# Fractional Liouville Equations and Calogero-Moser NLS

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# Abstract

We prove existence, uniqueness and radial symmetry of solutions for the nonlocal Liouville equation

$$(-\Delta)^{1/2}w = Ke^w \quad \text{in } \mathbb{R}$$

with finite total $Q$-curvature $\int_{\mathbb{R}} Ke^w \, \mathrm{d}x < +\infty$. Here the prescribed $Q$-curvature function $K = K(|x|) > 0$ is assumed to be a continuously differentiable, positive, symmetric-decreasing function satisfying suitable decay bounds. In particular, we obtain uniqueness of solutions in the Gaussian case with $K(x) = e^{-x^2}$.

Our existence and uniqueness proof exploits a connection of the nonlocal Liouville equation in one dimension to ground state solitons for Calogero-Moser derivative NLS of the form

$$i\partial_t \psi = -\partial_{xx}\psi + V\psi - \left((-\Delta)^{1/2}|\psi|^2\right)\psi + \frac{1}{4}|\psi|^4\psi \quad \text{in } \mathbb{R}.$$

As a consequence, in the case of the harmonic external potential $V(x) = x^2$ we obtain an explicit expression for the ground state energy. We also discuss existence and decay of solitons and excited states for more general potentials $V$.

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# Contents

# Chapter 1

# Introduction

In this thesis, we study the following one-dimensional nonlocal Liouville equation

$$(-\Delta)^{1/2} w = K e^{w} \quad \text{in } \mathbb{R} \tag{L}$$

subject to the finiteness condition

$$\lambda := \int_{\mathbb{R}} K(x) e^{w(x)} \, \mathrm{d}x < +\infty. \tag{1.1}$$

Here $K \colon \mathbb{R} \to \mathbb{R}_{>0}$ denotes a given continuously differentiable, symmetric-decreasing function, satisfying some suitable bounds on the decay, which will be specified below. Geometrically speaking, if $w$ is a solution of (L), then $K$ can be seen as the $Q$-curvature of the metric $g = e^{2w}|\mathrm{d}x|^2$ on $\mathbb{R}$ which is conformal to the standard metric $g_0 = |\mathrm{d}x|^2$ on $\mathbb{R}$. The quantity $\lambda$ then corresponds to the total $Q$-curvature of the metric $g$ on $\mathbb{R}$. We note that, by means of the stereographic projection, the nonlocal Liouville equation (L) can also be related to a prescribed $Q$-curvature problem on the unit circle. We refer to [12, 13] for more details on the geometric background on (L) and its relation to the generalized Riemann mapping theorem in the complex plane $\mathbb{C}$.

Existence and non-existence results of prescribed $Q$-curvatures problems in $\mathbb{R}^n$ for general dimensions $n \geq 1$ have recently attracted a great deal of attention, leading to the class of Liouville type equations given by

$$(-\Delta)^{n/2} w = K e^{nw} \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

In the case of $n = 2$ space dimensions, equation (1.2) then becomes the well-known Liouville equation which is a central object in nonlinear elliptic PDEs and geometric analysis; see [4, 9, 10, 23, 24, 27].

From an analytic point of view, a particularly challenging situation for equation (1.2) arises in odd space dimensions $n \in \{1, 3, 5, \ldots\}$ due to the nonlocal nature of the pseudo-differential operator $(-\Delta)^{n/2}$. Apart from the important special case of positive constant $Q$-curvature $K > 0$, where solutions $w$ are known in closed form (see [9, 12, 27]), the question of uniqueness of solutions $w$ has been out of reach so far in odd dimensions. In this thesis, we address the case of $n = 1$ space dimension.

We prove regularity, existence, uniqueness and radial-symmetry of solutions of (L) subject to the finiteness condition (1.1). Our analysis is based on a surprisingly strong connection to ground state solitons of the Calogero-Moser derivative NLS, which reads

$$\text{(CM)} \qquad i\partial_t \psi = -\partial_{xx}\psi + V\psi - \left((-\Delta)^{1/2}|\psi|^2\right)\psi + \frac{1}{4}|\psi|^4\psi \quad \text{in } \mathbb{R},$$

for a complex-valued field $\psi \colon [0,\infty) \times \mathbb{R} \to \mathbb{C}$. Here $V \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a continuous, nonnegative external potential. The natural choices are the external harmonic potential given by $V(x) = x^2$ and the case $V \equiv 0$, where no external potential is added. Both arise in the physical context of continuum limits of completely integrable many-body systems of Calogero-Moser type. For a formal derivation of (CM) in the physics literature, we refer to [1, 2].

The existence of ground states and excited states for (CM) can be shown by classical variational methods, for potentials that are monotone in $|x|$ obeying the growth condition $V(x) \to +\infty$ as $|x| \to +\infty$. In the above-mentioned case $V(x) = x^2$, the connection to the Liouville equation allows us to compute the ground state energy explicitly, while for $V \equiv 0$, which corresponds to the well-studied equation $(-\Delta)^{1/2} w = e^w$ with constant $Q$-curvature $K \equiv 1$, we actually derive a complete classification of ground states.

## 1.1 Main Results

We give a rigorous definition of the half-laplacian and introduce its natural space of definition.

### Definition of the Half-Laplacian

On the space of Schwartz functions $\mathcal{S}(\mathbb{R})$, the half-laplacian $(-\Delta)^{1/2}\varphi$ is defined in Fourier space by

$$\mathcal{F}((-\Delta)^{1/2}\varphi)(\xi) = |\xi|\hat{\varphi}(\xi).$$

This definition can be extended in a natural way to

$$L_{1/2}(\mathbb{R}) \coloneqq \left\{ w \in L^1_{\mathrm{loc}}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} \frac{|w(x)|}{1+x^2}\,\mathrm{d}x < +\infty \right\}.$$

Indeed, for $w \in L_{1/2}(\mathbb{R})$ the half-laplacian $(-\Delta)^{1/2} w$ can be defined as a tempered distribution as follows:

$$\langle (-\Delta)^{1/2} w, \varphi \rangle \coloneqq \langle w, (-\Delta)^{1/2}\varphi \rangle \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}),$$

where the integral on the right-hand side converges due to the decay

$$|(-\Delta)^{1/2}\varphi(x)| = O\left(\frac{1}{x^2}\right).$$

An elementary proof is given in Lemma A.4. For a more general decay result for $(-\Delta)^s$, see for instance [20].

For $f$ in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, we say that $w \in L_{1/2}(\mathbb{R})$ is a distributional solution to

$$(-\Delta)^{1/2} w = f$$

if

$$\langle (-\Delta)^{1/2} w, \varphi \rangle := \langle w, (-\Delta)^{1/2} \varphi \rangle = \langle f, \varphi \rangle \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}).$$

### Solitons of the Calogero-Moser DNLS

To establish our main results on the Calogero-Moser derivative NLS (CM) we first introduce our assumptions on the external potential $V$.

