Algebraic subgroups of groups of birational transformations

Inauguraldissertation
zur
Erlangung der Würde eines Doktors der Philosophie
vorgelegt der
Philosophisch-Naturwissenschaftlichen Fakultät
der Universität Basel

von

Pascal Fong

2023

Genehmigt von der Philosophisch-Naturwissenschaftlichen Fakultät

auf Antrag von

Prof. Dr. Jérémy Blanc,
Prof. Dr. Philipp Habegger,
Prof. Dr. Michel Brion und
Prof. Dr. Ivan Cheltsov.

Basel, den 20. September 2022.

Prof. Dr. Marcel Mayor, Dekan.

Contents

Acknowledgments

First, I am grateful to my advisor Jérémy Blanc for his guidance and his ideas all along the preparation of this thesis, which has been possible only thanks to his creativity and availability. Learning birational geometry under his supervision was a pleasure and not a necessity. I also thank Philipp Habegger for being my second advisor, Michel Brion and Ivan Cheltsov for being the external experts. Special thanks to Michel Brion and Ronan Terpereau, who have always been enthusiastic to teach me about algebraic groups, and discussions with them have been an immense help. Many thanks to Julie Déserti and Susanna Zimmermann, who have also been a precious help for me. I must also thank our irreplaceable secretary Barbara Fridez.

More generally, I would like to thank all the members of the groups of algebraic geometry and of number theory from Universität Basel, including the former members, for the pleasant atmosphere. Undoubtedly, I will miss Basel and Manor. I also thank my friends for existing, in particular: the ones I have met in Paris, the ones I have met during my studies, the ones I have met in the WG, and of course, the Greeks and the Spanish.

My deepest thanks go to my family, especially to my parents and my sisters, for their unconditional love and support.

Last but not least, thanks Nathalie!

Cette thèse est dédiée à André et Polo.

Introduction

A part of birational geometry is devoted to the study of groups of birational transformations. Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. If X is a variety over $\mathbf{k}$, a *birational transformation* of X is a rational map from X to itself, which also admits a rational map as inverse. The *group of birational transformations* of X is denoted by $\mathrm{Bir}(X)$. We denote by $\mathrm{Aut}(X)$ the group of *automorphisms* of X, which is a subgroup of $\mathrm{Bir}(X)$.

Let G be a smooth algebraic group and denote by $p_1\colon G \times X \to G$ the projection onto the first factor. Assume there exists a rational map $\rho\colon G \times X \dashrightarrow X$ such that the following hold:

- The birational transformation $\alpha = (p_1, \rho)\colon G \times X \dashrightarrow G \times X$ is dominant; and there exist open subsets $U, V \subset G \times X$ such that $\alpha_{|U}\colon U \to V$ is an isomorphism and the morphisms $p_{1|U}\colon U \to G$ and $p_{1|V}\colon V \to G$ are surjective.
- For all $g, h \in G$ and $x \in X$ such that $\alpha(h, x)$ and $\alpha(g, \rho(h, x))$ are defined, the element $\alpha(gh, x)$ is defined and equals $\alpha(g, \alpha(h, x))$.

The rational map ρ is called a *birational action* of G on X and ρ induces a group homomorphism $\mu_G\colon G(\mathbf{k}) \to \mathrm{Bir}(X)$, $g \mapsto \rho(g, \cdot)$. The image of μ_G is called an *algebraic subgroup* of $\mathrm{Bir}(X)$. If moreover α is an automorphism, the image of μ_G is called an *algebraic subgroup* of $\mathrm{Aut}(X)$. When μ_G is injective, we make the abuse to consider G as an algebraic subgroup of $\mathrm{Bir}(X)$, or of $\mathrm{Aut}(X)$.

Let X be a projective variety. The group $\mathrm{Aut}(X)$ is the group of $\mathbf{k}$-rational points of a group scheme by a classical result (see [MO67]); but generally, it does not have the structure of an algebraic group (e.g. the automorphism group of a K3 surface can be infinite and discrete). However, the connected component of the identity in $\mathrm{Aut}(X)$, which is denoted by $\mathrm{Aut}^\circ(X)$, is a normal subgroup of $\mathrm{Aut}(X)$ and is equipped with the structure of an algebraic group. In particular, $\mathrm{Aut}^\circ(X)$ is a connected algebraic subgroup of $\mathrm{Bir}(X)$. Assume furthermore in this paragraph that $\mathrm{char}(\mathbf{k}) = 0$. If X is a smooth projective curve or if X is a relatively minimal surface with non-negative Kodaira dimension, the group $\mathrm{Bir}(X)$ coincides with its subgroup $\mathrm{Aut}(X)$ (see e.g. [Han87, §1. (1.3)]). In general, when X is a minimal model, then $\mathrm{Bir}(X)$ is equipped with a structure of a group scheme and $\mathrm{Bir}^\circ(X) = \mathrm{Aut}^\circ(X)$ (see [Han87, (3.3) Theorem]). As Hanamura explains in the introduction of [Han87], we can define "naively" a structure of scheme on $\mathrm{Bir}(X)$. But generally, the scheme $\mathrm{Bir}(X)$ is not a group scheme, and in particular when X has Kodaira dimension $-\infty$ (see also [Bla17, Corollary 3.15]). Therefore, it is a challenging problem to describe the group of birational transformations of such varieties.

One possible way to study $\mathrm{Bir}(X)$ is to classify its algebraic subgroups. Over $\mathbb{C}$, Enriques studied the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ in [Enr93]. Then, with Fano, they state a classification of maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ in [EF98], which has been proven and detailed later by Umemura in a series of four articles [Ume80, Ume82a, Ume82b, Ume85], using analytic techniques. From the classification of Enriques and Umemura, it follows that every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathrm{Bir}(\mathbb{P}^3)$ is contained in a maximal one. In the pioneer article [Dem70], seen as the starting point of toric geometry, Demazure classified the algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ containing a torus of dimension n. Finite subgroups of groups of birational transformations have also been of central interest in birational geometry. The study of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ over $\mathbb{C}$ has been initiated by the work of Bertini, Kantor and Wiman (see [Ber77, Kan95, Wim97]), and pursued more recently by numerous algebraic geometers (over algebraically closed fields, see e.g. [Bla06, DI09]; and for non-closed fields, see e.g. [Yas22, CMYZ22]). In higher dimension, research on finite subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ is still ongoing (see e.g. [Pro12, Pro13, PS14, PS16, CS19], or also [CS16] and the references therein). When X is not rational, there are also studies on finite subgroups of $\mathrm{Bir}(X)$ (see e.g. [BZ17]).

However, less is known on infinite algebraic subgroups of $\mathrm{Bir}(X)$ when X is a non-rational variety of Kodaira dimension $-\infty$, and this will be the main setting of this thesis. A modern approach to study algebraic subgroups of $\mathrm{Bir}(X)$ is the following:

(1) find a smooth projective variety Y which is G-birationally equivalent to X and on which G acts regularly,

(2) run a G-equivariant Minimal Model Program on Y (or shortly, a G-MMP, and see [Pro21] for a reference),

(3) study the automorphism groups of the output, that is, of the G-Mori fibre space or of the G-minimal model (see e.g. [Pro21, Definitions 3.1.3 and 3.1.5])

The first step is achieved by the Weil regularization theorem [Wei55, Thm. p. 355] and its refined versions. Weil showed that X is G-birational to a normal variety Y on which the action of G is regular. In order to run a G-MMP in the second step, one would like to assume that Y is projective. When G is affine, this follows from a result of Sumihiro [Sum74, Sum75]. Then this result has been extended to the case where G is assumed to be connected, without the affineness assumption, by Brion in [Bri17, Corollary 3]. Eventually, the general case has been very recently covered by Brion in [Bri22b, Theorems 1 and 2]. One would also like to assume that Y is smooth (or regular in the general case, but we assume that $\mathbf{k}$ is algebraically closed). In dimension two, this follows from [Lip78, Remark B p.155]: the surface Y can be desingularized G-equivariantly by successive blowups and normalizations. In higher dimension, we need to assume furthermore that $\mathrm{char}(\mathbf{k}) = 0$ to use a G-equivariant resolution of singularities [Kol07, Thm. 3.36, Prop. 3.9.1].

Running a G-MMP (with the additional assumption that $\mathrm{char}(\mathbf{k}) = 0$ if X is of dimension at least 3), we get a G-Mori fibre space or a G-minimal model, and this conjugates G to an algebraic subgroup of the automorphism group of a G-Mori fibre space or a G-minimal model. Notice that if G is connected, then every MMP is G-equivariant [Flo20, Lemma 2.5]. This does not hold true when G is not connected, not even in dimension two, as we can get automorphism groups of conic bundles over a smooth projective curve, containing a non-trivial involution permuting the irreducible components of the singular fibres.

To classify the connected algebraic subgroups of $\mathrm{Bir}(X)$, the idea is to provide a family $\mathcal{F}$, consisting of pairs $(Z, \mathrm{Aut}^\circ(Z))$, where Z an output of a MMP, for which it is possible to describe all the $\mathrm{Aut}^\circ(Z)$, and such that each connected algebraic subgroup of $\mathrm{Bir}(X)$ is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(Z)$ for some pair $(Z, \mathrm{Aut}^\circ(Z)) \in \mathcal{F}$. A remaining work would be to study all the algebraic subgroups of the $\mathrm{Aut}^\circ(Z)$ such that $(Z, \mathrm{Aut}^\circ(Z)) \in \mathcal{F}$. Therefore, the notion of maximality emerges: a (connected) algebraic subgroup of $\mathrm{Bir}(X)$ is *maximal* if it is maximal among the (connected) algebraic subgroups with respect to the inclusion inside $\mathrm{Bir}(X)$. As mentioned earlier, every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ and $\mathrm{Bir}(\mathbb{P}^3)$ over $\mathbb{C}$ is contained in a maximal one. It is however unknown if this holds true for connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ when $n \geq 4$, but we will see in this thesis that this is false for $\mathrm{Bir}(C \times \mathbb{P}^n)$ when $n \geq 1$ and C is a curve of positive genus. In the non-connected case, the classification of pairs $(Z, \mathrm{Aut}(Z))$, where Z is an output of the G-MMP, fully makes sense if $\mathrm{Aut}(Z)$ is equipped with a structure of algebraic group. For example, this is true when Z is a conic bundle over a smooth curve. By the classification of algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ by Blanc [Bla09b], it follows again that every algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a maximal one. We will also see in this thesis that this is false for $\mathrm{Bir}(C \times \mathbb{P}^1)$, when C is a curve of positive genus.

The strategy presented above gave a new impetus to the study of algebraic subgroups of groups of birational transformations. In [Bla09b], Blanc classified the algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ over the field of complex numbers (although his results hold over any algebraically closed field of characteristic different than two). In higher dimension, Blanc, Fanelli and Terpereau recovered most of the classification of Umemura over any algebraically closed field $\mathbf{k}$ of characteristic zero, using algebraic methods (see [BFT21a, BFT21b]). A first strike in dimension four has been initiated by Blanc and Floris (see [BF20]). On another direction, infinite algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ over $\mathbb{R}$ have been studied by Robayo and Zimmermann in [RZ18]; then by Schneider and Zimmermann over any perfect field in [SZ21].

Let C be a smooth projective curve of positive genus. In this thesis, through three chapters, we study the (connected) algebraic subgroups of the group of birational transformations of some (non-rational) varieties:

I First, we prove the existence of increasing infinite sequences of connected algebraic subgroups in $\mathrm{Bir}(C \times \mathbb{P}^1)$ (see Theorem A), and in particular, there exist connected algebraic subgroups of $\mathrm{Bir}(C\times\mathbb{P}^1)$ of arbitrary large dimension. Then, we classify the maximal connected algebraic subgroups of $\mathrm{Bir}(C\times\mathbb{P}^1)$ (see Theorem B), and it turns out that they are of dimension at most 4, unlike maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ which can be of arbitrary large dimension.

   Let Z be a projective surface. We show that there exist connected algebraic subgroups of $\mathrm{Bir}(Z)$ which are not contained in maximal ones if and only if Z is birationally equivalent to $C\times\mathbb{P}^1$ (Theorem C). Finally, we obtain the classification of pairs $(Z, \mathrm{Aut}^\circ(Z))$, where $\mathrm{Aut}^\circ(Z)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(Z)$ for any surface Z (see Theorem D), but this last result only holds when $\mathrm{char}(\mathbf{k}) = 0$.

II Following the techniques of [Bla09b], we classify the maximal algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ under the assumption that $\mathrm{char}(\mathbf{k}) \neq 2$ (see Theorem E). As in the connected case, it follows from the classification that maximal algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ are of dimension at most 4, and we also get algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ which are not contained in a maximal one (see Corollary F).

III This is a joint work with Zikas. Under the assumption that $\mathrm{char}(\mathbf{k}) = 0$, we prove that for each $n \geq 1$, there exist connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^n)$ which are not contained in a maximal one (see Theorem G).

Theorem D obtained in the end of the first chapter can be extended in positive characteristic, with the notion of G-normal curves developed recently in an article of Brion [Bri22a, Proposition 5.6].

One possible future work is to classify the maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^2)$, in the spirit of [BFT21a, BFT21b]. Another interesting viewpoint is to extend the classification of pairs $(X, \mathrm{Aut}^\circ(X))$, where X is birational to $C \times \mathbb{P}^2$ and $\mathrm{Aut}^\circ(X)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^2)$, to a family $\mathcal{F}$ of pairs $(Z, \mathrm{Aut}^\circ(Z))$, in such a way that it is possible to describe all these pairs, and such that each connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^2)$ is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(Z)$ with $(Z, \mathrm{Aut}^\circ(Z)) \in \mathcal{F}$.

Connected algebraic groups acting on algebraic surfaces

I.1 Introduction

In this text, varieties are reduced and separated schemes of finite type over an algebraically closed field $\mathbf{k}$. Unless otherwise stated, curves are also assumed to be smooth, irreducible and projective. If X is a variety, we denote by $\mathrm{Bir}(X)$ the group of birational transformations of X. A subgroup $G \subset \mathrm{Bir}(X)$ is *algebraic* if there exists a structure of algebraic group (i.e. a smooth group scheme of finite type) on G such that the action $G \times X \dashrightarrow X$ induced by the inclusion of G into $\mathrm{Bir}(X)$ is a rational action (see Definition I.2.1). Moreover, G is a *maximal algebraic subgroup* of $\mathrm{Bir}(X)$ if there is no algebraic subgroup G' of $\mathrm{Bir}(X)$ which strictly contains G. If X is a projective variety, then the subgroup $\mathrm{Aut}(X) \subset \mathrm{Bir}(X)$ of automorphisms of X is a smooth group scheme (see [MO67]); and the connected component of the identity $\mathrm{Aut}^\circ(X)$ is an algebraic subgroup of $\mathrm{Bir}(X)$. In this paper we answer in Theorem D the following question when $\mathrm{char}(\mathbf{k}) = 0$.

Question. What are the maximal connected algebraic subgroups of $\mathrm{Bir}(X)$, when X is an algebraic surface (or equivalently, when X is a projective surface)?

The maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ have been studied by Enriques in [Enr93]: the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ are conjugate to $\mathrm{Aut}^\circ(\mathbb{P}^2)$ or to $\mathrm{Aut}^\circ(\mathbb{F}_n)$ for some $n \in \mathbb{N} \setminus \{1\}$, where $\mathbb{F}_n$ denotes the n-th *Hirzebruch surface* (i.e. the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ having a section of self-intersection $-n$). Furthermore, any connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is contained in a maximal connected algebraic subgroup. We will show in Theorem A that if S is a $\mathbb{P}^1$-bundle over a curve C of genus $g \geq 1$, it is not always true that $\mathrm{Aut}^\circ(S)$ is contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$. Our approach to prove Theorem A uses elementary tools like blowups and contractions, and a classification of automorphisms of ruled surfaces due to Maruyama in [Mar71].

Theorem A. *Let S be a non trivial $\mathbb{P}^1$-bundle over a curve C of genus g. We assume that $g \geq 2$, or that $g = 1$ and S admits a section of negative self-intersection number. Then there exists a family $(S_n)_{n \geq 1}$ of $\mathbb{P}^1$-bundles over C*

with birational maps $\phi_n\colon S \dashrightarrow S_n$ such that:

$$\mathrm{Aut}^\circ(S) \subset \phi_1^{-1}\mathrm{Aut}^\circ(S_1)\phi_1 \subset ... \subset \phi_n^{-1}\mathrm{Aut}^\circ(S_n)\phi_n \subset ...$$

is not a stationary sequence. In particular, the connected algebraic subgroup $\mathrm{Aut}^\circ(S)$ *of* $\mathrm{Bir}(S)$ *is not maximal.*

Then we study the connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ when C is a curve of genus 1. So assume in this paragraph that C is an elliptic curve. We denote the Atiyah ruled surfaces by $\mathcal{A}_0$ and $\mathcal{A}_1$ (see Theorem I.2.19) and if $z_1, z_2 \in C$ are distinct points, we denote by S_{z_1,z_2} the ruled surface $\mathbb{P}(\mathcal{O}_C(z_1) \oplus \mathcal{O}_C(z_2))$. A geometrical description of these surfaces via sequences of blowups and contractions from $C \times \mathbb{P}^1$ is given in Section I.2.4. Then we show that their automorphism groups are maximal connected algebraic subgroups. With Theorem A, this leads to Theorem B:

Theorem B. *Let C be a curve of genus g and G be a maximal connected algebraic subgroup of* $\mathrm{Bir}(C \times \mathbb{P}^1)$*. If $g \geq 2$ then G is conjugate to the maximal algebraic subgroup* $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$*, and if $g = 1$ then G is conjugate to one of the following:*

1. $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$*,*
2. $\mathrm{Aut}^\circ(S_{z_1,z_2})$ *where z_1 and z_2 are distinct points in C,*
3. $\mathrm{Aut}^\circ(\mathcal{A}_0)$*,*
4. $\mathrm{Aut}^\circ(\mathcal{A}_1)$*.*

The algebraic subgroups in $(1), (2), (3), (4)$ *are all maximal and are pairwise not conjugate. Moreover in case* (2)*, two algebraic subgroups* $\mathrm{Aut}^\circ(S_{z_1,z_2})$ *and* $\mathrm{Aut}^\circ(S_{z_1',z_2'})$ *are conjugate if and only if there exists* $f \in \mathrm{Aut}(C)$ *such that* $f(\{z_1, z_2\}) = \{z_1', z_2'\}$.

We have $\mathrm{Aut}^\circ(C\times\mathbb{P}^1) \simeq \mathrm{Aut}^\circ(C)\times\mathrm{PGL}_2$, which is isomorphic to $C\times\mathrm{PGL}_2$ if $g = 1$, or isomorphic to PGL_2 if $g \geq 2$. Hence the structure of $\mathrm{Aut}^\circ(C\times\mathbb{P}^1)$ as algebraic group is simple to understand. In Theorem I.3.23, we describe the other maximal connected algebraic groups of Theorem B as extensions of an elliptic curve by a linear group. The structures of $\mathrm{Aut}^\circ(\mathcal{A}_0)$ and $\mathrm{Aut}^\circ(\mathcal{A}_1)$ as extensions in Theorem I.3.23 are actually a direct consequence of Maruyama's theorem, and it has already been proven in a more general setting in [Lau20, Theorem 4.2, 2.(b) and 2.(c)]. However, our approach only uses elementary techniques of birational geometry to compute the kernel of the morphism $\mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ induced by Blanchard lemma (I.2.4), when S is isomorphic to $\mathcal{A}_0$ or S_{z_1,z_2} for some $z_1, z_2 \in C$. Moreover, we describe $\mathrm{Aut}^\circ(\mathcal{A}_1)$ as an extension by giving an explicit construction of the surface $\mathcal{A}_1$.

Combining Theorems A and B with general arguments from the theory of algebraic groups, we show the following equivalence:

Theorem C. *Let X be a surface. Then every connected algebraic subgroup of* $\mathrm{Bir}(X)$ *is contained in a maximal one if and only if X is not birationally equivalent to $C \times \mathbb{P}^1$ for some curve C of genus $g \geq 1$.*

Finally, we answer the question when the characteristic of $\mathbf{k}$ is 0, by giving the classification of all maximal connected algebraic subgroups in dimension 2 (Theorem D). If the characteristic is positive, we have a partial classification, see Proposition I.3.26 and remark I.3.27.

Theorem D. *Let X be a surface over a field $\mathbf{k}$ of characteristic 0. We denote by E the set of surfaces of the form $(C \times Y)/F$ where C is an elliptic curve, Y is a smooth curve of general type, and F is a finite subgroup of $\mathrm{Aut}^\circ(C)$ acting diagonally on $C \times Y$. The pairs $(X, \mathrm{Aut}^\circ(X))$ are classified as following:*

<table>
<tr><th>$\kappa(X)$</th><th>Representative of the birational class of X</th><th>$\mathrm{Aut}^\circ(X)$</th></tr>
<tr><td rowspan="2">$-\infty$</td><td>Rational surface</td><td>Maximal if and only if X is isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_n$ with $n \neq 1$. Else $\mathrm{Aut}^\circ(X)$ is conjugate to an algebraic subgroup of a maximal one.</td></tr>
<tr><td>Ruled surface (over a curve C of positive genus)</td><td>Maximal if and only if X is isomorphic to $C \times \mathbb{P}^1$, or $\mathcal{A}_0$, or $\mathcal{A}_1$, or S_{z_1,z_2} with $z_1, z_2 \in C$ (the three last cases happen only when C is an elliptic curve). Else $\mathrm{Aut}^\circ(X)$ is not maximal and fits into an infinite chain of strict inclusions.</td></tr>
<tr><td rowspan="4">0</td><td>Abelian surface</td><td>$\mathrm{Aut}^\circ(X) \simeq X$ if and only if X is an abelian surface; and in this case $\mathrm{Aut}^\circ(X)$ is maximal. Else, $\mathrm{Aut}^\circ(X)$ is trivial and is not maximal.</td></tr>
<tr><td>$K3$ surface</td><td rowspan="2">$\mathrm{Aut}^\circ(X)$ trivial and maximal.</td></tr>
<tr><td>Enriques surface</td></tr>
<tr><td>Bielliptic surface</td><td>$\mathrm{Aut}^\circ(X) \simeq C$ is an elliptic curve if and only if $X \simeq (C \times Y)/F$ where C, Y are elliptic curves and F is a finite group acting on C by translations, and acting also on Y not only by translations (equivalently, $Y/F \simeq \mathbb{P}^1$ and X is a bielliptic surface). In this case, $\mathrm{Aut}^\circ(X)$ is maximal. Else, $\mathrm{Aut}^\circ(X)$ is trivial and is not maximal.</td></tr>
<tr><td>1</td><td>Elliptic surface</td><td>$\mathrm{Aut}^\circ(X) \simeq C$ is an elliptic curve if and only if $X \simeq (C \times Y)/F$ where Y is a smooth curve of general type and F is a finite group acting diagonally on $C \times Y$ and by translations on C (i.e. $X \in E$). In this case, $\mathrm{Aut}^\circ(X)$ is maximal.
If X is birational to an element of E but $X \notin E$, then $\mathrm{Aut}^\circ(X)$ is trivial and not maximal. Else $\mathrm{Aut}^\circ(X)$ is trivial and maximal.</td></tr>
<tr><td>2</td><td>Surface of general type</td><td>$\mathrm{Aut}^\circ(X)$ trivial and maximal.</td></tr>
</table>

Acknowledgments. *I am thankful to Jérémy Blanc, Michel Brion, Serge Cantat, Gabriel Dill, Bruno Laurent, Julia Schneider, Ronan Terpereau, Immanuel van Santen, Egor Yasinsky, Sokratis Zikas, ShengYuan Zhao for interesting discussions and ideas, and to the referee for the careful lecture and many useful remarks.*

I.2 Preliminaries

I.2.1 Equivariance and maximality

In this section we reduce the question to the maximality of the automorphism groups of *minimal surfaces*, i.e. smooth projective surfaces without (-1)-curves. The idea has already been used in the rational case to study algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ (e.g. see [Bla09b], and [RZ18] when $\mathbf{k} = \mathbb{R}$).

Definition I.2.1. Let G be an algebraic group, X be a variety and $\alpha\colon G \to \mathrm{Bir}(X)$ be a group homomorphism.

(1) The map α is a *rational action* of G on X if there exists a non-empty open subset U of $G \times X$ such that:

  (i) The map $G \times X \dashrightarrow X$, $(g, x) \longmapsto \alpha(g)(x)$ is regular on U,

  (ii) For all $g \in G$, the open subset $U_g = \{x \in X, (g, x) \in U\}$ is dense in X and $\alpha(g)$ is regular on U_g.

(2) The map α is a *regular action* of G on X if the map $G \times X \to X$, $(g, x) \mapsto \alpha(g)(x)$ is a morphism of varieties.

If $G \subset \mathrm{Bir}(X)$ is an algebraic subgroup and $\phi\colon X \dashrightarrow Y$ is a birational map, there exists a unique rational action of G on Y which is induced by ϕ and such that the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, dashed] \arrow[d, dashed, "id\times\phi"'] & X \arrow[d, dashed, "\phi"] \\ G \times Y \arrow[r, dashed] & Y. \end{tikzcd}$$

A powerful and classical result on rational actions of algebraic groups is the *Regularization Theorem* due to Weil. A modern proof has been given in [Zai95] (see also [Kra18]).

Theorem I.2.2. [Wei55] *For every rational action of an algebraic group G on a variety X, there exists a variety Y and a birational map $X \dashrightarrow Y$ such that the induced action of G on Y is regular.*

We recall in Lemma I.2.4 the powerful Blanchard lemma.

Definition I.2.3. Let G be an algebraic group acting regularly on varieties X and Y. A birational map $\phi\colon X \dashrightarrow Y$ is *G-equivariant* if the following diagram is commutative:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, dashed, "id\times\phi"'] & X \arrow[d, dashed, "\phi"] \\ G \times Y \arrow[r] & Y. \end{tikzcd}$$

Lemma I.2.4. [BSU13, Proposition 4.2.1] *Let X and Y be varieties and $\phi\colon X \to Y$ be a proper morphism such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$. Let G be a connected algebraic group acting regularly on X. Then there exists a unique regular action of G on Y such that ϕ is G-equivariant.*

In this text, we will use Blanchard lemma in the case where X and Y are smooth projective surfaces or smooth curves (and more precisely, ϕ will be the contraction of (-1)-curves or the structure morphism of a $\mathbb{P}^1$-bundle). Then ϕ induces a morphism of algebraic groups $\phi_* : \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$. The following proposition is a classical result (see also [LU21, Proposition 3.11] for a modern proof using actions on CAT(0) cubes complexes):

Proposition I.2.5. *Let X be a surface and G be a connected algebraic subgroup of* $\mathrm{Bir}(X)$*. Then G is conjugate to an algebraic subgroup of* $\mathrm{Aut}^\circ(S)$*, where S is a minimal surface.*

Proof. First we can apply the Regularization Theorem of Weil on X to get a surface Y birationally equivalent to X and equipped with a regular action of G. Replace Y by its smooth locus and from [Bri17, Theorem 1], there exists a non empty open subset U of Y which is G-stable and quasi-projective. Then by [Bri17, Theorem 2], the open U admits a G-equivariant completion into a projective variety $\overline{Y}$ which can be desingularized: by [Lip78, Remark B p.155], there exists a birational morphism $\delta\colon \widetilde{Y} \to \overline{Y}$ such that $\widetilde{Y}$ is a smooth projective variety and δ is obtained by successive blowups of singular points and normalizations. Hence the action of G on $\overline{Y}$ lifts to $\widetilde{Y}$ so that δ is G-equivariant. The contraction of (-1)-curves of $\widetilde{Y}$ is G-equivariant from Blanchard lemma, so we conclude that G is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(S)$, where S is a minimal surface. □

Apply Proposition I.2.5 to a surface X birationally equivalent to $C\times\mathbb{P}^1$ with C a curve. Then from [Har77, Examples V.5.8.2, V.5.8.3 and Remark V.5.8.4], the minimal surface S is either $\mathbb{P}^2$ or a ruled surface over C. The following lemma will be useful to check if $\mathrm{Aut}^\circ(S)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$.

Lemma I.2.6. *Let S be a projective surface and G be a connected algebraic subgroup of* $\mathrm{Aut}(S)$*. Then the following hold:*

(1) *The algebraic subgroup G is maximal if and only if for every projective surface T and G-equivariant birational map $\phi\colon S \dashrightarrow T$, we have $\phi G\phi^{-1} =$* $\mathrm{Aut}^\circ(T)$.

(2) *For every projective surface T and G-equivariant birational map $\phi\colon S \dashrightarrow T$, there exist $\beta\colon X \to S$ and $\kappa\colon X \to T$ compositions of blowups of fixed points of the G-action such that $\phi = \kappa\beta^{-1}$.*

(3) *Assume moreover that S is a minimal surface and $\mathrm{Aut}^\circ(S)$ acts on S without fixed points, then every $\mathrm{Aut}^\circ(S)$-equivariant birational map $\phi\colon S \dashrightarrow T$ with T projective, is an isomorphism. In particular $\mathrm{Aut}^\circ(S)$ is maximal.*

Proof.

(1) If $\phi\colon S \dashrightarrow T$ is a G-equivariant birational map, we have $\phi G\phi^{-1} \subset \mathrm{Aut}^\circ(T)$ (see Definition I.2.3). Since G is maximal, the inclusion is an equality. Conversely assume by contraposition that G is not maximal, then it is strictly contained in a connected algebraic subgroup H of $\mathrm{Bir}(S)$. From Proposition I.2.5 there exists a minimal surface T and a birational map $\phi : S \dashrightarrow T$ such that H is conjugate to a connected algebraic subgroup of $\mathrm{Aut}^\circ(T)$, i.e. $\phi G\phi^{-1} \subsetneq \mathrm{Aut}^\circ(T)$.

(2) Every birational map $\phi\colon S \dashrightarrow T$ can be decomposed as $\phi = \kappa\beta^{-1}$ with $\beta\colon X \to S$ and $\kappa\colon X \to T$ compositions of blowups of smooth points, and we can assume that κ and β do not contract the same (-1)-curves in X. Then for all $g \in G$:

$$\mathrm{Bs}(\phi) = \mathrm{Bs}(g^{-1}\phi g) = \mathrm{Bs}(\phi g) = g^{-1}(\mathrm{Bs}(\phi)).$$

Therefore, the base points of ϕ are fixed points for the G-action, so β consists in the blowup of fixed points of the G-action, which is G-equivariant by the universal property of the blowup. Similarly, the morphism κ is also G-equivariant.

(3) Because S is minimal, there is no contraction and since $\mathrm{Aut}^\circ(S)$ has no fixed point, there is no $\mathrm{Aut}^\circ(S)$-equivariant blowup. Therefore, every $\mathrm{Aut}^\circ(S)$-equivariant map from S to a projective surface T is an isomorphism from (2). From (1), $\mathrm{Aut}^\circ(S)$ is maximal .

□

I.2.2 Generalities on ruled surfaces

First we want to classify algebraic subgroups of $\mathrm{Bir}(C\times\mathbb{P}^1)$ as stated in Theorem B. Then Proposition I.2.5 suggests studying the maximality of $\mathrm{Aut}^\circ(S)$ when S is a minimal surface birationally equivalent to $C \times \mathbb{P}^1$, i.e. a geometrically ruled surface. Since this object will play an important role, we recall in this subsection the definition and some basic properties.

Definition I.2.7. A *geometrically ruled surface*, or simply *ruled surface*, is a surface S equipped with a morphism $\pi\colon S \to C$ where C is a curve, and such that all fibers of π are isomorphic to $\mathbb{P}^1$. A *section* of S is a morphism $\sigma\colon C \to S$ such that $\pi\sigma = id$. Through misuse of language, the image of σ is also called a *section*.

Notice that Definition I.2.7 is equivalent to the definition of geometrically ruled surface given in [Har77, Section V.2], since Hartshorne mentions that the existence of a section is provided by Tsen's theorem.

Definition I.2.8. A $\mathbb{P}^1$-*bundle* S over a curve C is a morphism $\pi\colon S \to C$ endowed with an open cover $(U_i)_i$ of C with isomorphisms $g_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{P}^1$, such that for all i the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "g_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{P}^1 \arrow[dl, "p_1"] \\ & U_i, & \end{tikzcd}$$

where p_1 denotes the projection on the first factor. The morphism π is called the *structural morphism* and the open cover $(U_i, g_i)_i$ is called a *trivializing open cover* of C. We denote by U_{ij} the open subset $U_i \cap U_j$ and the *transition maps* are $\tau_{ij} \in \mathrm{PGL}_2(\mathcal{O}_C(U_{ij}))$ so that $g_i g_j^{-1}$ is equal to:

$$\begin{aligned} U_{ij} \times \mathbb{P}^1 &\to U_{ij} \times \mathbb{P}^1 \\ (x, [u:v]) &\mapsto (x, \tau_{ij}(x) \cdot [u:v]) \,. \end{aligned}$$

Let $\pi_1\colon S_1 \to C$ and $\pi_2\colon S_2 \to C$ be $\mathbb{P}^1$-bundles over C. A C-*isomorphism* (or an isomorphism of $\mathbb{P}^1$-bundles) $f\colon S_1 \to S_2$ is an isomorphism of varieties such that $\pi_1 = \pi_2 f$. If moreover $S_1 = S_2$ then f is called a C-*automorphism* of S. We denote by $\mathrm{Aut}_C(S) \subset \mathrm{Aut}(S)$ the subgroup of C-automorphisms of S.

From Definition I.2.8, we see that a $\mathbb{P}^1$-bundle over C is also a ruled surface over C. Conversely, ruled surfaces $\pi\colon S \to C$ are also $\mathbb{P}^1$-bundles over C (see e.g. [Har77, Proposition V.2.2]). If V is a vector bundle of rank 2 over C, we denote by $\mathbb{P}(V)$ the $\mathbb{P}^1$-bundle over C obtained by projectivization of V. Recall that all $\mathbb{P}^1$-bundles over C are obtained by projectivization of a vector bundle of rank 2 over C (see e.g. [Har77, II. Exercise 7.10]).

Definition I.2.9. Let V be a vector bundle of rank 2. We say that V is *decomposable* if $V \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$, for some $\mathcal{L}_1$ and $\mathcal{L}_2$ line subbundles of V. If V is not decomposable, we say that V is *indecomposable*. We also say that $\mathbb{P}(V)$ is *decomposable* (resp. *indecomposable*) if V is decomposable (resp. indecomposable).

Lemma I.2.10. *Let S be a $\mathbb{P}^1$-bundle over a curve C and let σ_1, σ_2, σ_3 be sections of S. The following hold:*

(1) *There exists a trivialization of S such that σ_1 is the infinity section: i.e. for all U_i trivializing open subsets of C we have $\sigma_{1|U_i}(x) = (x, [1:0])$ and the transition maps of S are upper triangular matrices:*

$$\begin{aligned} U_{ij} &\to \mathrm{PGL}_2(\mathcal{O}_C(U_{ij})) \\ x &\mapsto \begin{bmatrix} a_{ij}(x) & c_{ij}(x) \\ 0 & b_{ij}(x) \end{bmatrix}. \end{aligned}$$

(2) *If σ_1 and σ_2 are disjoint then there exists a trivialization of S such that σ_1 is the infinity section and σ_2 is the zero section, i.e. for all U_i trivializing*

open subset of C we have $\sigma_{2|U_i}(x) = (x, [0:1])$. Moreover, the transition maps of S are diagonal matrices:

$$\begin{aligned} U_{ij} &\to \mathrm{PGL}_2(\mathcal{O}_C(U_{ij})) \\ x &\mapsto \begin{bmatrix} a_{ij}(x) & 0 \\ 0 & b_{ij}(x) \end{bmatrix}. \end{aligned}$$

(3) *If σ_1, σ_2, σ_3 are pairwise disjoint then there exists a trivialization of S such that: σ_1 is the infinity section, σ_2 is the zero section, and σ_3 is the section defined on all U_i trivializing open subset of C as $\sigma_{3|U_i}(x) = (x, [1:1])$. Moreover, S is isomorphic to $C \times \mathbb{P}^1$.*

Proof. (1) Let $(U_i)_i$ be a trivializing open cover of C. For all i, we have $\sigma_{1|U_i}(x) = (x, [u_{1i}(x) : v_{1i}(x)])$ with $u_{1i}, v_{1i} \in \mathcal{O}_C(U_i)$. If u_{1i} and v_{1i} both vanish at $z \in U_i$ with respectively multiplicities m_u and m_v, $m = \min(m_u, m_v)$, and f is a local parameter at z, then $\sigma_{1U_i}(x) = (x, [u_{1i}(x)/f(x)^m : v_{1i}(x)/f(x)^m])$. We can assume that u_{1i} and v_{1i} do not vanish simultaneously and by refining the open cover $(U_i)_i$, we can also assume that either $u_{1i} \in \mathcal{O}_C(U_i)^*$ or $v_{1i} \in \mathcal{O}_C(U_i)^*$. Then one can compose $\sigma_{1|U_i}$ and the charts on the left by the automorphisms of $U_i \times \mathbb{P}^1$:

$$(x, [u:v]) \mapsto \begin{cases} \left(x, \begin{bmatrix} 1 & 0 \\ -v_{1i}(x) & u_{1i}(x) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}\right) \text{ if } u_{1i}(x) \neq 0 \text{ on } U_i, \\ \left(x, \begin{bmatrix} 0 & 1 \\ -v_{1i}(x) & u_{1i}(x) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}\right) \text{ if } v_{1i}(x) \neq 0 \text{ on } U_i. \end{cases}$$

Under this trivialization of S, the section σ_1 is the infinity section and $[1:0]$ is preserved by the transition maps, which have to be upper triangular matrices.

