Existence and stability of weak solutions of the Vlasov–Poisson system in localized Yudovich spaces

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EXISTENCE AND STABILITY OF WEAK SOLUTIONS OF THE VLASOV–POISSON SYSTEM IN LOCALIZED YUDOVICH SPACES

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Abstract. We consider the Vlasov–Poisson system both in the repulsive (electrostatic potential) and in the attractive (gravitational potential) cases. In our first main theorem, we prove the uniqueness and the quantitative stability of Lagrangian solutions $f = f(t, x, v)$ whose associated spatial density $\rho_f = \rho_f(t, x)$ is potentially unbounded but belongs to suitable uniformly-localized Yudovich spaces. This requirement imposes a condition of slow growth on the function $p \mapsto \|\rho_f(t, \cdot)\|_{L^p}$ uniformly in time. Previous works by Loeper, Miot and Holding–Miot have addressed the cases of bounded spatial density, i.e., $\|\rho_f(t, \cdot)\|_{L^p} \lesssim 1$, and spatial density such that $\|\rho_f(t, \cdot)\|_{L^p} \sim p^{1/\alpha}$ for $\alpha \in [1, +\infty)$. Our approach is Lagrangian and relies on an explicit estimate of the modulus of continuity of the electric field and on a second-order Osgood lemma. It also allows for iterated-logarithmic perturbations of the linear growth condition. In our second main theorem, we complement the aforementioned result by constructing solutions whose spatial density sharply satisfies such iterated-logarithmic growth. Our approach relies on real-variable techniques and extends the strategy developed for the Euler equations by the first and fourth-named authors. It also allows for the treatment of more general equations that share the same structure as the Vlasov–Poisson system. Notably, the uniqueness result and the stability estimates hold for both the classical and the relativistic Vlasov–Poisson systems.

1. Introduction

1.1. Framework. For some fixed $T \in (0, +\infty)$, we consider the *Vlasov–Poisson system*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E_f \cdot \nabla_v f = 0 & \text{in } (0,T) \times \mathbb{R}^{2d}, \\ E_f(t,x) = \kappa \displaystyle\int_{\mathbb{R}^d} K(x-y)\, \rho_f(t,y)\, \mathrm{d}y & \text{in } (0,T) \times \mathbb{R}^{d}, \\ \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, \mathrm{d}v & \text{in } (0,T) \times \mathbb{R}^{d}, \\ f(0,x,v) = f_0(x,v) & \text{in } \mathbb{R}^{2d}, \end{cases} \tag{1.1}$$

where $f_0 \in L^1(\mathbb{R}^{2d})$ is the initial datum, $f \in L^\infty([0,T]; L^1(\mathbb{R}^{2d}))$ is the unknown, $\rho_f \in L^\infty([0,T]; L^1(\mathbb{R}^d))$ is the spatial density associated with f, $\kappa \in \{-1, +1\}$ and $K\colon \mathbb{R}^d \to \mathbb{R}^d$

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is the *Riesz kernel*, given by

$$K(z) = \frac{x}{|x|^d}, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{1.2}$$

In particular, the vector field $E_f \in L^\infty([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ is well defined. For $d=3$, the Vlasov–Poisson system (1.1) describes the time evolution of the density f of plasma consisting of charged particles with long-range interaction, e.g., a repulsive Coulomb potential for $\kappa = 1$ or an attracting gravitational potential for $\kappa = -1$.

The Vlasov–Poisson system (1.1) has been extensively investigated. Existence and uniqueness of classical solutions of the system (1.1) under some regularity assumptions on the initial data go back to Iordanski [16] for $d = 1$ and to Okabe–Ukai [30] for $d = 2$. In any dimension, global existence of weak solutions with finite energy

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\kappa}{2}\int_{\mathbb{R}^d} |E_f(t,x)|^2\,\mathrm{d}x < +\infty$$

is due to Arsen'ev [2]. For $d = 3$, global existence and uniqueness have been addressed by Bardos–Degond [3] for classical solutions with small initial data, and then by Pfaffelmoser [25] and Lions–Perthame [19] using different methods. The main idea of [25] is to exploit *Lagrangian* techniques to prove global existence and uniqueness of classical solutions with compactly supported initial data. The approach of [19], instead, relies on an *Eulerian* point of view, yielding existence of global weak solutions with finite velocity moments. More precisely, for $d = 3$, if $f_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^{2d}} |v|^m f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v < +\infty \quad \text{for some } m > 3, \tag{1.3}$$

then there exists a corresponding weak solution $f \in L^\infty([0,+\infty); L^1(\mathbb{R}^{2d}))$ such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} |v|^m f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v < +\infty \quad \text{for any } T > 0.$$

For further developments concerning the propagation of moments and global existence of weak solutions of the Vlasov–Poisson system (1.1), we refer the reader to [5, 7, 9, 23, 24, 27].

Sufficient conditions for uniqueness of weak solutions of the Vlasov–Poisson system (1.1) have been first obtained in [19], provided that (1.3) holds with $m > 6$ and a technical assumption on the support of the initial data is satisfied. A simpler criterion has been then proposed by Robert [26] for compactly supported weak solutions, and later extended by Loeper [20] to measure-valued solutions f with spatial density such that

$$\rho_f \in L^\infty([0,T]; L^\infty(\mathbb{R}^d)). \tag{1.4}$$

Recently, Miot [22] generalized the uniqueness criterion of [19] to measure-valued solutions f with spatial density such that, for some $T > 0$,

$$\sup_{t\in[0,T]} \sup_{p\geq 1} \frac{\|\rho_f(t,\cdot)\|_{L^p}}{p} < +\infty. \tag{1.5}$$

The uniqueness condition (1.5) is satisfied by some non-trivial weak solutions with initial data having unbounded macroscopic density, see [22, Ths. 1.2 and 1.3]. Later, Holding–Miot [13] provided a uniqueness criterion interpolating between the conditions (1.4) and (1.5) by considering measure-valued solutions f with spatial density such that, for some $T > 0$ and $\alpha \in [1,+\infty)$,

$$\sup_{t\in[0,T]} \sup_{p\geq \alpha} \frac{\|\rho_f(t,\cdot)\|_{L^p}}{p^{1/\alpha}} < +\infty. \tag{1.6}$$

The case $\alpha = 1$ corresponds to (1.5), while the limiting case $\alpha = +\infty$ corresponds to (1.4). Condition (1.6) implies that ρ_f belongs to an *exponential Orlicz space*, see [13, Sec. 1.1.1].

Conditions (1.5) and (1.6) allow to consider initial data with compact support in velocity as well as *Maxwell–Boltzmann distributions* with exponential decay as $|v| \to +\infty$, see the comments below [22, Th. 1.2] and [13, Prop. 1.14].

1.2. **Yudovich spaces and modulus of continuity.** The main aim of the present paper is to establish existence and stability properties of weak solutions of the Vlasov–Poisson system (1.1), extending the results obtained in [13, 20, 22] to measure-valued solutions with spatial density belonging to *uniformly-localized Yudovich spaces*.

We consider solutions f of the system (1.1) whose spatial density ρ_f satisfies

$$\sup_{t\in[0,T]} \sup_{p\geq 1} \frac{\|\rho_f(t,\cdot)\|_{L^p}}{\Theta(p)} < +\infty \tag{1.7}$$

for some fixed increasing function $\Theta\colon [0,+\infty) \to (0,+\infty)$, called *growth function*. Note that (1.4) corresponds to Θ constant, (1.5) to $\Theta(p) = p$ and (1.6) to $\Theta(p) = p^{\frac{1}{\alpha}}$. Also notice that the behavior of $\Theta(p)$ as $p \to +\infty$ only matters. We call such densities *admissible* for the system (1.1), and we let

$$\mathcal{A}^\Theta([0,T]) = \Big\{ f \in L^\infty([0,T]; L^1(\mathbb{R}^{2d})) : \rho_f \in L^\infty([0,T]; Y^\Theta_{\mathrm{ul}}(\mathbb{R}^d)) \Big\}. \tag{1.8}$$

Here and in the following, we let

$$Y^\Theta_{\mathrm{ul}}(\mathbb{R}^d) = \left\{ f \in \bigcap_{p\in[1,+\infty)} L^p_{\mathrm{ul}}(\mathbb{R}^d)\colon \|f\|_{Y^\Theta_{\mathrm{ul}}} = \sup_{p\in[1,+\infty)} \frac{\|f\|_{L^p_{\mathrm{ul}}}}{\Theta(p)} < +\infty \right\} \tag{1.9}$$

be the *uniformly-localized Yudovich space*, where, for $p \in [1,+\infty)$,

$$L^p_{\mathrm{ul}}(\mathbb{R}^d) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)\colon \|f\|_{L^p_{\mathrm{ul}}} = \sup_{x\in\mathbb{R}^d} \|f\|_{L^p(B_1(x))} < +\infty \right\},$$

is the *uniformly-localized L^p space* on $\mathbb{R}^d$. We also define the *Yudovich space* $Y^\Theta(\mathbb{R}^d)$ as in (1.9) by dropping the subscript 'ul' everywhere. These spaces were first introduced by Yudovich [32] to provide uniqueness of unbounded weak solutions of incompressible inviscid 2-dimensional Euler's equations. We also refer to the recent works [4, 6, 28, 29].

Following [13, 20, 22], our starting point is the relation between the L^p growth condition (1.7) and the continuity of the vector field E_f, see Lemma 1.1 below. Our result encodes the log-Lipschitz regularity obtained in [20, Lem. 3.1] following from (1.4), as well as its more general version proved in [13, Lem. 2.1] concerning (1.5) and (1.6). As for Euler's equations [6], the main novelty here is that, once the spatial density ρ_f satisfies (1.7), then we can explicitly express the (*generalized*) *modulus of continuity* of E_f depending on the chosen growth function Θ, namely, $\varphi_\Theta\colon [0,+\infty) \to [0,+\infty)$ defined as

$$\varphi_\Theta(r) = \begin{cases} 0 & \text{for } r = 0, \\ r\,|\log r|\,\Theta(|\log r|) & \text{for } r \in (0, e^{-d-1}), \\ e^{-d-1}\,(d+1)\,\Theta(d+1) & \text{for } r \in [e^{-d-1}, +\infty) \end{cases} \tag{1.10}$$

(the choice of the constant e^{-d-1} is irrelevant and is made for convenience only, see below). With a slight abuse of notation, we set

$$C_b^{0,\varphi_\Theta}(\mathbb{R}^d;\mathbb{R}^d) = \left\{ E \in L^\infty(\mathbb{R}^d;\mathbb{R}^d) : \sup_{x\neq y} \frac{|E(x)-E(y)|}{\varphi_\Theta(|x-y|)} < +\infty \right\}.$$

Lemma 1.1 (Modulus of continuity)**.** *If $f \in \mathcal{A}^\Theta([0,T])$, then*

$$E_f \in L^\infty([0,T]; C_b^{0,\varphi_\Theta}(\mathbb{R}^d;\mathbb{R}^d)).$$

The proof of Lemma 1.1 revisits a classical strategy for proving Morrey's estimates for Riesz-type potential operators, see [21, Chap. 8] and [22, Lem. 2.2] (for strictly related results see [8, Ths. A and B]). Here we adopt the elementary approach proposed in [6, Sec. 2], generalizing the computations done in the 2-dimensional case to any dimension.

1.3. **Weak solutions and transport equation.** A simple but quite crucial byproduct of Lemma 1.1 is that $fE_f \in L^\infty([0,T];L^1(\mathbb{R}^{2d};\mathbb{R}^d))$ whenever $f \in \mathcal{A}^\Theta([0,T])$. This allows us to define weak solution of the system (1.1) among admissible densities, as follows.

