the contraction of $T \cong \mathbb{P}^2$ with $N_{T/Y_0} = \mathcal{O}_{\mathbb{P}^2}(-2)$. These modifications produce 2 quotient singularities of type $\frac{1}{2}(1, 1, 1)$, no singularities and another quotient singularity of type $\frac{1}{2}(1, 1, 1)$ respectively.
- #2. We have the $(1, 3)$-flip of a curve, followed by the flop of 5 curves and finally the contraction of $T \cong \mathbb{P}^2$ with $N_{T/Y_0} = \mathcal{O}_{\mathbb{P}^2}(-2)$. These modifications produce 1 quotient singularity of type $\frac{1}{3}(1, 1, 2)$, no singularities and a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ respectively.
- #3. We have the $(1, 2)$-flip of a curve, followed by the flop of 4 curves (notice that the line $l_{5,6}$ is a 4-secant) and finally the contraction of $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $N_{T/Y_0} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. These modifications produce 1 quotient singularity of type $\frac{1}{2}(1, 1, 1)$, no singularities and an ordinary double point respectively.
- #4. We have the $(1, 2)$-anti-flip of a curve, followed by flop of 5 curves and finally the contraction of $T \cong \mathbb{P}^2$ with $N_{T/Y_0} = \mathcal{O}_{\mathbb{P}^2}(-2)$. These modifications produce 1 quotient singularity of type $\frac{1}{2}(1, 1, 1)$, no singularities and a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ respectively.

Cube of $-K_Y$

Again following the Sarkisov diagram clockwise and using Lemmata 6.1 and 6.2 as well as Corollary 6.3 we may compute that

$$\begin{aligned}
&\#1.\ -K_X^3 = 8 \implies -K_{X_0}^3 = 9 \implies -K_{Y_0}^3 = 9 \implies -K_Y^3 = \tfrac{19}{2};\\
&\#2.\ -K_X^3 = 6 \implies -K_{X_0}^3 = \tfrac{26}{3} \implies -K_{Y_0}^3 = \tfrac{26}{3} \implies -K_Y^3 = \tfrac{55}{6};\\
&\#3.\ -K_X^3 = 10 \implies -K_{X_0}^3 = \tfrac{21}{2} \implies -K_{Y_0}^3 = \tfrac{21}{2} \implies -K_Y^3 = \tfrac{25}{2};\\
&\#4.\ -K_X^3 = 8 \implies -K_{X_0}^3 = \tfrac{17}{2} \implies -K_{Y_0}^3 = \tfrac{17}{2} \implies -K_Y^3 = 9.
\end{aligned}$$

Fano index of Y

We distinguish cases.
Cases 1 and 4: In those cases, the least common multiple among the indices of the singularities of Y is 2. This implies that $2\,\mathrm{WDiv}(Y) \subseteq \mathrm{CDiv}(Y)$ (see [Kaw88, Corollary 5.2]). Denote by r the Fano-Weil index of Y. That is precisely the number

$$r := \max\{q \in \mathbb{Z} \mid -K_Y \sim qA, A \text{ is a Weil divisor}\}.$$

Let A be a Weil divisor such that $-K_Y = rA$. Then, using Lemma 6.2, we have

$$rp^*(2A) = p^*(2rA) = p^*(-2K_Y) = -2K_{Y_0} + T,$$

which is the vector $(-2, 1)$ in the $\mathbb{Z}$-submodule $\langle K_{Y_0}, T\rangle \leq \mathrm{WDiv}(Y_0)$. By the calculations in Subsection 6.2 its strict transform is

$$r\chi_*^{-1}(p^*(2A)) = \begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix} = (11H - 3E)$$

which is not divisible in $\mathrm{WDiv}(X)$. Thus $r = 1$.
Case 2: Similarly, we have $6\,\mathrm{WDiv}(Y) \subseteq \mathrm{CDiv}(Y)$. We have

$$rp^*(6A) = p^*(6rA) = p^*(-6K_Y) = -6K_{Y_0} + 3T$$

and so

$$r\chi_*^{-1}(p^*(6A)) = \begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 33 \\ -9 \end{pmatrix} = (33H - 9E),$$

which is divisible by 3 in $\mathrm{WDiv}(X)$. Thus $r = 1$ or 3. However, the divisor $A = p_*(\chi_*(11H - 3E))$ has the property $-K_Y = 3A$, thus the index is 3.
Case 3: In the final case we have $2\,\mathrm{WDiv}(Y) \subseteq \mathrm{CDiv}(Y)$ and

$$rp^*(2A) = p^*(2rA) = p^*(-2K_Y) = -2K_{Y_0} + 2T.$$

We thus get

$$r\chi_*^{-1}(p^*(2A)) = \begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -4 \end{pmatrix} = (14H - 4E)$$

which is divisible by 2 in $\mathrm{WDiv}(X)$. As before we may conclude that the index is 2.

We summarize the data in the following table.

#	Type of the contraction p	Singularities of Y	$-K_Y^3$	Fano-Weil Index
1	$E5$	$3 \times \frac{1}{2}(1,1,1)$	$\frac{19}{2}$	1
2	$E5$	$\frac{1}{2}(1,1,1), \frac{1}{3}(1,1,2)$	$\frac{55}{6}$	3
3	$E3$	$\frac{1}{2}(1,1,1), \mathrm{odp}$	$\frac{25}{2}$	2
4	$E5$	$2 \times \frac{1}{2}(1,1,1)$	9	1

6.4. Some invariants of the targets of the links - Cases 5 and 6.

In cases 5 and 6 of the table, a similar argument as Proposition 5.9 together with Remark 5.10 shows that after anti-flipping the 5-secant lines we have the flop of any 4-secant lines. Using again Remark 5.10 we see that there are no irreducible curves between (the rays spanned by the classes of) the 4-secants and the 3-secants and so the next step in the link is the contraction of the 3-secant lines. This is given by the linear system of cubics containing C, thus the Sarkisov diagrams take respectively the forms

$$\begin{tikzcd} X \arrow[r, dotted, "\text{anti-flip}"] \arrow[d] & X_0 \arrow[r, dotted, "\text{flop}"] & Y_0 \arrow[d, "p"] \\ \mathbb{P}^3 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\ & pt & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, dotted, "\text{anti-flip}"] \arrow[d] & & Y_0 \arrow[d, "p"] \\ \mathbb{P}^3 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\ & pt & \end{tikzcd}$$

where $p\colon Y_0 \to \mathbb{P}^1$ is a del-Pezzo fibration.

The fibration $Y_0 \to \mathbb{P}^1$

As in the previous section, the restriction of $\chi\colon X \dashrightarrow Y_0$ to any cubic surface can be checked to be the contraction of the anti-flipped/flopped curve. We conclude that:

#5. In case 5, the restriction of χ on a cubic is the contraction of 2 curves. The strict transform of any cubic is thus a del-Pezzo surface of degree 5.

#6. In case 6, the restriction of χ on a cubic is the contraction of 1 curve. The strict transform of any cubic is thus a del-Pezzo surface of degree 4.

The singularities of Y_0

As before we will compute the singularities of Y_0 following the Sarkisov diagram clockwise.

#5. In case 5, we have the $(1,2)$-flip of 1 curve followed by the flop of 1 curve. These pseudo-isomorphisms produce 1 quotient singularity of type $\frac{1}{2}(1,1,1)$.

#6. In case 6, we have the $(1,2)$-flip of 1 curve which again produces 1 quotient singularity of type $\frac{1}{2}(1,1,1)$.

Cube of $-K_{Y_0}$

Once more we will use Lemma 6.1 and Corollary 6.3 and follow the diagram clockwise to compute $(-K_{Y_0})^3$. We have

#5. $-K_X^3 = 14 \implies -K_{X_0}^3 = \frac{29}{2} \implies -K_{Y_0}^3 = \frac{29}{2}$;

#6. $-K_X^3 = 12 \implies -K_{X_0}^3 = \frac{25}{2} \implies -K_{Y_0}^3 = \frac{25}{2}$.

#	Type of the contraction p	Singularities of Y	$-K_Y^3$
5	del-Pezzo fibration of degree 5	$\frac{1}{2}(1,1,1)$	$\frac{29}{2}$
6	del-Pezzo fibration of degree 4	$\frac{1}{2}(1,1,1)$	$\frac{25}{2}$

CHAPTER IV

Rigid birational involutions of $\mathbb{P}^3$ and cubic surfaces

Abstract

We construct families of birational involutions on $\mathbb{P}^3$ or a smooth cubic threefold which do not fit into a non-trivial elementary relation of Sarkisov links. As a consequence, we construct new homomorphisms from their group of birational transformations, effectively reproving their non-simplicity. We also prove that these groups admit a free product structure. Finally, we produce automorphisms of these groups that are not generated by inner and field automorphisms.

1. Introduction

1.1. Homomorphisms from the Cremona group and free product structure

The Cremona group $\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is the group of birational transformations of the projective space $\mathbb{P}^n$ over a field $\mathbf{k}$. The study of this group has been a classical problem dating back to 19th century.

The Cremona group in dimension 2 over any field $\mathbf{k}$ is known to be non-simple (see [CL13, Lon16]) i.e., it admits non-trivial homomorphisms to other groups. Recently, many families of such homomorphisms were constructed: for example in dimension 2 over a perfect field by [LZ20, Sch21] and over a subfield of the complex numbers by [BY20] in dimension 3 and by [BLZ21] in dimension greater or equal to 3. Among other important consequences, the examples in the latter case proved for the first time the non-simplicity of the Cremona group in dimension greater or equal to 3.

In this paper, we construct uncountable families of involutions of $\mathbb{P}^3$ over $\mathbb{C}$, which are Sarkisov links and do not fit into any non-trivial relations of Sarkisov links. These are links of Type II of the form

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[d] \arrow[dr] & & X \arrow[d] \arrow[dl] \\ \mathbb{P}^3 & Z & \mathbb{P}^3 \end{tikzcd}$$

where $X \to \mathbb{P}^3$ is a divisorial contraction to a curve C and the central model Z is a sextic double solid, whose covering map induces the involution. The link χ_C is completely determined by C and the families of these links are parametrized by the Hilbert schemes of these curves.

Using these links we obtain the following result:

Theorem B. *There exists a group homomorphism*

$$\psi\colon \ \mathrm{Cr}_3(\mathbb{C}) \to \mathop{\ast}_{I} \mathbb{Z}/2\mathbb{Z},$$

where

1. *the indexing set I parametrizes projective equivalence classes of curves and is uncountable;*
2. $\mathrm{PGL}_4(\mathbb{C})$ *lies in the kernel and*
3. *there exist elements $\chi_i \in \mathrm{Cr}_3(\mathbb{C})$ of degree 19 not in the kernel.*

Moreover, ψ admits a section giving the $\mathrm{Cr}_3(\mathbb{C})$ a semi-direct product structure.

This can be thought of as an counterpart to the homomorphisms constructed in [BLZ21]. It should be noted that their results hold in all dimensions greater than or equal to 3 and apply to many other classes of varieties; however, the advantage of our construction lies in the fact that it is quite explicit, thus proving the non-simplicity of $\mathrm{Cr}_3(\mathbb{C})$ in an effective way.

Theorem B also provides the first example of a surjective group homomorphism $\mathrm{Cr}_3(\mathbb{C}) \to \ast_I \mathbb{Z}/2\mathbb{Z}$, where we have specific examples of elements which are known to lie outside the kernel. This also contrasts the situation in dimension 2 over $\mathbb{C}$: in that case, for all known surjective group homomorphisms from $\mathrm{Cr}_2(\mathbb{C})$ to a non-trivial group, no elements of low degree are known to lie in the kernel.

Using a subset of the aforementioned involutions we obtain another structural result. More specifically, let J be the subset of I corresponding to curves which are fixed by no non-trivial automorphism of $\mathbb{P}^3$, and denote by G the subgroup of $\mathrm{Cr}_3(\mathbb{C})$ generated by all elements admitting a decomposition into Sarkisov links, none of them equivalent to χ_{C_j}, $j \in J$ (see Remark 2.8) . We then have the following:

Theorem C. *The Cremona group $\mathrm{Cr}_3(\mathbb{C})$ can be written as the free product*

$$\mathrm{Cr}_3(\mathbb{C}) = G \ast \left(\mathop{\ast}_{J} \langle \chi_{C_i} \rangle \right) \cong G \ast \left(\mathop{\ast}_{J} \mathbb{Z}/2\mathbb{Z} \right)$$

where the indexing set J is uncountable.

This is an analogue to [LZ20, Theorem C], where $\mathrm{Cr}_2(\mathbf{k})$ is shown to admit a similar free product structure when $\mathbf{k}$ is a perfect field that admits a Galois extension of degree 8.

We will now briefly discuss the techniques used to construct both ours, as well as the aforementioned examples. The basic idea is to use the Sarkisov program. This is essentially an algorithm which decomposes any birational map between Mori fiber spaces into a sequence of simpler maps called Sarkisov links. The algorithm was proven to hold in dimension 2, over perfect fields by [Isk96], in dimension 3, over $\mathbb{C}$ by [Cor95] and in dimension greater than or equal to 3, over $\mathbb{C}$ by [HM13].

