Survival and complete convergence for a branching annihilating random walk

Matthias Birkner, Alice Callegaro, Jiří Černý, Nina Gantert, Pascal Oswald

Departement Mathematik und Informatik
Fachbereich Mathematik
Universität Basel
CH-4051 Basel

Preprint No. 2023-10
June 2023

dmi.unibas.ch

SURVIVAL AND COMPLETE CONVERGENCE FOR A BRANCHING ANNIHILATING RANDOM WALK

BY MATTHIAS BIRKNER[1,a], ALICE CALLEGARO[1,2,b] JIŘÍ ČERNÝ[3,d]
NINA GANTERT[2,c] AND PASCAL OSWALD[1,3,e]

[1]*Johannes Gutenberg-Universität Mainz,* [a]*birkner@mathematik.uni-mainz.de*

[2]*Technical University of Munich,* [b]*alice.callegaro@tum.de;* [c]*nina.gantert@tum.de*

[3]*University of Basel,* [d]*jiri.cerny@unibas.ch;* [e]*pascalamadeus.oswald@unibas.ch*

We study a discrete-time branching annihilating random walk (BARW) on the d-dimensional lattice. Each particle produces a Poissonian number of offspring with mean μ which independently move to a uniformly chosen site within a fixed distance R from their parent's position. Whenever a site is occupied by at least two particles, all the particles at that site are annihilated. We prove that for any $\mu > 1$ the process survives when R is sufficiently large. For fixed R we show that the process dies out if μ is too small or too large. Furthermore, we exhibit an interval of μ-values for which the process survives and possesses a unique non-trivial ergodic equilibrium for R sufficiently large. We also prove complete convergence for that case.

CONTENTS

MSC2020 subject classifications: Primary 60K35; secondary 92D25.
Keywords and phrases: Branching annihilating random walk, survival, extinction, complete convergence, non-monotone interacting particle systems.

1. Introduction. As a model for a population evolving in space, one may consider branching random walks. These are systems of particles where the particles reproduce and move randomly in space, independently for different families. For instance, the children may take i.i.d. displacements from their mother particle or, in a more general model, the parent particle may generate a configuration of children according to some point process. Branching random walks are a very active research topic, we refer to [28] for an introduction.

Our goal is to model a population which competes for resources, hence a particle system in which particles reproduce, move randomly in space, and compete with each other locally. We chose here a rather radical form of interaction: whenever two or more particles are on the same site, they annihilate. The annihilation makes the system non-attractive in the sense of interacting particle systems, i.e. adding more particles initially can stochastically decrease the law of the configuration at later times.

A first question about branching random walks is if the system has a strictly positive survival probability. In the classical case, that is without annihilation, the answer is well-known since the number of particles at time n forms a Galton–Watson process. However, taking into account annihilation, the question is much more difficult and there are relatively few mathematical papers addressing it, see the discussion of related literature in Section 1.3 below.

Assuming that the parameters of the model are such that the survival probability is indeed strictly positive, the next question is about invariant measures and the convergence towards the invariant measure in the case of survival. As for the classical branching random walk or the contact process, it is clear that the Dirac measure on the empty configuration is invariant. We can show for our model that in a certain range of parameters there is complete convergence, i.e. there is exactly one non-trivial ergodic invariant measure and the law of the process, conditioned on survival, approaches this invariant measure.

Our model allows for a representation as a probabilistic cellular automaton. Questions about ergodicity and complete convergence are notoriously difficult for such systems, we refer to [25] for an introduction. If we consider the iteration of the expected number of particles at the sites of the lattice, we have a deterministic system, a coupled map lattice, see Section 1.4 below. This system is of independent interest and we expect that it admits, in a certain range of parameters, travelling wave solutions. Hence our model can be interpreted as a stochastic perturbation of the coupled map lattice, and this interpretation raises several interesting questions.

Let us now give a more precise definition of the model and describe our results. We study a process $\eta = (\eta_n(x) : x \in \mathbb{Z}^d, n \geq 0)$ evolving in discrete time on $\mathbb{Z}^d$, where $\eta_n(z)$ denotes the state of site z at time n. We write $\eta_n(z) = 1$ if site z is occupied by exactly one particle at time n and $\eta_n(z) = 0$ otherwise. We denote by $\|\cdot\|$ the sup-norm on $\mathbb{Z}^d$ and define $B_R(z) = \{x \in \mathbb{Z}^d : \|z - x\| \leq R\}$ to be the d-dimensional ball (box) of radius $R \in \mathbb{N}$ centred at $z \in \mathbb{Z}^d$. We set $V_R = 2R + 1$ to be its side length, so that its volume is V_R^d.

For fixed $R \in \mathbb{N}$, $\mu > 0$, and an initial particle configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, the configurations at later times are obtained recursively as follows. Given η_n, $n \geq 0$, for every $z \in \mathbb{Z}^d$ with $\eta_n(z) = 1$ the particle at z dies and gives birth to a Poisson number of children with mean μ. Each child moves independently to a uniformly chosen site in $B_R(z)$. Whenever there is more than one particle at a given site, then all the particles at that site are killed. This means that if two (or more) children of the same parent jump to the same site they will disappear, but also children coming from different parents who jump to the same site will annihilate. The particles remaining after the annihilation make up the configuration η_{n+1}.

The thinning and superposition properties of the Poisson distribution give the following equivalent description of the model, which is particularly convenient to carry out calculations.

For a configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $z \in \mathbb{Z}^d$, define first the (local) density of particles at z by

$$\delta_R(z;\eta) = V_R^{-d} \sum_{x \in B_R(z)} \eta(x). \tag{1.1}$$

Fix η_n and denote by $N_{n+1}(z)$ the number of newborn particles at z in the next generation before the annihilation occurs. This is given by the superposition of the offspring of all particles that can move to z, that is of all $x \in B_R(z)$ with $\eta_n(x) = 1$. Thus $N_{n+1}(z)$ is a Poisson random variable with mean $\mu\delta_R(z;\eta_n)$. Taking the annihilation into account, it then holds that

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{if } N_{n+1}(z) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Let

$$\varphi_\mu(w) = \mu w\, e^{-\mu w}, \quad w \in [0,\infty) \tag{1.3}$$

denote the probability that a Poisson random variable with mean μw equals 1. By construction, the random variables in the family $(\eta_{n+1}(z) : z \in \mathbb{Z}^d)$ are conditionally independent given η_n and by (1.2), (1.3) we can represent our system as

$$\eta_{n+1}(z) = \begin{cases} 1 & \text{with probability } \varphi_\mu(\delta_R(z;\eta_n)), \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

This gives a representation of η as a particular example of a probabilistic cellular automata. We point out that this representation is only possible because we choose a Poisson offspring distribution. For more detailed discussion of the assumptions of the model, see the discussion in Section 1.2 below.

1.1. *Main results.* We can now state the main results of this paper. For the intuition behind them, we find it useful to first point out a few properties of the function φ_μ introduced in (1.3) which governs the behaviour of the process:

(a) For $\mu \le 1$, φ_μ has a unique fixpoint at 0, which is attractive.
(b) For $\mu > 1$, φ_μ has two fixpoints, 0 and $\theta_\mu = \mu^{-1} \ln \mu$. In this case 0 is always repulsive.
(c) For $\mu \in (1, e^2)$, θ_μ is an attractive fixpoint.
(d) For $\mu > e^2$, there are no attractive fixpoints

In the case (d), the one point iteration $x \mapsto \varphi_\mu(x)$ has rich behaviour. Depending on the value of μ, there can be attractive periodic orbits or chaotic behaviour.

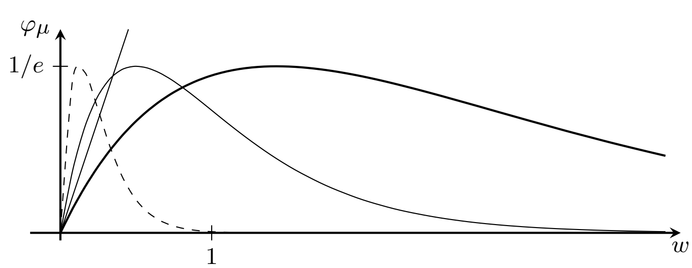

FIG 1. *Graphs of φ_μ for $\mu = 0.7$ (thick), 2 and 8 (dashed), together with the identity function.*

Extinction. Our first result identifies a range of parameters (μ, R) where the process dies out a.s. Here, we say that η goes extinct *locally* if $\lim_{n\to\infty} \eta_n(x) = 0$ for every $x \in \mathbb{Z}^d$, and that η goes extinct *globally*, if $\eta_n \equiv 0$ for all n large enough.

THEOREM 1.1. *For $R \in \mathbb{N}$, let $\mu_1(R), \mu_2(R)$ be the two real solutions of*

$$V_R^d \varphi_\mu(V_R^{-d}) = 1$$

with $1 < \mu_1(R) < \mu_2(R) < \infty$. If $\mu < \mu_1(R)$ or $\mu > \mu_2(R)$, then, for all initial conditions η goes locally extinct a.s., and for all initial conditions containing only a finite number of particles η goes extinct globally a.s. Furthermore, $\mu_1(R) \to 1$ and $\mu_2(R) \to +\infty$ as $R \to \infty$.

The result of the proposition is not optimal, we expect (based on simulations, see Figure 7 in Section 7 below) that the process goes extinct for many values (μ, R) outside of the specified range. On the other hand, its proof is relatively simple. It is given in Section 5 below, and is based on the observation that for (μ, R) in the specified range the killing by annihilation among siblings is already strong enough to make the expected number of "surviving" offspring of a single particle strictly smaller than one, and thus the branching effectively subcritical, even though $\mu > 1$.

REMARK 1.2. The two values $\mu_1(R)$ and $\mu_2(R)$ can be given explicitly as

$$\mu_1(R) = -V_R^d W_0(-V_R^{-d}), \quad \mu_2(R) = -V_R^d W_{-1}(-V_R^{-d}),$$

where W_0 and W_{-1} are the two real branches of the Lambert W function. This also describes their asymptotic behaviour as $R \to \infty$, see (5.2) and (5.3) in the proof of Theorem 1.1 below.

Survival. The second result identifies a range of parameters where it is possible that the process survives locally, by which we mean that for every $x \in \mathbb{Z}^d$ the set of times n when $\eta_n(x) = 1$ is unbounded. Similarly as in Theorem 1.1, the identified range is not optimal.

THEOREM 1.3. *For every $\mu > 1$ there exists $R_\mu \in \mathbb{N}$ such that η survives with positive probability from any non-trivial initial condition when $R \geq R_\mu$.*

REMARK 1.4. Inspection of the proof of Theorem 1.3 shows that there is $R_0 \in \mathbb{N}$ such that for every $R \geq R_0$ there exist two values $1 < \underline{\mu}_R < \overline{\mu}_R$ such that η survives with positive probability from any non-trivial initial condition when $\mu \in (\underline{\mu}_R, \overline{\mu}_R)$. Furthermore, on the event of survival, it holds that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \eta_j(x) > 0 \quad \text{a.s. for any } x \in \mathbb{Z}^d. \tag{1.5}$$

Ergodicity and complete convergence. The final set of results discusses the invariant measures of the process, in the case when the system survives. For this we equip the state space $\{0,1\}^{\mathbb{Z}^d}$ with the product topology and the corresponding Borel σ-algebra. In these results we restrict ourselves to $\mu \in (1, e^2)$, where the non-trivial fixpoint of φ_μ is attractive, as pointed out above.

THEOREM 1.5. *For every $\mu \in (1, e^2)$ there is $R'_\mu < \infty$ such that for every $R \geq R'_\mu$ the process η has two extremal invariant distributions: the first one is trivial and is concentrated on the empty configuration $\eta \equiv 0$, and the second one, $\nu_{\mu,R}$, is non-trivial, translation invariant, ergodic, and has exponential decay of correlations.*

Furthermore, starting from any non-trivial initial condition the process η, conditioned on non-extinction, converges in distribution in the weak topology to the non-trivial extremal invariant distribution $\nu_{\mu,R}$.

The driving result behind Theorem 1.5 is the following strong coupling property of the system η, which is of independent interest.

THEOREM 1.6. *Assume that $\mu \in (1, e^2)$ and $R \geq R'_\mu$ satisfy the assumptions of Theorem 1.5. Then there exists a speed $a = a(R, \mu, d) > 0$ such that for every pair of (possibly random) initial conditions $\eta_0^{(1)}, \eta_0^{(2)} \in \{0,1\}^{\mathbb{Z}^d}$ there exists a coupling of the processes $(\eta_n^{(i)})_{n \in \mathbb{N}_0}$, $i = 1, 2$, with the following property. For each $x \in \mathbb{Z}^d$ there is an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable T_x^{coupl} (whose exact law will in general depend on the initial conditions and on x) such that $\{\eta_n^{(i)} \not\equiv 0$ for all $n \in \mathbb{N}$, $i = 1, 2\} \subseteq \{T_x^{\mathrm{coupl}} < \infty\}$ a.s. and*

$$\eta_n^{(1)}(y) = \eta_n^{(2)}(y) \quad \text{for all } n > T_x^{\mathrm{coupl}} \text{ and } \|y - x\| \leq a(n - T_x^{\mathrm{coupl}}).$$

REMARK 1.7. It follows from the proof of Theorem 1.6 that when starting from a finite (or a half-space) initial condition, the system η, given that it survives, will expand into the "empty territory" at least at some (small) linear speed. Furthermore, simple comparison arguments with supercritical branching random walks show that this expansion cannot occur faster than linearly. However, identifying an actual linear speed or even an asymptotic profile of the expanding population near its tip remains a topic for future research.

REMARK 1.8. Denote by $\overline{\theta}_{\mu,R} = \mathbb{E}_{\nu_{\mu,R}}[\eta_0(0)] \in (0,1)$ the particle density of the non-trivial invariant measure $\nu_{\mu,R}$ from Theorem 1.5, where $\mu \in (1, e^2)$ and $R \geq R'_\mu$. By ergodicity, we have almost surely when $\eta_0 \sim \nu_{\mu,R}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \eta_n(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_R(x; \eta_n) = \overline{\theta}_{\mu,R} \quad \text{for every } x \in \mathbb{Z}^d. \tag{1.6}$$

By the coupling property from Theorem 1.6, (1.6) holds in fact a.s. for any initial condition given that the system survives.

Furthermore, for $1 < \mu < e^2$, inspection of the proof of Theorem 1.6 shows that for every $\varepsilon \in (0,1)$ there exists $R'_{\mu,\varepsilon} < \infty$ such that if $R \geq R'_{\mu,\varepsilon}$ then, conditionally on non-extinction,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{|\delta_R(x;\eta_n) - \theta_\mu| < \varepsilon\}} \geq 1 - \varepsilon \quad \text{almost surely for every } x \in \mathbb{Z}^d, \tag{1.7}$$

where we recall that θ_μ is the fixpoint of φ_μ (on "good" blocks, the particle density is close to θ_μ, see Definition 4.4 below, and good blocks are shown to occur with high space-time density).

Note that (1.6) and (1.7) together imply that $|\overline{\theta}_{\mu,R} - \theta_\mu| \leq 2\varepsilon$ for $R \geq R'_{\mu,\varepsilon}$. This corroborates the idea that for large R, the particle system's behaviour is close to that of the corresponding deterministic coupled map lattice, which we discuss in Section 1.4 below, and which, as shown in Proposition 1.9, converges to the configuration which is constant and equal to θ_μ.

1.2. *Possible generalisations.* The construction of our model might seem very rigid. Therefore, we discuss here the role of the different assumptions in the model and their possible generalisations.

The assumption that the particles jump distribution is uniform over a box of length V_R is non-essential and is made only for convenience of the notation. It can in principle be replaced by an arbitrary (centred) *finite range* transition kernel, and the proofs can be adapted by suitably replacing the particle density (1.1) by the convolution of this kernel with η.

The assumption that the number of offspring of a single particle has Poisson distribution is more important, as it allows for the essential representation (1.4), and also yields the conditional independence of $(\eta_{n+1}(x) : x \in \mathbb{Z}^d)$ given η_n. Replacing the Poisson distribution would thus require non-trivial modifications to our proofs. On the other hand, if we take (1.4) as the definition of the model, the particular form of the function φ_μ used there does not play a strong role. Our techniques will continue to work, if we replace φ_μ by another function of a "similar shape". In fact, for survival we only need that $\varphi_\mu : [0,1] \to [0,1]$ is continuously differentiable and strictly positive on $(0,1]$ with 0 as an unstable fixpoint. For the proof of the convergence result, we also need that there is a unique attracting fixpoint $\theta_\mu \neq 0$.

In a different direction, the "hard" annihilation constraint of at most one particle per site could be relaxed by replacing the definition (1.2) of η by $\eta_{n+1}(z) = N_{n+1}(z)\mathbb{1}_{\{N_{n+1}(z)\leq k\}}$ for some $k \in \mathbb{N}$. Since this modification retains the conditional Poisson and independence properties of the $N_n(z)$'s and the sums of truncated Poisson random variables have good concentration properties, we are hopeful that our proofs could be adapted to this scenario with some additional work.