(2) First apply (1) so that σ_1 is the infinity section. If $\sigma_{2|U_i}(x) = (x, [u_{2i}(x) : v_{2i}(x)])$, we also can assume $v_{2i} \in \mathcal{O}_C(U_i)^*$ as in (1) if needed. Then we compose by the following automorphisms of $U_i \times \mathbb{P}^1$:

$$(x, [u:v]) \mapsto \left(x, \begin{bmatrix} v_{2i}(x) & -u_{2i}(x) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}\right).$$

Under this trivialization of S, the section σ_1 remains the infinity section and σ_2 is the zero section. Moreover, $[1:0]$ and $[0:1]$ are preserved by the transition maps, which have to be diagonal matrices.

(3) First apply (2) so that σ_1 is the infinity section and σ_2 is the zero section. On the trivializing open subset U_i, we can write $\sigma_3(x) = (x, [u_{3i}(x) : v_{3i}(x)])$ with $u_{3i}, v_{3i} \in \mathcal{O}_C(U_i)^*$ as in (1). Then we compose by the following automorphisms of $U_i \times \mathbb{P}^1$:

$$(x, [u:v]) \mapsto \left(x, \begin{bmatrix} 1/u_{3i}(x) & 0 \\ 0 & 1/v_{3i}(x) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}\right).$$

Under this trivialization of S, the sections σ_1 and σ_2 remain respectively the infinity section and the zero section; and σ_3 is the constant section $x \mapsto (x, [1:$

1]) on every trivializing open subset of C. This implies that the transition maps of S are the identity matrices i.e. S is trivial.

$\square$

Lemma I.2.11. *Let $\mathbb{P}(V)$ be a $\mathbb{P}^1$-bundle over a curve C, and σ be a section of $\mathbb{P}(V)$ given locally by:*

$$\begin{aligned}\sigma_i\colon U_i &\to U_i \times \mathbb{P}^1\\ x &\mapsto (x, [u_i(x) : v_i(x)]).\end{aligned}$$

For all i, we define $\mathcal{L}_i = \{(x, (\lambda u_i(x), \lambda v_i(x))) \in U_i \times \mathbb{A}^2, \lambda \in \mathbf{k}\} \simeq U_i \times \mathbb{A}^1$ and the line subbundle $\pi\colon \mathcal{L}(\sigma) \to C$ of V such that $\pi^{-1}(U_i) = \mathcal{L}_i$. Then the following hold:

1. *The map $\sigma \mapsto \mathcal{L}(\sigma)$ is a bijection between the set of sections of $\mathbb{P}(V)$ and the set of line subbundles of V over C.*
2. *Two sections σ_1 and σ_2 are disjoint if and only if $\mathbb{P}(V)$ is C-isomorphic to $\mathbb{P}(\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2))$.*

Proof. (1) If $\mathcal{L}$ is a line subbundle of V, we have for all i an embedding $\mathcal{L}_{|U_i} \hookrightarrow V_{|U_i}$ which induces by projectivisation an embedding $\sigma_i\colon U_i \to U_i \times \mathbb{P}^1$. Since the family $(\mathcal{L}_{|U_i})_i$ glues into $\mathcal{L}$, the morphisms $(\sigma_i)_i$ glue into a section σ of $\mathbb{P}(V)$. This construction is the inverse of the map $\sigma \mapsto \mathcal{L}(\sigma)$.

(2) Let σ_1 and σ_2 be disjoint sections of $\mathbb{P}(V)$. From Lemma I.2.10 (2), we can assume that σ_1 is the infinity section and σ_2 is the zero section, and the transition maps of $\mathbb{P}(V)$ are:

$$\begin{aligned}U_{ij} &\to \mathrm{PGL}_2(\mathcal{O}_C(U_{ij}))\\ x &\mapsto \begin{bmatrix} a_{ij}(x) & 0 \\ 0 & b_{ij}(x)\end{bmatrix}.\end{aligned}$$

The coefficients a_{ij} and b_{ij} don't vanish on U_{ij}, and we can choose $x \mapsto a_{ij}(x)$ as the transition maps of $\mathcal{L}(\sigma_1)$ and $x \mapsto b_{ij}(x)$ as the transition maps of $\mathcal{L}(\sigma_2)$, i.e. $\mathbb{P}(V) \simeq \mathbb{P}(\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2))$. Conversely if we have $\mathbb{P}(V) \simeq \mathbb{P}(\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2))$, one can choose a trivializing open cover so that the transition maps are given by diagonal matrices. Under this choice, the section σ_1 is the zero section and σ_2 is the infinity section, thus they are disjoint.

$\square$

I.2.3 Segre invariant

In this subsection we recall the Segre invariant and its properties. This invariant has already been used by Maruyama in his classification of ruled surfaces [Mar70]. One can also check that the Segre invariant corresponds to $-e$, where e is the invariant defined in [Har77, V. Proposition 2.8]. If c and c' are curves in a smooth projective surface S, we denote by $c \cdot c'$ their intersection number and if $c = c'$, we denote it by c^2.

Definition I.2.12. Let $S \to C$ be a ruled surface. The *Segre invariant* $\mathfrak{S}(S)$ of S is defined as the quantity:

$$\min\{\sigma^2, \sigma \text{ section of } S\}.$$

A *minimal section* of S is a section σ of S such that $\sigma^2 = \mathfrak{S}(S)$.

The Segre invariant is well-defined since all ruled surfaces are obtained from $C \times \mathbb{P}^1$ by finitely many elementary transformations (see e.g. [Har77, Exercise V.5.5]) and $\mathfrak{S}(C \times \mathbb{P}^1) = 0$ (see Lemma I.2.14).

Definition I.2.13. A line subbundle $\mathcal{M}$ of a rank-2 vector bundle V is *maximal* if its degree is maximal among all line subbundles of V.

One can use Riemann-Roch theorem to show that the degree of line subbundles is bounded above, but it follows also from Proposition I.2.15 and the fact that the Segre invariant is well-defined.

In explicit computations, we will often use that the group of divisors up to numerical equivalence of a ruled surface S is generated by the class of a section σ and a fibre f, and they satisfy $f^2 = 0$ and $\sigma \cdot f = 1$ (see [Har77, Proposition V.2.3]). The next lemma is partially contained in [Har77, Exercise V.2.4].

Lemma I.2.14. *Let C be a curve of genus g and σ be a section of $C \times \mathbb{P}^1$ defined as $C \xrightarrow{\sigma} C \times \mathbb{P}^1$, $x \mapsto (x, g_\sigma(x))$ where $g_\sigma : C \to \mathbb{P}^1$ is a morphism. Then:*

$$\sigma^2 = 2\deg(g_\sigma),$$

and in particular, each section of $C \times \mathbb{P}^1$ has an even and non negative self-intersection number. In particular, $\mathfrak{S}(C \times \mathbb{P}^1) = 0$. Moreover, if $g > 0$ then there is no section of self-intersection 2, and if $g = 1$ then there exist sections of self-intersection 4.

Proof. Let σ be a section of $C \times \mathbb{P}^1$, and write $\sigma = (x \mapsto (x, g_\sigma(x)))$ where $g_\sigma \colon C \to \mathbb{P}^1$ is a morphism. The section σ is numerically equivalent to $a\sigma_c + bf$, where σ_c and f respectively denote the numerical class of a constant section and of a fibre, and $a, b \in \mathbb{Z}$ ([Har77, Proposition V.2.3]). Intersecting σ with f and with σ_c, one finds respectively that $a = 1$ and $b = \sigma_c \cdot \sigma$. Since all constant sections are linearly equivalent, and for a general constant section the quantity $\sigma_c \cdot \sigma$ corresponds to $\deg(g_\sigma)$, we get that $\sigma \equiv \sigma_c + \deg(g_\sigma) f$. Consequently, we have $\sigma^2 = 2\deg(g_\sigma) \geq 0$. Because constant sections have self-intersection 0, it follows that $\mathfrak{S}(C \times \mathbb{P}^1) = 0$. If moreover $g > 0$ then there does not exist a morphism $C \to \mathbb{P}^1$ of degree 1 and it implies that there is no section of self-intersection 2 in $C \times \mathbb{P}^1$. But if $g = 1$ then there exist morphisms $C \to \mathbb{P}^1$ of degree 2, hence there exist sections of self-intersection 4 in $C \times \mathbb{P}^1$. □

In [Mar70, Lemma 1.15] has been stated Corollary I.2.16 which provides an alternative way to compute the Segre invariant of a ruled surface. However, it is a consequence of the more general and useful statement given in Proposition I.2.15, which also follows from Maruyama's proof. We give a simple proof of Proposition I.2.15 based on a direct computation in local charts.

Proposition I.2.15. *Let $\pi\colon \mathbb{P}(V) \to C$ be a $\mathbb{P}^1$-bundle and σ be a section. Then the following equality holds:*

$$\sigma^2 = \deg(V) - 2\deg(\mathcal{L}(\sigma)),$$

where $\deg(V)$ *is the degree of the determinant line bundle of* V *and* $\mathcal{L}(\sigma)$ *is the line subbundle associated to* σ *(see Lemma I.2.11).*

Proof. From Lemma I.2.10 (1), we can assume that σ is the infinity section and the transition maps of $\mathbb{P}(V)$ are upper triangular matrices. Let (U_0, g_0) be a trivializing open subset of V, and let σ_0 and f_0 be defined as below:

$$\begin{aligned} \sigma_0\colon U_0 &\longrightarrow U_0 \times \mathbb{P}^1 & \qquad f_0\colon U_0 \times \mathbb{P}^1 &\dashrightarrow \mathbf{k} \\ x &\longmapsto (x, [0:1]), & (x, [u:v]) &\longmapsto \frac{u}{v}. \end{aligned}$$

The morphism σ_0 extends to a section defined over C which is disjoint from σ on U_0, and f_0 extends to a rational function f over $\mathbb{P}(V)$. We have:

$$\mathrm{div}(f) = \sigma_0 - \sigma + \sum_{z \in C\setminus U_0} \nu_{\pi^{-1}(z)}(f) \cdot \pi^{-1}(z),$$

where $\nu_{\pi^{-1}(z)}$ denotes the valuation along the fiber $\pi^{-1}(z)$. In consequence:

$$\sigma^2 = \sigma \cdot \sigma_0 + \sum_{z \in C\setminus U_0} \nu_{\pi^{-1}(z)}(f). \tag{I.1}$$

The subset $C \setminus U_0$ has finitely many points and for each $z \in C \setminus U_0$, we can choose a trivializing open neighborhood (U_z, g_z) of z and we denote by τ_{0z} the transition map defining $g_0 g_z^{-1}$:

$$\begin{aligned} \tau_{0z}\colon U_{0z} &\to \mathrm{PGL}_2(\mathcal{O}_C(U_{0z})) \\ x &\mapsto \begin{bmatrix} a_{0z}(x) & c_{0z}(x) \\ 0 & b_{0z}(x) \end{bmatrix}. \end{aligned}$$

The coordinates of the section σ_0 above $U_{0z} \subset U_z$ are solutions of the equation $\tau_{0z}(x) \cdot [u(x) : v(x)] = [0 : 1]$, i.e. $g_z g_0^{-1} \sigma_0(x) = (x, [-c_{0z}(x) : a_{0z}(x)])$. The rational map $x \mapsto [-c_{0z}(x) : a_{0z}(x)]$ extends to U_z and if we denote by ν_z the valuation in $\mathcal{O}_{C,z}$, then the two sections σ and σ_0 intersect above z if and only if $\nu_z(c_{0z}) < \nu_z(a_{0z})$. When they intersect, the intersection number equals $\nu_z(a_{0z}) - \nu_z(c_{0z})$: this quantity is independent of the choice of the trivializing open subset U_z and of the choice of a_{0z}, b_{0z}, c_{0z}. This implies that:

$$\sigma \cdot \sigma_0 = \sum_{z \in C\setminus U_0} \max(\nu_z(a_{0z}) - \nu_z(c_{0z}), 0). \tag{I.2}$$

Moreover, we have $f_{|U_z}(x, [u:v]) = f_0\left(x, \tau_{0z}(x) \cdot [u:v]\right) = \frac{a_{0z}(x)u + c_{0z}(x)v}{b_{0z}(x)v}$ since the following diagram is commutative:

$$\begin{tikzcd} U_{0z} \times \mathbb{P}^1 \arrow[rr, "g_0 g_z^{-1}"] \arrow[dr, dashed, "f_{|U_z}"'] & & U_{0z} \times \mathbb{P}^1 \arrow[dl, dashed, "f_0"] \\ & \mathbf{k}. & \end{tikzcd}$$

Let $r = \min(\nu_z(a_{0z}), \nu_z(c_{0z}))$ and $s = \nu_z(b_{0z})$. If t is a generator of the maximal ideal $\mathfrak{m}_{C,z} \subset \mathcal{O}_{C,z}$ then there exist $\widetilde{a}_{0z}, \widetilde{b}_{0z}, \widetilde{c}_{0z} \in \mathcal{O}_C(U_{0z})$ with $\widetilde{b}_{0z} \in \mathcal{O}^*_{C,z}$ and $\widetilde{a}_{0z}$ or $\widetilde{c}_{0z}$ in $\mathcal{O}^*_{C,z}$, such that $a_{0z} = t^r\widetilde{a}_{0z}$, $c_{0z} = t^r\widetilde{c}_{0z}$ and $b_{0z}(x) = t^s\widetilde{b}_{0z}$. We obtain:

$$\nu_{\pi^{-1}(z)}(f) = \nu_{\pi^{-1}(z)}\left(\frac{a_{0z}(x)u + c_{0z}(x)v}{b_{0z}(x)v}\right) = (r-s)+\nu_{\pi^{-1}(z)}\left(\frac{\widetilde{a}_{0z}(x)u + \widetilde{c}_{0z}(x)v}{\widetilde{b}_{0z}(x)v}\right),$$

and $\nu_p\left(\frac{\widetilde{a}_{0z}(x)u+\widetilde{c}_{0z}(x)v}{\widetilde{b}_{0z}(x)v}\right) = 0$ for a general point $p \in \pi^{-1}(z)$. Therefore:

$$\nu_{\pi^{-1}(z)}(f) = r - s = \min(\nu_z(a_{0z}), \nu_z(c_{0z})) - \nu_z(b_{0z}), \tag{I.3}$$

which is also independent of the choice of the trivializing open U_z and of the choice of a_{0z}, b_{0z}, c_{0z}. Then by substituting (I.2) and (I.3) in (I.1), we get:

$$\sigma^2 = \sum_{z\in C\setminus U_0} \nu_z(a_{0z}) - \nu_z(b_{0z}).$$

Since σ is the infinity section, the line subbundle $\mathcal{L}(\sigma)$ of V is defined by $\{(x,(\lambda,0)) \in U \times \mathbb{A}^2, \lambda \in \mathbf{k}\}$ on every trivializing open subset U. Hence we can choose the transition map of $\mathcal{L}(\sigma)$ on U_{0z} as $x \mapsto a_{0z}(x)$ and the transition map of $V/\mathcal{L}(\sigma)$ on U_{0z} as $x \mapsto b_{0z}(x)$. Let $a\colon C \to \mathcal{L}(\sigma) \subset V$ and $b\colon C \to V/\mathcal{L}(\sigma)$ be the rational sections defined by:

$$\begin{aligned} a\colon U_0 &\to \mathcal{L}(\sigma)_{|U_0} \\ x &\mapsto (x,1), \end{aligned} \qquad\qquad \begin{aligned} b\colon U_0 &\to (V/\mathcal{L}(\sigma))_{|U_0} \\ x &\mapsto (x,1). \end{aligned}$$

Up to a multiple, we have that a_{0z}^{-1} and b_{0z}^{-1} are respectively the coordinates of the sections a and b on U_z. Finally we have that $\sigma^2 = \sum_{z\in C} \nu_z(b) - \nu_z(a) = \deg(V/\mathcal{L}(\sigma)) - \deg(\mathcal{L}(\sigma))$. Using the additivity of the degree on the short exact sequence $0 \to \mathcal{L}(\sigma) \to V \to V/\mathcal{L}(\sigma) \to 0$, we deduce that

$$\sigma^2 = \deg(V) - 2\deg(\mathcal{L}(\sigma)).$$

□

Propositions I.2.11 (1) and I.2.15 imply that σ is a minimal section of $\mathbb{P}(V)$ if and only if $\mathcal{L}(\sigma)$ is a maximal line subbundle of V. In particular, we have the following corollary that can also be found in [Har77, Proposition V.2.9]:

Corollary I.2.16. *Let $S = \mathbb{P}(V)$ be a $\mathbb{P}^1$-bundle over a curve C and $\mathcal{M}$ be a maximal line subbundle of V. Then the following equality holds:*

$$\mathfrak{S}(S) = \deg(V) - 2\deg(\mathcal{M}),$$

where $\deg(V)$ *is the degree of the determinant line bundle of V.*

The main use of the Segre invariant is given in Proposition I.2.18, which is partially stated in [Mar70, Corollary 1.17] without proof, and partially proven in [Har77, Theorem V.2.12].

Lemma I.2.17. *Let S be a $\mathbb{P}^1$-bundle over a curve C. Two sections of S having the same self-intersection number are numerically equivalent.*

Proof. Let σ_1 and σ_2 be sections of S having the same self-intersection number. Let $\mathrm{Num}(S)$ be the group of divisor classes up to numerical equivalence. Then $\mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by K_S and a fiber f (see [Har77, Proposition V.2.3]), and from the arithmetic genus formula:

$$\frac{1}{2}(\mathrm{K}_S + \sigma_1) \cdot \sigma_1 + 1 = g(C) = \frac{1}{2}(\mathrm{K}_S + \sigma_2) \cdot \sigma_2 + 1,$$

it implies that $\mathrm{K}_S \cdot \sigma_1 = \mathrm{K}_S \cdot \sigma_2$. In particular, the sections σ_1 and σ_2 are numerically equivalent. $\square$

Proposition I.2.18. *Let $S = \mathbb{P}(V)$ be a $\mathbb{P}^1$-bundle over a curve C. The following assertions hold:*

(1) *if $\mathfrak{S}(S) > 0$ then S is indecomposable.*

(2) *if $\mathfrak{S}(S) < 0$ then S admits a unique minimal section.*

(3) *if $\mathfrak{S}(S) = 0$ then:*

 (i) *any two distinct minimal sections of S are disjoint,*

 (ii) *S is indecomposable if and only if S has a unique minimal section,*

 (iii) *S is decomposable and not trivial if and only if S has exactly two minimal sections,*

 (iv) *S is trivial if and only if S has at least three minimal sections.*

Proof. (1) Assume that $V \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$ is decomposable and let $\mathcal{M}$ be a maximal line subbundle of V. Then $\deg(V) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2) \leq 2\deg(\mathcal{M})$ and from Corollary $I.2.16$: $\mathfrak{S}(S) = \deg(V) - 2\deg(\mathcal{M}) > 0$ which is a contradiction.

(2) We assume that S admits two distinct minimal sections σ_1 and σ_2. From Lemma I.2.17, the sections σ_1 and σ_2 are numerically equivalent and therefore $\mathfrak{S}(S) = \sigma_2 \cdot \sigma_1 \geq 0$ and it is a contradiction.

(3) We assume $\mathfrak{S}(S) = 0$:

(i) Let σ_1 and σ_2 be distinct minimal sections. Since they are numerically equivalent from Lemma I.2.17: $0 = \mathfrak{S}(S) = \sigma_1^2 = \sigma_1 \cdot \sigma_2$ i.e. σ_1 and σ_2 are disjoint sections.

(ii) Assume by contraposition that S has two minimal sections: since they are disjoint from (i), it implies that S is decomposable. To prove the converse we assume by contraposition that S is decomposable, i.e. C-isomorphic to $\mathbb{P}(\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2))$ for some sections σ_1 and σ_2, and then we have $0 = \mathfrak{S}(S) = \deg(\mathcal{L}(\sigma_1)) + \deg(\mathcal{L}(\sigma_2)) - 2\deg(\mathcal{M})$. This implies that $\mathcal{L}(\sigma_1)$ and $\mathcal{L}(\sigma_2)$ are maximal line subbundles i.e. σ_1 and σ_2 are disjoint minimal sections of S. By Lemma I.2.11, S is decomposable.

(iii) If S is decomposable then S admits at least two minimal sections from (ii). Assume that S has three distinct minimal sections then they are pairwise disjoints from (i) and it implies that S is trivial from Lemma I.2.10 (3). So if S is decomposable and non trivial then S has exactly two minimal sections. Conversely, assume that S has exactly two minimal sections σ_1 and σ_2: it follows again from (i) and Lemma I.2.11 (2) that S is C-isomorphic to $\mathbb{P}(\mathcal{L}(\sigma_1)\oplus\mathcal{L}(\sigma_2))$ which is decomposable. Since the trivial bundle has infinitely many minimal sections, S cannot be the trivial bundle.

(iv) It follows from the equivalences of (ii) and (iii).

□

I.2.4 Construction of some ruled surfaces via elementary transformations

Let C be a curve and $z_1, z_2 \in C$. We denote by S_{z_1,z_2} the ruled surface $\mathbb{P}(\mathcal{O}_C(z_1) \oplus \mathcal{O}_C(z_2))$. If C is an elliptic curve, we denote by $\mathcal{A}_0$ and $\mathcal{A}_1$ the Atiyah $\mathbb{P}^1$-bundles defined in Theorem I.2.19. The classification of vector bundles over an elliptic curve given in [Ati57] is much more general than the statement we need for ruled surfaces, and the reader can also find it in [Har77, Example V.2.11.6 and Theorem V.2.15]:

Theorem I.2.19. [Ati57, Theorems 5, 6, 7, 10 and 11] *Let C be an elliptic curve and $p \in C$. There exist two indecomposable vector bundles $\mathcal{F}_0$ and $\mathcal{F}_1$ of rank 2 and respectively of degree 0 and 1, which fit into the exact sequences:*

$$0 \to \mathcal{O}_C \to \mathcal{F}_0 \to \mathcal{O}_C \to 0,$$
$$0 \to \mathcal{O}_C \to \mathcal{F}_1 \to \mathcal{O}_C(p) \to 0.$$

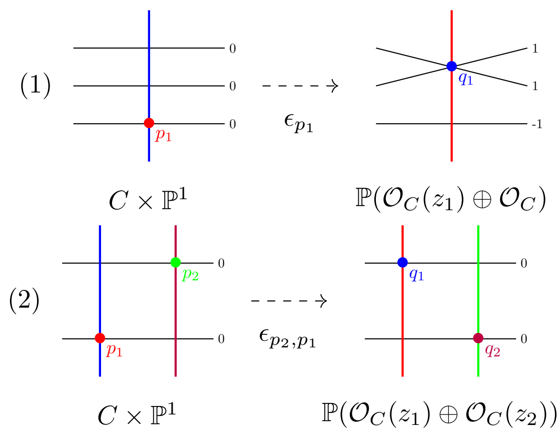
Moreover, the isomorphism class of $\mathbb{P}(\mathcal{F}_1)$ does not depend on the choice of p, and the Atiyah $\mathbb{P}^1$-bundles $\mathcal{A}_0 := \mathbb{P}(\mathcal{F}_0)$ and $\mathcal{A}_1 := \mathbb{P}(\mathcal{F}_1)$ are exactly the two C-isomorphism classes of indecomposable $\mathbb{P}^1$-bundles over C.

A ruled surface $S \to C$ is birationally equivalent to $C \times \mathbb{P}^1$. In this subsection, we give explicitly birational maps $C \times \mathbb{P}^1 \dashrightarrow S$ when S is isomorphic to $\mathcal{A}_0$, $\mathcal{A}_1$ or S_{z_1,z_2} for some $z_1, z_2 \in C$. We also deduce the Segre invariant of the Atiyah $\mathbb{P}^1$-bundles and of S_{z_1,z_2} for all $z_1, z_2 \in C$. Let S be a $\mathbb{P}^1$-bundle over a curve C, let $p \in S$ and f_p the fiber containing p. We denote by $\beta_p : \mathrm{Bl}_p(S) \to S$ the blowup of S at p and by E_p the exceptional divisor. The strict transform of f_p under the birational map β_p^{-1} is a (-1)-curve and we denote by $\kappa_p : \mathrm{Bl}_p(S) \to T$ its contraction. The *elementary transformation* of S centered on p is the birational map $\epsilon_p = \kappa_p\beta_p^{-1} \colon S \dashrightarrow T$. If $p_1, p_2 \in C \times \mathbb{P}^1$ are distinct points such that p_1 and p_2 are not on the same fiber, we denote by ϵ_{p_2,p_1} the blowups of p_1 and p_2 followed by the contractions of their respective fibers.

Proposition I.2.20. *Let C be an elliptic curve and $\pi \colon C \times \mathbb{P}^1 \to C$ be the projection on the first factor. Let $p_1, p_2 \in C \times \mathbb{P}^1$, we denote by $z_1 = \pi(p_1)$ and $z_2 = \pi(p_2)$. The following hold:*

(1) *The surface* $\epsilon_{p_1}(C\times\mathbb{P}^1)$ *is isomorphic to* $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$ *and* $\mathfrak{S}(\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)) = -1$. *Moreover, the base point* q_1 *of* $\epsilon_{p_1}^{-1}$ *is the unique point where all the sections of self-intersection* 1 *meet.*

(2) *Assume moreover that* p_1 *and* p_2 *are not in the same fiber and not in the same constant section. Then the surface* $\epsilon_{p_2,p_1}(C\times\mathbb{P}^1)$ *is isomorphic to* $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C(z_2))$ *and it has exactly two disjoint sections of self-intersection* 0. *We have* $\mathfrak{S}(\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C(z_2))) = 0$ *and if* q_1, q_2 *are the base points of* ϵ_{p_2,p_1}^{-1}, *then every section of self-intersection* 2 *passing through* q_1 *also passes through* q_2.



Proof. (1) Up to a C-automorphism of $C\times\mathbb{P}^1$ we can assume that $p_1 = (z_1,[1:0])$ and let U_1 be an open neighborhood of z_1. Let $f\in\mathbf{k}(C)^*$ which has a zero of order 1 at z_1, we can also assume that U_1 does not contain any zeros and poles of f except at z_1. Let $U_0 = C\setminus z_1$, we define:

$$\begin{aligned}\phi_0 : U_0\times\mathbb{P}^1 &\longrightarrow U_0\times\mathbb{P}^1 \\ (x,[u:v]) &\longrightarrow (x,[u:v])\end{aligned}\qquad\qquad \begin{aligned}\phi_1 : U_1\times\mathbb{P}^1 &\dashrightarrow U_1\times\mathbb{P}^1 \\ (x,[u:v]) &\longrightarrow (x,[f(x)u:v]).\end{aligned}$$

The domains of ϕ_0,ϕ_1 glue into $C\times\mathbb{P}^1$ and the codomains glue into a $\mathbb{P}^1$-bundle over C through the transition map:

$$\begin{aligned}U_0\cap U_1 &\to \mathrm{PGL}_2(\mathcal{O}_C(U_0\cap U_1)) \\ x &\mapsto \begin{bmatrix} f(x) & 0 \\ 0 & 1\end{bmatrix}.\end{aligned}$$

So ϕ_0 and ϕ_1 glue onto a birational map $\phi\colon C\times\mathbb{P}^1 \dashrightarrow \mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$. Since ϕ has p_1 as unique base point of order 1, one can check by describing the blowups in local charts that ϕ is the elementary transformation ϵ_{p_1}. Moreover, the strict transform by ϵ_{p_1} of the infinity section is the unique section of self-intersection number -1 and all the other sections in $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$ have self-intersection number at least 1, so $\mathfrak{S}(\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)) = -1$. Because C is an elliptic curve, there is no section of self-intersection 2 in $C\times\mathbb{P}^1$ from Lemma I.2.14 and the sections of self-intersection 1 in $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$ are exactly the strict transforms by ϵ_{p_1} of the constant sections of $C\times\mathbb{P}^1$, so they all pass through q_1 and it is their unique common intersection.

(2) Similarly we can assume $p_1=(z_1,[1:0])$ and $p_2=(z_2,[0:1])$. Let U_1 and U_2 be open trivializing neighborhoods of z_1 and z_2. Let $f,g\in\mathbf{k}(C)^*$ having a zero of order one respectively at z_1 and at z_2. We can assume that U_1 does not contain z_2 and any zeros or poles of f except z_1, and similarly we can assume that U_2 does not contain z_1 and any zeros or poles of g except z_2. Let $U_0=C\setminus\{z_1,z_2\}$ then we take ϕ_0 and ϕ_1 as in (1) and we define:

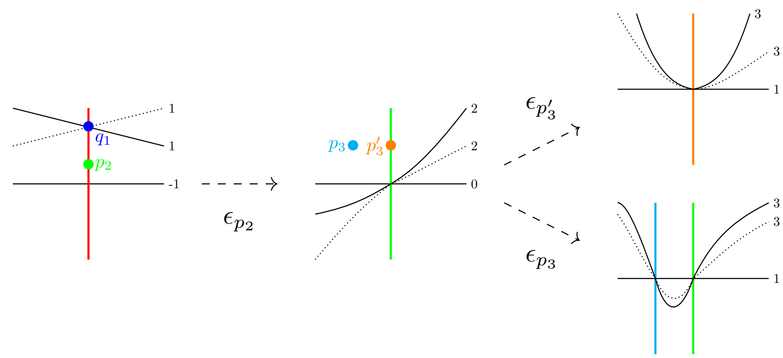
$$\begin{aligned}\phi_2: U_2\times\mathbb{P}^1 &\dashrightarrow U_2\times\mathbb{P}^1\\ (x,[u:v]) &\longmapsto (x,[u:g(x)v]).\end{aligned}$$

The maps ϕ_0,ϕ_1,ϕ_2 glue into a birational map $\phi: C\times\mathbb{P}^1\dashrightarrow\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C(z_2))$. Since ϕ has exactly two base points p_1 and p_2 of order 1, one can check by local equations of blowups that ϕ equals ϵ_{p_2,p_1}. From Lemma I.2.14 there is no section of self-intersection 2 in $C\times\mathbb{P}^1$, so the strict transform by ϵ_{p_2,p_1} of the infinity section and of the zero section of $C\times\mathbb{P}^1$ are the only sections of self-intersection number 0 in $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C(z_2))$, and all the other sections have self-intersection number at least 2. Therefore $\mathfrak{S}(\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C(z_2)))=0$. Finally, a section σ of self-intersection 2 passing through q_1 is the strict transform of a constant section in $C\times\mathbb{P}^1$ which also intersects the fiber of p_2. Thus σ also passes through q_2.

$\square$

Proposition I.2.21. *Let C be an elliptic curve. Let $p_1\in C\times\mathbb{P}^1$ such that $\pi(p_1)=z_1$ and let $\epsilon_{p_1}: C\times\mathbb{P}^1\dashrightarrow\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$. We denote by q_1 be the unique base point of $\epsilon_{p_1}^{-1}$. Then the following hold:*

1. *For all $p_2\in\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$ in the same fiber as q_1, such that $p_2\neq q_1$ and p_2 does not belong to the unique (-1)-section of $\mathbb{P}(\mathcal{O}_C(z_1)\oplus\mathcal{O}_C)$, the surface $\epsilon_{p_2}\epsilon_{p_1}(C\times\mathbb{P}^1)$ is C-isomorphic to $\mathcal{A}_0$. Moreover, $\mathcal{A}_0$ has a unique section σ_0 of self-intersection number 0 and all the other sections have self-intersection number at least 2. In particular $\mathfrak{S}(\mathcal{A}_0)=0$.*
2. *For all $p_3\in\mathcal{A}_0\setminus\sigma_0$, the surface $\epsilon_{p_3}(\mathcal{A}_0)$ is C-isomorphic to $\mathcal{A}_1$ and $\mathfrak{S}(\mathcal{A}_1)=1$.*



$\mathbb{P}(\mathcal{O}_C(z_1) \oplus \mathcal{O}_C)$ $\qquad$ $\mathcal{A}_0$ $\qquad$ $\mathcal{A}_1$

Proof. We prove (1) and (2) simultaneously. Let $p_2 \in \mathbb{P}(\mathcal{O}_C(z_1) \oplus \mathcal{O}_C)$ be as in the statement of the proposition. Because $\mathbb{P}(\mathcal{O}_C(z_1) \oplus \mathcal{O}_C)$ has a unique (-1)-section which does not contain p_2 and all sections of self-intersection number 1 pass through q_1, the strict transform under ϵ_{p_2} of the unique (-1)-section is the unique section σ_0 of self-intersection number 0 and all the other sections have self-intersection at least 2. In particular, $\mathfrak{S}(\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)) = 0$ and $\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)$ is indecomposable from Proposition I.2.18 (3)(ii). Then for all $p_3 \in \epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1) \setminus \sigma_0$, we have $\mathfrak{S}(\epsilon_{p_3}\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)) = 1$ so $\epsilon_{p_3}\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)$ is indecomposable from Proposition I.2.18 (1). At this stage, we know that $\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)$ and $\epsilon_{p_3}\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1)$ are the Atiyah $\mathbb{P}^1$-bundles defined in Theorem I.2.19. Moreover, we know that $\mathcal{A}_0 = \mathbb{P}(\mathcal{F}_0)$ and the indecomposable vector bundle $\mathcal{F}_0$ of rank 2 and degree 0 satisfies:

$$0 \to \mathcal{O}_C \to \mathcal{F}_0 \to \mathcal{O}_C \to 0.$$

From Corollary I.2.16, we have that $\mathfrak{S}(\mathcal{A}_0) = \deg(\mathcal{F}_0) - 2\deg(\mathcal{M}) = -2\deg(\mathcal{M})$ where $\mathcal{M}$ is a maximal line subbundle of $\mathcal{F}_0$. Since $\mathcal{O}_C$ is a line subbundle of $\mathcal{F}_0$, it follows that $\deg(\mathcal{M}) \geq 0$ and $\mathfrak{S}(\mathcal{A}_0) \leq 0$. Therefore $\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1) \simeq \mathcal{A}_0$ and $\epsilon_{p_3}\epsilon_{p_2}\epsilon_{p_1}(C \times \mathbb{P}^1) \simeq \mathcal{A}_1$. $\square$

I.3 The classification

I.3.1 Infinite inclusion chains of automorphism groups

In this subsection, we prove Theorem A. Let S be a $\mathbb{P}^1$-bundle over a curve C of genus g and $\mathrm{Aut}_C(S)$ be the subgroup of $\mathrm{Aut}(S)$ which induces the identity on C. If $g \geq 2$, it is known that $\mathrm{Aut}^\circ(C)$ is trivial (see e.g. [Har77, Exercise IV.2.5]) and this implies that $\mathrm{Aut}^\circ(S)$ is a subgroup of $\mathrm{Aut}_C(S)$. When $g = 1$ and $\mathfrak{S}(S) < 0$, it is still true that $\mathrm{Aut}^\circ(S)$ is a subgroup of $\mathrm{Aut}_C(S)$ by the following result:

Lemma I.3.1. [Mar71, Lemma 7] *If S is a $\mathbb{P}^1$-bundle over an elliptic curve C with $\mathfrak{S}(S) < 0$ then the image of $\mathrm{Aut}(S) \to \mathrm{Aut}(C)$ is a finite group.*

In [Mar71, Theorem 2], the C-automorphism groups of ruled surfaces over C are classified. We will not need the entire classification but we will use:

Lemma I.3.2. [Mar71, Theorem 2 (1) and case (b) p.92] *Let $S = \mathbb{P}(V)$ be a $\mathbb{P}^1$-bundle over a curve C, let σ be a section of S and $\mathcal{L}(\sigma)$ be the line subbundle of V associated to σ (see Lemma I.2.11). We choose trivializations of S such that σ is the infinity section (Lemma I.2.10 (1)). The following holds true:*

(1) *If $\mathfrak{S}(S) > 0$, then $\mathrm{Aut}_C(S)$ is finite.*

(2) *If $\mathfrak{S}(S) < 0$ and $\gamma \in \Gamma(C, \det(V)^{-1} \otimes \mathcal{L}(\sigma)^2)$, then the local isomorphisms:*

$$U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$$
$$(x, [u : v]) \mapsto \left(x, \begin{bmatrix} 1 & \gamma_{|U_i} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}\right)$$

glue into a C-automorphism f_γ of S.

Remark I.3.3. In the proof of Theorem A, we show that if $\mathfrak{S}(S) < 0$ is small enough, then there exists a non-zero $\gamma \in \Gamma(C, \det(V)^{-1} \otimes \mathcal{L}(\sigma)^2)$.

Remark I.3.4. The automorphism f_γ plays a crucial role in the proof of Theorem A, hence we recall Maruyama's construction. First we write the transition maps of S as $s_{ij} \colon U_j \times \mathbb{P}^1 \dashrightarrow U_i \times \mathbb{P}^1$, $(x, [u : v]) \mapsto (x, [a_{ij}u + c_{ij}v : b_{ij}v])$ where a_{ij} are the transition maps of the line bundle $\mathcal{L}(\sigma)$. Then b_{ij} are the transition maps of the line bundle $\det(V) \otimes \mathcal{L}(\sigma)^{-1}$. The local isomorphisms $f_{\gamma_i} \colon U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, $(x, [u : v]) \mapsto (x, [u + \gamma_i v : v])$, where $\gamma_i \in \mathcal{O}_C(U_i)$, glue into a C-automorphism of S if and only if $s_{ij} f_{\gamma_j} = f_{\gamma_i} s_{ij}$, and a direct computation shows that it is equivalent to the condition $a_{ij} b_{ij}^{-1} \gamma_j = \gamma_i$. In particular, $(\gamma_i)_i$ defines a section of the line bundle $\det(V)^{-1} \otimes \mathcal{L}(\sigma)^2$.