Using the Sarkisov program we get a set of generators, not quite for $\mathrm{Cr}_n(\mathbf{k})$, but for the groupoid $\mathrm{BirMori}_{\mathbf{k}}(\mathbb{P}^n)$. This is a groupoid whose objects are Mori fiber spaces birational to $\mathbb{P}^n$ and whose morphisms are birational maps between them. Once we have a set of generators, we want to know the relations between them. This is made possible by the machinery of *rank r fibrations* developed in [BLZ21] based on ideas from [Kal13]. This gives us a presentation of the groupoid $\mathrm{BirMori}_{\mathbf{k}}(\mathbb{P}^n)$, where relations are induced by rank 3 fibrations. Once we have a presentation, we can construct groupoid homomorphisms to groups or groupoids and restrict them to get group homomorphisms from $\mathrm{Cr}_n(\mathbf{k})$.

1.2. Non-generation of $\mathrm{Aut}(\mathrm{Cr}_3(\mathbb{C}))$ by inner and field automorphisms

The group of field automorphisms of $\mathbf{k}$ acts on $\mathbb{P}^n_{\mathbf{k}}$ naturally: given $\tau \in \mathrm{Aut}(\mathbf{k})$ we may define the map a_τ as

$$\begin{array}{ccc} \mathbb{P}^n & \to & \mathbb{P}^n \\ (x_0 : \ldots : x_n) & \mapsto & \big(\tau(x_0) : \ldots : \tau(x_n)\big). \end{array}$$

Note that this is not a morphism defined over $\mathrm{Spec}(\mathbf{k})$. However, $\mathrm{Aut}(\mathbf{k})$ acts on the group $\mathrm{Cr}_n(\mathbf{k})$ by conjugation. Given a $\tau \in \mathrm{Aut}(\mathbf{k})$ we define a group automorphism b_τ as

$$\begin{array}{ccc} \mathrm{Cr}_n(\mathbf{k}) & \to & \mathrm{Cr}_n(\mathbf{k}) \\ f & \mapsto & a_\tau \circ f \circ (a_\tau)^{-1}. \end{array}$$

A quick calculation yields that if $f = (f_0 : \ldots : f_n)$, where f_i are homogeneous polynomials of the same degree having no common factor, then $b_\tau(f) = ({f_0}^\tau : \ldots : {f_n}^\tau)$, where if $f_j = \sum a_I x^I$, then ${f_j}^\tau = \sum \tau(a_I) x^I$.

In [Dés06], the group $\mathrm{Aut}(\mathrm{Cr}_2(\mathbb{C}))$ was shown to be generated by inner and field automorphisms, that is if $\phi\colon \mathrm{Cr}_2(\mathbb{C}) \to \mathrm{Cr}_2(\mathbb{C})$ is a group homomorphism, then there exists a field automorphism τ of $\mathbb{C}$ and an element $g \in \mathrm{Cr}_2(\mathbb{C})$ such that for every $f \in \mathrm{Cr}_2(\mathbb{C})$ we have

$$\phi(f) = g \circ b_\tau(f) \circ g^{-1}.$$

It is therefore a natural question to ask whether such a result is true in higher dimensions or over other fields. In this text, we give a negative answer in dimension 3 over $\mathbb{C}$:

Theorem D. *There exists uncountably many automorphism of* $\mathrm{Cr}_3(\mathbb{C})$ *of arbitrary order which are not generated by inner and field automorphisms.*

These automorphisms are constructed using the free product structure on $\mathrm{Cr}_3(\mathbb{C})$ of Theorem C. They act on the generators by exchanging two elements of the form χ_{C_j} and $\chi_{C_{j'}}$. The fact that such an automorphism is not inner boils down to the fact that these involutions do not fit into a non-trivial relation of Sarkisov links, while a correct choice of C_j and $C_{j'}$ shows that the automorphism is not a field automorphism up to inner ones.

Finally, in [UZ21], the authors prove that any homeomorphism of $\mathrm{Cr}_3(\mathbf{k})$, with respect to either the Zariski or the Euclidean topology, is a composition of an inner and a field automorphism for $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$. Thus our examples constitutes, to our knowledge, the first examples of non-continuous automorphisms of $\mathrm{Cr}_3(\mathbb{C})$.

1.3. Extensions of our results to cubic 3-folds

All three of our theorems extend to the case of the group of birational automorphisms of a smooth cubic 3-fold Y.

For Theorems B and C, the same construction applies to any smooth cubic 3-fold unconditionally. In the case of Theorem B, we note again that the results of [BLZ21] still apply to the case of $\mathrm{Bir}_{\mathbb{C}}(Y)$. Again, the advantage of our result lies in its explicit nature. For instance, our approach provides examples of elements of order as low as 11 not in the kernel of the homomorphism $\mathrm{Bir}_{\mathbb{C}}(Y) \to \ast_I \mathbb{Z}/2\mathbb{Z}$.

Finally, for Theorem D the action of a field automorphism τ on $\mathrm{Aut}(\mathrm{Bir}_{\mathbb{C}}(Y))$ is well defined if and only if τ preserves Y, that is $a_\tau(Y) = Y$. Thus the statement of the corresponding theorem must be modified accordingly.

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2. Preliminaries

In the rest of the paper all varieties and birational maps between them are defined over $\mathbb{C}$.

2.1. Rank r fibrations and elementary relations

Here, we give a brief account of the theory developed in [BLZ21, Sections 3 and 4]. Any proofs provided here are sketches of the actual proofs found there.

Definition 2.1. *Let X/B be a Mori fiber space with singularities not worse that terminal (terminal Mori fiber space for short). We define* **BirMori**(X) *to be the groupoid whose objects are terminal Mori fiber spaces, birational to X and the morphisms between them to be birational maps.*

Definition 2.2. *Let X/B and X'/B' be Mori fiber spaces. An isomorphism between X and X' is called an* ***isomorphism of Mori fiber spaces*** *if there exists an isomorphism between B and B' that makes the induced diagram commute.*

Definition 2.3. *Let $r \geq 1$ be an integer. A morphism $\eta\colon X \longrightarrow B$ is a* ***rank r fibration*** *if the following conditions hold:*

1. *the fiber space X/B given by η is a relative Mori Dream Space (see [BLZ21, Definition 2.2]);*
2. $\dim X > \dim B$ *and $\rho(X/B) = r$;*
3. *X is $\mathbb{Q}$-factorial and terminal and for any divisor D on X, the output of any D-MMP over B is still $\mathbb{Q}$-factorial and terminal.*
4. *There exists an effective $\mathbb{Q}$-divisor Δ_B such that the pair (B, Δ_B) is klt.*
5. *The anticanonical divisor of X is η-big.*

We say that a rank r fibration X/B ***dominates*** *a rank r' fibration X'/B' if we have a commutative diagram*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, dashed] & & B \\ & X' \arrow[r] & B' \arrow[u] \end{tikzcd}$$

where $X \dashrightarrow X'$ is a birational contraction and $B' \longrightarrow B$ is a morphism with connected fibres.

Remark 2.4. *A rank 1 fibration $\eta\colon X \longrightarrow B$ is a terminal Mori fibre space. Indeed, the only thing left to check is the relative ampleness of the anti-canonical divisor. However, since $-K_X$ is η-big, we may write*

$$-K_X \equiv A + E,$$

where A is η-ample and E is effective. Since $\rho(X/B) = 1$, E is either η-nef or η-anti-nef. Since the contracted curves cover X, an effective divisor cannot be η-anti-nef, thus E is η-nef and subsequently, $-K_X$ is η-ample.

Similarly, rank 2 fibrations correspond to Sarkisov links between two Mori fibre spaces in the following manner:

If X/B is a rank 2 fibration then we may run a $(-A)$-MMP over B for any ample divisor A. Then since $\rho(X/B) = 2$, at the first step we have a choice between 2 rays to contract giving us 2 *different MMPs. Since $\kappa(-A) = -\infty$, the output of both MMPs must be rank 1 fibrations, which correspond to Mori fibre spaces.*

On the other hand, let

$$\begin{tikzcd}
X_m \arrow[d] & \cdots \arrow[l, dashed] \arrow[ddrr] & X_0 \arrow[l, dashed] \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & Y_0 \arrow[r, dashed] \arrow[dl] \arrow[ddl] & \cdots \arrow[r, dashed] \arrow[ddll] & Y_n \arrow[d] \arrow[ddlll] \\
{} \arrow[drrr] & & & Z \arrow[d] & & & {} \arrow[dlll] \\
& & & B & & &
\end{tikzcd}$$

be a Sarkisov diagram, where $X_0 \dashrightarrow Y_0$ is either a flop or an isomorphism. Then X_0/B is weak Fano thus a Mori Dream Space. Moreover X_0 is $\mathbb{Q}$-factorial and terminal and the output of any MMP is among the maximal dimensional varieties appearing in the diagram, which by assumption are all $\mathbb{Q}$-factorial and terminal. Finally, the fact that B is klt is proven in [Fuj99, Corollary 4.6].

The correspondence above is not one-to-one, *namely a rank 2 fibration gives rise to a Sarkisov link and its inverse, up to Mori fiber space isomorphisms. On the other hand, in the Sarkisov diagram above, all X_i/B and Y_i/B are rank 2 fibrations.*

Proposition 2.5 ([BLZ21, Proposition 4.3])**.** *Let $X \longrightarrow B$ be a rank 3 fibration. Then there are only finitely many rank 2 fibrations, corresponding to Sarkisov links χ_i, dominated by X/B, and they fit in a relation*

$$\chi_t \circ \cdots \circ \chi_1 = id.$$

Definition 2.6. *A **trivial relation** between Sarkisov links is a relation of one of the following forms*

$$\phi^{-1} = \psi \quad \text{and} \quad \alpha \circ \phi \circ \beta = \psi,$$

where ϕ, ψ are Sarkisov links and α, β are isomorphisms of Mori fiber spaces.

*An **elementary relation** between Sarkisov links is one that arises from a rank 3 fibration (see Proposition 2.5).*

Theorem 2.7 ([HM13, Theorem 1.1], [BLZ21, Theorem 4.28])**.** *Let X/B be a terminal Mori fibre space.*

1. *The groupoid* $\mathrm{BirMori}(X)$ *is generated by Sarkisov links and isomorphisms of Mori fiber spaces.*
2. *Any relation between Sarkisov links in* $\mathrm{BirMori}(X)$ *is generated by trivial and elementary relations.*

Remark 2.8. *The first part of the theorem is due to [HM13]. The original version does not mention the isomorphisms of Mori fiber spaces, which are however implicit in their proof. Note that an isomorphism between the total spaces of two Mori fiber spaces which is not a Mori fiber space isomorphism is a non-trivial Sarkisov link.*

Similarly, the second part of the original theorem in [BLZ21] does not mention the trivial relations as generators. These are indeed "trivial" from a birational point of view. For our purposes though, we will need a slightly more accurate statement and so we explain the subtleties.

For the first type of trivial relation, a rank 2 fibration corresponds to a unique Sarkisov diagram up to composition with Mori fiber space isomorphisms on the left and right. However, as already discussed in Remark 2.4, a Sarkisov diagram is not directed and thus corresponds to both a link and its inverse. The second type of relation is just a by-product of not working up to Mori fiber space isomorphism.

*With that in mind, we will say that two Sarkisov links ϕ and ψ are **equivalent** if there exist α, β, isomorphisms of Mori fiber spaces such that $\alpha \circ \phi \circ \beta = \psi$.*

2.2. Weighted blowups

Definition 2.9. *Let $\mathbf{w} = (w_1, \dots, w_n)$ be positive integers. Define the $\mathbb{C}^*$-action on $\mathbb{A}^{n+1}$ by*

$$\lambda \cdot (u, x_1, \dots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n).$$

The morphism from the geometric quotient $T := \mathbb{A}^{n+1}/\mathbb{C}^* \to \mathbb{A}^n$ defined by

$$\begin{array}{ccc} T & \longrightarrow & \mathbb{A}^n \\ (u : x_1 : \cdots : x_n) & \mapsto & (u^{w_1}x_1, \dots, u^{w_n}x_n) \end{array}$$

*is called the **standard** $\mathbf{w}$**-blowup** of $\mathbb{A}^n$ at the origin.*

*Let $f\colon E \subset Y \to p \in X$ be a morphism contracting a divisor E to a smooth point p. We say that f is a $\mathbf{w}$**-blowup** of Y at p if there exists an analytic neighbourhood $(U, p) \cong (\mathbb{A}^n, 0)$ of p such that the restriction $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is the standard $\mathbf{w}$-blowup of $\mathbb{A}^n$ at 0.*

Lemma 2.10. *Let $p \in X$ be a smooth point of a 3-fold and let $\pi(Y, E) \longrightarrow (X, p)$ be a $(1, a, b)$-blowup of X at p. Then the ramification formula takes the form*

$$K_Y = \pi^* K_X + (a + b)E.$$

Proof. Since this is something that can be checked locally, up to local analytic isomorphism we may assume that $(X, p) = (\mathbb{A}^3, 0)$, Y is the quotient $\mathbb{A}^4/\mathbb{C}^*$ under the action

$$\lambda \cdot (u, x_1, x_2, x_3) = (\lambda^{-1}u, \lambda x_1, \lambda^a x_2, \lambda^b x_3)$$

and π is given by $(u : x_1 : x_2 : x_3) \mapsto (ux_1, u^a x_2, u^b x_3)$.