**Assumption (A).** *We assume that $V : \mathbb{R} \to \mathbb{R}$ satisfies the following properties.*

(i) *$V$ is nonnegative.*

(ii) *$V$ is monotone increasing in $|x|$ and $\lim_{|x| \to +\infty} V(x) = +\infty$.*

(iii) *$V$ is continuous.*

The natural space to define weak solutions of (CM) is given by the complex-valued Hilbert space

$$X := \left\{ v \in H^1(\mathbb{R}) \;\middle|\; \sqrt{V} v \in L^2(\mathbb{R}) \right\},$$

endowed with the norm

$$\|v\|_X^2 = \|v\|_{L^2(\mathbb{R})}^2 + \|\partial_x v\|_{L^2(\mathbb{R})}^2 + \|\sqrt{V} v\|_{L^2(\mathbb{R})}^2,$$

which turns out to be compactly embedded in $L^2(\mathbb{R})$ due to the additional decay condition $\sqrt{V} v \in L^2(\mathbb{R})$.

Our first main result is as follows.

**Theorem.** *For any given $N \in (0, 2\pi)$, the following existence results hold.*

(i) ***Existence of Ground States:*** *There exists a ground state $v \in X$ of (CM) with $L^2$-mass $\|v\|_{L^2(\mathbb{R})}^2 = N$.*

(ii) ***Existence of infinitely many Excited States:*** *There exist infinitely many excited states $v \in X$ of (CM) with $L^2$-mass $\|v\|_{L^2(\mathbb{R})}^2 = N$.*

*Remark.* Our approach is strongly related to the strict bound on the $L^2$- mass $N < 2\pi$. It remains an open question whether or not there exist solitons and excited states for $L^2$-mass $N \geq 2\pi$.

Whereas the existence statements of the theorem above can be treated by well-known variational methods including suitably critical point theory, the proofs of the following theorems adressing the cases $V \equiv 0$ and $V(x) = x^2$ strongly exploit the connection to the nonlocal Liouville equation.

The second main theorem is about the particular case of no external potential. In fact by the results on the shape of solutions of $(-\Delta)^{1/2} w = e^w$, given in [9, 12, 27], we derive a complete classification of ground states. For an alternative self-contained proof we refer to [17].

**Theorem (Classification of Ground States).** *The ground states of*

$$i\partial_t \varphi = -\partial_{xx}\varphi - \left((-\Delta)^{1/2}|\varphi|^2\right)\varphi + \frac{1}{4}|\varphi|^4\varphi$$

*in $H^1(\mathbb{R})$ are of the form*

$$v(x) = e^{i\alpha}\sqrt{\frac{2\lambda}{1+\lambda^2(x-x_0)^2}}$$

*for arbitrary constants $\alpha, x_0 \in \mathbb{R}$ and $\lambda > 0$. In particular every ground state is of $L^2$-mass $\|v\|^2_{L^2(\mathbb{R})} = 2\pi$. Moreover the ground state energy is 0.*

The third main theorem adresses the harmonic Calogero-Moser DNLS

$$i\partial_t \psi = -\partial_{xx}\psi + x^2\psi - \left((-\Delta)^{1/2}|\psi|^2\right)\psi + \frac{1}{4}|\psi|^4\psi. \tag{1.3}$$

**Theorem (Ground State Energy).** *Every ground state $v \in X$ of (1.3) with $L^2$-mass $N = \|v\|^2_{L^2(\mathbb{R})} \in (0, 2\pi)$ is a radial-symmetric decreasing function (up to a phase $e^{i\alpha}$) and the corresponding ground state energy is given by*

$$E(v) = \frac{1}{4\pi}N\left(2\pi - N\right).$$

## Solutions of the Nonlocal Liouville Equation

Throughout this thesis we always assume that a solution $w : \mathbb{R} \to \mathbb{R}$ to (L) belongs to the space of real-valued functions in $L_{1/2}(\mathbb{R})$, which is the natural space to define distributional solutions of as we have just seen above.

In order to state the main results on the fractional Liouville equation (L), we will impose the following conditions on the $Q$-curvature function $K$.

**Assumption (B).** *We assume that $K : \mathbb{R} \to \mathbb{R}$ has the following properties.*

(i) *$K$ is strictly positive, even and monotone decreasing in $|x|$.*

(ii) *$K$ is continuously differentiable.*

(iii) *There exist $C > 0$ and $\delta > 0$ such that $K$ satisfies the pointwise bound*

$$\sqrt{K(x)} + |x\partial_x\sqrt{K(x)}| \le C\langle x\rangle^{-1/2-\delta},$$

*where $\langle x\rangle = \sqrt{1+x^2}$.*

Important examples for admissible functions are $K(x) = e^{-x^2}$ and $K(x) = \langle x\rangle^{-1-2\delta}$ for some $\delta > 0$. In Chapter 4, we will see that imposing regularity and decay conditions on the square root $\sqrt{K}$ of the $Q$-curvature function becomes natural due to our approach that is based on a connection to solitons for the Calogero-Moser derivative NLS discussed below.

The first main result on (L) is now as follows.

**Theorem.** *Suppose $K$ satisfies Assumption* ***(B)*** *and let $w \in L_{1/2}(\mathbb{R})$ be a solution of (L) satisfying (1.1). Then the following properties hold.*

*(i)* ***Regularity and Universal Bound on $\lambda$:*** *We have $w \in \mathcal{C}^{1,1/2}_{\mathrm{loc}}(\mathbb{R})$ and $\lambda = \int_{\mathbb{R}} Ke^w \,\mathrm{d}x$ satisfies $0 < \lambda < 2\pi$.*

*(ii)* ***Symmetry and Monotonicity:*** *$w$ is even and decreasing in $|x|$, i. e., it holds $w(-x) = w(x)$ for all $x \in \mathbb{R}$ and $w(x) \geq w(y)$ whenever $|x| \leq |y|$.*

*(iii)* ***Existence:*** *For every $w_0 \in \mathbb{R}$, there exists a solution $w \in L_{1/2}(\mathbb{R})$ of (L) with $w(0) = w_0$ such that (1.1) holds.*

The next main result establishes uniqueness of solutions for the fractional Liouville equation (L). In fact, despite the nonlocal nature of the problem, we obtain the following Cauchy–Lipschitz ODE type uniqueness result stating that the initial value $w(0) = w_0$ completely determines the solution $w$ in all of $\mathbb{R}$.

**Theorem (Global Uniqueness).** *Suppose $K$ satisfies Assumption* ***(B)****. If $w, \widetilde{w} \in L_{1/2}(\mathbb{R})$ are solutions of (L) satisfying (1.1), then it holds*

$$\widetilde{w}(0) = w(0) \quad \Rightarrow \quad \widetilde{w} \equiv w.$$

*Remarks.* 1) In view of existing techniques, we consider the uniqueness theorem above to be the most original contribution of this thesis. Further below, we will comment in more detail on the strategy behind its proof.

2) It remains an interesting open question whether – instead of prescribing the initial value $w(0)$ – we also have uniqueness of solutions $w$ determined by the value of the total $Q$-curvature $\lambda$. That is, if for two solutions $\widetilde{w}, w \in L_{1/2}(\mathbb{R})$ of (L) such that

$$\int_{\mathbb{R}} Ke^{\widetilde{w}} \,\mathrm{d}x = \int_{\mathbb{R}} Ke^w \,\mathrm{d}x$$

we necessarily have that the identity $\widetilde{w} \equiv w$ holds.