1.3. *Discussion of related results.* One of the first models of branching annihilating random walks was introduced and studied by Bramson and Gray [7]. They considered a particle system on $\mathbb{Z}$, in which sites can be occupied as the result of the following mechanisms: particles can either *jump* to one of the two neighbouring sites at a certain rate or *branch* into two by giving birth to a new particle on one of the neighbouring sites. On top of this, particles behave independently except when they land on a site which is already occupied, in which case both particles disappear, *annihilate*. The authors show that, starting from any finite number of particles, the system survives with positive probability if the jumping rate is small compared to the branching rate and that the population dies out almost surely if the jumping rate is sufficiently high. This process is an interacting particle system in the sense of [21, 22] but it is not attractive. The authors use contour arguments which rely on the one-dimensional model they chose.

Very general interacting particle systems on $\mathbb{Z}$ are considered in [30], where pairwise interactions among neighbours can produce annihilation, birth, coalescence, and exclusion and single individuals can die. Conditions on the rates which ensure positive probability of survival are given by making use of self-duality (which has been proved by the same author in [31]) and supermartingale arguments. In [6] instead, processes on $\mathbb{Z}^d$ with nearest-neighbour birth at rate 1, annihilation and spontaneous death at rate δ have been considered. An extinction result for the branching annihilating process started from one particle at the origin is obtained by comparison with the contact process. On the other hand, survival when δ is small is proved through comparison with oriented percolation.

In cases where survival can be established, natural questions concern the existence of stationary distributions and weak convergence. Sudbury [29] considers a version of Bramson and Gray's model in $\mathbb{Z}^d$ in the case of no random walk and shows that the product measure with rate $1/2$ is the only non-empty limiting distribution. In the case of a double branching and annihilating process on $\mathbb{Z}$ (where each particle can place offspring on both of its neighbouring sites), a richer variety of limiting measures is exhibited. In [6], the authors prove that when $\delta = 0$ the product measure with density $1/2$ is stationary and is the limiting measure, thus obtaining independently the same result proved in [29]. Furthermore the authors show that for any δ there are at most two extremal translation invariant stationary distributions, and if δ is small there exists a non-trivial stationary distribution.

Another question of interest is whether branching processes with annihilation satisfy duality relations. Athreya and Swart [2] consider processes in continuous time where particles can annihilate, branch, coalesce or die. They show that annihilation does not play a key role in a duality relation: the process with annihilation is dual to a system of interacting Wright-Fisher diffusions, and this result holds also if annihilation is suppressed (but in the case of annihilation the duality function is different and more complicated). It would be highly interesting to find a useful duality relation for our model as well.

Versions of branching annihilating processes in discrete time are generally more difficult to deal with, since continuous time implies that changes in the configuration can only occur one site at a time, sequentially as opposed to in parallel. A discrete-time analogous of [7] has been considered in [1] for a model on $\mathbb{Z}$, where particles at each time move with probability $1-\varepsilon$ or branch with probability ε, with the rule that two particles occupying the same site will annihilate. The authors show that, if the branching probability is small enough, for any finite initial configuration of particles the probability $p(t)$ that at least one site is occupied at time t decays exponentially fast in t.

Perl, Sen and Yadin [27] consider a branching annihilating random walk on the complete graph which evolves in discrete time, where the number of offspring is Poisson distributed with mean μ and each one of them independently moves to one of the neighbouring sites of their parent. This corresponds to our model on the complete graph. Since on a finite graph there is always a positive probability of total annihilation in one step, the system eventually dies out at some finite time. They show that if $\mu > 1$, then the process on the complete graph with N vertices has an exponentially long lifetime in N and that, conditional on extinction, its last excursion from the "equilibrium value" $\theta_\mu N$ before it reaches the zero state is logarithmic in N.

Besides systems where particles can annihilate, recent research directions have also been focusing on spatial branching systems in which the interaction among particles is regulated by a competition kernel which can reduce the average reproductive success of an individual at a given site. In this case, rather than annihilating particles in areas with high particle density, the existing particles will produce fewer offspring. Spatial models with local competition are for example investigated in [11, 5, 4, 13, 26, 24]. The two papers most related to our present work are [4, 24].

Birkner and Depperschmidt [4] consider a discrete time branching system with a finite range (and thus *local*) competition kernel. They show that the system survives with positive probability if the competition term is small enough and obtain complete convergence of the system to a non-trivial equilibrium for some choices of the model parameters. The strategy used in [4] to prove survival is building a comparison with an oriented percolation model. We will use similar ideas to show survival for our branching annihilating random walk, as well as complete convergence.

In a more recent paper, Maillard and Penington [24] work in continuous time and consider non-local competition kernels, where the range of interaction can be arbitrary, even infinite. Using a contour argument, they prove that in the low competition regime the system survives globally. In the same regime, they also provide a shape theorem, showing that the asymptotic spreading speed of the population is the same as in the branching random walk without competition.

Since we work in discrete time, our model is not an interactive particle system in the sense of [21, 22] but rather a probabilistic cellular automaton. We refer to [25] for a survey on probabilistic cellular automata. Ergodicity and complete convergence for probabilistic cellular automata is a notoriously difficult topic where a lot of the proof techniques are model-dependent. For attractive systems there are still some general tools as monotonicity and subadditivity, see [14]. We refer to [12] for a collection of recent results.

1.4. *Auxiliary coupled map lattice.* Our work also raises questions about coupled map lattices which are deterministic versions of the probabilistic cellular automata, see (1.4), and which, in our examinations of the BARW, serve as an intuitional guide for the proofs of the survival and the complete convergence. This coupled map lattice is a deterministic $[0,e^{-1}]^{\mathbb{Z}^d}$-valued process Ξ_n (note that $\max_{w\geq 0}\varphi_\mu(w)=e^{-1}$) defined, given any initial condition Ξ_0, by the iteration of

$$\Xi_{n+1}(x)=\varphi_\mu(\delta_R(x;\Xi_n)). \tag{1.8}$$

At least for R large, locally, the dynamics of this process is a good approximation for the dynamics of the "density profile" $\delta_R(\cdot;\eta_n)$ of η, as can be heuristically seen from (1.4) and the law of large numbers.

We will prove and exploit the fact that in the regime when φ_μ has the unique attractive fixpoint θ_μ, that is for $\mu\in(1,e^2)$, when starting from a non-zero initial condition, the coupled map lattice converges locally to θ_μ, and the region where it is close to this value expands.

PROPOSITION 1.9. *Let $\mu\in(1,e^2)$ and assume that $\Xi_0(0)>0$. Then*

$$\lim_{n\to\infty}\Xi_n(z)=\theta_\mu \qquad \text{for all } z\in\mathbb{Z}^d,$$

and for every $\varepsilon>0$ there is a speed $a=a(\mu,\varepsilon,\Xi_0(0))>0$ such that $\Xi_n(x)\in(\theta_\mu-\varepsilon,\theta_\mu+\varepsilon)$ for all $|x|\leq an$.

We believe that for localised or half-space initial conditions, the process Ξ will approach a "travelling wave". While there is a rich literature addressing travelling waves, we were not able to find results which literally apply in our context, in particular since our model has discrete time and space. We thus prove the above (weaker and non-optimal) proposition by rather bare hand arguments, which involve a construction of a "travelling wave sub-solution", see Section 2.3 below. Travelling waves in the context of PDEs have been widely studied, also with a view of biological applications. In the context of discrete time, *continuous space* models, the existence of travelling waves has also been considered quite extensively, see e.g. [34, 20, 16, 17]. In particular, in situations where φ_μ in (1.8) is replaced by an increasing (and hence monotone) function, existence of such travelling waves has been shown [14, 34].

The regime $\mu>e^2$ is also very interesting. In this regime the iteration of φ_μ does not converge to a single point but to a stable orbit, which as μ increases beyond e^2 will increase its number of elements. In this case, we are not aware of results in the literature covering the coupled map lattice model. But even given such results, the behaviour of the stochastic system might be different and more difficult to control than in the stable-fixpoint case treated here. We leave these questions for future work.

2. Preliminary results and tools.

In this section we collect some preliminary results that will be used throughout the paper.

2.1. *A general coupling construction.*

We will frequently make use of the following construction allowing to define the process η for all initial conditions simultaneously and also allowing to compare η with other particle systems, in particular with monotone ones.

Let $U(x,n)$, $x\in\mathbb{Z}^d$, $n\in\mathbb{N}_0$, be a collection of i.i.d. uniform random variables on $[0,1]$. Recall the definition of the function φ_μ from (1.3), and let $\psi:[0,1]\to\mathbb{R}_+$ be any non-decreasing function satisfying

$$\psi(w)\leq\varphi_\mu(w) \qquad \text{for all } w\in[0,1]\cap V_R^{-d}\mathbb{Z}, \tag{2.1}$$

that is, for all possible values of the density $\delta_R(\cdot;\eta_n)$. Then, for any initial conditions η_0, $\widetilde{\eta}_0$, define, recursively for $n\geq 0$,

$$\eta_{n+1}(x)=\mathbb{1}_{\{U(x,n+1)\leq\varphi_\mu(\delta_R(x;\eta_n))\}}, \tag{2.2}$$

$$\widetilde{\eta}_{n+1}(x)=\mathbb{1}_{\{U(x,n+1)\leq\psi(\delta_R(x;\widetilde{\eta}_n))\}}. \tag{2.3}$$

The construction (2.2) of η is morally the analogue of the common graphical construction of an interacting particle system in our context, and can be viewed as a stochastic flow on the configuration space $\{0,1\}^{\mathbb{Z}^d}$. The next lemma gives its main properties.

LEMMA 2.1 (General coupling construction). (a) *The process η defined by* (2.2) *has the law of the branching-annihilating random walk with parameters μ and R and initial condition η_0.*

(b) *If $\widetilde{\eta}_0(x) \le \eta_0(x)$ for all $x \in \mathbb{Z}^d$, then $\widetilde{\eta}_n(x) \le \eta_n(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$.*

PROOF. Part (a) follows immediately from (1.4). To see part (b) assume $\widetilde{\eta}_1(x) = 1$ for some $x \in \mathbb{Z}^d$. Then, by construction $U(x,1) \le \psi(\delta_R(x;\widetilde{\eta}_0))$. Since $\widetilde{\eta}_0 \le \eta_0$ and ψ is non-decreasing, and φ_μ dominates ψ, this yields $U(x,1) \le \varphi_\mu(\delta_R(x;\eta_0))$, and so $\eta_1(x) = 1$. It follows that $\widetilde{\eta}_1 \le \eta_1$, and by iteration, $\widetilde{\eta}_n \le \eta_n$. □

In what follows we always assume that η is constructed as in (2.2) and define the filtration

$$\mathcal{F}_n := \sigma\big(\eta_0(x) : x \in \mathbb{Z}^d\big) \vee \sigma\big(U(x,i) : x \in \mathbb{Z}^d, i \le n\big) \supseteq \sigma\big(\eta_i(x) : x \in \mathbb{Z}^d, i \le n\big). \tag{2.4}$$

2.2. *Concentration and comparison with deterministic profiles.* As remarked under (1.3), the random variables $(\eta_{n+1}(x))_{x\in\mathbb{Z}^d}$ are conditionally independent given η_n. Therefore, the density $\delta_R(x;\eta_{n+1})$ should concentrate, at least for R large. We need estimates providing quantitative control of this concentration. These estimates involve certain sequences of functions $\xi_k^\pm$ on $\mathbb{Z}^d$, which we call *comparison density profiles*, that have the property that if at some time t the local density of η_t is controlled by $\xi_k^\pm$, then, at least locally, the density of η_{t+1} is controlled by $\xi_{k+1}^\pm$. In fact, the sequences ξ_k^- and ξ_k^+ that we use later can be regarded as a travelling wave sub- and super-solution, respectively, of the coupled map lattice iteration (1.8).

DEFINITION 2.2. For a given $\varepsilon, \delta > 0$, *comparison density profiles* are deterministic functions $\xi_k^-, \xi_k^+ : \mathbb{Z}^d \to [0,\infty)$, $k = 0,1,\dots,k_0$, satisfying:

(i) For every $k = 0,\dots,k_0$, $\xi_k^-(\cdot) \le \xi_k^+(\cdot)$.

(ii) For every $k = 0,\dots,k_0$, $\mathrm{Supp}(\xi_k^-) := \{x \in \mathbb{Z}^d : \xi_k^-(x) > 0\}$ is finite, and $\xi_k^-(x) \ge \varepsilon$ for every $x \in \mathrm{Supp}(\xi_k^-)$.

(iii) For every $k = 0,\dots,k_0-1$, and $x \in \mathrm{Supp}(\xi_k^-)$ it holds that if $\zeta : B_R(x) \to \mathbb{R}$ satisfies $\zeta(y) \in [\xi_k^-(y), \xi_k^+(y)]$ for all $y \in B_R(x)$, then

$$(1+\delta)\xi_{k+1}^-(x) \le V_R^{-d} \sum_{y \in B_R(x)} \varphi_\mu(\zeta(y)) \le (1-\delta)\xi_{k+1}^+(x). \tag{2.5}$$

Note that ξ_k^-, ξ_k^+ will in general depend on R, μ, ε and δ, but we do not make this explicit in the notation (in fact, δ, ε could also depend on R and μ).

LEMMA 2.3. (a) *For comparison density profiles $\xi_k^\pm$, if for some $x \in \mathbb{Z}^d$ and $k \in \{0,\dots,k_0-1\}$*

$$\delta_R(y;\eta_k) \in \big[\xi_k^-(y), \xi_k^+(y)\big] \quad \textit{for all } y \in B_R(x), \tag{2.6}$$

then

$$\mathbb{P}\Big(\xi_{k+1}^-(x) \le \delta_R(x;\eta_{k+1}) \le \xi_{k+1}^+(x) \,\Big|\, \mathcal{F}_k\Big) \ge 1 - 2\exp(-cV_R^d), \tag{2.7}$$

where $c = (\delta\varepsilon)/\big(1/(2\delta\varepsilon) + 2/3\big)$.

(b) *If, in* (2.5)*, φ_μ is replaced by any ψ satisfying* (2.1)*, then statement (a) holds for the monotone dynamics $\widetilde{\eta}$ defined in* (2.3) *in place of η.*

REMARK 2.4. If only a lower bound is required, as e.g. in the proof of survival, one can use the "trivial" upper bound for ξ_n^+, namely $\xi_n^+(\cdot) \equiv \max(\varphi_\mu)/(1-\delta) = e^{-1}/(1-\delta)$, and then apply (2.7) only for the lower bound.

PROOF OF LEMMA 2.3. We only show (a), the proof of (b) is completely analogous. We consider first the lower bound, that is we want to show that the conditional probability of the event $\{\delta_R(x;\eta_{k+1}) < \xi_{k+1}^-(x)\}$ is small. Note that, by (2.6) and (2.5),

$$\sum_{y\in B_R(x)} \mathbb{E}[\eta_{k+1}(y) \mid \mathcal{F}_k] = \sum_{y\in B_R(x)} \varphi_\mu\big(\delta_R(y;\eta_k)\big) \geq (1+\delta)V_R^d \xi_{k+1}^-(x).$$

Therefore,

$$\begin{aligned}&\mathbb{P}\Big(\delta_R(x;\eta_{k+1}) < \xi_{k+1}^-(x)\,\Big|\,\mathcal{F}_k\Big)\\ &\quad\leq \mathbb{P}\Big(\textstyle\sum_{y\in B_R(x)}\big(\eta_{k+1}(y) - \mathbb{E}[\eta_{k+1}(y)\,|\,\mathcal{F}_k]\big) < -\delta V_R^d \xi_{k+1}^-(x)\,\Big|\,\mathcal{F}_k\Big)\end{aligned} \tag{2.8}$$

and

$$\operatorname{Var}\big(\delta_R(x;\eta_{k+1})\,\big|\,\mathcal{F}_k\big) = V_R^{-2d}\sum_{y\in B_R(x)} \varphi_\mu\big(\delta_R(y;\eta_k)\big)\big(1-\varphi_\mu\big(\delta_R(y;\eta_k)\big)\big) \leq \frac{1}{4}V_R^{-d}.$$

We now apply the Bernstein inequality (which we recall in Lemma A.1 in the Appendix) to the right-hand side of (2.8) with $n = V_R^d$, $\sigma_n \leq V_R^{-d/2}/2$, $m_n \leq 1$ and $w = \delta V_R^d \xi_{k+1}^-(x) \geq \delta\varepsilon V_R^d$ (since, by assumption (ii) $\xi_{k+1}^-(x) \geq \varepsilon$ if $\xi_{k+1}^-(x) > 0$, and there is nothing to prove if $\xi_{k+1}^-(x) = 0$). The expression in the exponent of the right-hand side of (A.1) then satisfies

$$\frac{w^2}{2\sigma_n^2 + (2/3)m_n w} = \frac{w}{2\sigma_n^2/w + (2/3)m_n} \geq \frac{w}{V_R^d/(2w) + 2/3} \geq \frac{\delta\varepsilon}{1/(2\delta\varepsilon)+2/3}V_R^d,$$

which completes the proof of the lower bound in (2.7).