Definition 1.2 (Admissible weak solution)**.** We say that $f \in \mathcal{A}^\Theta([0,T])$ is an *admissible weak solution* of the system (1.1) starting from the initial datum $f_0 \in L^1(\mathbb{R}^{2d})$ if

$$\int_0^T \int_{\mathbb{R}^{2d}} (\partial_t \psi + v \cdot \nabla_x \psi + E_f \cdot \nabla_v \psi)\, f \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = -\int_{\mathbb{R}^{2d}} \psi(0,\cdot)\, f_0 \,\mathrm{d}x\,\mathrm{d}v$$

for any $\psi \in C_c^\infty([0,T) \times \mathbb{R}^{2d})$.

Due to the structure of the system (1.1), one is tempted to look for weak solutions $f \in \mathcal{A}^\Theta([0,T])$ transported along the flow of the vector field $b_f \colon [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$,

$$b_f(t,x,v) = (v, E_f(t,x)) \quad \text{for } t \in [0,T],\ x,v \in \mathbb{R}^d. \tag{1.11}$$

The Cauchy problem corresponding to the vector field b_f in (1.11) is in fact a second-order ODE that can be rewritten in the form

$$\begin{cases} \dot X = V, & \text{for } t \in (0,T), \\ \dot V = E_f(t,X), & \text{for } t \in (0,T), \\ X(0) = x,\ V(0) = v, \end{cases} \tag{1.12}$$

where $t \mapsto (X(t),V(t))$ is any flow line starting from the initial datum $(x,v) \in \mathbb{R}^{2d}$. Since the modulus of continuity of b_f in (1.11) uniquely depends on φ_Θ in (1.10), which, in turn, only depends on the choice of Θ, here and in the rest of the paper we make the following

Assumption 1.3. *The growth function Θ is such that φ_Θ is continuous on* $[0,+\infty)$.

Consequently, given a weak solution $f \in \mathcal{A}^\Theta([0,T])$, in virtue of Lemma 1.1 and Peano's Theorem, the Cauchy problem (1.12) is well posed and admits a (classical) globally-defined, possibly non-unique, flow $\Gamma_f \colon [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$.

Definition 1.4 (Admissible Lagrangian weak solution)**.** We say that $f \in \mathcal{A}^\Theta([0,T])$ is an admissible *Lagrangian* weak solution of the system (1.1) starting from the initial datum $f_0 \in L^1(\mathbb{R}^{2d})$ if f is as in Definition 1.2 and, moreover,

$$f(t,\cdot) = (\Gamma_f(t,\cdot))_\# f_0 \quad \text{for all } t \geq 0, \tag{1.13}$$

where Γ_f is any flow solving the Cauchy problem (1.12).

A natural way to ensure the well-posedness of the ODE in (1.12) is to impose the *Osgood condition* on the modulus of (spatial) continuity of b_f in (1.11). However, due to the special second-order structure of (1.12), such condition can be considerably relaxed.

Theorem 1.5 (ODE well-posedness)**.** *Under Assumption 1.3, problem* (1.12) *admits a globally-defined classical solution. Moreover, if $\Phi_\Theta \colon [0,+\infty) \to [0,+\infty)$, given by*

$$\Phi_\Theta(r) = \int_0^r \varphi_\Theta(s)\,\mathrm{d}s \quad \textit{for all } r \geq 0, \tag{1.14}$$

satisfies

$$\int_{0^+} \frac{\mathrm{d}r}{\sqrt{\Phi_\Theta(r)}} = +\infty, \tag{1.15}$$

then the solution of problem (2.8) *is unique and the induced flow is a measure-preserving homeomorphism on* $\mathbb{R}^{2d}$ *at each time.*

Assumption (1.15) imposes the Osgood condition on $\sqrt{\Phi_\Theta}$ and can be seen as a second-order-type Osgood condition on φ_Θ. Indeed, taking $d=1$, $X(0)=V(0)=0$ and $E_f(t,x)=\varphi_\Theta(x)$ in (1.12) for simplicity, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot X^2}{2}=\varphi_\Theta(X)\,\dot X\quad\text{for } t\in(0,T),$$

so that, by integrating and changing variables, we get

$$\dot X^2(t)=2\int_0^t\varphi_\Theta(X(s))\,\dot X(s)\,\mathrm{d}s=2\Phi_\Theta(X(t))\quad\text{for all } t\in(0,T). \tag{1.16}$$

Hence uniqueness of solutions of the ODE (1.12) should follow as soon as

$$\int_{0+}\frac{\dot X(t)\,\mathrm{d}t}{\sqrt{\Phi_\Theta(X(t))}}=\int_{0+}\frac{\mathrm{d}r}{\sqrt{\Phi_\Theta(r)}}=+\infty,$$

leading to (1.15). Note that (1.16) involves the (square of the) velocity $V=\dot X$ of the trajectory, besides its position X, since in fact X solves a second-order ODE, namely, $\ddot X=E_f(t,X)$. This explains why (1.15) should be seen as a second-order Osgood condition on the modulus of continuity of the vector field E_f.

1.4. **Lagrangian stability.** Our first main result exploits the ODE well-posedness in Theorem 1.5 to provide stability of admissible Lagrangian weak solutions of the Vlasov–Poisson system (1.1), see Theorem 1.6 below, generalizing [22, Th. 1.1] and [13, Th. 1.9].

Due to the physical meaning of the problem (1.1) when $d=3$, we restrict our attention to non-negative densities $f\ge 0$ and, up to (non-linearly) rescaling all estimates, we shall work with probability densities. More precisely, we operate within the space of *probability measures with finite* 1*-moment* on $\mathbb{R}^{2d}$,

$$\mathscr{P}_1(\mathbb{R}^{2d})=\left\{\mu\in\mathscr{P}(\mathbb{R}^{2d}):\int_{\mathbb{R}^{2d}}|p|\,\mathrm{d}\mu(p)<+\infty\right\}.$$

Such space can be naturally endowed with the 1-*Wasserstein distance*, given by

$$\mathsf{W}_1(\mu_1,\mu_2)=\inf\left\{\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}|p-q|\,\mathrm{d}\pi(p,q):\pi\in\mathsf{Plan}(\mu_1,\mu_2)\right\} \tag{1.17}$$

for $\mu_1,\mu_2\in\mathscr{P}_1(\mathbb{R}^{2d})$. Here

$$\mathsf{Plan}(\mu_1,\mu_2)=\left\{\pi\in\mathscr{P}(\mathbb{R}^{2d}\times\mathbb{R}^{2d}):(\mathsf{p}_i)_\#\pi=\mu_i,\ i=1,2\right\}$$

denotes the set of *plans* (or *couplings*) between μ_1 and μ_2, where $\mathsf{p}_i\colon\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ is the projection on the i-th component. As well-known [1], there exist *optimal* plans $\pi\in\mathsf{Plan}(\mu_1,\mu_2)$, i.e., plans attaining the infimum in (1.17), and the resulting 1-*Wasserstein space* $(\mathscr{P}_1(\mathbb{R}^{2d}),\mathsf{W}_1)$ is a complete and separable metric space.

Theorem 1.6 (Lagrangian stability)**.** *Assume that* φ_Θ *is concave on* $[0,+\infty)$ *and* Φ_Θ *satisfies* (1.15)*. There is* $\Omega_{\Theta,T}\colon[0,+\infty)\to[0,+\infty)$ *continuous, with* $\Omega_{\Theta,T}(0)=0$*, satisfying the following property. Let* $i=1,2$ *and let* $f_i\in\mathcal{A}^\Theta([0,T])$ *be a Lagrangian weak solution of the Vlasov–Poisson system* (1.1) *starting from the initial datum* $f_0^i\in L^1(\mathbb{R}^{2d})$*. If* $\mu_0^i=f_0^i\mathscr{L}^{2d}\in\mathscr{P}_1(\mathbb{R}^{2d})$*, then also* $\mu_i(t,\cdot)=f_i(t,\cdot)\mathscr{L}^{2d}\in\mathscr{P}_1(\mathbb{R}^{2d})$ *for all* $t\in[0,T]$ *and*

$$\sup_{t\in[0,T]}\mathsf{W}_1(\mu_1(t,\cdot),\mu_2(t,\cdot))\le\Omega_{\Theta,T}(\mathsf{W}_1(\mu_0^1,\mu_0^2)).$$

In particular, if $f_0^1=f_0^2$*, then also* $f_1(t,\cdot)=f_2(t,\cdot)$ *for all* $t\in[0,T]$*.*

The function $\Omega_{\Theta,T}$ appearing in Theorem 1.6 can be actually made more explicit and, basically, it depends on the inverse of the function $\Psi_{\Theta,\delta,c}\colon [0,+\infty)\to[0,+\infty)$,

$$\Psi_{\Theta,\delta,c}(t)=\int_0^t \frac{\mathrm{d}s}{\delta+\sqrt{2c\,\Phi_\Theta(s)}}\quad \text{for all } t\ge 0,$$

for suitably chosen parameters $\delta, c>0$.

The proof of Theorem 1.6 follows the elementary strategy introduced in [6] for the well-posedness of 2-dimensional Euler's equations (we also refer to recent applications of this method to *transport–Stokes equations* [14] and to systems of *non-local* continuity equations [15]). Basically, to control the distance between two Lagrangian weak solutions of the system (1.1) in $\mathcal{A}^\Theta([0,T])$, in view of (1.13), we just need to control the time evolution of the distance between the initial data along the flows of the corresponding Cauchy problem (1.12) via a Grönwall-type argument, exploiting both the stability of trajectories solving the associated ODE (1.12) given by Theorem 1.5 and the modulus of continuity of the vector field provided by Lemma 1.1.

Actually, our approach is more general and in fact provides stability of admissible Lagrangian weak solutions for a large family of system like (1.1). More precisely, we can deal with *generalized Vlasov–Poisson equations* of the form

$$\begin{cases} \partial_t f+F\cdot\nabla_x f+E_f\cdot\nabla_v f=0 & \text{in } (0,T)\times\mathbb{R}^{2d},\\ E_f(t,x)=\displaystyle\int_{\mathbb{R}^d} K(x,y)\,\rho_f(t,y)\,\mathrm{d}y & \text{for } t\in[0,T],\ x\in\mathbb{R}^d,\\ \rho_f(t,x)=\displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v & \text{for } t\in[0,T],\ x\in\mathbb{R}^d,\\ f(0,\cdot)=f_0 & \text{on } \mathbb{R}^{2d}, \end{cases} \tag{1.18}$$

where $F\in L^\infty([0,T];C(\mathbb{R}^{2d};\mathbb{R}^d))$ satisfies

$$\operatorname*{ess\,sup}_{t\in[0,T]}|F(t,x,v)-F(t,y,w)|\le L\,[|x-y|+|v-w|]\quad \text{for all } x,y,v,w\in\mathbb{R}^d$$

for some $L\ge 0$, and $K\colon\mathbb{R}^{2d}\to\mathbb{R}^d$ is any sufficiently well-behaved antisymmetric kernel.

The choice $F(t,x,v)=\frac{v}{\sqrt{1+|v|^2}}$ for $t\in[0,T]$ and $x,v\in\mathbb{R}^d$ in (1.18) corresponds to the *relativistic* Vlasov–Poisson equations. The well-posedness theory in the relativistic framework is less understood. For $d=3$ and only in the attractive case, global existence of solutions has been established in [10–12, 17, 31] for radially symmetric initial data. For both the attractive and the repulsive case, well-posedness—global for $d=2$ and only local for $d=3$—and propagation of regularity for general initial data have been recently obtained in [18] via propagation of velocity moments.

1.5. **Existence of Lagrangian solutions.** Our second main result provides existence of admissible Lagrangian weak solutions of the Vlasov–Poisson system (1.1), generalizing the constructions in [22, Ths. 1.2 and 1.3] and [13, Prop. 1.14].