3) We remark that our uniqueness result is non-perturbative, since no smallness condition on either the $Q$-curvature $K$ nor the initial value $w(0)$ is imposed.

### Comments on the Existence and Uniqueness Proof for the Nonlocal Liouville Equation

We briefly describe the strategy behind proving the existence and uniqueness results stated in theorems above. We start by applying the Hilbert transform H to both sides of (L), to derive the equivalent equation

$$-\partial_x w = \mathrm{H}(Ke^w). \tag{1.4}$$

Next, we introduce the positive function $v \colon \mathbb{R} \to \mathbb{R}_{>0}$ given by

$$v = \sqrt{Ke^w}. \tag{1.5}$$

In terms of $v$, equation (1.4) takes the form

$$\partial_x v - \partial_x (\log \sqrt{K}) v + \frac{1}{2}\mathrm{H}(v^2) v = 0 \quad \text{in } \mathbb{R}. \tag{1.6}$$

Here the function $-\partial_x(\log \sqrt{K})$ plays the role of a given external potential. In fact, equation (1.6) and its solutions $v$ naturally arise in the study of solitons for the Calogero-Moser derivative NLS (CM); see below.

Despite the nonlocality of the Hilbert transform H, it turns out that (1.6) becomes more amenable to the study of existence and uniqueness for solutions $v$ parametrized by its initial value $v_0 = v(0)$. To this end, we recast (1.6) once more into the corresponding integral equation

$$v(x) = v_0 \sqrt{\frac{K(x)}{K(0)}} e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\, \mathrm{d}y}, \tag{1.7}$$

where $v_0 > 0$ enters as a parameter.

The existence result now follows by a suitable version of Schauder's fixed point theorem (see for instance [26]), applied on the set of symmetric-decreasing functions in $H^1(\mathbb{R})$, satisfying some additional integrability condition, which is expedient to arrive at the required compactness result.

As an essential step towards proving our theorem on global uniqueness, we establish a local uniqueness result around any given solution $v$ of (1.7) by constructing a locally unique branch parametrized by $v_0$ using the implicit function theorem. To achieve this, we show that the invertibility of the relevant linearized (and nonlocal) operator is tantamount to ruling out non-trivial solutions $\psi \in \dot{H}^1_{\mathrm{even}}(\mathbb{R})$ with $\psi(0) = 0$ that satisfy

$$(-\Delta)^{1/2}\psi - v^2\psi = 0 \quad \text{in } \mathbb{R}.$$

Here the use of a monotonicity formula for the fractional Laplacian $(-\Delta)^{1/2}$ found in [5,6] (and applied for the spectral analysis related to nonlinear ground states in [16]) becomes the key ingredient. However, in contrast to these works, we develop a different approach which completely avoids the use of the harmonic extension to the upper half-plane $\mathbb{R}^2_+$. Instead, we directly work with the singular integral expression for $(-\Delta)^{1/2}$ and we thus obtain expressions which relate to the classical theory of Carleman-Hankel operators on the half-line; see Section 4.4 for more details. We believe that this novel approach for monotonicity formulas can lead to further general insights into spectral and uniqueness problems involving the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ and other suitable pseudo-differential operators (but which may not be seen as Dirichlet-to-Neumann maps).

Once the local uniqueness for solutions $v$ of (1.7) is established, we complete the proof of global uniqueness by a-priori bounds allowing us to make a global continuation argument linking to the limit $v_0 \to 0^+$, which finally shows that there exists only one global branch of solutions $v$ parametrized by $v(0) = v_0$.

### Connection between the nonlocal Liouville equation and solitons for the Calogero-Moser DNLS

We now sketch the connection between the nonlocal Liouville equation (L) and solitons for the Calogero-Moser derivative NLS (CM) in the cases of hamiltonian external potential $V(x) = x^2$ and no external potential $V \equiv 0$. The centerpiece here, is to show that (CM) stems from a Hamiltonian energy functional $E$ which admits a factorization into a complete square of first-order terms; see Chapters 2 and 3 for details. We also refer to [17] here. More precisely, the Hamiltonian energy is found to be

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left|\partial_x v + \sqrt{V} v + \frac{1}{2}\mathrm{H}(|v|^2) v\right|^2 \, \mathrm{d}x + C \tag{1.8}$$

where $C$ is a constant only depending on the $L^2$-mass of $v$. Evidently solutions of

$$\partial_x v + \sqrt{V} v + \frac{1}{2}\mathrm{H}(|v|^2) v = 0,$$

(provided they exist) minimize the energy $E$ and hence are actually ground state solitons to (CM).

Comparing this first order differential equation to (1.6) we find that adding an external harmonic potential $V = x^2$ in (CM), corresponds to the choice of a prescribed $Q$-curvature in (L) given by the Gaussian function $K(x) = e^{-x^2}$, which clearly obeys Assumption **(B)**.

In the case of no external potential $V \equiv 0$, we receive that $K \equiv 1$, a positive constant. As mentionend above here all solutions for (L) are known in closed form. In fact they are given by

$$w(x) = \log\left(\frac{2\lambda}{1 + \lambda^2 (x - x_0)^2}\right),$$

for arbitrary $\lambda > 0$ and $x_0 \in \mathbb{R}$. Translating this back via (1.5), this shows that all real-valued minimizers of the energy $E$, must be of the explicit form

$$v(x) = \pm\sqrt{\frac{2\lambda}{1 + \lambda^2 (x - x_0)^2}}.$$

Another self-contained proof of this fact, based on Hardy-space techniques, that completely avoids exploiting the relation to the fractional Liouville equation (L), can be found in [17].

## 1.2 Structure of the Thesis

We give a brief overview of the contents of each chapter.

### Chapter 2: Solitons of the Calogero-Moser Derivative NLS

The aim of Chapter 2 is to give an explicit expression for the ground state solitons of the Calogero-Moser DNLS without an external potential. We find the corresponding Hamiltonian energy functional to be

$$\widetilde{E}(v) = \frac{1}{2} \|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2) v\|^2_{L^2(\mathbb{R})} \geq 0,$$

and determine its vanishing points, which turns out to be (up to a phase constant) of the form $v = \sqrt{e^w}$ for the well-known solutions $w \in L_{1/2}(\mathbb{R})$ of $(-\Delta)^{1/2} w = e^w$. To rule out the existence of further ground states, we argue that the infimum of $\widetilde{E}$ on the set of $H^1$-functions of a given fixed $L^2$-norm, is always 0.

### Chapter 3: Solitons of the CM DNLS with external potential

In Chapter 3 we study the Calogero-Moser derivative NLS (CM) with an external potential $V$ satisfying Assumption **(A)**.

We prove existence of ground state solitons for a given $L^2$-mass $N \in (0, 2\pi)$ by classical variational methods. We establish weak lower semicontinuity of the nonnegative Hamiltonian energy $E$ and use our knowledge of Chapter 2 to retrieve boundedness of its minimizing sequences in $X$. We then conclude by the compactness of $X \subset L^2(\mathbb{R})$.

We furthermore show the existence of infintely many excited states, again under the assumption $N \in (0, 2\pi)$. We prove that the energy functional $E$ satisfies the Palais-Smale condition. Critical-point theory for even functionals, based on the definition of the Krasonelskii genus, then implies our desired existence result.