The proof of the upper bound, that is showing that the probability (conditional on η_k) of the event $\{\delta_R(x;\eta_{k+1}) > \xi_{k+1}^+(x)\}$ is small, is completely analogous. □

2.3. *Lower bounds on travelling waves.* The goal of this section is to construct explicit comparison density profiles ξ_k^- which can later be used as the lower bounds on $\delta_R(\cdot;\eta)$ in the proofs of survival and complete convergence. As pointed out before, these can be viewed as travelling wave sub-solutions to the iteration (1.8).

We start by providing the basic building block for this construction. To this end we concentrate first on the one-dimensional setting. For parameters $a > 1$, $\varepsilon_0 \in (0,1)$, $w > 0, s > 0$ and $R \in \mathbb{N}$ we say that a non-decreasing function $f : \mathbb{Z} \to [0,\infty)$ is a linear travelling wave shape with width $\lceil wR\rceil$, shift $\lceil sR\rceil$, growth factor a and minimal step size ε_0 if it fulfils

$$f(x) = 0 \text{ for } x < 0, \qquad f(0) = \varepsilon_0, \qquad f(x) = 1 \text{ for } x \geq \lceil wR\rceil \tag{2.9}$$

and

$$a\delta_R(x;f) \geq f\big(x + \lceil sR\rceil\big) \quad \text{for all } x \in \mathbb{Z}. \tag{2.10}$$

In this parametrisation, we think of a "wave profile" which, when subjected to one iteration of the operation $f(\cdot) \mapsto a\delta(\cdot;f)$, moves to the left by at least $\lceil sR\rceil$ in each time step. Note that by construction, one necessarily has that $s \leq 1$.

We now show that such a function f exists for any $a > 1$ and R large.

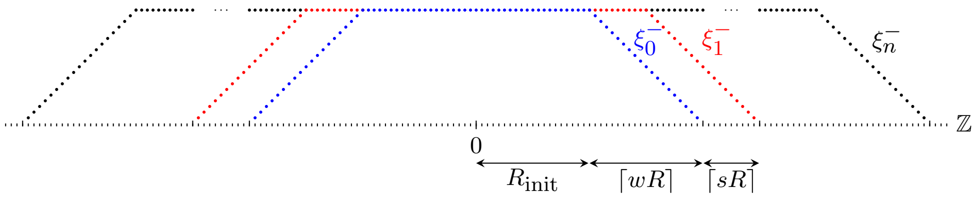

FIG 2. *The one-dimensional deterministic comparison density profile ξ_n^- built from the linear travelling wave shape f, with fronts of width $\lceil wR\rceil$ that get shifted outward by $\lceil sR\rceil$ in every time step.*

LEMMA 2.5. *For every $a>1$, there is $w\ge 2$, $\varepsilon_0\in(0,1)$, $s\in(0,1)$, and $R_0\in\mathbb{N}$ such that the function*

$$f(x)=\min\left\{(\varepsilon_0+x/\lceil wR\rceil)\mathbb{1}_{x\ge 0},1\right\}$$

satisfies (2.9), (2.10) *for all $R\ge R_0$.*

The proof of Lemma 2.5 is a straightforward, albeit somewhat lengthy computation, and is given in Section 6.1. In fact, with even lengthier computations it could be shown that the lemma holds for any $R\ge 1$.

Using this travelling wave shape we can now define the desired comparison density profile ξ_n^-. For this, fix $R_{\text{init}}\in\mathbb{N}$ with $R_{\text{init}}>2R$ and set, for $x\in\mathbb{Z}$,

$$\widetilde{\xi}_n(x)=f\big(R_{\text{init}}+n\lceil sR\rceil+\lceil wR\rceil-|x|\big) \tag{2.11}$$

with f from Lemma 2.5, see Figure 2 for an illustration. Note that by construction, $\widetilde{\xi}_n(\cdot)\equiv 1$ on $B_{R_{\text{init}}+n\lceil sR\rceil}(0)$ and $\operatorname{Supp}(\widetilde{\xi}_n)=B_{R_{\text{init}}+n\lceil sR\rceil+\lceil wR\rceil}(0)$. Furthermore, using (2.10), $a\delta_R(x;\widetilde{\xi}_n)\ge\widetilde{\xi}_{n+1}(x)$ for all $x\in\mathbb{Z}$, and $\widetilde{\xi}_n(x)>0$ implies $\widetilde{\xi}_n(x)\ge\varepsilon_0$.

Finally, for any $d\ge 1$, write $x=(x_1,\dots,x_d)$ and set

$$\xi_n^-(x):=b\prod_{i=1}^d\widetilde{\xi}_n(x_i),\quad x\in\mathbb{Z}^d,n\in\mathbb{N}_0 \tag{2.12}$$

with some $0<b\le 1$ that will be suitably tuned later. Note that ξ_n^- implicitly depends on d, R, R_{init}, a and b but our notation does not make this explicit. We summarise the relevant properties of ξ_n^- in the following lemma.

LEMMA 2.6. *The functions ξ_n^- have the following properties:*

(i) $0\le\xi_n^-(x)\le b$ *for every $n\in\mathbb{N}_0$ and $x\in\mathbb{Z}^d$,*
(ii) *for every $n\in\mathbb{N}_0$, $\xi_n^-(\cdot)\equiv b$ on $B_{R_{\text{init}}+n\lceil sR\rceil}(0)$ and $\operatorname{Supp}(\xi_n^-)=B_{R_{\text{init}}+n\lceil sR\rceil+\lceil wR\rceil}(0)$,*
(iii) $a^d\delta_R(x;\xi_n^-)\ge\xi_{n+1}^-(x)$ *for all $n\in\mathbb{N}_0$, $x\in\mathbb{Z}^d$,*
(iv) $\xi_n^-(x)>0$ *implies $\xi_n^-(x)\ge b\varepsilon_0^d$.*

PROOF. The properties (i), (ii) and (iv) follow directly from (2.9), (2.11) and (2.12). Using (2.10), (2.11), (2.12), it follows moreover that

$$\begin{aligned}a^d\delta_R(x;\xi_n^-)&=a^dV_R^{-d}\sum_{y\in B_R(0)}\xi_n^-(y+x)\\&=ba^dV_R^{-d}\sum_{y_1=-R}^R\cdots\sum_{y_d=-R}^R\prod_{i=1}^d\widetilde{\xi}_n(x_i+y_i)\end{aligned}$$

$$= b\prod_{i=1}^d \left(aV_R^{-1} \sum_{y=-R}^{R} \widetilde{\xi}_n(x_i+y)\right) = b\prod_{i=1}^d \left(a\delta_R(x_i;\widetilde{\xi}_n)\right)$$

$$\geq b\prod_{i=1}^d \widetilde{\xi}_{n+1}(x_i) = \xi^-_{n+1}(x),$$

which shows (iii) and completes the proof. □

3. Survival for large R: Proof of Theorem 1.3. In this section we prove Theorem 1.3, stating that the system survives for any $\mu > 1$, given that R is chosen sufficiently large. The proof is based on the comparison with a monotone system $\widetilde{\eta}$, which in turn is shown to survive using a comparison with finite range oriented percolation. The latter is a by now classical technique for interacting particle systems, we refer to [8], [18] or [32] for recent and reader-friendly introductions.

The monotone system $\widetilde{\eta}$ is constructed as in Section 2.1: we first fix parameters $\widetilde{a} \in (1,\mu)$ and $b \in (0,1)$, so that the function ψ defined by

$$\psi(w) := \widetilde{a}(w \wedge b) \tag{3.1}$$

satisfies (2.1). This is possible since $\mu > 1$. With this ψ, we define $\widetilde{\eta}$ as in (2.3) and simultaneously η as in (2.2) on the probability space supporting the i.i.d. uniform random variables $(U(x,n))_{x\in\mathbb{Z}^d, n\in\mathbb{N}_0}$.

We then fix $a > 1$ such that $a^d < \widetilde{a}$, and for this choice of a, we fix R_0, w, s and ε_0 according to Lemma 2.5. For $R \geq R_0$, we set $R_{\text{init}} := \lceil wR/2 \rceil$ and define ξ_n^- as in (2.12). We claim that $\xi_n^-(x)$ (and the trivial ξ_n^+, as explained in Remark 2.4) is a comparison density profile in the sense of Definition 2.2 with $\delta = (\widetilde{a}/a^d) - 1$ and $\varepsilon = b\varepsilon_0^d$. Moreover the lower bound of (2.5) even holds with ψ in place of φ_μ. Indeed, (i) is trivially true, (ii) follows from Lemma 2.6(iv). To show (iii), that is (2.5) (with ψ in place of φ_μ), let $\zeta = (\zeta(y)) \in [0,1]^{\mathbb{Z}^d}$ be such that $\zeta(\cdot) \geq \xi_n^-(\cdot)$ for some $n \in \mathbb{N}_0$. Then, using Lemma 2.6(iii) for the inequality,

$$\begin{aligned} V_R^{-d} \sum_{y\in B_R(x)} \psi\big(\zeta(y)\big) &= V_R^{-d} \sum_{y\in B_R(x)} \widetilde{a}\big(\zeta(y)\wedge b\big) \\ &\geq \frac{\widetilde{a}}{a^d}\cdot a^d V_R^{-d} \sum_{y\in B_R(x)} \xi_n^-(y) \\ &= \frac{\widetilde{a}}{a^d}\cdot a^d \delta_R(x;\xi_n^-) \geq \frac{\widetilde{a}}{a^d}\cdot \xi^-_{n+1}(x), \end{aligned}$$

as required. As a consequence, we will later be able to apply the concentration result of Lemma 2.3(b) to the process $\tilde{\eta}$.

Define $R'_{\text{block}} = \lceil wR/2 \rceil$. To set up the comparison with oriented percolation, we coarse-grain the system by using blocks spaced by $L'_{\text{block}} := 2R'_{\text{block}}$, of side length $L_{\text{block}} := 5L'_{\text{block}}$ and temporal size $T_{\text{block}} := \big\lceil \lceil wR \rceil / \lceil sR \rceil \big\rceil$. Since we often refer to radii rather than block lengths, it is convenient to define $R_{\text{block}} = L_{\text{block}}/2$.

For (z,t) in the sub-lattice $\mathbb{L} := L'_{\text{block}}\mathbb{Z}^d \times T_{\text{block}}\mathbb{N}_0$, we define

$$\texttt{Block}(z,t) = \big\{(x,n) \in \mathbb{Z}^d \times \mathbb{N}_0 : \|x-z\| \leq R_{\text{block}},\, t \leq n \leq t + T_{\text{block}}\big\}.$$

Note that blocks in the same time-layer have non-trivial overlap with their neighbours but the number of overlapping neighbours in $\mathbb{L}$ per block does not grow with R. In the time direction, only the top time slice of a given block coincides with the bottom layer of the next block(s).

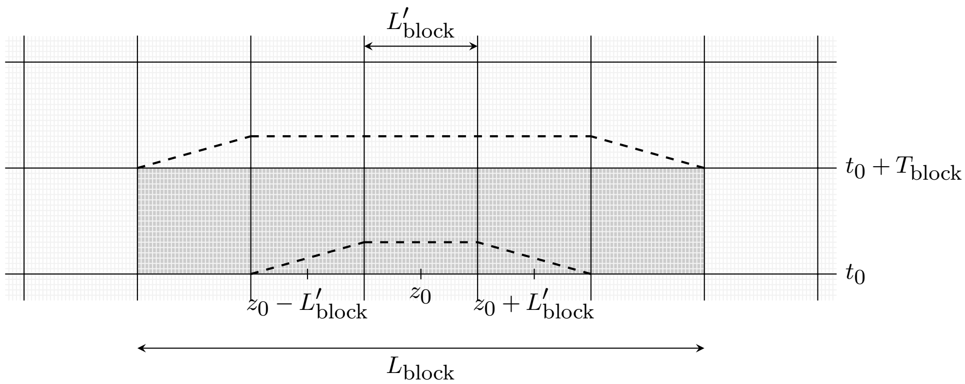

FIG 3. *Sketch of a* $\texttt{Block}(z_0,t_0)$ *(dark grey), centred at the coarse-grained space-time lattice point* (z_0,t_0)*. The thick dashed lines depict the deterministic comparison density profiles* $\xi^-_{t_0}(\cdot)$ *and* $\xi^-_{t_0+T_{\text{block}}}(\cdot)$ *which have to be dominated by the density* $\delta_R(\cdot;\widetilde{\eta}_n)$*,* $t_0 \le n \le t_0 + T_{\text{block}}$ *in order for the block to be* good*. Note the picture is not drawn to scale:* L_{block} *and* L'_{block} *are both growing linearly in* R *while* T_{block} *does not grow with* R*.*

DEFINITION 3.1. We call $\texttt{Block}(z,t)$ *well-started* if the density of the monotone system $\widetilde{\eta}$ dominates the (suitably shifted) density profile ξ^-_0 at the bottom of the block, that is

$$\delta_R\big(x;\widetilde{\eta}_t\big) \ge \xi^-_0(x-z) \quad \text{for } \|x-z\| \le R_{\text{block}}. \tag{3.2}$$

Note that for any $(z,t) \in \mathbb{L}$ the event $\{\texttt{Block}(z,t) \text{ is well-started}\}$ is measurable with respect to the filtration $\mathcal{F}_t$, which was defined in (2.4).

DEFINITION 3.2. $\texttt{Block}(z,t)$ is called *good* if it is well-started and the random variables $U(x,n)$ are such that the domination property of (3.2) propagates over the block. That is, $\texttt{Block}(z,t)$ is *good* if it holds that

$$\delta_R\big(x;\widetilde{\eta}_{t+n}\big) \ge \xi^-_n(x-z) \quad \text{for } \|x-z\| \le R_{\text{block}},\ n = 0,\dots,T_{\text{block}}.$$

The properties of the comparison density profiles ξ^-_n, see Lemma 2.6, enforce

$$\{\texttt{Block}(z,t) \text{ is good}\} \subseteq \bigcap_{\substack{z' \in L'_{\text{block}}\mathbb{Z}^d:\\ \|z-z'\| \le L'_{\text{block}}}} \{\texttt{Block}(z',t+T_{\text{block}}) \text{ is well-started}\}. \tag{3.3}$$

In particular, the process $\widetilde{\eta}$ survives up to time $t+T_{\text{block}}$ in a good $\texttt{Block}(z,t)$ and the region of the desired density control by the profiles ξ^-_n expands, see Figure 3.

By the construction (2.3) of $\widetilde{\eta}$, given $\mathcal{F}_t$, if $\texttt{Block}(z,t)$ is well-started, it can be decided whether or not the event $\{\texttt{Block}(z,t) \text{ is good}\}$ occurs for $(z,t) \in \mathbb{L}$ by inspecting (only) the values of

$$\big(U(x,n) \in \mathbb{Z}^d \times \mathbb{N}_0 : \|x-z\| \le R_{\text{block}} + T_{\text{block}}R,\ t < n \le t + T_{\text{block}}\big) \tag{3.4}$$

(in fact, strictly speaking it suffices to observe the values of U's at the space-time points $\{(x,n) : \|x-z\| \le R_{\text{block}} + (t+T_{\text{block}}-n)R,\ t < n \le t + T_{\text{block}}\}$). Note that for $(z,t) \in \mathbb{L}$ and $(z',t) \in \mathbb{L}$ with

$$\|z'-z\| > L_{\text{block}} + 2T_{\text{block}}R \approx (5+2/s)L'_{\text{block}} \quad \text{(when } R \text{ is large)} \tag{3.5}$$

the space-time regions corresponding to (3.4) will be disjoint.

Furthermore, by invoking Lemma 2.3(b) we can uniformly bound the probability of the density of $\widetilde{\eta}$ dominating the comparison density profile ξ^- for all space-time sites in $\texttt{Block}(z,t)$, which in turn yields

(3.6) $$\mathbb{P}\big(\texttt{Block}(z,t) \text{ is good} \,\big|\, \mathcal{F}_t\big) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\big(1-q(T_{\text{block}},R)\big)$$

with

(3.7) $$q(T_{\text{block}},R) = 2\big|\texttt{Block}(z,t)\big|e^{-cV_R^d},$$

which tends to 0 as $R\to\infty$, since $|\texttt{Block}(z,t)|$ grows only polynomially in R.

In order to make the comparison with oriented percolation, we define random variables

(3.8) $$Y(z,t) := \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is good}\}}, \quad (z,t)\in\mathbb{L},$$

and say that $(z,t)\in\mathbb{L}$ is *connected to infinity* in Y if there is a path $((z_i,t+iT_{\text{block}}) : i\in\mathbb{N}_0)$ in $\mathbb{L}$ with $z_0=z$ and $\|z_i-z_{i-1}\|\leq L'_{\text{block}}$ for all $i\in\mathbb{N}$, such that $Y(z_i,t+iT_{\text{block}})=1$ for all $i\in\mathbb{N}_0$ (such a path is called *open* in Y). By the argument above, it follows that if (z,t) is well-started and connected to infinity in Y, then the process $\widetilde{\eta}$ survives.