Let U_1 be the open subset $\{x_1 \neq 0\} \subset Y$, isomorphic to $\mathbb{A}^3$. If we denote the composition

$$\begin{array}{ccccc} \mathbb{A}^3 & \longrightarrow & U_1 \subset Y & \longrightarrow & \mathbb{A}^3 \\ (v, y_1, y_2) & \mapsto & (v : 1 : y_1 : y_2) & \mapsto & (v, y_1 v^a, y_2 v^b). \end{array}$$

by ψ then we may calculate that

$$\psi^*(1\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3) = v^{a+b}\,\mathrm{d}v \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2.$$

Taking the divisor of this 3-form we conclude. ☐

Lemma 2.11. *Let $p \in X$ be a smooth point of a 3-fold and let $\pi\colon E \subset Y \dashrightarrow p \in X$ be a $(1,a,b)$-blowup of X at p. Let Γ be a curve in X which is a complete intersection in an affine neighbourhood U of p. Choose generators f_1 and f_2 for the ideal of regular functions on U vanishing along Γ. We then have*

$$E \cdot \tilde{\Gamma} = \frac{v_E(f_1) \cdot v_E(f_2)}{ab},$$

where $\tilde{\Gamma}$ denotes the strict transform of Γ, v_E is the divisorial valuation defined by E and f_1 and f_2 are considered as rational functions on X.

Proof. Again we will work in a local analytic neighbourhood and assume that $(X,p) = (\mathbb{A}^3, 0)$ and π is given by $(u : x_1 : x_2 : x_3) \mapsto (ux_1, u^a x_2, u^b x_3)$. We may write

$$f_n = \sum_{i=k_n}^{d_n} h_{n,i}(v, y_1, y_2),$$

for $n = 1, 2$, where $h_{n,i}$ are homogeneous polynomials with respect to the grading $(1,a,b)$ and $h_{n,k_n} \neq 0$. Then, pulling back under π we get

$$\pi^*(f_n) = f_n(ux_1, u^a x_2, u^b x_3) = u^{k_n}\left(\sum_{i=k_n}^{d_n} u^{i-k_n} h_{n,i}(x_1, x_2, x_3)\right),$$

which shows that $v_E(f_n) = k_n$. Moreover the ideal of $\tilde{C}$ is generated by $\tilde{f}_1$ and $\tilde{f}_2$ with

$$\tilde{f}_n = \sum_{i=k_n}^{d_n} u^{i-k_n} h_{n,i}(x_1, x_2, x_3),$$

for $n = 1, 2$. Finally, using the fact that $E \cong \mathbb{P}(1,a,b)$ and that $\tilde{\Gamma}$ is given by the vanishing of the f_n, $n = 1, 2$, we may compute that

$$E \cdot \tilde{\Gamma} = \mathbb{V}(\tilde{f}_1)|_E \cdot \mathbb{V}(\tilde{f}_2)|_E = \mathbb{V}(h_{1,k_1}) \cdot_E \mathbb{V}(h_{2,k_2}) = \frac{k_1 \cdot k_2}{ab}.$$

☐

3. The construction

Throughout this section, Y will denote either $\mathbb{P}^3$ or a smooth cubic 3-fold in $\mathbb{P}^4$. Denote by $\mathcal{H}^Y_{g,d}$ the Hilbert scheme of curves of arithmetic genus g and degree d in Y.

Proposition 3.1. *Consider the following pairs (g, d) depending on Y:*

Y	(g, d)
$\mathbb{P}^3$	$(2, 8), (6, 9), (10, 10), (14, 11)$
Cubic 3*-fold*	$(0, 5), (2, 6)$

Then $\mathcal{H}^Y_{g,d}$ is non-empty (see Lemma 4.1 for an estimation of its dimension).

Let C be a smooth general element of $\mathcal{H}^Y_{g,d}$. If $X \dashrightarrow Y$ is the blowup of Y along C then:

1. *X is a smooth weak-Fano 3-fold, there are finitely many $(-K_X)$-trivial curves and $|-K_X|$ is base-point free;*
2. *The anti-canonical model $Z := \mathrm{Proj}\left(\oplus_{n\geq 0} H^0(X, -nK_X)\right)$ of X is a sextic double solid, that is a double cover of $\mathbb{P}^3$ ramified along a sextic hypersurface;*
3. *Any curve γ contracted by $X \dashrightarrow Z$ is rational. If $W \to X$ denotes the blowup of X along γ, then $-K_W$ is nef.*

Proof. For the non-emptiness of the Hilbert schemes we refer to [BL12, Section 5.1] and [BL15, Section 3.3] for the cases of $\mathbb{P}^3$ and a smooth cubic 3-fold respectively.

Similarly, the proof of (1) can be found in [BL12, Proposition 5.11] and [BL15, Proposition 3.7] for the two cases respectively.

As for (2), we first note that in all cases, using the formula

$$(-K_X)^3 = (-K_Y)^3 + 2K_Y \cdot C + 2g - 2$$

we get $(-K_X)^3 = 2$. By the Hirzebruch-Riemann-Roch theorem (see [Har77, pg. 437, Ex. 6.7]) together with the Kawamata-Viehweg vanishing theorem we get

$$h^0(X, -nK_X) = \frac{n(n+1)(2n+1)}{12}(-K_X^3) + 2n + 1 = \frac{n(n+1)(2n+1)}{6} + 2n + 1.$$

For $n = 1$ we get $h^0(X, -K_X) = 4$; we write x_0, x_1, x_2, x_3 for the generators. By (1) the linear system $|-K_X|$ is base-point free and the associated morphism $X \to \mathbb{P}(H^0(X, -K_X))$ contracts finitely many curves and is thus dominant. In particular, the x_i's satisfy no polynomial relation. Moreover, since a general element of $|-K_X|$ is the pullback of a general hyperplane and $(-K_X)^3 = 2$, we may conclude that $X \to \mathbb{P}(H^0(X, -K_X))$ is generically 2 to 1. For $n = 2$ we get $h^0(X, -2K_X) = 10 = \dim S^2 H^0(X, -K_X)$. Since there is no relation between the x_i's, we get the equality of these two spaces. For $n = 3$ we get $h^0(X, -3K_X) = 15 = \dim S^3 H^0(X, -K_X) + 1$. Again using the fact that there is no relation between the x_i's, we get that we only have one new generator. That is

$$H^0(X, -3K_X) = S^3 H^0(X, -K_X) \oplus \langle t \rangle.$$

We now consider the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & X' \arrow[dl] \\ & \mathbb{P}^3 & \end{tikzcd}$$

with $X' = \mathrm{Proj}(R)$, where R is the graded algebra generated by $x_0, \ldots, x_3$ with degrees 1 and t with degree 3 and $X' \to \mathbb{P}^3$ is the projection to the first three factors. Note that $X \to X'$ and $X \to \mathbb{P}^3$ both contract the $(-K_X)$-trivial curves. Moreover, if $X' \to \mathbb{P}^3$ were generically one to one, it would be a bijection and thus an isomorphism from Zariski's Main Theorem. Thus $X' \to \mathbb{P}^3$ is two to one, which implies that $X \to X'$ has connected fibers. For $n \geq 4$, the morphism given by $|-nK_X|$ for $n \geq 4$ contracts the same curves as $X \to X'$ and has connected fibers. Thus by [Deb01, Proposition 1.14], these two morphisms are the same up to isomorphism. In particular, there is no new generator for any $n \geq 4$. Finally, since we know that the algebra $\oplus_{n\geq 0} H^0(X, -nK_X)$ is generated by $x_0, \ldots, x_3, t$, we only have to calculate the dimensions of the graded components to see that we have only one relation in degree 6.

Finally for (3), since X is weak-Fano, it is log-Fano and thus a Mori Dream Space (see [BCHM10, Corollary 1.3.2]). This implies that the contraction of the ray generated by the $(-K_X)$-trivial curves exists and moreover, these curves are rational (see [Mat02, Theorem 10-3-1]).

As for the nefness of $-K_W$, we consider the diagram

$$\begin{tikzcd} W \arrow[d, "g"'] & \\ X \arrow[dr, "f"'] & F \arrow[d, "r"] \\ & \mathbb{P}^3 \end{tikzcd}$$

where $f\colon X \to \mathbb{P}^3$ is the morphism given by $|-K_X|$ and $r\colon F \to \mathbb{P}^3$ is the blowup of the image $p \in \mathbb{P}^3$ of γ under f. Since the preimage of p under $f \circ g$ is a Cartier divisor, $f \circ g$ factors through r via $s\colon W \to F$. Finally, sections of $-K_W$ are pullbacks of hyperplanes of $\mathbb{P}^3$ through the p. Thus the previous diagram completes to the following:

$$\begin{tikzcd} W \arrow[d, "g"'] \arrow[dr, "s"] \arrow[ddrr, bend left, "|-K_W|"] & & \\ X \arrow[dr, "f"'] & F \arrow[d, "r"] \arrow[dr] & \\ & \mathbb{P}^3 \arrow[r, dashed] & \mathbb{P}^2, \end{tikzcd}$$

where $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ denotes the projection from the point p. This shows that $-K_W$ is the pullback of an ample divisor, thus nef.

$\square$

Remark 3.2. *In the setting of Proposition 3.1, the normal bundle of any curve γ contracted by $X \to Z$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, with $(a,b) = (-1,-1)$ or $(0,-2)$.*

Indeed, let E denote the exceptional divisor of $X \to Z$, so that E is isomorphic to the Hirzebruch surface $\mathbb{F}_{a-b}$. Then by [IP99, Lemma 2.2.14] we have

$$a + b = \deg\left(N_{\gamma/X}\right) = (-K_X) \cdot \gamma + 2g(\gamma) - 2 = -2.$$

Moreover, using adjunction formula as well as the formulas in [Zik20, Lemma 5.5] we may compute

$$-K_W|_E = -K_E + E|_E = -\frac{1}{2}K_E,$$

which shows that $-K_E$ is nef. Thus $E \cong \mathbb{F}_n$ with $n = 0, 1$ or 2, that is $a - b = 0, 1$ or 2. The only integer solutions to the two equations are $(a,b) = (-1,-1)$ and $(0,-2)$.

Remark 3.3. *The construction above induces a birational self-map of Y in the following way: denote by η the rational map $Y \dashrightarrow X \longrightarrow Z$ and by p the deck transformation of Z over $\mathbb{P}^3$. Then $\chi_C := \eta^{-1} \circ p \circ \eta \colon Y \dashrightarrow Y$ defines a birational map. Note that χ_C is an involution. Schematically, we have the diagram*

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[dd] \arrow[dr] & & X \arrow[dd] \arrow[dl] \\ & Z \arrow[loop below, "p"] & \\ Y & & Y. \end{tikzcd}$$

This also gives a birational map from the sextic double solid Z to the Mori fiber space $Y \to$ $\mathrm{Spec}(\mathbb{C})$, showing that Z is not birationally (super) rigid. Birational rigidity of sextic double solids has been studied in [CP10] where the authors show that a nodal $\mathbb{Q}$-factorial sextic double solid is birationally super rigid.

The sextic double solids arising from the construction above generically have nodal singularities: their singularities arise from the small contraction $X \to Z$ and their type is determined by the normal bundles of the contracted curves; if the normal bundle is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ then the resulting singularity is a node.

While it is an open condition on $\mathcal{H}^Y_{g,d}$ for the contracted curves to have a normal of $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, with $(a,b) = (-1,-1)$, it should be possible to construct examples with $(a,b) = (0,-2)$. Birational rigidity of sextic double solids with singularities other than nodes is much less understood (see [KOPP22]).

Proposition 3.4. *Let Y, C and X be as above and let H denote a hyperplane if Y is $\mathbb{P}^3$ and a hyperplane section otherwise. Then the degree of χ_C with respect to H is*

$$\deg(\chi_C) = (r^2 H^3 - d)r - 1,$$

where d the degree of C and r is the index of Y.

Proof. We consider the induced diagram

$$\begin{tikzcd} X \arrow[rr, dotted, "\phi"] \arrow[d] \arrow[dr] & & X \arrow[dl] \arrow[d] \\ Y & Z & Y \end{tikzcd}$$

where ϕ is a flop over Z. Fix the basis (K_X, H) for the $\mathbb{Q}$-vector space $N^1(X)$, where, by abuse of notation, we denote again by H the class of the pullback H. Then ϕ induces an automorphism of $N^1(X)$, by pullback, and since K_X is an eigenvector for it, the associated matrix has the form

$$\phi^* = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

Since $\phi^2 = id_X$, $b = -1$. Thus $\phi^* H = aK_X - H$.