At last we adapt well-known concepts to establish some decay bounds. For a strictly positive potential $V > 0$ we obtain $L^2$-exponential decay of excited states. If $V$ is of some polynomial growth we even derive a pointwise superexponential decay bound.

### Chapter 4: Main Results for the Nonlocal Liouville equation in $\mathbb{R}$

Chapter 4 is the main part of the thesis.

We prove regularity in the sense that solutions $w \in L_{1/2}(\mathbb{R})$ of (L) belong to the Hölder space $\mathcal{C}^{1,1/2}_{\text{loc}}(\mathbb{R})$. Furthermore we find the integral representation,

$$w(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \log(|x-y|) K(y) e^{w(y)} \, \mathrm{d}y + C$$

which allows us to establish the asymptotics

$$w(x) = -\frac{\lambda}{\pi} \log|x| + O(1) \quad \text{as} \quad |x| \to +\infty,$$

as well as the total $Q$-curvature bound $0 < \lambda < 2\pi$.

We use a suitable moving planes method, again for the integral equation of $w$, to argue that a solution $w \in L_{1/2}(\mathbb{R})$ of (L) has to be even symmetric and monotone decreasing in $|x|$.

In order to derive existence and uniqueness of solutions of (L), we introduce the equivalent fixed-point equation (1.7) in terms of $v = \sqrt{Ke^w} \in H^1(\mathbb{R})$. Due to the fact that we may reduce to even functions (or actually positive symmetric decreasing functions whenever necessary), we are able to prove that for every positive initial value $v_0 > 0$ there exists a unique solution $v \in H^1(\mathbb{R})$ of (1.7) satisfying $v(0) = v_0$. Translating this back via (1.5) we retrieve the desired existence and uniqueness results for (L).

As an important byproduct of our uniqueness proof, we derive a self-sufficing result on equation

$$(-\Delta)^{1/2}\psi + W\psi = 0 \quad \text{in } \mathbb{R},$$

where $W\colon \mathbb{R} \to R$ is a given $\mathcal{C}^1$-function with nonnegtive derivative on the upper half line $\mathbb{R}_{\geq 0}$. We prove that a solution $\psi \in \dot{H}^1_{\text{even}}(\mathbb{R})$ with initial value $\psi(0) = 0$ is trivial, i.e. $\psi \equiv 0$.

### Chapter 5: Harmonic CM DNLS

In Chapter 5 we study the harmonic Calogero-Moser DNLS.

We find an explicit expression for the ground state energy in terms of the $L^2$-mass $N \in (0, 2\pi)$. We use the formula

$$E(v) = \frac{1}{2}\|\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \frac{1}{4\pi}N\,(2\pi - N)\,,$$

established in Chapter 3 and apply a suitable Schauder's fixed-point argument to prove existence of solutions to $\partial_x v + xv + \frac{1}{2}\mathrm{H}(|v|^2)v = 0$ subject to the condition $\|v\|^2_{L^2(\mathbb{R})} = N$.

In one last step we elaborate the connection of the harmonic Calogero-Moser DNLS to the equation without any external potential, using the Lens transform. This enables to give an outlook on the well-posedness of the time-evolution on a subset, however showing this for a dense subset remains an open problem.

# Chapter 2

# Solitons of the Calogero-Moser Derivative NLS

This chapter is devoted to the proof of a complete classification of ground states of the one-dimensional Calogero-Moser derivative NLS without an additional external potential.

Our proof points out the strong connection between solitons of the Calogero-Moser derivative NLS and the nonlocal Liouville equation, which will be of major significance in Chapter 4 and 5 in a more gerenal setting.

In this chapter, we study the Hamiltonian PDE

$$i\partial_t \varphi = -\partial_{xx}\varphi - \left((-\Delta)^{1/2}|\varphi|^2\right)\varphi + \frac{1}{4}|\varphi|^4\varphi \quad \text{in } \mathbb{R}, \tag{2.1}$$

for the complex-valued field $\varphi\colon [0,\infty)\times\mathbb{R}\to\mathbb{C}$, where the map $t\mapsto\varphi(t,\cdot)$ belongs to $\mathcal{C}^0([0,\infty);H^1(\mathbb{R}))$. Our aim is to give a complete classification of its ground states in $H^1(\mathbb{R})$.

The corresponding Hamiltonian energy functional of (2.1) is given by

$$\widetilde{E}(v) = \frac{1}{2}\|\partial_x v\|^2_{L^2(\mathbb{R})} - \frac{1}{4}\left\langle |v|^2, (-\Delta)^{1/2}|v|^2\right\rangle + \frac{1}{24}\|v\|^6_{L^6(\mathbb{R})}. \tag{2.2}$$

We briefly summarize the arguments from which we derive our main result stated in Theorem 2.1, below. First, we will show that the energy functional $\widetilde{E}$ admits a factorization into a complete square of first-order terms and hence is nonnegative (see Lemma 2.3). To be more specific, it turns out that vanishing points $v\in H^1(\mathbb{R})$ of $\widetilde{E}$ are given by the solutions of the nonlinear first-order differential equation $\partial_x v + \frac{1}{2}\mathrm{H}(|v|^2)v = 0$. To characterize its real-valued, positive solutions, we introduce the function $w\colon\mathbb{R}\to\mathbb{R}$ by $w=\log(v^2)$. The equation $\partial_x v + \frac{1}{2}\mathrm{H}(v^2)v=0$ can be rewritten in terms of $w$ as the one-dimensional Liouville equation

$$(-\Delta)^{1/2} w = e^w, \quad \text{with} \quad \int_{\mathbb{R}} e^{w(x)}\,\mathrm{d}x < +\infty \tag{2.3}$$

in

$$L_{1/2}(\mathbb{R}) := \Big\{ w \in L^1_{\text{loc}}(\mathbb{R}) \;\Big|\; \int_{\mathbb{R}} \frac{|w(x)|}{1+x^2}\, \mathrm{d}x < \infty \Big\}.$$

The solutions of (2.3) are explicitly known to be

$$w(x) = \log\left( \frac{2\lambda}{1+\lambda^2(x-x_0)^2} \right), \tag{2.4}$$

for any $x_0 \in \mathbb{R}$ and $\lambda > 0$. For a proof we refer to [9, 12, 27]. In particular the solutions are unique up to symmetry of the problem.

The argument above, which will be laid out in greater detail in Lemma 2.3 and Proposition 2.4, leads to the following result.

**Theorem 2.1.** *The energy $\widetilde{E}$ attains its minimum on $H^1(\mathbb{R}) \setminus \{0\}$ and it holds*

$$\min\big\{ \widetilde{E}(v) \;\big|\; v \in H^1(\mathbb{R}) \setminus \{0\} \big\} = 0.$$

*Moreover all nontrivial minimizers are given by*

$$v(x) = e^{i\alpha} \sqrt{\frac{2\lambda}{1+\lambda^2(x-x_0)^2}} \tag{2.5}$$

*for arbitrary constants $\alpha$, $x_0$ in $\mathbb{R}$ and $\lambda > 0$.*

Theorem 2.1 shows that functions of the form (2.5) are in fact ground states of (2.1). To rule out that further ground states exist, we establish the following corollary.