Theorem 1.7 (Existence)**.** *Let $d=2,3$. Let $\theta\in Y^\Theta(\mathbb{R}^d)$ be such that*

$$\theta\not\equiv 0,\quad \theta\ge 0\quad \textit{and}\quad \int_{\mathbb{R}^d}(1\vee|x|)\,\theta(x)\,\mathrm{d}x<+\infty, \tag{1.19}$$

There exists a Lagrangian weak solution $f\in\mathcal{A}^\Theta([0,T])$ of the Vlasov–Poisson system (1.1)*, starting from the initial datum*

$$f_0(x,v)=\frac{\mathbf{1}_{(-\infty,0]}\left(|v|^2-\theta(x)^{\frac{2}{d}}\right)}{|B_1|\,\|\theta\|_{L^1}},\quad \textit{for } x,v\in\mathbb{R}^d,$$

such that $f(t,\cdot)\,\mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$ *for all* $t \in [0,T]$ *and*

$$C\,\|\theta\|_{L^p} \le \|\rho_f\|_{L^\infty([0,T];L^p)} \le C_T \|\theta\|_{L^p} \quad \textit{for all } p \in [1,+\infty)\,,$$

for some constants $C, C_T > 0$*, where* C_T *depends on* T*.*

The construction behind Theorem 1.7 builds upon the proofs of [22, Ths. 1.2 and 1.3] and essentially applies the existence result proved in [19, Th. 1] to a suitable initial datum depending on the chosen function $\theta \in Y^\Theta(\mathbb{R}^d)$.

Note that any (non-zero) non-negative bounded and compactly supported function satisfies (1.19). Hence Theorem 1.7 becomes truly interesting if θ also satisfies

$$\inf_{p\ge 1} \frac{\|\theta\|_{L^p}}{\Theta(p)} > 0, \tag{1.20}$$

that is, the L^p norm of θ grows as fast as Θ. In view of Theorem 1.6, we may restrict our attention only to growth functions Θ for which φ_Θ is concave and condition (1.15) is met. This is in fact the case for a countable family of growth functions of iterated-logarithmic type defined as follows. For each $m \in \mathbb{N}$, we let $\Theta_m \colon [0,+\infty) \to [0,+\infty)$ be given by

$$\Theta_m(p) = \begin{cases} p\,|\log_1(p)|^2\,|\log_2(p)|^2 \cdots |\log_m(p)|^2 & \text{for } p \ge \exp_m(1), \\ \Theta_m(\exp_m(1)) & \text{for } p \in [0, \exp_m(1)], \end{cases}$$

where $\exp_0(1) = 1$ and $\exp_{m+1}(1) = e^{\exp_m(1)}$ recursively, and

$$\log_m = \begin{cases} \mathrm{id} & \text{for } m = 0 \\ \underbrace{\log\log\cdots\log}_{(m-1)\ \text{times}} |\log| & \text{for } m \ge 1. \end{cases} \tag{1.21}$$

Proposition 1.8 (Saturation of Θ_m)**.** *For each* $m \in \mathbb{N}_0$*,* φ_{Θ_m} *is concave,* Φ_{Θ_m} *satisfies* (1.15) *and there is* $\theta_m \in Y^{\Theta_m}(\mathbb{R}^d)$ *with compact support satisfying* (1.19) *and* (1.20)*.*

Theorem 1.7, together with Proposition 1.8, yield that the class of admissible Lagrangian weak solutions considered in Theorem 1.6 is non-empty for $d \in \{2,3\}$ and $\Theta = \Theta_m$ for some $m \in \mathbb{N}_0$. When $m = 0$, our results embed the example given in the proof of [22, Th. 1.3]. Actually, the functions θ_m in Proposition 1.8 are modeled on a well-known example due to Yudovich (see [32, Eq. (3.7)], [28, Rem. 1(i)] and the discussion around [6, Eq. (1.12)]) concerning 2-dimensional Euler equations in vorticity form.

1.6. **Organization of the paper.** In Section 2 we provide an abstract approach to achieve the well-posedness of the Cauchy problem (1.12) and the stability of admissible Lagrangian weak solutions of the system (1.1), considering the generalized Vlasov–Poisson equations (1.18). We refer the reader to Theorem 2.2 and Theorem 2.8, respectively. In Section 3, we detail the proofs of the results presented above.

2. Lagrangian stability for a generalized Vlasov–Poisson system

In this section, we provide an abstract approach to obtain stability properties for Lagrangian solutions of (a generalized version of) the Vlasov–Poisson system (1.1). Our stability result is stated in Theorem 2.8 and exploits the well-posedness of the corresponding second-order Cauchy problem provided by Theorem 2.2.

2.1. **Notation.** Throughout this section, we consider

$$\varphi \in C([0,+\infty);[0,+\infty)), \quad \text{with } \varphi(t)>0 \text{ for } t>0. \tag{2.1}$$

We also let $\Phi\colon [0,+\infty)\to[0,+\infty)$ be given by

$$\Phi(t)=\int_0^t \varphi(s)\,\mathrm{d}s \quad \text{for all } t\ge 0. \tag{2.2}$$

Note that Φ is a non-negative and non-decreasing C^1 function. For certain results we will also assume that Φ satisfies the additional condition

$$\int_{0^+} \frac{\mathrm{d}t}{\sqrt{\Phi(t)}} = +\infty, \tag{2.3}$$

i.e., the function $\sqrt{\Phi}$ satisfies the Osgood condition. Clearly, condition (2.3) implies that $\varphi(0)=0$. Given $\delta, c>0$, we also define the function $\Psi_{\delta,c}\colon[0,+\infty)\to[0,+\infty)$ by

$$\Psi_{\delta,c}(t)=\int_0^t \frac{\mathrm{d}s}{\delta+\sqrt{2c\,\Phi(s)}} \quad \text{for all } t\ge 0.$$

To keep the notation short, we set $\Psi_\delta=\Psi_{\delta,1}$. Note that $\Psi_{\delta,c}$ is a non-negative and strictly increasing C^1 function with bounded derivative. In particular, $\Psi_{\delta,c}$ is invertible, with continuous and strictly-increasing inverse. Note that, if (2.3) is assumed, then

$$\lim_{\delta\to0^+}\Psi_{\delta,c}(t)=+\infty \quad \text{and} \quad \lim_{\delta\to0^+}\Psi_{\delta,c}^{-1}(t)=0 \quad \text{for all } t,c>0.$$

2.2. **Second-order Grönwall's inequality.** We begin with the following result, which may be considered as a Grönwall-type lemma for a second-order differential inequality.

Lemma 2.1 (Grönwall)**.** *Let $u\in W^{2,\infty}([0,T])$ be such that $u,u'\ge0$. If*

$$u''\le cu'+\varphi(u) \quad \text{a.e. in } [0,T] \tag{2.4}$$

for some $c>0$ and $u'(0)\le\delta$ for some $\delta>0$, then

$$u'(t)\le e^{ct}\big(\delta+\sqrt{2\Phi(u(t))}\big) \quad \text{and} \quad u(t)\le\Psi_\delta^{-1}\Big(\Psi_\delta(u(0))+e^{ct}-1\Big)$$

for all $t\in[0,T]$.

Proof. Multiplying (2.4) by $u'\ge0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[(u')^2\Big]\le 2c(u')^2+2\varphi(u)u' \quad \text{a.e. in } [0,T].$$

Integrating and changing variables, we can estimate

$$\begin{aligned}(u'(t))^2 &\le (u'(0))^2+2\Phi(u(t))-2\Phi(u(0))+2c\int_0^t(u'(s))^2\,\mathrm{d}s\\ &\le \delta^2+2\Phi(u(t))+2c\int_0^t(u'(s))^2\,\mathrm{d}s\end{aligned}$$

for all $t\in[0,T]$. Since $t\mapsto\Phi(u(t))$ is non-decreasing, by Grönwall's inequality we get

$$(u'(t))^2\le e^{2ct}\Big(\delta^2+2\Phi(u(t))\Big) \quad \text{for all } t\in[0,T],$$

so that

$$\frac{u'(t)}{\delta+\sqrt{2\Phi(u(t))}}\le e^{ct} \quad \text{for all } t\in[0,T].$$

Integrating the above inequality, we conclude that

$$\Psi_\delta(u(t))-\Psi_\delta(u(0))\le e^{ct}-1 \quad \text{for all } t\in[0,T],$$

from which the conclusion follows immediately. □

2.3. **Second-order Cauchy problem.** We let $b\colon [0,T]\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ be given by

$$b(t,x,v) = \big(F(t,x,v), E(t,x)\big) \quad \text{for } t\in[0,T],\ x,v\in\mathbb{R}^d, \tag{2.5}$$

where $E\in L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))$ satisfies

$$\operatorname*{ess\,sup}_{t\in[0,T]} |E(t,x)-E(t,y)| \le \varphi(|x-y|) \quad \text{for all } x,y\in\mathbb{R}^d \tag{2.6}$$

with φ as in (2.1) and $F\in L^\infty([0,T];C(\mathbb{R}^{2d};\mathbb{R}^d))$ satisfies

$$\operatorname*{ess\,sup}_{t\in[0,T]} |F(t,x,v)-F(t,y,w)| \le L\,[|x-y|+|v-w|] \quad \text{for all } x,y,v,w\in\mathbb{R}^d \tag{2.7}$$

for some fixed $L\in[0,+\infty)$. For any given $x,v\in\mathbb{R}^d$, we consider the Cauchy problem

$$\begin{cases} \dot\gamma_{x,v} = b(t,\gamma_{x,v}), & \text{for } t\in(0,T),\\ \gamma(0)=(x,v). \end{cases} \tag{2.8}$$

Note that (2.8) is in fact a second-order Cauchy problem and can be rewritten as

$$\begin{cases} \dot X = F(t,X,V), & \text{for } t\in(0,T),\\ \dot V = E(t,X), & \text{for } t\in(0,T),\\ X(0)=x,\ V(0)=v, \end{cases} \tag{2.9}$$

denoting $\gamma_{x,v}(t) = (X(t,x,v),V(t,x,v))$ for $t\in[0,T]$, $x,v\in\mathbb{R}^d$.

Theorem 2.2 (ODE well-posedness). *Problem* (2.8) *admits a globally-defined classical solution* $\gamma_{x,v}\in W^{1,\infty}([0,T];\mathbb{R}^{2d})$ *for all* $x,v\in\mathbb{R}^d$. *Moreover, if* Φ *in* (2.2) *satisfies condition* (2.3), *then the solution of* (2.8) *is unique for all* $x,v\in\mathbb{R}^d$. *Finally, letting*

$$\Gamma\colon [0,T]\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}$$

with $\Gamma(t,x,v)=\gamma_{x,v}(t)$ *for* $t\in[0,T]$, $x,v\in\mathbb{R}^d$, *be the associated flow map, if* $\operatorname{div}_x F=0$, *then* $\Gamma(t,\cdot)$ *is a measure-preserving homeomorphism on* $\mathbb{R}^{2d}$ *for all* $t\in[0,T]$.

Since $b\in L^\infty([0,T];C(\mathbb{R}^{2d};\mathbb{R}^{2d}))$ has at most linear growth, the first part of Theorem 2.2 concerning the global existence of at least one solution of (2.8) follows by standard ODE Theory (namely, by Peano's Theorem and Grönwall's inequality). Instead, the validity of the second part of Theorem 2.2 concerning the uniqueness of the solution of (2.8) and the measure-preserving property of the associated flow map follows from the following result.