Using the formulas in [IP99, Lemma 2.2.4], we may compute that

$$(K_X)^2 \cdot H = r^2 H^3 - d,$$

where r is the index of Y and d the degree of C, and

$$(\phi^* K_X)^2 \cdot \phi^* H = (K_X)^2 \cdot (aK_X - H) = a(K_X)^3 - K_X \cdot H = -2a - (r^2 H^3 - d).$$

Equating the above formulas we get $a = -(r^2 H^3 - d)$. Thus

$$\phi^* H = -(r^2 H^3 - d) K_X - H = ((r^2 H^3 - d) r - 1) H - (r^2 - d) E$$

from which we conclude that ${\chi_C}^*(H) = ((r^2 H^3 - d) r - 1) H$. □

For the pairs of genus and degree of Proposition 3.1, we obtain the following values for the degree of χ_C:

$\mathbb{P}^3$	(g, d)	$(2, 8)$	$(6, 9)$	$(10, 10)$	$(14, 11)$
	$\deg(\chi_C)$	31	27	23	19
Cubic 3-fold	(g, d)	(0,5)		(2,6)	
	$\deg(\chi_C)$	13		11	

Proposition 3.5. *Let X be as in Proposition 3.1 and $\pi\colon (W, E) \longrightarrow (X, z)$ be a divisorial contraction with W $\mathbb{Q}$-factorial and terminal. Then $-K_W$ is not big.*

Proof. We first note that since W is terminal and X is smooth, by [Tzi03, Proposition 1.2] and [Kaw01, Theorem 1.2], $W \longrightarrow X$ is either the regular blowup of a curve or a $(1, a, b)$-blowup of a point, with a, b coprime.

We distinguish 3 cases based on the geometry of the center z:

Case 1: z is a point and $W \longrightarrow X$ is a $(1, a, b)$-blowup.

Suppose for contradiction that $-K_W$ is big and let $S_W \in |-nK_W|$ be a general element, for $n \gg 1$. Denote by S_X the image of S_W in X, by H_X the pullback of a general hyperplane section H_Z of Z containing the image of z and $\Gamma \subset X$ the intersection of S_X with H_X. First notice that $S_X \in |-nK_X|$ and $H_X \in |-K_X|$. We thus have

$$(-K_X) \cdot \Gamma = n(-K_X)^3 = 2n.$$

If we denote by Γ_W the strict transform of Γ in W, then by Lemma 2.11 we have

$$E \cdot \Gamma_W = \frac{v_E(S_X) \cdot v_E(H_X)}{ab} = \frac{n(a+b)}{ab} v_E(H_X),$$

where the second equality follows from the ramification formula of Lemma 2.10. Again, using the same formula we may compute that

$$(-K_W) \cdot \Gamma_W = n \left(2 - \frac{(a+b)^2}{ab} v_E(H_X) \right).$$

Since we chose H_X to be the pullback of a hyperplane containing the image of z, $v_E(H_X) \geq 1$. The quantity $\frac{(a+b)^2}{ab}$ is always strictly greater than 2, and so $(-K_W) \cdot \Gamma_W < 0$. Finally, since we assumed that $-K_W$ is big then the sections of $-nK_W$ cover W for sufficiently large n and so do the curves Γ chosen as above. This gives us a dense subset of W covered by $(-K_W)$-negative curves, which contradicts the bigness of $-K_W$.

Case 2: z is a curve not contracted by $X \longrightarrow Z$.

We have

$$-nK_W = \pi^*(-nK_X) - nE.$$

Sections of $-nK_W$ are pullbacks of degree n hypersurface sections of Z vanishing along the curve $C := \pi(z)$ with multiplicity n. Let $h = 0$ be such a hypersurface section and $I = (f_1, \ldots, f_k)$ be the ideal of C. Then $h \in I^n$ and since $\deg(h) = n$, h can only be a linear combination of degree n monomials in the linear elements in I. Thus for $-nK_W$ to be big, we need to have at least 4 linear elements in I which is a contradiction.

Case 3: z is a curve contracted by $X \longrightarrow Z$.

In this case, by Proposition 3.1(3), $-K_W$ is nef. Moreover, we have the formula

$$(-K_W)^3 = (-K_X)^3 + 2K_X \cdot z - 2 + 2g_z = 0$$

(see [BL12, Lemma 2.4]). Thus $-K_W$ being a nef divisor with zero top self intersection, it is not big (see [Laz04, Theorem 2.2.16]). □

Corollary 3.6. *Using the notations of Proposition 3.1 and Remark 3.3, there exists no rank 3 fibration dominating the rank 2 fibration* $X \longrightarrow Y \longrightarrow \operatorname{Spec}(\mathbb{C})$. *Consequently, there are no non-trivial relations in* $\operatorname{BirMori}(Y)$ *involving* χ_C.

Proof. Let $W' \longrightarrow B$ be a rank 3 fibration dominating $X \longrightarrow \operatorname{Spec}(\mathbb{C})$. Then $B = \operatorname{Spec}(\mathbb{C})$ and we have a diagram of the form

$$\begin{tikzcd} W' \arrow[rr] \arrow[dr, dashed, "f"'] & & \operatorname{Spec}(\mathbb{C}) \\ & X \arrow[ur] & \end{tikzcd}$$

By the definition of a rank 3 fibration W' is a Mori Dream Space. Let a be an ample divisor on X. Then there exists a composition of log-flips $g\colon W \dashrightarrow W'$ so that $g_*f^*(A)$ is nef on W'. With W' being a Mori Dream Space itself, $g_*f^*(A)$ is semi-ample and the associated contraction gives rise to the diagram

$$\begin{tikzcd} W' \arrow[r, dotted, "g"] \arrow[dr, dashed, "f"'] & W \arrow[d] \\ & X. \end{tikzcd}$$

By property (3) of Definition 2.3 W is also terminal. Thus by Proposition 3.5, $-K_W$ is not big. However this would also imply that $-K_{W'}$ is not big which contradicts property (1) of Definition 2.3. The second claim follows directly from Theorem 2.7. □

Remark 3.7. *The trivial relations involving χ_C are:*

$$(\chi_C)^2 = \mathrm{id} \qquad \textit{and} \qquad a \circ \chi_C \circ b \circ \psi^{-1} = \mathrm{id}$$

where a, b^{-1} are any Mori fiber space isomorphisms starting from Y and ψ is the Sarkisov link given by the composition $a \circ \chi_C \circ b$. Moreover, in the second type of relation, if $a, b \in \operatorname{Aut}(Y)$ with $b = a^{-1}$, then $a \circ \chi_C \circ a^{-1} = \chi_{a(C)}$.

4. Consequences

In what follows we will stick to the notation introduced in section 3: Y will denote either $\mathbb{P}^3$ or a smooth cubic 3-fold in $\mathbb{P}^4$ and $\mathcal{H}^Y_{g,d}$ will denote the Hilbert scheme of curves of arithmetic genus g and degree d in Y.

4.1. Homomorphism and semi-direct product structure.

We now construct a group homomorphism from $\operatorname{Bir}(Y)$ to a free product $\ast_I \mathbb{Z}/2\mathbb{Z}$, where the indexing set I is uncountable. To do so, we will first construct a groupoid homomorphism from $\operatorname{BirMori}(Y)$ to the same target and then restrict it to $\operatorname{Bir}(Y)$.

Let (g, d) be one of the pairs of Proposition 3.1. We define the set $I_{g,d}$ to be the set of elements $\mathcal{H}^Y_{g,d}$ up to automorphisms of Y and I to be the disjoint union of all $I_{g,d}$ for all pairs (g, d) considered in Proposition 3.1. The following lemma shows that $I_{g,d}$ and thus I is uncountable.

Lemma 4.1. *For all pairs (g, d) and $C \in \mathcal{H}^Y_{g,d}$ satisfying the generality conditions of Proposition 3.1,*

$$-K_Y \cdot C \leq \dim \mathcal{H}^Y_{g,d} \leq -K_Y \cdot C + 1.$$

In particular, $\dim\left(\mathcal{H}^Y_{g,d}\right) > \dim\left(\operatorname{Aut}(Y)\right)$.

Proof. By [BL12, Proposition 2.8] and [BL15, Proposition 3.7] a general anti-canonical section S containing C is a smooth K3 surface (see [BL12, Proposition 2.8] and [BL15, Proposition 2.9]). The normal bundle sequence for the embeddings $C \subset S \subset Y$ gives

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/Y} \longrightarrow N_{S/Y}|_C \longrightarrow 0.$$

The long exact sequence and the fact that $(C^2)_S = 2g-2$ yield

$$\begin{aligned} 0 &\longrightarrow H^0\left(C, \mathcal{O}_C(2g-2)\right) \longrightarrow H^0(C, N_{C/Y}) \longrightarrow H^0\left(C, \mathcal{O}_C(-K_X \cdot C)\right) \\ &\longrightarrow H^1\left(C, \mathcal{O}_C(2g-2)\right) \longrightarrow H^1(C, N_{C/Y}) \longrightarrow H^1\left(C, \mathcal{O}_C(-K_X \cdot C)\right) \longrightarrow 0. \end{aligned}$$

By Serre duality, $h^1\left(C, \mathcal{O}_C(-K_X \cdot C)\right) = h^0\left(C, \mathcal{O}_C(2g-2+K_X \cdot C)\right)$. For the six cases of (g,d) and Y of Proposition 3.1, we get the following values for $2g-2+K_X\cdot C$: $-30, -26, -22, -18$ and $-12, -10$, thus $h^1\left(C, \mathcal{O}_C(-K_X \cdot C)\right) = 0$. Similarly $h^1\left(C, \mathcal{O}_C(2g-2)\right) = h^0\left(C, \mathcal{O}_C\right) = 1$, thus $h^1(C, N_{C/Y})$ is either 0 or 1. Moreover, using the additivity of the Euler characteristic on short exact sequences and the Riemann-Roch theorem to compute we get

$$h^0(C, N_{C/Y}) - h^1(C, N_{C/Y}) = -K_X \cdot C \implies -K_X \cdot C \leq h^0(C, N_{C/Y}) \leq -K_X \cdot C + 1.$$

Since C represents a general and thus smooth point of $\mathcal{H}^Y_{g,d}$ we get that $\dim \mathcal{H}^Y_{g,d} = h^0(C, N_{C/Y})$.

For the last assertion, if Y is a cubic 3-fold, then $\dim \operatorname{Aut}(Y) = 0$ (see [MM64]) and we are automatically done. If $Y = \mathbb{P}^3$, then in all cases $d \geq 8$ and so $-K_Y \cdot C = 4d > 15 = \dim \operatorname{Aut}(\mathbb{P}^3)$. $\square$

Theorem 4.2. *There exists a surjective group homomorphism* $\psi\colon \operatorname{Bir}(Y) \to \ast_I \mathbb{Z}/2\mathbb{Z}$*, which admits a section, giving the* $\operatorname{Bir}(Y)$ *a semidirect product structure*

$$\operatorname{Bir}(Y) = N \rtimes \mathop{\ast}_{I} \mathbb{Z}/2\mathbb{Z},$$

where N *is the kernel of* ψ*.*

Proof. We will first define a groupoid homomorphism $\Psi\colon \operatorname{BirMori}(Y) \to \ast_I \mathbb{Z}/2\mathbb{Z}$. To do so, for each i in I, we fix an element $C_i \in \mathcal{H}^Y_{g,d}$ in the equivalence class corresponding to $i \in I_{g,d}$. The groupoid $\operatorname{BirMori}(Y)$ is generated by Sarkisov links and isomorphisms of Mori fiber spaces, and relations are generated by trivial and elementary ones (see Theorem 2.7). Thus to define a groupoid homomorphism from $\operatorname{BirMori}(Y)$ it is enough to define it on the generators and check that all relators are mapped to the natural element. With that in mind we define Ψ as follows: on the level of objects, Ψ maps everything to the unique object of $\ast_I \mathbb{Z}/2\mathbb{Z}$ (when considered as a groupoid). On the level of Sarkisov links and automorphisms, for each $i \in I$, Ψ maps all links equivalent to χ_{C_i} (see Remark 2.8) to the non-zero element of the factor i. All other links and isomorphisms are mapped to the neutral element. Any relator not involving any link equivalent to

χ_{C_i} is automatically sent to the neutral element, and the same is true for both relators of Remark 3.7.

Define $\psi\colon \mathrm{Bir}(Y) \ast_I \mathbb{Z}/2\mathbb{Z}$ to be the restriction of Ψ to the subgroup $\mathrm{Bir}(Y)$ of $\mathrm{BirMori}(Y)$. Since ψ is the restriction of a groupoid homomorphism, it is a group homomorphism itself. Let 1_k be the non-zero element of the k-th factor of $\ast_I \mathbb{Z}/2\mathbb{Z}$, $k \in I$. Then $\psi(C_k) = 1_k$, thus the homomorphism is surjective. Conversely, we may define a section by sending 1_k to χ_{C_k}.