**Corollary 2.2.** *Consider the constrained minimizing problem*

$$I(N) := \inf\big\{ \widetilde{E}(v) \;\big|\; v \in H^1(\mathbb{R}),\ \|v\|^2_{L^2(\mathbb{R})} = N \big\}.$$

*Then $I(N) = 0$ for every $N > 0$. In particular all the ground states of (2.1) are given by (2.5). Moreover the infimum $I(N)$ is attained if and only if $N = 2\pi$.*

The proof of Corollary 2.2 will be given at the end of this chapter.

**Lemma 2.3.** *The energy functional $\widetilde{E} \colon X \to \mathbb{R}$ can be written as*

$$\widetilde{E}(v) = \frac{1}{2} \|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})}, \tag{2.6}$$

*where $\mathrm{H}$ denotes the Hilbert transform given on $L^2(\mathbb{R})$ by $\widehat{\mathrm{H}(f)}(\xi) = -i\operatorname{sgn}(\xi)\hat{f}(\xi)$.*

In the proof we use several basic facts about the Hilbert transform, which are collected in Lemma A.1 in the appendix.

*Proof.* To prove equation (2.6) we compare the energy expression given in (2.2)

$$\widetilde{E}(v) = \frac{1}{2}\|\partial_x v\|^2_{L^2(\mathbb{R})} - \frac{1}{4}\left\langle |v|^2, (-\Delta)^{1/2}|v|^2 \right\rangle + \frac{1}{24}\|v\|^6_{L^6(\mathbb{R})}$$

with

$$\frac{1}{2}\|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|_{L^2(\mathbb{R})}^2 = \frac{1}{2}\|\partial_x v\|_{L^2(\mathbb{R})}^2 + \frac{1}{2}\Re\left\langle \partial_x v, \mathrm{H}(|v|^2)v\right\rangle + \frac{1}{8}\|\mathrm{H}(|v|^2)v\|_{L^2(\mathbb{R})}^2.$$

We will conclude the proof by showing

$$\Re\left\langle \partial_x v, \mathrm{H}(|v|^2)v\right\rangle = -\frac{1}{2}\left\langle |v|^2, (-\Delta)^{1/2}|v|^2\right\rangle \quad \text{and} \tag{2.7}$$

$$\|\mathrm{H}(|v|^2)v\|_{L^2(\mathbb{R})}^2 = \frac{1}{3}\|v\|_{L^6(\mathbb{R})}^6. \tag{2.8}$$

Using integration by parts, we readily check that

$$\Re\left\langle \partial_x v, \mathrm{H}(|v|^2)v\right\rangle = -\frac{1}{2}\Re\int_{\mathbb{R}} |v|^2 \partial_x \mathrm{H}(|v|^2)\, \mathrm{d}x.$$

Moreover since $\mathrm{H}(|v|^2)$ is real-valued and the identity $\partial_x \mathrm{H}(f) = (-\Delta)^{1/2} f$ holds for $f \in H^1(\mathbb{R})$, we deduce

$$\Re\left\langle \partial_x v, \mathrm{H}(|v|^2)v\right\rangle = -\frac{1}{2}\int_{\mathbb{R}} |v|^2 (-\Delta)^{1/2}|v|^2\, \mathrm{d}x,$$

which proves (2.7).

To show (2.8) we denote the real-valued functions $f$ and $g$ by $f = |v|^2$ and $g = \mathrm{H}(|v|^2) = \mathrm{H}(f)$. The Hilbert transform is anti self-adjoint on $L^2(\mathbb{R})$ and thus we find

$$\|\mathrm{H}(|v|^2)v\|_{L^2(\mathbb{R})}^2 = \langle \mathrm{H}(f), fg\rangle = -\langle f, \mathrm{H}(fg)\rangle. \tag{2.9}$$

To rewrite the scalar product we use a corollary to Cotlar's identity (see [19]), that

$$\mathrm{H}(fg) = \mathrm{H}(f)g + f\mathrm{H}(g) + \mathrm{H}(\mathrm{H}(f)\mathrm{H}(g)),$$

for $f, g \in H^1(\mathbb{R})$, which can be obtained by a polarization argument. Since the Hilbert transform is an anti-involution and therefore in particular $\mathrm{H}(g) = \mathrm{H}^2(f) = -f$ holds, this simplifies to

$$\mathrm{H}(fg) = g^2 - f^2 - \mathrm{H}(fg),$$

or respectively

$$2\mathrm{H}(fg) + f^2 = g^2.$$

We integrate this equation against $f$ and use the fact that $f$ and $g$ are real-valued, to obtain

$$2\langle f, \mathrm{H}(fg)\rangle + \left\langle f, f^2\right\rangle = \left\langle f, g^2\right\rangle = \langle g, fg\rangle = \langle \mathrm{H}(f), fg\rangle = -\langle f, \mathrm{H}(fg)\rangle.$$

Now this and (2.9) lead directly to

$$\|\mathrm{H}(|v|^2)v\|_{L^2(\mathbb{R})}^2 = -\langle f, \mathrm{H}(fg)\rangle = \frac{1}{3}\left\langle f, f^2\right\rangle = \frac{1}{3}\|v\|_{L^6(\mathbb{R})}^6,$$

which proves (2.8). □

The next proposition will use the expression we found in (2.6) to give a complete classification of all nontrivial vanishing points of $\widetilde{E}$ in $H^1(\mathbb{R})$. In particular this will conclude the proof of Theorem 2.1.

**Proposition 2.4.** *The nontrivial solutions $v$ in $H^1(\mathbb{R})$ of*

$$\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v = 0 \tag{2.10}$$

*are given by*

$$v(x) = e^{i\alpha}\sqrt{\frac{2\lambda}{1+\lambda^2(x-x_0)^2}}$$

*for constants $\alpha$, $x_0$ in $\mathbb{R}$ and $\lambda > 0$.*

*Proof.* Consider a solution $v \neq 0$ of (2.10) in $H^1(\mathbb{R})$. The Hilbert transform is an isometry on $H^1(\mathbb{R})$, which implies that $\mathrm{H}(|v|^2)v$ also belongs to $H^1(\mathbb{R})$. Thus, since $v$ solves (2.10), we conclude that $v$ is actually an element of $H^2(\mathbb{R})$ and therefore by Sobolev embedding theorem belongs to $\mathcal{C}^{1,1/2}(\mathbb{R})$.

By this higher regularity of $v$, (2.10) is equivalent to

$$v(x) = re^{i\alpha}e^{-\frac{1}{2}\int_0^x \mathrm{H}(|v|^2)(y)\mathrm{d}y},$$

for some $\alpha \in \mathbb{R}$ and $r > 0$. Therefore up to the phase $\alpha$ we assume $v$ to be positive and we can define the $\mathcal{C}^1$ function $w\colon \mathbb{R} \to \mathbb{R}$ by $w(x) = \log(v(x)^2)$. In terms of $w$, equation (2.10) is equivalent to

$$\partial_x w = \frac{2\partial_x v}{v} = -\mathrm{H}(v^2) = -\mathrm{H}(e^w).$$

If $w \in L_{1/2}(\mathbb{R})$ and $e^w \in L^2(\mathbb{R})$ hold, we can apply Lemma A.2, to obtain the Liouville equation

$$(-\Delta)^{1/2}w = e^w.$$

If in addition $w$ satisfies the finiteness condition $e^w \in L^1(\mathbb{R})$, the solutions of $(-\Delta)^{1/2}w = e^w$ are given by (2.4). This upon using $v = \sqrt{e^w}$ concludes the proof.