Proof of Theorem A. Assume first that $g \geq 2$ and $\mathfrak{S}(S) > 0$, then it follows from Lemma I.3.2 (1) that $\mathrm{Aut}^\circ(S)$ is trivial. If $g \geq 2$ and $\mathfrak{S}(S) = 0$ then from Lemma I.2.18 (3)(ii) and (iii), we know that S has at most two minimal sections because S is not isomorphic to $C \times \mathbb{P}^1$. Let p be a point on a minimal section, then every automorphism of $\mathrm{Aut}^\circ(S)$ has to fix p because $\mathrm{Aut}^\circ(S)$ is connected and $\mathrm{Aut}^\circ(C)$ is trivial. Therefore the elementary transformation $\epsilon_p : S \dashrightarrow T$ is $\mathrm{Aut}^\circ(S)$-equivariant, i.e. $\epsilon_p \mathrm{Aut}^\circ(S) \epsilon_p^{-1} \subset \mathrm{Aut}^\circ(T)$, and we have $\mathfrak{S}(T) = -1$. So when $g \geq 2$, it suffices to prove the theorem when $\mathfrak{S}(S) < 0$. In the statement of Theorem A, we suppose that $g \geq 2$, or $g = 1$ and $\mathfrak{S}(S) < 0$. From now on we assume that $\mathfrak{S}(S) < 0$ and $g \geq 1$. Then from Lemma I.2.18 (2), the ruled surface S has a unique minimal section σ and from Lemma I.3.1, the algebraic group $\mathrm{Aut}^\circ(S)$ is a subgroup of $\mathrm{Aut}_C(S)$. So any point p_0 of σ is fixed by the action of $\mathrm{Aut}^\circ(S)$ and it implies that $\epsilon_{p_0} : S \dashrightarrow S_1$ is $\mathrm{Aut}^\circ(S)$-equivariant. By induction, there exist $\mathbb{P}^1$-bundles S_n having a unique minimal section σ_n and p_n on σ_n such that $\epsilon_{p_n} : S_n \dashrightarrow S_{n+1}$ is $\mathrm{Aut}^\circ(S_n)$-equivariant. By denoting $\phi_n = \epsilon_{p_{n-1}} ... \epsilon_{p_1} \epsilon_{p_0}$, we get a sequence:

$$\mathrm{Aut}^\circ(S) \subseteq \phi_1^{-1} \mathrm{Aut}^\circ(S_1) \phi_1 \subseteq ... \subseteq \phi_n^{-1} \mathrm{Aut}^\circ(S_n) \phi_n \subseteq ...$$

and it remains to prove that the obtained sequence is not stationary. Suppose $S_{n+1} = \mathbb{P}(V_{n+1})$, we define $\mathcal{L}_{n+1} = \det(V_{n+1})^{-1} \otimes \mathcal{L}(\sigma_{n+1})^2$. Let q_n be the unique base point of $\epsilon_{p_n}^{-1}$ and we can assume that $q_n = (z, [0:1])$ over a trivializing open subset. If n is large enough then $\deg(\mathcal{L}_{n+1}) = -\mathfrak{S}(S_{n+1})$ is large enough, and it implies that $\mathrm{h}^1(C, \mathcal{L}_{n+1}) = \mathrm{h}^1(C, \mathcal{L}_{n+1} - z) = 0$ by Serre duality. From Riemann-Roch theorem: $\mathrm{h}^0(C, \mathcal{L}_{n+1} - z) = \deg(\mathcal{L}_{n+1}) - g < \deg(\mathcal{L}_{n+1}) - g + 1 = \mathrm{h}^0(C, \mathcal{L}_{n+1})$, and therefore z is not a base point of the complete linear system $|\mathcal{L}_{n+1}|$. Then there exists $\gamma \in \Gamma(C, \mathcal{L}_{n+1})$ such that $\gamma(z) \neq 0$, i.e. f_γ defines an automorphism of S_{n+1} which does not fix q_n. As a consequence, the automorphism f_γ defined in Lemma I.3.2 (2) does not belong to $\epsilon_n \mathrm{Aut}^\circ(S_n)\epsilon_n^{-1}$ and $\epsilon_n \mathrm{Aut}^\circ(S_n)\epsilon_n^{-1} \neq \mathrm{Aut}^\circ(S_{n+1})$ when n is taken large enough. $\square$

I.3.2 Maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$

In this subsection, we prove Theorem B. Let S be a ruled surface over a curve C of genus g such that $\mathrm{Aut}^\circ(S)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$. If $g \geq 2$, then S is isomorphic to $C \times \mathbb{P}^1$ by Theorem A. If $g = 1$, then $\mathfrak{S}(S) \geq 0$ by Theorem A. The following lemma determines the remaining cases when $g = 1$:

Lemma I.3.5. *Let S be a ruled surface over an elliptic curve C. If $\mathfrak{S}(S) \geq 0$ then S is isomorphic to one of the following:*

(1) $C \times \mathbb{P}^1$,

(2) $\mathcal{A}_0$,

(3) $\mathcal{A}_1$,

(4) S_{z_1,z_2} *for some distinct points* $z_1, z_2 \in C$.

Let $z_1, z_2 \in C$ be distinct points, then the surfaces $C \times \mathbb{P}^1$, $\mathcal{A}_0$, $\mathcal{A}_1$ and S_{z_1,z_2} are pairwise non-isomorphic.

Proof. If S is indecomposable, it follows from Theorem I.2.19 that S is isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$, and their Segre invariant is non-negative (Proposition I.2.21). If S is decomposable and $\mathfrak{S}(S) \geq 0$, it follows from Proposition I.2.18 that $\mathfrak{S}(S) = 0$ and S is C-isomorphic to $\mathbb{P}(\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2))$ where σ_1 and σ_2 are disjoint minimal sections. Tensoring $\mathcal{L}(\sigma_1) \oplus \mathcal{L}(\sigma_2)$ by a line bundle with degree $(-\deg(\mathcal{L}(\sigma_1)) + 1)$, it follows that S is isomorphic to $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$ as $\mathbb{P}^1$-bundle, and where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles of degree 1. Since C is an elliptic curve, the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively isomorphic to $\mathcal{O}_C(z_1)$ and $\mathcal{O}_C(z_2)$ for some $z_1, z_2 \in C$. Indeed, a line bundle of degree 1 over C corresponds to a divisor of degree 1 on C, and its complete linear system is a unique point by Riemann-Roch formula. If $z_1 = z_2$ then S is isomorphic to $C \times \mathbb{P}^1$, otherwise S is isomorphic to S_{z_1,z_2}. Finally, the Atiyah ruled surfaces are not isomorphic to each other from Theorem I.2.19, and they cannot be isomorphic to a decomposable $\mathbb{P}^1$-bundle. Since the surface S_{z_1,z_2} has exactly two sections of self-intersection 0 from Proposition I.2.20 (2), it cannot be isomorphic to $C \times \mathbb{P}^1$. □

If S is isomorphic to $C \times \mathbb{P}^1$ with C an elliptic curve, then $\mathrm{Aut}^\circ(S) \simeq C \times \mathrm{PGL}_2$ is a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$. From Proposition I.2.5, Lemma I.3.5 and Theorem A, we are left with studying the maximality of $\mathrm{Aut}^\circ(S)$ when S is $\mathcal{A}_0$, $\mathcal{A}_1$ and S_{z_1,z_2} for $z_1, z_2 \in C$.

Lemma I.3.6. *Let C be a curve of genus $g \geq 1$, let $\pi\colon S \to C$ and $\pi'\colon S' \to C$ be ruled surfaces. Then an isomorphism from S to S' induces an automorphism of C. If $S = S'$, we have a morphism of group schemes:*

$$\pi_*\colon \mathrm{Aut}(S) \to \mathrm{Aut}(C).$$

The restriction of π_* to the connected components of identity coincides with the morphism of algebraic groups $\mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ induced by Blanchard lemma (see Lemma I.2.4).

Proof. If $g\colon S \to S'$ is an isomorphism and f is a fiber in S, then $\pi' g_{|f}$ is a morphism from $f \simeq \mathbb{P}^1$ to C. Hence it is constant and the image of f by g is a fiber f' in S'. Then the isomorphism g induces a bijection of C. If $S = S'$, we get a morphism $\pi_*\colon \mathrm{Aut}(S) \to \mathrm{Bij}(C)$, where $\mathrm{Bij}(C)$ denotes the set of bijections of C. Let σ be a section of π and $g \in \mathrm{Aut}(S)$. Then $\pi_*(g) = \pi g \sigma$, and in particular $\pi_*(g)$ is a morphism. Since g is an automorphism, it follows that $\pi_*(g)$ is also an automorphism and the image of π_* is contained in $\mathrm{Aut}(C)$. The restriction of π_* induces a morphism of algebraic groups $\mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$. Then π is $\mathrm{Aut}^\circ(S)$-equivariant, with $\mathrm{Aut}^\circ(S)$ acting on C by $(g,x) \mapsto \pi_*(g)(x)$. In particular, π_* coincides with the morphism induced by Blanchard lemma by the unicity part of Lemma I.2.4. □

The following proposition will be useful.

Proposition I.3.7. *Let C be a curve and $\pi\colon S \to C$ be a $\mathbb{P}^1$-bundle. Then $\mathrm{Aut}(S)$ is an algebraic group.*

Proof. Since S is a ruled surface, the adjunction formula gives $-\mathrm{K}_S \cdot f = 2$ for all fibers f. In particular, $-\mathrm{K}_S$ is π-ample and if A denotes an ample divisor on C, then the divisor $D = -\mathrm{K}_S + m\pi^*(A)$ is ample for m positive and large enough (see e.g. [Sta21, Lemma 0892 (1)], but it is also a consequence of Nakai ampleness criterion). Moreover, the numerical class of D is fixed by $\mathrm{Aut}(S)$ since K_S and $\pi^*(A)$ are fixed. From [Bri19, Theorem 2.10], the group scheme $\mathrm{Aut}(S)$ has finitely many connected components and thus it is an algebraic group. □

We will use the following proposition to show that the automorphism groups of the Atiyah ruled surfaces are maximal.

Proposition I.3.8. *Let C be an elliptic curve and $\pi\colon \mathcal{A}_i \to C$ be the structure morphism. For $i \in \{0,1\}$, the morphism of algebraic groups induced by Blanchard lemma (or by Lemma I.3.6):*

$$\pi_*\colon \mathrm{Aut}^\circ(\mathcal{A}_i) \to \mathrm{Aut}^\circ(C)$$

is surjective.

Proof. Let $g \in \mathrm{Aut}(C)$. Then the pullback $\pi^* \colon g^*(\mathcal{A}_i) \to C$ is an indecomposable $\mathbb{P}^1$-bundle over C. Since the Atiyah bundles are unique up to C-isomorphism from Theorem I.2.19, it follows that $g^*(\mathcal{A}_i)$ is C-isomorphic to $\mathcal{A}_i$, so the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}_i \arrow[r, "\sim"] \arrow[rr, bend left, "\tilde{g}"] \arrow[dr, "\pi"'] & g^*(\mathcal{A}_i) \arrow[r] \arrow[d, "\pi^*"'] & \mathcal{A}_i \arrow[d, "\pi"] \\
 & C \arrow[r, "g"'] & C.
\end{tikzcd}$$

In particular, there exists $\tilde{g} \in \mathrm{Aut}(\mathcal{A}_i)$ such that $\pi_*(\tilde{g}) = g$, i.e. the morphism $\pi_* \colon \mathrm{Aut}(\mathcal{A}_i) \to \mathrm{Aut}(C)$ from Lemma I.3.6 is surjective. Let $H = (\pi_*)^{-1}(\mathrm{Aut}^\circ(C))$, then we can write $H = \bigsqcup_{j\in J} H_j$ where H_j are the connected components of H and J is finite from Proposition I.3.7. Because H contains $\mathrm{Aut}^\circ(\mathcal{A}_i)$, we can assume that $H_0 = \mathrm{Aut}^\circ(\mathcal{A}_i)$. Then $\pi_*(H_0)$ is a connected algebraic subgroup of $\mathrm{Aut}^\circ(C) \simeq C$ and hence it has to be $\mathrm{Aut}^\circ(C)$ or a point. If $\pi_*(H_0)$ is a point then $\pi_*(H_j)$ is also a point because $h_j \cdot H_0 = H_j$ for all $h_j \in H_j$. Then $\pi_*(H)$ is finite and it is a contradiction because $\mathrm{Aut}^\circ(C)$ is infinite. In consequence $\pi_*(H_0) = \mathrm{Aut}^\circ(C)$ i.e. π_* induces a surjective morphism of algebraic groups $\mathrm{Aut}^\circ(\mathcal{A}_i) \to \mathrm{Aut}^\circ(C)$. □

We will use Proposition I.3.10 to show that for distinct points $z_1, z_2 \in C$, the algebraic subgroup $\mathrm{Aut}^\circ(S_{z_1,z_2})$ is maximal. To prove Proposition I.3.10, we first prove the following lemma:

Lemma I.3.9. *Let C be an elliptic curve and $f \in \mathbf{k}(C)^*$ such that $\mathrm{div}(f) = y_1 + z_1 - y_2 - z_2$ with y_1, y_2, z_1, z_2 distinct points of C. We define:*

$$\begin{aligned} \phi_f \colon C \times \mathbb{P}^1 &\dashrightarrow C \times \mathbb{P}^1 \\ (x, [u:v]) &\longmapsto (x, [f(x)u : v]). \end{aligned}$$

Then ϕ_f is the birational map consisting in the blowup of $C \times \mathbb{P}^1$ at $p_1 = (y_1, [1:0]), q_1 = (z_1, [1:0]), p_2 = (y_2, [0:1]), q_2 = (z_2, [0:1])$; followed by the contraction of the strict transforms of their fibers.

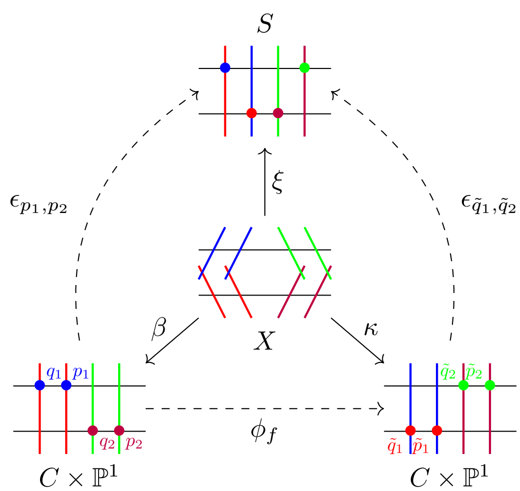
Proof. First ϕ_f is birational because $\phi_f^{-1} = \phi_{1/f}$. The base points of ϕ_f are exactly p_1, q_1, p_2 and q_2 and have all order 1, so one can check by blowups in local charts that ϕ_f corresponds to the blowups at p_1, p_2, q_1, q_2 followed by the contraction of the strict transforms of their fibers. □

Proposition I.3.10. *Let C be an elliptic curve, let $z_1, z_2 \in C$ be distinct points and t be a translation of C. Then S_{z_1,z_2} is C-isomorphic to $S_{t(z_1),t(z_2)}$ and* *moreover, the morphism of algebraic groups induced by Blanchard lemma (or by Lemma I.3.6):*

$$\pi_* \colon \mathrm{Aut}^\circ(S_{z_1,z_2}) \to \mathrm{Aut}^\circ(C)$$

is surjective.

Proof. As $z_1 - z_2$ is linearly equivalent to $t(z_1) - t(z_2)$, there exists $f \in \mathbf{k}(C)^*$ such that $\mathrm{div}(f) = z_1 + t(z_2) - t(z_1) - z_2$. We define $\phi_f \colon C \times \mathbb{P}^1 \dashrightarrow C \times \mathbb{P}^1$ as in Lemma I.3.9, and we know that $\phi_f = \kappa\beta^{-1}$, where β is the blowup $\beta \colon X \to C \times \mathbb{P}^1$ at $p_1 = (z_1, [1:0]), q_1 = (t(z_1), [1:0]), p_2 = (z_2, [0:1]), q_2 = (t(z_2), [0:1])$ and $\kappa \colon X \to C \times \mathbb{P}^1$ is the contraction of the strict transforms of their fibers. Let E_{q_1} and E_{q_2} be the exceptional divisors from respectively the blowups of q_1 and q_2, and let $\tilde{f}_{p_1}$ and $\tilde{f}_{p_2}$ be strict transforms under β^{-1} of the fibers f_{p_1} and f_{p_2} containing respectively p_1 and p_2. We denote by $\xi \colon X \to S$ the contraction of E_{q_1}, E_{q_2}, $\tilde{f}_{p_1}$ and $\tilde{f}_{p_2}$, i.e. $\epsilon_{p_1,p_2} = \xi\beta^{-1}$. Denote by $\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2$ the base points of ϕ_f^{-1}, respectively from the elementary transformations centered at p_1, p_2, q_1, q_2. Similarly, we have $\epsilon_{\tilde{q}_1,\tilde{q}_2} = \xi\kappa^{-1}$, and the following diagram is commutative:



Therefore the surfaces S, S_{z_1,z_2} and $S_{t(z_1),t(z_2)}$ are C-isomorphic. It implies in particular that every translation of C can be lifted to an automorphism of S_{z_1,z_2}. Therefore, we have a morphism of algebraic groups $\pi_* \colon \mathrm{Aut}(S_{z_1,z_2}) \to \mathrm{Aut}(C)$ such that $\mathrm{Aut}^\circ(C)$ is contained in the image of π_*. The proof ends in the same way as the proof of Proposition I.3.8. Let $H = (\pi_*)^{-1}(\mathrm{Aut}^\circ(C))$, then we can write $H = \bigsqcup_{j \in J} H_j$ where H_j are the connected components of H and J is finite from Proposition I.3.7. The image of $\mathrm{Aut}^\circ(S_{z_1,z_2})$ by π_* cannot be a point because it would imply that the image of H is finite and it is a contradiction, therefore $\pi_*(\mathrm{Aut}^\circ(S_{z_1,z_2})) = \mathrm{Aut}^\circ(C)$. $\square$

Unlike the ruled surfaces $\mathcal{A}_0$ and $\mathcal{A}_1$ which are unique up to C-isomorphism, the surfaces S_{z_1,z_2} depend on the choice of the points $z_1, z_2 \in C$. In Lemma

I.3.11, we determine the C-isomorphism classes in the family $\{S_{z_1,z_2}\}_{z_1,z_2\in C}$. In Lemma I.3.12, we determine the isomorphism classes in the family $\{S_{z_1,z_2}\}_{z_1,z_2\in C}$ and the conjugacy classes in the family $\{\mathrm{Aut}^\circ(S_{z_1,z_2})\}_{z_1,z_2\in C}$ as algebraic subgroups of $\mathrm{Bir}(C\times\mathbb{P}^1)$.

Lemma I.3.11. *Let C be an elliptic curve and $z_1,z_1',z_2,z_2'\in C$ such that $z_1\neq z_2$ and $z_1'\neq z_2'$. Let $\pi\colon S_{z_1,z_2}\to C$ and $\pi'\colon S_{z_1',z_2'}\to C$ be the structure morphisms. Let σ_1 and σ_2 be the two disjoint sections of self-intersection 0 in S_{z_1,z_2}; and let σ_1' and σ_2' be the two disjoint sections of self-intersection 0 in $S_{z_1',z_2'}$. The following hold:*

(1) *Let $q_1\in\sigma_1$. Then every section of S_{z_1,z_2} of self-intersection 2 passing through q_1 also passes through the unique point $q_2\in\sigma_2$ such that $\pi(q_2)-\pi(q_1)=z_2-z_1$.*

(2) *Let $q_1,q_2\in S_{z_1,z_2}$ as in (1). If there exist $q_1'\in\sigma_1'$, $q_2'\in\sigma_2'$ such that $\pi'(q_2')-\pi'(q_1')=z_2-z_1$ and if there exists a section of $S_{z_1',z_2'}$ of self-intersection 2 passing through q_1' and q_2', then $S_{z_1',z_2'}$ is C-isomorphic to S_{z_1,z_2}.*

(3) *The ruled surfaces S_{z_1,z_2} and $S_{z_1',z_2'}$ are C-isomorphic if and only if there exists a translation $t\in\mathrm{Aut}^\circ(C)$ such that $t(\{z_1,z_2\})=\{z_1',z_2'\}$.*

Proof. (1) Let $p_1\in\sigma_1,p_2\in\sigma_2$ such that $\pi(p_1)=z_1$ and $\pi(p_2)=z_2$. Let $q_1\in\sigma_1,q_2\in\sigma_2$ and assume there exists a section σ of S_{z_1,z_2} of self-intersection 2 passing through q_1 and q_2. The translation of C sending $\pi(q_1)$ to z_1 lifts to $f\in\mathrm{Aut}^\circ(S_{z_1,z_2})$ (Proposition I.3.10), and since σ_1 is $\mathrm{Aut}^\circ(S_{z_1,z_2})$-invariant, f sends q_1 to p_1. Then the section $f(\sigma)$ of self-intersection 2 passes through p_1, and hence it also passes through p_2 from Proposition I.2.20 (2). Therefore, the automorphism f sends q_2 to p_2 and $\pi(q_2)-\pi(q_1)=z_2-z_1$. In particular, all sections of self-intersection 2 passing through q_1 also pass through q_2.

(2) It follows from (1) that $z_2'-z_1'=z_2-z_1$. Let t be the translation of C by $z_1'-z_1$. Then $t(z_1)=z_1'$ and $t(z_2)=z_2'$. From Proposition I.3.10, the surfaces S_{z_1,z_2} and $S_{z_1',z_2'}$ are C-isomorphic.

(3) Assume that S_{z_1,z_2} is C-isomorphic to $S_{z_1',z_2'}$. From Proposition I.3.10, there exists $f\in\mathrm{Aut}^\circ(S_{z_1',z_2'})$ such that $\pi_*(f)$ is the translation of C sending z_1' to z_1, and $S_{z_1',z_2'}$ is C-isomorphic to $S_{z_1,z_2''}$ where $z_2''=\pi_*(f)(z_2')$. From (1), we have that $z_2''=z_2$ and this proves the direct implication. Let $t\in\mathrm{Aut}^\circ(C)$ such that $t(\{z_1,z_2\})=\{z_1,z_2\}$. Without lost of generality, we can assume that $z_1'=t(z_1)$ and $z_2'=t(z_2)$. Then S_{z_1,z_2} is C-isomorphic to $S_{z_1',z_2'}$ from Proposition I.3.10.

□

Lemma I.3.12. *Let C be an elliptic curve. Denote by $\pi_1\colon C\times\mathbb{P}^1\to C$ the projection on the first factor and let $z_1,z_1',z_2,z_2'\in C$ such that $z_1\neq z_2$, $z_1'\neq z_2'$. Let $p_1,p_1',p_2,p_2'\in C\times\mathbb{P}^1$ with p_1,p_1' on the zero section, p_2,p_2' on the infinity section and such that $\pi_1(p_1)=z_1$, $\pi_1(p_1')=z_1'$, $\pi_1(p_2)=z_2$, $\pi_1(p_2')=z_2'$. Then the following assertions are equivalent:*

(1) *There exists* $f \in \mathrm{Aut}(C)$ *such that* $f(\{z_1, z_2\}) = \{z'_1, z'_2\}$.

(2) *The surfaces* S_{z_1,z_2} *and* $S_{z'_1,z'_2}$ *are isomorphic.*

(3) *The algebraic subgroups* $\mathrm{Aut}^\circ(S_{z_1,z_2})$ *and* $\mathrm{Aut}^\circ(S_{z'_1,z'_2})$ *are conjugate.*

Proof. (1) ⇒ (2) Assume there exists $f \in \mathrm{Aut}(C)$ such that $f(\{z_1, z_2\}) = \{z'_1, z'_2\}$ and $\pi\colon S_{z_1,z_2} \to C$ is the structure morphism. Then the ruled surface $f\pi : S_{z_1,z_2} \to C$ has 0 as Segre invariant and is decomposable, thus it has two disjoint minimal sections σ_1 and σ_2. Let $q_1 \in \sigma_1, q_2 \in \sigma_2$ such that $f\pi(q_1) = z'_1$ and $f\pi(q_2) = z'_2$, then there is a section of self-intersection 2 passing through q_1 and q_2. From Lemma I.3.11 (2), the ruled surfaces $f\pi\colon S_{z_1,z_2} \to C$ and $\pi'\colon S_{z'_1,z'_2} \to C$ are C-isomorphic, and the following diagram is commutative:

$$\begin{tikzcd} S_{z_1,z_2} \arrow[r, "\sim"] \arrow[d, "\pi"'] \arrow[dr, "f\pi"] & S_{z'_1,z'_2} \arrow[d, "\pi'"] \\ C \arrow[r, "f"'] & C. \end{tikzcd}$$

Hence the surfaces S_{z_1,z_2} and $S_{z'_1,z'_2}$ are isomorphic.

(2) ⇒ (1) From Proposition I.3.10, there exists $f \in \mathrm{Aut}^\circ(S_{z_1,z_2})$ such that $\pi_*(f)$ is the translation from z_1 to z'_1. We can then assume that $z_1 = z'_1$. From Lemma I.3.6, an isomorphism from S_{z_1,z_2} to S_{z_1,z'_2} induces an automorphism of C. From Proposition I.2.20 (2), this automorphism of C sends z_2 to z'_2 and fixes z_1, i.e. is a group map with z_1 taken as the neutral element of the elliptic curve. Therefore, an isomorphism from S_{z_1,z_2} to $S_{z'_1,z'_2}$ induces an automorphism of C sending z_1 to z'_1 and z_2 to z'_2.

(2) ⇒ (3) If $\phi : S_{z_1,z_2} \to S_{z'_1,z'_2}$ is an isomorphism then $\phi \mathrm{Aut}^\circ(S_{z_1,z_2})\phi^{-1} = \mathrm{Aut}^\circ(S_{z'_1,z'_2})$.

(3) ⇒ (2) Let $\phi : S_{z_1,z_2} \dashrightarrow S_{z'_1,z'_2}$ be a birational map such that $\phi \mathrm{Aut}^\circ(S_{z_1,z_2})\phi^{-1} = \mathrm{Aut}^\circ(S_{z'_1,z'_2})$. Because $\mathrm{Aut}^\circ(S_{z_1,z_2})$ acts transitively on the base curve C from Proposition I.3.10, the action of $\mathrm{Aut}^\circ(S_{z_1,z_2})$ on S_{z_1,z_2} has no fixed points and it follows from Lemma I.2.6 (3) that ϕ is an isomorphism. □

Proof of Theorem B. Let S be a ruled surface over C. If $g(C) \geq 2$ or $g(C) = 1$ and $\mathfrak{S}(S) < 0$, Theorem A implies that every maximal connected algebraic subgroup of $\mathrm{Bir}(S)$ is conjugate to $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$ which is isomorphic to PGL_2 if $g \geq 2$, and isomorphic to $C \times \mathrm{PGL}_2$ is $g = 1$. Therefore, the algebraic subgroup $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$ is maximal from Lemma I.2.6 (3). We have proved in Lemma I.3.5 that it remains to consider the case $g = 1$ and show the maximality of $\mathrm{Aut}^\circ(S)$ when S is isomorphic to $\mathcal{A}_0$, or $\mathcal{A}_1$, or S_{z_1,z_2} for distinct points $z_1, z_2 \in C$. Assume S is one of these surfaces, then from Propositions I.3.8 and I.3.10, there is a surjective morphism of algebraic groups $\mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$. Lemma I.2.6 (3) implies that $\mathrm{Aut}^\circ(S)$ is maximal. Moreover, for distinct points $z_1, z_2 \in C$, the surfaces $C \times \mathbb{P}^1$, $\mathcal{A}_0$, $\mathcal{A}_1$ and S_{z_1,z_2} are not isomorphic to each other from

Lemma I.3.5. Hence the algebraic groups $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$, $\mathrm{Aut}^\circ(\mathcal{A}_0)$, $\mathrm{Aut}^\circ(\mathcal{A}_1)$ and $\mathrm{Aut}^\circ(S_{z_1,z_2})$ are not conjugate to each other from Lemma I.2.6 (3). Finally, Lemma I.3.12 tells us that $\mathrm{Aut}^\circ(S_{z_1,z_2})$ is conjugate to $\mathrm{Aut}^\circ(S_{z_1',z_2'})$ if and only if there exists $f \in \mathrm{Aut}(C)$ such that $f(\{z_1, z_2\}) = \{z_1', z_2'\}$. $\square$

I.3.3 Description of the maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ as extensions

In this subsection, the curve C always denotes an elliptic curve and we prove Theorem I.3.23.

The algebraic groups $\mathrm{Aut}^\circ(S_{z_1,z_2})$

Proposition I.3.13. *Let C be an elliptic curve and z_1, z_2 be distinct points in C. The group homomorphism π_* of Lemma I.3.10 gives rise to an exact sequence of algebraic groups:*

$$1 \to \mathbb{G}_m \to \mathrm{Aut}^\circ(S_{z_1,z_2}) \xrightarrow{\pi_*} \mathrm{Aut}^\circ(C) \to 1.$$

Proof. From Proposition I.3.10, it suffices to prove that $\ker(\pi_*) \simeq \mathbb{G}_m$. Let $p_1, p_2 \in C \times \mathbb{P}^1$ respectively on the fibers of z_1 and z_2 such that p_1 is on the zero section and p_2 is on the infinity section; and let $\epsilon_{p_1,p_2} \colon C \times \mathbb{P}^1 \dashrightarrow S_{z_1,z_2}$ be the blowups of p_1, p_2 followed by the contractions of their fibers. We denote respectively by q_1 and q_2 the base points of ϵ_{p_1,p_2}^{-1} which belong to the fibers of z_1 and z_2. The automorphisms ϕ_α with $\alpha \neq 0$ defined by:

$$\begin{aligned} \phi_\alpha \colon C \times \mathbb{P}^1 &\to C \times \mathbb{P}^1 \\ (x, [u : v]) &\mapsto (x, [\alpha u : v]), \end{aligned}$$

form a subgroup of $\mathrm{Aut}(C \times \mathbb{P}^1)$ which we denote by $\mathrm{Aut}_{0,\infty}$. Since p_1 and p_2 are fixed by $\mathrm{Aut}_{0,\infty}$, the birational map ϵ_{p_1,p_2} is $\mathrm{Aut}_{0,\infty}$-equivariant i.e. $\mathbb{G}_m \subset \ker(\pi_*)$. Conversely, let $f \in \mathrm{Aut}^\circ(S_{z_1,z_2})$ be such that $\pi_*(f) = id$. Then f fixes q_1 and q_2 because they belong to one of the two minimal sections. Hence $\epsilon_{p_1,p_2}^{-1} f \epsilon_{p_1,p_2}$ is a C-automorphism of $C \times \mathbb{P}^1$ which fixes p_1 and p_2, so it equals ϕ_α for some $\alpha \in \mathbb{G}_m$. Therefore $\ker(\pi_*) \simeq \mathbb{G}_m$. $\square$

The algebraic group $\mathrm{Aut}^\circ(\mathcal{A}_0)$

Lemma I.3.14. *Let C be an elliptic curve, $\pi_1 \colon C \times \mathbb{P}^1 \to C$ be the projection on the first factor and σ_∞ be the infinity section in $C \times \mathbb{P}^1$. For all $p, p' \in \sigma_\infty$, there exists a section σ such that $\sigma^2 = 4$, $\sigma \cap \sigma_\infty = \{p, p'\}$ and they intersect transversely at p and p'. Suppose σ is such a section, then a section σ' satisfies the same properties if and only if there exist $\alpha \in \mathbb{G}_m$ and $\gamma \in \mathbb{G}_a$ such that σ' is the image of σ by the automorphism:*

$$\begin{aligned} \phi_{\alpha,\gamma} \colon C \times \mathbb{P}^1 &\to C \times \mathbb{P}^1 \\ (x, [u : v]) &\mapsto (x, [\alpha u + \gamma v : v]). \end{aligned}$$

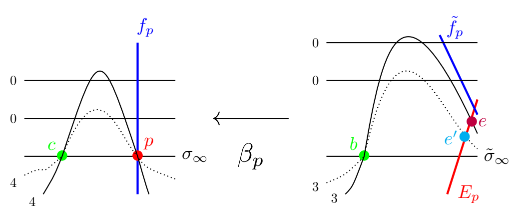
Moreover $\phi_{\alpha,\gamma}$ is the unique C-automorphism of $C \times \mathbb{P}^1$ which sends σ to σ'.

Proof. Let $z = \pi_1(p), z' = \pi_1(p')$ and $D = z + z'$. From Riemann-Roch theorem and Serre duality, we have $\dim(\Gamma(C, D)) = 2$. Because $1 \in \Gamma(C, D)$, there is a section $\sigma \in \Gamma(C, D)$ with exactly two poles of order 1 at z and z', i.e. σ intersect transversely σ_∞ at p and p', and $\{1, \sigma\}$ is a basis for $\Gamma(C, D)$. Since σ is given by a morphism $g_\sigma : C \to \mathbb{P}^1$, we know from Lemma I.2.14 that $\sigma^2 = 2\deg(g_\sigma) = 4$. Let $\phi_{\alpha,\gamma}$ be an automorphism of $C \times \mathbb{P}^1$ defined as in the statement, then the section $\phi_{\alpha,\gamma}(\sigma)$ intersects transversely σ_∞ at exactly p and p', and $\phi_{\alpha,\gamma}(\sigma)^2 = 4$. Conversely if σ' is a section which satisfies the same properties, then $\sigma' \in \Gamma(C, D)$. In particular if $\sigma \colon x \mapsto (x, [u(x) : v(x)])$, then there exist $\alpha, \gamma \in \mathbf{k}$ such that $\sigma'(x) = (x, [\alpha u(x) + \gamma v(x) : v(x)]) = \phi_{\alpha,\gamma}(\sigma(x))$. Finally, the C-automorphisms of $C \times \mathbb{P}^1$ fixing the infinity section are of the form $\phi_{\alpha,\gamma}$ for some $\alpha \in \mathbb{G}_m$ and $\gamma \in \mathbb{G}_a$, and the image of σ uniquely determines α and γ. Therefore $\phi_{\alpha,\gamma}$ is the unique C-automorphism of $C \times \mathbb{P}^1$ sending σ to σ'. $\square$

The group of all automorphisms $\phi_{\alpha,\gamma}$ is denoted Aut_∞ and it is isomorphic to $\mathbb{G}_a \rtimes \mathbb{G}_m$. In particular, Aut_∞ is connected.

Lemma I.3.15. *Let C be an elliptic curve, σ_∞ be the infinity section in $C \times \mathbb{P}^1$ and $\beta_p \colon \mathrm{Bl}_p(C \times \mathbb{P}^1) \to C \times \mathbb{P}^1$ be the blowup of $p = (z, [1 : 0])$. Let $\tilde{\sigma}_\infty$ and $\tilde{f}_p$ be the strict transforms under β_p^{-1} of respectively σ_∞ and the fiber f_p containing p in $C \times \mathbb{P}^1$. Then β_p^{-1} is Aut_∞-equivariant and $\beta_p^{-1}\mathrm{Aut}_\infty\beta_p$ induces a simply transitive action of $\mathbb{G}_m$ on $E_p \setminus \{\tilde{\sigma}_\infty, \tilde{f}_p\}$. More precisely, the following hold for all $b \in \tilde{\sigma}_\infty \setminus E_p$:*

1. *There exists $e \in E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$ and a section $\sigma \subset \mathrm{Bl}_p(C \times \mathbb{P}^1)$ of self-intersection 3 passing through b and e.*
2. *For all $e' \in E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$ there exists a unique $\alpha \in \mathbb{G}_m$ such that the sections of self-intersection 3 and passing through b and e' are the image of σ by the automorphism $\beta_p^{-1}\phi_{\alpha,\gamma}\beta_p$ for some $\gamma \in \mathbb{G}_a$.*



$C \times \mathbb{P}^1$ $\qquad$ $\mathrm{Bl}_p(C \times \mathbb{P}^1)$

Proof. (1) Let $c = \beta_p(b)$ and from Lemma I.3.14, there exists a section s of self-intersection 4 passing with multiplicity 1 through c and p. Then σ be the strict transform of s under β_p^{-1}, it is a section of self-intersection 3 passing through b and some point $e \in E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$.

(2) Let U be an open neighborhood of z and $V = U \times (\mathbb{P}^1 \setminus [0:1]) \subset C \times \mathbb{P}^1$. For all $\alpha \in \mathbb{G}_m$ and $\gamma \in \mathbb{G}_a$, the automorphisms $\phi_{\alpha,\gamma}$ restricted on V gives an isomorphism $V \to \phi_{\alpha,\gamma}(V)$, $(x,t) \mapsto (x, t/(\alpha+\gamma t))$; which extends to an isomorphism $\tilde{\phi}_{\alpha,\gamma} = \beta_p^{-1}\phi_{\alpha,\gamma}\beta_p$ defined on $\mathrm{Bl}_p V = \{(x,t),[u:v] \in V \times \mathbb{P}^1 : tu = vf(x)\}$, with f a local parameter of $\mathcal{O}_{C,z}$, as:

$$\begin{aligned} \tilde{\phi}_{\alpha,\gamma} \colon \mathrm{Bl}_p V &\to \mathrm{Bl}_p(\phi_{\alpha,\gamma}(V)) \\ ((x,t),[u:v]) &\mapsto \left(\left(x, \frac{t}{\alpha+\gamma t}\right), [u(\alpha+\gamma t):v]\right). \end{aligned}$$

In particular, the restriction of $\tilde{\phi}_{\alpha,\gamma}$ on E_p is obtained by substituting (x,t) by $(z,0)$ and we get:

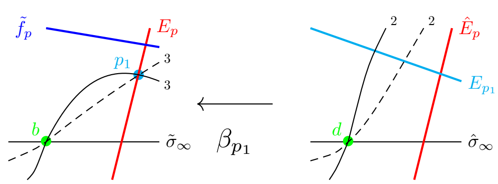
$$\begin{aligned} E_p &\to E_p \\ [u:v] &\mapsto [u\alpha : v]. \end{aligned}$$

The automorphisms $\phi_{\alpha,\gamma}$ induce an action of $\mathbb{G}_m$ on the exceptional divisor E_p, with only fixed points $[0:1]$ and $[1:0]$ which correspond to the intersection of E_p with $\tilde{f}_p$ and $\tilde{\sigma}_\infty$. So $\mathbb{G}_m$ acts simply transitively on $E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$. In particular if σ' is a section of self-intersection 3 passing through $e' \in E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$ and b, then there exists a unique $\alpha \in \mathbb{G}_m$ and there exists $\gamma \in \mathbb{G}_a$ such that σ' is the image of σ by $\tilde{\phi}_{\alpha,\gamma}$.