Proposition 2.3 (ODE stability). *Let* $i=1,2$, *let* $b_i=(F_i,E_i)$ *be as in* (2.5), *with* $E_i\in L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))$ *satisfying* (2.6) *and* $F_i\in L^\infty([0,T];C(\mathbb{R}^{2d};\mathbb{R}^d))$ *satisfying* (2.7), *and let* $\gamma_i=(X_i,V_i)\in W^{1,\infty}([0,T];\mathbb{R}^{2d})$ *be a solution of* (2.8) *with initial condition* $(x_i,v_i)\in\mathbb{R}^{2d}$. *If*

$$L|x_1-x_2| + L|v_1-v_2| + L\|E_1-E_2\|_{L^\infty(C)} + \|F_1-F_2\|_{L^\infty(C)} \le \delta$$

for some $\delta>0$, *then*

$$\begin{aligned} \|\gamma_1-\gamma_2\|_{L^\infty} \le |v_1-v_2| + \|E_1-E_2\|_{L^\infty} + \Psi_{\delta,L}^{-1}\left(\Psi_{\delta,L}(|x_1-x_2|)+e^{LT}-1\right)\\ + T\varphi\left(\Psi_{\delta,L}^{-1}\left(\Psi_{\delta,L}(|x_1-x_2|)+e^{LT}-1\right)\right). \end{aligned}$$

Proof. In the following, we drop the spatial variables to keep the notation short. In virtue of (2.7) and (2.9), we can estimate

$$\begin{aligned}|X_1(t) - X_2(t)| &\le |x_1 - x_2| + \int_0^t |F_1(s, X_1(s), V_1(s)) - F_2(s, X_2(s), V_2(s))| \,\mathrm{d}s \\ &\le |x_1 - x_2| + \int_0^t |F_1(s, X_1(s), V_1(s)) - F_1(s, X_2(s), V_2(s))| \,\mathrm{d}s \\ &\quad + \int_0^t |F_1(s, X_2(s), V_2(s)) - F_2(s, X_2(s), V_2(s))| \,\mathrm{d}s \\ &\le |x_1 - x_2| + L\int_0^t |X_1(s) - X_2(s)| \,\mathrm{d}s + L\int_0^t |V_1(s) - V_2(s)| \,\mathrm{d}s + t\|F_1 - F_2\|_{L^\infty}\end{aligned} \tag{2.10}$$

for all $t \in [0, T]$. Because of (2.6) and again of (2.9), we can also estimate

$$\begin{aligned}|V_1(s) - V_2(s)| &\le |v_1 - v_2| + \int_0^s |E_1(r, X_1(r)) - E_2(r, X_2(r))| \,\mathrm{d}r \\ &\le |v_1 - v_2| + \int_0^s |E_1(r, X_1(r)) - E_1(r, X_2(r))| \,\mathrm{d}r \\ &\quad + \int_0^s |E_1(r, X_2(r)) - E_2(r, X_2(r))| \,\mathrm{d}r \\ &\le |v_1 - v_2| + \|E_1 - E_2\|_{L^\infty} + \int_0^s \varphi(|X_1(r) - X_2(r)|) \,\mathrm{d}r\end{aligned} \tag{2.11}$$

for all $s \in [0, T]$. Therefore, we obtain that

$$\begin{aligned}|X_1(t) - X_2(t)| &\le |x_1 - x_2| + t\,[L|v_1 - v_2| + L\|E_1 - E_2\|_{L^\infty} + \|F_1 - F_2\|_{L^\infty}] \\ &\quad + L\int_0^t |X_1(s) - X_2(s)| \,\mathrm{d}s + L\int_0^t \int_0^s \varphi(|X_1(r) - X_2(r)|) \,\mathrm{d}r \,\mathrm{d}s\end{aligned} \tag{2.12}$$

for all $t \in [0, T]$. Letting $u \in W^{2,\infty}([0, T])$ be the function in the right-hand side of (2.12), we observe that $u \ge 0$, $u(0) = |x_1 - x_2|$,

$$\begin{aligned}u'(t) &= L|v_1 - v_2| + L\|E_1 - E_2\|_{L^\infty} + \|F_1 - F_2\|_{L^\infty} + L|X_1(t) - X_2(t)| \\ &\quad + L\int_0^t \varphi(|X_1(s) - X_2(s)|) \,\mathrm{d}s,\end{aligned} \tag{2.13}$$

for all $t \in [0, T]$ and so, in particular,

$$u'(0) = L|x_1 - x_2| + L|v_1 - v_2| + L\|E_1 - E_2\|_{L^\infty} + \|F_1 - F_2\|_{L^\infty} \le \delta.$$

We also observe that

$$u''(t) \le L|\dot{X}_1(t) - \dot{X}_2(t)| + L\varphi(|X_1(t) - X_2(t)|) \quad \text{for a.e. } t \in [0, T]. \tag{2.14}$$

We now estimate the right-hand side of (2.14) in terms of u. Exploiting (2.7), (2.9) and the estimate in (2.11), we have

$$\begin{aligned}|\dot{X}_1(t) - \dot{X}_2(t)| &= |F_1(t, X_1(t), V_1(t)) - F_2(t, X_2(t), V_2(t))| \\ &\le \|F_1(t) - F_2(t)\|_C + L|X_1(t) - X_2(t)| + L|V_1(t) - V_2(t)| \\ &\le \|F_1 - F_2\|_{L^\infty} + L|X_1(t) - X_2(t)| + L|v_1 - v_2| \\ &\quad + L\|E_1 - E_2\|_{L^\infty} + L\int_0^t \varphi(|X_1(s) - X_2(s)|) \,\mathrm{d}s \\ &= u'(t)\end{aligned}$$

for all $t \in [0, T]$ in virtue of (2.13). We thus get that u satisfies

$$u'' \le Lu' + L\varphi(u) \quad \text{a.e. in } [0, T],$$

as in (2.4) in Lemma 2.1, from which we immediately get that

$$|X_1(t) - X_2(t)| \le \Psi^{-1}_{\delta,L}\left(\Psi_{\delta,L}(|x_1 - x_2|) + e^{Lt} - 1\right)$$

for all $t \in [0,T]$. Consequently, by (2.11), we also find that

$$|V_1(t) - V_2(t)| \le |v_1 - v_2| + \|E_1 - E_2\|_{L^\infty} + t\,\varphi\left(\Psi^{-1}_{\delta,L}\left(\Psi_{\delta,L}(|x_1 - x_2|) + e^{LT} - 1\right)\right)$$

for all $t \in [0,T]$, from which the conclusion immediately follows. □

From Proposition 2.3, we plainly deduce the following approximation result.

Corollary 2.4 (ODE convergence)**.** *Let $n \in \mathbb{N}$, let $b = (F,E), b_n = (F_n, E_n)$ be as in* (2.5)*, with $E, E_n \in L^\infty([0,T]; C_b(\mathbb{R}^d;\mathbb{R}^d))$ satisfying* (2.6) *and $F, F_n \in L^\infty([0,T]; C(\mathbb{R}^{2d};\mathbb{R}^d))$ satisfying* (2.7)*, and let $\gamma_n = (X_n, V_n) \in W^{1,\infty}([0,T];\mathbb{R}^{2d})$ be a solution of* (2.8) *with initial condition $(x,v) \in \mathbb{R}^{2d}$. If Φ in* (2.2) *satisfies* (2.3) *and*

$$\lim_{n\to+\infty} \|b_n - b\|_{L^\infty} = 0, \tag{2.15}$$

then $(\gamma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]\times\mathbb{R}^{2d})$, and each of its limit points $\gamma = (X,V)$ is a solution of (2.8) *relative to $b = (F,E)$ with initial condition (x,v).*

Proof. By Proposition 2.3, we immediately infer that

$$\|\gamma_m - \gamma_n\|_{L^\infty} \le \delta_{m,n} + \Psi^{-1}_{\delta_{m,n},L}(e^{LT} - 1) + T\varphi\left(\Psi^{-1}_{\delta_{m,n},L}(e^{LT} - 1)\right).$$

for all $m, n \in \mathbb{N}$, where

$$\delta_{m,n} = \|E_m - E_n\|_{L^\infty} + \|F_m - F_n\|_{L^\infty} + \tfrac{1}{m} + \tfrac{1}{n}.$$

Since $\delta_{m,n} \to 0^+$ as $m,n \to +\infty$, by (2.3) we infer that $\Psi^{-1}_{\delta_{m,n},L}(e^{LT} - 1) \to 0^+$ as $m,n \to +\infty$, easily yielding the conclusion. □

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. We just need to deal with the second part of the statement concerning the uniqueness of the solution of (2.8) and the measure-preserving property of the associated flow map. The uniqueness part is an immediate consequence of Proposition 2.3. Indeed, if γ_1 and γ_2 are two solutions of (2.8) relative to b starting from the same initial datum (x,v), with $x, v \in \mathbb{R}^n$, then Proposition 2.3 implies that

$$\|\gamma_1 - \gamma_2\|_{L^\infty} \le \Psi^{-1}_{\delta,L}(e^{LT} - 1) + T\varphi\left(\Psi^{-1}_{\delta,L}(e^{LT} - 1)\right)$$

for all $\delta > 0$. Since $\Psi^{-1}_{\delta,L}(e^{LT} - 1) \to 0^+$ as $\delta \to 0^+$, we get $\gamma_1 = \gamma_2$. The measure-preserving property of the associated flow map, instead, follows from an approximation argument and Corollary 2.4. We leave the simple details to the reader. □

2.4. **Generalized Vlasov–Poisson system.** From now on, we fix a measurable function $K\colon \mathbb{R}^{2d} \to \mathbb{R}^d$, that we call *kernel*, which is assumed to be antisymmetric, i.e., $K(x,y) = -K(x,y)$ for a.e. $x, y \in \mathbb{R}^d$. We thus consider the associated Vlasov–Poisson-type system

$$\begin{cases} \partial_t f + F\cdot\nabla_x f + E_f\cdot\nabla_v f = 0 & \text{in } (0,T)\times\mathbb{R}^{2d},\\[1ex] E_f(t,x) = \displaystyle\int_{\mathbb{R}^d} K(x,y)\,\rho_f(t,y)\,\mathrm{d}y & \text{for } t\in[0,T],\ x\in\mathbb{R}^d,\\[1ex] \rho_f(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v & \text{for } t\in[0,T],\ x\in\mathbb{R}^d,\\[1ex] f(0,\cdot) = f_0 & \text{on } \mathbb{R}^{2d}, \end{cases} \tag{2.16}$$

where the unknown density is $f \in L^\infty([0,T];L^1(\mathbb{R}^{2d}))$ and the initial datum is $f_0 \in L^1(\mathbb{R}^{2d})$. The function $F \in L^\infty([0,T];C(\mathbb{R}^{2d};\mathbb{R}^d))$ in the first line of (2.16) always satisfies (2.7), and may be additionally assumed to satisfy $\operatorname{div}_x F = 0$. If $F(t,x,v) = v$, then (2.16) reduces to the classical Vlasov–Poisson system, while, if $F(t,x,v) = \frac{v}{\sqrt{1+|v|^2}}$, then (2.16) becomes the relativistic Vlasov–Poisson system.

Definition 2.5 (Weak φ-solution)**.** We say that $f \in L^\infty([0,T];L^1(\mathbb{R}^{2d}))$ is a *weak φ-solution* of (2.16) with initial datum $f_0 \in L^1(\mathbb{R}^{2d})$ if

$$(t,x) \mapsto \int_{\mathbb{R}^d} |K(x,z)|\,|\rho_f(t,z)|\,\mathrm{d}z \in L^\infty([0,T]\times\mathbb{R}^d), \tag{2.17}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^d} |K(x,z)-K(y,z)|\,|\rho_f(t,z)|\,\mathrm{d}z \le \varphi(|x-y|) \quad \text{for all } x,y\in\mathbb{R}^d \tag{2.18}$$

and

$$\int_0^T \int_{\mathbb{R}^{2d}} (\partial_t\psi + F\cdot\nabla_x\psi + E_f\cdot\nabla_v\psi)\, f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = -\int_{\mathbb{R}^{2d}} \psi(0,\cdot)\, f_0\,\mathrm{d}x\,\mathrm{d}v \tag{2.19}$$

for all $\psi \in C_c^\infty([0,T)\times\mathbb{R}^{2d})$, where E_f, ρ_f are as in (2.16).

Note that, if f is a weak φ-solution of (2.16) as in Definition 2.5, then (2.17) and (2.18) lead to $E_f \in L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))$ satisfying (2.6). In particular, the equation (2.19) is well defined, since $fE_f \in L^\infty([0,T];L^1(\mathbb{R}^{2d};\mathbb{R}^d))$ thanks to (2.17).