Clearly $e^w = v^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ holds true as $v$ belongs to $H^1(\mathbb{R})$. To prove $w \in L_{1/2}(\mathbb{R})$ we need the following inequality which we will also use later on.

$$\|\mathrm{H}(v^2)\|_{L^2(\mathbb{R})} = \|v^2\|_{L^2(\mathbb{R})} = \|v\|^2_{L^4(\mathbb{R})} \lesssim \|v\|^2_{H^1(\mathbb{R})}.$$

We use this and the expression $v(x) = re^{-\frac{1}{2}\int_0^x \mathrm{H}(|v|^2)(y)\mathrm{d}y}$ to obtain the estimate

$$\begin{aligned} |w(x)| = |\log(v(x)^2)| &\leq 2|\log(r)| + \int_0^{|x|} |\mathrm{H}(v^2)(y)|\mathrm{d}y \\ &\leq 2|\log(r)| + |x|^{1/2}\|\mathrm{H}(v^2)\|_{L^2(\mathbb{R})} \lesssim (1+|x|^{1/2}), \end{aligned}$$

which proves that $|w(x)|(1+x^2)^{-1}$ is integrable and therefore $w$ belongs to $L_{1/2}(\mathbb{R})$. □

*Proof of Corollary 2.2.* We first assume $I(N) = 0$ holds true for every $N > 0$. So the infimum $I(N)$ is attained if and only if there exists a solution $v \in H^1(\mathbb{R})$ to $\widetilde{E}(v) = 0$ satisfying $\|v\|^2_{L^2(\mathbb{R})} = N$. By Theorem 2.1 vanishing points of $\tilde{E}$ acquire the form (2.5) and in particular $N = 2\pi$ must hold.

By Lemma 2.3 we already know that $\widetilde{E}$ is nonnegative and hence $I(N) \geq 0$. To prove that it is actually zero we take an arbitrary $v \in H^1(\mathbb{R})$ satisfying $\|v\|^2_{L^2(\mathbb{R})} = N$. Notice that for any $\lambda > 0$, the $L^2$-norm is preserved by the dilation $v_\lambda(x) = \lambda^{1/2} v(\lambda x)$. Moreover

$$\mathrm{H}(|v_\lambda|^2)(x) = \lambda \mathrm{H}(|v|^2)(\lambda x).$$

In particular using (2.6) we obtain

$$\widetilde{E}(v_\lambda) = \frac{\lambda^3}{2} \int_{\mathbb{R}} |\partial_x v(\lambda x) + \tfrac{1}{2}\mathrm{H}(|v|^2)(\lambda x) v(\lambda x)|^2 \, \mathrm{d}x = \lambda^2 \widetilde{E}(v).$$

Summarized we have that

$$\|v_\lambda\|^2_{L^2(\mathbb{R})} = \|v\|^2_{L^2(\mathbb{R})} = N \quad \text{and} \quad \widetilde{E}(v_\lambda) = \lambda^2 \widetilde{E}(v) \to 0 \quad \text{as } \lambda \to 0$$

and therefore $I(N) \leq 0$, whence it follows that $I(N) = 0$. □

# Chapter 3

# Solitons of the CM DNLS with external potential

In this chapter we will use a classical variational approach to study the one-dimensional Calogero-Moser derivative NLS with nonnegative, continuous external potential $V$, where we assume $V$ to be monotone increasing in $|x|$ and to satisfy $\lim_{|x|\to+\infty} V(x) = +\infty$. We recall Equation (CM) from the introduction for the reader's convenience:

$$\text{(CM)} \qquad i\partial_t\psi = -\partial_{xx}\psi + V\psi - \left((-\Delta)^{1/2}|\psi|^2\right)\psi + \frac{1}{4}|\psi|^4\psi,$$

for a complex-valued field $\psi\colon [0,+\infty)\times\mathbb{R}\to\mathbb{C}$.

We will prove the existence of ground states for $L^2$-mass strictly smaller than $2\pi$. Moreover we will use critical point theory based on the definition of Krasonelskii genus to show that there exist infinitely many excited states. In addition we will prove some regularity results and give an explicit bound on the decay. The natural choice for $V$ will be the harmonic external potential $V(x) = x^2$. In this particular case we find a connection to the nonlocal Liouville equation, similar to the case $V \equiv 0$, which we treated in Chapter 2. This leads to another approach for proving existence of ground states, which also allows us to compute the zero point energy explicitly, as we will see in Chapter 5.

Throughout this chapter we work on the Hilbert space

$$X := \left\{ v \in H^1(\mathbb{R}) \;\middle|\; \sqrt{V}v \in L^2(\mathbb{R}) \right\},$$

endowed with the norm

$$\|v\|_X^2 = \|v\|_{L^2(\mathbb{R})}^2 + \|\partial_x v\|_{L^2(\mathbb{R})}^2 + \|\sqrt{V}v\|_{L^2(\mathbb{R})}^2,$$

which is the natural space to define weak solutions $\psi(t,\,\cdot\,)\colon \mathbb{R}\to\mathbb{C}$ of (CM). Moreover we always assume that the external potential $V\colon\mathbb{R}\to\mathbb{R}$ satisfies the properties listed in Assumption **(A)**, which we also recall here.

**Assumption (A).** *We assume that* $V\colon \mathbb{R} \to \mathbb{R}$ *satisfies the following properties.*

(i) $V$ *is nonnegative.*

(ii) $V$ *is monotone increasing in* $|x|$ *and* $\lim_{|x|\to+\infty} V(x) = +\infty$.

(iii) $V$ *is continuous.*

Notice that the corresponding Hamiltonian energy functional of (CM) is given by

$$
\begin{aligned}
E(v) = {} & \frac{1}{2}\|\partial_x v\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\sqrt{V}v\|^2_{L^2(\mathbb{R})} \\
& - \frac{1}{4}\left\langle |v|^2, (-\Delta)^{1/2}|v|^2 \right\rangle + \frac{1}{24}\|v\|^6_{L^6(\mathbb{R})}.
\end{aligned}
\tag{3.1}
$$

## 3.1 Existence of Solitons

The aim of this section is to prove the following theorem, which shows the existence of ground states for fixed $L^2$-mass $\|v\|^2_{L^2(\mathbb{R}^2)} = N \in (0, 2\pi)$.

**Theorem 3.1.** *Consider the constrained minimizing problem*

$$
I(N) \coloneqq \inf\left\{E(v) \;\middle|\; v \in X,\ \|v\|^2_{L^2(\mathbb{R})} = N\right\}.
$$

*If* $N \in (0, 2\pi)$ *the infimum is attained.*

The classical approach to prove the existence of ground states for (CM) is to establish weak lower semicontinuity of the energy functional $E$ and to prove boundedness of minimizing sequences subject to the constraint that the mass is fixed in $(0, 2\pi)$. Our proof is closely related to the existence of ground states of the Calogero-Moser NLS without external potential (2.1), which are explicitly known by Theorem 2.1 and are of $L^2$-norm $\sqrt{2\pi}$. In fact we will use that an unbounded minimizing sequence can be rescaled in a way such that it weakly converges to a ground state of equation (2.1). This clearly leads to a contradiction if the $L^2$-norm of our minimizing sequence is fixed by a constant smaller than $\sqrt{2\pi}$. The existence of ground states of (CM) with mass greater or equal than $2\pi$ can not be treated with this approach and is still unknown.