In order to show that the latter event occurs with positive probability, we iteratively construct a coupling between the $Y(z,t)$'s from (3.8) and a family $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ of i.i.d. Bernoulli random variables with parameter $p(R)$ which satisfies $p(R)\to 1$ as $R\to\infty$ such that we have

(3.9) $$Y(z,t) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\widetilde{Y}(z,t) \quad \text{for all } (z,t)\in\mathbb{L}.$$

We construct $\widetilde{Y}(\cdot,t)$ inductively over t and begin with a slightly informal description of this construction: Assume that for some $t'\in T_{\text{block}}\mathbb{N}$, a coupling satisfying (3.9) has been achieved for all $(z,t)\in\mathbb{L}$ with $T_{\text{block}}\mathbb{N}\ni t<t'$. We then work conditionally on $\mathcal{F}_{t'}$. The (random) set of nodes

$$W(t') := \{z'\in L'_{\text{block}}\mathbb{Z}^d : \texttt{Block}(z',t') \text{ is well-started}\},$$

viewed as a graph where z' and z'' are connected by an edge if the space-time regions from (3.4) centred at (z',t') and at (z'',t'), respectively, overlap, is a locally finite graph with uniformly bounded degrees. In fact, we see from (3.4) that we have irrespective of the realisation of $\widetilde{\eta}_{t'}$ the deterministic bound $(11+4/s)^d$ on the degree of any node (up to rounding, see (3.5)). Thus, by (3.4)–(3.7), using well known stochastic domination arguments for percolation models with finite-range dependencies [23], it follows that the family $(Y(z,t'))_{z\in W(t')}$ stochastically dominates a family $(\widetilde{Y}(z,t'))_{z\in W(t')}$ of i.i.d. Bernoulli random variables with parameter $p(R)$, where $p(R)\to 1$ as $R\to\infty$ and the $(\widetilde{Y}(z,t'))_{z\in W(t')}$ are independent of $\mathcal{F}_{t'}$ given $W(t')$, i.e. (3.9) holds for all $z\in W(t')$. In fact, $p(R)$ is a function of the maximal degree $(11+4/s)^d$ of the dependence graph and the minimal guaranteed density $1-q(T_{\text{block}},R)$ of good blocks, see Theorem 1.3 in [23]. For $z\notin W(t')$, (3.9) imposes no condition at all on $\widetilde{Y}(z,t')$. Thus, we can simply define $\widetilde{Y}(z,t')=\widehat{Y}(z,t')$ for $z\notin W(t')$ where $(\widehat{Y}(z,t))_{(z,t)\in\mathbb{L}}$ is an independent family of i.i.d. Bernoulli($p(R)$) random variables.

In order to formalise this construction and, in particular, to show that the random variables $\widetilde{Y}(z,t)$ are independent over different time layers, note that by the construction (2.2) from Lemma 2.1, we can write

$$Y(\cdot,t') = g\big(\eta_{t'},(U(\cdot,n) : t'<n\leq t'+T_{\text{block}})\big)$$

for some deterministic function $g:\{0,1\}^{\mathbb{Z}^d}\times[0,1]^{\mathbb{Z}^d\times\{1,\dots,T_{\text{block}}\}}\to\{0,1\}^{L'_{\text{block}}\mathbb{Z}^d}$, furthermore $W(t')=W(\eta_{t'})=\{z\in L'_{\text{block}}\mathbb{Z}^d : \texttt{Block}(z,t') \text{ is well started}\}$. For every $\zeta=$

$(\zeta(z))_{z\in\mathbb{Z}^d} \in \{0,1\}^{\mathbb{Z}^d}$, [23, Thm. 1.3] and the discussion above provides a coupling ν_ζ of $\mathcal{L}(Y(\cdot,t') \,|\, \eta_{t'} = \zeta)$ and $\mathrm{Ber}(p(R))^{\otimes \mathbb{Z}^d}$ with the desired properties. We can then disintegrate this joint law with respect to its first marginal and describe the joint law ν_ζ in a two-step procedure. It is convenient to describe this via an auxiliary function $h(\zeta;\cdot,\cdot)$ using additional independent randomness and obtain that given $\eta_{t'} = \zeta$,

$$Y(\cdot,t') = g\big(\zeta, (U(\cdot,n) : t' < n \le t' + T_{\text{block}})\big), \quad \widetilde{Y}(\cdot,t') = h\big(\zeta; Y(\cdot,t'), \widetilde{U}_{t'}\big)$$

where $\widetilde{U}_{t'}$ is independent of everything else and uniformly distributed on $[0,1]$ (see, for example, Theorem 5.10 in [15]). By construction, since $U(\cdot,n), n > t'$ and $\widetilde{U}_{t'}$ are independent of $\mathcal{F}_{t'}$, we have for $A \in \mathcal{F}_{t'}$ and measurable $B \subseteq \{0,1\}^{L'_{\text{block}}\mathbb{Z}^d}$

$$\begin{aligned}\mathbb{P}\big(A \cap \{\widetilde{Y}(\cdot,t') \in B\}\big) &= \mathbb{E}\Big[\mathbb{1}_A \mathbb{P}\big(h(\eta_{t'}; Y(\cdot,t'), \widetilde{U}_{t'}) \in B \,|\, \mathcal{F}_{t'}\big)\Big] \\ &= \mathbb{P}(A)\mathrm{Ber}(p(R))^{\otimes L'_{\text{block}}\mathbb{Z}^d}(B).\end{aligned}$$

This shows the required independence of $\widetilde{Y}$ and completes the induction step.

We see from (3.8), (3.9) and (3.3) that every open path in $\widetilde{Y}(\cdot,\cdot)$ is automatically also an open path in $Y(\cdot,\cdot)$. Furthermore, by well known properties of oriented site percolation, we have

$$\mathbb{P}\big((z,t) \text{ is connected to infinity in } \widetilde{Y}\big) = \mathbb{P}\big((0,0) \text{ is connected to infinity in } \widetilde{Y}\big) > 0$$

if $p(R)$ is sufficiently close to 1, i.e. for all R large enough.

To conclude, let η_0 be any initial configuration containing at least one particle, and let $\widetilde{\eta}_0 = \eta_0$. It is then easy to see (as this involves requiring only finitely many random variables $U(x,n)$ to be sufficiently small), that one can find $(z,t) \in \mathbb{L}$, so that the probability that $\texttt{Block}(z,t)$ is well-started is positive.

Therefore, due to the above properties,

$$\begin{aligned}\mathbb{P}(\eta \text{ survives}) &\ge \mathbb{P}(\widetilde{\eta} \text{ survives}) \\ &\ge \mathbb{E}\big[\mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}} \mathbb{1}_{\{(z,t) \text{ is connected to infinity in } Y\}}\big] \\ &\ge \mathbb{P}\big(\texttt{Block}(z,t) \text{ is well-started}\big)\mathbb{P}\big((z,t) \text{ is connected to infinity in } \widetilde{Y}\big) \\ &> 0 \qquad \text{for all } R \text{ large enough,}\end{aligned}$$

which completes the proof of Theorem 1.3.

4. Complete convergence. In this section we show our main results in the regime where the particle system survives with a positive probability and is well approximated by the deterministic coupled map lattice introduced in Section 1.4. In particular, we assume that $\mu \in (1, e^2)$ and R is large enough. In Section 4.1, we start with Theorem 1.6 providing the coupling of processes started with different initial conditions. Theorem 1.5 is then shown in Section 4.3.

4.1. *Coupling construction: Proof of Theorem 1.6.* As in Section 3, the central ingredient will be a block construction and then a suitable comparison with oriented percolation. The definition of "good blocks" will be more involved than in Section 3 and is inspired by the construction in [4, Section 5].

In brief, the construction of a good block around z is as follows. We consider a (large) ball B around z and assume that $\eta^{(1)}$ and $\eta^{(2)}$ agree on B and the respective R-densities of the two processes are close to θ_μ. On an even larger ball B' we add milder and milder requirements (as the distance from the centre increases) on the densities of the processes. The contraction property of φ_μ, see Lemma 4.1 below, together with the concentration property of the densities of $\eta^{(i)}$ guaranteed by Lemma 2.3 then ensure that the area in which the $\eta^{(1)}$ and $\eta^{(2)}$ are coupled expands in time with high probability. In order to guarantee survival of the processes we also require that the respective densities of $\eta^{(1)}, \eta^{(2)}$ dominate the deterministic comparison density profile as defined in (2.12) (the latter was also used in Section 3).

We now proceed with the formal definitions. Throughout this section, we again use the coupling construction from Section 2.1: Given two initial conditions $\eta_0^{(1)}$ and $\eta_0^{(2)}$, we construct both $(\eta_n^{(1)})_n$ and $(\eta_n^{(2)})_n$ using (2.2) with the same $U(x,n)$'s, that is, we set

$$\eta_{n+1}^{(i)}(x) = \mathbb{1}_{\{U(x,n+1)\le \varphi_\mu(\delta_R(x;\eta_n^{(i)}))\}}, \quad i\in\{1,2\}, \quad (x,n)\in\mathbb{Z}^d\times\mathbb{N}_0. \tag{4.1}$$

Since we are from now on interested in two copies of the branching annihilating process, we redefine the filtration $(\mathcal{F}_n)$ from (2.4) by including both initial conditions, i.e.

$$\mathcal{F}_n := \sigma\big(\eta_0^{(i)}(x) : x\in\mathbb{Z}^d, i=1,2\big) \vee \sigma\big(U(x,j) : x\in\mathbb{Z}^d, j\le n\big).$$

It is clear that this updated filtration is finer than the natural filtration of the two processes, in the sense that for all $n\ge 0$, it holds that $\mathcal{F}_n \supseteq \sigma\big(\eta_j^{(i)}(x) : x\in\mathbb{Z}^d, j\le n, i=1,2\big)$.

In order to define the comparison density profiles that are used to determine whether a block is good, we need a simple lemma which gives some useful properties of the function φ_μ in the vicinity of its non-trivial fixpoint θ_μ. The result is fairly standard, we provide a proof for completeness' sake in Section 6.2 (cf. also [4, Proof of Lemma 12]).

LEMMA 4.1. *For every $\mu\in(1,e^2)$ there is $\varepsilon>0$ and $\kappa(\mu,\varepsilon)<1$ such that φ_μ is a contraction on $[\theta_\mu-\varepsilon,\theta_\mu+\varepsilon]$, that is,*

$$|\varphi_\mu(w_1)-\varphi_\mu(w_2)| \le \kappa(\mu,\varepsilon)|w_1-w_2| \qquad \text{for } w_1,w_2\in[\theta_\mu-\varepsilon,\theta_\mu+\varepsilon].$$

Moreover, there exist a strictly increasing sequence $\alpha_m\uparrow\theta_\mu$ and a strictly decreasing sequence $\beta_m\downarrow\theta_\mu$ such that $\varphi_\mu([\alpha_m,\beta_m])\subseteq(\alpha_{m+1},\beta_{m+1})$ for all $m\in\mathbb{N}$. Furthermore, it is possible to choose $\alpha_1>0$ arbitrarily small and $\beta_1>1/e$.

We now take b as in (3.1) and fix ε, $\kappa(\mu,\varepsilon)<1$, as well as sequences $\alpha_m\uparrow\theta_\mu$, $\beta_m\downarrow\theta_\mu$ as in Lemma 4.1, with $\alpha_1=b$ and $\beta_1>1/e$. Then we choose m_0 such that $\beta_m-\alpha_m<\varepsilon$ for every $m\ge m_0$. These choices will remain fixed throughout the remainder of this section.

Next define the size of the blocks

$$L'_{\text{block}} = 2\lceil R\log R\rceil, \qquad L_{\text{block}} = c_{\text{space}}L'_{\text{block}} \quad \text{and} \quad T_{\text{block}} = c_{\text{time}}\lceil\log R\rceil, \tag{4.2}$$

where $c_{\text{time}} > -(d+1)/\log\kappa(\mu,\varepsilon)$ and $c_{\text{space}} = 4(1+c_{\text{time}})$ are integer constants. Remark 4.5 below explains these choices. As in Section 3, we introduce $R'_{\text{block}} = L'_{\text{block}}/2$ and $R_{\text{block}} = L_{\text{block}}/2$ for the radii of the blocks, and, for (z,t) in the sub-lattice $\mathbb{L} := L'_{\text{block}}\mathbb{Z}^d\times T_{\text{block}}\mathbb{N}_0$, we define

$$\texttt{Block}(z,t) = \big\{(x,n)\in\mathbb{Z}^d\times\mathbb{N}_0 : \|x-z\|\le R_{\text{block}},\ t\le n\le t+T_{\text{block}}\big\}.$$

Further, let us specify the radius for which the strongest form of density control, alluded to in the above informal description, holds. More precisely set $c_{\text{dens}} = 1+2c_{\text{time}}$ and $R_{\text{dens}} := 2c_{\text{dens}}R'_{\text{block}}$. Again, the discussion on the choice of c_{dens} is postponed to Remark 4.5.

Recall the functions $\xi_n^-(x)$ defined in (2.12). We use them here with $R_{\text{init}} = R_{\text{dens}}+m_0R$ in (2.11). For $k\in\{0,\dots,T_{\text{block}}\}$ set $R_{\text{dens}}(k) = R_{\text{dens}}+k\lceil sR\rceil$, then let

$$\zeta_k^-(x) := \begin{cases} \alpha_{m_0} & \text{if } \|x\|\le R_{\text{dens}}(k) \\ \alpha_{m_0-j+1} & \text{if } R_{\text{dens}}(k)+(j-1)R<\|x\|\le R_{\text{dens}}(k)+jR,\ 1\le j\le m_0 \\ \xi_k^-(x) & \text{if } \|x\|>R_{\text{dens}}(k)+m_0R, \end{cases}$$

and

$$\zeta_k^+(x) := \begin{cases} \beta_{m_0} & \text{if } \|x\| \le R_{\text{dens}}(k) \\ \beta_{m_0-j+1} & \text{if } R_{\text{dens}}(k) + (j-1)R < \|x\| \le R_{\text{dens}}(k) + jR,\ 1 \le j \le m_0 \\ 1 \vee \beta_1 & \text{if } \|x\| > R_{\text{dens}}(k) + m_0 R. \end{cases}$$

See also Figure 4.

The functions $\zeta_k^-(\cdot) < \zeta_k^+(\cdot)$ are comparison density profiles in the sense of Definition 2.2, in particular, they satisfy the following analogue of (2.5).

LEMMA 4.2. *There exists $\delta > 0$ with the following property: For $k \in \mathbb{N}_0$ and any $(\zeta(x))_{x \in \mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d}$ satisfying $\zeta_k^- \le \zeta \le \zeta_k^+$ on $\text{Supp}(\zeta_k^-)$, it follows that*

$$(1+\delta)\zeta_{k+1}^-(x) \le V_R^{-d} \sum_{y \in B_R(x)} \varphi_\mu\big(\zeta(y)\big) \le (1-\delta)\zeta_{k+1}^+(x) \quad \textit{for all } x \in \text{Supp}(\zeta_{k+1}^-).$$

PROOF. For x such that $\zeta_k^-(x)$ agrees with the previously defined profile $\xi_k^-(x)$ the lower bound in the statement follows easily from Lemma 2.6.