Definition 2.6 (Lagrangian weak φ-solution)**.** We say that $f \in L^\infty([0,T];L^1(\mathbb{R}^{2d}))$ is a *Lagrangian weak φ-solution* of (2.16) with initial datum $f_0 \in L^1(\mathbb{R}^{2d})$ if f is a weak φ-solution of (2.16) as in Definition 2.5 and, moreover,

$$f(t,\cdot) = \Gamma(t,\cdot)_\# f_0 \quad \text{for all } t\in[0,T], \tag{2.20}$$

where Γ is any flow map associated to the Cauchy problem (2.8) with $b=(F,E)$.

The following result collects two basic features of Lagrangian weak φ-solutions of (2.16) that will be useful in the sequel.

Lemma 2.7 (Sign and moment preservation)**.** *Assume* $\operatorname{div}_x F = 0$ *and* Φ *in* (2.2) *satisfies* (2.3)*. Let* $f \in L^\infty([0,T];L^1(\mathbb{R}^{2d}))$ *be a Lagrangian weak φ-solution of* (2.16) *with initial datum* $f_0 \in L^1(\mathbb{R}^{2d})$*. If* $f_0 \ge 0$*, then also* $f(t,\cdot)\ge 0$ *for all* $t\in[0,T]$*. Moreover, if* $\mu_0 = f_0\,\mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$*, then also* $\mu(t,\cdot) = f(t,\cdot)\,\mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$ *for all* $t\in[0,T]$*.*

Proof. Fix $t\in[0,T]$. Since $\Gamma(t,\cdot)$ is a measure-preserving homeomorphism by Proposition 2.3, then from (2.20) we easily deduce that

$$\begin{aligned}\mathscr{L}^{2d}\left(\left\{z\in\mathbb{R}^{2d} : f(t,z)<0\right\}\right) &= \mathscr{L}^{2d}\left(\left\{z\in\mathbb{R}^{2d} : f(t,\Gamma(t,z))<0\right\}\right)\\ &= \mathscr{L}^{2d}\left(\left\{z\in\mathbb{R}^{2d} : f_0(z)<0\right\}\right) = 0,\end{aligned}$$

so that $f(t,\cdot)\ge 0$. In addition, if

$$\int_{\mathbb{R}^{2d}} |z|\,\mathrm{d}\mu_0(z) = \int_{\mathbb{R}^{2d}} |z|\, f_0(z)\,\mathrm{d}z < +\infty,$$

then again by (2.20) we get

$$\int_{\mathbb{R}^{2d}} |z|\,\mathrm{d}\mu(t,z) = \int_{\mathbb{R}^{2d}} |z|\, f(t,z)\,\mathrm{d}z = \int_{\mathbb{R}^{2d}} |\Gamma(t,z)|\, f_0(z)\,\mathrm{d}z < +\infty,$$

since $|\Gamma(t,z)| \le C|z|e^{CT}$ for all $t\in[0,T]$ and $z\in\mathbb{R}^{2d}$, for some $C>0$ depending on $\|E_f\|_{L^\infty}$ and $\|F\|_{L^\infty(\mathrm{Lip})}$ only, by standard ODE Theory, in virtue of (2.7) and (2.17). $\square$

We can now state and prove the main result of this section, providing a stability property for Lagrangian weak φ-solutions of the Vlasov–Poisson-type system (2.16). The proof of Theorem 2.8 adopts the elementary point of view of [6] and extends the approaches exploited in the proofs of [22, Th. 1.1] and [13, Th. 1.9].

Theorem 2.8 (Lagrangian stability)**.** *Let $i = 1, 2$, let $\mu_i \in L^\infty([0,T]; \mathscr{P}_1(\mathbb{R}^{2d}))$ be such that $\mu_i = f_i \mathscr{L}^{2d}$, where $f_i \in L^\infty([0,T]; L^1(\mathbb{R}^{2d}))$ is a Lagrangian weak φ-solution of* (2.16)*, relative to (F_i, E_i), $E_i = E_{f_i}$, with $F_i \in L^\infty([0,T]; C(\mathbb{R}^d; \mathbb{R}^d))$ satisfying* (2.7) *for some $L \in [1, +\infty)$ and $\operatorname{div}_x F_i = 0$, with initial datum $f_0^i \in L^1(\mathbb{R}^{2d})$. Assume that φ in* (2.1) *is concave and Φ in* (2.2) *satisfies* (2.3)*. If*

$$2L\,\mathsf{W}_1(\mu_0^1, \mu_0^2) + \|F_1 - F_2\|_{L^\infty} < \delta$$

for some $\delta > 0$, then

$$\begin{aligned}\mathsf{W}_1(\mu_1(t,\cdot), \mu_2(t,\cdot)) &\le \Psi_{\delta,2L}^{-1}\big(\Psi_{\delta,2L}(\mathsf{W}_1(\mu_0^1, \mu_0^2)) + e^{Lt} - 1\big) \\ &\quad + e^{Lt}\left(\delta + \sqrt{4L\Phi\big(\Psi_{\delta,2L}^{-1}\big(\Psi_{\delta,2L}(\mathsf{W}_1(\mu_0^1,\mu_0^2)) + e^{Lt} - 1\big)\big)}\right)\end{aligned}$$

for all $t \in [0,T]$. In particular, if $f_0^1 = f_0^2$ and $F_1 = F_2$, then $f_1 = f_2$.

Proof. Let $\pi_0 \in \mathsf{Plan}(\mu_0^1, \mu_0^2)$ be an optimal plan. By Definition 2.6, we can write $\mu_i(t,\cdot) = \Gamma_i(t,\cdot)_\# \mu_0^i$ for $t \in [0,T]$ and $i = 1, 2$, so that

$$\pi(t,\cdot) = (\Gamma_1(t, \mathsf{p}_1), \Gamma_2(t, \mathsf{p}_2))_\# \pi_0 \in \mathsf{Plan}(\mu_1(t,\cdot), \mu_2(t,\cdot)) \tag{2.21}$$

for all $t \in [0,T]$. Since $\Gamma_i = (X_i, V_i)$, $i = 1, 2$, we define

$$\begin{aligned}\mathcal{X}(t) &= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |X_1(t,p) - X_2(t,q)| \,\mathrm{d}\pi_0(p,q) \\ \mathcal{V}(t) &= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |V_1(t,p) - V_2(t,q)| \,\mathrm{d}\pi_0(p,q)\end{aligned} \tag{2.22}$$

for all $t \in [0,T]$, where $p = (x,v)$ and $q = (y,w)$. Arguing as in (2.10), we can estimate

$$\begin{aligned}|X_1(t,p) - X_2(t,q)| &\le |x-y| + L\int_0^t |X_1(s,p) - X_2(s,q)|\,\mathrm{d}s + L\int_0^t |V_1(s,p) - V_2(s,q)|\,\mathrm{d}s \\ &\quad + t\|F_1 - F_2\|_{L^\infty}\end{aligned}$$

for all $t \in [0,T]$, so that

$$\begin{aligned}\mathcal{X}(t) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |x-y|\,\mathrm{d}\pi_0(p,q) + t\|F_1 - F_2\|_{L^\infty} + L\int_0^t \mathcal{X}(s)\,\mathrm{d}s + L\int_0^t \mathcal{V}(s)\,\mathrm{d}s \\ &\le \mathsf{W}_1(\mu_0^1, \mu_0^2) + t\|F_1 - F_2\|_{L^\infty} + L\int_0^t \mathcal{X}(s)\,\mathrm{d}s + L\int_0^t \mathcal{V}(s)\,\mathrm{d}s\end{aligned}$$

Similarly arguing as in (2.11), we also get that

$$|V_1(t,p) - V_2(t,q)| \le |v - w| + \int_0^t |E_1(s, X_1(s,p)) - E_2(s, X_2(s,q))|\,\mathrm{d}s$$

for all $t \in [0,T]$, so that

$$\begin{aligned}\mathcal{V}(t) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |v-w|\,\mathrm{d}\pi_0(p,q) \\ &\quad + \int_0^t \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |E_1(s, X_1(s,p)) - E_2(s, X_2(s,q))|\,\mathrm{d}\pi_0(p,q)\,\mathrm{d}s \\ &\le \mathsf{W}_1(\mu_0^1, \mu_0^2) + \int_0^t \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |E_1(s, X_1(s,p)) - E_2(s, X_2(s,q))|\,\mathrm{d}\pi_0(p,q)\,\mathrm{d}s\end{aligned} \tag{2.23}$$

for all $t \in [0,T]$ and so, in particular,

$$\begin{aligned}\mathcal{X}(t) &\le (1+Lt)\,\mathsf{W}_1(\mu_0^1,\mu_0^2) + t\|F_1-F_2\|_{L^\infty} + L\int_0^t \mathcal{X}(s)\,\mathrm{d}s \\ &\quad + L\int_0^t\int_0^s\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |E_1(r,X_1(r,p)) - E_2(r,X_2(r,q))|\,\mathrm{d}\pi_0(p,q)\,\mathrm{d}r\,\mathrm{d}s\end{aligned}$$

for all $t \in [0,T]$. Now we have

$$\begin{aligned}|E_1(r,X_1(r,p)) - E_2(r,X_2(r,q))| &\le |E_1(r,X_1(r,p)) - E_1(r,X_2(r,q))| \\ &\quad + |E_1(r,X_2(r,q)) - E_2(r,X_2(r,q))|.\end{aligned}$$

On the one side, since f_1 is a weak φ-solution of (2.16) with respect to (F_1,E_1), by (2.18) E_1 satisfies (2.6), and thus we can estimate

$$|E_1(r,X_1(r,p)) - E_1(r,X_2(r,q))| \le \varphi(|X_1(r,p) - X_2(r,q)|).$$

On the other side, again since f_1 and f_2 are weak φ-solutions of (2.16), we can write

$$\begin{aligned}&|E_1(r,X_2(r,q)) - E_2(r,X_2(r,q))| \\ &\quad = \left|\int_{\mathbb{R}^d} K(X_2(r,q),z)\,\rho_1(r,z)\,\mathrm{d}z - \int_{\mathbb{R}^d} K(X_2(r,q),z')\,\rho_2(r,z')\,\mathrm{d}z'\right| \\ &\quad = \left|\int_{\mathbb{R}^{2d}} K(X_2(r,q),z)\,f_1(r,z,u)\,\mathrm{d}z\,\mathrm{d}u - \int_{\mathbb{R}^d} K(X_2(r,q),z')\,f_2(r,z',u')\,\mathrm{d}z'\,\mathrm{d}u'\right| \\ &\quad = \left|\int_{\mathbb{R}^{2d}} K(X_2(r,q),X_1(r,o))\,f_0^1(o)\,\mathrm{d}o - \int_{\mathbb{R}^d} K(X_2(r,q),X_2(r,o'))\,f_0^2(o')\,\mathrm{d}o'\right|\end{aligned}$$

where in the last equality we changed variables, in virtue of (2.20), letting $o = (z,u)$ and $o' = (z',u')$ for brevity. Since $\pi_0 \in \mathsf{Plan}(\mu_0^1,\mu_0^2)$, we can thus write

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2d}} K(X_2(r,q),X_1(r,o))\,f_0^1(o)\,\mathrm{d}o - \int_{\mathbb{R}^d} K(X_2(r,q),X_2(r,o))\,f_0^2(o')\,\mathrm{d}o'\right| \\ &\qquad = \left|\int_{\mathbb{R}^{2d}} K(X_2(r,q),X_1(r,o))\,\mathrm{d}\mu_0^1(o) - \int_{\mathbb{R}^d} K(X_2(r,q),X_2(r,o))\,\mathrm{d}\mu_0^2(o')\right| \\ &\qquad = \left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \big(K(X_2(r,q),X_1(r,o)) - K(X_2(r,q),X_2(r,o'))\big)\,\mathrm{d}\pi_0(o,o')\right| \\ &\qquad \le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |K(X_2(r,q),X_1(r,o)) - K(X_2(r,q),X_2(r,o'))|\,\mathrm{d}\pi_0(o,o')\end{aligned}$$