□

Remark 4.3. *Using Proposition 3.4, the degree of an involution χ_{C_i} and thus of an element not in the kernel of ψ, can be as low as* 19 *in the case $Y = \mathbb{P}^3$ and* 11 *in the case Y is a cubic* 3*-fold.*

4.2. Free product structure

We now show that $\mathrm{Bir}(Y)$ admits a free product structure $G \ast (\ast_J \mathbb{Z}/2\mathbb{Z})$. The indexing set J is defined similarly to the indexing set I of the previous section: we first define $J^Y_{g,d}$ to be the set of elements of $\mathcal{H}^Y_{g,d}$ that are fixed by no non-trivial automorphism of Y, up to projective equivalence; then we define J^Y to be the disjoint union over all pairs (g, d) of Proposition 3.1 corresponding to Y.

A priori, it is not clear that J^Y is uncountable or even non-empty. Thus we first set out to prove that J^Y is uncountable. First we treat the case $Y = \mathbb{P}^3$.

Lemma 4.4. *Let C be a curve of genus $g \geq 2$, and let D be a very ample divisor on C, such that* $\dim|D| \geq 5$*.*

Then for $n \geq 3$, a general $(n+1)$-dimensional subsystem V of $|D|$ defines an embedding of C in $\mathbb{P}^n$ that admits no projective automorphisms. Moreover, for every such V, there are only finitely many other subsystems of the same dimension which are projectively equivalent.

Proof. The complete linear system $|D|$ defines an embedding to $\mathbb{P}^N$ for some $N \geq 4$. Maps given by $(n+1)$-dimensional subsystems correspond to composition of the embedding with projections from $\mathbb{P}^N$ to n-dimensional linear subspaces. Thus since $n \geq 3$, a general $(n+1)$-dimensional subsystem defines an embedding (see [Har77, Propositions 3.4 and 3.5]).

Recall that since the genus of C is greater than or equal to 2, by a classical theorem of Hurwitz (see [Hur92]) its automorphism group $\mathrm{Aut}(C)$ is a finite group. Denote by G the subgroup $\{g \in \mathrm{Aut}(C) \mid g^*D \sim D\}$ of $\mathrm{Aut}(C)$. Let V be an n-dimensional subspace of $|D|$. Then the automorphisms of $\mathbb{P}^n$ acting on C are exactly the elements of G that leave V invariant. If G is trivial, we are done. If G is non-trivial, then by considering the non-trivial action of G on the Grassmanian $G(n+1, |D|)$, we see that being invariant under G is a closed condition.

Finally, two embeddings corresponding to two subspaces V_1 and V_2 are projectively equivalent if and only if there exists $g \in G$ such that $g(V_1) = V_2$. Since G is a finite group, so is the orbit of every element in $G(n+1, |D|)$, proving the second claim. □

Lemma 4.5. *For $(g,d) \in \{(2,8),(6,9)\}$ and any curve C of genus g, there exists uncountably many non-projectively equivalent curves in $\mathcal{H}_{g,d}^{\mathbb{P}^3}$, isomorphic to C, that admit no non-trivial projective automorphisms.*

Consequently, $J_{2,8}^{\mathbb{P}^3}$, $J_{6,9}^{\mathbb{P}^3}$ and thus $J^{\mathbb{P}^3}$ are uncountable.

Proof. We will do this case by case. For $(g,d) = (2,8)$, let D be a divisor of degree 8. Since $8 \geq 4 = 2g$, D is very ample and non-special and by Riemann-Roch, $\dim|D| = 7$. By Lemma 4.4, a general 4-dimensional subspace of $|D|$ defines an embedding in $\mathbb{P}^3$ such that C admits no non-trivial projective automorphism. A general choice of two such subspaces gives non-projectively equivalent embeddings.

For $(g,d) = (6,9)$, we start with an abstract curve of genus 6, choose a point p and define the divisor $D = K_C - p$, which is of degree 9. By the Riemann-Roch theorem we have

$$h^0(C,\mathcal{O}_C(D)) = 9 - 6 + 1 + h^0(C,\mathcal{O}_C(K_C - D)) = 4 + h^0(C,\mathcal{O}_C(p)) = 5.$$

We will now show that D is very ample which is equivalent to showing that for any two points r,s on C, $h^0(C,\mathcal{O}_C(D-r-s)) = h^0(C,\mathcal{O}_C(D)) - 2 = 3$. Suppose for contradiction that $h^0(C,\mathcal{O}_C(D-r-s)) = h^0(C,\mathcal{O}_C(K_C - p - r - s)) = 4$. We consider the canonical embedding $C_\kappa \subset \mathbb{P}^5$ of C. The fact that $h^0(C,\mathcal{O}_C(K_C - p - r - s)) = 4$ implies that the three points p, r and s are collinear in the canonical embedding. Write $x_0, \dots, x_5$ for the generators of $H^0(C,\mathcal{O}_C(K_C))$. Then we may compute that $h^0(C,\mathcal{O}_C(2K_C)) = 15$, while $S^2\left(H^0(C,\mathcal{O}_C(K_C))\right) = 21$. This implies that there are at least 6 relations among $x_0, \dots, x_5$ in degree 2. Thus, C is contained in the complete intersection of 4 quadrics and by Bézout's theorem so is any tri-secant line. Consequently, there are finitely many tri-secant lines. Choosing a point p which does not lie on any tri-secant line we get that for any $r,s \in C$, $h^0(C,\mathcal{O}_C(D-r-s)) = 3$ and thus D is very ample. Finally, we may apply Lemma 4.4 to the divisor D to obtain the desired result.

□

We now treat the case Y is a smooth cubic 3-fold.

Lemma 4.6. *Let Y be a smooth cubic 3-fold. For $(g,d) \in \{(0,5),(2,6)\}$, a general element $C \in \mathcal{H}_{g,d}^Y$ is fixed by no non-trivial automorphism of Y and thus J^Y is uncountable.*

Proof. Define $G := \mathrm{Aut}(Y) \setminus \{\mathrm{id}\}$ and consider the correspondence

$$\mathcal{F} = \left\{(C,a) \in \mathcal{H}_{g,d}^Y \times G \,|\, a(C) = C\right\}$$

together with the projections p_1 and p_2 to the two factors. Notice that the subset of $\mathcal{H}_{g,d}^Y$ of curves which are fixed by some automorphism coincides with the subscheme

$$F := \bigcup_{a \in G} p_1\left(p_2^{-1}(a)\right) \subset \mathcal{H}_{g,d}^Y.$$

By [MM64] $\mathrm{Aut}(Y)$ is finite and so is G. Thus G and consequently p_1 are projective. This implies that F is closed as the finite union of the closed subschemes $p_1\left(p_2^{-1}(a)\right)$, $a \in G$. We will now show that $F \neq \mathcal{H}^Y_{g,d}$, more precisely, we will show that for any $a \in G$, there exists a $C \in \mathcal{H}^Y_{g,d}$ not fixed by a.

We briefly recall a construction from [BL15, Section 3.3]: let p be a general point in Y and S a general hyperplane section of Y not containing p. Define the rational map $\phi\colon S \dashrightarrow Y$ by sending a point q to the third point of intersection of the line through p and q and Y. Then $Q := \overline{\phi(S)}$ is a hyperquadric section of Y singular at the point p. Moreover, Q is isomorphic to the blowup of $\mathbb{P}^2$ along 12 points, all lying on a cubic curve Γ, followed by the contraction of Γ. Using this construction, the authors provide examples of curves of genus g and degree d with $(g,d) \in \{(0,5),(2,6)\}$ lying on Q and passing though p, satisfying the generality conditions of Proposition 3.1.

Now let p be a general point in Y such that $a(p) = q \neq p$. Choose a general hyperplane section S of Y not containing p, such that the hyperquadric section Q of the previous construction does not contain q. Then for $(g,d) \in \{(0,5),(2,6)\}$, we may find a curve $C \in \mathcal{H}^Y_{g,d}$ lying on Q and passing though p. We have $p \in C$ but $a(p) \notin C$, thus $a(C) \neq C$. □

Theorem 4.7. *For each j in J, we fix an element $C_j \in \mathcal{H}^Y_{g,d}$ in the projective equivalence class corresponding to $j \in I_{g,d}$.*

Let G be the subgroup of $\mathrm{Bir}(Y)$ *generated by elements admitting a decomposition into Sarkisov links none of them equivalent to χ_{C_j} (see Remark 2.8). We then have*

$$\mathrm{Bir}(Y) = G * \left(\mathop{\ast}_{J^Y} \langle \chi_{C_j} \rangle\right) \cong G * \left(\mathop{\ast}_{J^Y} \mathbb{Z}/2\mathbb{Z}\right),$$

where the indexing set J is uncountable.

Proof. The groupoid $\mathrm{BirMori}(Y)$ is generated by Sarkisov links and isomorphisms of Mori fiber spaces, and relations are generated by trivial and elementary ones (see Theorem 2.7). Every link equivalent to χ (see Remark 2.8) is of the form $a \circ \chi \circ b$ and is thus redundant in the generation of the groupoid. Thus we may take as generators Mori fiber space isomorphisms, as well as all Sarkisov links that are either χ_{C_j} or they are not equivalent to χ_{C_j} for any $j \in J$. Moreover, by replacing links equivalent to χ_{C_j} by $a \circ \chi_{C_j} \circ b$ in all relations, for any $j \in J$, we see that the only generating relations involving χ_{C_j} of Remark 3.7 are $\chi^2_{C_j} = \mathrm{id}_Y$ and $a \circ \chi_{C_j} \circ b = \chi_{C_j}$. In the second relation, the target and the source of χ_{C_j} being Y, implies that $a, b \in \mathrm{Aut}(Y)$. However, by comparing base loci, we see that a and b must fix the curve C_j, which by our choice of J^Y, implies that $a = d = \mathrm{id}_Y$. Thus the only relation among our new set of generators, involving χ_{C_j} is $\chi^2_{C_j} = \mathrm{id}_Y$.

To show that $\mathrm{Bir}(Y) = G * \left(\ast_{J^Y} \langle \chi_{C_j} \rangle\right)$, we have to show that:

1. each element of $\mathrm{Bir}(Y)$ can be written as a product of elements in the factors of $G * \left(\ast_{J^Y} \langle \chi_{C_j} \rangle\right)$;

2. generating relations involve only elements from a single factor of $G * \left(\ast_{J^Y} \langle \chi_{C_j} \rangle\right)$.

For the former, given any element of $\mathrm{Bir}(Y)$ we may decompose it into Sarkisov links using the generators chosen in the previous paragraph. Then factoring this decomposition by isolating all elements χ_{C_j}, we get a product of elements in G and $\langle \chi_{C_j} \rangle$, $j \in J$.

As for the latter, let $r = \mathrm{id}_Y$ be a relator in $\mathrm{Bir}(Y)$. As previously, r is a product of conjugates of the generating relations chosen in the first paragraph, these are precisely elements of the form ${\chi_{C_j}}^2$, $j \in J$ and R, with $R = \mathrm{id}_W$ is a relator in $\mathrm{BirMori}(Y)$ involving none of the χ_{C_j}. Again factoring by isolating all expressions ${\chi_{C_j}}^2$ we get that r is a product of conjugates of elements of the form r_G and ${\chi_{C_j}}^2$, where $r_G = \mathrm{id}_Y$ is a relation in G. Thus r may be generated by relators involving only elements of G or $\langle \chi_{C_j} \rangle$, $j \in J$.

For the last assertion, if $Y = \mathbb{P}^3$ we conclude by Corollary 4.5 and otherwise by Lemma 4.6. □

Remark 4.8. *The construction of the isomorphism above depends on the choice of a curve in each projective equivalence class of $\mathcal{H}^Y_{g,d}$. Different choices give rise to different isomorphisms.*

4.3. Inner and Field Automorphisms

We now construct a group automorphism of $\mathrm{Bir}(Y)$ which we show that is not generated by inner and field automorphisms.

We first fix an isomorphism $\mathrm{Bir}(Y) \cong G * (\ast\, \mathbb{Z}/2\mathbb{Z})$ among the ones constructed in the previous section (see Remark 4.8). Choose a non-trivial permutation ρ of J, such that there exists $j_0 \in J^Y_{g,d}$ with ${j_0}' := \rho(j_0) \in J^Y_{g',d'}$ and $(g,d) \neq (g', d')$. We note that, whether Y is $\mathbb{P}^3$ or a smooth cubic 3-fold, such a choice is always possible.

We now define an automorphism $\phi = \phi(\rho)$ of $\mathrm{Bir}(Y)$ by sending the factor of the free product with index j to that with index $\rho(j)$. More precisely, we define the automorphism $\phi = \phi(\rho)$ on the generators of the free product by sending χ_{C_j} with $\chi_{C_{\rho(j)}}$ and fixing all generators in G.