Let $x \in \text{Supp}(\zeta_{k+1}^-)$ with $\zeta_{k+1}^-(x) = \alpha_j$, for some $2 \le j \le m_0$. Then

$$\zeta_k^-(y) = \alpha_j \text{ if } y \in B_R(x) \cap \Upsilon_R \qquad \text{and} \qquad \zeta_k^-(y) \ge \alpha_{j-1} \text{ if } y \in B_R(x) \cap \Upsilon_R^c,$$

where $\Upsilon_R := \{z : R_{\text{dens}}(k) + (j-1)R \le \|z\| \le R_{\text{dens}}(k) + jR\}$. Note that $|\Upsilon_R \cap B_R(x)| \ge cV_R^d$ for some $c > 0$, uniformly in the x we consider here. The properties of sequences α_m, β_m from Lemma 4.1 then imply that there exists $\delta > 0$ (depending on $(\alpha_m)_{m \le m_0}$, $(\beta_m)_{m \le m_0}$ and d) such that

$$V_R^{-d} \sum_{y \in B_R(x)} \varphi_\mu(\zeta(y)) \ge \alpha_{j+1} |B_R(x) \cap \Upsilon_R| V_R^{-d} + \alpha_j |B_R(x) \cap \Upsilon_R^c| V_R^{-d} \ge \alpha_j(1+\delta)$$

and similarly for the upper bound. This completes the claim for the remaining parts of the profile (those in orange in Figure 4). □

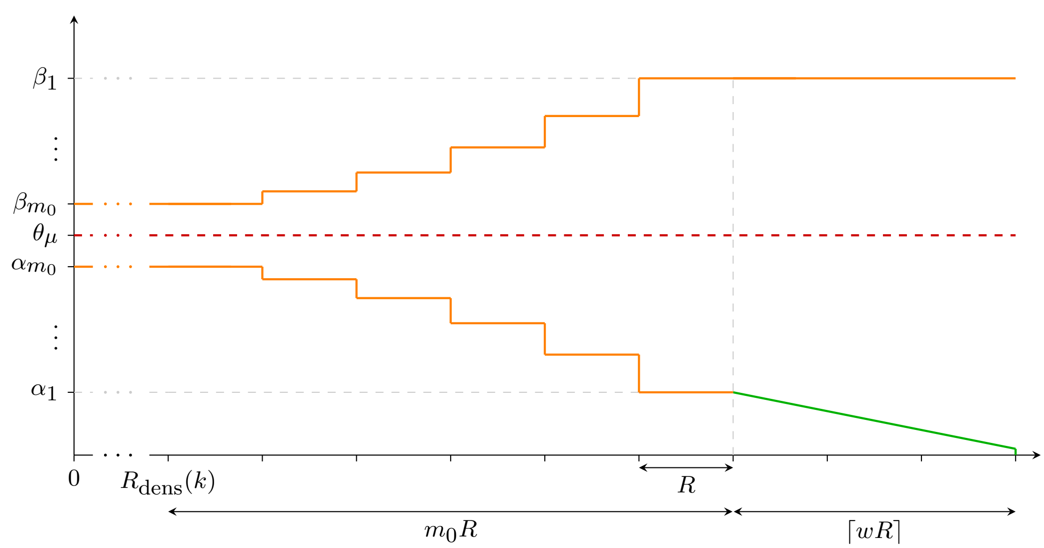

FIG 4. *The part of the deterministic comparison density profiles ζ_k^+ and ζ_k^- (in orange and green) left from $R_{\text{dens}}(k)$. In a* good *block the densities of both $\eta^{(1)}$ and $\eta^{(2)}$ stay between the union of the orange lines and the green profile, which is glued to the bottom orange profile. The green line is the (suitably recentred and shifted) profile of ξ^-, which we introduced to prove survival.*

We proceed in a similar fashion as in Section 3 and introduce a new notion of well-started and of good blocks. These updated definitions involve two copies $\eta^{(1)}$, $\eta^{(2)}$ of the system. A well-started block is now determined by the local density of the true system being controlled by the ζ_k^+, ζ_k^- profiles, in addition to which we require agreement of the true processes in the central part of the block.

DEFINITION 4.3. A $\texttt{Block}(z,t)$ based at $(z,t) \in \mathbb{L}$ is *well-started* if

$$\delta_R(x;\eta_t^{(i)}) \in \left[\zeta_0^-(x-z), \zeta_0^+(x-z)\right] \quad \text{for all } x \in z + \mathrm{Supp}(\zeta_0^-),\ i = 1,2 \tag{4.3}$$

and

$$\eta_t^{(1)}(x) = \eta_t^{(2)}(x) \quad \text{for all } x \in B_{R'_{\mathrm{block}}}(z). \tag{4.4}$$

Again as in Section 3 we use this as the starting point off of which we base our notion of goodness as the spreading of the control given by well-startedness to neighbouring regions.

DEFINITION 4.4. We call a $\texttt{Block}(z,t)$ based at $(z,t) \in \mathbb{L}$ *good* if

(i) $\texttt{Block}(z,t)$ is *well-started*,
(ii) $\eta^{(1)}_{t+T_{\mathrm{block}}}(x) = \eta^{(2)}_{t+T_{\mathrm{block}}}(x)$ for $\|x - z\| \le 3R'_{\mathrm{block}}$,
(iii) $\left(\eta^{(1)}_{t+T_{\mathrm{block}}}, \eta^{(2)}_{t+T_{\mathrm{block}}}\right)$ satisfy (4.3) around $z + L'_{\mathrm{block}}e$ for all $e \in B_1(0)$.

Property (iii) implies that if $\texttt{Block}(z,t)$ is good, then $\texttt{Block}(z + L'_{\mathrm{block}}e, t + T_{\mathrm{block}})$ will be well-started for all $\|e\| \le 1$.

REMARK 4.5. Let us now comment on our choice of the constants $c_{\mathrm{space}}, c_{\mathrm{dens}}, c_{\mathrm{time}}$. It is instructive to first give c_{space} as a function of c_{dens}, then c_{dens} as a function of c_{time}, and ultimately fixing c_{time} large enough.

(i) Note first that $\zeta_0^{\pm}$ are constant on a box of size R_{dens} (which is of order $R\log R$) and then increase (resp. decrease) on boxes with length of order R. It follows readily that $\mathrm{Supp}(\zeta_0^-) \subseteq B_{2R_{\mathrm{dens}}}(0)$ for large enough R. Therefore $2c_{\mathrm{dens}}$ blocks of size L'_{block} fully cover the spatial region determining whether a block is well-started. Furthermore, we need to to provide additional space for the well-started configurations to spread to in time T_{block}. This warrants the choice $c_{\mathrm{space}} = 2c_{\mathrm{dens}} + 2$. Note in this context that a much smaller L_{block} would suffice, but defining it to be a multiple of L'_{block} gives a more convenient notation.
(ii) In order to have a *well-started* block at (z,t) for which property (4.4) spreads to a region of radius $3R'_{\mathrm{block}}$ around z in time T_{block}, the region of space around z for which the densities of $\eta^{(1)}, \eta^{(2)}$ are near θ_μ must be large enough. As will be seen later on (see Section 4.2) this is due to the crucial role that the contraction property of Lemma 4.1 plays in the expansion of the coupling and translates loosely to R_{dens} being large enough, namely

$$R_{\mathrm{dens}} > R'_{\mathrm{block}} + T_{\mathrm{block}}\lceil sR \rceil + T_{\mathrm{block}}R.$$

This can also be seen as an incentive for taking T_{block} to be of order $\log R$ and R_{dens} to be of order $R\log R$. Further it shows that c_{dens} needs to be chosen suitably large; it suffices to take $c_{\mathrm{dens}} = 1 + 2c_{\mathrm{time}}$.

(iii) Assume that on the event that a block at (z,t) is well started, property (ii) of Definition 4.4 does not hold, i.e. there is a site at the top of the block at which $\eta^{(1)}$ and $\eta^{(2)}$ disagree. As will be seen in Section 4.2, the probability of the two processes disagreeing at a site (in a well-started block) decays by a factor of $\kappa(\mu,\varepsilon)$ at each time step, when tracing the unsuccessful coupling backwards in time though the block. By a union bound, it follows that the probability that a well-started block at (z,t) does not satisfy (ii), is bounded by $\kappa(\mu,\varepsilon)^{T_{\text{block}}}$ multiplied by the number of sites that are within distance $R'_{\text{block}} + T_{\text{block}}\lceil sR \rceil$ of z. For this probability to decay in R, the constant c_{time} must satisfy $c_{\text{time}} > -(d+1)/\log\kappa(\mu,\varepsilon)$.

In order to set up comparison with oriented percolation, in the same fashion as in Section 3, we need to show that the *good* blocks have high density and that the block dependencies have finite range that does not depend on R. To this end, note first that the event $\{\texttt{Block}(z,t) \text{ is good}\}$ depends (only) on $\{\eta_t^{(i)}(x), x \in B_{R_{\text{block}}}(z), i=1,2\}$ and $\{U(y,t+k) : y \in B_{3R_{\text{block}}}(z), k=1,2,\dots,T_{\text{block}}\}$.

LEMMA 4.6. *For $(z,t)\in\mathbb{L}$,*

$$\mathbb{P}\big(\texttt{Block}(z,t) \text{ is good} \,\big|\, \mathcal{F}_t\big) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\big(1-q(R,\mu)\big)$$

with $q(R,\mu)\to 0$ as $R\to\infty$.

See Section 4.2 for the proof.

Armed with Lemma 4.6 we can repeat the comparison construction from Section 3 and obtain the analogues of (3.8) and (3.9) in our context. That is we define $Y(z,t) = \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is good}\}}$, and then couple $(\eta^{(1)},\eta^{(2)})$ with a (high density) i.i.d. Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ such that

$$Y(z,t) \geq \mathbb{1}_{\{\texttt{Block}(z,t) \text{ is well-started}\}}\widetilde{Y}(z,t) \quad \text{for all } (z,t)\in\mathbb{L}$$

and $p(R) = \mathbb{P}(\widetilde{Y}(z,t)=1)\to 1$ as $R\to\infty$.

This shows that the density of good blocks (and thus also the density of space-time sites where $\eta^{(1)}$ and $\eta^{(2)}$ agree) will be high. In order to conclude that in fact $\eta^{(1)}$ and $\eta^{(2)}$ will agree a.s. from some time on in a growing space-time region, we invoke the fact that "dry" ($\widehat{=}$ "uncoupled") clusters of blocks do not percolate when $p(R)$ is close to 1. More precisely we set

$$C_0 := \left\{ (z,t)\in\mathbb{L} : \begin{array}{l} \text{There exists a path } (z_0,0),(z_1,T_{\text{block}}),\dots,(z_t,t) \text{ in } \mathbb{L} \\ \text{with } z_0=0, z_t=z \text{ such that } \|z_i - z_{i-1}\| \leq L'_{\text{block}} \text{ and} \\ \widetilde{Y}(z_i, iT_{\text{block}}) = 1 \text{ for } i\in\{1,\dots,t/T_{\text{block}}\} \end{array} \right\}$$

to be the cluster of sites which are connected to the origin by an open path in the Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$. Further we say that a space-time point $(z,t)\in\mathbb{L}$ is C_0-*exposed* if there is an arbitrary path from it to the zero-time slice, which entirely avoids C_0, i.e. if there is a path $(z_0,0),\dots,(z_t,t)$ in $\mathbb{L}$ with $z_t = z$ such that $\|z_k - z_{k-1}\| \leq L'_{\text{block}}$ and $(z_k, kT_{\text{block}}) \notin C_0, k=1,\dots,t/T_{\text{block}}$.

It follows from [10, Section 3] that there is a truncated cone originating from the origin in which there exist no C_0-exposed sites. The exact statement we are interested in is a direct reformulation of [4, Lemma 14].

LEMMA 4.7 ([4, Lemma 14]). *If $p(R)$ is sufficiently close to 1, then there is a positive constant $c>0$ and an almost surely finite random time τ, such that conditioned on $\{|C_0|=\infty\}$ there are no C_0-exposed sites in $\{(z,t)\in\mathbb{L} : \|z\| \leq ct, t\geq\tau\}$.*

For large enough R the Bernoulli field $(\widetilde{Y}(z,t))_{(z,t)\in\mathbb{L}}$ contains an infinite cluster of open sites with probability one. Similarly to Section 3, because a *good* block will be created with positive probability from any non-trivial initial condition, we can assume without loss of generality that this cluster contains the origin and that the block at the origin is good.

Lemma 4.7 together with Lemma 4.6 imply that for sufficiently large R, on $\{|C_0|=\infty\}$ there is a (random) time $\tau>0$ and a constant $c>0$ such that no sites in $\{(z,t)\in\mathbb{L}:\|z\|\le ct, t\ge\tau\}$ are C_0-exposed. We show that this implies that $\eta^{(1)}$ agrees with $\eta^{(2)}$ on the space-time cone $A=\{(z,t)\in\mathbb{Z}^d\times\mathbb{N}:\|z\|\le c(t-\tau),\, t\ge\tau\}$ centered at $(0,\tau)$. Indeed, assume to the contrary that there exists $(z,s)\in A$ such that $\eta_s^{(1)}(z)\neq\eta_s^{(2)}(z)$. Then we can find a path $(z,s),(x_{s-1},s-1),\dots,(x_0,0)$ in $\mathbb{Z}^d\times\mathbb{N}_0$ such that $x_u\in B_R(x_{u+1})$ and $\eta_u^{(1)}(x_u)\neq\eta_u^{(2)}(x_u)$ for all $0\le u\le s-1$. By disregarding all u's which are not a multiple of $T=T_{\text{block}}$, there exists some integer k and a sub-path $(z,s),(x_{kT},kT),\dots,(x_0,0)$ in $\mathbb{Z}^d\times\mathbb{N}_0$ "backwards in time". Assume without loss of generality that s is a multiple of T and associate to the sub-path the nearest neighbour path $\big((Z,k+1),(X_k,k),\dots,(X_0,0)\big)\subseteq\mathbb{L}$ where $Z,X_k\in L'_{\text{Block}}\mathbb{Z}^d$ are the respective closest grid-points to z and x_{kT} in the coarse-grained lattice. In particular $\|X_k-x_{kT}\|\le R'_{\text{Block}}$ for $k=0,\dots,s$. By definition $\widetilde{Y}(X_k,kT)=0$ for $k=0,\dots,s$, whence (Z,s) is a C_0-exposed site, contradicting Lemma 4.7 and yielding that in fact $\eta_s^{(1)}(z)=\eta_s^{(2)}(z)$. As $(z,s)\in A$ was chosen arbitrarily, the claim of Theorem 1.6 follows with $T^{\text{coupl}}=\tau$ and $a(R,\mu,d)=c$.

4.2. *Proof of Lemma 4.6.* The key step in proving Theorem 1.6 is showing that the coupled region in the *well-started* configuration of a *good* block expands to the neighbouring sites with high probability. In order to keep the notation lighter we only show this property for a block centred at the origin at time 0. That is, we show that for some $q=q(R,\mu)\to 0$ as $R\to\infty$,

$$\mathbb{P}\big(\mathtt{Block}(0,0)\text{ is good}\,\big|\,\mathcal{F}_0\big)\ge\mathbb{1}_{\{\mathtt{Block}(0,0)\text{ is well-started}\}}\big(1-q(R,\mu)\big) \tag{4.5}$$

Shifting the block yields the desired property for blocks centred at arbitrary space-time sites. Note that we still condition on $\mathcal{F}_0$, as we allow for possibly random initial configurations $\eta_0^{(i)}, i=1,2$. As was already anticipated in Remark 4.5, in order to see the spreading of the coupling after T_{block} steps, we need a large number of sites within distance R_{dens} of the origin for which the densities of both $\eta^{(1)},\eta^{(2)}$ are close to θ_μ. This is made precise by the following auxiliary events, where the densities have the prescribed behaviour on balls whose radii decrease by R at each time step.

Recall that $T_{\text{block}}=c_{\text{time}}\lceil\log R\rceil$ and write $R'(k):=R'_{\text{block}}+k\lceil sR\rceil$. For $n\in\mathbb{N}$ let

$$\Psi_n=\Big\{|\delta_R(x;\eta_j^{(i)})-\theta_\mu|<\varepsilon,\ \forall x:\|x\|\le R'(n)+(n-j)R,\ \forall j\in\{1,\dots,n\},\ i\in\{1,2\}\Big\}.$$

(Recall also that ε was chosen at the beginning of Section 4.1, above (4.2).)

Note that in a *well-started* configuration around the origin we have $|\delta_R(x;\eta_0^{(i)})-\theta_\mu|<\varepsilon$ for every x such that $\|x\|\le R'(T_{\text{block}})+T_{\text{block}}R$ (in fact, this holds for all x within distance R_{dens} from the origin and $R_{\text{dens}}\ge R'(T_{\text{block}})+T_{\text{block}}R$). The sites, where the densities of $\eta^{(1)},\eta^{(2)}$ are close to θ_μ due to the well-startedness, encompass the entire $n=0$ (bottom) level of the space-time pyramid $\Psi_{T_{\text{block}}}$, see also Figure 5. Due to this the the event $\Psi_{T_{\text{block}}}$ holds with high probability. Indeed, by defining the events $A_0=\emptyset$ and

$$A_j=\big\{\exists z\in B_{R'(T_{\text{block}})+(T_{\text{block}}-j)R}(0):|\delta_R(z;\eta_j^{(1)})-\theta_\mu|>\varepsilon\big\},$$

we see that on the event $\{\mathtt{Block}(0,0)\text{ is well-started}\}$

$$\mathbb{P}\big(\Psi^c_{T_{\text{block}}}\big|\mathcal{F}_0\big)\le 2\,\mathbb{P}\left(\bigcup_{j=1}^{T_{\text{block}}}A_j\middle|\mathcal{F}_0\right)\le 2\sum_{j=1}^{T_{\text{block}}}\mathbb{P}\big(A_j\cap A^c_{j-1}\big|\mathcal{F}_0\big)\le 2\sum_{j=1}^{T_{\text{block}}}\mathbb{P}(A_j\mid A^c_{j-1},\mathcal{F}_0).$$

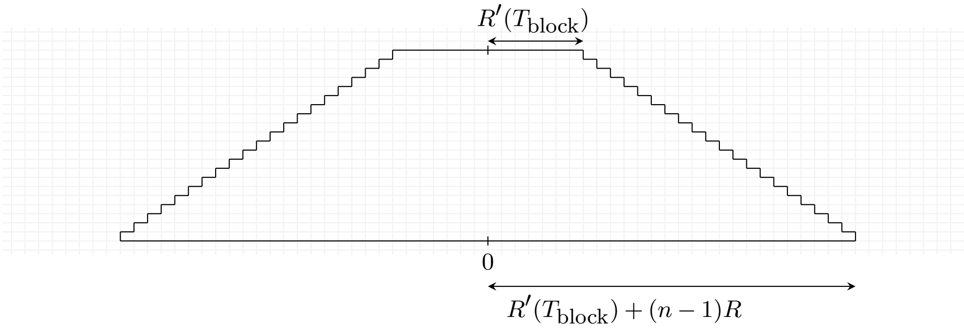

FIG 5. *The event Ψ_n occurs if the local density of $\eta^{(i)}, i = 1, 2$ is within ε distance of the fixed point θ_μ for all space-time points in the above pyramid. For convenience of presentation the spatial axis in the sketch is scaled by R, while the temporal axis is not scaled.*