□

Lemma I.3.16. *Under the same notations as in Lemma I.3.15, let $p_1 \in E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$, $\beta_{p_1} \colon \mathrm{Bl}_{p_1}(\mathrm{Bl}_p(C \times \mathbb{P}^1)) \to \mathrm{Bl}_p(C \times \mathbb{P}^1)$ be the blowup of $\mathrm{Bl}_p(C \times \mathbb{P}^1)$ at p_1 and $\beta = \beta_{p_1}\beta_p$. Let E_{p_1} be the exceptional divisor of β_{p_1}, let $\hat{E}_p$ and $\hat{\sigma}_\infty$ be the strict transforms under $\beta_{p_1}^{-1}$ of respectively E_p and $\tilde{\sigma}_\infty$. Then we have a simply transitive action of $\mathbb{G}_a$ on $E_{p_1} \setminus \hat{E}_p$ and more precisely for all $d \in \hat{\sigma}_\infty \setminus \hat{E}_p$:*

1. *There exists $e \in E_{p_1} \setminus \hat{E}_p$ and a unique section σ of self-intersection 2 passing through d and e.*
2. *For all $e' \in E_{p_1} \setminus \hat{E}_p$, there exists a unique $\gamma \in \mathbb{G}_a$ such that $\beta^{-1}\phi_{1,\gamma}\beta(\sigma)$ is the unique section of self-intersection 2 passing through d and e'.*



$\mathrm{Bl}_p(C \times \mathbb{P}^1)$ $\qquad$ $\mathrm{Bl}_{p_1}(\mathrm{Bl}_p(C \times \mathbb{P}^1))$

Proof. (1) Let $b = \beta_{p_1}(d)$ and from Lemma I.3.15 (1), there exists a section s of self-intersection 3 passing through b and p_1. Then the strict transform σ of s under $\beta_{p_1}^{-1}$ is a section of self-intersection 2 passing through d and a point $e \in E_{p_1} \setminus \hat{E}_p$.

(2) From Lemma I.3.15, we know $\mathbb{G}_m$ acts transitively on $E_p \setminus \{\tilde{f}_p, \tilde{\sigma}_\infty\}$ so we can assume p_1 has coordinates $((z,0),[1:1])$ in $\mathrm{Bl}_p(V)$. We choose an open subset W of $\mathrm{Bl}_p(V) = \{(x,t),[u:v] \in V \times \mathbb{P}^1 : tu = vf(x)\}$ containing p_1 and such that $u \neq 0$. By the change of variable $v \mapsto v/u$, we have $t = vf(x)$ and the isomorphism $\tilde{\phi}_{1,\gamma}$ restricted on W gives:

$$\begin{aligned} W &\to \tilde{\phi}_{1,\gamma}(W) \\ ((x,vf(x)),[1:v]) &\mapsto \left(\left(x, \frac{vf(x)}{1+\gamma vf(x)}\right), \left[1 : \frac{v}{1+\gamma vf(x)}\right]\right). \end{aligned} \tag{I.4}$$

Since W is isomorphic to an open subset of $\mathbb{A}^2$ by the map

$$(x,v) \mapsto ((x,(v+1)f(x)),[1:v+1])$$

which sends $(z,0)$ to p_1, we can rewrite (I.4) as:

$$(x,v) \mapsto \left(x, \frac{v+1}{1+\gamma f(x)(v+1)} - 1\right),$$

which sends $(z,0)$ to $(z,0)$. The isomorphism $\tilde{\phi}_{1,\gamma}$ extends to $\mathrm{Bl}_{(z,0)}(W)$ by:

$$((x,v),[u_1:u_2]) \longmapsto \left(\left(x, \frac{v - \gamma f(x)(v+1)}{1+\gamma f(x)(v+1)}\right), \left[u_1 : \frac{u_2 - u_1\gamma(v+1)}{1+\gamma f(x)(v+1)}\right]\right),$$

and restricted on $E_{(z,0)}$ one gets: $[u_1 : u_2] \mapsto [u_1 : u_2 - u_1\gamma]$. In particular $\mathbb{G}_a$ acts on the exceptional divisor $E_{(z,0)}$ and the action has a unique fixed point $[0:1]$ which is the intersection of E_{p_1} with $\hat{E}_p$. Therefore the action of $\mathbb{G}_a$ on $E_{p_1} \setminus \hat{E}_p$ is simply transitive. As a consequence, if $e' \in E_{p_1} \setminus \hat{E}_p$ then there exists a unique $\gamma \in \mathbb{G}_a$ such that $\beta^{-1}\phi_{1,\gamma}\beta(\sigma)$ is the unique section of self-intersection 2 passing through d and e'.

□

Lemma I.3.17. *Let C be an elliptic curve and σ_0 be the unique minimal section of $\mathcal{A}_0$. For all $a \in \sigma_0$, $b \in \mathcal{A}_0 \setminus \sigma_0$ with a and b not in the same fiber, there exists a unique section σ passing through a and b such that $\sigma^2 = 2$. Moreover the subgroup $\{\phi_{1,\gamma}\}_{\gamma \in \mathbb{G}_a}$ of Aut_∞ induces a simply transitive action of $\mathbb{G}_a$ on $f \setminus \sigma_0$, where f is any fiber of $\mathcal{A}_0$.*

Proof. Let $q = \sigma_0(\pi(b))$ and $\epsilon_q \colon \mathcal{A}_0 \dashrightarrow \mathbb{P}(\mathcal{O}_C(\pi(q)) \oplus \mathcal{O}_C)$ be an elementary transformation centered on a. From Proposition I.2.20 (1), there is a unique point $q_1 \in \mathbb{P}(\mathcal{O}_C(\pi(q)) \oplus \mathcal{O}_C)$ where all the sections of self-intersection 1 meet and we have $\epsilon_{q_1} : \mathbb{P}(\mathcal{O}_C(\pi(q)) \oplus \mathcal{O}_C) \dashrightarrow C \times \mathbb{P}^1$. Let $\psi = \epsilon_{q_1}\epsilon_q$, then $p = \psi(b)$ belongs to the same constant section as $c = \psi(a)$. Up to an automorphism of $C \times \mathbb{P}^1$ we can assume that c and p lie on the infinity section and apply Lemmas I.3.15 and I.3.16. Then using the notation of Lemma I.3.16, the contraction $\mathrm{Bl}_{p_1}(\mathrm{Bl}_p(C \times \mathbb{P}^1)) \to \mathcal{A}_0$ of $\hat{E}_p$ and of the strict transform $\hat{f}_p$ of f_p is $\mathbb{G}_a$-equivariant, so there exists a unique section σ of self-intersection 2 passing through a and b. Moreover for all $b' \in f_q \setminus \sigma_0$, it also follows from Lemma I.3.16 that there exists a unique $\gamma \in \mathbb{G}_a$ such that $\psi^{-1}\phi_{1,\gamma}\psi(\sigma)$ is the unique section of self-intersection 2 passing through a and b'. □

Proposition I.3.18. *Let C be an elliptic curve. Then $\mathrm{Aut}_C(\mathcal{A}_0)$ is isomorphic to $\mathbb{G}_a$ and the following sequence of algebraic groups is exact:*

$$1 \to \mathbb{G}_a \to \mathrm{Aut}^\circ(\mathcal{A}_0) \to \mathrm{Aut}^\circ(C) \to 1.$$

Proof. From Lemma I.3.17, we know that $\mathbb{G}_a$ is one-to-one to a subgroup of $\mathrm{Aut}_C(\mathcal{A}_0)$. Conversely, let $a \in \sigma_0$, $b \in \mathcal{A}_0 \setminus \sigma_0$ with a and b not in the same fiber. Then an automorphism $f \in \mathrm{Aut}_C(\mathcal{A}_0)$ sends a section of self-intersection 2 passing through a and b to a section of self-intersection 2 passing through a and a point b' in the same fiber as b. Let ψ, p, and c be as defined in the proof of Lemma I.3.17, then the automorphism $\psi^{-1} f \psi$ sends a section in $C \times \mathbb{P}^1$ of self-intersection 4 passing through p and c to another section of self-intersection 4 passing through p and c. From Lemma I.3.14, it follows that $\psi^{-1} f \psi = \phi_{1,\gamma}$ for some γ and therefore $\mathrm{Aut}_C(\mathcal{A}_0)$ is isomorphic to $\mathbb{G}_a$. Since $\mathbb{G}_a$ is connected, we get the exact sequence given in the statement. □

The algebraic group $\mathrm{Aut}^\circ(\mathcal{A}_1)$ when $\mathrm{char}(\mathbf{k}) \neq 2$

In this paragraph we assume that the characteristic of $\mathbf{k}$ is different than 2. Let $\Delta = \{1, d_1, d_2, d_3\}$ be the subgroup of two torsion points of C which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Then Δ acts on C by translation and on $\mathbb{P}^1$ in the following way:

$$\begin{aligned} 1 &\colon [u:v] \mapsto [u:v], \\ d_1 &\colon [u:v] \mapsto [-u:v], \\ d_2 &\colon [u:v] \mapsto [v:u], \\ d_3 &\colon [u:v] \mapsto [-v:u]. \end{aligned}$$

We denote by $\mathcal{E}$ the quotient $(C \times \mathbb{P}^1)/\Delta$ given by the diagonal action:

$$\begin{aligned} \Delta \times (C \times \mathbb{P}^1) &\to C \times \mathbb{P}^1 \\ (d_i, (x, [u:v])) &\mapsto (d_i + x, d_i \cdot [u:v]). \end{aligned}$$

Lemma I.3.19. *Let Δ be a finite group acting on a irreducible variety X. Then $\mathbf{k}(X)^\Delta$ is isomorphic to $\mathbf{k}(X/\Delta)$.*

Proof. Since G is finite, we can find an affine Δ-invariant open subset $U \subset X$. Because $\mathcal{O}_X(U)^\Delta \subset \mathcal{O}_X(U)$, we have an extension $\mathrm{Frac}(\mathcal{O}_X(U)^\Delta) \to \mathrm{Frac}(\mathcal{O}_X(U))^\Delta$. Let $f/g \in \mathrm{Frac}(\mathcal{O}_X(U))^\Delta$, write $\Delta = \{d_0, d_1, ..., d_n\}$ where d_0 is the neutral element and consider $g' = \prod_{i=1}^n d_i \cdot g$. Then $f/g = (fg')/(gg')$ and gg' are Δ-invariant, hence fg' as well. Thus $\mathrm{Frac}(\mathcal{O}_X(U)^\Delta) \simeq \mathrm{Frac}(\mathcal{O}_X(U))^\Delta$, i.e. $\mathbf{k}(X/\Delta)$ is isomorphic to $\mathbf{k}(X)^\Delta$ by definition of the quotient. □

Lemma I.3.20. *Let C be an elliptic curve. Then the following hold:*

(1) *The surface $\mathcal{E}$ is a $\mathbb{P}^1$-bundle over C/Δ with the following structure morphism:*

$$\begin{aligned} \pi\colon \mathcal{E} &\longrightarrow C/\Delta \\ (x,[u:v]) \bmod \Delta &\longmapsto x \bmod \Delta. \end{aligned}$$

(2) *If $q\colon C \to C/\Delta$ is the quotient map for the action of Δ on C by translations of order 2, then the pullback bundle $q^*(\mathcal{E})$ is C-isomorphic to $C\times\mathbb{P}^1$.*

Proof. (1) First one can check that π is well-defined. Let $d\colon C\times\mathbb{P}^1 \to \mathcal{E}$ be the quotient map for the diagonal action of Δ on $C\times\mathbb{P}^1$, then the following diagram is commutative:

$$\begin{tikzcd} C\times\mathbb{P}^1 \arrow[r, "d"] \arrow[d, "\pi_1"'] & \mathcal{E} \arrow[d, "\pi"] \\ C \arrow[r, "i"'] & C/\Delta. \end{tikzcd}$$

Every fiber of π corresponds to the gluing of 4 disjoint fibers of $C\times\mathbb{P}^1 \to C/\Delta$. Since every fiber of π is isomorphic to $\mathbb{P}^1$, it follows that $\pi\colon \mathcal{E} \to C/\Delta$ is a ruled surface.

(2) Since the diagram in (1) is commutative, there exists $\alpha\colon C\times\mathbb{P}^1 \to i^*(\mathcal{E})$ such that the following diagram is commutative:

$$\begin{tikzcd} C\times\mathbb{P}^1 \arrow[dr, "\alpha"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "\pi_1"'] & & \\ & q^*(\mathcal{E}) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{E} \arrow[d, "\pi"] \\ & C \arrow[r, "q"'] & C/\Delta. \end{tikzcd}$$

From Lemma I.3.19, we have that $\mathbf{k}(\mathcal{E}) \simeq \mathbf{k}(C\times\mathbb{P}^1)^\Delta$ and hence Δ is the Galois group of the extension $d^*\colon \mathbf{k}(\mathcal{E}) \to \mathbf{k}(C\times\mathbb{P}^1)$ (see e.g. [Jac85, Theorem 4.7]). In particular, $[\mathbf{k}(C\times\mathbb{P}^1):\mathbf{k}(\mathcal{E})] = \#\Delta = 4$. Since p_2 is 4-to-1, it follows that α^* is a $\mathbf{k}$-isomorphism i.e. α is a birational morphism. Because α is also bijective, it follows from Zariski's main theorem (see e.g. [Gro67, Corollary 18.12.13]) that α is an isomorphism.

$\square$

Lemma I.3.21. *The ruled surface $\mathcal{E}$ is isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$ (see Theorem I.2.19).*

Proof. Let $q\colon C \to C/\Delta$ and assuming that $\mathcal{E}$ admits two disjoint sections σ_1 and σ_2, we will derive a contradiction. For $k \in \{1,2\}$, the pullback sections $q^*\sigma_k$ defined as:

$$\begin{aligned} C &\to q^*(\mathcal{E}) \\ x &\to (x, \sigma_k(x \bmod \Delta)) \end{aligned}$$

induce two disjoint sections $\alpha^{-1}(q^*\sigma_k)$ of $C\times\mathbb{P}^1$ since $C\times\mathbb{P}^1$ is C-isomorphic to $q^*(\mathcal{E})$ by Lemma I.3.20 (2). Then it implies that $\alpha^{-1}(q^*\sigma_1)$ and $\alpha^{-1}(q^*\sigma_2)$ are constant sections. Then for $k \in \{1,2\}$, there exists a constant $[u:v] \in \mathbb{P}^1$ such that $\alpha^{-1}(q^*\sigma_k)$ is defined as $C \to C\times\mathbb{P}^1$, $x \mapsto (x,[u:v])$. This implies that σ_k is given by $C/\Delta \to \mathcal{E}$, $x \bmod \Delta \mapsto (x,[u:v]) \bmod \Delta$, which is not well-defined. Therefore, constant sections of $C\times\mathbb{P}^1$ are not obtained by pulling back sections of π. Thus, there are no disjoint sections of π and $\mathcal{E}$ is an indecomposable $\mathbb{P}^1$-bundle over C/Δ. Finally Δ is the kernel of the multiplication by 2 in C, hence C is isomorphic to C/Δ, so $\mathcal{E}$ is isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$. ☐

Proposition I.3.22. *The following sequence is exact:*

$$0 \to \Delta \to \mathrm{Aut}^\circ(\mathcal{E}) \to \mathrm{Aut}^\circ(C/\Delta) \to 0.$$

In particular, the ruled surface $\mathcal{E}$ is C-isomorphic to $\mathcal{A}_1$.

Proof. First we have an injective morphism of algebraic groups $j\colon \mathrm{Aut}^\circ(C) \to \mathrm{Aut}^\circ(\mathcal{E})$, $t \mapsto ((x,[u:v]) \bmod \Delta \mapsto (t(x),[u:v]) \bmod \Delta)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}^\circ(C) \arrow[rr, hook, "j"] \arrow[dr, two heads] & & \mathrm{Aut}^\circ(\mathcal{E}) \arrow[dl, "\pi_*"] \\ & \mathrm{Aut}^\circ(C/\Delta). & \end{tikzcd}$$

In particular the morphism $\pi_*\colon \mathrm{Aut}^\circ(\mathcal{E}) \to \mathrm{Aut}^\circ(C/\Delta)$ is also surjective. Let $i\colon C \to C/\Delta$ then $\ker(\pi_*)$ is a subgroup of $\mathrm{Aut}(i^*(\mathcal{E}))$ by the embedding $f \mapsto (id,f)$. Moreover $i^*(\mathcal{E})$ is isomorphic to $C\times\mathbb{P}^1$ from Lemma I.3.20 (2) and the automorphism (id,f) of $\mathrm{Aut}^\circ(i^*(\mathcal{E}))$ corresponds to a C-automorphism of $C\times\mathbb{P}^1$, i.e. of the form (id,M) where $M \in \mathrm{PGL}_2$. For such an automorphism to be compatible with the Δ-action, it has to send an orbit to an orbit for the action of Δ and a direct computation shows that M belongs to one the following matrices:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix};$$

and conversely they all define automorphisms of $\mathcal{E}$. It follows that $\ker(\pi_*)$ is isomorphic to Δ and we get the exact sequence in the statement. Since $\mathrm{Aut}^\circ(\mathcal{E})$ is a 1-dimensional algebraic variety and $\mathcal{E}$ is an Atiyah bundle (Lemma I.3.21), and we know from Proposition I.3.18 that $\mathrm{Aut}^\circ(\mathcal{A}_0)$ is 2-dimensional algebraic group, it follows from Theorem I.2.19 that $\mathcal{E}$ is C-isomorphic to $\mathcal{A}_1$. ☐

Description of the maximal automorphism groups

We have proven the following result, which also follows from [Mar71, Theorem 3].

Theorem I.3.23. *Let C be an elliptic curve. Then for all distinct points $z_1, z_2 \in C$, we have the following exact sequences of algebraic groups:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{Aut}^\circ(S_{z_1,z_2}) & \longrightarrow & C & \longrightarrow & 0, \\ 0 & \longrightarrow & \mathbb{G}_a & \longrightarrow & \mathrm{Aut}^\circ(\mathcal{A}_0) & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

Moreover, the surfaces $\mathrm{Aut}^\circ(S_{z_1,z_2})$ and $\mathrm{Aut}^\circ(\mathcal{A}_0)$ are commutative algebraic groups and $\mathrm{Aut}^\circ(\mathcal{A}_0)$ is not isomorphic to a semidirect product $\mathbb{G}_a \rtimes \mathrm{Aut}^\circ(C)$. Finally, if the characteristic of $\mathbf{k}$ is different than 2 and if $\Delta \simeq (\mathbb{Z}/2\mathbb{Z})^2$ denotes the subgroup of 2-torsion points of C, then the following sequence of algebraic groups is exact:

$$0 \longrightarrow \Delta \longrightarrow \mathrm{Aut}^\circ(\mathcal{A}_1) \longrightarrow C \longrightarrow 0.$$

Proof. The three exact sequences in the statement are proven in Propositions I.3.13, I.3.18, I.3.22. Moreover, the algebraic groups $\mathrm{Aut}^\circ(\mathcal{A}_0)$ and $\mathrm{Aut}^\circ(S_{z_1,z_2})$ are commutative from [Ros56, Corollary 2 p. 433]. Because there is no non-trivial morphism from C to $\mathrm{Aut}(\mathbb{G}_a) \simeq \mathbf{k}^*$ (or because $\mathrm{Aut}^\circ(\mathcal{A}_0)$ is commutative), the algebraic group $\mathrm{Aut}^\circ(\mathcal{A}_0)$ is not isomorphic to $\mathbb{G}_a \rtimes \mathrm{Aut}^\circ(C)$. $\square$

Remark I.3.24.

(1) From [Ser84, VII.16, Theorem 6] (see also [BSU13, Example 1.1.2]), the extensions of C by $\mathbb{G}_m$ are classified by C itself. Let $z_1, z_2 \in C$ be distinct points and G be the $\mathbb{G}_m$-bundle defined as the complement of the zero section in $\mathcal{O}_C(z_1 - z_2)$. Then we have a morphism $\pi \colon G \to C$ with kernel $\mathbb{G}_m$, i.e. an exact sequence:

$$0 \to \mathbb{G}_m \to G \xrightarrow{\pi} C \to 0.$$

Let S be the quotient of $(G \times \mathbb{P}^1)$ by $\mathbb{G}_m$, given by the following action of $\mathbb{G}_m$ on $G \times \mathbb{P}^1$: $t \cdot (g, [u : v]) \mapsto (g \cdot t^{-1}, [g \cdot u, v])$. Then this gives a morphism $S \to G/\mathbb{G}_m \simeq C$ which endows S with a structure of $\mathbb{P}^1$-bundle over C. One can check by computing in local charts that $S \to C$ is C-isomorphic to $S_{z_1,z_2} \to C$, hence the extension $0 \to \mathbb{G}_m \to \mathrm{Aut}^\circ(S_{z_1,z_2}) \to C \to 0$ corresponds to the point $z_1 - z_2 \in C$.

(2) From [Ser84, VII. 17, Theorem 7] (see also [BSU13, Example 1.1.2]), the extensions of C by $\mathbb{G}_a$ are classified by $H^1(C, \mathcal{O}_C) \simeq \mathbf{k}$. Since the extension $0 \to \mathbb{G}_a \to \mathrm{Aut}^\circ(\mathcal{A}_0) \to C \to 0$ does not split, it corresponds to a non zero element of $\mathbf{k}$.

(3) Serre shows in [Ser84, VII. 15, Theorem 5] that the algebraic groups $\mathrm{Aut}^\circ(\mathcal{A}_0)$ and $\mathrm{Aut}^\circ(S_{z_1,z_2})$ are respectively endowed with a canonical structure of $\mathbb{G}_a$-principal bundle and $\mathbb{G}_m$-principal bundle over C. From [Mar71, Theorem

3.(3)], the connected algebraic group $\mathrm{Aut}^\circ(\mathcal{A}_0)$ is isomorphic to $\mathcal{A}_0 \setminus \sigma_0$, where σ_0 is the unique minimal section of $\mathcal{A}_0$. From [Mar71, Theorem 3.(2)], the connected algebraic group $\mathrm{Aut}^\circ(S_{z_1,z_2})$ is isomorphic to $S_{z_1,z_2} \setminus \{\sigma_1, \sigma_2\}$, where σ_1, σ_2 are the two minimal sections of S_{z_1,z_2}. A natural problem is to describe geometrically their group laws, and a formula has been computed explicitly for the group law of $\mathcal{A}_0 \setminus \sigma_0$ when $\mathbf{k} = \mathbb{C}$ in [LMP09, § 3.3 p.251].

I.3.4 Proof of Theorems C and D

Proposition I.3.25. [†] *Let X be a surface and $G = \mathrm{Aut}^\circ(X)$. If X is not birationally equivalent to $C \times \mathbb{P}^1$, for some curve C, then G is an abelian variety and exactly one of the following cases holds:*

(1) *G is an abelian surface and $G \simeq X$.*

(2) *G is isomorphic to an elliptic curve and moreover, there exist a not necessarily reduced curve Y which is connected, a finite subgroup scheme F of G and a G-equivariant isomorphism:*

$$X \simeq (G \times Y)/F.$$

The quotient $(G \times Y)/F$ is given by a diagonal action of F on $G \times Y$, $f \cdot (g, y) \mapsto (g \cdot f^{-1}, f \cdot y)$.

(3) *G is trivial.*

In case (2), if the characteristic of $\mathbf{k}$ is zero then F is reduced and Y is smooth.

Proof. From Chevalley's structure theorem (see e.g. [BSU13, Theorem 1.1.1]), there exists an exact sequence $0 \to \mathcal{L} \to G \to A \to 0$ where $\mathcal{L}$ is a linear algebraic group and A is an abelian variety. If $\mathcal{L}$ is not trivial, it contains a $\mathbb{G}_a$ or a $\mathbb{G}_m$ and this implies that X is birationally equivalent to $C \times \mathbb{P}^1$ for some curve C (this follows from [Ros56, Theorems 2 and 10], see also [BFT21b, Proposition 2.5.1]). Thus $\mathcal{L}$ is trivial, i.e. G is isomorphic to A.

First suppose that G has an open orbit O in X which is isomorphic to $G/\mathrm{Stab}(x)$ for some $x \in O$. Since G is commutative and acts faithfully on O, it follows that $\mathrm{Stab}(x)$ is trivial and hence $O \simeq G$. Because O is also the image of the projective morphism $G \to X$, $g \mapsto g \cdot x$, then O is closed in X. Therefore, we have $G \simeq O = X$ and G is an abelian surface acting on itself by translation.

Otherwise suppose that G has an orbit O of dimension 1 and then for all $x \in O$, the subgroup $\mathrm{Stab}(x)$ is finite (see [BSU13, Proposition 2.2.1]). Therefore G is an elliptic curve. From [BSU13, Theorem 2.2.2 and the paragraph following the theorem], there exist a positive integer n and a G-equivariant isomorphism $\mathrm{h}\colon X \to (G \times \tilde{Y})/G_n$, where G_n denotes the finite subgroup scheme of n-torsion points of G and $\tilde{Y}$ is a closed subscheme of X of dimension 1. The projection to the first factor $G \times \tilde{Y} \to G$ induces a morphism $g\colon (G \times \tilde{Y})/G_n \to G/G_n$ which is G-equivariant. The Stein factorization of $f = gh$ gives morphisms

[†] The idea of the proof is due to Michel Brion.

$u\colon X \to Z$ and $v\colon Z \to G/G_n$, such that u has connected fibers, v is finite and $f = vu$. From Blanchard lemma, there exists an action of G on Z such that u is G-equivariant. Since u is also surjective and f is G-equivariant, it follows that v is G-equivariant. Moreover, Z is a curve because v is finite, hence it is an orbit of the G-action and the stabilizer F of a point is finite. Therefore $Z \simeq G/F$ and $u\colon X \to Z \simeq G/F$ is a G-equivariant morphism. From [Bri17, Section 2.5, paragraph following Lemma 2.10] (see also [BSU13, Paragraph following Example 6.1.2]), X is isomorphic to the F-torsor $(G\times Y)/F$ where $Y = u^{-1}(F/F)$ is connected and the quotient is given by the diagonal action $f\cdot(g,y) \mapsto (g\cdot f^{-1}, f\cdot y)$ for all $g \in G, f \in F, y \in Y$. Since X is a surface, it implies that Y is a not necessarily reduced curve in positive characteristic. In characteristic zero, G_n is reduced because the multiplication by n is an étale endomorphism of G, and it follows that the finite subgroup scheme F of G_n is also reduced. Hence $G \times Y \to (G\times Y)/F \simeq X$ is an étale finite morphism from [Mum08, Theorem in Section 2.7, p.63]. Because X is smooth, it follows that Y is also smooth.

Finally if the orbits of G have dimension 0 then G is trivial. □

Proof of Theorem C. Let C be a curve of genus $g \geq 1$. From Theorem A, not every connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ is contained in a maximal one. If X is rational, then every connected algebraic subgroup of $\mathrm{Bir}(X)$ is contained in $\mathrm{Aut}^\circ(\mathbb{P}^2)$ or some $\mathrm{Aut}^\circ(\mathbb{F}_n)$ for $n \neq 1$. If X is not a ruled surface and is not rational, then Proposition I.3.25 implies that $\mathrm{Aut}^\circ(X)$ is contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(X)$. □

Proposition I.3.26. *Let X be a surface over $\mathbf{k}$ and G be a maximal connected algebraic subgroup of $\mathrm{Bir}(X)$. If X is birationally equivalent to $C \times \mathbb{P}^1$ with C a curve of genus g, then G is conjugate to one of the following:*

(1) $\mathrm{Aut}(\mathbb{P}^2)$ *or* $\mathrm{Aut}^\circ(\mathbb{F}_n)$ *with* $n \neq 1$*, if* $g = 0$.

(2) $\mathrm{Aut}^\circ(C\times\mathbb{P}^1)$*, or* $\mathrm{Aut}^\circ(\mathcal{A}_0)$*, or* $\mathrm{Aut}^\circ(\mathcal{A}_1)$*, or* $\mathrm{Aut}^\circ(S_{z_1,z_2})$ *for some* $z_1, z_2 \in C$*, if* $g = 1$.

(3) $\mathrm{Aut}^\circ(C \times \mathbb{P}^1)$*, if* $g \geq 2$.

If X is not birationally equivalent to $C \times \mathbb{P}^1$ then up to conjugation we have $G = \mathrm{Aut}^\circ(X)$ and one of the following holds:

(4) *G is isomorphic to X, which is an abelian surface.*

(5) *G is isomorphic to an elliptic curve and moreover, there exist a not necessarily reduced curve Y which is connected, a finite subgroup scheme F and a G-equivariant isomorphism:*

$$X \simeq (G \times Y)/F.$$

The quotient $(G \times Y)/F$ is given by a diagonal action of F on $G \times Y$, $f \cdot (g, y) \mapsto (g \cdot f^{-1}, f \cdot y)$.

(6) *G is trivial.*

In case (5)*, if the characteristic of* $\mathbf{k}$ *is zero then* F *is reduced and* Y *is smooth.*

Proof. Let X be a surface and $G = \mathrm{Aut}^\circ(X)$ a maximal algebraic subgroup of $\mathrm{Bir}(X)$. From Proposition I.2.5, G is conjugate to $\mathrm{Aut}^\circ(S)$ with S a minimal surface birationally equivalent to X.

If X is birationally equivalent to $C\times\mathbb{P}^1$ with C a curve, it follows that S is $\mathbb{P}^2$ or a ruled surface over C from [Har77, Examples V.5.8.2, V.5.8.3 and Remark V.5.8.4]. If X is rational then G is conjugate to $\mathrm{Aut}^\circ(\mathbb{P}^2) \simeq \mathrm{PGL}_3$ which is maximal from Lemma I.2.6 (3), or to $\mathrm{Aut}^\circ(\mathbb{F}_n)$ for some integer $n \neq 1$. From [Bla09b, §4.2] there exists a surjective group homomorphism $\mathrm{Aut}^\circ(\mathbb{F}_n) \to \mathrm{PGL}_2$, and hence $\mathrm{Aut}^\circ(\mathbb{F}_n)$ is also maximal from Lemma I.2.6 (3). If X is not rational, the statement follows from Theorem B.

Otherwise X is not birationally equivalent to $C \times \mathbb{P}^1$ and the statement follows from Proposition I.3.25. □

Proof of Theorem D. Assume that $\mathbf{k}$ is a field of characteristic 0, the first two columns of the table are given by the classification of algebraic surfaces. For the last column, the case $\kappa(X) = -\infty$ follows from Proposition I.3.26 and Theorem A.

Assume that X is a surface isomorphic to $(C \times Y)/F$, where C is a elliptic curve, Y is a smooth curve and F is a finite subgroup of $\mathrm{Aut}^\circ(C)$ acting diagonally on $C \times Y$ (in particular, F acts on C by translations). First notice that we have a morphism $X \to C/F$ with all fibres isomorphic to Y, because the pullback of $X \to C/F$ by the quotient morphism $C \to C/F$ is $C \times Y$. Moreover, the curve C/F is an elliptic curve because F is a finite subgroup of $\mathrm{Aut}^\circ(C)$. If $Y \simeq \mathbb{P}^1$, then X is a ruled surface over C/F. If Y is an elliptic curve, it follows that X is a quotient of an abelian surface by a finite group. If $Y/F \simeq \mathbb{P}^1$, then X is a bielliptic surface [Bea96, Definition VI.19]. Else F acts on Y only by translations, then F is an abelian subgroup of $(C \times Y)$ and X is again an abelian surface. If Y is a smooth curve of general type, then $\kappa(X) \geq \kappa(Y) + \kappa(C/F) = 1$ [Fuj20, Theorem 6.1.1]. Because $\mathrm{Aut}^\circ(C)$ acts on X on the left factor, it is an algebraic subgroup of $\mathrm{Aut}^\circ(X)$ and X cannot be a surface of general type. Thus $\kappa(X) = 1$.

We denote by E the set of all surfaces of the form $(C \times Y)/F$, where C is a elliptic curve, Y is a smooth curve of general type and F is a finite group of $\mathrm{Aut}^\circ(C)$ acting diagonally on $C \times Y$. We have shown that E is included in the set of surfaces of Kodaira dimension 1. Let X' be a surface of Kodaira dimension 1 which is not in E, then $\mathrm{Aut}^\circ(X')$ is trivial by Proposition I.3.26 (6). If the minimal model X of X' is an element of E, then $\mathrm{Aut}^\circ(X)$ is not trivial and this implies that $\mathrm{Aut}^\circ(X')$ is not maximal. If X is not an element of E, then $\mathrm{Aut}^\circ(X')$ is maximal.

If X is an abelian surface, then $\mathrm{Aut}^\circ(X) \simeq X$ is maximal. If X' is not an abelian surface but is birationally equivalent to an abelian surface X, then $\mathrm{Aut}^\circ(X')$ is trivial by Proposition I.3.26 (6) (since we have also shown that X' does not correspond to a surface given by Proposition I.3.26 (5)). Thus $\mathrm{Aut}^\circ(X')$ is trivial and is not maximal.

Let $X = (C \times Y)/F$ be a bielliptic surface, with C, Y elliptic curves, and F a finite group acting on C as a group of translations and acting also on Y not only by translations. Then $\mathrm{Aut}^\circ(X) \simeq C$ [BM90, section 3] is maximal. Else assume that X' is not a bielliptic surface but is birational to a bielliptic surface X, then $\mathrm{Aut}^\circ(X')$ is trivial by Proposition I.3.26 (6) and is conjugated to the trivial subgroup of $\mathrm{Aut}^\circ(X)$. Thus $\mathrm{Aut}^\circ(X')$ is not maximal.

From Proposition I.3.26 (6), if X is an Enriques surface, or a $K3$ surface, or a surface of general type, then $\mathrm{Aut}^\circ(X)$ is trivial. Moreover, it is maximal since $\mathrm{Aut}^\circ(X')$ is also trivial for any representative X' of the birational class of X. $\square$

Remark I.3.27. Let X be a surface and G be a maximal connected algebraic subgroup of $\mathrm{Bir}(X)$.

(1) In positive characteristic, Proposition I.3.26 (1), (2), (3), (4), (6) still provides pairs $(X, \mathrm{Aut}^\circ(X))$ where $\mathrm{Aut}^\circ(X)$ is maximal.

(2) In Proposition I.3.26 (5), we show that there exists a finite subgroup scheme $F \subset G$ and a connected curve Y which is not necessarily reduced, such that X is isomorphic to $(G \times Y)/F$ and $G \simeq \mathrm{Aut}^\circ(X)$. To extend Theorem D in positive characteristic, it remains to determine which curves Y are possible. Recently, Brion shows that Y is G-normal [Bri22a, Definition 4.1, Proposition 5.6].

Algebraic subgroups of the group of birational transformations of ruled surfaces

II.1 Introduction

In this article, all varieties are defined over an algebraically closed field $\mathbf{k}$, algebraic groups are smooth group schemes of finite type (or equivalently, reduced group schemes of finite type), and C denotes a smooth projective curve of genus g. The main results, namely Theorem E and Corollary F, hold when the characteristic of $\mathbf{k}$ is different than two. When X is a projective variety, the automorphism group of X is the group of $\mathbf{k}$-rational points of a group scheme (see [MO67]), and we only consider its reduced structure.

The study of algebraic subgroups of the group of birational transformations has started with [Enr93], where the author has classified the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. More recently, the maximal algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ have been classified [Bla09b]. The purpose of this text is to study the algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ when $g \geq 1$, which will complete the classification for surfaces of Kodaira dimension $-\infty$.

Let G be an algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$. The strategy is classical: first regularize the action of G, find a G-equivariant completion and run a G-equivariant minimal model program to embed G in the automorphism group of a G-minimal fibration. The equivariant completion from Sumihiro [Sum74, Sum75] works for linear algebraic groups, therefore his results cannot be applied in our setting. Recently, Brion proved the existence of an equivariant completion for connected algebraic groups (not necessarily linear) acting birationally on integral varieties [Bri17, Corollary 3]. Using his results, we find an equivariant completion for algebraic groups (not necessarily linear or connected) acting on surfaces (see Proposition II.2.5). Then we reprove the G-equivariant MMP (Proposition II.2.6), which is a folklore result (see e.g. [KM98, Example 2.18]), by using only elementary arguments. We are left with studying the automorphism groups of conic bundles. Following the ideas of [Bla09b], we prove Propositions II.2.17 and II.3.11 which reduce the study to the cases of ruled surfaces, exceptional conic bundles and $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles (see II.2.1 for definitions). The Segre invariant $\mathfrak{S}(X)$ of a ruled surface X (see Definition II.2.9) has been introduced in [Mar70] and [Mar71] for the classification of ruled surfaces and their automorphisms. The ruled surfaces $\mathcal{A}_0$ and $\mathcal{A}_1$ in

Theorem E (4) and (5) are the only indecomposable $\mathbb{P}^1$-bundles over C up to C-isomorphism, when C is an elliptic curve (see Definition II.2.8 and [Ati57, Theorem 11], or [Har77, Theorem V.2.15]). Combining techniques from Blanc and results of Maruyama, we prove the following theorem:

Theorem E. *Let $\mathbf{k}$ be an algebraically closed field of characteristic different than two, and let C be a smooth projective curve of genus $g \geq 1$. The following algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ are maximal:*

(1) $\mathrm{Aut}(C \times \mathbb{P}^1) \simeq \mathrm{Aut}(C) \times \mathrm{PGL}(2, \mathbf{k})$.