Proposition 4.9. *The automorphism ϕ of $\mathrm{Bir}(Y)$ defined above is not the composition of a field automorphism σ of $\mathbb{C}$ preserving Y and an inner automorphism of $\mathrm{Bir}(Y)$.*

Proof. Suppose the contrary. Then for any $f \in \mathrm{Bir}(Y)$ we have

$$\phi(f) = \beta_\tau \left(g \circ f \circ g^{-1}\right),$$

where $g \in \mathrm{Bir}(Y)$ and τ is a field automorphism of $\mathbb{C}$. For $f = \chi_{C_{j_0}}$ we get

$$\chi_{C_{{j_0}'}} = \beta_\tau \left(g \circ \chi_{C_{j_0}} \circ g^{-1}\right) \iff \beta_\sigma(\chi_{C_{{j_0}'}}) = g \circ \chi_{C_{j_0}} \circ g^{-1},$$

where $\sigma = \tau^{-1}$. However, by the description of relations involving $\chi_{C_{j_0}}$, the only possible choice would be for $g = \mathrm{id}_Y$. We would then have

$$\beta_\sigma(\chi_{C_{{j_0}'}}) = \chi_{C_{j_0}}.$$

By comparing base loci, we get that $\alpha_\sigma(C_{j_0'}) = C_{j_0}$. However, $\alpha_\sigma(C_{j_0'})$ is abstractly isomorphic to $C_{j_0'}$ which cannot be isomorphic to C_{j_0} as they have different genera, which is a contradiction. $\square$

Corollary 4.10. *The group* $\mathrm{Bir}(Y)$ *is not generated by inner automorphisms and field automorphisms preserving* Y*.*

Moreover, there exist elements of any order which do not lie in the subgroup generated by inner and field automorphisms: the order of $\phi(\rho)$ *is equal to the order of* ρ *and since* J *is infinite we can find permutations of any order.*

Remark 4.11. *For* $Y = \mathbb{P}^3$ *and any* ρ *as above, the group automorphism* $\phi(\rho)\colon \mathrm{Cr}_3(\mathbb{C}) \to \mathrm{Cr}_3(\mathbb{C})$ *is not a homeomorphism with respect to either the Zariski or the Euclidian topology on* $\mathrm{Cr}_3(\mathbb{C})$*. Indeed by the results of [UZ21], any homeomorphism of* $\mathrm{Cr}_3(\mathbb{C})$*, with respect to either of the two topologies, is the composition of a field automorphism with an inner automorphism.*

4.4. Extensions of the construction

All results proven in the previous sections rely on the involutions constructed in Proposition 3.1 and their rigidity proved in Proposition 3.5. These involutions have appeared before in the literature in [CM13], [BL12] and [BL15].

The rigidity of these involutions essentially boils down to the fact that they are dominated by a smooth weak-Fano 3-fold of anti-canonical degree 2, which is the smallest degree possible. However, among the lists of [CM13] there are several other examples of involutions of Fano 3-folds which have the same property. It is a natural question whether the whole construction extends to these cases as well.

Another approach would be to work with a group action. Many examples of Sarkisov links are naturally G-equivariant for some group G (cf [CS19, Proposition 5.27]). In general however there are many G-equivariant Sarkisov links which are not Sarkisov links (cf [CS22]). Working in that setting would allow for more freedom in the choice of the links while enforcing more restrictions (based on the group G) on the rank 3 fibrations one needs to rule out.

CHAPTER V

Connected algebraic subgroups of Bir(X) not contained in a maximal one

Abstract

We prove that for each $n \geq 2$, there exist a ruled variety X of dimension n and a connected algebraic subgroup of $\mathrm{Bir}(X)$ which is not contained in a maximal one.

1. Introduction

Let $\mathbf{k}$ be an algebraically closed field. The classification of algebraic subgroups of groups of birational transformations was initiated in [Enr93], where Enriques shows that each connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(S)$, with S isomorphic to $\mathbb{P}^2$ or to the n-th Hirzebruch surface $\mathbb{F}_n$ for $n \neq 1$; and these are all maximal, with respect to the inclusion, among the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. The connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ have been classified over $\mathbf{k} = \mathbb{C}$ by Umemura in a series of four papers [Ume80, Ume82a, Ume82b, Ume85] and it follows again from his classification that each connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3)$ is contained in a maximal one (see also [BFT21a, BFT21b] for a modern approach). However, it is an open problem whether every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ is contained in a maximal one when $n \geq 4$.

On the other hand, it is proven in [Fon21b, Theorem C] that there exist connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ not contained in a maximal one when C is a smooth curve of positive genus. The proof of this result is based on the existence of infinite increasing sequences of connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ (see [Fon21b, Theorem A]), and on the fact that the dimension of a maximal connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ is bounded by 4 (see [Fon21b, Theorem B] and [Mar71, Theorem 3]). Our main result in this note is a higher dimensional analogue of [Fon21b, Theorem C]:

Theorem A. *Let $\boldsymbol{k}$ be an algebraically closed field of characteristic* 0*. Let $n \geq 1$ and C be a smooth curve of positive genus. Then there exists a connected algebraic subgroup of* $\mathrm{Bir}(C \times \mathbb{P}^n)$ *which is not contained in a maximal one.*

The idea of the proof is to consider the connected algebraic subgroup $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$, where S is a ruled surface such that $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$, and to show that it cannot be contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. Since $\mathrm{Aut}^\circ(S \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S) \times \mathrm{PGL}_{n+1}(\mathbf{k})$ by [BSU13, Corollary 4.2.7], the existence of infinite increasing sequences of connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ is an immediate consequence of [Fon21b, Theorem A]. From this alone, it is nonetheless insufficient to deduce that one of the connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ appearing in the infinite increasing sequences is not contained in a maximal one (see Remark 2.7), and classifying all connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ seems out of reach at the moment.

This article is organized as follows. Section 2 contains two results, namely Lemmas 2.5 and 2.6, which are important for the proof of the higher dimensional case. As a consequence of these two lemmas, we also get a new and short proof of the dimension two case (see Proposition 2.8), without using the classification of the maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ ([Fon21b, Theorem B]). In Section 3, we prove the higher dimensional case and we need to assume furthermore that $\mathrm{char}(\mathbf{k}) = 0$, in view of using the machinery of the MMP and the G-Sarkisov program. The latter has been developped by Floris in [Flo20], building upon results of Hacon and McKernan in [HM13]. More precisely, if G is a connected algebraic group, then every G-equivariant birational map between Mori fibre spaces decomposes into G-Sarkisov links (see [Flo20, Theorem 1.2]). We study the possible links in Lemmas 3.4 and 3.5. Combining Proposition 2.8 and Theorem 3.6, we get Theorem A.

It is very natural to also ask whether for all $n \geq 2$, there exists a variety X of dimension n such that $\mathrm{Bir}(X)$ contains algebraic subgroups which are not lying in a maximal one, without the connectedness assumption. If $n = 2$, the answer is also affirmative (see [Fon21a, Lemma 3.1, Corollary B]), and the proof is analogous to that of the connected case. Since the G-Sarkisov program is known for connected algebraic groups, it is not clear if the proof presented in this article could be adapted for the non-connected case in higher dimension.

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2. Some preliminaries and the case of dimension two

From now on, C will always denote a smooth curve of genus g over a field $\mathbf{k}$. In this section, $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic. The following invariant was used by Maruyama in [Mar70, Mar71] for his classification of ruled surfaces and their automorphisms.

Definition 2.1. *Let $\tau\colon S \to C$ be a ruled surface. We define the* Segre invariant *of S as*

$$\mathfrak{S}(S) = \min\{\sigma^2, \sigma \textit{ section of } \tau\}.$$

Remark 2.2. *Let $\tau\colon S \to C$ be a ruled surface.*

1. *Let $p \in S$ and σ be a section of τ. Recall that the blow-up of S at p followed by the contraction of the strict transform of the fibre passing through p, yields a ruled surface $\tau'\colon S' \to C$ and a birational map $\epsilon\colon S \dashrightarrow S'$ called the* elementary transformation of S centered at p *(see e.g. [Har77, V. Example 5.7.1]). Let σ' be the strict transform of σ by ϵ. If $p \in \sigma$, then $\sigma'^2 = \sigma^2 - 1$. Else, $\sigma'^2 = \sigma^2 + 1$.*
2. *As S is obtained by finitely many elementary transformations from $C \times \mathbb{P}^1$ (see e.g. [Har77, V. Exercise 5.5]) and $\mathfrak{S}(C \times \mathbb{P}^1) = 0$ (see e.g. [Fon21b, Lemma 2.14]), it follows that $\mathfrak{S}(S) > -\infty$. If moreover $\mathfrak{S}(S) < 0$, then there exists a unique section with negative self-intersection number (see e.g. [Fon21a, Lemma 2.10. (1)]).*
3. *The Segre invariant $\mathfrak{S}(S)$ equals $-e$, where e is the invariant defined in [Har77, V. Proposition 2.8]. If S is indecomposable, then by [Har77, V. Theorem 2.12. (b)], we get $\mathfrak{S}(S) \geq 2-2g = -\deg(K_C)$. In particular, if $\mathfrak{S}(S) < -\deg(K_C)$, then S is decomposable.*

We recall the statement of Blanchard's lemma and its corollary (see [BSU13, Proposition 4.2.1, Corollary 4.2.6]):

Proposition 2.3. *Let $f\colon X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and let G be a connected group scheme acting on X. Then there exists a unique action of G on Y such that f is G-equivariant.*

Corollary 2.4. *Let $f\colon X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then f induces a homomorphism of group schemes $f_*\colon \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$.*

In the next two lemmas, we compute $\mathrm{Aut}^\circ(S)$ and its orbits for a ruled surface S with $\mathfrak{S}(S) < -(1+\deg(K_C))$ (which is decomposable by Remark 2.2 3).

Lemma 2.5. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S = \mathbb{P}(V) \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$. Let σ be the minimal section of τ and $L(\sigma)$ be the line subbundle of V associated to σ. We choose trivializations of τ such that σ is the infinity section. Then the following hold:*

1. *The group* $\mathrm{Aut}^\circ(S)$ *is isomorphic to* $\mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$*, where* $\det(V)$ *denotes the determinant line bundle of* V*. This isomorphism associates* $\alpha \in \mathbb{G}_m$ *and* $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$*, to the element* $\mu_{\alpha,\gamma} \in \mathrm{Aut}^\circ(S)$ *obtained by gluing the automorphisms:*

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to U_i \times \mathbb{P}^1 \\ (x, [u : v]) &\mapsto \left(x, [\alpha u + \gamma_{|U_i}(x) v : v]\right). \end{aligned}$$

2. *The* $\mathrm{Aut}^\circ(S)$*-orbits in* S *are* $\{p\}$ *and* $\tau^{-1}(\tau(p)) \setminus \{p\}$ *for* $p \in \sigma$*.*

Proof. 1. The proof follows from the computation made in [Mar71, case (b) p.92]. For the sake of self-containess, we recall it below. Since τ is decomposable, we can write its transition maps as $t_{ij} \colon U_j \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, $(x, [u : v]) \mapsto (x, [a_{ij}(x)u : b_{ij}(x)v])$, where $a_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$ denotes the transition maps of the line bundle $L(\sigma)$ and $b_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$. Let $\mu \in \mathrm{Aut}^\circ(S)$. The morphism induced by Blanchard's lemma $\tau_* \colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial by [Mar71, Lemma 7]. Moreover, σ is fixed by $\mathrm{Aut}^\circ(S)$ as it is the unique minimal section. Therefore, for each trivializing open subset $U_i \subset C$, μ induces an automorphism $\mu_i \colon U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, given by $(x, [u : v]) \mapsto (x, [\alpha_i(x)u + \gamma_i(x)v : v])$, where $\alpha_i \in \mathcal{O}_C(U_i)^*$ and $\gamma_i \in \mathcal{O}_C(U_i)$. The condition $\mu_i t_{ij} = t_{ij}\mu_j$ implies that $\alpha_i = \alpha_j = \alpha \in \mathbb{G}_m$ and $\gamma_i = b_{ij}^{-1} a_{ij} \gamma_j$. Since $a_{ij} b_{ij}$ are the transition maps of the line bundle $\det(V)$, and a_{ij} denote the transition maps of $L(\sigma)$, it implies that $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. The data of $\alpha \in \mathbb{G}_m$ and $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ determine uniquely the automorphism μ, this proves that we have an embedding $\mathrm{Aut}^\circ(S) \hookrightarrow \mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. Conversely, one can check that the automorphisms defined in the statement commute with the transition maps, hence their gluing defines an automorphism of S. Because $\mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ is also connected, we get that it is isomorphic to $\mathrm{Aut}^\circ(S)$.