Together with Lemma 2.3 it follows with some constants $c_1, c_2 > 0$ that

$$\mathbb{1}_{\{\texttt{Block}(0,0)\text{ is well-started}\}}\mathbb{P}\big(\Psi^c_{T_{\text{block}}}\big|\mathcal{F}_0\big) \le c_1 T_{\text{block}}(L_{\text{block}})^d \exp(-c_2\, V_R^d). \tag{4.6}$$

In order to utilise the control guaranteed by the pyramids Ψ_n we introduce events that describe properties (ii) and (iii) in Definition 4.4:

$$\mathcal{C} = \big\{\eta^{(1)}_{T_{\text{block}}}(x) = \eta^{(2)}_{T_{\text{block}}}(x) \text{ for } \|x\| \le 3R'_{\text{block}}\big\}$$
$$\mathcal{D} = \big\{\big(\eta^{(1)}_{T_{\text{block}}}, \eta^{(2)}_{T_{\text{block}}}\big) \text{ satisfy (4.3) around } L'_{\text{block}}e \text{ for all } \|e\| \le 1\big\}.$$

We are interested in the conditional probability $\mathbb{P}(\mathcal{C}\cap\mathcal{D}|\mathcal{F}_0)$ on the event that $\texttt{Block}(0,0)$ is well-started. Clearly it holds that

$$\mathbb{P}\big(\mathcal{C}^c \cup \mathcal{D}^c\big|\mathcal{F}_0\big) \le \mathbb{P}\big(\mathcal{C}^c \cap \Psi_{T_{\text{block}}}\big|\mathcal{F}_0\big) + \mathbb{P}\big(\Psi^c_{T_{\text{block}}}\big|\mathcal{F}_0\big) + \mathbb{P}\big(\mathcal{D}^c\big|\mathcal{F}_0\big). \tag{4.7}$$

By (4.6) the second term in (4.7) decays in R for well-started configurations. To deal with the third term, note that it follows from Lemma 4.2 and Lemma 2.3 that for some constants $c_3, c_4 > 0$

$$\mathbb{1}_{\{\texttt{Block}(0,0)\text{ is well-started}\}}\mathbb{P}(\mathcal{D}^c|\mathcal{F}_0) \le c_3 T_{\text{block}}(L_{\text{block}})^d \exp(-c_4\, V_R^d). \tag{4.8}$$

It remains to find a bound for $\mathbb{P}\big(\mathcal{C}^c \cap \Psi_{T_{\text{block}}}\big|\mathcal{F}_0\big)$. To this end fix $k \in \{1, \dots, T_{\text{block}}\}$. By a union bound and Markov's inequality

$$\begin{aligned}
&\mathbb{P}\big(\{\exists\, |x| \le R'(k) \text{ such that } \eta^{(1)}_k(x) \ne \eta^{(2)}_k(x)\} \cap \Psi_k \big|\mathcal{F}_0\big)\\
&\quad\le \sum_{x\in B_{R'(k)}(0)} \mathbb{E}\big[\mathbb{1}_{\Psi_k}|\eta^{(1)}_k(x) - \eta^{(2)}_k(x)|\big|\mathcal{F}_0\big]\\
&\quad= \mathbb{E}\Big[\sum_{x\in B_{R'(k)}(0)} \mathbb{1}_{\Psi_{k-1}}\mathbb{E}\big[|\eta^{(1)}_k(x) - \eta^{(2)}_k(x)|\big|\mathcal{F}_{k-1}\big]\Big|\mathcal{F}_0\Big].
\end{aligned} \tag{4.9}$$

In light of the coupling (4.1), we have

$$\begin{aligned}
\mathbb{E}\big[|\eta^{(1)}_k(x) - \eta^{(2)}_k(x)|\big|\,\mathcal{F}_{k-1}\big] &= \mathbb{P}\Big(U(x,k) \le \big|\varphi_\mu(\delta_R(x;\eta^{(1)}_{k-1})) - \varphi_\mu(\delta_R(x;\eta^{(2)}_{k-1}))\big|\,\Big|\,\mathcal{F}_{k-1}\Big)\\
&= \big|\varphi_\mu(\delta_R(x;\eta^{(1)}_{k-1})) - \varphi_\mu(\delta_R(x;\eta^{(2)}_{k-1}))\big|.
\end{aligned}$$

Now $\delta_R(x;\eta^{(i)}_{k-1}) \in [\theta_\mu - \varepsilon, \theta_\mu + \varepsilon]$ for $i = 1,2$ on the event Ψ_{k-1} and by Lemma 4.1, φ_μ is a contraction with Lipschitz constant $\kappa(\mu,\varepsilon) < 1$ on this interval. Therefore

$$\begin{aligned}\mathbb{1}_{\Psi_{k-1}}\mathbb{E}\big[|\eta^{(1)}_k(x) - \eta^{(2)}_k(x)|\,\big|\,\mathcal{F}_{k-1}\big] \leq& \mathbb{1}_{\Psi_{k-1}}\kappa(\mu,\varepsilon)\big|\delta_R(x;\eta^{(1)}_{k-1}) - \delta_R(x;\eta^{(2)}_{k-1})\big| \\ \leq& \mathbb{1}_{\Psi_{k-1}}\kappa(\mu,\varepsilon)V_R^{-d}\sum_{y\in B_R(x)}\big|\eta^{(1)}_{k-1}(y) - \eta^{(2)}_{k-1}(y)\big|.\end{aligned}$$

Plugging this back into (4.9) yields

$$\begin{aligned}&\mathbb{P}\big(\{\exists\,|x| \leq R'(k) \text{ such that } \eta^{(1)}_k(x) \neq \eta^{(2)}_k(x)\} \cap \Psi_k \big| \mathcal{F}_0\big) \\ &\leq \kappa(\mu,\varepsilon)V_R^{-d}\sum_{x\in B_{R'(k)}(0)}\sum_{y\in B_R(x)}\mathbb{E}\Big[\mathbb{1}_{\Psi_{k-1}}\big|\eta^{(1)}_{k-1}(y) - \eta^{(2)}_{k-1}(y)\big|\Big|\mathcal{F}_0\Big].\end{aligned}$$

By inductively repeating this step another $k-1$ times, we can upper bound the right hand side of the last display by

$$\kappa(\mu,\varepsilon)^k V_R^{-dk}\,\mathbb{E}\Big[\sum_{x\in B_{R'(k)}(0)}\sum_{y_1\in B_R(x)}\sum_{y_2\in B_R(y_1)}\cdots\sum_{y_k\in B_R(y_{k-1})}\big|\eta^{(1)}_0(y_k) - \eta^{(2)}_0(y_k)\big|\Big|\mathcal{F}_0\Big].$$

Since $\big|\eta^{(1)}_0(y_k) - \eta^{(2)}_0(y_k)\big| \leq 1$, with $k = T_{\text{block}}$ we obtain

$$\mathbb{1}_{\{\texttt{Block}(0,0)\text{ is well-started}\}}\mathbb{P}\big(\mathcal{C}^c \cap \Psi_{T_{\text{block}}}\big|\mathcal{F}_0\big) \leq \kappa(\mu,\varepsilon)^{T_{\text{block}}}V^d_{R'(T_{\text{block}})}. \tag{4.10}$$

The choice $c_{\text{time}} > -(d+1)/\log\kappa$ guarantees that this probability tends to zero as R goes to infinity. Combining (4.10) together with (4.8) and (4.6) gives that, on the event that $\texttt{Block}(0,0)$ is well-started, all the terms on the right-hand side of (4.7) tend to zero as R goes to infinity, thus proving (4.5).

4.3. *Proof of Theorem 1.5.* We now have all required tools to prove complete convergence of the BARW. Given these tools, the proof is relatively standard and thus it is kept brief.

PROOF. As the Dirac measure concentrated around $\eta \equiv 0$ is an invariant distribution for η we only need to show existence of a unique non-trivial limiting invariant measure which does not charge the empty configuration. To this end, let ν_0 be the product measure on $\mathbb{Z}^d$ such that, for all $x \in \mathbb{Z}^d$, $\eta_0(x) = 1$ with probability $p > 0$ and $\eta_0(x) = 0$ otherwise. For any $n \geq 1$, denote by ν_n the distribution of η_n given that η_0 is distributed as ν_0.

Since the set of all probability measures on $\{0,1\}^{\mathbb{Z}^d}$ is compact, there exists a subsequence along which $\frac{1}{N}\sum_{n=0}^N \nu_n$ converges to some probability measure ν on $\{0,1\}^{\mathbb{Z}^d}$. From a standard result for interacting particle systems, see e.g. [21, Proposition 1.8], any such subsequential limit ν must be invariant for the process η.

To show that ν is non-trivial (and actually gives zero mass to the empty configuration $\eta \equiv 0$), it suffices to show that η survives almost surely. As we chose ν_0 to be a product measure and since for any fixed R the blocks defined in Section 3 depend only on finitely many sites, it follows that at time 0 there are almost surely infinitely many *well-started* blocks and hence by (3.6) infinitely many *good* blocks. By the correspondence of the blocks with supercritical oriented site percolation and the fact that supercritical oriented site percolation starting from infinitely many occupied sites does not die out (see e.g. [22, Theorem B24]), we have $\mathbb{P}_{\nu_0}(\exists n \geq 1 : \eta_n \equiv 0) = 0$.

Furthermore, the measure ν is extremal, because any limiting invariant distribution ν' which gives zero mass to $\eta \equiv 0$ must be unique. Indeed, if two stationary distributions existed with this property, then by Theorem 1.6 they would coincide on finite subsets of $\mathbb{Z}^d$, and would therefore be equal. Furthermore, under ν, η has exponentially decaying correlations in space and in time, which in particular implies ergodicity w.r.t. spatial shifts. Indeed, using the construction of good blocks from the proof of Theorem 1.6 below, this can be deduced from the corresponding property of supercritical oriented percolation in a fairly straightforward way, see for example the analogous construction in [9, Section 3.4] for the related model of a locally regulated population from [4].

Finally, in order to verify the complete convergence, consider any (fixed) initial condition $\widetilde{\eta}_0 \in \{0,1\}^{\mathbb{Z}^d}$, a finite box $B \subset \mathbb{Z}^d$ centred at the origin and a configuration $\zeta \in \{0,1\}^B$. With $\mathcal{S} := \{\eta_m \not\equiv 0 \text{ for all } m \in \mathbb{N}\}$ we have to check that

$$\lim_{n\to\infty} \mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \mathcal{S}) = \mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S})\nu(\{\eta_0|_B = \zeta\}). \tag{4.11}$$

Pick $\varepsilon > 0$. The coupling construction from the proof of Theorem 1.6 and standard properties of supercritical oriented percolation show that one can pick $L' \in \mathbb{N}$ and $T' \in \mathbb{N}$ large so that $\big|\mathbb{P}_{\eta_0'}(\{\eta_m|_B = \zeta\}) - \nu(\{\eta_0|_B = \zeta\})\big| \le \varepsilon$ for all $m \ge T'$ and all starting configurations

$$\eta_0' \in G' := \left\{ \widetilde{\eta} \in \{0,1\}^{\mathbb{Z}^d} : \begin{array}{l} \text{The density of well-started sub-boxes, where the local} \\ \text{density of } \widetilde{\eta} \text{ satisfies (4.3) from Definition 4.3, in a box} \\ \text{of radius } L'R_{\text{block}} \text{ is at least } 1/2. \end{array} \right\}.$$

Furthermore, since starting from any non-trivial initial condition there is a positive chance of producing a well-started box in a finite number of steps, a "restart" argument together with the construction from Theorem 1.6 shows that $\mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S}\Delta\{\eta_n \in G'\}) \le \varepsilon$ for all large enough n. Thus

$$\begin{aligned} &\big|\mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \mathcal{S}) - \mathbb{P}_{\widetilde{\eta}_0}(\mathcal{S})\nu(\{\eta_0|_B = \zeta\})\big| \\ &\quad \le \big|\mathbb{P}_{\widetilde{\eta}_0}(\{\eta_n|_B = \zeta\} \cap \{\eta_{n/2} \in G'\}) - \mathbb{P}_{\widetilde{\eta}_0}(\{\eta_{n/2} \in G'\})\nu(\{\eta_0|_B = \zeta\})\big| + 2\varepsilon \\ &\quad \le \mathbb{E}_{\widetilde{\eta}_0}\Big[\mathbb{1}_{\{\eta_{n/2}\in G'\}}\big|\mathbb{P}_{\widetilde{\eta}_0}(\eta_n|_B = \zeta \,|\, \mathcal{F}_{n/2}) - \nu(\{\eta_0|_B = \zeta\})\big|\Big] + 2\varepsilon \le 3\varepsilon. \end{aligned}$$

Taking $n \to \infty$ and then $\varepsilon \downarrow 0$ proves (4.11). □

5. Extinction results.

We provide here a simple proof of Theorem 1.1 describing the extinction regime.

PROOF OF THEOREM 1.1. Let $R \in \mathbb{N}$ and $\mu > 0$ be such that

$$\widetilde{\mu} := V_R^d \varphi_\mu\big(V_R^{-d}\big) = \mu e^{-\mu V_R^{-d}} < 1. \tag{5.1}$$

Then $\psi(w) := \widetilde{\mu} w$ fulfils $\varphi_\mu(w) \le \psi(w)$ on $[0,1] \cap V_R^{-1}\mathbb{Z}$ (note that if $w \ge V_R^{-1}$, we have $\varphi_\mu(w) = \mu w \exp(-\mu w) \le \mu w \exp(-\mu V_R^{-d}) = \widetilde{\mu} w$ and $\varphi_\mu(0) = \psi(0)$).

Thus, we can define a process $(\widetilde{\eta}_n)_{n\in\mathbb{N}_0}$ with $\widetilde{\eta}_0 = \eta_0$ using this ψ as in (2.3). By the coupling construction from Section 2.1 and specifically Lemma 2.1(b) we conclude that $\eta_n(x) \le \widetilde{\eta}_n(x)$ holds for all $n \in \mathbb{N}, x \in \mathbb{Z}^d$. Since ψ is a linear function, we have

$$\mathbb{E}[\widetilde{\eta}_n(x)] = \widetilde{\mu} V_R^{-d} \sum_{y \in B_R(x)} \mathbb{E}[\widetilde{\eta}_{n-1}(y)]$$

Iterating this n times shows

$$\mathbb{E}[\widetilde{\eta}_n(x)] = \widetilde{\mu}^n \sum_{z \in \mathbb{Z}^d} p^{(n)}(x,z)\mathbb{E}[\eta_0(z)] \le \widetilde{\mu}^n$$

where $p^{(n)}$ is the n-fold convolution of the uniform transition kernel on $B_R(0)$ with itself. Since $\widetilde{\mu} < 1$ this combined with the coupling shows that $\sum_{n=1}^{\infty} \mathbb{P}(\eta_n(x) > 0) < \infty$ so that indeed for every $x \in \mathbb{Z}^d$

$$\mathbb{P}(\eta_n(x) = 0 \text{ for all } n \text{ large enough}) = 1.$$

Next note that the equation $\mu \exp(-\mu V_R^{-d}) = 1$, i.e. the equality in (5.1), has two positive real solutions μ_1, μ_2 such that $1 < \mu_1 < \mu_2 < \infty$ when $R \geq 1$ (when $R = 0$ there is always extinction). The function $\mu \mapsto \mu \exp(-\mu V_R^{-d})$ is unimodal and vanishes at 0 as well as at $+\infty$, so if $\mu < \mu_1$ or $\mu > \mu_2$ there is extinction.

We can rewrite $\mu \exp(-\mu V_R^{-d}) = 1$ as $ye^y = x$ where $y = -\mu V_R^{-d}$ and $x = -V_R^{-d}$. When $x \in [-1/e, 0)$, this equation has two real solutions $y_1 = W_0(x)$ and $y_2 = W_{-1}(x)$, where W_0 and W_{-1} are two branches of the Lambert W function. Since $\mu = -V_R^d y$, the two solutions of $\mu \exp(-\mu V_R^{-d}) = 1$ are

$$\mu_1 = -V_R^d W_0(-V_R^{-d}), \qquad \mu_2 = -V_R^d W_{-1}(-V_R^{-d}).$$

Since $x \in [-1/e, 0)$, we can express $W_0(x)$ with its Taylor series centred at 0, which has radius of convergence $1/e$, that is

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \cdots$$

This gives

$$\mu_1 = -V_R^d W_0(-V_R^{-d}) = 1 + V_R^{-d} + \frac{3}{2} V_R^{-2d} + \cdots \tag{5.2}$$

For the second solution, we use that

$$-1 - \sqrt{2u} - u < W_{-1}(-e^{-u-1}) < -1 - \sqrt{2u} - \frac{2u}{3}$$

for every $u > 0$. Take $u = d \log V_R - 1$. Then the formula above gives

$$\begin{aligned} -\sqrt{2d \log V_R - 2} - d \log V_R &< W_{-1}(-V_R^{-d}) \\ &< -\frac{1}{3} - \sqrt{2d \log V_R - 2} - \frac{2d}{3} \log V_R, \end{aligned} \tag{5.3}$$

which gives the result. □

6. Auxiliary results. We prove here the auxiliary technical results that were omitted in the previous sections. Section 6.1 deals with Lemma 2.5 which was used in the construction of the comparison density profiles ξ_n^- in Section 3. In Section 6.2, we then show Lemma 4.1 used in the proof of the complete convergence in Section 4. Finally, in Section 6.3 we provide a proof of Proposition 1.9.