(2) $\mathrm{Aut}(X)$*, where X is an exceptional conic bundle over C, which is the blowup of a decomposable ruled surface $\pi\colon \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C) \to C$ along $F = \{p_1, p_2, \cdots, p_{2\deg(D)}\}$ lying in two disjoint sections s_1 and s_2 of π, and such that $-2D$ is linearly equivalent to*

$$\sum_{p \in s_1 \cap F} \pi(p) - \sum_{p \in s_2 \cap F} \pi(p).$$

Then $\mathrm{Aut}(X)$ fits into an exact sequence

$$1 \to \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(X) \to H,$$

where H is the finite subgroup of $\mathrm{Aut}(C)$ preserving the image of the singular fibres.

(3) $\mathrm{Aut}(X)$*, where X is a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle with at least one singular fibre. Then $\mathrm{Aut}(X)$ fits into an exact sequence*

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathrm{Aut}(X) \to H,$$

where H is the finite subgroup of $\mathrm{Aut}(C)$ preserving the image of the singular fibres.

(4) $\mathrm{Aut}(X)$ *where X is a $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface (in consequence, $\mathfrak{S}(X) > 0$). Then $\mathrm{Aut}(X)$ fits into an exact sequence*

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathrm{Aut}(X) \to \mathrm{Aut}(C).$$

Moreover, if $g = 1$, there exists an unique $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface over C denoted $\mathcal{A}_1$, which satisfies $\mathfrak{S}(\mathcal{A}_1) = 1$ and $\mathrm{Aut}(\mathcal{A}_1)$ fits into an exact sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathrm{Aut}(\mathcal{A}_1) \to \mathrm{Aut}(C) \to 1.$$

(5) $\mathrm{Aut}(\mathcal{A}_0)$*, where $\mathcal{A}_0$ is the unique indecomposable ruled surface over C with Segre invariant zero when $g = 1$. Then there exists an exact sequence*

$$1 \to \mathbb{G}_a \to \mathrm{Aut}(\mathcal{A}_0) \to \mathrm{Aut}(C) \to 1.$$

(6) $\mathrm{Aut}(X)$, *where* $X \simeq \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$ *is a non trivial decomposable ruled surface over* C *with* $\deg(D) = 0$ *(or equivalently,* $\mathfrak{S}(X) = 0$*), with the additional assumption that* $2D$ *is principal if* $g \geq 2$. *Then* $\mathrm{Aut}(X)$ *fits into an exact sequence*

$$1 \to G \to \mathrm{Aut}(X) \to \mathrm{Aut}(C),$$

where $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $2D$ *is principal, else* $G = \mathbb{G}_m$.

Moreover, any maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ *is conjugate to one in the list above.*

By Corollary II.3.7, there exist exceptional conic bundles $X \to C$, where C is a curve of positive genus, such that $\mathrm{Aut}(X)$ is not a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$. This does not happen when the base curve is rational: the automorphism group of an exceptional conic bundle over $\mathbb{P}^1$ is always maximal (if the number of singular fibres is at least four, see [Bla09b, Theorem 1. (2)], else the number of singular fibres equals two and the result follows from [Bla09b, Theorem 2. (3)]). Moreover, the cases (4), (5) and (6) of Theorem E do not exist when the base curve is rational: the Segre invariant of a ruled surface $\pi \colon S \to \mathbb{P}^1$ is always non-positive (see [HM82] and Proposition I.2.18 (1)), and equals zero if and only if π is trivial.

From the classification of Blanc, it follows that every algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to a subgroup of a maximal one. This does not hold anymore for algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ when C has positive genus. The following corollary is an analogue of Theorem C for surfaces of Kodaira dimension $-\infty$.

Corollary F. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic different than two and let* X *be a surface of Kodaira dimension* $-\infty$. *Then, every algebraic subgroup of* $\mathrm{Bir}(X)$ *is contained in a maximal one if and only if* X *is rational.*

Acknowledgments. *The author is thankful to Jérémy Blanc, Michel Brion, Ronan Terpereau and Sokratis Zikas for helpful discussions and comments. The author is also grateful to the anonymous referees for their careful reading and interesting remarks.*

Conventions. *Unless otherwise stated, all varieties are smooth and projective, and* C *is a smooth projective curve.*

II.2 Preliminaries

II.2.1 Regularization and relative minimal fibrations

Definition II.2.1. Let C be a curve.

(1) A *ruled surface* over C is a morphism $\pi \colon S \to C$ such that each fibre is isomorphic to $\mathbb{P}^1$.

(2) A *conic bundle* over C is a morphism $\kappa\colon X \to C$ such that all fibres are isomorphic to $\mathbb{P}^1$, except finitely many (possibly zero) which are called *singular fibres* and are transverse unions of two (-1)-curves.

(3) A conic bundle $\kappa\colon X \to C$ is an *exceptional conic bundle* over C if there exists $n \geq 1$ such that κ has exactly $2n$ singular fibres and two sections of self-intersection $-n$.

(4) If $\kappa\colon X \to C$ is a conic bundle, we denote by $\mathrm{Bir}_C(X)$ the subgroup of $\mathrm{Bir}(X)$ which consists of the elements $f \in \mathrm{Bir}(X)$ such that $\kappa f = \kappa$. We also define $\mathrm{Aut}_C(X) = \mathrm{Aut}(X) \cap \mathrm{Bir}_C(X)$.

(5) A conic bundle $\kappa\colon X \to C$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-*conic bundle* over C if $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and each non trivial involution in this group fixes pointwise an irreducible curve, which is a 2-to-1 cover of C ramified above an even positive number of points.

(6) A $(\mathbb{Z}/2\mathbb{Z})^2$-*ruled surface* over C is a ruled surface over C which is also a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle over C.

Remark II.2.2. Assume that C has positive genus and let $\pi\colon X \to C$ be a conic bundle. Let f be a smooth fibre of π and $\alpha \in \mathrm{Aut}(X)$. Since $(\pi\alpha)_{|f}\colon f \simeq \mathbb{P}^1 \to C$ is constant, it follows that $\alpha(f)$ is also a smooth fibre of π. The set of singular fibres is preserved by $\mathrm{Aut}(X)$ and π induces a morphism of group schemes $\pi_*\colon \mathrm{Aut}(X) \to \mathrm{Aut}(C)$. This implies that every automorphism of X preserves the conic bundle structure.

Definition II.2.3. Let X be a surface and G be an algebraic subgroup of $\mathrm{Aut}(X)$.

(1) A birational map $\phi\colon X \dashrightarrow Y$ is G-*equivariant* if $\phi G \phi^{-1} \subset \mathrm{Aut}(Y)$.

(2) The pair (G, X) is *minimal* if every G-equivariant birational morphism $X \to X'$, where X' is a surface, is an isomorphism.

The classical approach to study algebraic subgroups uses the regularization theorem of Weil [Wei55] (see also [Zai95] or [Kra18] for modern proofs). By [Bri22b, Theorem 1], the regularization of X contains a G-stable dense open subset U which is smooth and quasi-projective. Then by [Bri22b, Theorem 2], U admits a G-equivariant completion by a normal projective G-variety, that we can assume smooth by a G-equivariant desingularization (see [Lip78]).

In the following lemma and proposition, we give an elementary proof of the existence of an equivariant completion for surfaces equipped with the action of an algebraic group G, not necessarily connected or linear, without using results of [Bri22b].

Lemma II.2.4. *Let X be a surface and G be an algebraic subgroup of $\mathrm{Bir}(X)$, such that G° acts regularly on X. Denote by $\mathrm{Bs}(X)$ the set of base points of the G-action, including the infinitely near ones. Then $\mathrm{Bs}(X)$ is finite and the action of G lifts to a regular action on the blowup of X at $\mathrm{Bs}(X)$.*

Proof. The set G/G° is finite, we can write $G = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_n$ where $G^\circ = G_0$ and each G_i is a connected component of G. For each i, we fix $g_i \in G_i$ and get that $g_i G^\circ = G_i$. Let $\mathrm{Bs}(g_i)$ be the set of base points of g_i, including the infinitely near ones, which is finite because X is smooth and projective. The subgroup $G^\circ \subset G$ is normal, it follows that every element $g_i' \in G_i$ equals gg_i for some $g \in G^\circ$. Since G° acts regularly, $\mathrm{Bs}(G_i) = \mathrm{Bs}(g_i)$ for each i, and this implies that $\mathrm{Bs}(X) = \bigcup_{i=1\cdots n} \mathrm{Bs}(g_i)$ is also finite. Besides, for each $g \in G^\circ$, there exists $\tilde{g} \in G^\circ$ such that $g_i g = \tilde{g} g_i$. Then $g^{-1}(\mathrm{Bs}(g_i)) = \mathrm{Bs}(g_i)$ and this implies that G° acts trivially on $\mathrm{Bs}(X)$.

If $\mathrm{Bs}(X)$ is empty, the result holds. Suppose that $\mathrm{Bs}(X) \neq \emptyset$. Let $p \in \mathrm{Bs}(X) \cap X$ be a proper base point, $\eta\colon X_p \to X$ be the blowup of X at p. We consider the action of G on X_p obtained by conjugation. As p is fixed by G°, the algebraic group G° still acts regularly on X_p. We prove that each element $q \in \mathrm{Bs}(X_p)$ corresponds via η to an element of $\mathrm{Bs}(X)$. Let $q \in \mathrm{Bs}(X_p)$, there exists a surface Y such that $q \in Y$, and a birational morphism $\pi\colon Y \to X_p$ such that q is a base point of $\eta^{-1} g_i \eta\pi$. Let W be a smooth projective surface, with birational morphisms $\alpha\colon W \to Y$ and $\beta\colon W \to X_p$ such that $\beta\alpha^{-1}$ is a minimal resolution of $\eta^{-1} g_i \eta\pi$. The following diagram is commutative:

$$\begin{tikzcd}
 & & W \arrow[dll, "\alpha"] \arrow[ddr, "\beta"] & \\
q \in Y \arrow[dr, "\pi"] \arrow[ddr, "\eta\pi"'] & & & \\
 & X_p \arrow[rr, dashed, "\eta^{-1}g_i\eta"'] \arrow[d, "\eta"] & & X_p \arrow[d, "\eta"] \\
 & X \arrow[rr, dashed, "g_i"'] & & X.
\end{tikzcd}$$

Since q is a base point of $\eta^{-1} g_i \eta\pi$, it must be blown up by α. There exists a (-1)-curve $\widetilde{C}$ in W contracted by α to q, and such that its image by β is a curve C in X_p. If the image of C by η is a curve in X, then q is a base point of g_i. Else C is contracted by η, i.e. C is the exceptional divisor of η. As $C^2 = \widetilde{C}^2 = -1$, the morphism β is an isomorphism $\widetilde{U} \to U$ where $\widetilde{U} \subset W$, $U \subset X_p$ are open subsets containing $\widetilde{C}$ and C. Let j be such that p is a proper base point of g_j. Let W' be a smooth projective surface with birational morphisms $\alpha'\colon W' \to X_p$ and $\beta'\colon W' \to X$ such that $\beta'\alpha'^{-1}\eta^{-1}$ is a minimal resolution of g_j. We obtain the following commutative diagram:

$$\begin{tikzcd}
 & & W \arrow[dll, "\alpha"] \arrow[ddrr, "\beta"] & & W' \arrow[dd, "\alpha'"] \arrow[dddrr, "\beta'"] & & \\
q \in Y \arrow[dr, "\pi"] \arrow[ddr, "\eta\pi"'] & & & & & & \\
 & X_p \arrow[rrr, dashed, "\eta^{-1}g_i\eta"'] \arrow[d, "\eta"] & & & X_p \arrow[d, "\eta"] & & \\
 & X \arrow[rrr, dashed, "g_i"'] & & & X \arrow[rr, dashed, "g_j"'] & & X.
\end{tikzcd}$$

There exists a (-1)-curve C' in W' contracted to p by $\eta\alpha'$, and such that its image by β' is a curve in X. The image of C' by α' is C or a point of C. Since $\beta\colon \tilde{U} \to U$ is an isomorphism, this implies that $\beta^{-1}\alpha'\colon W' \dashrightarrow W$ is defined on a neighborhood of C' and sends C' on either $\tilde{C}$ or a point of $\tilde{C}$. Hence $\alpha\beta^{-1}\alpha'\colon W' \dashrightarrow Y$ contracts C' onto q, so $q \in \mathrm{Bs}(g_j g_i)$.

Let $\mathrm{Bs}(X_p)$ be the set of base points of $\eta^{-1}G\eta$, including the infinitely near ones. We have shown that if $q \in \mathrm{Bs}(X_p)$ then $q \in \mathrm{Bs}(X)$. The map $\mathrm{Bs}(X_p) \to \mathrm{Bs}(X) \setminus \{p\}$, sending the infinitesimal base point (q, π) to $(q, \eta\pi)$ is injective. Conversely, if $q \in \mathrm{Bs}(g_i)$ and $q \neq p$, then $\eta^{-1}(q) \in \mathrm{Bs}(\eta^{-1}g_i\eta)$. Therefore $\mathrm{Bs}(X_p) \simeq \mathrm{Bs}(X) \setminus \{p\}$. Proceeding by induction, the blowup of all elements of Bs gives rise to a surface on which G acts regularly. □

Proposition II.2.5. *Let X be a surface and G be an algebraic subgroup of* $\mathrm{Bir}(X)$*. Then there exists a smooth projective surface Y with a birational map $\psi\colon X \dashrightarrow Y$ such that $\psi G\psi^{-1}$ is an algebraic subgroup of* $\mathrm{Aut}(Y)$*.*

Proof. Apply [Bri17, Corollary 3] to X equipped with the action of the connected component of the identity $G^\circ \subset G$. There exists a normal projective surface Z with a birational map $\phi\colon X \dashrightarrow Z$, such that $\phi G^\circ \phi^{-1} \subset \mathrm{Aut}^\circ(Z)$. By an equivariant desingularization, we can also assume that Z is smooth [Lip78]. Let $H = \phi G\phi^{-1}$ and $\eta\colon Y \to Z$ be the blowup of Z at $\mathrm{Bs}(Z)$. By Lemma II.2.4, the action of H lifts to a regular action on Y. Then $\eta^{-1}H\eta \subset \mathrm{Aut}(Y)$ is a closed subgroup which is an algebraic subgroup of $\mathrm{Bir}(Y)$. Take $\psi = \eta^{-1}\phi$, we get that $\psi G\psi^{-1}$ is an algebraic subgroup of $\mathrm{Aut}(Y)$. □

The next result is also known, see e.g. [KM98, Example 2.18], we reprove it in our specific situation using elementary arguments.

Proposition II.2.6. *Let C be a curve of positive genus, and let X be a surface birationally equivalent to $C \times \mathbb{P}^1$. Let G be an algebraic subgroup of* $\mathrm{Aut}(X)$*. If (G, X) is minimal (see Definition II.2.3), then X is a conic bundle over C.*

Proof. Since X is birational to $C \times \mathbb{P}^1$, there exists a morphism $\kappa\colon X \to C$ and a birational map $\phi\colon C \times \mathbb{P}^1 \dashrightarrow X$ such that $\kappa\phi = p_1$, where $p_1\colon C \times \mathbb{P}^1 \to C$ denotes the projection on the first factor. In particular, ϕ is a finite composite of blowups and contractions, and there exists a non empty open $U \subset C$ such that $\phi_{|U \times \mathbb{P}^1}$ is an isomorphism. Let $p \in C \setminus U$, it remains to see that $\kappa^{-1}(p)$ is isomorphic to $\mathbb{P}^1$ or is the transverse union of two (-1)-curves. Since X is the blowup of a ruled surface S in finitely many points (maybe infinitely close), we can write $\kappa^{-1}(p) = E_1 \cup \cdots . \cup E_n$ where:

- Each E_i is isomorphic to $\mathbb{P}^1$.
- For all i, j distinct, E_i and E_j intersect transversely at a point or are disjoint.

If $n = 1$, $\kappa^{-1}(p)$ is a smooth fibre isomorphic to $\mathbb{P}^1$. If $n = 2$, then E_1 and E_2 intersect transversely in one point. Because there is a contraction to the ruled surface S, either E_1 or E_2 can be contracted. Therefore, $E_1^2 = E_2^2 = -1$.

Assume from now on that $n \geq 3$. First, $E_i^2 < 0$ for all i. The contraction of any collection of disjoint (-1)-curves permuted transitively by G is G-equivariant. Since (G, X) is minimal, there exist $k, l \in \{1, \cdots, n\}$ with $k \neq l$ such that E_k and E_l are two (-1)-curves in the same G-orbit and $E_k \cap E_l \neq \emptyset$. The image of E_l by the contraction of E_k has self-intersection zero, and in particular it cannot be contracted. By assumption that $n \geq 3$, we can contract other (-1)-curves in $\kappa^{-1}(p)$, which increases the self-intersection. This contradicts the existence of a contraction of X to the ruled surface S, where $f^2 = 0$ for any fibre f. Therefore, we must have $n \leq 2$. $\square$

The previous proposition motivates the study of automorphism groups of conic bundles. The next lemma can be used as a criterion of maximality of their automorphism groups.

Lemma II.2.7. *Let C be a curve of positive genus. Let $\kappa\colon X \to C$ and $\kappa'\colon X' \to C$ be conic bundles. Let G be an algebraic group acting on X and X' such that (G, X) and (G, X') are minimal, and let $\phi\colon X \dashrightarrow X'$ be a G-equivariant birational map which is not an isomorphism. Then $\phi = \phi_n \cdots \phi_1$ where each ϕ_j is the blowup of a finite G-orbit of a point, which is contained in the complement of the singular fibres and does not contain two points on the same smooth fibre, followed by the contractions of the strict transforms of the fibres through the points of the G-orbit. In particular, κ and κ' have the same number of singular fibres.*

Proof. Take a minimal resolution of ϕ, i.e. a surface Z with G-equivariant birational morphisms $\eta\colon Z \to X$ and $\eta'\colon Z \to X'$ satisfying $\eta' = \phi\eta$, and such that there is no (-1)-curve contracted by η and η'. Let $E_1, \cdots, E_m \subset Z$ be a G-orbit of (-1)-curves contracted by η' and let $p_i = \kappa\eta(E_i)$. Denote by $\widetilde{E}_i$ the images of E_i by η, which are contained in the fibres $\kappa^{-1}(p_i)$. Since (G, X) is minimal, η must blowup a G-orbit of points Ω contained in $\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_m$. Hence, $\widetilde{E}_i^2 \geq 0$ for all i. In particular, $\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_m$ is not contained in the set of singular fibres of κ and $\widetilde{E}_i^2 = 0$ for each i.

Then, $\widetilde{E}_1 \cup \cdots \cup \widetilde{E}_m$ is contained in the complement of the singular fibres. As $\widetilde{E}_i^2 = 0$ and $E_i^2 = -1$ for each i, no distinct points of Ω lie in the same smooth fibre. Because (G, X') is minimal, we can contract the strict transforms of the fibres, which yields a G-equivariant birational map $\phi_1\colon X \dashrightarrow X_1$ such that ϕ factorizes through ϕ_1. By induction, we find G-equivariant birational maps $\phi_j\colon X_{j-1} \dashrightarrow X_j$ such that $\phi = \phi_n \cdots \phi_1$, where each ϕ_j is as we wanted.

Finally, applying elementary transformations in the complement of the set of singular fibres does not change the number of singular fibres. $\square$

II.2.2 Generalities on ruled surfaces and their automorphisms

Definition II.2.8. A ruled surface π is *decomposable* if it admits two disjoint sections. Else, π is *indecomposable.*

The following notion has been already used in [Mar70, Mar71] and in Chapter I.

Definition II.2.9. The *Segre invariant* $\mathfrak{S}(S)$ of a ruled surface $\pi\colon S \to C$ is the integer $\min\{\sigma^2, \sigma \text{ section of } \pi\}$. A section σ of π such that $\sigma^2 = \mathfrak{S}(S)$ is called a *minimal section*.

Lemma II.2.10. *Let $\pi\colon S \to C$ be a decomposable ruled surface with $\mathfrak{S}(S) = 0$. If two sections are disjoint then they are both minimal sections.*

Proof. Let σ be a minimal section. Let $\mathrm{Num}(S)$ be the group of divisors of S, up to numerical equivalence. Then $\mathrm{Num}(S)$ is generated by the classes of σ and f, where σ is a minimal section and f is a fibre [Har77, Proposition V.2.3]. Let s_1 and s_2 be disjoint sections. In particular, $s_1 \equiv \sigma + b_1 f$ and $s_2 \equiv \sigma + b_2 f$ for some $b_1, b_2 \in \mathbb{Z}$. Since we have $s_1 \cdot \sigma \geq 0$ and $s_2 \cdot \sigma \geq 0$, it follows that $b_1 \geq 0$ and $b_2 \geq 0$. Since $s_1 \cdot s_2 = 0$, this implies that $b_1 = b_2 = 0$. □

Lemma II.2.11. *Let $S \to C$ be a ruled surface such that $\mathfrak{S}(S) = -n < 0$. The following hold:*

(1) *there exists a unique section of negative self-intersection and all other sections have self-intersection at least n,*
(2) *two sections are disjoint if and only if one is the $(-n)$-section and the other has self-intersection n,*
(3) *there exists a section of self-intersection n if and only if S is decomposable.*

Proof. (1) By assumption, there exists a section s_{-n} of self-intersection $s_{-n}^2 = -n < 0$. Let $s \neq s_{-n}$ be a section, then s is numerically equivalent to $s_{-n} + bf$ for some integer b and $0 \leq s \cdot s_{-n} = s_{-n}^2 + b$. Therefore $b \geq -s_{-n}^2 = n$ and it follows that $s^2 = s_{-n}^2 + 2b \geq n$.

(2) Denote by s_1, s_2 two disjoint sections. Then $s_1 \equiv s_{-n} + b_1 f$ and $s_2 \equiv s_{-n} + b_2 f$ for some $b_1, b_2 \in \mathbb{Z}$. We get that $0 = s_1 \cdot s_2 = -n + b_1 + b_2$. If s_1 and s_2 are both different from s_{-n}, then $b_1 \geq n$ and $b_2 \geq n$ by (1), and this contradicts the equality $0 = -n + b_1 + b_2$. Then we can assume that $s_1 = s_{-n}$ and $b_1 = 0$. It follows that $b_2 = n$ and $s_2^2 = s_{-n}^2 + 2n = n$. Conversely, if s is a section of self-intersection n then $s \equiv s_{-n} + nf$ and $s_{-n} \cdot s = 0$.

(3) Let s be a section such that $s^2 = n$, then $s \equiv s_{-n} + nf$ and $s \cdot s_{-n} = 0$, i.e. s and s_{-n} are disjoint sections. In particular, S is decomposable. Conversely, if S is decomposable, there exist two disjoints sections and one of them has self-intersection n by (2).

□

Definition II.2.12. Let $f \in \mathrm{Bir}_C(C \times \mathbb{P}^1) \simeq \mathrm{PGL}(2, \mathbf{k}(C))$. The *determinant* of f, denoted $\det(f)$, is the element of $\mathbf{k}(C)^*/(\mathbf{k}(C)^*)^2$ defined as the class of the determinant of a representative of f in $\mathrm{GL}(2, \mathbf{k}(C))$.

A non trivial decomposable ruled surface S of Segre invariant zero admits exactly two minimal sections. In [Mar71, Theorem 2, (3) and (4)], a necessary and sufficient condition for such surfaces to have an automorphism permuting two minimal sections is given. We provide below a revisited version of this result, that we prove by computations in local charts:

Lemma II.2.13. *Let C be a curve. Let $\pi\colon S = \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C) \to C$ be a decomposable $\mathbb{P}^1$-bundle and $p_1\colon C \times \mathbb{P}^1 \to C$ be the trivial $\mathbb{P}^1$-bundle over C. Then $\mathfrak{S}(S) = 0$ if and only if $\deg(D) = 0$. Moreover, if $\mathfrak{S}(S) = 0$ and π is not trivial then the following hold:*

(1) *π has exactly two minimal sections s_1 and s_2 of self-intersection 0.*
(2) *$\mathrm{Aut}_C(S) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ if $2D$ is principal. In this case, for each element $\iota \in \mathrm{Aut}_C(S)$ permuting s_1 and s_2, there exists a birational map $\xi\colon S \dashrightarrow C \times \mathbb{P}^1$ such that $\pi = p_1 \xi$ and $\xi\iota\xi^{-1} = \begin{bmatrix} 0 & \beta \\ 1 & 0 \end{bmatrix}$, with $\beta \in \mathbf{k}(C)^*$ and $\mathrm{div}(\beta) = 2D$. In particular, ι is not a square in $\mathrm{Bir}_C(S)$.*
(3) *$\mathrm{Aut}_C(S) \simeq \mathbb{G}_m$ if D is 2-torsion.*

Proof. We prove first that $\mathfrak{S}(S) = 0$ if and only if $\deg(D) = 0$. Assume that $\mathfrak{S}(S) = 0$. By [Mar70, Lemma 1.15] (see also Corollary I.2.16), we get $0 = \deg(\mathcal{O}_C(D) \oplus \mathcal{O}_C) - 2\deg(M)$, where M is a line subbundle of $\mathcal{O}_C(D) \oplus \mathcal{O}_C$ of maximal degree. By additivity of the degree, $\deg(D) - 2\deg(M) = 0$. Since $\deg(M) \geq 0$ and $\deg(M)$ is maximal, $\deg(D) = \deg(M) = 0$. Conversely, we have that $\mathfrak{S}(S) \leq 0$ by Proposition I.2.18 (1). Moreover, S admits two disjoint sections corresponding to the line subbundles $\mathcal{O}_C$ and $\mathcal{O}_C(D)$ of $\mathcal{O}_C \oplus \mathcal{O}_C(D)$, and they both have self-intersection zero by Proposition I.2.15. The case $\mathfrak{S}(S) < 0$ is ruled out by Lemma II.2.11 (1), and thus $\mathfrak{S}(S) = 0$.

We now assume that $\mathfrak{S}(S) = 0$ and that π is not trivial, and prove (1), (2), (3). The proof of (1) can be found in [Mar71, Lemma 2. (2)], or Proposition I.2.18 (3.iii). Then we prove (2) and (3). Let A be a very ample divisor on C. For an integer m large enough, the divisor $B = D + mA$ is also very ample. In particular, we can find $B' \sim B$ and $A' \sim mA$ such that $\mathrm{Supp}(B') \cap \mathrm{Supp}(D) = \emptyset$ and $\mathrm{Supp}(A') \cap \mathrm{Supp}(D) = \emptyset$. Let $E = A' - B'$, then $D + E \sim 0$, and there exists $f \in \mathbf{k}(C)$ such that $\mathrm{div}(f) = D + E$ and $\mathrm{Supp}(D) \cap \mathrm{Supp}(E) = \emptyset$. Choose $U = C \setminus \mathrm{Supp}(E)$ and $V = C \setminus \mathrm{Supp}(D)$ as trivializing open subsets of π, and local trivializations of π such that s_1 and s_2 are respectively the zero and the infinity sections. The transition map of S can be written as:

$$\begin{aligned} s_{vu}\colon U \times \mathbb{P}^1 &\dashrightarrow V \times \mathbb{P}^1 \\ x, [y_0 : y_1] &\longmapsto x, [f(x) y_0 : y_1]. \end{aligned}$$

By (1), an element of $\mathrm{Aut}_C(S)$ either fixes pointwise s_1 and s_2, or otherwise permutes s_1 and s_2. If $\phi \in \mathrm{Aut}_C(S)$ fixes s_1 and s_2, then it induces automorphisms $\phi_u\colon U \times \mathbb{P}^1 \to U \times \mathbb{P}^1$, $x, [y_0 : y_1] \mapsto x, [\alpha_u(x) y_0 : y_1]$ and $\phi_v\colon V \times \mathbb{P}^1 \to V \times \mathbb{P}^1$, $x, [y_0 : y_1] \mapsto x, [\alpha_v(x) y_0 : y_1]$ with $\alpha_u \in \mathcal{O}_C(U)^*$, $\alpha_v \in \mathcal{O}_C(V)^*$. The condition $\phi_v s_{vu} = s_{vu} \phi_u$ is equivalent to $\alpha_u = \alpha_v = \alpha \in \mathbb{G}_m$. We have shown that $\left\{\begin{bmatrix}\alpha & 0\\ 0 & 1\end{bmatrix}, \alpha\in\mathbb{G}_m\right\}$ is the algebraic subgroup of $\mathrm{Aut}_C(S)$ fixing s_1 and s_2. If $\iota\in\mathrm{Aut}_C(S)$ permutes s_1 and s_2, then ι induces automorphisms $\iota_u\colon U\times\mathbb{P}^1\to U\times\mathbb{P}^1$, $x,[y_0:y_1]\mapsto x,[\beta_u(x)y_1:y_0]$ and $\iota_v\colon V\times\mathbb{P}^1\to V\times\mathbb{P}^1$, $x,[y_0:y_1]\mapsto x,[\beta_v(x)y_1:y_0]$ with $\beta_u\in\mathcal{O}_C(U)^*$, $\beta_v\in\mathcal{O}_C(V)^*$. The condition $\iota_v s_{vu}=s_{vu}\iota_u$ is now equivalent to $f^2\beta_u=\beta_v$. In particular, $\mathrm{div}(\beta_v)=2D$ and $2D$ is a principal divisor. Conversely, if $2D$ is a principal divisor, there exists $\beta\in\mathbf{k}(C)^*$ such that $\mathrm{div}(\beta)=2D$. Choose $\beta_v=\beta$ and $\beta_u=f^{-2}\beta_v$, the automorphisms ι_u, ι_v glue back to a C-automorphism ι of S of order two which permutes s_1 and s_2. Thus, $\mathrm{Aut}_C(S)\simeq\mathbb{G}_m\rtimes\mathbb{Z}/2\mathbb{Z}$ if and only if $2D$ is principal, and ι_v induces the birational map $\xi\colon S\dashrightarrow C\times\mathbb{P}^1$ given in the statement.

Finally, assume ι is a square in $\mathrm{Bir}(S)$. Then $\xi\iota\xi^{-1}$ is a square in $\mathrm{Bir}(C\times\mathbb{P}^1)$ and has determinant $-\beta$. Since π is not trivial by assumption, it follows that D is not principal and $\mathrm{div}(\beta)=2D$, this implies that $-\beta=\det(\xi\iota\xi^{-1})$ is not a square, which is a contradiction. □

Every element of $\mathrm{Aut}_{\mathbb{P}^1}(\mathbb{F}_n)$ fixes pointwise a section of $\mathbb{F}_n$. This is not true when we consider $\mathbb{P}^1$-bundles over a non-rational curve C, as we have seen in Lemma II.2.13 (2). The following lemma shows that it is the only exception up to conjugation:

Lemma II.2.14. *Let C be a curve. Let $\pi\colon S\to C$ be a ruled surface, let $p_1\colon C\times\mathbb{P}^1\to C$ be the trivial $\mathbb{P}^1$-bundle and $f\in\mathrm{Aut}_C(S)$. Then f satisfies one of the following:*

1. *f fixes pointwise a section of π,*
2. *f does not fix any section of π, and there exists a birational map $\xi\colon S\dashrightarrow C\times\mathbb{P}^1$ such that $\pi=p_1\xi$ and $\xi f\xi^{-1}=\begin{bmatrix}0 & \beta\\ 1 & 0\end{bmatrix}$, with $\mathrm{div}(\beta)=2D$ for some divisor D which is not principal.*

Moreover, if f satisfies (2) then f is not a square in $\mathrm{Bir}_C(S)$.

Proof. First we deal with the case $\mathfrak{S}(S)\leq 0$. If $\mathfrak{S}(S)<0$, or $\mathfrak{S}(S)=0$ and S is indecomposable, then S has a unique minimal section which is $\mathrm{Aut}_C(S)$-invariant (see Lemma II.2.11 (1) and [Mar71, Lemma 2. (1.ii)], or Proposition I.2.18 (3.ii)). If S is trivial, then $\mathrm{Aut}_C(S)=\mathrm{PGL}(2,\mathbf{k})$ and every element fixes pointwise a section. In particular, f satisfies the condition (1). Else $\mathfrak{S}(S)=0$, S is decomposable and S is not trivial. Then $S=\mathbb{P}(\mathcal{O}_C(D)\oplus\mathcal{O}_C)$ for some divisor D of degree 0 and by Lemma II.2.13, $\mathrm{Aut}_C(S)\simeq\mathbb{G}_m$ or $\mathbb{G}_m\rtimes\mathbb{Z}/2\mathbb{Z}$. In particular, the automorphism f fixes the two minimal sections of S and satisfies (1), or permutes them and satisfies (2).

Assume $\mathfrak{S}(S)>0$. Then S is indecomposable and $\mathrm{Aut}_C(S)$ is finite [Mar71, Theorem 2 (1)]. Let s be a section of S. If $f(s)=s$, we are done. Else s and $f(s)$ intersect in finitely many points which are fixed by f. Blow up these points and contract the strict transforms of their fibres, and repeat the process until that the strict transforms of the sections are disjoint. This yields a f-equivariant birational map $\phi\colon S \dashrightarrow S'$ with S' decomposable. By Proposition I.2.18 (1), it follows that $\mathfrak{S}(S') \leq 0$. Moreover, the strict transforms of s and $f(s)$ by ϕ are disjoint and permuted by $\phi f \phi^{-1}$, hence $\mathfrak{S}(S') = 0$ (see Lemma II.2.11 (2)). Then Lemma II.2.13 implies that f satisfies (2). Since $\phi f \phi^{-1}$ is a not a square in $\mathrm{Bir}_C(S)$ by Lemma II.2.13 (2), it also follows that f is also not a square in $\mathrm{Bir}_C(S)$. □

II.2.3 Reduction of cases

The following lemma is an analogue of [Bla09a, Lemma 6.1. (1) ⇔ (2)] for conic bundles, not necessarily rational. The proof is slightly more difficult, due to the case (2) of Lemma II.2.14 which does not exist in the rational case.

Lemma II.2.15. *Let C be a curve. Let $\kappa\colon X \to C$ be a conic bundle with at least one singular fibre and let $f \in \mathrm{Aut}_C(X)$ permuting the irreducible components of at least one singular fibre. Then f has order two.*

Proof. Let $\eta\colon X \to S$ be the contraction of one irreducible component in each singular fibre. The automorphism f^2 preserves all the irreducible components of the singular fibres, hence η is f^2-equivariant. Let $g = \eta f^2 \eta^{-1} \in \mathrm{Aut}_C(S)$, which is a square in $\mathrm{Bir}_C(S)$, then g fixes pointwise a section (Lemma II.2.14). Let s'_{inv} be a g-invariant section of S, and s_{inv} its strict transform by η which is f^2-invariant. As f exchanges the irreducible components of at least one singular fibre, the section s_{inv} is not f-invariant. The sections s_{inv} and $f(s_{inv})$ meet a general fibre in two points which are exchanged by the action of f. Thus f has order two. □

Lemma II.2.16. *Let C be a curve. Let $\kappa\colon X \to C$ be a conic bundle with at least one singular fibre, such that its two irreducible components are exchanged by an element $\rho \in \mathrm{Aut}(X)$. Let G be the normal subgroup of $\mathrm{Aut}_C(X)$ which leaves invariant each irreducible component of the singular fibres. The following hold:*

1. *If G fixes a section σ_1 of κ and G is not trivial, then κ is an exceptional conic bundle.*
2. *If there exists a contraction $\eta\colon X \to S$ such that $\mathfrak{S}(S) \leq 0$ and S is indecomposable, then G is trivial.*

Proof. (1) The subgroup $G \subset \mathrm{Aut}(X)$ is normal, hence $\rho G \rho^{-1} = G$ and the section $\sigma_2 = \rho\sigma_1 \neq \sigma_1$ is also G-invariant. Let $\eta\colon X \to S$ be the contraction of one irreducible component in each singular fibre of κ, namely the one intersecting σ_2, then it is a G-equivariant birational morphism. Let $H = \eta G \eta^{-1} \subset \mathrm{Aut}_C(S)$, which is not trivial. The images of σ_1 and σ_2 by η are H-invariant sections s_1 and s_2 of S. Assume that s_1 and s_2 intersect. Choose another section s_3. Apply elementary transformations centered on $\{s_i \cap s_j, i, j \in \{1,2,3\}, i \neq j\}$, and repeat until that the strict transforms of s_1, s_2, s_3 are disjoint. This yields an H-equivariant birational map $\psi\colon S \dashrightarrow C \times \mathbb{P}^1$. The group $\psi H \psi^{-1}$ is an algebraic subgroup of $\mathrm{PGL}(2,\mathbf{k})$ which fixes the strict transforms of s_1, s_2 and the base points of ψ^{-1}. The base points of ψ^{-1} coming from the contraction of the strict transforms of the fibres passing through the intersections of s_1 and s_2 are outside of the strict transforms of s_1 and s_2. Then H is conjugate to a subgroup of $\mathrm{PGL}(2,\mathbf{k})$ fixing three distinct points on $\mathbb{P}^1$, which implies that H is trivial and it is a contradiction. Therefore, s_1 and s_2 are disjoint sections of S, and it follows that S is decomposable and $\mathfrak{S}(S) \leq 0$ by Proposition I.2.18 (1). Thus σ_1 and σ_2 are also disjoint sections of X which pass through different irreducible components in each singular fibre.

Since $\sigma_2 = \rho\sigma_1$, it follows that $\sigma_1^2 = \sigma_2^2$. Then $s_1^2 < s_2^2$. If $\mathfrak{S}(S) = 0$, then s_1 and s_2 are both minimal section as they are disjoint, and this contradicts the inequality $s_1^2 < s_2^2$. Therefore, $\mathfrak{S}(S) < 0$ and by Lemma II.2.11 (2), it follows that $s_1^2 = -n < 0$ and $s_2^2 = n > 0$ for $n = -\mathfrak{S}(S)$. In particular, η is the blow-up of $2n$ points on σ_2. Then κ has $2n$ singular fibres and two disjoint $(-n)$-sections, i.e. it is an exceptional conic bundle.