Therefore, again changing variables in virtue of (2.20), we get

$$\begin{aligned}&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |E_1(r,X_2(r,q)) - E_2(r,X_2(r,q))|\,\mathrm{d}\pi_0(p,q) \\ &\quad \le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |K(X_2(r,q),X_1(r,o)) - K(X_2(r,q),X_2(r,o'))|\,\mathrm{d}\pi_0(p,q)\,\mathrm{d}\pi_0(o,o') \\ &\quad = \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |K(h,X_1(r,o)) - K(h,X_2(r,o'))|\,\rho_2(t,h)\,\mathrm{d}h\,\mathrm{d}\pi_0(o,o') \\ &\quad \le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \varphi(|X_1(r,o) - X_2(r,o')|)\,\mathrm{d}\pi_0(o,o').\end{aligned}$$

Recalling that φ is concave, by Jensen's inequality we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |E_1(r,X_1(r,p)) - E_2(r,X_2(r,q))|\,\mathrm{d}\pi_0(p,q) \\ &\qquad \le 2\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \varphi(|X_1(r,p) - X_2(r,q)|)\,\mathrm{d}\pi_0(p,q) \le 2\,\varphi(\mathcal{X}(r)),\end{aligned}$$

so that

$$\mathcal{X}(t) \le (1+Lt)\,\mathsf{W}_1(\mu_0^1,\mu_0^2) + t\|F_1-F_2\|_{L^\infty} + L\int_0^t \mathcal{X}(s)\,\mathrm{d}s + 2L\int_0^t\int_0^s \varphi(\mathcal{X}(r))\,\mathrm{d}r\,\mathrm{d}s \tag{2.24}$$

for all $t\in[0,T]$. In addition, recalling (2.23), we also get that

$$\mathcal{V}(t) \le \mathsf{W}_1(\mu_0^1,\mu_0^2) + 2\int_0^t \varphi(\mathcal{X}(s))\,\mathrm{d}s \tag{2.25}$$

for all $t\in[0,T]$. Now, letting $u\in W^{2,\infty}([0,T])$ be the function on the right-hand side of (2.24), we immediately get that $u,u'\ge 0$ with $u(0)=\mathsf{W}_1(\mu_0^1,\mu_0^2)$ and

$$u'(t) = L\,\mathsf{W}_1(\mu_0^1,\mu_0^2) + \|F_1-F_2\|_{L^\infty} + L\mathcal{X}(t) + 2L\int_0^t \varphi(\mathcal{X}(s))\,\mathrm{d}s \tag{2.26}$$

for all $t\in[0,T]$, so that, $u'(0)\le 2L\,\mathsf{W}_1(\mu_0^1,\mu_0^2)+\|F_1-F_2\|_{L^\infty}$. Furthermore, we have

$$u''(t) = L\dot{\mathcal{X}}(t) + 2L\varphi(\mathcal{X}(t))$$

for a.e. $t\in(0,T)$. Note that, in virtue of the definition in (2.22) and of problem (2.9),

$$\begin{aligned}
\dot{\mathcal{X}}(t) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |\dot{X}_1(t,p)-\dot{X}_2(t,q)|\,\mathrm{d}\pi_0(p,q)\\
&= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |F_1(t,X_1(t,p),V_1(t,p)) - F_2(t,X_2(t,q),V_2(t,q))|\,\mathrm{d}\pi_0(p,q)\\
&\le \|F_1-F_2\|_{L^\infty},
\end{aligned}$$

so that, recalling (2.24) and (2.26) and since φ is non-decreasing,

$$u''(t) \le L\|F_1-F_2\|_{L^\infty} + 2L\varphi(\mathcal{X}(t)) \le Lu'(t) + 2L\varphi(u(t))$$

for a.e. $t\in(0,T)$. Thanks to Lemma 2.1, we thus conclude that, if

$$2L\,\mathsf{W}_1(\mu_0^1,\mu_0^2) + \|F_1-F_2\|_{L^\infty} < \delta$$

for some $\delta>0$, then

$$\mathcal{X}(t) \le \Psi_{\delta,2L}^{-1}\big(\Psi_{\delta,2L}(\mathsf{W}_1(\mu_0^1,\mu_0^2)) + e^{Lt}-1\big)$$

for all $t\in[0,T]$. Moreover, from (2.25) and (2.26), we also get that $\mathcal{V}(t)\le u'(t)$, so that

$$\mathcal{V}(t) \le e^{Lt}\big(\delta + \sqrt{4L\Phi(\mathcal{X}(t))}\big) \le e^{Lt}\left(\delta + \sqrt{4L\Phi\big(\Psi_{\delta,2L}^{-1}(\Psi_{\delta,2L}(\mathsf{W}_1(\mu_0^1,\mu_0^2)) + e^{Lt}-1)\big)}\right)$$

for all $t\in[0,T]$, in virtue of Lemma 2.1. To conclude, we simply note that, by (2.21),

$$\begin{aligned}
\mathsf{W}_1(\mu_1(t,\cdot),\mu_2(t,\cdot)) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |p-q|\,\mathrm{d}\pi(t,p,q)\\
&= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |\Gamma_1(t,p)-\Gamma_2(t,q)|\,\mathrm{d}\pi_0(p,q) \le \mathcal{X}(t)+\mathcal{V}(t)
\end{aligned}$$

for all $t\in[0,T]$, readily ending the proof. □

3. Proofs of the main results

3.1. **Proof of Lemma 1.1.** We begin with the proof Lemma 1.1. Actually, we achieve the following slightly stronger result. Here and in the following, the kernel K is as in (1.2).

Proposition 3.1 (Mapping properties of K)**.** *There is a dimensional constant $C_d > 0$ with the following property. If $\rho \in L^1(\mathbb{R}^d) \cap Y^\Theta_{\mathrm{ul}}(\mathbb{R}^d)$, then $K * \rho \in C^{0,\varphi_\Theta}_b(\mathbb{R}^d)$, with*

$$\|K * \rho\|_{L^\infty} \le C_d \left(\|\rho\|_{L^1} + \|\rho\|_{Y^\Theta_{\mathrm{ul}}} \right), \tag{3.1}$$

$$\int_{\mathbb{R}^d} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z \le C_d \left(\|\rho\|_{L^1} + \|\rho\|_{Y^\Theta_{\mathrm{ul}}} \right) \varphi_\Theta(|x-y|) \quad \forall x, y \in \mathbb{R}^d. \tag{3.2}$$

To prove Proposition 3.1, we need the following simple estimate, which generalizes [6, Eq. (2.2)] to any dimension $d \ge 2$.

Lemma 3.2 (Oscillation)**.** *There exists a dimensional constant $C_d > 0$ such that*

$$|K(x-z) - K(y-z)| \le C_d \left(\frac{1}{|x-z||y-z|^{d-1}} + \frac{1}{|y-z||x-z|^{d-1}} \right) |x-y| \tag{3.3}$$

for all $x, y, z \in \mathbb{R}^d$ with $x, y \neq z$.

Proof. We can assume $z = 0$ without loss of generality. For $x, y \in \mathbb{R}^d \setminus \{0\}$, we have

$$\left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right|^2 = \frac{1}{|x|^{2(d-1)}} + \frac{1}{|y|^{2(d-1)}} - \frac{2(x \cdot y)}{|x|^d |y|^d} = \left[\frac{|x|x|^{d-2} - y|y|^{d-2}|}{|x|^{d-1}|y|^{d-1}} \right]^2,$$

so that

$$\left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right| = \frac{|x|x|^{d-2} - y|y|^{d-2}|}{|x|^{d-1}|y|^{d-1}}$$

for all $x, y \in \mathbb{R}^d \setminus \{0\}$. Letting $F_d(\xi) = \xi|\xi|^{d-2}$ for all $\xi \in \mathbb{R}^d$, we have $|\nabla F_d(\xi)| \le C_d |\xi|^{d-2}$ for all $\xi \in \mathbb{R}^d$, where $C_d > 0$ is a dimensional constant. Hence

$$\left| x|x|^{d-2} - y|y|^{d-2} \right| \le |x-y| \sup_{t \in [0,1]} |\nabla F_d(x + t(x-y))| \le C_d\, |x-y| \sup_{t \in [0,1]} |x + t(x-y)|^{d-2}$$

for all $x, y \in \mathbb{R}^d$. Since $d \ge 2$, and thus the function $\xi \mapsto |\xi|^{d-2}$ is convex, we can estimate

$$|x + t(x-y)|^{d-2} \le (1-t)|x|^{d-2} + t|y|^{d-2} \le |x|^{d-2} + |y|^{d-2}$$

for all $x, y \in \mathbb{R}^d$. Therefore, we get that

$$\left| \frac{x}{|x|^d} - \frac{y}{|y|^d} \right| = \frac{|x|x|^{d-2} - y|y|^{d-2}|}{|x|^{2(d-1)}|y|^{d-1}} \le C_d\, |x-y| \left[\frac{|x|^{d-2} + |y|^{d-2}}{|x|^{d-1}|y|^{d-1}} \right]$$

for all $x, y \in \mathbb{R}^d \setminus \{0\}$, yielding (3.3) for $z = 0$. $\square$

We can now prove Proposition 3.1. We follow the strategy of the proofs of [6, Th. 2.2 and Cor. 2.4]. We also refer to the proofs of [13, Lem. 2.1] and [8, Ths. A and B].

Proof of Proposition 3.1. We write $K = K^1 + K^\infty$, with $K^1 = K\mathbf{1}_{B_1} \in L^{\frac{d+1}{d}}(\mathbb{R}^d)$ and $K^\infty = K\mathbf{1}_{B_1^c} \in L^\infty(\mathbb{R}^d)$. Since $\rho \in L^1 \cap L^{d+1}_{\mathrm{ul}}(\mathbb{R}^d)$, we can estimate

$$\begin{aligned} |K| * \rho(x) &\le |K^1| * \rho(x) + |K^\infty| * \rho(x) \le \|K^1\|_{L^{\frac{d+1}{d}}} \|\rho\|_{L^{d+1}(B_1(x))} + \|K^\infty\|_{L^\infty} \|\rho\|_{L^1} \\ &\le \max\left\{ \|K^1\|_{L^{\frac{d+1}{d}}}, \|K^\infty\|_{L^\infty} \right\} \left(\|\rho\|_{L^{d+1}_{\mathrm{ul}}} + \|\rho\|_{L^1} \right) \le C_d \left(\|\rho\|_{L^{d+1}_{\mathrm{ul}}} + \|\rho\|_{L^1} \right) \\ &\le C_d \left(\Theta(d+1)\, \|\rho\|_{Y^\Theta_{\mathrm{ul}}} + \|\rho\|_{L^1} \right) \le C_d \left(\|\rho\|_{Y^\Theta_{\mathrm{ul}}} + \|\rho\|_{L^1} \right) \end{aligned}$$

for all $x \in \mathbb{R}^d$, yielding (3.1). To prove (3.2), fix $x, y \in \mathbb{R}^d$ and set $\varepsilon = |x-y|$. Due to (3.1), we can assume $\varepsilon < e^{-d-1}$ without loss of generality. We write

$$\int_{\mathbb{R}^d} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z$$

$$= \left(\int_{B_2(x)^c} + \int_{B_2(x)\setminus B_{2\varepsilon}(x)} + \int_{B_{2\varepsilon}(x)} \right) |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z.$$

By Lemma 3.2, we can estimate the first integral as

$$\begin{aligned}
&\int_{B_2(x)^c} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z \\
&\qquad \le C_d\, |x-y| \int_{B_2(x)^c} \left(\frac{1}{|x-z||y-z|^{d-1}} + \frac{1}{|y-z||x-z|^{d-1}} \right) \rho(z)\, \mathrm{d}z \\
&\qquad \le C_d\, |x-y|\, \|\rho\|_{L^1}.
\end{aligned}$$