6.1. *Proof of Lemma 2.5.* Recall that $f : \mathbb{Z} \to [0,1]$ is defined by

$$f(x) = \min\{(\varepsilon_0 + x/\lceil wR \rceil)\mathbb{1}_{x \geq 0}, 1\}.$$

It is immediate that this function satisfies the properties in (2.9). Therefore, it only remains to show that (2.10) holds for a suitable choice of parameters.

It is clear that the larger the growth factor a is, the easier it is for (2.10) to be satisfied. Setting $\varepsilon_0 = \min\{(a-1)^2, 1/100\}$, it follows that

$$a\delta_R(x; f) \geq (1 + \sqrt{\varepsilon_0})\delta_R(x; f) \quad \text{for all } x \in \mathbb{Z},$$

which lets us reduce to the case where $1 < a < 11/10$ and $a = 1 + \sqrt{\varepsilon_0}$.

We now set

$$w = 1/\sqrt{\varepsilon_0}, \tag{6.1}$$

and define

$$C_0 = \{y \in \mathbb{Z} : y < 0\}, \qquad C_1 = \{y \in \mathbb{Z} : y \geq \lceil wR \rceil\},$$

so that $f(y) = 0$ for every $y \in C_0$ and $f(y) = 1$ for every $y \in C_1$. Since $w > 1$, exactly one of the two sets $B_R(x) \cap C_0$ and $B_R(x) \cap C_1$ can be non-empty. Clearly (2.10) holds when $B_R(x) \subseteq C_0$, or $B_R(x) \subseteq C_1$.

When $B_R(x) \cap (C_0 \cup C_1) = \emptyset$, then $f(y) = \varepsilon_0 + y/\lceil wR \rceil$ for every $y \in B_R(x)$ and thus $\delta_R(x; f) = f(x)$, so (2.10) holds as well.

The remaining two cases are more delicate. When $B_R(x) \cap C_0 \neq \emptyset$ and $B_R(x) \not\subseteq C_0$, that is when $-R \leq x < R$, then the density of f around x can be written as

$$\begin{aligned}\delta_R(x; f) = V_R^{-1} \sum_{y=0}^{x+R} f(y) &= V_R^{-1} \sum_{y=0}^{x+R} \Big(\varepsilon_0 + \frac{y}{\lceil wR \rceil}\Big) \\ &= V_R^{-1}\Big((x + R + 1)\varepsilon_0 + \frac{1}{2\lceil wR \rceil}(x + R)(x + R + 1)\Big).\end{aligned}$$

Using this, (2.10) is equivalent to

$$\begin{aligned}a(x + R + 1)\varepsilon_0 + \frac{a}{2\lceil wR \rceil}(x + R)(x + R + 1) &\geq V_R f(x + \lceil sR \rceil) \\ &= V_R\Big(\varepsilon_0 + \frac{x}{\lceil wR \rceil} + \frac{\lceil sR \rceil}{\lceil wR \rceil}\Big).\end{aligned}$$

Rearranging terms, we arrive at a quadratic inequality

$$\alpha x^2 + \beta x + \gamma \geq 0, \tag{6.2}$$

where

$$\begin{aligned}\alpha &= \frac{a}{2\lceil wR \rceil}, \\ \beta &= a\varepsilon_0 + \frac{R}{\lceil wR \rceil}\Big(\frac{a}{2} - 1\Big)\Big(2 + \frac{1}{R}\Big), \\ \gamma &= a\varepsilon_0 R\Big(1 + \frac{1}{R}\Big) + \frac{aR^2}{2\lceil wR \rceil}\Big(1 + \frac{1}{R}\Big) - R\Big(2 + \frac{1}{R}\Big)\Big(\varepsilon_0 + \frac{\lceil sR \rceil}{\lceil wR \rceil}\Big).\end{aligned}$$

As $\alpha > 0$ for our choices of parameters, (6.2) and hence (2.10) follow immediately if the polynomial $\alpha x^2 + \beta x + \gamma$ has no real roots. The discriminant of (6.2) is given by

$$\begin{aligned}\beta^2 - 4\alpha\gamma &= \Big(a\varepsilon_0 + \frac{2}{w}\Big(\frac{a}{2} - 1\Big)\Big)^2 - \frac{2a}{w}\Big(a\varepsilon_0 + \frac{a}{2w} - 2\Big(\varepsilon_0 + \frac{s}{w}\Big)\Big) + O(R^{-1}) \\ &= (a\varepsilon_0)^2 + \frac{4}{w^2}(1 - a + as) + O(R^{-1}).\end{aligned} \tag{6.3}$$

We now choose

$$s = \frac{\sqrt{\varepsilon_0}}{1 + \sqrt{\varepsilon_0}} - \varepsilon_0,$$

which is clearly positive for $\varepsilon_0 \in (0, 1/100)$. Recalling also (6.1) and that $a = 1 + \sqrt{\varepsilon_0}$, the right-hand side of (6.3) (without the error term) equals

$$\varepsilon_0^2(\sqrt{\varepsilon_0} - 3)(1 + \sqrt{\varepsilon_0})$$

which is clearly negative. As consequence, the quadratic inequality (6.2) holds for all R big enough, depending only on a, and thus (2.10) holds also in this case.

For the final case, when $B_R(x) \cap C_1 \neq \emptyset$ that is $\lceil wR \rceil - R \leq x \leq \lceil wR \rceil - R$, we observe that the right-hand side of (2.10) is bounded by one and the left-hand side is increasing in x. It is thus sufficient to show that $a\delta_R(\lceil wR \rceil - R - 1) \geq 1$. Using again the fact that f is linear in the R-neighbourhood of $\lceil wR \rceil - R - 1$, this is equivalent to showing $af(\lceil wR \rceil - R - 1) \geq 1$. Recalling the definitions of a, w and s in terms of ε_0, we have

$$\begin{aligned} af(\lceil wR \rceil - R - 1) &= a\Big(\varepsilon_0 + \frac{\lceil wR \rceil - R - 1}{\lceil wR \rceil}\Big) \\ &= a\big(\varepsilon_0 + 1 - w^{-1} + O(R^{-1})\big) \\ &= (1 + \sqrt{\varepsilon_0})\big(\varepsilon_0 + 1 - \sqrt{\varepsilon_0} + O(R^{-1})\big) \\ &= 1 + \varepsilon_0^{3/2} + O(R^{-1}), \end{aligned}$$

and thus the required inequality is satisfied for R large enough. □

6.2. *Proof of Lemma 4.1.* We now prove Lemma 4.1, exploiting properties of φ_μ in the vicinity of its fixpoint θ_μ.

PROOF OF LEMMA 4.1. To prove that φ_μ is a contraction in the vicinity of its critical point $\theta_\mu = \mu^{-1} \log \mu$, it suffices to observe that $|\varphi_\mu'(w)| < 1$ in some neighbourhood of θ_μ. Since $\varphi_\mu'(w) = \mu e^{-\mu w}(1 - \mu w)$, it holds that $|\varphi_\mu'(\theta_\mu)| = |1 - \log \mu| < 1$ if $\mu \in (1, e^2)$. The statement then follows by the continuity of the derivative.

To find the sequences α_m and β_m, note first that φ_μ is increasing on $[0, 1/\mu]$ and decreasing on $[1/\mu, \infty]$. It is convenient to consider three cases (cf. also Figure 6):

(1) If $\mu \in (1, e)$, then $\theta_\mu < 1/e < 1/\mu$, and thus φ_μ is a strictly increasing on $[0, 1/e] \ni \theta_\mu$, and $\varphi_\mu(w) > w$ if $w < \theta_\mu$, and $\varphi_\mu(w) < w$ when $w \in (\theta_\mu, 1/e]$. Pick $\alpha_1 < \theta_\mu$ and $\beta_1 > 1/\mu$ satisfying $\varphi_\mu(\beta_1) \geq \varphi_\mu(\alpha_1)$. Put $\alpha_2 = (\alpha_1 + \varphi_\mu(\alpha_1))/2$, $\beta_2 = (e^{-1} + \mu^{-1})/2$, then we have indeed $\varphi_\mu([\alpha_1, \beta_1]) \subseteq (\alpha_2, \beta_2)$. From here on, we can simply iterate by setting

$$\alpha_{m+1} = \frac{\alpha_m + \varphi_\mu(\alpha_m)}{2}, \qquad \beta_{m+1} = \frac{\beta_m + \varphi_\mu(\beta_m)}{2}, \quad m \geq 2. \tag{6.4}$$

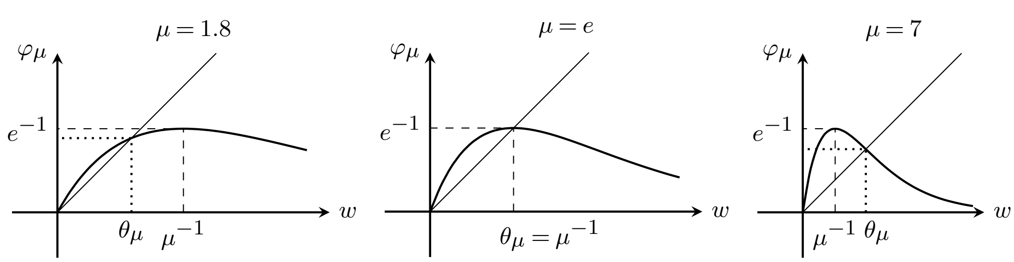

FIG 6. *The function φ_μ, its fixpoint θ_μ and its maximum in the case $\mu < e$ (left), $\mu = e$ (middle) and $e < \mu < e^2$ (right)*

This defines two sequences converging to θ_μ. Furthermore $\alpha_m < \alpha_{m+1} < \varphi_\mu(\alpha_m)$ and $\varphi_\mu(\beta_m) < \beta_{m+1} < \beta_m$, so $(\alpha_m)_{m\geq 1}$ is strictly increasing and $(\beta_m)_{m\geq 1}$ is strictly decreasing. Since φ_μ is strictly increasing on $[0,1/e]$ we also have $\varphi_\mu([\alpha_m,\beta_m]) \subseteq (\alpha_{m+1},\beta_{m+1})$ for every $m \geq 0$, as required.

(2) Consider now the case $\mu = e$, that is when $\theta_\mu = 1/e$ and $\varphi'_\mu(\theta_\mu) = 0$. Pick any $\alpha_1 < 1/e$ and $\beta_1 = 1/e$ such that $\varphi_\mu(\beta_1) \geq \varphi_\mu(\alpha_1)$; then build the sequence $(\alpha_m)_{m\geq 1}$ in the same way as in the case $\mu \in (1,e)$, i.e. as in (6.4), using $m \geq 1$ there. By construction, this sequence is strictly increasing and converges to θ_μ. For every $m \geq 2$, let β_m be the largest solution of $\varphi_\mu(x) = \varphi_\mu(\alpha_m)$. Since $w \mapsto \varphi_\mu(w)$ is (strictly) increasing if and only if $w \in [0,1/e]$, this defines a strictly decreasing sequence $(\beta_m)_{m\geq 1}$ converging to θ_μ and such that

$$\varphi_\mu([\alpha_m,\beta_m]) \subseteq [\varphi_\mu(\alpha_m),1/e] \subseteq (\alpha_{m+1},\beta_{m+1}),$$

as required.

(3) Finally, let $\mu \in (e,e^2)$, which implies $1/\mu < \theta_\mu$ and $\varphi'_\mu(\theta_\mu) \in (-1,0)$. For the initial piece, pick $\alpha_1 < 1/\mu$ and $\lambda > 0$ so small that $\mu^{-1} + \lambda e^{-1} < \theta_\mu$. Define, similarly to (6.4),

$$\alpha_{m+1} = \lambda\varphi_\mu(\alpha_m) + (1-\lambda)\alpha_m,\ m \leq m_0 - 1,$$

where m_0 is the smallest integer satisfying $\alpha_{m_0} > 1/\mu$. Note that by construction and the choice of λ, since φ_μ is strictly increasing on $[0,1/\mu]$ and bounded by $1/e$, we have $\alpha_1 < \alpha_2 < \cdots < \alpha_{m_0-1} \leq 1/\mu < \alpha_{m_0} < \theta_\mu$. Choose $\beta_1 > \beta_2 > \cdots > \beta_{m_0} > 1/e\ (> \theta_\mu)$ so that $\varphi_\mu(\beta_m) > \varphi_\mu(\alpha_m)$ for $m = 1,\ldots,m_0$, then we have $\varphi_\mu([\alpha_m,\beta_m]) \subseteq (\alpha_{m+1},\beta_{m+1})$ for $m = 1,\ldots,m_0-1$.

Since $\alpha_{m_0} > 1/\mu$, the iteration has reached the decreasing part of φ_μ after m_0 steps and we thus must swap the roles of the upper and the lower boundary in each step: Set for $m \geq m_0$

$$\alpha_{m+1} = \frac{\varphi_\mu(\beta_m)+\alpha_m}{2}, \qquad \beta_{m+1} = \frac{\varphi_\mu(\alpha_m)+\beta_m}{2}.$$

We note that if $\varphi_\mu(\alpha_m) < \beta_m$ and $\varphi_\mu(\beta_m) > \alpha_m$ then the same holds for α_{m+1} and β_{m+1}. Indeed $\varphi_\mu(\beta_m) > \alpha_m$ implies that $\alpha_{m+1} > \alpha_m$ and since φ_μ is decreasing then $\varphi_\mu(\alpha_{m+1}) < \varphi_\mu(\alpha_m)$. Similarly $\varphi_\mu(\alpha_m) < \beta_m$ implies that $\beta_{m+1} < \beta_m$ and so $\varphi_\mu(\alpha_m) = 2\beta_{m+1} - \beta_m < \beta_{m+1}$. Combining the two gives $\varphi_\mu(\alpha_{m+1}) < \varphi_\mu(\alpha_m) < \beta_{m+1}$. In the same way we can prove that $\varphi_\mu(\beta_{m+1}) > \alpha_{m+1}$. Hence for $m \geq m_0$

$$\varphi_\mu((\alpha_m,\beta_m)) \subseteq [\varphi_\mu(\beta_m),\varphi_\mu(\alpha_m)] \subseteq (\alpha_{m+1},\beta_{m+1}).$$

It is clear from the construction that in each one of the three cases α_1 can be chosen arbitrarily small and $\beta_1 > 1/e$ (if a large β_1 is required, this can be achieved by decreasing α_1 appropriately). □

6.3. *Proof of Proposition 1.9.* Note again that since $\max_{w\geq 0}\varphi_\mu(w) = 1/e$, for every initial condition $\Xi_0 \in \mathbb{R}_+^{\mathbb{Z}^d}$ of the coupled map lattice defined in (1.8) we have $\Xi_1 \in [0,1/e]^{\mathbb{Z}^d}$. Thus we can assume without loss of generality that $0 \leq \Xi_0(z) \leq 1/e$ for every $z \in \mathbb{Z}^d$. Assume moreover that $\Xi_0(z_0) > 0$ for some $z_0 \in \mathbb{Z}^d$, as it otherwise obviously holds that $\Xi_n \equiv 0$ for all n. The proof follows ideas from Section 4 in [4].

PROOF OF PROPOSITION 1.9. Fix $\varepsilon > 0$ and let $a > 1$ and $b > 0$ be such that $\psi(w) = aw \wedge b$ satisfies $\varphi_\mu(w) \geq \psi(w)$ for every $w \in [0,1]$. Since θ_μ is a stable fixpoint when $\mu \in (1,e^2)$, we can choose sequences $(\alpha_m)_{m\geq 0}$, $(\beta_m)_{m\geq 0}$ as in Lemma 4.1 with $\alpha_1 < b/2$ and a suitable $\beta_1 > 1/e$, such that $\varphi_\mu([\alpha_m,\beta_m]) \subseteq (\alpha_{m+1},\beta_{m+1})$ and $\beta_{m^*} - \alpha_{m^*} < \varepsilon$ for some $m^* \in \mathbb{N}$.