(2) If $\mathfrak{S}(S) \leq 0$ and π is indecomposable, then S has a unique minimal section which is $\mathrm{Aut}(S)$-invariant [Mar71, Lemma 2 (1), (i) and (ii)] (or Proposition I.2.18 (2) and (3), and its strict transform by η is a G-invariant section of κ. If G is not trivial, it follows from (1) that X is an exceptional conic bundle. This implies that S admits two disjoint sections, which is a contradiction.

□

The key result of this section is the following proposition, analogue of [Bla09b, Lemma 4.3.5], which will be useful to reduce to the study of automorphism groups of ruled surfaces, exceptional conic bundles and $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles.

Proposition II.2.17. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve. Let $\kappa\colon X \to C$ be a conic bundle with at least one singular fibre, such that its two irreducible components are exchanged by an element of* $\mathrm{Aut}(X)$. *Let G be the normal subgroup of* $\mathrm{Aut}_C(X)$ *which leaves invariant every irreducible component of the singular fibres. If G is not trivial and if there exists a contraction $\eta\colon X \to S$ with S a decomposable $\mathbb{P}^1$-bundle over C, then κ is an exceptional conic bundle. Else,* $\mathrm{Aut}_C(X)$ *is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \{0,1,2\}$.*

Proof. If G is trivial, then every element of $\mathrm{Aut}_C(X)$ is an involution. This implies that $\mathrm{Aut}_C(X)$ is a finite subgroup of $\mathrm{PGL}(2,\mathbf{k}(C))$ and the statement follows. Assume that G is not trivial and let $\eta\colon X \to S$ be a contraction where $\pi\colon S \to C$ is a $\mathbb{P}^1$-bundle. Then η is G-equivariant and $H = \eta G \eta^{-1} \subset \mathrm{Aut}_C(S)$ is not trivial. Three cases arise:

(1) Assume first that $\mathfrak{S}(S) < 0$. Then S admits a unique minimal section, and its strict transform by η is a G-invariant section of κ. By Lemma II.2.16, S is decomposable and κ is an exceptional conic bundle.

(2) Assume that $\mathfrak{S}(S) = 0$. If a section of S of self-intersection 0 passes through at least one of the points blown-up by η, its strict transform is a section s of X of negative self-intersection. Contracting in each fibre the irreducible

component not intersecting s gives a birational morphism $\eta'\colon X \to S'$ with $\mathfrak{S}(S') < 0$, reducing to the previous case. We now assume that no section of S of self-intersection 0 passes through any point blown-up by η. Firstly, $\pi\colon S \to C$ is not a trivial bundle, as otherwise sections of S of self-intersection 0 would cover S. From Lemma II.2.16 (2), S is decomposable. Moreover, by Lemma II.2.13 (1) there exist exactly two disjoint sections s_1', s_2' of S of self-intersection 0. Furthermore, $\eta G \eta^{-1}$ is a non-trivial subgroup of $\mathrm{Aut}_C(S)$, isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ (see Lemma II.2.13), that fixes the base-points of η^{-1}, not lying on s_1' or s_2'. We now prove that no non-trivial element of $\eta G \eta^{-1}$ can lie in $\mathbb{G}_m$: taking a trivializing open subset $U \subseteq C$ of $\pi\colon S \to C$ containing the image of a base-point, and taking an isomorphism $\pi^{-1}(U) \simeq U \times \mathbb{P}^1$ sending s_1', s_2' onto the zero and infinity sections, the action of $\mathbb{G}_m$ on $U \times \mathbb{P}^1$ is $(x, [u : v]) \mapsto (x, [\alpha u : v])$ and thus no non-trivial element of $\mathbb{G}_m$ fixes any point outside of s_1', s_2'. Then $\eta G \eta^{-1} \cap \mathbb{G}_m = \{1\}$ and G has order 2. By Lemma II.2.15, every element of $\mathrm{Aut}_C(X)$ is an involution. As $\mathrm{Aut}_C(X)$ is a finite subgroup of $\mathrm{PGL}(2, \mathbf{k}(C))$, this implies that $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \{0, 1, 2\}$.

(3) Assume that $\mathfrak{S}(S) > 0$. In particular, S is indecomposable (Proposition I.2.18 (1)). Then from [Mar71, Lemma 3], $\mathrm{Aut}_C(S)$ is isomorphic to a subgroup of $\mathrm{Pic}^0(C)[2]$. In particular, it is a finite subgroup of $\mathrm{PGL}(2, \mathbf{k}(C))$ such that every element is an involution. Hence $\mathrm{Aut}_C(S) \simeq (\mathbb{Z}/2\mathbb{Z})^s$ for some $s \in \{0, 1, 2\}$. It follows that every element of G is an involution, and by Lemma II.2.15 every element of $\mathrm{Aut}_C(X)$ is an involution. Since $\mathrm{Aut}_C(X)$ is a finite subgroup of $\mathrm{PGL}(2, \mathbf{k}(C))$, it follows that $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \{0, 1, 2\}$.

□

II.3 Automorphism groups of irrational conic bundles

II.3.1 Infinite increasing sequence of automorphism groups

We first prove the following lemma which is a generalization of Theorem A, and the proof works essentially the same, based on an explicit automorphism of ruled surfaces computed in [Mar71].

Lemma II.3.1. *Let C be a curve of positive genus and $\pi\colon S \to C$ be a ruled surface such that $\mathfrak{S}(S) < 0$. Then there exists an infinite family $\{S_i, \phi_i\}_{i \geq 1}$, where $\pi_i\colon S_i \to C$ are ruled surfaces and $\phi_i\colon S \dashrightarrow S_i$ are $\mathrm{Aut}(S)$-equivariant birational maps, such that:*

$$\mathrm{Aut}(S) \subsetneq \phi_1^{-1}\mathrm{Aut}(S_1)\phi_1 \subsetneq \cdots \subsetneq \phi_n^{-1}\mathrm{Aut}(S_n)\phi_n \subsetneq \cdots$$

is an infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$. In particular, $\mathrm{Aut}(S)$ is not a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$.

Proof. Since $\mathfrak{S}(S) < 0$, there exists a unique negative section (Lemma II.2.11 (1)), which is $\mathrm{Aut}(S)$-invariant. From [Mar71, Lemmas 6 and 7], the morphism of algebraic groups $\pi_*\colon \mathrm{Aut}(S) \to \mathrm{Aut}(C)$ has a finite image. Let p be a point on the minimal section, its orbit by the $\mathrm{Aut}(S)$-action is a finite subset of the minimal section. The blowing up of the orbit of p followed by the contractions of the strict transforms of the fibres, defines an $\mathrm{Aut}(S)$-equivariant birational map $\eta_1\colon S \dashrightarrow S_1$ with $\mathfrak{S}(S_1) < \mathfrak{S}(S)$. Repeating this process gives rise to a family of ruled surfaces $\{\pi_i\colon S_i \to C\}_{i\geq 1}$ with an infinite sequence

$$\mathrm{Aut}(S) \subset \phi_1^{-1}\mathrm{Aut}(S_1)\phi_1 \subset \cdots \subset \phi_n^{-1}\mathrm{Aut}(S_n)\phi_n \subset \cdots \qquad (\dagger)$$

where $\phi_i = \eta_i \cdots \eta_1$, and we will see that this sequence is not stationary.

Take n large and let $z = \pi(p)$. By a choice of trivialization $\pi_n^{-1}(U) \simeq U \times \mathbb{P}^1$, we can assume that $q = (z, [0:1]) \in U \times \mathbb{P}^1$ is a base point of $\eta_n^{-1}\colon S_n \dashrightarrow S_{n-1}$. Let V be a vector bundle of rank two over C such that $\mathbb{P}(V) = S_n$, and let $L \subset V$ be the line subbundle associated to the minimal section in S_n. Let $\mathcal{L} = \det(V)^{-1} \otimes L^2$, it follows from Corollary I.2.16 that $\deg(\mathcal{L}) = -\mathfrak{S}(S_n)$. Since n is chosen large, we can assume that $\mathfrak{S}(S_n) < 0$ is small enough, such that $\mathrm{h}^1(C, \mathcal{L}) = \mathrm{h}^1(C, \mathcal{L} \otimes \mathcal{O}_C(z)^{-1}) = 0$. By Riemann-Roch, we get that $h^0(C, \mathcal{L} \otimes \mathcal{O}_C(z)^{-1}) < h^0(C, \mathcal{L})$, i.e. z is not a base point of the complete linear system $|\mathcal{L}|$. Therefore, there exists $\gamma \in \mathrm{H}^0(C, \mathcal{L})$ such that $\gamma(z) \neq 0$.

Let $(U_i)_i$ be a trivializing open subsets of π_n, the automorphisms:

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to U_i \times \mathbb{P}^1 \\ (x, [y_0 : y_1]) &\mapsto (x, [y_0 + y_1 \gamma_{|U_i}(x) : y_1]) \end{aligned}$$

glue into a C-automorphism f_γ of S_n (see [Mar71, case (b) p.92]) such that f_γ does not fix q and $\mathrm{Aut}(S_{n-1}) \subsetneq \eta_n^{-1}\mathrm{Aut}(S_n)\eta_n$. We have proved that the sequence (†) is not stationary. Removing in the sequence the groups which are not strictly bigger than the previous term and renaming the elements accordingly, yields the increasing sequence of the statement. □

Remark II.3.2. Notice that the proof of Lemma II.3.1 gives back Theorem A. Let $\gamma \in \Gamma(C, \det(V)^{-1} \otimes L^2)$ as above. For any $t \in \mathbb{G}_a$, the automorphisms

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to U_i \times \mathbb{P}^1 \\ (x, [y_0 : y_1]) &\mapsto (x, [y_0 + y_1 t \gamma_{|U_i}(x) : y_1]) \end{aligned}$$

glue into an C-automorphism $f_{t\gamma}$. In particular, each automorphism f_γ belongs to the connected component of the identity. Restricting the infinite chain (†) to the connected components, one gets that

$$\mathrm{Aut}^\circ(S) \subsetneq \phi_1^{-1}\mathrm{Aut}^\circ(S_1)\phi_1 \subsetneq \cdots \subsetneq \phi_n^{-1}\mathrm{Aut}^\circ(S_n)\phi_n \subsetneq \cdots,$$

and in particular, $\dim(\mathrm{Aut}^\circ(S_n)) < \dim(\mathrm{Aut}^\circ(S_{n+1}))$ for all n.

II.3.2 Exceptional conic bundles

The following lemma is a generalization of [Bla09b, Lemma 4.3.1], for exceptional conic bundles which are not necessarily rational.

Lemma II.3.3. *Let C be a curve and let $\kappa\colon X \to C$ be a conic bundle with $2n \geq 0$ singular fibres. The following assertions are equivalent:*

(1) *π is exceptional,*

(2) *there exist exactly two sections s_1, s_2 with self-intersection $-n$, which are disjoint and intersect different irreducible components of each singular fibre,*

(3) *there exists a birational morphism $\eta_n\colon X \to S$ where S is a decomposable $\mathbb{P}^1$-bundle over C with $\mathfrak{S}(S) = -n$, which consists in the blowup of $2n$ points on a section of self-intersection n in S,*

(4) *there exists a birational morphism $\eta_0\colon X \to S$ where S is a decomposable $\mathbb{P}^1$-bundle over C with $\mathfrak{S}(S) = 0$, which consists in the blowup of $2n$ points, such that no two points are in the same fibre, n are chosen on a section of self-intersection 0 and the other n are chosen on a another section of self-intersection 0.*

Proof. (1) $\Longrightarrow$ (2), (3) Assume κ is exceptional and let s_1, s_2 be sections of self-intersection $-n$. Contracting in each singular fibre the irreducible component which does not meet s_1 yields a birational morphism $\eta_n\colon X \to S$ where S is a ruled surface over C. Denote by s_1' and s_2' the images of s_1 and s_2 by η_n, then $s_1'^2 = -n$ and $s_2'^2 \leq n$. The case $s_2'^2 < n$ cannot happen (Lemma II.2.11 (1)), and the equality implies s_1 and s_2 pass through different irreducible components of each singular fibre. Then the sections s_1 and s_2 are disjoint (Lemma II.2.11 (2)), and S is decomposable (Lemma II.2.11 (3)). Assume there exists a third section s_3 of self-intersection $(-n)$ on X'. By the same argument, s_3 has to pass through different irreducible components than s_1 and s_2. Since each singular fibre contains exactly two irreducible components, this is not possible.

(2) $\Longrightarrow$ (4) Contract in n singular fibres the irreducible components meeting s_1, and contract in the other singular fibres the irreducible components meeting s_2. This defines a birational morphism $\eta_0\colon X \to S$ such that the images of s_1 and s_2 by η_0 are disjoint sections of S with self-intersection zero. In particular, S is decomposable and by Lemma II.2.11 (1), $\mathfrak{S}(S) = 0$.

(2), (3), (4) $\Longrightarrow$ (1) The implication (2) $\Longrightarrow$ (1) is trivial. The strict transforms by η_n of the sections of S with self-intersection n and $(-n)$ are two sections of κ of self-intersection $(-n)$. The strict transforms by η_0 of the two sections of S with self-intersection zero are two sections of κ of self-intersection $(-n)$. This proves that (3) and (4) imply (1). $\square$

In [Bla09b, Lemma 4.3.3 (1)], it is proven that $\mathrm{Aut}_{\mathbb{P}^1}(X) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ when X is an exceptional conic bundle over $\mathbb{P}^1$, which implies that $\mathrm{Aut}(X)$ is maximal. We see below that automorphism groups of exceptional conic bundles over a non rational curve do not always contain an involution permuting the two $(-n)$-sections (Proposition II.3.5), and are not always maximal (Lemma II.3.4).

Lemma II.3.4. *Let C be a curve of positive genus and let $\kappa\colon X \to C$ be an exceptional conic bundle. If $\mathrm{Aut}_C(X)$ contains a non-trivial involution permuting the irreducible components of the singular fibres, then $\mathrm{Aut}_C(X) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(X)$ is maximal. Else, $\mathrm{Aut}_C(X) \simeq \mathbb{G}_m$ and $\mathrm{Aut}(X)$ can be embedded in a infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$.*

Proof. Denote by s_1, s_2 the two $(-n)$-sections of κ. Let G be the subgroup of $\mathrm{Aut}_C(X)$ which leaves invariant the irreducible components of each singular fibre and let $\eta_0\colon X \to S$ be a birational morphism which contracts an irreducible component in each singular fibre and such that $\mathfrak{S}(S) = 0$ (Lemma II.3.3 (4)). Let $\pi\colon S \to C$ be the morphism such that $\kappa = \pi\eta_0$. Let s_1', s_2' be respectively the images of s_1 and s_2 by η_0. Then η_0 is G-equivariant and $\eta_0 G \eta_0^{-1}$ is an algebraic subgroup of $\mathrm{Aut}_C(S)$ which leaves invariant s_1' and s_2'. If π is trivial, then $\eta_0 G \eta_0^{-1}$ is an algebraic subgroup of $\mathrm{PGL}(2, \mathbf{k})$ fixing at least two points on a fibre, and thus is contained in $\mathbb{G}_m \subset \mathrm{PGL}(2, \mathbf{k})$. If π is not trivial, then $\eta_0 G \eta_0^{-1}$ is an algebraic subgroup of $\mathbb{G}_m$ or $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ by Lemma II.2.13. Since the element of order two in $\mathbb{Z}/2\mathbb{Z}$ permutes s_1' and s_2', it follows that $\eta_0 G \eta_0^{-1}$ is also contained in $\mathbb{G}_m$ in the second case. Hence $\eta_0 G \eta_0^{-1}$ is contained in a subgroup of $\mathrm{Aut}_C(S)$ isomorphic to $\mathbb{G}_m$. Conversely, every element of $\mathbb{G}_m$ fixes s_1', s_2' (the images of s_1, s_2 by η_0), hence $\eta_0 G \eta_0^{-1} = \mathbb{G}_m$.

The exceptional conic bundle κ has exactly two $(-n)$-sections, which are left invariant or are permuted by the elements of $\mathrm{Aut}(X)$. Assume that $\mathrm{Aut}_C(X)$ contains an element ι which permutes the two $(-n)$ sections of κ (or equivalently, the irreducible components of each singular fibre by Lemma II.3.3 (2)). The automorphism ι acts on a general fibre by permuting two points, which implies that ι is an involution. If $f \in \mathrm{Aut}_C(X)$ permutes the two $(-n)$-sections then ιf does not, i.e. $\iota f \in G$. This implies that $\mathrm{Aut}_C(X) = G \rtimes \langle \iota \rangle$. Moreover, there is no ι-equivariant contraction from X and all $\mathrm{Aut}(X)$-orbits in the complement of the singular fibres are infinite, or contain two points on a smooth fibre. Hence there is no $\mathrm{Aut}(X)$-equivariant birational map from X (Lemma II.2.7) and $\mathrm{Aut}(X)$ is maximal.

Else $\mathrm{Aut}_C(X) = G \simeq \mathbb{G}_m$. Since there exist exactly two $(-n)$-sections, an element of $\mathrm{Aut}(X)$ fixes them or permutes them. The contraction of $\mathrm{Aut}(X)$-orbits of (-1)-curves yields an $\mathrm{Aut}(X)$-equivariant birational morphism $\eta_n\colon X \to S$ where $\mathfrak{S}(S) < 0$ (see Lemma II.3.3 (3)). Then use Lemma II.3.1 to conclude. $\square$

Proposition II.3.5. *Let C be a curve. Let $\kappa\colon X \to C$ be a conic bundle with two disjoint sections s_1 and s_2 passing through different irreducible components of each singular fibre. Let $\eta\colon X \to S$ be the contraction of an irreducible component in each singular fibre. This yields a ruled surface $\pi\colon S \to C$ such that $\kappa = \pi\eta$. Denote by s_1' and s_2' the images of s_1 and s_2 by η. The following hold:*

1. *π is decomposable, i.e. there exists $D \in \mathrm{Pic}(C)$ such that $S = \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$.*
2. *The birational morphism η is the blow-up of finite sets $Z \subset s_1'$ and $P \subset s_2'$.*

(3) *The group* $\mathrm{Aut}_C(X)$ *contains a non-trivial involution permuting the irreducible components of each singular fibre if and only if the divisor* $-2D$ *is linearly equivalent to*
$$\sum_{z\in\pi(Z)} z - \sum_{p\in\pi(P)} p.$$

(4) *If one of the condition of* (3) *holds, then* κ *is an exceptional conic bundle.*

Proof. (1) Each fibre of π is isomorphic to $\mathbb{P}^1$. It follows that π is a ruled surface, which is decomposable because s_1' and s_2' are disjoint.

(2) Each irreducible component of a singular fibre intersects s_1 or s_2. It follows that η is the blow-up of finitely many points lying in s_1' or s_2'.

(3) Up to a choice on the trivialization of S, we can also assume that s_1' is the zero section and s_2' is the infinity section. Replacing D by another divisor of its linear class, we can assume that $\mathrm{Supp}(D)\cap\pi(Z\cup P)=\emptyset$. Let $U\subset C$ be a trivializing open subset of π containing $\pi(Z\cup P)$ and such that $\mathrm{Supp}(D)\subset C\setminus U$.

Assume first that there exists $f\in\mathbf{k}(C)$ such that
$$\mathrm{div}(f)=\sum_{z\in\pi(Z)} z - \sum_{p\in\pi(P)} p + 2D.$$

Then define the birational map $\phi_u\colon U\times\mathbb{P}^1 \dashrightarrow U\times\mathbb{P}^1$, $(x,[y_0:y_1])\mapsto(x,[f(x)y_1:y_0])$, which is involutive and has base points at $Z\cup P$. Take another trivializing open subset V with a trivialization map such that transition function of π equals $g_{uv}\colon V\times\mathbb{P}^1\dashrightarrow U\times\mathbb{P}^1$, $(x,[y_0:y_1])\mapsto(x,[\alpha_{uv}(x)y_0:y_1])$, where $\alpha_{uv}\in\mathbf{k}(C)^*$ denotes the transition function of $\mathcal{O}_C(D)$. Denote by ν_q the multiplicity at $q\in C$, then $\nu_q(\alpha_{uv}^{-2}f)=\nu_q(f)-2\nu_q(\alpha_{uv})=0$ for all $q\in V\setminus U$. This implies that $\phi_v=g_{uv}^{-1}\phi_u g_{uv}\colon(x,[y_0:y_1])\mapsto(x,[\alpha_{uv}^{-2}(x)f(x)y_1:y_0])$ extends to a birational map defined at $(V\setminus U)\times\mathbb{P}^1$. Hence ϕ_u extends to a C-birational map ϕ of S, and $\eta^{-1}\phi\eta\in\mathrm{Aut}_C(X)$ is an involution permuting the irreducible components of the singular fibres of κ.

Conversely assume there exists an involution $\psi\in\mathrm{Aut}_C(X)$ permuting the irreducible components of the singular fibres. Then $\eta\psi\eta^{-1}\in\mathrm{Bir}(S)$ acts trivially on C and permutes s_1 and s_2, hence there exists $f\in\mathbf{k}(C)$ such that the restriction of $\eta\psi\eta^{-1}$ on $\pi^{-1}(U)$ yields a birational map
$$\begin{aligned}\phi_u\colon U\times\mathbb{P}^1 &\dashrightarrow U\times\mathbb{P}^1\\ (x,[y_0:y_1]) &\longmapsto (x,[f(x)y_1:y_0]).\end{aligned}$$

Since the set of base points of $\eta\psi\eta^{-1}$ is exactly $Z\cup P$, the rational function $f_{|U}$ has zero in $\pi(Z)$ and poles in $\pi(P)$. Computing in local charts, one can check that $\nu_q(f)=1$ if $q\in\pi(Z)$ and $\nu_q(f)=-1$ if $q\in\pi(P)$. Conjugate as before by the transition maps of π gives $\phi_v=g_{uv}^{-1}\phi_u g_{uv}\colon V\times\mathbb{P}^1\dashrightarrow V\times\mathbb{P}^1$, $(x,[y_0:y_1])\mapsto(x,[\alpha_{uv}^{-2}(x)f(x)y_1:y_0])$. The birational map $\eta\psi\eta^{-1}$ is biregular on $\pi^{-1}(C\setminus U)$, hence $\nu_q(\alpha_{uv}^{-2}f)=0$ for all $q\in V\setminus U$. Thus $\mathrm{div}(f)=\sum_{z\in\pi(Z)}z-\sum_{p\in\pi(P)}p+2D$.

(4) Let m be the number of singular fibres of κ and denote by $\iota \in \mathrm{Aut}_C(X)$ a non-trivial involution permuting the irreducible components of each singular fibre. Without loss of generality, we can replace s_2 by $\iota(s_1)$ and it follows that $s_1^2 = s_2^2$. Contracting the irreducible components intersecting s_1 in each singular fibre of κ gives a birational morphism $\eta\colon X \to S$ where S is decomposable with two disjoint sections s_1' and s_2'. Then $\mathfrak{S}(S) \leq 0$ (see [Mar70, Corollary 1.17], or Proposition I.2.18 (1)), and $\mathfrak{S}(S) \neq 0$ by definition of η and by Lemma II.2.10 as $S \to C$ is decomposable by (1). This implies that $s_1'^2 = -\mathfrak{S}(S)$ and $s_2'^2 = \mathfrak{S}(S)$ (Lemma II.2.11 (2)). On the other hand, we have $s_2^2 = s_2'^2$ and $s_1^2 = s_1'^2 - m$. Thus $m = s_1'^2 - s_1^2 = -2\mathfrak{S}(S) > 0$. In particular, η corresponds to a birational map η_n as in Lemma II.3.3 (3) and κ is an exceptional conic bundle over C.

□

Combining the previous lemma and proposition, we get the main result of this section:

Proposition II.3.6. *Let C be a curve of positive genus. Let $\kappa\colon X \to C$ be an exceptional conic bundle with two $(-n)$-sections s_1 and s_2. The contraction of an irreducible component in each singular fibre gives a birational morphism $\eta\colon X \to S$ where $\pi\colon S = \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C) \to C$ is a decomposable ruled surface, for some $D \in \mathrm{Pic}(C)$. In particular, $\kappa = \pi\eta$. Denote by s_1', s_2' the images of s_1, s_2 by η, and by $Z \subset s_1'$, $P \subset s_2'$ the set of base points of η^{-1}. The algebraic group $\mathrm{Aut}(X)$ is maximal if and only if $-2D$ is linearly equivalent to*

$$\sum_{z \in \pi(Z)} z - \sum_{p \in \pi(P)} p,$$

and in this case, $\mathrm{Aut}(X)$ fits into an exact sequence of algebraic groups:

$$0 \longrightarrow \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Aut}(X) \xrightarrow{\kappa_*} H,$$

where H denotes the subgroup of $\mathrm{Aut}(C)$ which fixes the finite subset $\pi(Z \cup P)$. Else, $\mathrm{Aut}(X)$ is not maximal and can be embedded in an infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$.

Proof. The structure morphism κ induces a morphism of algebraic groups $\kappa_*\colon \mathrm{Aut}(X) \to \mathrm{Aut}(C)$, and an element in the image of κ_* must preserves $\pi(Z \cup P)$ which is the set of points in C having singular fibres. The rest of the statement follows from Lemma II.3.4 and Proposition II.3.5. □

Corollary II.3.7. *Let C be a curve of positive genus. Then there exist exceptional conic bundles $X \to C$ such that $\mathrm{Aut}(X)$ is not a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$.*

Proof. Let X be an exceptional conic bundle over C, which is not the blowup of a decomposable ruled surface $\pi\colon \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C) \to C$ along

$F = \{p_1, p_2, \cdots, p_{2\deg(D)}\}$ lying in two disjoint sections s_1 and s_2 such that $-2D$ is not linearly equivalent to

$$\sum_{p \in s_1 \cap F} \pi(p) - \sum_{p \in s_2 \cap F} \pi(p).$$

By Proposition II.3.6, $\mathrm{Aut}(X)$ is not a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$. $\square$

In Proposition II.3.6, the morphism $\kappa_* \colon \mathrm{Aut}(X) \to H$ can be surjective: it is always the case if $C = \mathbb{P}^1$ (see [Bla09b, Lemma 4.3.3 (1)]), or if C is a curve of genus $g \geq 2$ with a trivial automorphism group. We give an example where this surjectivity fails.

Example II.3.8. *Let C be an elliptic curve over $\mathbb{C}$ with neutral element p_0. Choose a 4-torsion point $p_1 \in C$ such that $p_2 = 2p_1 \neq p_0$, and denote by $\Delta = \{p_0, p_1, p_2, p_3\}$ the subgroup generated by p_1. Define the ruled surface $\pi \colon S = \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C) \to C$ where $D = p_0 + p_1$. The line subbundle $\mathcal{O}_C(D) \subset \mathcal{O}_C(D) \oplus \mathcal{O}_C$ corresponds to a section with self-intersection $-\deg(D) = -2$ (see Proposition I.2.15). By Lemma II.2.11 (1), it follows that σ is the unique section of π with negative self-intersection, and therefore $\mathfrak{S}(S) = -2$. Let s_1', s_2' be two disjoint sections with $s_1'^2 = -2$ and $s_2'^2 = 2$. Denote by $\eta \colon X \to S$ the blowup of $s_2' \cap \pi^{-1}(\Delta)$, and by s_1, s_2 the strict transforms of s_1' and s_2' by η. Then $\kappa = \pi\eta$ is a conic bundle. Moreover,*

$$(p_0+p_1+p_2+p_3)-(2D) = -p_0-p_1+p_2+p_3 = -(p_1-p_0)+(p_2-p_0)+(p_3-p_0) \sim 0$$

implies that $(-2D) \sim -(p_0+p_1+p_2+p_3)$ and it follows that $\mathrm{Aut}(X)$ is maximal with $\mathrm{Aut}_C(X) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ (Proposition II.3.6).

Let $f \in \mathrm{Aut}(C)$ be the translation $x \mapsto x + p_1$ which preserves Δ, i.e. $f \in H$. Denote by $\widetilde{H}$ the subgroup of $\mathrm{Aut}(X)$ which fixes s_1 and s_2. Notice that η is $\widetilde{H}$-equivariant and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Aut}(X) \arrow[ddrr, "\kappa_*"'] & \supset & \widetilde{H} \arrow[dd, "\kappa_*"] \arrow[dr, "\eta_*"] & \\
 & & & \mathrm{Aut}(S) \arrow[dl, "\pi_*"] \\
\mathrm{Aut}(C) & \supset & H. &
\end{tikzcd}$$

Assume that $f \in \mathrm{Aut}(C)$ lifts to an element of $\mathrm{Aut}(X)$, then it can also be lifted in $\widetilde{H}$ (if the lifting permutes s_1 and s_2, compose it with the non-trivial involution to get an element in $\widetilde{H}$), and a fortiori can be lifted in $\mathrm{Aut}(S)$. This is not the case because $f^*(D) = p_0 + p_3$ is not linearly equivalent to D .

II.3.3 $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles

The key result in this section is Proposition II.3.11. We will also need [Bla09b, Lemmas 4.4.1, 4.4.3, 4.4.4], and their proofs are left as exercises in the original article. For the sake of self-containess, we reprove them below (see Lemmas II.3.9, II.3.10, II.3.13).

Lemma II.3.9. *Let C be a curve. Every element of order two in $\mathrm{PGL}(2,\mathbf{k}(C))$ is conjugate to an element of the form $\sigma_f = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$ where $f \in \mathbf{k}(C)^*$. Moreover, σ_f and σ_g are conjugate if and only if f/g is a square in $\mathbf{k}(C)^*$.*

Proof. Let $\sigma \in \mathrm{PGL}(2,\mathbf{k}(C))$ be an element of order two and let $v \in \mathbb{P}^1$ such that $\sigma(v) \neq v$. Since σ is of order two, there exists $f \in \mathbf{k}(C)^*$ such that the matrix of σ with respect to the basis $\{v, \sigma(v)\}$ is $\sigma_f = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$. Let $f, g \in \mathbf{k}(C)^*$. Assume σ_f and σ_g are conjugate. Take $\widetilde{\sigma_f}, \widetilde{\sigma_g}$ be their respective representatives in $\mathrm{GL}(2,\mathbf{k}(C))$ having 1 as the lower left coefficient. Then there exists $\lambda \in \mathbf{k}(C)$, $P \in \mathrm{GL}(2,\mathbf{k}(C))$ such that $P\widetilde{\sigma_f}P^{-1} = \lambda\widetilde{\sigma_g}$. Taking the determinant in the last equality gives $f/g = \lambda^2$. Conversely, assume that $f/g = \lambda^2$ for some $\lambda \in \mathbf{k}(C)^*$. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \cdot \sigma_g \cdot \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} = \sigma_f.$$

□

Lemma II.3.10. *Assume that $\mathrm{char}(\mathbf{k}) \neq 2$. Let C be a curve. Let $\sigma = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix} \in \mathrm{PGL}(2,\mathbf{k}(C))$, where $f \in \mathbf{k}(C)^*$ is not a square. Let N_σ be the normalizer of $\langle\sigma\rangle$ in $\mathrm{PGL}(2,\mathbf{k}(C))$. Then*

$$N_\sigma = \left\{ \begin{bmatrix} a & bf \\ b & a \end{bmatrix}, a, b \in \mathbf{k}(C) \right\} \cup \left\{ \begin{bmatrix} a & -bf \\ b & -a \end{bmatrix}, a, b \in \mathbf{k}(C) \right\} \simeq N_\sigma^\circ \rtimes \mathbb{Z}/2\mathbb{Z},$$

where $N_\sigma^\circ = \left\{ \begin{bmatrix} a & bf \\ b & a \end{bmatrix}, a, b \in \mathbf{k}(C) \right\}$ is isomorphic to $\mathbf{k}(C)[\sqrt{f}]^*/\mathbf{k}(C)^*$ via the group homomorphism $\begin{bmatrix} a & bf \\ b & a \end{bmatrix} \mapsto [a + b\sqrt{f}]$, and $\mathbb{Z}/2\mathbb{Z}$ is generated by the diagonal involution. The action of $\mathbb{Z}/2\mathbb{Z}$ on N_σ° sends $\begin{bmatrix} a & bf \\ b & a \end{bmatrix}$ on $\begin{bmatrix} a & -bf \\ -b & a \end{bmatrix}$.

Proof. Since σ has order two, the normalizer of $\langle\sigma\rangle$ equals the centralizer of $\langle\sigma\rangle$. Then it is a straightforward computation in $\mathrm{PGL}(2,\mathbf{k}(C))$ to check that matrices commuting with σ are of the form $\begin{bmatrix} a & bf \\ b & a \end{bmatrix}$ or $\begin{bmatrix} a & -bf \\ b & -a \end{bmatrix}$ for some $a, b \in \mathbf{k}(C)$, and N_σ° is a normal subgroup of N_σ. Since $N_\sigma^\circ \cap \left\{ I_2, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} =$

$\{I_2\}$ and $N_\sigma^\circ \cdot \left\{I_2, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\} = N_\sigma$, it follows that $N_\sigma \simeq N_\sigma^\circ \rtimes \mathbb{Z}/2\mathbb{Z}$. For all $a_1, a_2, b_1, b_2 \in \mathbf{k}(C)$,

$$\begin{bmatrix}a_1 & b_1 f\\ b_1 & a_1\end{bmatrix} \cdot \begin{bmatrix}a_2 & b_2 f\\ b_2 & a_2\end{bmatrix} = \begin{bmatrix}a_1a_2 + b_1b_2f & (a_1b_2 + a_2b_1)f\\ a_2b_1 + a_1b_2 & a_1a_2 + b_1b_2f\end{bmatrix}$$

and $(a_1 + b_1\sqrt{f})(a_2 + b_2\sqrt{f}) = (a_1a_2 + b_1b_2f) + (a_1b_2 + a_2b_1)\sqrt{f}$. Hence N_σ° is isomorphic to $\mathbf{k}(C)[\sqrt{f}]^*/\mathbf{k}(C)^*$ via $\begin{bmatrix}a & bf\\ b & a\end{bmatrix} \mapsto [a + b\sqrt{f}]$. Finally,

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \cdot \begin{bmatrix}a & bf\\ b & a\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}a & -bf\\ -b & a\end{bmatrix},$$

i.e. the action of $\mathbb{Z}/2\mathbb{Z}$ on N_σ° sends $\begin{bmatrix}a & bf\\ b & a\end{bmatrix}$ on $\begin{bmatrix}a & -bf\\ -b & a\end{bmatrix}$. $\square$

The following key proposition is an analogue of [Bla09b, Proposition 5.2.2] for non rational ruled surfaces. We prove it by copying, mutatis mutandis, the proof of [Bla09b, Proposition 5.2.2].

Proposition II.3.11. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve of positive genus and let G be a finite subgroup of* $\mathrm{Bir}(C \times \mathbb{P}^1) = \mathrm{PGL}(2, \mathbf{k}(C)) \rtimes \mathrm{Aut}(C)$. *Denote by $G' \subset G$ and $H \subset \mathrm{Aut}(C)$ the kernel and the image of the action of G on the base of the fibration $C \times \mathbb{P}^1 \to C$. Then the following hold:*

1. *if $G' = \{1\}$ then G is conjugate to H in* $\mathrm{Bir}(C \times \mathbb{P}^1)$.
2. *if $G' \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by an involution with a non trivial determinant, then G normalizes a group $V \subset \mathrm{PGL}(2, \mathbf{k}(C))$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and containing G'.*

Proof. By Tsen theorem, $\mathbf{k}(C)$ is a C_1-field. Then $H^1(H, \mathrm{GL}(2, \mathbf{k}(C)))$ and $H^2(H, \mathbf{k}(C)^*)$ are trivial ([Ser68, Chap. X, Propositions 3, 10 and 11]) and this implies that $H^1(H, \mathrm{PGL}(2, \mathbf{k}(C)))$ is also trivial.

(1) If $G' = \{1\}$, then G is isomorphic to H and there exists a section $s\colon H \to G$. Let s_c be the homomorphism $H \to \mathrm{PGL}(2, \mathbf{k}) \rtimes H$, $h \mapsto (1, h)$ and f be the homomorphism $H \to \mathrm{Bir}(C \times \mathbb{P}^1)$, $h \mapsto s(h)s_c(h)^{-1}$. Denote by $\pi\colon \mathrm{Bir}(C \times \mathbb{P}^1) \to \mathrm{Aut}(C)$ the projection on $\mathrm{Aut}(C)$. For all $h \in H$, $\pi f(h) = \pi(s(h))\pi(s_c(h)^{-1}) = 1$, i.e. there exists a homomorphism $\tilde{f}\colon H \to \mathrm{PGL}(2, \mathbf{k}(C))$, such that $f(h) = (\tilde{f}(h), 1)$. In particular, $s(h) = (\tilde{f}(h), h)$. For all $h_1, h_2 \in H$, $(\tilde{f}(h_1h_2), 1) = f(h_1h_2) = s(h_1)s(h_2)s_c(h_2)^{-1}s_c(h_1)^{-1} = (\tilde{f}(h_1)h_1 \cdot \tilde{f}(h_2), 1)$, i.e. $\tilde{f}$ is a cocycle. Since $H^1(H, \mathrm{PGL}(2, \mathbf{k}(C)))$ is trivial, f is conjugate to the trivial cocycle and this implies that s and s_c are conjugate up to an element of $\mathrm{PGL}(2, \mathbf{k}(C))$. Thus G and H are conjugate.

(2) Let σ be the element of order two of G'. From Lemma II.3.9, we can assume up to conjugation that $\sigma = \begin{bmatrix}0 & f\\ 1 & 0\end{bmatrix}$, for some $f \in \mathbf{k}(C)^*$ which is not a

square (by assumption the determinant of σ is not trivial). We denote by N_σ the normalizer of σ. By Lemma II.3.10, $N_\sigma \simeq N_\sigma^\circ \rtimes \mathbb{Z}/2\mathbb{Z}$ with N_σ° isomorphic to $\mathbf{k}(C)[\sqrt{f}]^*/\mathbf{k}(C)^*$ and $\mathbb{Z}/2\mathbb{Z}$ acts on N_σ° by sending $\begin{bmatrix} a & bf \\ b & a \end{bmatrix}$ on $\begin{bmatrix} a & -bf \\ -b & a \end{bmatrix}$.