Concerning the second integral, since

$$|y-z| \ge \frac{1}{2}|x-z| \quad \text{for all } z \in B_2(x) \setminus B_{2\varepsilon}(x),$$

again by Lemma 3.2 we can estimate

$$\begin{aligned}
&\int_{B_2(x)\setminus B_{2\varepsilon}(x)} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z \\
&\le C_d\, |x-y| \int_{B_2(x)\setminus B_{2\varepsilon}(x)} \left(\frac{1}{|x-z||y-z|^{d-1}} + \frac{1}{|y-z||x-z|^{d-1}} \right) \rho(z)\, \mathrm{d}z \\
&\le C_d\, |x-y| \int_{B_2(x)\setminus B_{2\varepsilon}(x)} \frac{\rho(z)}{|x-z|^d}\, \mathrm{d}z \le C_d\, |x-y|\, \|\rho\|_{L^p(B_2(x))} \left(\int_{2\varepsilon}^2 r^{-dp'+d-1}\, \mathrm{d}r \right)^{\frac{1}{p'}} \\
&\le C_d\, |x-y|\, \|\rho\|_{L^p_{\mathrm{ul}}} \left(\tfrac{2^{-dp'+d}(1-\varepsilon^{-dp'+d})}{-dp'+d} \right)^{\frac{1}{p'}} \le C_d\, |x-y|\, \|\rho\|_{L^p_{\mathrm{ul}}}\, 2^{-\frac{d}{p}} (\varepsilon^{-\frac{d}{p-1}} - 1)^{\frac{p-1}{p}} \left(\tfrac{p-1}{d} \right)^{\frac{p-1}{p}} \\
&\le C_d\, |x-y|\, \|\rho\|_{L^p_{\mathrm{ul}}}\, p\, \varepsilon^{-\frac{d}{p}} \le C_d\, p\, \Theta(p)\, \|\rho\|_{Y^\Theta_{\mathrm{ul}}}\, |x-y|^{1-\frac{d}{p}}.
\end{aligned}$$

for any $p > d+1$, with p' the conjugate of p. Finally, regarding the third and last integral, since $B_{2\varepsilon}(x) \subset B_{3\varepsilon}(y)$, we can estimate

$$\begin{aligned}
\int_{B_{2\varepsilon}(x)} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z &\le \int_{B_{2\varepsilon}(x)} \frac{\rho(z)}{|x-z|^{d-1}}\, \mathrm{d}z + \int_{B_{3\varepsilon}(z)} \frac{\rho(z)}{|y-z|^{d-1}}\, \mathrm{d}z \\
&\le C_d\, \|\rho\|_{L^p_{\mathrm{ul}}} \left(\int_0^{3\varepsilon} r^{(-d+1)p'+d-1}\, \mathrm{d}r \right)^{\frac{1}{p'}} \le C_d\, \|\rho\|_{L^p_{\mathrm{ul}}} \left(\tfrac{(3\varepsilon)^{(-d+1)p'+d}}{(-d+1)p'+d} \right)^{\frac{1}{p'}} \\
&\le C_d\, \|\rho\|_{L^p_{\mathrm{ul}}} (3\varepsilon)^{1-\frac{d}{p}} \left(\tfrac{p-1}{p-d} \right)^{\frac{p-1}{p}} \le C_d\, p\, \Theta(p)\, \|\rho\|_{Y^\Theta_{\mathrm{ul}}}\, |x-y|^{1-\frac{d}{p}}
\end{aligned}$$

again for $p > d+1$. Putting everything altogether, we conclude that

$$\int_{\mathbb{R}^d} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z \le C_d \left(\|\rho\|_{L^1(\mathbb{R}^d)} + \|\rho\|_{Y^\Theta_{\mathrm{ul}}} \right) p\, \Theta(p)\, |x-y|^{1-\frac{d}{p}}$$

for $x, y \in \mathbb{R}^d$ with $|x-y| < e^{-d-1}$ and $p > d+1$. In particular, choosing $p = -\log|x-y|$, since $r^{\frac{d}{\log(r)}} = e^d$ for $r \in (0,1)$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |K(x-z) - K(y-z)|\, \rho(z)\, \mathrm{d}z \\
&\qquad \le C_d\, (\|\rho\|_{L^1} + \|\rho\|_{Y^\Theta_{\mathrm{ul}}})\, |x-y|\, |\log|x-y||\, \Theta(|\log|x-y||)\, |x-y|^{\frac{d}{\log|x-y|}} \\
&\qquad \le C_d\, (\|\rho\|_{L^1} + \|\rho\|_{Y^\Theta_{\mathrm{ul}}})\, \varphi_\Theta(|x-y|)
\end{aligned}$$

for $x, y \in \mathbb{R}^d$ with $|x-y| < e^{-d-1}$, completing the proof of (3.2). $\square$

3.2. **Proof of Theorem 1.6.** In view of Theorem 2.8, we just have to check that, if $f \in \mathcal{A}^\Theta([0,T])$ is a Lagrangian weak solution of (1.1) in the sense of Definition 1.4, then f is a Lagrangian weak φ_Θ-solution of (2.16) with $F(t,x,v) = v$ for $t \in [0,T]$ and $x, v \in \mathbb{R}^d$ and $E_f = K * \rho_f$, where K is as in (1.2). Indeed, we just need to check the validity of (2.17) and (2.18), but these respectively follow from (3.1) and (3.2) in Proposition 3.1. □

Remark 3.3 (Relativistic case). Note that the above argument *verbatim* applies to the relativistic setting, that is, choosing $F(t,x,v) = \frac{v}{\sqrt{1+|v|^2}}$ for $t \in [0,T]$ and $x, v \in \mathbb{R}^d$.

3.3. **Proof of Theorem 1.7.** From now on, we assume $d \in \{2,3\}$. We begin with the following result, providing a suitable initial datum for the construction of the weak solution in Theorem 1.7.

Lemma 3.4 (Datum)**.** *If $\theta \colon \mathbb{R}^d \to \mathbb{R}$ satisfies* (1.19)*, then $f_0 \colon \mathbb{R}^{2d} \to [0,+\infty)$ given by*

$$f_0(x,v) = \frac{\mathbf{1}_{(-\infty,0]}\left(|v|^2 - \theta(x)^{\frac{2}{d}}\right)}{|B_1|\,\|\theta\|_{L^1}}, \quad \text{for } x, v \in \mathbb{R}^d, \tag{3.4}$$

satisfies $f_0 \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$, $f_0 \mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$ and, for some constant $C > 0$,

$$\int_{\mathbb{R}^{2d}} |v|^p f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \le \frac{\|\theta\|_{L^{\frac{p}{d}+1}}^{\frac{p}{d}+1}}{\|\theta\|_{L^1}} \quad \text{for all } p \in [1,+\infty). \tag{3.5}$$

Proof. Note that $|v| \le \theta(x)^{\frac{1}{d}}$ for all $(x,v) \in \operatorname{supp} f_0$. We thus have

$$\rho_0(x) = \int_{\mathbb{R}^d} f_0(x,v)\,\mathrm{d}v = \frac{\mathscr{L}^d\left(\left\{v \in \mathbb{R}^d : |v| \le \theta(x)^{\frac{1}{d}}\right\}\right)}{|B_1|\|\theta\|_{L^1}} = \frac{\theta(x)}{\|\theta\|_{L^1}} \tag{3.6}$$

for all $x \in \mathbb{R}^d$. Consequently, we can estimate

$$\int_{\mathbb{R}^{2d}} |v|^p f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \le \int_{\mathbb{R}^{2d}} |\theta(x)|^{\frac{p}{d}} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} |\theta(x)|^{\frac{p}{d}} \rho_0(x)\,\mathrm{d}x = \frac{\|\theta\|_{L^{\frac{p}{d}+1}}^{\frac{p}{d}+1}}{\|\theta\|_{L^1}},$$

readily yielding the conclusion. □

We can now prove Theorem 1.7. Actually, we prove the following more precise result.

Proposition 3.5 (Existence)**.** *Assume that $\theta \in Y^\Theta(\mathbb{R}^d)$ satisfies* (1.19)*. There exists a Lagrangian weak solution*

$$f \in C([0,T]; L^p(\mathbb{R}^{2d})) \cap L^\infty([0,T] \times \mathbb{R}^{2d}) \cap \mathcal{A}^\Theta([0,T]) \quad \text{for all } p \in [1,+\infty)$$

of the system (1.1) *starting from f_0 in* (3.4) *of Lemma 3.4 such that $f(t,\cdot)\mathscr{L}^{2d} \in \mathscr{P}_1(\mathbb{R}^{2d})$,*

$$\rho_f \in C([0,T]; L^p(\mathbb{R}^d)) \quad \text{for all } p \in [1,+\infty) \tag{3.7}$$

and, for some constant $C_T > 0$ depending on T,

$$\frac{\|\theta\|_{L^q}}{\|\theta\|_{L^1}} \le \|\rho_f\|_{L^\infty([0,T];L^q)} \le C_T \|\theta\|_{L^q} \quad \text{for all } q \in [1,+\infty)\,. \tag{3.8}$$

Proof. By [19, Th. 1] (for $d = 3$, the case $d = 2$ being similar, see [13,22]), there exists

$$f \in C([0,+\infty); L^p(\mathbb{R}^{2d})) \cap L^\infty([0,+\infty) \times \mathbb{R}^{2d}) \quad \text{for all } p \in [1,+\infty)$$

a weak solution of the system (1.1) starting from f_0 in (3.4) of Lemma 3.4 and such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} |v|^p f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v < +\infty \quad \text{for all } p \in [1,+\infty). \tag{3.9}$$

Note that the notion of weak solution here is well-posed in the sense of Definition 1.2, since $E_f \in L^\infty([0,T]\times\mathbb{R}^d)$ in virtue of (3.9) and [19, Eq. (16)]. Moreover, f is constant along characteristic curves of (1.12) which are defined almost everywhere. Finally, by [19, Eq. (8)] and (3.5), we get (3.7). Thus, we just need to show (3.8), so that $f \in \mathcal{A}^\Theta([0,T])$ in particular. For the first inequality in (3.8), we observe that

$$\|\rho_f\|_{L^\infty(L^q)} \ge \|\rho_f(0,\cdot)\|_{L^q} = \|\rho_0\|_{L^q} = \frac{\|\theta\|_{L^q}}{\|\theta\|_{L^1}}$$

because of (3.6) and (3.7). For the second inequality in (3.8), we argue as in [22, Sec. 3]. By [19, Eq. (14)], we can estimate

$$\|\rho_f(t,\cdot)\|_{L^{\frac{p}{d}+1}} \le C M_p(t)^{\frac{d}{p+d}} \quad \text{for } t\in[0,T],$$

for some constant $C_T>0$ independent of p and $t\in[0,T]$, but dependent on $T>0$, which may vary from line to line in what follows, where

$$M_p(t) = \int_{\mathbb{R}^{2d}} |v|^p\, f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Exploiting (1.12) and the fact that f is constant along characteristics, we can estimate

$$M_p(t) \le M_p(0) + C_T\, p \int_0^t M_p(s)^{1-\frac{1}{p}}\,\mathrm{d}s$$

By a simple Grönwall-type argument, we infer that

$$\sup_{t\in[0,T]} M_p(t) \le M_p(0) + C_T^p \quad \text{for all } t\in[0,T].$$

Since $f(0,\cdot)=f_0$, by (3.5) we get

$$M_p(t)^{\frac{d}{p+d}} \le \left(\frac{\|\theta\|_{L^{\frac{p}{d}+1}}^{\frac{p}{d}+1}}{\|\theta\|_{L^1}} + C_T^p\right)^{\frac{d}{p+d}} \le C_T\,\|\theta\|_{L^{\frac{p}{d}+1}},$$

proving the second inequality in (3.8) and ending the proof. □

3.4. **Proof of Proposition 1.8.** We need some notation and the preliminary Lemma 3.6 below. For each $m\in\mathbb{N}$, we define $\ell_m\colon[0,+\infty)\to[0,+\infty)$ by letting

$$\ell_m(r) = \mathbf{1}_{(0,\varepsilon_m)}(r)\,\log_m(r) \quad \text{for all } r\ge 0, \tag{3.10}$$

where $\varepsilon_m\in(0,1)$ is such that $\log_m(\varepsilon_m)=-1$ (recall the notation in (1.21)).