For a fixed $z \in \mathbb{Z}^d$ we show that there exists $n_0 > m^*$ such that $\Xi_n(z) \in [\alpha_{m^*}, \beta_{m^*}]$ for all $n \geq n_0$. We start by showing that

$$\Xi_n(z) \geq \sum_{y \in \mathbb{Z}^d} p^{(n)}(z,y)\Big[\big(a^n \Xi_0(y)\big) \wedge b\Big], \tag{6.5}$$

where $p^{(n)}(\cdot,\cdot)$ are the n-step transition probabilities of a random walk whose steps are uniformly distributed in $B_R(0) \cap \mathbb{Z}^d$. We can check (6.5) by induction. Using Jensen's inequality, it holds that

$$\begin{aligned}\Xi_{n+1}(z) = \varphi_\mu(\delta_R(z;\Xi_n)) &\geq \psi\Bigg(V_R^{-d} \sum_{x \in B_R(0)} \Xi_n(z+x)\Bigg) \\ &\geq V_R^{-d} \sum_{x \in B_R(0)} \psi\big(\Xi_n(z+x)\big).\end{aligned}$$

Using the inductive assumption,

$$\begin{aligned}\psi\big(\Xi_n(z+x)\big) &\geq \Bigg[a \sum_{y \in \mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^n \Xi_0(y)\big) \wedge b\Big)\Bigg] \wedge b \\ &= \sum_{y \in \mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big) \wedge ab\Big) \wedge b \\ &\geq \sum_{y \in \mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big) \wedge b\Big) \wedge b \\ &= \sum_{y \in \mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big) \wedge b\Big),\end{aligned}$$

so

$$\Xi_{n+1}(z) \geq V_R^{-d} \sum_{x \in B_R(0)} \sum_{y \in \mathbb{Z}^d} p^{(n)}(z+x,y)\Big(\big(a^{n+1}\Xi_0(y)\big) \wedge b\Big)$$

and the conclusion follows from the fact that

$$V_R^{-d} \sum_{x \in B_R(0)} p^{(n)}(z+x,y) = \sum_{x \in B_R(0)} p(z,z+x) p^{(n)}(z+x,y) = p^{(n+1)}(z,y).$$

For our fixed choice of z, we show that

$$\Xi_n(x) \in [\alpha_1, \beta_1] \text{ for all } n \geq n_0 \text{ and } \|x - z\| \leq 2Rm^*. \tag{6.6}$$

Take $n_1 > \big(4Rm^* + 2\|z - z_0\|\big)^2 \vee \big(\big(\ln(b) - \ln(\Xi_0(z_0))\big)/\ln(a)\big)$ large enough. By a local central limit theorem for symmetric finite range random walks, cf. [19, Theorem 2.1.1] there exists $c > 0$ such that $p^{(n_1)}(y, z_0) \geq c n_1^{-d/2}$ if $\|y - z_0\| \leq \sqrt{n_1}$. By letting $n_1 > \big(\ln(b) - \ln(\Xi_0(z_0))\big)/\ln(a)$ it holds that $a^{n_1}\Xi_0(z_0) \wedge b = b$ and hence it follows with (6.5) that

$$\Xi_{n_1}(y) \geq \sum_{w \in \mathbb{Z}^d} p^{(n_1)}(y,w)\Big[\big(a^{n_1}\Xi_0(w)\big) \wedge b\Big] \geq p^{(n_1)}(y,z_0)\Big[\big(a^{n_1}\Xi_0(z_0)\big) \wedge b\Big] \geq c n_1^{-d/2} b.$$

Using (6.5) again, we deduce that for any $n_2 < \sqrt{n_1}/2$

$$\begin{aligned}\Xi_{n_1+n_2}(x) &\geq \sum_{y \in \mathbb{Z}^d} p^{(n_2)}(x,y)\Big[\big(a^{n_2}\Xi_{n_1}(y)\big) \wedge b\Big] \\ &\geq \sum_{y \in B_{\sqrt{n_1}}(z_0)} p^{(n_2)}(x,y)\Big(\big(a^{n_2} c n_1^{-d/2}\big) \wedge 1\Big) b.\end{aligned}$$

Choosing $n_2 = d\log n_1 - 2\log c$ gives that $\left(a^{n_2} c n_1^{-d/2}\right) \wedge 1 = 1$ and, since $B_{n_2}(x) \subseteq B_{\sqrt{n_1}}(z_0)$ when $n_1 > \left(4Rm^* + 2\|z - z_0\|\right)^2$, the above is larger than b.

Since $b > 2\alpha_1$ and trivially $\varphi_\mu(w) \leq 1/e < \beta_1$ for every $w \geq 0$, this shows (6.6). It follows that

$$\Xi_{n+1}(x) = \varphi_\mu(\delta_R(x;\Xi_n)) \in [\alpha_2, \beta_2] \text{ for all } n \geq n_0 \text{ and } \|x - z\| \leq (2m^* - 1)R$$

and iterating m^* steps shows that

$$\Xi_{n+m^*-1}(x) \in [\alpha_m, \beta_m] \text{ for all } n \geq n_0 \text{ and } \|x - z\| \leq m^* R.$$

Take $x = z$ to conclude that $\Xi_n(z) \in [\alpha_m, \beta_m]$ for $n \geq n_0 + m^*$. □

7. Open Questions. We collect here some natural follow-up questions to our results, several of them were already mentioned in the text.

- Is there a sharp transition? That is, for given R, is the survival region a (possible empty) interval of values of μ? See also Figure 7.
- Is there always extinction for small values of R? Simulations suggest that in $d = 1$ for $R \leq 2$ the process dies out for all values of μ, see Figure 7 again.
- Can one give results for "soft" annihilation, allowing multiple occupancy of the sites? Of course, instead of the strong competition we consider, one could look at truncation, keeping for instance at most N particles per site at the same time and removing the others. Theorem 1.1 in [26] implies for this truncation in our model that there is, for each $\mu > 1$ and all R, a critical value $N_c \in \{2, 3, 4, \dots\}$ such that the survival probability is 0 for $N \leq N_c$ and strictly positive for $N > N_c$.
- What is the speed for the stochastic "travelling waves" in our model? Is there a shape theorem?
- The representation (1.4) suggests an interesting connection to spread-out oriented site percolation: let each site be open with probability p and closed with probability $1 - p$, where $p = \min\{\varphi_\mu((2R+1)^{-d}), \varphi_\mu(1)\}$. Connect the open sites at time $n+1$ to their "parent"

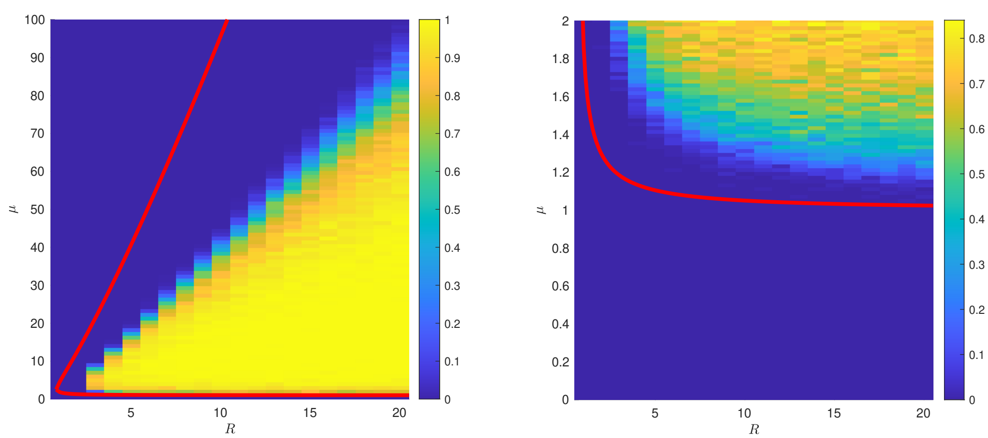

FIG 7. *Simulations of the "phase diagram" for a one-dimensional BARW on $\mathbb{Z}/1000\mathbb{Z}$ with initial condition $\eta_0 = \delta_0$, showing a Monte Carlo estimate of the survival probability as a function of R and μ. On the left,* 200 *iterations of this process were run and the proportion of realisations that survived the first* 250 *generations is shown. Dark blue colour corresponds to no surviving realisations and yellow to only surviving realisations. The right image zooms in the region of small μ's. In both cases the red line is our theoretical bound for extinction from Theorem 1.1.*

(with distance $\le R$) at time n, provided it is open. Then the "wet" sites at time n are a lower bound for η_n.

Let $p_c(d,R)$ be the percolation threshold for the event that there is an infinite connected cluster. How does the percolation threshold in directed space-time percolation behave for $R\to\infty$?

We have the following conjecture, based on the analogy with "spread-out oriented bond percolation", see [33]:

$$\lim_{R\to\infty}(2R+1)^d p_c(d,R)=1 \quad \text{for every } d>4.$$

It is plausible since the lattice should be more and more tree-like in high dimensions but we could not find a proof in the literature. Since $\varphi'_\mu(0)>1$, this conjecture would lead to an alternative proof of survival for large R in $d>4$.

APPENDIX

For completeness and ease of reference, we state the following concentration estimate for sums of independent Bernoulli random variables, which is a straightforward consequence of Bernstein's inequality.

LEMMA A.1. *Let $(X_i)_{i=1,\dots,n}$ be independent Bernoulli random variables with $p_i=\mathbb{P}(X_i=1)$, and let $S_n:=X_1+\cdots+X_n$. Then, setting $\mu_n:=\mathbb{E}[S_n]=\sum_{i=1}^n p_i$, $\sigma_n^2:=\operatorname{Var}S_n=\sum_{i=1}^n p_i(1-p_i)$, and $m_n:=\max_{1\le i\le n}\max\{p_i,1-p_i\}=\max_{1\le i\le n}\operatorname{ess\,sup}|X_i-\mathbb{E}[X_i]|$ (≤ 1), we have*

$$\mathbb{P}(S_n-\mu_n\ge w)\le \exp\Big(-\frac{w^2}{2\sigma_n^2+(2/3)m_n w}\Big),\quad w\ge 0, \tag{A.1}$$

and the same bound applies to $\mathbb{P}(S_n-\mu_n\le w)$ for $w\le 0$.

PROOF. By Bernstein's inequality (see e.g. [3, Ineq. (8)]), for every $t\ge 0$,

$$\mathbb{P}(S_n\ge \mu_n+t\sigma_n)\le \exp\Big(-\frac{t^2}{2+2m_n t/(3\sigma_n)}\Big)=\exp\Big(-\frac{(\sigma_n t)^2}{2\sigma_n^2+(2/3)m_n t\sigma_n}\Big).$$

Reparametrising $t\sigma_n=w$ (and implicitly assuming $\sigma_n>0$, otherwise the problem becomes trivial) we can rewrite this as (A.1).

Applying the argument to the $1-X_i$'s gives the same bound for $\mathbb{P}(S_n-\mu_n\le w)$, for $w\le 0$. □

Funding. The work was supported by Deutsche Forschungsgemeinschaft through DFG project no. 443869423 and by Schweizerischer Nationalfonds through SNF project no. 200021E_193063 in the context of a joint DFG-SNF project within in the framework of DFG priority programme SPP 2265 Random Geometric Systems.

REFERENCES

[1] ALILI, S. and IGNATIOUK-ROBERT, I. (2001). On the surviving probability of an annihilating branching process and application to a nonlinear voter model. *Stochastic Process. Appl.* **93** 301–316. https://doi.org/10.1016/S0304-4149(00)00101-0 MR1828777

[2] ATHREYA, S. R. and SWART, J. M. (2012). Systems of branching, annihilating, and coalescing particles. *Electron. J. Probab.* **17** no. 80, 32. https://doi.org/10.1214/EJP.v17-2003 MR2981905

[3] BENNETT, G. (1962). Probability Inequalities for the Sum of Independent Random Variables. *Journal of the American Statistical Association* **57** 33–45.

[4] BIRKNER, M. and DEPPERSCHMIDT, A. (2007). Survival and complete convergence for a spatial branching system with local regulation. *The Annals of Applied Probability* **17** 1777–1807.

[5] BLATH, J., ETHERIDGE, A. and MEREDITH, M. (2007). Coexistence in locally regulated competing populations and survival of branching annihilating random walk. *The Annals of Applied Probability* **17** 1474–1507.

[6] BRAMSON, M., DING, W. D. and DURRETT, R. (1991). Annihilating branching processes. *Stochastic Process. Appl.* **37** 1–17. https://doi.org/10.1016/0304-4149(91)90056-I MR1091690

[7] BRAMSON, M. and GRAY, L. (1985). The survival of branching annihilating random walk. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **68** 447–460.

[8] CZUPPON, P. (2016). Phenotypic heterogeneity in bacterial populations : a mathematical study, PhD thesis, Albert-Ludwigs-Universität Freiburg.

[9] DEPPERSCHMIDT, A. (2008). Survival, complete convergence and decay of correlations for a spatial branching system with local regulation, PhD thesis, TU Berlin. https://doi.org/10.14279/depositonce-1966

[10] DURRETT, R. (1992). Multicolor particle systems with large threshold and range. *Journal of Theoretical Probability* **5** 127–152.

[11] ETHERIDGE, A. M. (2004). Survival and extinction in a locally regulated population. *Ann. Appl. Probab.* **14** 188–214. https://doi.org/10.1214/aoap/1075828051 MR2023020

[12] FERNÁNDEZ, R., LOUIS, P.-Y. and NARDI, F. R. (2018). Overview: PCA models and issues. In *Probabilistic cellular automata*. *Emerg. Complex. Comput.* **27** 1–30. Springer, Cham. MR3793679

[13] FINKELSHTEIN, D., KONDRATIEV, Y. and KUTOVIY, O. (2009). Individual based model with competition in spatial ecology. *SIAM J. Math. Anal.* **41** 297–317. https://doi.org/10.1137/080719376 MR2505861

[14] HAMMERSLEY, J. M. (1974). Postulates for subadditive processes. *Ann. Probability* **2** 652–680. https://doi.org/10.1214/aop/1176996611 MR370721

[15] KALLENBERG, O. (1997). *Foundations of modern probability*. *Probability and its Applications (New York)*. Springer-Verlag, New York. MR1464694

[16] KOT, M. (1992). Discrete-time travelling waves: ecological examples. *J. Math. Biol.* **30** 413–436. https://doi.org/10.1007/BF00173295 MR1154698

[17] KOT, M. and SCHAFFER, W. M. (1986). Discrete-time growth-dispersal models. *Math. Biosci.* **80** 109–136. https://doi.org/10.1016/0025-5564(86)90069-6 MR849698

[18] LANCHIER, N. (2017). *Stochastic modeling*. Springer.

[19] LAWLER, G. F. and LIMIC, V. (2010). *Random walk: a modern introduction* **123**. Cambridge University Press.

[20] LI, B., LEWIS, M. A. and WEINBERGER, H. F. (2009). Existence of traveling waves for integral recursions with nonmonotone growth functions. *J. Math. Biol.* **58** 323–338. https://doi.org/10.1007/s00285-008-0175-1 MR2470192

[21] LIGGETT, T. M. (1985). *Interacting particle systems*. *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **276**. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4613-8542-4 MR776231

[22] LIGGETT, T. M. (1999). *Stochastic interacting systems: contact, voter and exclusion processes* **324**. Springer science & Business Media.

[23] LIGGETT, T. M., SCHONMANN, R. H. and STACEY, A. M. (1997). Domination by product measures. *The Annals of Probability* **25** 71–95.

[24] MAILLARD, P. and PENINGTON, S. (2022). Branching random walk with non-local competition. *arXiv preprint arXiv:2209.14653*.

[25] MAIRESSE, J. and MARCOVICI, I. (2014). Around probabilistic cellular automata. *Theoretical Computer Science* **559** 42–72.

[26] MÜLLER, S. (2015). Interacting growth processes and invariant percolation. *Ann. Appl. Probab.* **25** 268–286. https://doi.org/10.1214/13-AAP995 MR3297773

[27] PERL, I., SEN, A. and YADIN, A. (2015). Extinction window of mean field branching annihilating random walk. *The Annals of Applied Probability* **25** 3139–3161.

[28] SHI, Z. (2015). *Branching random walks*. *Lecture Notes in Mathematics* **2151**. Springer, Cham Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. https://doi.org/10.1007/978-3-319-25372-5 MR3444654

[29] SUDBURY, A. (1990). The branching annihilating process: an interacting particle system. *Ann. Probab.* **18** 581–601. MR1055421

[30] SUDBURY, A. (2000). The survival of nonattractive interacting particle systems on **Z**. *Ann. Probab.* **28** 1149–1161. https://doi.org/10.1214/aop/1019160329 MR1797307

[31] SUDBURY, A. (2000). Dual families of interacting particle systems on graphs. *J. Theoret. Probab.* **13** 695–716. https://doi.org/10.1023/A:1007806427774 MR1785526

[32] SWART, J. M. (2017). A course in interacting particle systems. *arXiv preprint arXiv:1703.10007*.

[33] VAN DER HOFSTAD, R. and SAKAI, A. (2005). Critical points for spread-out self-avoiding walk, percolation and the contact process above the upper critical dimensions. *Probab. Theory Related Fields* **132** 438–470. https://doi.org/10.1007/s00440-004-0405-4 MR2197108

[34] WEINBERGER, H. F. (1978). Asymptotic behavior of a model in population genetics. In *Nonlinear partial differential equations and applications (Proc. Special Sem., Indiana Univ., Bloomington, Ind., 1976–1977). Lecture Notes in Math., Vol. 648* 47–96. Springer, Berlin. MR0490066