All elements of the form $\begin{bmatrix} a & -bf \\ b & -a \end{bmatrix}$ have order two in $\mathrm{PGL}(2, \mathbf{k}(C))$: the goal is to find one of them which generates with σ a subgroup V isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and normalized by G.

Let $h \in H$. Then $\pi^{-1}(h) \cap G = \{(\gamma, h), (\sigma\gamma, h)\}$ for some $\gamma \in \mathrm{PGL}(2, \mathbf{k}(C))$. The element σ has order two and G' is normal in G, this implies that $(\sigma, 1)$ is in the center of G. In particular, $(\gamma, h)(\sigma, 1)(h^{-1} \cdot \gamma^{-1}, h^{-1}) = (\sigma, 1)$, and it follows that $\gamma(h \cdot \sigma)\gamma^{-1} = \sigma$. By Lemma II.3.9, there exists $\mu \in \mathbf{k}(C)^*$ such that $\mu^2 = f/(h \cdot f)$. Let $\beta = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix}$ and $\alpha = \gamma\beta^{-1}$. Then $\beta(h \cdot \sigma)\beta^{-1} = \sigma$ and $\alpha\sigma\alpha^{-1} = \sigma$. Therefore, $\alpha \in N_\sigma$ and replacing μ by $-\mu$ if needed, we can assume that $\alpha \in N_\sigma^\circ$. Under this last further condition, μ and α^2 are uniquely determined by h, since $(\sigma\alpha)^2 = \alpha^2$. By associating h to $\rho_h = \alpha^2$ and $\mu_h = \mu$, this yields the following well-defined maps:

$$\begin{aligned} \rho \colon H &\to N_\sigma^\circ \\ h &\mapsto \rho_h \end{aligned} \qquad\qquad \begin{aligned} \mu \colon H &\to \mathbf{k}(C)^* \\ h &\mapsto \mu_h. \end{aligned}$$

We show that μ is a cocycle, and ρ is also a cocycle after conjugating by some element of $\mathrm{PGL}(2, \mathbf{k}(C))$. Let $h_1, h_2 \in H$ and $h_3 = h_1h_2$. For $i \in \{1, 2, 3\}$, choose as previously $(\alpha_i\beta_i, h_i) \in G$ where $\alpha_i \in N_\sigma^\circ$, $\beta_i = \begin{bmatrix} \mu_i & 0 \\ 0 & 1 \end{bmatrix}$, $\mu_i^2 = f/(h_i \cdot f)$. We can also choose $\alpha_3\beta_3$ such that $(\alpha_1\beta_1, h_1)(\alpha_2\beta_2, h_2) = (\alpha_3\beta_3, h_3)$, which implies that

$$\alpha_3 = \alpha_1\beta_1(h_1 \cdot (\alpha_2\beta_2))\beta_3^{-1} = \alpha_1(\beta_1(h_1 \cdot \alpha_2)\beta_1^{-1})\beta_1(h_1 \cdot \beta_2)\beta_3^{-1}. \qquad (\dagger)$$

Writing explicitly $\alpha_2 = \begin{bmatrix} a & bf \\ b & a \end{bmatrix}$, it follows that

$$\beta_1(h_1 \cdot \alpha_2)\beta_1^{-1} = \begin{bmatrix} \mu_1(h_1 \cdot a) & (h_1 \cdot b)f \\ h \cdot b & \mu_1(h_1 \cdot a) \end{bmatrix} \in N_\sigma^\circ$$

. Then $\beta_1(h_1 \cdot \beta_2)\beta_3^{-1} \in N_\sigma^\circ$, which is a diagonal matrix, and thus equals identity. This implies that $\mu_3 = \mu_1(h_1 \cdot \mu_2)$, i.e. μ is a cocycle. The group $H^1(H, \mathbf{k}(C)^*)$ is trivial ([Ser68, Chap. X, Propositions 10 and 11]), there exists $\nu \in \mathbf{k}(C)^*$ such that $\mu_h = \nu/(h \cdot \nu)$ for all $h \in H$. Then f/ν^2 is H-invariant. Conjugating G by $\begin{bmatrix} 1 & 0 \\ 0 & \nu \end{bmatrix}$, we can assume that f is H-invariant, which is equivalent to $\mu_h = 1$ for all $h \in H$. From the equation $(\dagger)$, it follows that $\alpha_3 = \alpha_1(h_1 \cdot \alpha_2)$, which implies that ρ is a cocycle.

The H-equivariant exact sequence $1 \to \mathbf{k}(C)^* \to \mathbf{k}(C)[\sqrt{f}]^* \to N_\sigma^\circ \to 1$ with the equalities $H^1(H, \mathbf{k}(C)[\sqrt{f}]^*) = \{1\}$ and $H^2(H, \mathbf{k}(C)^*) = \{1\}$ (see [Ser68, Chap. X, Propositions 10 and 11]) imply that $H^1(H, N_\sigma^\circ) = \{1\}$. Therefore, ρ is conjugate to the trivial cocycle, i.e. there exists $\tau \in N_\sigma^\circ$ such that $\rho_h = \tau \cdot (h \cdot \tau)^{-1}$ for all $h \in H$. The element $T = (\tau, -1) \in N_\sigma^\circ \rtimes \mathbb{Z}/2\mathbb{Z}$ has order two and is different from σ (because $\sigma \in N_\sigma^\circ$). Besides, every element of G is of the form $((\alpha, 1), h)$ with $\alpha \in N_\sigma^\circ$ and $h \in H$ such that $\alpha^2 = \rho_h$, and $((\alpha, 1), h)(T, 1)(h^{-1} \cdot (\alpha^{-1}, 1), h^{-1}) = ((\alpha, 1)(h \cdot \tau, -1)(\alpha^{-1}, 1), 1) = ((\alpha^2(h \cdot \tau), -1), 1) = (T, 1)$ in G. The subgroup generated by σ and $(T, 1)$ is normalized by G and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

□

Under the assumptions of Proposition II.3.11, the automorphism group of a conic bundle X such that $\mathrm{Aut}_C(X) \simeq \mathbb{Z}/2\mathbb{Z}$ is not maximal. Below, we see that a conic bundle X such that $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is always a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle and it has a maximal automorphism group (Lemmas II.3.15 and II.3.16).

Lemma II.3.12. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve and let $\kappa \colon X \to C$ be a conic bundle having at least one singular fibre. Suppose there exists a non trivial involution $f \in \mathrm{Aut}_C(X)$ fixing pointwise two sections s_1 and s_2. Then in each singular fibre, the sections s_1 and s_2 pass through different irreducible components.*

Proof. Assume there is a singular fibre $\kappa^{-1}(p)$, where s_1 and s_2 pass through the same irreducible component. Since f fixes pointwise s_1 and s_2, the contraction of the other irreducible component gives a f-equivariant birational morphism $\eta \colon \kappa^{-1}(U) \to U \times \mathbb{P}^1$, where U is an open neighborhood of p. The C-automorphism $\eta f \eta^{-1}$ has order two and this implies that there exist $a, b, c \in \mathcal{O}_C(U)$ such that

$$\eta f \eta^{-1} : (x, [u : v]) \mapsto (x, [au + bv : cu - av]). \qquad (\star)$$

On the other hand, $\eta f \eta^{-1}$ fixes the point contracted by η, and the sections $\eta(s_1)$ and $\eta(s_2)$. In particular, it fixes three distinct points in the fibre $p_1^{-1}(p)$ where $p_1 \colon U \times \mathbb{P}^1 \to U$ denotes the first projection. Therefore $\eta f \eta^{-1}_{|p_1^{-1}(p)}$ equals identity. It implies from $(\star)$ that $b(p) = c(p) = 0$ and $a(p) = -a(p) \neq 0$, which is a contradiction, since $\mathrm{char}(\mathbf{k}) \neq 2$. □

Lemma II.3.13. *Let C be a curve. Let $\sigma \in \mathrm{PGL}(2, \mathbf{k}(C))$ be a non trivial involution with* $\det(\sigma) \in \mathbf{k}(C)^*/(\mathbf{k}(C)^*)^2$.

(1) *If* $\det(\sigma) = 1$, *then σ is diagonalisable and fixes pointwise two sections.*

(2) *If* $\det(\sigma) \neq 1$, *then σ is not diagonalisable and fixes pointwise an irreducible curve, which is birational to a 2-to-1 cover of C ramified above an even positive number of points.*

Proof. (1) If $\det(\sigma)=1$ then σ is conjugate to the matrix $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ by Lemma II.3.9, and

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

In particular, σ is diagonalizable and fixes two sections.

(2) If $\det(\sigma)\neq 1$ then σ is conjugate to $\sigma_f=\begin{bmatrix}0 & f\\ 1 & 0\end{bmatrix}$, where $f\in\mathbf{k}(C)^*$ is not a square by Lemma II.3.9. Assume σ_f is conjugate to a diagonal matrix $\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$. By taking the determinant, there exists $\lambda\in\mathbf{k}(C)^*$ such that $\lambda^2\alpha\beta=f$. Since σ has order two, $\alpha^2=\beta^2$ and this implies that $f^2=\lambda^4\alpha^4$. This contradicts the assumption that f is not a square.

The equation $[f(x)v:u]=[u:v]$ is equivalent to $u^2-v^2f(x)=0$ which defines an irreducible curve Q in $C\times\mathbb{P}^1$. The first projection of $C\times\mathbb{P}^1$ restricted to Q is a 2-to-1 cover of C, and Q is birational to a curve having an even positive number of ramification points by Hurwitz formula [Har77, IV. Corollary 2.4]. □

Corollary II.3.14. *Let C be a curve. Let $\pi\colon S\to C$ be a ruled surface such that $\mathfrak{S}(S)<0$, let $p_1\colon C\times\mathbb{P}^1\to C$ be the first projection and let $f\colon S\dashrightarrow C\times\mathbb{P}^1$ be a birational map such that $p_1f=\pi$. Then $f\operatorname{Aut}_C(S)f^{-1}\subset\operatorname{PSL}(2,\mathbf{k}(C))$.*

Proof. Let $\sigma\in\operatorname{Aut}_C(S)$. Then $f\sigma f^{-1}\in\operatorname{PGL}(2,\mathbf{k}(C))$. Since $\mathfrak{S}(S)<0$, there exists a section $s\colon C\to S$ of negative self-intersection which is fixed by σ. Assume that $\det(\sigma)\neq 1$, then σ also fixes pointwise an irreducible curve which is a 2-to-1 cover of C by Lemma II.3.13 (2). Then σ fixes three distinct points in a general fibre, hence σ equals identity, which contradicts $\det(\sigma)\neq 1$. □

Lemma II.3.15. *Let C be a curve of positive genus. Let $\kappa\colon X\to C$ be a ruled surface, or a conic bundle with at least one singular fibre such that its two irreducible components are exchanged by an element of $\operatorname{Aut}(X)$. If $\operatorname{Aut}_C(X)\simeq(\mathbb{Z}/2\mathbb{Z})^r$ with $r\in\{1,2\}$, then $\det(\sigma)\neq 1$ for all $\sigma\in\operatorname{Aut}_C(X)\setminus\{1\}$. In particular, if $r=2$ then κ is a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle.*

Proof. Assume first that κ has at least one singular fibre and let $\eta\colon X\to S$ be the contraction of an irreducible component in each singular fibre. Let G be the normal subgroup of $\operatorname{Aut}_C(X)$ which leaves invariant each irreducible component of the singular fibres.

Suppose that $r=1$. Let $\sigma_1\in\operatorname{Aut}_C(X)$ be the element of order two and assume that $\det(\sigma_1)=1$. By Lemma II.3.13 (1), the automorphism σ_1 fixes pointwise two sections, which do not pass through the same irreducible components in each singular fibre (Lemma II.3.12), and η is σ_1-equivariant. This implies that $\eta G\eta^{-1}$ has an invariant section and by Lemma II.2.16 (1), κ is an exceptional conic bundle, which contradicts the assumption that $\operatorname{Aut}_C(X)$ is finite (Lemma II.3.4).

Suppose that $r = 2$. Let $\sigma_1, \sigma_2, \sigma_3$ be the elements of order two in $\mathrm{Aut}_C(X)$. If $\det(\sigma_i) \neq 1$ for all $i \in \{1, 2, 3\}$, then κ is a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle by Lemma II.3.13 (2). Without loss of generality, we can assume by contradiction that $\det(\sigma_1) = 1$. By Lemma II.3.13 (1), the automorphism σ_1 fixes two sections s_1 and s_2, which do not pass through the same irreducible components in each singular fibre (Lemma II.3.12), and η is σ_1-equivariant. In particular, $\sigma_1 \in G$ and G is not trivial. Let $\sigma_j \in G$ and denote by $\mathrm{Fix}(\sigma_1)$ the set of fixed points of σ_1. Then $\sigma_j(\mathrm{Fix}(\sigma_1)) = \mathrm{Fix}(\sigma_1)$ for $j = \{2, 3\}$. It follows that σ_j permutes s_1 and s_2, or leaves them invariant. Since we assume that $\sigma_j \in G$, it follows that σ_j must leave them invariant. This implies that G also leaves s_1 and s_2 invariant. Therefore κ is an exceptional conic bundle by Lemma II.2.16 (1), which contradicts the assumption that $\mathrm{Aut}_C(X)$ is finite (Lemma II.3.4).

Assume from now on that κ has no singular fibre and $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^r$ for $r \in \{1, 2\}$. Then X is a ruled surface with $\mathfrak{S}(X) > 0$ [Mar71, Theorem 2]. Let σ_1 be an element of order two with $\det(\sigma_1) = 1$, then σ_1 fixes two sections s_1 and s_2 (Lemma II.3.13 (1)) which intersect because X is indecomposable by Proposition I.2.18 (1). Applying elementary transformations centered on the intersections yields a σ_1-equivariant birational map $\phi\colon X \dashrightarrow S$ where $\pi\colon S \to C$ is a decomposable ruled surface. The automorphism $\phi\sigma_1\phi^{-1}$ fixes the strict transforms of s_1 and s_2 which are disjoint sections of π, and the base points of ϕ^{-1} which are in the complement of $s_1 \cup s_2$. Choose trivializations $(U_i)_i$ of π such that the strict transforms of s_1 and s_2 are the zero and infinity sections of π. Then $(\phi\sigma_1\phi^{-1})_{|U_i}\colon (x, [u : v]) \mapsto (x, [\alpha_i(x)u : v])$ for some $\alpha_i \in \mathcal{O}_C(U_i)^*$ and the transition maps are of the form: $s_{ij}\colon U_j \times \mathbb{P}^1 \dashrightarrow U_i \times \mathbb{P}^1$, $(x, [u : v]) \mapsto (x, [t_{ij}(x)u : v])$ for some $t_{ij} \in \mathcal{O}_C(U_{ij})^*$. The condition $(\phi\sigma_1\phi^{-1})_{|U_i} s_{ij} = s_{ij}(\phi\sigma_1\phi^{-1})_{|U_j}$ implies that there exists $\alpha \in \mathbb{G}_m$ such that $\alpha_i = \alpha_j = \alpha$ for all i, j. Since $\phi\sigma_1\phi^{-1}$ fixes the base points of ϕ^{-1} which do not lie in $s_1 \cup s_2$, this implies that $\alpha = 1$, i.e. σ_1 equals identity, which is a contradiction. $\square$

Lemma II.3.16. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve of positive genus and let $\kappa\colon X \to C$ be a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle. Then* $\mathrm{Aut}(X)$ *is a maximal algebraic subgroup of* $\mathrm{Bir}(C \times \mathbb{P}^1)$. *Moreover,* $\mathrm{Aut}(X)$ *fits into an exact sequence*

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathrm{Aut}(X) \xrightarrow{\pi_*} \mathrm{Aut}(C),$$

where the image of π_* equals $\mathrm{Aut}(C)$ *if $X \simeq \mathcal{A}_1$ (or equivalently, C is an elliptic curve and κ is the only $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface over C), or equals a finite subgroup of* $\mathrm{Aut}(C)$ *preserving the set of singular fibres of κ (possibly empty, if κ is a ruled surface over a curve C of genus ≥ 2).*

Proof. Let $p \in C$ such that $\kappa^{-1}(p)$ is a smooth fibre. The group $\mathrm{Aut}_C(X) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ acts on $\kappa^{-1}(p)$. Any $\sigma \in \mathrm{Aut}_C(X)$ of order two is the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ for some $a, b, c \in \mathcal{O}_C(U)$ where U is an open neighborhood of p. If σ equals identity on $\kappa^{-1}(p)$, then $b(p) = c(p) = 0$ and $a(p) = -a(p) \neq 0$. Hence σ does not act trivially on $\kappa^{-1}(p)$, i.e. the action of $(\mathbb{Z}/2\mathbb{Z})^2$ over $\kappa^{-1}(p)$ is faithful. Besides, any $(\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{PGL}(2, \mathbf{k})$ acts without fix points and by Lemma II.2.7, the subgroup $\mathrm{Aut}(X)$ is maximal.

It remains to prove the exact sequence. By definition, the kernel of π_* is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Assume first that κ is a ruled surface. If $g = 1$, the assumption that $\mathrm{Aut}_C(S)$ is finite implies that $\mathfrak{S}(X) > 0$ (see [Mar71, Theorem 2]) and κ is an indecomposable ruled surface (see Proposition I.2.18 (1)). Then X is C-isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$ (see [Har77, Theorem V.2.15]), which respectively satisfy $\mathfrak{S}(\mathcal{A}_0) = 0$ and $\mathfrak{S}(\mathcal{A}_1) = 1$ (see Proposition I.2.21). Therefore, X is isomorphic to $\mathcal{A}_1$ and the exact sequence follows from [Mar71, Theorem 3. (4)]. If $g \geq 2$, the statement holds because $\mathrm{Aut}(C)$ is a finite group ([Har77, Exercise IV.2.5]). Assume that κ has a singular fibre. Then any element of $\mathrm{Aut}(X)$ has to preserve to set of singular fibres, which is finite. It follows that the morphism $\kappa_* \colon \mathrm{Aut}(X) \to \mathrm{Aut}(C)$ has a finite image. □

II.3.4 Ruled surfaces

Proposition II.3.17. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve of genus $g \geq 1$ and $\pi \colon S \to C$ be a ruled surface. The following hold:*

(1) *If π is trivial, then $\mathrm{Aut}(S)$ is maximal.*

(2) *If $\mathfrak{S}(S) = 0$, π is not trivial and $S \simeq \mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$ is decomposable, then $\mathrm{Aut}(S)$ is maximal if and only if $g = 1$, or $g \geq 2$ and $2D$ is principal. If $g \geq 2$ and $2D$ is not principal, then $\mathrm{Aut}(S)$ can be embedded in an infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$.*

(3) *If $\mathfrak{S}(S) = 0$ and S is indecomposable, then $\mathrm{Aut}(S)$ is maximal if and only if $g = 1$. If $g \geq 2$, then $\mathrm{Aut}(S)$ can be embedded in an infinite increasing sequence of algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$.*

(4) *If $\mathfrak{S}(S) > 0$ then $\mathrm{Aut}(S)$ is maximal if and only if S is a $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface.*

Proof. (1) If S is trivial, each $\mathrm{Aut}(S)$-orbit contains at least a fibre. Hence $\mathrm{Aut}(S)$ is maximal by Lemma II.2.7.

(2) If $g = 1$, then the morphism $\mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is surjective Proposition I.3.10. There is no $\mathrm{Aut}(S)$-orbit of finite dimension, hence $\mathrm{Aut}(S)$ is maximal.

Assume that $g \geq 2$. If $2D$ is principal, then $\mathrm{Aut}_C(S) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ (Lemma II.2.13 (2)), and the group $\mathrm{Aut}_C(S)$ acts on a fibre with two orbits: one isomorphic to $\mathbb{G}_m$ and the other one is made of two points exchanged by the involution. From Lemma II.2.7, $\mathrm{Aut}(S)$ is a maximal algebraic subgroup. If $2D$ is not principal, $\mathrm{Aut}(S) \simeq \mathbb{G}_m$ (Lemma II.2.13 (3)). Take a point in a minimal section, its $\mathrm{Aut}(S)$-orbit is finite and contains at most one point in each fibre. The blowup of this orbit followed by the contraction of the strict transforms of the fibres, gives an $\mathrm{Aut}(S)$-equivariant birational map $S \dashrightarrow S'$ where S' is a ruled surface with $\mathfrak{S}(S') < 0$. Then apply Lemma II.3.1.

(3) From Proposition I.2.18 (3.ii), S has a unique minimal section which is $\mathrm{Aut}(S)$-invariant. If $g \geq 2$, take a point on this minimal section and blowup its orbit (which consists of finitely many points on the minimal section), then contract the strict transforms of the fibres. It defines an $\mathrm{Aut}(S)$-equivariant map $S \dashrightarrow S'$ with $\mathfrak{S}(S') < 0$. Then apply Lemma II.3.1. If $g = 1$ then $S \simeq \mathcal{A}_0$ and $\pi_* \colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is surjective Proposition I.3.8. In particular, there is no $\mathrm{Aut}(S)$-orbit of dimension 0 and no $\mathrm{Aut}(S)$-equivariant map. Thus $\mathrm{Aut}(S)$ is maximal.

(4) Assume that $\mathfrak{S}(S) > 0$. In particular, π is indecomposable (see e.g. Proposition I.2.18 (1)). If $g = 1$, then $S \simeq \mathcal{A}_1$ is a $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface and $\mathrm{Aut}(\mathcal{A}_1)$ is maximal by Lemma II.3.16. From now on, we assume that $g \geq 2$. By [Mar71, Lemma 3, Theorem 2], $\mathrm{Aut}_C(S)$ is isomorphic to a subgroup of $\mathrm{Pic}^0(C)[2]$. In particular, it is a finite subgroup of $\mathrm{PGL}(2, \mathbf{k})$ such that every element is an involution. Hence $\mathrm{Aut}_C(S) \simeq (\mathbb{Z}/2\mathbb{Z})^s$ for some $s \in \{0, 1, 2\}$. Moreover, $\mathrm{Aut}(C)$ is finite, and this implies that $\mathrm{Aut}(S)$ is also finite.

By Lemma II.3.15, each non-trivial element of $\mathrm{Aut}_C(S)$ has a non-trivial determinant. If $s = 0$, then $\mathrm{Aut}(S)$ is conjugate to a finite subgroup of $\mathrm{Aut}(C) \subsetneq \mathrm{Aut}(C \times \mathbb{P}^1)$ by Lemma II.3.11 (1). If $s = 1$, then by Lemma II.3.11 (2), $\mathrm{Aut}(S)$ normalizes a group $V \simeq (\mathbb{Z}/2\mathbb{Z})^2$ containing $\mathrm{Aut}_C(S)$, i.e. there exists a finite subgroup $G \subset \mathrm{Bir}(C \times \mathbb{P}^1)$ containing $\mathrm{Aut}(S)$ such that V is the kernel of the action of G on C. In particular, $\mathrm{Aut}(S) \subsetneq G$. Therefore, we get that $\mathrm{Aut}(S)$ is not maximal if $s \in \{0, 1\}$. If $s = 2$, then S is a $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface. Conversely, the automorphism group of a $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface is maximal by Lemma II.3.16.

□

II.3.5 Examples of $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles

If C is an elliptic curve, the Atiyah bundle $\mathcal{A}_1$ is the only $(\mathbb{Z}/2\mathbb{Z})^2$-ruled surface. We give below examples of $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles over any curve C of genus ≥ 2. If X is a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle over $\mathbb{P}^1$ with at least one singular fibre, then every element of order two in $\mathrm{Aut}_{\mathbb{P}^1}(X)$ acts non trivially on $\mathrm{Pic}(X)$ by permuting the irreducible components of a singular fibre (by [Bla09b, Lemma 4.3.5]. If there exists an element of $\mathrm{Aut}_{\mathbb{P}^1}(X) \setminus \{1\}$ acting trivially on $\mathrm{Pic}(X)$, then $X \to \mathbb{P}^1$ is an exceptional conic bundle, and thus not a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle by [Bla09b, Lemmas 4.3.3. (1) and 4.3.5]). The following example also shows that this does not hold anymore when C has positive genus.

Example II.3.18. *Assume that* $\mathrm{char}(\mathbf{k}) \neq 2$. *Let C be a curve of genus $g \geq 1$ and D be a non principal divisor such that $2D$ is principal. Let S be the decomposable ruled surface $\mathbb{P}(\mathcal{O}_C(D) \oplus \mathcal{O}_C)$. From Lemma II.2.13,* $\mathrm{Aut}_C(S) = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ *and the element σ of order two that generates $\mathbb{Z}/2\mathbb{Z}$ is conjugate to* $\begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}$ *for some $f \in \mathbf{k}(C)^*$ such that* $\mathrm{div}(f) = 2D$. *Since D is not principal, f is not a square. In particular,* $\det(\sigma) \neq 1$ *and σ fixes pointwise an irreducible* *curve birational to a 2-to-1 cover of C and ramified above an even positive number of points (Lemma II.3.13 (2)).*

The matrix $\tau = \begin{bmatrix} a & -bf \\ b & -a \end{bmatrix}$ *of order two has determinant* $-a^2+b^2f = -N(a+b\sqrt{f})$*, where* $N\colon \mathbf{k}(C)[\sqrt{f}] \to \mathbf{k}(C)$ *is the norm, which is surjective (see* [Ser68, X.7, Propositions 10 and 11]*). Choose a and b such that* $\det(\tau)$ *has a pole with odd multiplicity at a point where f is regular. Then* $\det(\tau) \neq 1$ *and* $\det(\sigma\tau) \neq 1$.

Since $\sigma\tau = \tau\sigma$, the subgroup of $\mathrm{PGL}(2, \mathbf{k}(C))$ *generated by σ and τ is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$*. Apply Propositions II.2.6 and II.2.17, there exists a* $\mathbb{Z}/2\mathbb{Z}$*-equivariant birational map from S to a conic bundle X such that* $(\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{Aut}_C(X)$*; and X is a ruled surface, or an exceptional conic bundle, or a* $(\mathbb{Z}/2\mathbb{Z})^2$*-conic bundle. Assume that X is not a* $(\mathbb{Z}/2\mathbb{Z})^2$*-conic bundle. Notice that X cannot be neither a ruled surface with* $\mathfrak{S}(X) < 0$ *by Corollary II.3.14 or* $\mathfrak{S}(X) > 0$ *by* [Mar71, Lemma 3]*. It implies X is either a ruled surface with* $\mathfrak{S}(X) = 0$*, or an exceptional conic bundle, such that* $\mathrm{Aut}_C(X) \simeq \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ *(Lemmas II.2.13 and II.3.4), with* $\det((-1,0)) = 1$*. Since* $\det(\sigma)$, $\det(\tau)$ *and* $\det(\sigma\tau)$ *are all non trivial, it is a contradiction. Therefore, X is a* $(\mathbb{Z}/2\mathbb{Z})^2$*-conic bundle.*

II.4 Proofs of the results

Proof of Theorem E. Each algebraic group in the list is a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ by Lemma II.3.16 and Propositions II.3.6, II.3.17. Conversely, let G be a maximal algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$, where C is a curve of genus $g \geq 1$. Using the regularization theorem (Proposition II.2.5) and the G-equivariant MMP (Proposition II.2.6), it follows that G is conjugate to $\mathrm{Aut}(X)$ for some conic bundle $\kappa\colon X \to C$. If κ has no singular fibre, apply directly Proposition II.3.17. Else κ has at least a singular fibre. If there is no element of $\mathrm{Aut}(X)$ permuting two irreducible components of a singular fibre, then there exists an $\mathrm{Aut}(X)$-equivariant contraction $X \to S$, where S is a ruled surface, and apply Proposition II.3.17 to conclude. Else, apply Proposition II.2.17 with Proposition II.3.11, and it follows that X is an exceptional conic bundle, or $\mathrm{Aut}(X)$ is conjugate to a subgroup of $\mathrm{Aut}(C \times \mathbb{P}^1)$ or $\mathrm{Aut}(X')$ where X' is a $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundle. To conclude, apply Proposition II.3.6 for the case of exceptional conic bundles, and apply Lemma II.3.16 for the case of $(\mathbb{Z}/2\mathbb{Z})^2$-conic bundles. Finally, the exact sequences of (4) in case $g = 1$ and (5) are taken from [Mar71, Theorem 3]. □

Proof of Corollary F. From [Bla09b, Theorem 1], every algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is included in a maximal one. From Theorem E, every m aximal algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ has dimension at most four. By Remark II.3.2, there exist algebraic subgroups of arbitrary large dimension, and they cannot be subgroups of the maximal ones. □

Connected algebraic subgroups of groups of birational transformations not contained in a maximal one (j.w. Sokratis Zikas)

III.1 Introduction

Let $\mathbf{k}$ be an algebraically closed field. The classification of algebraic subgroups of groups of birational transformations was initiated in [Enr93], where Enriques shows that each connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(S)$, with S isomorphic to $\mathbb{P}^2$ or to the n-th Hirzebruch surface $\mathbb{F}_n$ for $n \neq 1$; and these are all maximal, with respect to the inclusion, among the connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$. The connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ have been classified over $\mathbf{k} = \mathbb{C}$ by Umemura in a series of four papers [Ume80, Ume82a, Ume82b, Ume85] and it follows again from his classification that each connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3)$ is contained in a maximal one (see also [BFT21a, BFT21b] for a modern approach). However, it is an open problem whether every connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ is contained in a maximal one when $n \geq 4$.

On the other hand, it is proven in Theorem C that there exist connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ not contained in a maximal one when C is a smooth curve of positive genus. The proof of this result is based on the existence of infinite increasing sequences of connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ (see Theorem A), and on the fact that the dimension of a maximal connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ is bounded by 4 (see Theorem B and [Mar71, Theorem 3]). Our main result in this note is a higher dimensional analogue of Theorem C:

Theorem G. *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let $n \geq 1$ and C be a smooth curve of positive genus. Then there exists a connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^n)$ which is not contained in a maximal one.*

The idea of the proof is to consider the connected algebraic subgroup $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$, where S is a ruled surface such that $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$, and to show that it cannot be contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. Since $\mathrm{Aut}^\circ(S \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S) \times \mathrm{PGL}_{n+1}(\mathbf{k})$ by [BSU13, Corollary 4.2.7], the existence of infinite increasing sequences of connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ is an immediate consequence of Theorem A. From this alone, it is nonetheless insufficient to deduce that one of the connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ appearing in the infinite increasing sequences is not contained in a maximal one (see Remark III.2.8), and classifying all connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ seems out of reach at the moment.

This article is organized as follows. Section 2 contains two results, namely Lemmas III.2.6 and III.2.7, which are important for the proof of the higher dimensional case. As a consequence of these two lemmas, we also get a new and short proof of the dimension two case (see Proposition III.2.9), without using the classification of the maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ (Theorem B). In Section 3, we prove the higher dimensional case under the extra assumption that $\mathrm{char}(\mathbf{k}) = 0$, in view of using the machinery of the MMP and the G-Sarkisov program. The latter has been developped by Floris in [Flo20], building upon results of Hacon and McKernan in [HM13]. More precisely, if G is a connected algebraic group, then every G-equivariant birational map between Mori fibre spaces decomposes into G-Sarkisov links (see [Flo20, Theorem 1.2]). We study the possible links in Lemmas III.3.4 and III.3.5. Combining Proposition III.2.9 and Theorem III.3.6, we get Theorem G.

It is very natural to also ask whether for all $n \geq 2$, there exists a variety X of dimension n such that $\mathrm{Bir}(X)$ contains algebraic subgroups which are not lying in a maximal one, without the connectedness assumption. If $n = 2$, the answer is also affirmative (see Lemma II.3.1, Corollary F), and the proof is analogous to that of the connected case. Since the G-Sarkisov program is known only for connected algebraic groups, it is not clear if the proof presented in this article could be adapted for the non-connected case in higher dimension.

Acknowledgments. *We are thankful to Jérémy Blanc, Enrica Floris, Ronan Terpereau, Susanna Zimmermann for interesting discussions and remarks. Special thanks to Enrica Floris for pointing out a mistake in the proof of a preliminary version. We are also grateful to the anonymous referees for their careful reading and useful comments.*

III.2 Some preliminaries and the case of dimension two

From now on, C will always denote a smooth curve of genus g over a field $\mathbf{k}$. In this section, $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic. The following invariant was used by Maruyama in [Mar70, Mar71] for his classification of ruled surfaces and their automorphisms.

Definition III.2.1. Let V be a rank-2 vector bundle over C and $\tau \colon S = \mathbb{P}(V) \to C$ be a ruled surface. We say that τ is *decomposable* if V is the direct sum of two line bundles over C. Otherwise, we say that τ is *indecomposable.*

We define the *Segre invariant* of S as

$$\mathfrak{S}(S) = \min\{\sigma^2, \sigma \text{ section of } \tau\}.$$

Remark III.2.2. Let $\tau\colon S \to C$ be a ruled surface.

(1) Let $p \in S$ and σ be a section of τ. Recall that the blow-up of S at p followed by the contraction of the strict transform of the fibre passing through p, yields a ruled surface $\tau'\colon S' \to C$ and a birational map $\epsilon\colon S \dashrightarrow S'$ called the *elementary transformation of S centered at p* (see e.g. [Har77, V. Example 5.7.1]). Let σ' be the strict transform of σ by ϵ. If $p \in \sigma$, then $\sigma'^2 = \sigma^2 - 1$. Else, $\sigma'^2 = \sigma^2 + 1$.

(2) Applying an elementary transformation on a point lying on the minimal section of τ yields a ruled surface S' such that $\mathfrak{S}(S') = \mathfrak{S}(S) - 1$. Repeating this process gives ruled surfaces with arbitrary small Segre invariants.

(3) As S is obtained by finitely many elementary transformations from $C \times \mathbb{P}^1$ (see e.g. [Har77, V. Exercise 5.5]) and $\mathfrak{S}(C \times \mathbb{P}^1) = 0$ (see e.g. Lemma I.2.14), it follows that $\mathfrak{S}(S) > -\infty$. If moreover $\mathfrak{S}(S) < 0$, then there exists a unique section with negative self-intersection number (see e.g. Lemma II.2.11).

(4) The Segre invariant $\mathfrak{S}(S)$ equals $-e$, where e is the invariant defined in [Har77, V. Proposition 2.8]. If τ is indecomposable, then by [Har77, V. Theorem 2.12. (b)], we get $\mathfrak{S}(S) \geq 2 - 2g = -\deg(K_C)$. In particular, if $\mathfrak{S}(S) < -\deg(K_C)$, then τ is decomposable.

We recall the statement of Blanchard lemma and its corollary (see [BSU13, Proposition 4.2.1, Corollary 4.2.6]):

Proposition III.2.3. *Let $f\colon X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and let G be a connected group scheme acting on X. Then there exists a unique action of G on Y such that f is G-equivariant.*

Corollary III.2.4. *Let $f\colon X \to Y$ be a proper morphism of projective schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then f induces a homomorphism of group schemes $f_*\colon \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$.*

Remark III.2.5. Let $\tau\colon S \to C$ be a decomposable ruled surface. Assume that C has genus $g = 1$ and $\mathfrak{S}(S) \neq 0$, or that $g \geq 2$. Then by [Mar71, Lemma 7], the morphism induced by Blanchard lemma $\tau_*\colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial.

In the next two lemmas, we compute $\mathrm{Aut}^\circ(S)$ and its orbits for a ruled surface $\tau\colon S \to C$ with $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Such ruled surfaces exist and are decomposable by Remark III.2.2 (2) and (4)).

Lemma III.2.6. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S = \mathbb{P}(V) \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Let σ be the minimal section of τ and $L(\sigma)$ be the line subbundle of V associated to σ. We choose trivializations of τ such that σ is the infinity section. Then the following hold:*

(1) *The group* $\mathrm{Aut}^\circ(S)$ *is isomorphic to* $\mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$*, where* $\det(V)$ *denotes the determinant line bundle of* V*. This isomorphism associates* $\alpha \in \mathbb{G}_m$ *and* $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$*, to the element* $\mu_{\alpha,\gamma} \in \mathrm{Aut}^\circ(S)$ *obtained by gluing the automorphisms:*

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to U_i \times \mathbb{P}^1 \\ (x, [u:v]) &\mapsto \left(x, [\alpha u + \gamma_{|U_i}(x) v : v]\right). \end{aligned}$$

(2) *The* $\mathrm{Aut}^\circ(S)$*-orbits in* S *are* $\{p\}$ *and* $\tau^{-1}(\tau(p)) \setminus \{p\}$ *for* $p \in \sigma$*.*

Proof. (1) The proof follows from the computation made in [Mar71, case (b) p.92]. For the sake of self-containess, we recall it below. Since τ is decomposable, we can write its transition maps as $t_{ij} \colon U_j \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, $(x, [u:v]) \mapsto (x, [a_{ij}(x)u : b_{ij}(x)v])$, where $[u:v]$ denotes the coordinates of $\mathbb{P}^1$, $a_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$ denotes the transition maps of the line bundle $L(\sigma)$ and $b_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$. Let $\mu \in \mathrm{Aut}^\circ(S)$. The morphism induced by Blanchard lemma $\tau_* \colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial (Remark III.2.5). Moreover, σ is fixed by $\mathrm{Aut}^\circ(S)$ as it is the unique minimal section. Therefore, for each trivializing open subset $U_i \subset C$, μ induces an automorphism $\mu_i \colon U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, given by $(x, [u:v]) \mapsto (x, [\alpha_i(x)u + \gamma_i(x)v : v])$, where $\alpha_i \in \mathcal{O}_C(U_i)^*$ and $\gamma_i \in \mathcal{O}_C(U_i)$. The condition $\mu_i t_{ij} = t_{ij}\mu_j$ implies that $\alpha_i = \alpha_j = \alpha \in \mathbb{G}_m$ and $\gamma_i = b_{ij}^{-1} a_{ij} \gamma_j$. Since $a_{ij} b_{ij}$ are the transition maps of the line bundle $\det(V)$, and a_{ij} denote the transition maps of $L(\sigma)$, this implies that $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. The data of $\alpha \in \mathbb{G}_m$ and $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ determine uniquely the automorphism μ, this proves that we have an embedding $\mathrm{Aut}^\circ(S) \hookrightarrow \mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. Conversely, one can check that the automorphisms defined in the statement commute with the transition maps, hence their gluing defines an automorphism of S. Because $\mathbb{G}_m \rtimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ is also connected, we get that it is isomorphic to $\mathrm{Aut}^\circ(S)$.