Lemma 3.6. *For $m\in\mathbb{N}$, there are $p_m\in[1,+\infty)$ and $0<a_m<b_m<+\infty$ such that*

$$a_m \log_{m-1}(p) \le \|\ell_m(|\cdot|)\|_{L^p} \le b_m \log_{m-1}(p) \quad \textit{for all } p\ge p_m. \tag{3.11}$$

Proof. Given $p\ge\log(1/\varepsilon_m)$, we can easily estimate

$$\|\ell_m(|\cdot|)\|_{L^p}^p = \int_{B_{\varepsilon_m}} |\log_m(|x|)|^p\,\mathrm{d}x \ge \int_{B_{e^{-p}}} |\log_m(|x|)|^p\,\mathrm{d}x \ge C_d\, e^{-dp} |\log_{m-1}(p)|^p \tag{3.12}$$

for all $m\in\mathbb{N}$, proving the lower bound in (3.11). For the upper bound in (3.11), we argue by induction. If $m=1$, then by direct computation we have

$$\|\ell_1(|\cdot|)\|_{L^p}^p = \int_{B_1} |\log(|x|)|^p\,\mathrm{d}x = C_d\int_0^1 (-\log r)^p\, r^{d-1}\,\mathrm{d}r = C_d\, d^{-(p+1)}\,\Gamma(p+1)$$

and the desired upper bound readily follows by Stirling's formula. If $m \geq 2$, then

$$\begin{aligned}\|\ell_m(|\cdot|)\|_{L^p} &= \left(\int_{B_{\varepsilon_m}} |\log_m(|x|)|^p \,\mathrm{d}x\right)^{1/p}\\ &= \frac{|B_{\varepsilon_m}|^{1/p}}{p}\left(\fint_{B_{\varepsilon_m}} \left|\log\left(\log_{m-1}(|x|)\right)^p\right|^p \mathrm{d}x\right)^{1/p}.\end{aligned}$$

Now $r \mapsto (\log r)^p$ is concave on $\left[e^{p-1}, +\infty\right)$. Since $\log_{m-1}(\varepsilon_m) = -e$, for $p \geq 2$ we have

$$\begin{aligned}\fint_{B_{\varepsilon_m}} \left|\log\left(\log_{m-1}(|x|)\right)^p\right|^p \mathrm{d}x &\leq \left(\log\left(\fint_{B_{\varepsilon_m}} |\log_{m-1}(|x|)|^p \,\mathrm{d}x\right)\right)^p\\ &\leq p^p\Big(\log\left(|B_{\varepsilon_m}|^{-1/p}\,\|\ell_{m-1}(|\cdot|)\|_{L^p}\right)\Big)^p\end{aligned}$$

by Jensen's inequality, so that

$$\|\ell_m(|\cdot|)\|_{L^p} \leq |B_{\varepsilon_m}|^{1/p} \log\left(|B_{\varepsilon_m}|^{-1/p}\,\|\ell_{m-1}(|\cdot|)\|_{L^p}\right),$$

readily yielding the conclusion. □

Proof of Proposition 1.8. For each $m \in \mathbb{N}$, there exists $\delta_m > 0$ such that

$$\varphi_{\Theta_m}(r) = r\,|\log r|\,\Theta_m(|\log r|) = \Theta_{m+1}(r) \quad \text{for all } r \in [0, \delta_m].$$

Hence φ_{Θ_m} is concave on $[0, \delta_m]$ with $\varphi_{\Theta_m}(0) = 0$. Therefore, we can estimate

$$\Phi_{\Theta_m}(t) = \int_0^t \varphi_{\Theta_m}(s)\,\mathrm{d}s \leq t\,\varphi_{\Theta_m}(t) = t\,\Theta_{m+1}(t) \quad \text{for all } t \in [0, \delta_m].$$

In particular, we readily infer that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_\varepsilon^{\delta_m} \frac{\mathrm{d}t}{\sqrt{\Phi_{\Theta_m}(t)}} &\geq \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\delta_m} \frac{\mathrm{d}t}{\sqrt{t\,\Theta_{m+1}(t)}}\\ &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\delta_m} \frac{\mathrm{d}t}{t\,|\log t|\,|\log_2(t)|\cdots|\log_{m+1}(t)|} = +\infty,\end{aligned}$$

so that Φ_{Θ_m} satisfies (1.15). To conclude, we define $\theta_m \colon \mathbb{R}^d \to [0, +\infty)$ as

$$\theta_m(x) = \ell_1(|x|)\,\ell_2(|x|)^2 \ldots \ell_{m+1}(|x|)^2 \quad \text{for } x \in \mathbb{R}^d.$$

On the one side, arguing as in (3.12), we easily see that

$$\begin{aligned}\|\theta_m\|^p_{L^p} &\geq \int_{B_{e^{-p}}} |\log_1(|x|)|^p\,|\log_2(|x|)|^{2p} \ldots |\log_{m+1}(|x|)|^{2p}\,\mathrm{d}x\\ &\geq C_d\,e^{-dp}\,p^p\,|\log_1(p)|^{2p} \ldots |\log_m(p)|^{2p} = C_d\,e^{-dp}\,\Theta_m(p)^p\end{aligned}$$

for all $p \in [1, +\infty)$. On the other side, by Lemma 3.6 and Hölder's inequality, we get

$$\begin{aligned}\|\theta_m\|_{L^p} &\leq \|\ell_1(|\cdot|)\|_{L^{(m+1)p}}\,\|\ell_2(|\cdot|)^2\|_{L^{(m+1)p}} \ldots \|\ell_{m+1}(|\cdot|)^2\|_{L^{(m+1)p}}\\ &= \|\ell_1(|\cdot|)\|_{L^{(m+1)p}}\,\|\ell_2(|\cdot|)\|^2_{L^{2(m+1)p}} \ldots \|\ell_{m+1}(|\cdot|)\|^2_{L^{2(m+1)p}}\\ &\leq C_m\,p\,\log_1(p)^2 \ldots \log_m(p)^2 = C_m\,\Theta_m(p)\end{aligned}$$

for all $p \geq p_m$ for some constant $C_m > 0$ depending on m only, yielding the conclusion. □

Remark 3.7 (Saturation of $\Theta_\alpha(p) = p^{1/\alpha}$). Fix $\alpha \in [1, \infty)$. Arguing as above, one can easily see that $\theta_\alpha(x) = \ell_1(|x|)^{1/\alpha}$, for $x \in \mathbb{R}^d$, saturates the growth function $\Theta_\alpha(p) = p^{1/\alpha}$ in the sense of Proposition 1.8, giving an alternative proof of [13, Prop. 1.14].

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[2] A. A. Arsen'ev, *Global existence of a weak solution of Vlasov's system of equations*, USSR Computational Mathematics and Mathematical Physics **15** (1975), no. 1, 131–143.

[3] C. Bardos and P. Degond, *Global existence for the Vlasov-Poisson equation in* 3 *space variables with small initial data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 2, 101–118.

[4] Q. Chen, C. Miao, and X. Zheng, *The two-dimensional Euler equation in Yudovich and bmo-type spaces*, Rev. Mat. Iberoam. **35** (2019), no. 1, 195–240.

[5] G. Crippa, S. Ligabue, and C. Saffirio, *Lagrangian solutions to the Vlasov-Poisson system with a point charge*, Kinet. Relat. Models **11** (2018), no. 6, 1277–1299.

[6] G. Crippa and G. Stefani, *An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces* (2021). Preprint, available at arXiv:2110.15648v2.

[7] L. Desvillettes, E. Miot, and C. Saffirio, *Polynomial propagation of moments and global existence for a Vlasov-Poisson system with a point charge*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **32** (2015), no. 2, 373–400.

[8] R. Garg and D. Spector, *On the role of Riesz potentials in Poisson's equation and Sobolev embeddings*, Indiana Univ. Math. J. **64** (2015), no. 6, 1697–1719.

[9] I. Gasser, P.-E. Jabin, and B. Perthame, *Regularity and propagation of moments in some nonlinear Vlasov systems*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 6, 1259–1273.

[10] R. T. Glassey and J. Schaeffer, *On symmetric solutions of the relativistic Vlasov-Poisson system*, Comm. Math. Phys. **101** (1985), no. 4, 459–473.

[11] ———, *On global symmetric solutions to the relativistic Vlasov-Poisson equation in three space dimensions*, Math. Methods Appl. Sci. **24** (2001), no. 3, 143–157.

[12] M. Hadžić and G. Rein, *Global existence and nonlinear stability for the relativistic Vlasov-Poisson system in the gravitational case*, Indiana Univ. Math. J. **56** (2007), no. 5, 2453–2488.

[13] T. Holding and E. Miot, *Uniqueness and stability for the Vlasov-Poisson system with spatial density in Orlicz spaces*, Mathematical analysis in fluid mechanics—selected recent results, 2018, pp. 145–162.

[14] M. Inversi, *Lagrangian solutions to the transport-Stokes system* (2023). Preprint, available at arXiv:2303.05797.

[15] M. Inversi and G. Stefani, *Lagrangian stability for a system of non-local continuity equations under Osgood condition* (2023). Preprint, available at arXiv:2301.11822.

[16] S. V. Iordanskiĭ, *The Cauchy problem for the kinetic equation of plasma*, Trudy Mat. Inst. Steklov. **60** (1961), 181–194.

[17] M. K.-H. Kiessling and A. S. Tahvildar-Zadeh, *On the relativistic Vlasov-Poisson system*, Indiana Univ. Math. J. **57** (2008), no. 7, 3177–3207.

[18] N. Leopold and C. Saffirio, *Propagation of moments for large data and semiclassical limit to the relativistic Vlasov equation*, SIAM J. Math. Anal. (2022). To appear, preprint available at arXiv:2203.03031.

[19] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the* 3*-dimensional Vlasov-Poisson system*, Invent. Math. **105** (1991), no. 2, 415–430.

[20] G. Loeper, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl. (9) **86** (2006), no. 1, 68–79.

[21] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002.

[22] E. Miot, *A uniqueness criterion for unbounded solutions to the Vlasov-Poisson system*, Comm. Math. Phys. **346** (2016), no. 2, 469–482.

[23] C. Pallard, *Moment propagation for weak solutions to the Vlasov-Poisson system*, Comm. Partial Differential Equations **37** (2012), no. 7, 1273–1285.

[24] ———, *Space moments of the Vlasov-Poisson system: propagation and regularity*, SIAM J. Math. Anal. **46** (2014), no. 3, 1754–1770.

[25] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, J. Differential Equations **95** (1992), no. 2, 281–303.

[26] R. Robert, *Unicité de la solution faible à support compact de l'équation de Vlasov-Poisson*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 8, 873–877.

[27] D. Salort, *Transport equations with unbounded force fields and application to the Vlasov-Poisson equation*, Math. Models Methods Appl. Sci. **19** (2009), no. 2, 199–228.

[28] Y. Taniuchi, *Uniformly local L^p estimate for 2-D vorticity equation and its application to Euler equations with initial vorticity in* **bmo**, Comm. Math. Phys. **248** (2004), no. 1, 169–186.

[29] Y. Taniuchi, T. Tashiro, and T. Yoneda, *On the two-dimensional Euler equations with spatially almost periodic initial data*, J. Math. Fluid Mech. **12** (2010), no. 4, 594–612.

[30] S. Ukai and T. Okabe, *On classical solutions in the large in time of two-dimensional Vlasov's equation*, Osaka Math. J. **15** (1978), no. 2, 245–261.

[31] X. Wang, *Global solution of the 3D relativistic Vlasov-Poisson system for a class of large data* (2020). Preprint, available at arXiv:2003.14191v3.

[32] V. I. Yudovich, *Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid*, Math. Res. Lett. **2** (1995), no. 1, 27–38.

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