We divide the proof of Theorem 3.1 in the following lemmas.

**Lemma 3.2.** *The energy functional* $E\colon X \to \mathbb{R}$ *can be written as*

$$
E(v) = \widetilde{E}(v) + \frac{1}{2}\|\sqrt{V}v\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\sqrt{V}v\|^2_{L^2(\mathbb{R})},
\tag{3.2}
$$

*where* $\widetilde{E}$ *is defined in (2.2).*
*Furthermore in the special case that* $V$ *is given by the harmonic potential* $V(x) = x^2$, *the energy* $E$ *satisfies*

$$
\begin{aligned}
E(v) &= \frac{1}{2}\|\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \frac{1}{4\pi}\|v\|^2_{L^2(\mathbb{R})}\left(2\pi - \|v\|^2_{L^2(\mathbb{R})}\right) \\
&= \frac{1}{2}\|\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \frac{1}{4\pi}N\left(2\pi - N\right),
\end{aligned}
\tag{3.3}
$$

*for* $L^2$*-mass* $\|v\|^2_{L^2(\mathbb{R})} = N$.

From this lemma we readily deduce that the energy is nonnegative. A question that arises naturally is, whether we are able to compute the zero point energy $I(N)$ explicitly. Whereas we cannot answer this question for a general external potential $V$ satisfying Assumption **(A)**, we find an approach to treat the case $V(x) = x^2$. We will do this in detail in Chapter 5. For now we will only give a small outlook: In terms of equation (3.3) for fixed $L^2$-mass $N$ the energy is bounded below by $\frac{1}{4\pi}N\,(2\pi - N)$. We will prove that for every $N \in (0, 2\pi)$ there exists $v \in X$ with $L^2$-mass $N$ solving $\partial_x v + xv + \frac{1}{2}\mathrm{H}(|v|^2)v = 0$. So in particular $v$ is a ground state and $I(N) = E(v) = \frac{1}{4\pi}N\,(2\pi - N)$ is given by this lower bound.

*Proof. (Lemma 3.2)* The first identity directly follows from Lemma 2.3.

To prove the second identity we rewrite (3.2) as

$$\begin{aligned}2E(v) &= \|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \|xv\|^2_{L^2(\mathbb{R})}\\ &= \|\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} - 2\Re\langle \partial_x v, xv\rangle - \Re\left\langle xv, \mathrm{H}(|v|^2)v\right\rangle.\end{aligned}$$

By integration by parts we obtain

$$-2\Re\langle \partial_x v, xv\rangle = \|v\|^2_{L^2(\mathbb{R})}.$$

Therefore to prove (3.3) it remains to show that

$$\Re\left\langle xv, \mathrm{H}(|v|^2)v\right\rangle = \frac{1}{2\pi}\|v\|^4_{L^2(\mathbb{R})}. \tag{3.4}$$

To simplify notation we write $f = |v|^2 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. By the Parseval formula and the Fourier representation of the Hilbert transform $\mathcal{F}\mathrm{H}(f)(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi)$ the following identity holds true.

$$\begin{aligned}\Re\left\langle xv, \mathrm{H}(|v|^2)v\right\rangle &= \Re\langle xf, \mathrm{H}(f)\rangle = \Re\int_{\mathbb{R}} \overline{i\partial_\xi \hat{f}(\xi)}(-i)\mathrm{sgn}(\xi)\hat{f}(\xi)\;\mathrm{d}\xi\\ &= -\int_{\mathbb{R}} \Re\left(\overline{\partial_\xi \hat{f}(\xi)}\hat{f}(\xi)\right)\mathrm{sgn}(\xi)\;\mathrm{d}\xi = -\frac{1}{2}\int_{\mathbb{R}} \partial_\xi|\hat{f}(\xi)|^2\mathrm{sgn}(\xi)\;\mathrm{d}\xi.\end{aligned}$$

Now (3.4) is a direct consequence of the Riemann-Lebesgue lemma as follows

$$\begin{aligned}\Re\left\langle xv, \mathrm{H}(|v|^2)v\right\rangle &= -\frac{1}{2}\int_{\mathbb{R}} \partial_\xi|\hat{f}(\xi)|^2\mathrm{sgn}(\xi)\;\mathrm{d}\xi\\ &= |\hat{f}(0)|^2 = \frac{1}{2\pi}\|f\|^2_{L^1(\mathbb{R})} = \frac{1}{2\pi}\|v\|^4_{L^2(\mathbb{R})}.\end{aligned}$$

As mentioned above, this concludes the proof. □

In the following lemma we prove weak lower semicontinuity of $\widetilde{E}$ in $H^1(\mathbb{R})$. Thus, by the energy rewriting (3.2) we find that $E = \tilde{E} + \frac{1}{2}\|\sqrt{V}v\|^2_{L^2(\mathbb{R})}$ is weakly lower semicontinuous in $X$.

**Lemma 3.3.** *$\widetilde{E}$ is weakly lower semicontinuous in $H^1(\mathbb{R})$, i.e.*

$$\liminf_{n\to+\infty} \widetilde{E}(v_n) \geq \widetilde{E}(v) \quad \textit{as } v_n \rightharpoonup v \textit{ in } H^1(\mathbb{R}).$$

*Proof.* In the ensuing proof we assume $(v_n)$ to be a sequence that weakly converges to $v$ in $H^1(\mathbb{R})$. We have to verify that $\liminf_{n\to+\infty} \widetilde{E}(v_n) \geq \widetilde{E}(v)$. By Lemma 2.3 this is equivalent to

$$\liminf_{n\to+\infty} \frac{1}{2}\|\partial_x v_n + \tfrac{1}{2}\mathrm{H}(|v_n|^2)v_n\|^2_{L^2(\mathbb{R})} \geq \frac{1}{2}\|\partial_x v + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})}.$$

We pass to a subsequence if necessary, to replace the limit inferior by the limit. This allows us to freely pass to subsequences in the following.

By the weak lower semicontinuity of the $L^2$-norm and since $(\partial_x v_n)$ weakly converges to $\partial_x v$ in $L^2(\mathbb{R})$, it suffices to show that

$$\mathrm{H}(|v_n|^2)v_n \rightharpoonup \mathrm{H}(|v|^2)v \quad \text{in } L^2(\mathbb{R})$$

up to subsequences.