2. Since the morphism induced by Blanchard's lemma $\tau_* \colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial ([Mar71, Lemma 7]), each $\mathrm{Aut}^\circ(S)$-orbit is contained in a fibre of τ. As σ is the unique section with negative self-intersection number, it is fixed pointwise by $\mathrm{Aut}^\circ(S)$. It remains to see that $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for each p lying on σ.

Let $L = \det(V)^\vee \otimes L(\sigma)^{\otimes 2}$. It follows from [Fon21b, Proposition 2.15] that $\deg(L) = -\mathfrak{S}(S) > 1 + \deg(K_C)$. Let $p \in \sigma$ and let $\tau(p) = z$. We get by Serre duality that

$$h^1(C, L) = h^0(C, K_C \otimes L^\vee) = 0,$$

where the last equality follows from the fact that $\deg(K_C \otimes L^\vee) < -1$. Similarly we get the equality $h^1(C, L \otimes \mathcal{O}_C(z)^\vee) = 0$. By Riemann-Roch, $h^0(C, L \otimes \mathcal{O}_C(z)^\vee) = \deg(L) - g < \deg(L) - g + 1 = h^0(C, L)$. Therefore, z is not a base point of the complete linear system $|L|$, i.e. there exists $\gamma \in H^0(C, L)$ such that $\gamma(z) \neq 0$, and the subgroup $\mathbb{G}_a \simeq \{\mu_{1,\lambda\gamma}; \lambda \in \mathbf{k}\}$ acts transitively on $\tau^{-1}(z) \setminus \{p\}$ (see 1 for the definition of $\mu_{1,\lambda\gamma}$).

□

Let S be a ruled surface as in Lemma 2.5, and $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map. In the following lemma, we compute the fixed points of the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S'.

Lemma 2.6. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$. If $\tau'\colon S' \to C$ is a ruled surface and there exists an $\mathrm{Aut}^\circ(S)$-equivariant birational map $\phi\colon S \dashrightarrow S'$ which is not an isomorphism, then $\mathfrak{S}(S') < \mathfrak{S}(S)$ and $\phi\mathrm{Aut}^\circ(S)\phi^{-1} \subsetneq \mathrm{Aut}^\circ(S')$. The fixed points of the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S' are the points lying on the minimal section of τ' and the base points of ϕ^{-1}. Moreover, we can write ϕ as a product of $\mathrm{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections.*

Proof. By [DI09, Theorem 7.7], we can write $\phi = \phi_n \cdots \phi_1$ where each ϕ_i is an $\mathrm{Aut}^\circ(S)$-equivariant elementary transformation. Without loss of generality, we can assume that this decomposition is minimal (i.e. the number of elementary transformations n is minimal among all possible factorizations), and we prove the statement by induction on $n \geq 1$.

Let σ be the minimal section of τ. By Lemma 2.5 2, the algebraic group $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for every $p \in \sigma$. Since ϕ_1 is $\mathrm{Aut}^\circ(S)$-equivariant, it follows that $\phi_1\colon S \dashrightarrow S_1$ is an elementary transformation centered on a point $p_1 \in \sigma$. The strict transform of σ by ϕ_1 is the minimal section σ_1 of the ruled surface $\tau_1\colon S_1 \to C$, and so $\mathfrak{S}(S_1) = \mathfrak{S}(S) - 1$. Since the base point q_1 of ϕ_1^{-1} does not lie on the minimal section σ_1 of τ_1, it follows by Lemma 2.5 2 that q_1 is not fixed by $\mathrm{Aut}^\circ(S_1)$. Since q_1 is fixed by $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$, we have the strict inequality $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subsetneq \mathrm{Aut}^\circ(S_1)$. In the complement of the fibres $f_{p_1} \subset S$ and $f_{q_1} \subset S_1$ containing the points p_1 and q_1 respectively, ϕ_1 is an isomorphism. Therefore, by Lemma 2.5, the only fixed points of $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$ that lie in the complement of f_{q_1} are the points on the minimal section σ_1. It remains to check that the only fixed points on f_{q_1} are the point $q_1' \in \sigma_1$ and the base point q_1 of ϕ^{-1}. Let U be a trivializing open subset of τ with $\tau(p_1) \in U$, and let $f \in \mathcal{O}_C(U)$ such that $\mathrm{div}(f)_{|U} = \tau(p_1)$. We also choose trivializations of τ such that σ is the infinity section. Up to isomorphisms at the source and the target, $\phi_{1|U}$ equals $(x,[u:v]) \mapsto (x,[f(x)u:v])$. By Lemma 2.5 1, there is an action of $\mathbb{G}_m$ on S given locally by $(x,[u:v]) \mapsto (x,[\alpha u:v])$. It implies that there is an action of $\phi_1\mathbb{G}_m\phi_1^{-1}$ on S_1, given locally by $(x,[u:v]) \mapsto (x,[\alpha f(x)u : f(x)v]) = (x,[\alpha u:v])$. Therefore, $\phi_1\mathbb{G}_m\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$ acts transitively on $f_{q_1} \setminus \{q_1, q_1'\}$. Since $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$ acts fibrewise (see [Mar71, Lemma 7]) and is connected, we get that q_1 and q_1' are the fixed points of the action of $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$ on f_{q_1}.

Assume the statement holds for the birational map $\psi = \phi_i \cdots \phi_1\colon S \dashrightarrow S_i$, for some $i \geq 1$, and where $\tau_i\colon S_i \to C$ is a ruled surface with a minimal section σ_i. We now prove that the statement is then true for $\phi_{i+1}\psi$. By induction, the fixed points of $\psi\mathrm{Aut}^\circ(S)\psi^{-1}$ on S_i are the points lying on the minimal section σ_i and the base points of ψ^{-1}. Assume that ϕ_{i+1} is centered on a base point of ψ^{-1}, which is (the image of) the base point of the inverse of a previous elementary transformation ϕ_j. A local calculation yields that we may cancel both ϕ_j and ϕ_{i+1}, which contradicts the minimality of the factorization of ϕ. So ϕ_{i+1} is centered on a point lying on the minimal section σ_i. Hence $\mathfrak{S}(S_{i+1}) = \mathfrak{S}(S_i) - 1 < \mathfrak{S}(S)$ by induction, and $\phi_{i+1}(\psi\mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subset \mathrm{Aut}^\circ(S_{i+1})$. The base point of ϕ_{i+1} is fixed by $\phi_{i+1}(\psi \mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1}$, but is not fixed by $\mathrm{Aut}^\circ(S_i)$ (by Lemma 2.5). Thus, we get the strict inclusion $\phi_{i+1}(\psi \mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subsetneq \mathrm{Aut}^\circ(S_{i+1})$. □

The infinite increasing sequences of automorphism groups given in [Fon21b, Theorem A] can be obtained from Lemma 2.6, but they do not imply that $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup. As it is explained below, we can get an infinite increasing sequence of connected algebraic subgroups, where each of them is included in a maximal one, which a fortiori cannot be the same for all of them.

Remark 2.7. *Let $n \geq d \geq 2$. Define the connected algebraic groups*

$$G_d = \{\mathbb{A}^2 \to \mathbb{A}^2, (x,y) \mapsto (x, y + p(x)), p \in \boldsymbol{k}[x]_{\leq d}\},$$

acting regularly on $\mathbb{A}^2$, and then birationally on $\mathbb{P}^2$ via any embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Then $G_d \subsetneq G_{d+1}$ for all d. On the other hand, using an explicit description of $\mathrm{Aut}^\circ(\mathbb{F}_n)$ from [Bla09, §4.2], we get for all $n \geq d$ that G_d is a subgroup of $\mathrm{Aut}^\circ(\mathbb{F}_n)$, which is a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.

Notice that for any variety X, using Remark 2.7, we may produce an infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(X \times \mathbb{P}^2)$. In particular, for $n \geq 2$ and C a curve of positive genus, the same is true for $\mathrm{Bir}(C \times \mathbb{P}^n) \simeq \mathrm{Bir}(C \times \mathbb{P}^{n-2} \times \mathbb{P}^2)$.

We reprove below partially [Fon21b, Theorem C], without using [Fon21b, Theorem B].

Proposition 2.8. *Let C be a curve of genus $g \geq 1$ and let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$. Then $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$.*

Proof. Assume that $\mathrm{Aut}^\circ(S)$ is contained in a maximal connected algebraic subgroup G of $\mathrm{Bir}(S)$. Then G acts regularly on a variety Y by Weil regularization theorem (see [Wei55], or [Zai95,Kra18] for a modern proof). By [Bri17, Corollary 3], we can choose Y to be normal and projective. Using an equivariant resolution of singularities (see [Lip78, Remark B, p.155]), we can also assume Y to be smooth. Then by Blanchard's lemma (see Proposition 2.3), the successive contractions of the (-1)-curves gives rise to a ruled surface S' such that the induced birational morphism $Y \to S'$ is G-equivariant. Since G is maximal and connected, it follows that $G \simeq \mathrm{Aut}^\circ(S')$. The induced birational map $\phi\colon S \dashrightarrow S'$ is $\mathrm{Aut}^\circ(S)$-equivariant. If ϕ is an isomorphism, then $\mathfrak{S}(S) = \mathfrak{S}(S')$. Else ϕ factorises as product of $\mathrm{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections and $\mathfrak{S}(S') < \mathfrak{S}(S)$ (by Lemma 2.6). In both cases, we have $\mathfrak{S}(S') \leq \mathfrak{S}(S)$. Let $\epsilon\colon S' \dashrightarrow S''$ be an elementary transformation centered on the minimal section of $\tau'\colon S' \to C$. Then again by Lemma 2.6, it follows that $\epsilon \mathrm{Aut}^\circ(S')\epsilon^{-1} \subsetneq \mathrm{Aut}^\circ(S'')$, which contradicts the maximality of G as a connected algebraic subgroup of $\mathrm{Bir}(S)$. □

3. Higher dimensional case

In order to utilize the machinery of the G-Sarkisov program, from now on we furthermore assume that $\mathrm{char}(\mathbf{k}) = 0$. The G-Sarkisov program is a non-deterministic algorithm that decomposes every G-equivariant birational map between two G-Mori fibre spaces as a product of simpler maps called G-Sarkisov links. Its non-equivariant version was proven by Hacon and McKernan in [HM13] and, building on their result, Floris proved the G-equivariant version in [Flo20]. We follow the strategy of the proof of Proposition 2.8, and in view of using G-Sarkisov program, we recall first the definition:

Definition 3.1. *Let G be a connected algebraic group. A G-*Mori fibre space *is a Mori fibre space with a regular action of G. Let $\pi_1\colon X_1 \to B_1$ and $\pi_2\colon X_2 \to B_2$ be two birational G-Mori fibre spaces. A G-*Sarkisov diagram *between X_1/B_1 and X_2/B_2 is a commutative diagram of the form*

$$\begin{tikzcd}
Y_1 \arrow[rr, dotted, "\chi"] \arrow[d, "\alpha_1"'] & & Y_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[dr, "s_1"'] & & B_2 \arrow[dl, "s_2"] \\
& R &
\end{tikzcd}$$

which satisfies the following properties:

1. *all morphisms appearing in the diagram are either isomorphisms or outputs of some G-equivariant MMP on a $\mathbb{Q}$-factorial klt G-pair (Z, Φ) (recall that a G-pair is a pair (Z, Φ) such that G acts regularly on Z and there is an induced regular action on Φ),*
2. *maximal dimensional varieties have $\mathbb{Q}$-factorial and terminal singularities,*
3. *α_1 and α_2 are G-equivariant divisorial contractions or isomorphisms,*
4. *s_1 and s_2 are G-equivariant extremal contractions or isomorphisms,*
5. *χ is an isomorphism or a composition of G-equivariant anti-flips/flop/flips (in that order),*
6. *the relative Picard rank $\rho(Z/R)$ of any variety Z in the diagram is at most 2.*

We call R the base *of the diagram.*

Property 6 implies that α_1 is a divisorial contraction if and only if s_1 is an isomorphism. A similar statement holds for the right hand side of the diagram. Depending whether s_1 or s_2 is an isomorphism, we get four types of Sarkisov diagrams:

Type I

$$\begin{tikzcd} Y_1 \arrow[r, dashed] \arrow[d] & X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[d, "\pi_1"'] & B_2 \arrow[dl] \\ B_1 = R & \end{tikzcd}$$

Type II

$$\begin{tikzcd} Y_1 \arrow[rr, dashed] \arrow[d] & & Y_2 \arrow[d] \\ X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, phantom, "="] & R \arrow[r, phantom, "="] & B_2 \end{tikzcd}$$

Type III

$$\begin{tikzcd} X_1 \arrow[r, dashed] \arrow[d, "\pi_1"'] & Y_2 \arrow[d] \\ B_1 \arrow[dr] & X_2 \arrow[d, "\pi_2"] \\ & R = B_2 \end{tikzcd}$$

Type IV

$$\begin{tikzcd} X_1 \arrow[rr, dashed] \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[dr] & & B_2 \arrow[dl] \\ & R & \end{tikzcd}.$$

*The birational map $\psi = \alpha_2\chi\alpha_1^{-1}$ between X_1 and X_2 is called a G-*Sarkisov link.