(2) Since the morphism induced by Blanchard lemma $\tau_* \colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial (Remark III.2.5), each $\mathrm{Aut}^\circ(S)$-orbit is contained in a fibre of τ. As σ is the unique section with negative self-intersection number, it is fixed pointwise by $\mathrm{Aut}^\circ(S)$. It remains to see that $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for each p lying on σ.

Let $L = \det(V)^\vee \otimes L(\sigma)^{\otimes 2}$. It follows from Proposition I.2.15 that $\deg(L) = -\mathfrak{S}(S) > 1 + \deg(K_C)$. Let $p \in \sigma$ and let $\tau(p) = z$. We get by Serre duality that

$$h^1(C, L) = h^0(C, K_C \otimes L^\vee) = 0,$$

where the last equality follows from the fact that $\deg(K_C \otimes L^\vee) < -1$. Similarly we get the equality $h^1(C, L \otimes \mathcal{O}_C(z)^\vee) = 0$. By Riemann-Roch, $h^0(C, L \otimes \mathcal{O}_C(z)^\vee) = \deg(L) - g < \deg(L) - g + 1 = h^0(C, L)$. Therefore, z is not a base point of the complete linear system $|L|$, i.e. there exists $\gamma \in H^0(C, L)$ such that $\gamma(z) \neq 0$, and the subgroup $\mathbb{G}_a \simeq \{\mu_{1,\lambda\gamma}; \lambda \in \mathbf{k}\}$ acts transitively on $\tau^{-1}(z) \setminus \{p\}$ (see (1) for the definition of $\mu_{1,\lambda\gamma}$).

□

Let S be a ruled surface as in Lemma III.2.6, and $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map (i.e. $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$ acts regularly on S'). In the following lemma, we compute the fixed points of the action of $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$ on S'.

Lemma III.2.7. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$. If $\tau'\colon S' \to C$ is a ruled surface and there exists an $\mathrm{Aut}^\circ(S)$-equivariant birational map $\phi\colon S \dashrightarrow S'$ which is not an isomorphism, then $\mathfrak{S}(S') < \mathfrak{S}(S)$ and $\phi\mathrm{Aut}^\circ(S)\phi^{-1} \subsetneq \mathrm{Aut}^\circ(S')$. The fixed points of the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S' are the points lying on the minimal section of τ' and the base points of ϕ^{-1}. Moreover, we can write ϕ as a product of $\mathrm{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections.*

Proof. By [DI09, Theorem 7.7], we can write $\phi = \phi_n \cdots \phi_1$ where each ϕ_i is an $\mathrm{Aut}^\circ(S)$-equivariant elementary transformation. Without loss of generality, we can assume that this decomposition is minimal (i.e. the number of elementary transformations n is minimal among all possible factorizations), and we prove the statement by induction on $n \geq 1$.

Let σ be the minimal section of τ. By Lemma III.2.6 (2), the algebraic group $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for every $p \in \sigma$. Since ϕ_1 is $\mathrm{Aut}^\circ(S)$-equivariant, it follows that $\phi_1\colon S \dashrightarrow S_1$ is an elementary transformation centered on a point $p_1 \in \sigma$. The strict transform of σ by ϕ_1 is the minimal section σ_1 of the ruled surface $\tau_1\colon S_1 \to C$, and so $\mathfrak{S}(S_1) = \mathfrak{S}(S) - 1$. Since the base point q_1 of ϕ_1^{-1} does not lie on the minimal section σ_1 of τ_1, it follows by Lemma III.2.6 (2) that q_1 is not fixed by $\mathrm{Aut}^\circ(S_1)$. Since q_1 is fixed by $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$, we have the strict inclusion $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subsetneq \mathrm{Aut}^\circ(S_1)$. In the complement of the fibres $f_{p_1} \subset S$ and $f_{q_1} \subset S_1$ containing the points p_1 and q_1 respectively, ϕ_1 is an isomorphism. Therefore, by Lemma III.2.6, the only fixed points of $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$ that lie in the complement of f_{q_1} are the points on the minimal section σ_1. It remains to check that the only fixed points on f_{q_1} are the point $q_1' \in \sigma_1$ and the base point q_1 of ϕ^{-1}. Let U be a trivializing open subset of τ with $\tau(p_1) \in U$, and let $f \in \mathcal{O}_C(U)$ such that $\mathrm{div}(f)_{|U} = \tau(p_1)$. We also choose trivializations of τ such that σ is the infinity section. Up to isomorphisms at the source and the target, $\phi_{1|U}$ equals $(x,[u:v]) \mapsto (x,[f(x)u:v])$. By Lemma III.2.6 (1), there is an action of $\mathbb{G}_m$ on S given locally by $(x,[u:v]) \mapsto (x,[\alpha u : v])$. This implies that there is an action of $\phi_1\mathbb{G}_m\phi_1^{-1}$ on S_1, given locally by $(x,[u:v]) \mapsto (x,[\alpha f(x)u : f(x)v]) = (x,[\alpha u : v])$. Therefore, $\phi_1\mathbb{G}_m\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$ acts transitively on $f_{q_1} \setminus \{q_1, q_1'\}$. Since $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$ acts fibrewise (Remark III.2.5) and is connected, we get that q_1 and q_1' are the fixed points of the action of $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$ on f_{q_1}.

Assume the statement holds for the birational map $\psi = \phi_i \cdots \phi_1\colon S \dashrightarrow S_i$, for some $i \geq 1$, and where $\tau_i\colon S_i \to C$ is a ruled surface with a minimal section σ_i. We now prove that the statement is then true for $\phi_{i+1}\psi$. By induction, the fixed points of $\psi\mathrm{Aut}^\circ(S)\psi^{-1}$ on S_i are the points lying on the minimal section σ_i and the base points of ψ^{-1}.

Assume that ϕ_{i+1} is centered on a base point of ψ^{-1}, which is (the image of) the base point of the inverse of a previous elementary transformation ϕ_j. A local calculation yields that we may cancel both ϕ_j and ϕ_{i+1}, which contradicts the minimality of the factorization of ϕ. So ϕ_{i+1} is centered on a point lying on the minimal section σ_i. Hence $\mathfrak{S}(S_{i+1}) = \mathfrak{S}(S_i) - 1 < \mathfrak{S}(S)$ by induction, and $\phi_{i+1}(\psi \mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subset \mathrm{Aut}^\circ(S_{i+1})$. The base point of ϕ_{i+1} is fixed by $\phi_{i+1}(\psi \mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1}$, but is not fixed by $\mathrm{Aut}^\circ(S_i)$ (by Lemma III.2.6). Thus, we get the strict inclusion $\phi_{i+1}(\psi \mathrm{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subsetneq \mathrm{Aut}^\circ(S_{i+1})$. □

The infinite increasing sequences of automorphism groups given in Theorem A can be obtained from Lemma III.2.7, but they do not imply that $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup. As it is explained below, we can get an infinite increasing sequence of connected algebraic subgroups, where each of them is included in a maximal one, which a fortiori cannot be the same for all of them.

Remark III.2.8. Let $n \geq d \geq 2$. Define the connected algebraic groups

$$G_d = \{\mathbb{A}^2 \to \mathbb{A}^2, (x,y) \mapsto (x, y + p(x)), p \in \mathbf{k}[x]_{\leq d}\},$$

acting regularly on $\mathbb{A}^2$, and then birationally on $\mathbb{P}^2$ via any embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Then $G_d \subsetneq G_{d+1}$ for all d. On the other hand, using an explicit description of $\mathrm{Aut}^\circ(\mathbb{F}_n)$ from [Bla09b, §4.2], we get for all $n \geq d$ that G_d is a subgroup of $\mathrm{Aut}^\circ(\mathbb{F}_n)$, which is a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2)$.

Notice that for any variety X, using Remark III.2.8, we may produce an infinite increasing sequence of connected algebraic subgroups of $\mathrm{Bir}(X \times \mathbb{P}^2)$. In particular, for $n \geq 2$ and C a curve of positive genus, the same is true for $\mathrm{Bir}(C \times \mathbb{P}^n) \simeq \mathrm{Bir}(C \times \mathbb{P}^{n-2} \times \mathbb{P}^2)$.

We reprove below partially Theorem C, without using Theorem B.

Proposition III.2.9. *Let C be a curve of genus $g \geq 1$ and let $\tau \colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Then $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$.*

Proof. Assume that $\mathrm{Aut}^\circ(S)$ is contained in a maximal connected algebraic subgroup G of $\mathrm{Bir}(S)$. Then G acts regularly on a surface Y by Weil regularization theorem (see [Wei55], or [Zai95, Kra18] for a modern proof). By [Bri17, Corollary 3], we can choose Y to be normal and projective. Using an equivariant resolution of singularities (see [Lip78, Remark B, p.155]), we can also assume Y to be smooth. Then by Blanchard lemma (see Proposition III.2.3), the successive contractions of the (-1)-curves gives rise to a ruled surface S' such that the induced birational morphism $Y \to S'$ is G-equivariant. Since G is maximal and connected, it follows that $G \simeq \mathrm{Aut}^\circ(S')$. The induced birational map $\phi \colon S \dashrightarrow S'$ is $\mathrm{Aut}^\circ(S)$-equivariant. If ϕ is an isomorphism, then $\mathfrak{S}(S) = \mathfrak{S}(S')$. Else ϕ factorises as product of $\mathrm{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections and $\mathfrak{S}(S') < \mathfrak{S}(S)$ (by Lemma III.2.7). In both cases, we have $\mathfrak{S}(S') \leq \mathfrak{S}(S)$. Let $\epsilon\colon S' \dashrightarrow S''$ be an elementary transformation centered on the minimal section of $\tau'\colon S' \to C$. Then again by Lemma III.2.7, it follows that $\epsilon \mathrm{Aut}^\circ(S')\epsilon^{-1} \subsetneq \mathrm{Aut}^\circ(S'')$, which contradicts the maximality of G as a connected algebraic subgroup of $\mathrm{Bir}(S)$. □

III.3 Higher dimensional case

In what follows, we would like to utilize the machinery of the G-Sarkisov program for a connected algebraic group G. Thus from now on, we furthermore assume that $\mathrm{char}(\mathbf{k}) = 0$. The G-Sarkisov program is a non-deterministic algorithm that decomposes every G-equivariant birational map between two G-Mori fibre spaces as a product of simpler maps called G-Sarkisov links. Its non-equivariant version was proven by Hacon and McKernan in [HM13] and, building on their result, Floris proved the G-equivariant version in [Flo20]. We follow the strategy of the proof of Proposition III.2.9, and in view of using G-Sarkisov program, we recall first the definition:

Definition III.3.1. Let G be a connected algebraic group. A *G-Mori fibre space* is a Mori fibre space with a regular action of G. Let $\pi_1\colon X_1 \to B_1$ and $\pi_2\colon X_2 \to B_2$ be two birational G-Mori fibre spaces. A *G-Sarkisov diagram* between X_1/B_1 and X_2/B_2 is a commutative diagram of the form

$$\begin{tikzcd} Y_1 \arrow[rr, dotted, "\chi"] \arrow[d, "\alpha_1"'] & & Y_2 \arrow[d, "\alpha_2"] \\ X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[dr, "s_1"'] & & B_2 \arrow[dl, "s_2"] \\ & R & \end{tikzcd}$$

which satisfies the following properties:

(1) all morphisms appearing in the diagram are either isomorphisms or outputs of some G-equivariant MMP on a $\mathbb{Q}$-factorial klt G-pair (Z, Φ) (recall that a G-pair is a pair (Z, Φ) such that G acts regularly on Z and there is an induced regular action on Φ),

(2) maximal dimensional varieties have $\mathbb{Q}$-factorial and terminal singularities,

(3) α_1 and α_2 are G-equivariant divisorial contractions or isomorphisms,

(4) s_1 and s_2 are G-equivariant extremal contractions or isomorphisms,

(5) χ is an isomorphism or a composition of G-equivariant anti-flips/flop/flips (in that order),

(6) the relative Picard rank $\rho(Z/R)$ of any variety Z in the diagram is at most 2.

We call R the *base* of the diagram.

Property (6) implies that α_1 is a divisorial contraction if and only if s_1 is an isomorphism. A similar statement holds for the right hand side of the diagram. Depending whether s_1 or s_2 is an isomorphism, we get four types of Sarkisov diagrams:

Type I *Type II* *Type III* *Type IV*

$$\begin{tikzcd} Y_1 \arrow[rr, dashed] \arrow[d] & & X_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[d, "\pi_1"'] & & B_2 \arrow[dll] \\ B_1 = R & & \end{tikzcd}
\quad
\begin{tikzcd} Y_1 \arrow[rr, dashed] \arrow[d] & & Y_2 \arrow[d] \\ X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\ B_1 & = R = & B_2 \end{tikzcd}
\quad
\begin{tikzcd} X_1 \arrow[rr, dashed] \arrow[d, "\pi_1"'] & & Y_2 \arrow[d] \\ B_1 \arrow[drr] & & X_2 \arrow[d, "\pi_2"] \\ & & R = B_2 \end{tikzcd}
\quad
\begin{tikzcd} X_1 \arrow[rr, dashed] \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[dr] & & B_2 \arrow[dl] \\ & R & \end{tikzcd} .$$

The birational map $\psi = \alpha_2 \chi \alpha_1^{-1}$ between X_1 and X_2 is called a *G-Sarkisov link*.

Remark III.3.2. Property (2) does not follow directly from the original definition of a (G-)Sarkisov diagram of [HM13] and [Flo20]. For a proof, see [BLZ21, Proposition 4.25].

In subsequent proofs we are going to make heavy use of the following elementary but useful observation:

Remark III.3.3. Let Z be one of the varieties appearing in a G-Sarkisov diagram, such that the relative Picard rank $\rho(Z/R)$ is 2. Then the G-Sarkisov diagram is uniquely determined by the datum of $Z \to R$, by a process known as the 2-*ray game* (see [BLZ21, section 2.F]).

More specifically, the 2-ray game is a deterministic process that assigns to any such $Z \to R$ a G-Sarkisov diagram. Moreover any G-Sakrisov diagram can be recovered by the 2-ray game on any of its relative Picard rank 2 morphisms. Thus, up to orientation of the diagram, there is a unique G-Sarkisov diagram that contains $Z \to R$.

Lemma III.3.4. *Let $n \geq 1$ and C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$ with minimal section σ and let $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map (possibly the identity) to a $\mathbb{P}^1$-bundle $\tau'\colon S' \to C$. Let $\pi' = \tau' \times id_{\mathbb{P}^n}\colon S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ and $\pi_1'\colon S' \times \mathbb{P}^n \to S'$ be the projection to the first factor. Then the following hold:*

(1) *The only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi'\colon S' \times \mathbb{P}^n \to$*

$C \times \mathbb{P}^n$ is the LHS Mori fibre space, are the following ones:

$$\begin{tikzcd}
T \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\alpha"'] & T \times \mathbb{P}^n \arrow[d, "\beta"] \\
S' \times \mathbb{P}^n \arrow[d, "\pi'"'] & S'' \times \mathbb{P}^n \arrow[d, "\pi''"] \\
C \times \mathbb{P}^n \arrow[r, equal] & C \times \mathbb{P}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \times \mathbb{P}^n \arrow[rr, equal] \arrow[d, "\pi'"'] & & S' \times \mathbb{P}^n \arrow[d, "\pi'_1"] \\
C \times \mathbb{P}^n \arrow[dr, "p_1"'] & & S' \arrow[dl, "\tau'"] \\
& C &
\end{tikzcd} .$$

In the first case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S'' \times \mathbb{P}^n$ is equal to $\psi \times id_{\mathbb{P}^n}$, where $\psi\colon S' \dashrightarrow S''$ is an elementary transformation of $\mathbb{P}^1$-bundles whose center p is a point fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$, and T is the blow-up of S' at p. In the second case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ is equal to $id_{S' \times \mathbb{P}^n}$.

(2) *The only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi'_1\colon S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{tikzcd}
T \times \mathbb{P}^n \arrow[r, equal] \arrow[d, "\eta \times id_{\mathbb{P}^n}"'] & T \times \mathbb{P}^n \arrow[d, "\pi''_1"] \\
S' \times \mathbb{P}^n \arrow[d, "\pi'_1"'] & T \arrow[dl, "\eta"] \\
S' &
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \times \mathbb{P}^n \arrow[rr, equal] \arrow[d, "\pi'_1"'] & & S' \times \mathbb{P}^n \arrow[d, "\pi'"] \\
S' \arrow[dr, "\tau"'] & & C \times \mathbb{P}^n \arrow[dl, "p_1"] \\
& C &
\end{tikzcd} .$$

The induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow T \times \mathbb{P}^n$ is equal to $\eta^{-1} \times id_{\mathbb{P}^n}$ in the former case and $id_{S' \times \mathbb{P}^n}$ in the latter, where $\eta\colon T \to S'$ is the blowup of S' at point p fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$.

Proof. (1) We distinguish between two cases depending on the base R of the diagram: if $R = C \times \mathbb{P}^n$ then we have a link of Type I or II and so the first step of the link is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y \to S' \times \mathbb{P}^n$. Note that by [BSU13, Corollary 4.2.7], it follows that $(\phi \times id_{\mathbb{P}^n})\mathrm{Aut}^\circ(S \times \mathbb{P}^n)(\phi \times id_{\mathbb{P}^n})^{-1} \simeq \phi \mathrm{Aut}^\circ(S)\phi^{-1} \times \mathrm{PGL}_{n+1}(\mathbf{k})$. Let $(q, x) \in S' \times \mathbb{P}^n$ be a point in the center of α. If q is not point fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$, then and by Lemma III.2.6 and the description of $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$, the closure of the orbit of (q, x) is a Cartier divisor and thus α is an isomorphism, contradicting the assumption that α is a divisorial contraction.

Thus we may assume that q is fixed by $\phi \mathrm{Aut}^\circ(S)\phi^{-1}$. In that case the orbit of (q, x) is precisely $\{q\} \times \mathbb{P}^n$. Notice that the codimension of $\{q\} \times \mathbb{P}^n$ is 2 and so by [BLZ21, Lemma 2.13]

$$\alpha = (\eta \times id_{\mathbb{P}^n})\colon T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,$$

where $\eta\colon T \to S'$ is the blowup of S' at q. By Remark III.3.3, the unique Sarkisov diagram containing $T\times\mathbb{P}^n \to C\times\mathbb{P}^n$ is the one given in the statement.

We now consider the case when $R \neq C \times \mathbb{P}^n$. Then we have a contraction $C \times \mathbb{P}^n \to R$ of relative Picard rank 1. Since $\rho(C \times \mathbb{P}^n) = 2$, the cone of curves $\overline{\mathrm{NE}}(C\times\mathbb{P}^n)$ has two extremal rays and so there are only two such contractions, namely the projections to the two factors: $C \times \mathbb{P}^n \to C$ and $C \times \mathbb{P}^n \to \mathbb{P}^n$. However, by property (1) of Definition III.3.1, $C \times \mathbb{P}^n \to \mathbb{P}^n$ would have to be an output of some MMP on a klt pair (Z, Φ), and thus by [HM07] its exceptional locus would be rationally connected, a contradiction. Thus $R = C$ and again we conclude by Remark III.3.3 for $S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$.

(2) We again proceed by a similar distinction of cases. If $R = S'$ then, as in the proof of (1), the first step is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\eta \times id_{\mathbb{P}^n}\colon T \times \mathbb{P}^n \to S' \times \mathbb{P}^n$, where $\eta\colon T \to S'$ is the blow-up of a point of S' fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, and we conclude by Remark III.3.3.

If $R \neq S'$, then $S' \to R$ is one of the two morphisms $S' \to C$ or $S' \to \check{S}'$, where the latter is the contraction of the minimal section. Again, by [HM07] we may exclude the latter case since its exceptional locus is not rationally connected. Finally, Remark III.3.3, once again, guarantees that the Sarkisov diagram is the one in the statement.

□

Lemma III.3.5. *Let $n \geq 1$ and C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$ with minimal section σ. Let $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map, with S' being a smooth projective surface which is not minimal. Denote by $\pi_1'\colon S'\times\mathbb{P}^n \to S'$ the projection to the first factor. Then the only non-trivial $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-Sarkisov diagrams, where $\pi_1'\colon S'\times\mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{tikzcd}
T\times\mathbb{P}^n \arrow[r, equal] \arrow[d, "\eta\times id_{\mathbb{P}^n}"'] & T\times\mathbb{P}^n \arrow[d, "\pi_1''"] \\
S'\times\mathbb{P}^n \arrow[d, "\pi_1'"'] & T \arrow[dl, "\eta"] \\
S' &
\end{tikzcd}
\qquad
\begin{tikzcd}
S'\times\mathbb{P}^n \arrow[r, equal] \arrow[d, "\pi_1'"'] & S'\times\mathbb{P}^n \arrow[d, "\kappa\times id_{\mathbb{P}^n}"] \\
S' \arrow[dr, "\kappa"'] & T\times\mathbb{P}^n \arrow[d, "\pi_1''"] \\
& T.
\end{tikzcd}$$

In the first case, $\eta\colon T \to S'$ is the blow-up of a point p fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$. In the second case, $\kappa\colon S' \to T$ is the contraction of a (-1)-curve l. In both cases, π_1'' denotes the projection to the first factor.

Proof. We again distinguish between two cases depending on the base R of the Sarkisov diagram: if $R = S'$ then the first step of the link is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y \to S' \times \mathbb{P}^n$. We follow the same strategy of the proof of Lemma III.3.4: first by [BSU13, Corollary 4.2.7], $(\phi \times id_{\mathbb{P}^n})\mathrm{Aut}^\circ(S\times\mathbb{P}^n)(\phi\times id_{\mathbb{P}^n})^{-1} = \phi\mathrm{Aut}^\circ(S)\phi^{-1}\times\mathrm{PGL}_{n+1}(\mathbf{k})$. This again implies that α has to be an extraction with center of the form $\{q\}\times\mathbb{P}^n$, where q is a point fixed by the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S'. Since the center is of codimension 2, again using [BLZ21, Lemma 2.13], we conclude that

$$a = \eta \times id_{\mathbb{P}^n} : T \times \mathbb{P}^n \to S' \times \mathbb{P}^n,$$

where $\eta\colon T \to S'$ is the blow-up of q. By Remark III.3.3, the diagram is the one given in the statement.

If $R \neq S'$, we have a morphism $S' \to R$ of relative Picard rank 1. Since S' is not minimal, its Picard rank is greater or equal to 3 which already implies that $R = T$ is a surface. Again, using Remark III.3.3 we may conclude that the diagram is the one proposed in the statement. Moreover, by property (2) of Definition III.3.1, $T \times \mathbb{P}^n$ has to have terminal singularities. Thus the singular locus of $T \times \mathbb{P}^n$ has codimension at least 3 (see [KM98, Corollary 5.18]). If $q \in T$ is singular, then $\{q\} \times \mathbb{P}^n$ is singular and has codimension 2 in $T \times \mathbb{P}^n$. This implies that T is smooth and consequently, $S' \to T$ is the contraction of a (-1)-curve. □

We prove below the higher dimensional analog of Proposition III.2.9.

Theorem III.3.6. *Let $n \geq 1$. Let C be a curve of genus $g \geq 1$, let S be a decomposable $\mathbb{P}^1$-bundle over C such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Then $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$.*

Proof. Assume that $\mathrm{Aut}^\circ(S\times\mathbb{P}^n)$ is contained in a maximal connected algebraic subgroup $G \subset \mathrm{Bir}(S \times \mathbb{P}^n)$. By [Bri17, Corollary 3], there exists a normal and projective variety Y, G-birationally equivalent to $S \times \mathbb{P}^n$, and on which G acts regularly. Then we use an equivariant resolution of singularities (see [Kol07, Thm. 3.36, Prop. 3.9.1]) to furthermore assume that Y is smooth. Running an MMP, which is G-equivariant by [Flo20, Lemma 2.5], we get an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\chi\colon S\times\mathbb{P}^n \dashrightarrow Y$ such that $G \simeq \mathrm{Aut}^\circ(Y)$ and $Y \to B$ is a Mori fibre space. By [Flo20, Theorem 1.2], χ decomposes as a product of $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant Sarkisov links. By Lemmas III.3.4 and III.3.5, it follows that $Y = T \times \mathbb{P}^n$ for some surface T and χ is of the form $\psi \times id_{\mathbb{P}^n}$, where $\psi\colon S \dashrightarrow T$ is an $\mathrm{Aut}^\circ(S)$-equivariant birational map. Up to possibly performing an extra link of Type IV (namely the RHS link in Lemma III.3.4 (1)), we may assume that $B = T$ and θ is given by the projection to the first factor. Contracting successively all (-1)-curves in T yields an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\phi \times id_{\mathbb{P}^n}\colon S \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ (by Blanchard lemma, see Proposition III.2.3), where ϕ is $\mathrm{Aut}^\circ(S)$-equivariant and S' is a ruled surface. Two cases arise: either ϕ is an isomorphism and $\mathfrak{S}(S) = \mathfrak{S}(S')$, or ϕ is not an isomorphism and $\mathfrak{S}(S') < \mathfrak{S}(S)$ by Lemma III.2.7. In both cases, $\mathfrak{S}(S') \leq \mathfrak{S}(S)$ and since G is maximal, G is isomorphic to $\mathrm{Aut}^\circ(S' \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S') \times \mathrm{PGL}_{n+1}(\mathbf{k})$ ([BSU13, Corollary 4.2.7]). Let $\phi'\colon S' \dashrightarrow S''$ be an elementary transformation of S' centered at a point on the minimal section. Then $\phi'\mathrm{Aut}^\circ(S')\phi'^{-1} \subsetneq \mathrm{Aut}^\circ(S'')$ by Lemma III.2.6. Thus $(\phi' \times id_{\mathbb{P}^n})\mathrm{Aut}^\circ(S' \times \mathbb{P}^n)(\phi' \times id_{\mathbb{P}^n})^{-1} \subsetneq \mathrm{Aut}^\circ(S'' \times \mathbb{P}^n)$, which contradicts the maximality of G as connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. $\square$

Proof of Theorem G. Let C be a curve of positive genus and $S \to C$ be a ruled surface. As S is birational to $C \times \mathbb{P}^1$, we get for all $n \geq 1$ that $\mathrm{Bir}(C \times \mathbb{P}^n) \simeq \mathrm{Bir}(S \times \mathbb{P}^{n-1})$. We conclude with Proposition III.2.9 for $n = 1$ and Theorem III.3.6 for $n \geq 2$. $\square$

Bibliography

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957. Text available on https://math.berkeley.edu/~nadler/atiyah.classification.pdf.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[Ber77] E. Bertini. Ricerche sulle trasformazioni univoche involutorie nel piano. *Annali di Mat. (2)*, 8:254–287, 1877.

[BF20] Jérémy Blanc and Enrica Floris. Connected algebraic groups acting on Fano fibrations over $\mathbb{P}^1$, 2020. Text available on https://arxiv.org/abs/2011.04940.

[BFT21a] Jérémy Blanc, Andrea Fanelli, and Ronan Terpereau. Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces, 2021. Text available on https://arxiv.org/abs/1707.01462.

[BFT21b] Jérémy Blanc, Andrea Fanelli, and Ronan Terpereau. Connected algebraic groups acting on three-dimensional Mori fibrations. *International Mathematics Research Notices*, Oct 2021.

[Bla06] Jérémy Blanc. *Finite abelian subgroups of the Cremona group of the plane*. Genève: Univ. de Genève, Faculté des Sciences (Dissertation), 2006.

[Bla09a] Jérémy Blanc. Linearisation of finite abelian subgroups of the Cremona group of the plane. *Groups Geom. Dyn.*, 3(2):215–266, 2009. Text available on https://ems.press/content/serial-article-files/2054.

[Bla09b] Jérémy Blanc. Sous-groupes algébriques du groupe de Cremona. *Transform. Groups*, 14(2):249–285, 2009. Text available on https://link.springer.com/content/pdf/10.1007/s00031-008-9046-5.pdf.

[Bla17] Jérémy Blanc. Algebraic structures of groups of birational transformations. In *Algebraic groups: structure and actions*, volume 94 of *Proc. Sympos. Pure Math.*, pages 17–30. Amer. Math. Soc., Providence, RI, 2017. Text available on https://algebra.dmi.unibas.ch/blanc/articles/structurebir.pdf.

[BLZ21] Jérémy Blanc, Stéphane Lamy, and Susanna Zimmermann. Quotients of higher-dimensional Cremona groups. *Acta Math.*, 226(2):211–318, 2021. Text available on https://www.intlpress.com/site/pub/files/_fulltext/journals/acta/2021/0226/0002/ACTA-2021-0226-0002-a001.pdf.

[BM90] Curtis Bennett and Rick Miranda. The automorphism groups of the hyperelliptic surfaces. *Rocky Mountain J. Math.*, 20(1):31–37, 1990.

[Bri17] M. Brion. Algebraic group actions on normal varieties. *Trans. Moscow Math. Soc.*, 78:85–107, 2017. Text available on https://arxiv.org/abs/1703.09506.

[Bri19] M. Brion. Notes on automorphism groups of projective varieties, 2019. Text available on http://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf.

[Bri22a] Michel Brion. Actions of finite group schemes on curves, 2022. Text available on https://arxiv.org/abs/2207.08209.

[Bri22b] Michel Brion. On models of algebraic group actions, 2022. Text available on https://arxiv.org/abs/2202.04352.

[BSU13] Michel Brion, Preena Samuel, and V. Uma. *Lectures on the structure of algebraic groups and geometric applications*, volume 1 of *CMI Lecture Series in Mathematics*. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013. Text available on http://www-fourier.univ-grenoble-alpes.fr/~mbrion/chennai.pdf.

[BZ17] Tatiana Bandman and Yuri G. Zarhin. Jordan groups, conic bundles and abelian varieties. *Algebr. Geom.*, 4(2):229–246, 2017.

[CMYZ22] Ivan Cheltsov, Frédéric Mangolte, Egor Yasinsky, and Susanna Zimmermann. Birational involutions of the real projective plane, 2022. Text available on https://arxiv.org/abs/2208.00217.

[CS16] Ivan Cheltsov and Constantin Shramov. *Cremona groups and the icosahedron*. Monogr. Res. Notes Math. Boca Raton, FL: CRC Press, 2016.

[CS19] Ivan Cheltsov and Constantin Shramov. Finite collineation groups and birational rigidity. *Sel. Math., New Ser.*, 25(5):68, 2019. Id/No 71.

[Dem70] Michel Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970. Text available on http://archive.numdam.org/article/ASENS_1970_4_3_4_507_0.pdf.

[DI09] Igor V. Dolgachev and Vasily A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston, Boston, MA, 2009.

[EF98] F. Enriques and G. Fano. Sui gruppi continui di trasformazioni Cremoniane dello spazio. *Annali di Mat. (2)*, 26:59–98, 1898.

[Enr93] F. Enriques. Sui gruppi continui di trasformazioni cremoniane nel piano. *Rom. Acc. L. Rend. (5)*, 2(1):468–473, 1893.

[Flo20] Enrica Floris. A note on the G-Sarkisov program. *Enseign. Math.*, 66(1-2):83–92, 2020. Text available on https://ems.press/content/serial-article-files/11876.

[Fuj20] Osamu Fujino. *Iitaka conjecture*. SpringerBriefs in Mathematics. Springer, Singapore, [2020] ©2020. An introduction.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967. Text available on http://www.numdam.org/item/PMIHES_1967__32__5_0.pdf.

[Han87] Masaki Hanamura. On the birational automorphism groups of algebraic varieties. *Compositio Math.*, 63(1):123–142, 1987. Text available on http://www.numdam.org/item/CM_1987__63_1_123_0.pdf.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HM82] Michiel Hazewinkel and Clyde F. Martin. A short elementary proof of Grothendieck's theorem on algebraic vector bundles over the projective line. *J. Pure Appl. Algebra*, 25(2):207–211, 1982.

[HM07] Christopher D. Hacon and James Mckernan. On Shokurov's rational connectedness conjecture. *Duke Math. J.*, 138(1):119–136, 2007.

[HM13] Christopher D. Hacon and James McKernan. The Sarkisov program. *J. Algebraic Geom.*, 22(2):389–405, 2013.

[Jac85] Nathan Jacobson. *Basic algebra. I*. W. H. Freeman and Company, New York, second edition, 1985.

[Kan95] S. Kantor. Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene. Berlin. Mayer & Müller. 111 S. gr. 8° (1895)., 1895.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol07] János Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.

[Kra18] Hanspeter Kraft. Regularization of rational group actions, 2018.

[Lau20] Bruno Laurent. Almost homogeneous varieties of albanese codimension one, 2020. Text available on https://arxiv.org/abs/2003.08825.

[Lip78] Joseph Lipman. Desingularization of two-dimensional schemes. *Ann. Math. (2)*, 107(1):151–207, 1978.

[LMP09] Frank Loray and David Marín Pérez. Projective structures and projective bundles over compact Riemann surfaces. *Astérisque*, (323):223–252, 2009.

[LU21] Anne Lonjou and Christian Urech. Actions of Cremona groups on cat(0) cube complexes, 2021. Text available on https://arxiv.org/abs/2001.00783.

[Mar70] Masaki Maruyama. *On classification of ruled surfaces*, volume 3 of *Lectures in Mathematics, Department of Mathematics, Kyoto University*. Kinokuniya Book-Store Co., Ltd., Tokyo, 1970. Text available on https://www.math.kyoto-u.ac.jp/library/lmku/lmku03.pdf.

[Mar71] Masaki Maruyama. On automorphism groups of ruled surfaces. *J. Math. Kyoto Univ.*, 11:89–112, 1971. Text available on https://projecteuclid.org/journals/kyoto-journal-of-mathematics/volume-11/issue-1/On-automorphism-groups-of-ruled-surfaces/10.1215/kjm/1250523688.full.

[MO67] Hideyuki Matsumura and Frans Oort. Representability of group functors, and automorphisms of algebraic schemes. *Invent. Math.*, 4:1–25, 1967.

[Mum08] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[Pro12] Yuri Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebr. Geom.*, 21(3):563–600, 2012.

[Pro13] Yu. G. Prokhorov. On birational involutions of $\mathbb{P}^3$. *Izv. Math.*, 77(3):627–648, 2013.

[Pro21] Yu. G. Prokhorov. Equivariant minimal model program. *Uspekhi Mat. Nauk*, 76(3(459)):93–182, 2021. Translation available on `https://iopscience.iop.org/article/10.1070/RM9990/pdf`.

[PS14] Yuri Prokhorov and Constantin Shramov. Jordan property for groups of birational selfmaps. *Compos. Math.*, 150(12):2054–2072, 2014.

[PS16] Yuri Prokhorov and Constantin Shramov. Jordan property for Cremona groups. *Am. J. Math.*, 138(2):403–418, 2016.

[Ros56] Maxwell Rosenlicht. Some basic theorems on algebraic groups. *Amer. J. Math.*, 78:401–443, 1956.

[RZ18] Maria Fernanda Robayo and Susanna Zimmermann. Infinite algebraic subgroups of the real Cremona group. *Osaka J. Math.*, 55(4):681–712, 2018.

[Ser68] Jean-Pierre Serre. *Corps locaux*. Publications de l'Université de Nancago, No. VIII. Hermann, Paris, 1968. Deuxième édition.

[Ser84] Jean-Pierre Serre. *Groupes algébriques et corps de classes*, volume 7 of *Publications de l'Institut Mathématique de l'Université de Nancago [Publications of the Mathematical Institute of the University of Nancago]*. Hermann, Paris, second edition, 1984. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], 1264.

[Sta21] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2021.

[Sum74] Hideyasu Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.

[Sum75] Hideyasu Sumihiro. Equivariant completion. II. *J. Math. Kyoto Univ.*, 15(3):573–605, 1975.

[SZ21] Julia Schneider and Susanna Zimmermann. Algebraic subgroups of the plane Cremona group over a perfect field. *Épijournal Géom. Algébrique*, 5:Art. 14, 48, 2021. Text available on https://epiga.episciences.org/8709/pdf.

[Ume80] Hiroshi Umemura. Sur les sous-groupes algébriques primitifs du groupe de Cremona à trois variables. *Nagoya Math. J.*, 79:47–67, 1980.

[Ume82a] Hiroshi Umemura. Maximal algebraic subgroups of the Cremona group of three variables. Imprimitive algebraic subgroups of exceptional type. *Nagoya Math. J.*, 87:59–78, 1982.

[Ume82b] Hiroshi Umemura. On the maximal connected algebraic subgroups of the Cremona group. I. *Nagoya Math. J.*, 88:213–246, 1982.

[Ume85] Hiroshi Umemura. On the maximal connected algebraic subgroups of the Cremona group. II. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 349–436. North-Holland, Amsterdam, 1985.

[Wei55] André Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955.

[Wim97] A. Wiman. Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene. *Math. Ann.*, 48:195–240, 1897.

[Yas22] Egor Yasinsky. Automorphisms of real del Pezzo surfaces and the real plane Cremona group. *Ann. Inst. Fourier (Grenoble)*, 72(2):831–899, 2022.

[Zai95] Dmitri Zaitsev. Regularization of birational group operations in the sense of Weil. *J. Lie Theory*, 5(2):207–224, 1995.