Remark 3.2. *Property 2 does not follow directly from the original definition of a (G-)Sarkisov diagram of [HM13] and [Flo20]. For a proof, see [BLZ21, Proposition 4.25].*

In subsequent proofs we are going to make heavy use of the following elementary but useful observation:

Remark 3.3. *Let Z be a one of the varieties appearing in a G-Sarkisov diagram, such that the relative Picard rank $\rho(Z/R)$ is 2. Then the G-Sarkisov diagram is uniquely determined by the datum of $Z \to R$, by a process known as the* 2-ray game *(see [BLZ21, section 2.F]).*

More specifically, the 2-ray game is a deterministic process that assigns to any such $Z \to R$ a G-Sarkisov diagram. Moreover any G-Sakrisov diagram can be recovered by the 2-ray game on any of its relative Picard rank 2 morphisms. Thus, up to orientation of the diagram, there is a unique G-Sarkisov diagram that contains $Z \to R$.

Lemma 3.4. *Let $n \geq 1$ and C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$ with minimal section σ and let $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map (possibly the identity) to a $\mathbb{P}^1$-bundle $\tau'\colon S' \to C$. Let $\pi' = \tau' \times id_{\mathbb{P}^n}\colon S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ and $\pi'_1\colon S' \times \mathbb{P}^n \to S'$ be the projection to the first factor. Then the following hold:*

1. *The only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi'\colon S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ is the LHS Mori fibre space, are the following ones:*

$$\begin{tikzcd} T \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\alpha"'] & T \times \mathbb{P}^n \arrow[d, "\beta"] \\ S' \times \mathbb{P}^n \arrow[d, "\pi'"'] & S'' \times \mathbb{P}^n \arrow[d, "\pi''"] \\ C \times \mathbb{P}^n \arrow[r, equal] & C \times \mathbb{P}^n \end{tikzcd} \qquad \begin{tikzcd} S' \times \mathbb{P}^n \arrow[rr, equal] \arrow[d, "\pi'"'] & & S' \times \mathbb{P}^n \arrow[d, "\pi'_1"] \\ C \times \mathbb{P}^n \arrow[dr, "p_1"'] & & S' \arrow[dl, "\tau'"] \\ & C & \end{tikzcd}.$$

In the first case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S'' \times \mathbb{P}^n$ is equal to $\psi \times id_{\mathbb{P}^n}$, where $\psi\colon S' \dashrightarrow S''$ is an elementary transformation of $\mathbb{P}^1$-bundles whose center p is a point fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, and T is the blow-up of S' at p. In the second case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ is equal to $id_{S'\times\mathbb{P}^n}$.

2. *The only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi_1'\colon S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{tikzcd} T \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\eta\times id_{\mathbb{P}^n}"'] & T \times \mathbb{P}^n \arrow[d, "\pi_1''"] \\ S' \times \mathbb{P}^n \arrow[d, "\pi_1'"'] & T \arrow[dl, "\eta"] \\ S' & \end{tikzcd} \qquad \begin{tikzcd} S' \times \mathbb{P}^n \arrow[rr, equal] \arrow[d, "\pi_1'"'] & & S' \times \mathbb{P}^n \arrow[d, "\pi'"] \\ S' \arrow[dr, "\tau"'] & & C \times \mathbb{P}^n \arrow[dl, "p_1"] \\ & C & \end{tikzcd}.$$

The induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow T \times \mathbb{P}^n$ is equal to $\eta^{-1} \times id_{\mathbb{P}^n}$ in the former case and $id_{S'\times\mathbb{P}^n}$ in the latter, where $\eta\colon T \to S'$ is the blowup of S' at point p fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$.

Proof. 1. We distinguish between two cases depending on the base R of the diagram: if $R = C \times \mathbb{P}^n$ then we have a link of Type I or II and so the first step of the link is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y \to S' \times \mathbb{P}^n$. Note that by [BSU13, Corollary 4.2.7], it follows that $(\phi \times id_{\mathbb{P}^n})\mathrm{Aut}^\circ(S \times \mathbb{P}^n)(\phi \times id_{\mathbb{P}^n})^{-1} \simeq \phi\mathrm{Aut}^\circ(S)\phi^{-1} \times \mathrm{PGL}_{n+1}(\mathbf{k})$. Let $(q, x) \in S' \times \mathbb{P}^n$ be a point in the center of α. If q is not point fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, then and by Lemma 2.5 and the description of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, the closure of the orbit of (q, x) is a Cartier divisor and thus α is an isomorphism, contradicting the assumption that α is a divisorial contraction.

Thus we may assume that q is fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$. In that case the orbit of (q, x) is precisely $\{q\} \times \mathbb{P}^n$. Notice that the codimension of $\{q\} \times \mathbb{P}^n$ is 2 and so by [BLZ21, Lemma 2.13]

$$\alpha = (\eta \times id_{\mathbb{P}^n})\colon T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,$$

where $\eta\colon T \to S'$ is the blowup of S' at q. By Remark 3.3, the unique Sarkisov diagram containing $T \times \mathbb{P}^n \to C \times \mathbb{P}^n$ is the one given in the statement.

We now consider the case when $R \neq C \times \mathbb{P}^n$. Then we have a contraction $C \times \mathbb{P}^n \to R$ of relative Picard rank 1. Since $\rho(C \times \mathbb{P}^n) = 2$, the cone of curves $\mathrm{NE}(C \times \mathbb{P}^n)$ has two extremal rays and so there are only two such contractions, namely the projections to the two factors: $C \times \mathbb{P}^n \to C$ and $C \times \mathbb{P}^n \to \mathbb{P}^n$. However, by property 1 of Definition 3.1, $C \times \mathbb{P}^n \to \mathbb{P}^n$ would have to be an output of some MMP on a klt pair (Z, Φ), and thus by [HM07] its exceptional locus would be rationally connected, a contradiction. Thus $R = C$ and again we conclude by Remark 3.3 for $S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$.

2. We again proceed by a similar distinction of cases. If $R = S'$ then, as in the proof of 1, the first step is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\eta \times id_{\mathbb{P}^n} : T \times \mathbb{P}^n \to S' \times \mathbb{P}^n$, where $\eta\colon T \to S'$ is the blow-up of a point of S' fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$, and we conclude by Remark 3.3.

If $R \neq S'$, then $S' \to R$ is one of the two morphisms $S' \to C$ or $S' \to \check{S}'$, where the latter is the contraction of the minimal section. Again, by [HM07] we may exclude the latter case since its exceptional locus is not rationally connected. Finally, Remark 3.3, once again, guarantees that the Sarkisov diagram is the one in the statement.

□

Lemma 3.5. *Let $n \geq 1$ and C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$ with minimal section σ. Let $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map, with S' being a smooth projective surface which is not minimal. Denote by $\pi_1'\colon S' \times \mathbb{P}^n \to S'$ the projection to the first factor. Then the only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi_1'\colon S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{tikzcd}
T \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\eta \times id_{\mathbb{P}^n}"'] & T \times \mathbb{P}^n \arrow[d, "\pi_1''"] \\
S' \times \mathbb{P}^n \arrow[d, "\pi_1'"'] & T \arrow[dl, "\eta"] \\
S' &
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\pi_1'"'] & S' \times \mathbb{P}^n \arrow[d, "\kappa \times id_{\mathbb{P}^n}"] \\
S' \arrow[dr, "\kappa"'] & T \times \mathbb{P}^n \arrow[d, "\pi_1''"] \\
& T.
\end{tikzcd}$$

In the first case, $\eta\colon T \to S'$ is the blow-up of a point p fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$. In the second case, $\kappa\colon S' \to T$ is the contraction of a (-1)-curve l. In both cases, π_1'' denotes the projection to the first factor.

Proof. We again distinguish between two cases depending on the base R of the Sarkisov diagram: if $R = S'$ then the first step of the link is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y \to S' \times \mathbb{P}^n$. We follow the same strategy of the proof of Lemma 3.4: first by [BSU13, Corollary 4.2.7], $(\phi \times id_{\mathbb{P}^n})\mathrm{Aut}^\circ(S \times \mathbb{P}^n)(\phi \times id_{\mathbb{P}^n})^{-1} = \phi \mathrm{Aut}^\circ(S)\phi^{-1} \times \mathrm{PGL}_{n+1}(\mathbf{k})$. This again implies that α has to be an extraction with center of the form $\{q\} \times \mathbb{P}^n$, where q is a point fixed by the action of $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$ on S'. Since the center is of codimension 2, again using [BLZ21, Lemma 2.13], we conclude that

$$a = \eta \times id_{\mathbb{P}^n} : T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,$$

where $\eta\colon T \to S'$ is the blow-up of q. By Remark 3.3, the diagram is the one given in the statement.

If $R \neq S'$, we have a morphism $S' \to R$ of relative Picard rank 1. Since S' is not minimal, its Picard rank is greater or equal to 3 which already implies that $R = T$ is a surface. Again, using Remark 3.3 we may conclude that the diagram is the one proposed in the statement. Moreover, by property 2 of Definition 3.1, $T \times \mathbb{P}^n$ has to have terminal singularities. Thus the singular locus of $T \times \mathbb{P}^n$ has codimension at least 3 (see [KM98, Corollary 5.18]). If $q \in T$ is singular, then $\{q\} \times \mathbb{P}^n$ is singular and has codimension 2 in $T \times \mathbb{P}^n$. This implies that T is smooth and consequently, $S' \to T$ is the contraction of a (-1)-curve. □

We prove below the higher dimensional case of Proposition 2.8.

Theorem 3.6. *Let $n \geq 1$. Let C be a curve of genus $g \geq 1$, let S be a decomposable $\mathbb{P}^1$-bundle over C such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Then $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$.*

Proof. Assume that $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$ is contained in a maximal connected algebraic subgroup $G \subset \mathrm{Bir}(S \times \mathbb{P}^n)$. By [Bri17, Corollary 3], there exists a normal and projective variety Y, G-birationally equivalent to $S \times \mathbb{P}^n$, and on which G acts regularly. Then we use an equivariant resolution of singularities (see [Kol07, Thm. 3.36, Prop. 3.9.1]) to furthermore assume that Y is smooth. Running an MMP, which is G-equivariant by [Flo20, Lemma 2.5], we get a $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\chi\colon S \times \mathbb{P}^n \dashrightarrow Y$ such that $G \simeq \mathrm{Aut}^\circ(Y)$ and $Y \to B$ is a Mori fibre space. By [Flo20, Theorem 1.2], χ decomposes as a product of $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant Sarkisov links. By Lemmas 3.4 and 3.5, it follows that $Y = T \times \mathbb{P}^n$ for some surface T and χ is of the form $\psi \times id_{\mathbb{P}^n}$, where $\psi\colon S \dashrightarrow T$ is an $\mathrm{Aut}^\circ(S)$-equivariant birational map. Up to possibly performing an extra link of Type IV (namely the RHS link in Lemma 3.4 1), we may assume that $B = T$ and θ is given by the projection to the first factor. Contracting successively all (-1)-curves in T yields an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\phi \times id_{\mathbb{P}^n}\colon S \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ (by Blanchard's lemma, see Proposition 2.3), where ϕ is $\mathrm{Aut}^\circ(S)$-equivariant and S' is a ruled surface. Two cases arise: either ϕ is an isomorphism and $\mathfrak{S}(S) = \mathfrak{S}(S')$, or ϕ is not an isomorphism and $\mathfrak{S}(S') < \mathfrak{S}(S)$ by Lemma 2.6. In both cases, $\mathfrak{S}(S') \leq \mathfrak{S}(S)$ and since G is maximal, G is isomorphic to $\mathrm{Aut}^\circ(S' \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S') \times \mathrm{PGL}_{n+1}(\mathbf{k})$ ([BSU13, Corollary 4.2.7]). Let $\phi'\colon S' \dashrightarrow S''$ be an elementary transformation of S' centered at a point on the minimal section. Then $\phi' \mathrm{Aut}^\circ(S') \phi'^{-1} \subsetneq \mathrm{Aut}^\circ(S'')$ by Lemma 2.5. Thus $(\phi' \times id_{\mathbb{P}^n}) \mathrm{Aut}^\circ(S' \times \mathbb{P}^n)(\phi' \times id_{\mathbb{P}^n})^{-1} \subsetneq \mathrm{Aut}^\circ(S'' \times \mathbb{P}^n)$, which contradicts the maximality of G as connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. □

Proof of Theorem A. Let C be a curve of positive genus and $S \to C$ be a ruled surface. As S is birational to $C \times \mathbb{P}^1$, we get for all $n \geq 1$ that $\mathrm{Bir}(C \times \mathbb{P}^n) \simeq \mathrm{Bir}(S \times \mathbb{P}^{n-1})$. We conclude with Proposition 2.8 and Theorem 3.6. □

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