To do so we will simply use the definition of weak convergence and prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \mathrm{H}(|v_n|^2)v_n\varphi \,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{H}(|v|^2)v\varphi \,\mathrm{d}x \tag{3.5}$$

for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$. This can be done by dominated convergence theorem as follows. First, by Hölder's inequality, Sobolev embedding theorem and the fact that the Hilbert transform is an isometry on $H^1(\mathbb{R})$, we obtain the estimate

$$\begin{aligned}\|\mathrm{H}(|v_n|^2)v_n\|_{L^\infty(\mathbb{R})} &\leq \|v_n\|_{L^\infty(\mathbb{R})}\|\mathrm{H}(|v_n|^2)\|_{L^\infty(\mathbb{R})}\\ &\lesssim \|v_n\|_{H^1(\mathbb{R})}\|\mathrm{H}(|v_n|^2)\|_{H^1(\mathbb{R})} = \|v_n\|_{H^1(\mathbb{R})}\||v_n|^2\|_{H^1(\mathbb{R})} \lesssim \|v_n\|^3_{H^1(\mathbb{R})} \lesssim 1.\end{aligned}$$

From this we deduce the uniform bound

$$|\mathrm{H}(|v_n|^2)v_n\varphi| \leq \|\mathrm{H}(|v_n|^2)v_n\|_{L^\infty(\mathbb{R})}|\varphi| \lesssim |\varphi| \in L^1(\mathbb{R}).$$

Therefore to prove (3.5) it suffices to show that

$$\mathrm{H}(|v_n|^2)(x)v_n(x) \to \mathrm{H}(|v|^2)(x)v(x) \quad \text{for almost every } x \in \mathbb{R}. \tag{3.6}$$

Notice that by the weak convergence $v_n \rightharpoonup v$ in $H^1(\mathbb{R})$, there exists a subsequence of $(v_n)$ which converges pointwise almost everywhere to $v$. This fact follows by Rellich-Kondrachov theorem and a diagonal argument. Since $|v_n|^2$ belongs to $H^1(\mathbb{R})$ and therefore by Sobolev embedding theorem to the Hölder space $\mathcal{C}^{0,1/2}(\mathbb{R})$, we may write the Hilbert transform in terms of the absolutely convergent integral

$$\mathrm{H}(|v_n|^2)(x) = \frac{1}{\pi}\int_0^\infty \frac{|v_n|^2(x-t) - |v_n|^2(x+t)}{t}\,\mathrm{d}t. \tag{3.7}$$

Our aim is to argue that this sequence of integrals converges to $\mathrm{H}(|v|^2)(x)$ for every fixed $x$. For $R > 0$ we split this integral into two parts.

The integral over $(0, R)$ can be handled by dominated convergence theorem again. Clearly we have almost everywhere convergence of the integrand of (3.7). Moreover there exists the uniform bound

$$\left|\frac{|v_n|^2(x-t) - |v_n|^2(x+t)}{t}\right| \leq \frac{2^{1/2}\|v_n^2\|_{\mathcal{C}^{0,1/2}(\mathbb{R})}}{t^{1/2}} \lesssim \frac{\|v_n\|^2_{H^1(\mathbb{R})}}{t^{1/2}},$$

which belongs to $L^1((0,R))$. So for every fixed $R > 0$,

$$\frac{1}{\pi}\int_0^R \frac{|v_n|^2(x-t)-|v_n|^2(x+t)}{t}\,\mathrm{d}t \to \frac{1}{\pi}\int_0^R \frac{|v|^2(x-t)-|v|^2(x+t)}{t}\,\mathrm{d}t$$

as $n \to +\infty$.

The integral over $(R,+\infty)$ can be bounded uniformly by

$$\begin{aligned}\left|\frac{1}{\pi}\int_R^\infty \frac{|v_n|^2(x-t)-|v_n|^2(x+t)}{t}\,\mathrm{d}t\right| &\le \frac{1}{\pi}\left(\int_R^\infty t^{-2}\,\mathrm{d}t\right)^{1/2}\|v_n^2\|_{L^2(\mathbb{R})}\\ &\lesssim \frac{1}{R}\|v_n\|^2_{H^1(\mathbb{R})},\end{aligned}$$

which becomes arbitrarily small (independent of $n$) as we choose $R$ correspondingly large.

So we have seen that for every $x$ in $\mathbb{R}$ the integral in (3.7) really converges to $\mathrm{H}(|v|^2)(x)$ and therefore (3.6) holds true, which concludes the proof. □

The next lemma implies that minimizing sequences of $E$ with fixed $L^2$- mass $N \in (0, 2\pi)$, are always bounded in $X$. In fact we will state something slightly more general.

**Lemma 3.4.** *Let $(v_n)$ be a sequence in $H^1(\mathbb{R})$ satisfying the two a-priori bounds*

$$\widetilde{E}(v_n) \le C \quad \textit{and} \quad \|v_n\|^2_{L^2(\mathbb{R})} \le N \quad \textit{for every } n \in \mathbb{N}$$

*for some given constants $C > 0$ and $N \in (0, 2\pi)$. Then $(\|\partial_x v_n\|_{L^2(\mathbb{R})})$ is bounded and in particular $(v_n)$ is a bounded sequence in $H^1(\mathbb{R})$.*

*Proof.* We argue by contradiction. We assume the conditions of the lemma are satisfied, but $\|\partial_x v_n\|_{L^2(\mathbb{R})} \to +\infty$ up to subsequences. We split the proof into the following three steps in order to reveal the contradiction.

**Step 1.** We define the sequence $(w_n)$ by a dilation of $(v_n)$, preserving the $L^2$-norm, as follows.

$$w_n(x) = \lambda_n^{1/2} v_n(\lambda_n x), \quad \text{with} \quad \lambda_n = \|\partial_x v_n\|^{-1}_{L^2(\mathbb{R})} \to 0 \quad \text{as } n \to +\infty.$$

In particular

$$\|w_n\|^2_{L^2(\mathbb{R})} = \|v_n\|^2_{L^2(\mathbb{R})} \le N \quad \text{and} \quad \|\partial_x w_n\|^2_{L^2(\mathbb{R})} = \lambda_n^2\|\partial_x v_n\|^2_{L^2(\mathbb{R})} = 1 \tag{3.8}$$

for every $n \in \mathbb{N}$.

As in the proof of Corollary 2.2 with $w_n = v_{\lambda_n}$

$$\widetilde{E}(w_n) = \lambda_n^2 \widetilde{E}(v_n)$$

holds. Since by (2.6) $\widetilde{E}$ is nonnegative and by assumption $(\widetilde{E}(v_n))$ is bounded from above this proves that

$$\lim_{n\to+\infty} \widetilde{E}(w_n) = 0. \tag{3.9}$$

**Step 2.** We show that there exists a sequence $(y_n)$ in $\mathbb{R}$ and a function $w \equiv 0$ in $H^1(\mathbb{R})$, such that $w_n(\,\cdot + y_n)$ weakly converges to $w$ in $H^1(\mathbb{R})$. This is a direct consequence of the $p, q, r$ theorem and the nonzero weak convergence after translations (see [22] [Exercise 2.22 and Theorem 8.10], we recall the statements in Theorem A.5 and A.6), with $p = 2$, $q = 6$ and $r = 8$. In this step of the proof we verify the assumptions of those theorems.

First we notice that by (3.8)

$$\|w_n\|_{L^2(\mathbb{R})} \leq \sqrt{N} \quad \text{and} \quad \|w_n\|_{L^8(\mathbb{R})} \lesssim \|w_n\|_{H^1(\mathbb{R})} \leq \sqrt{N+1}.$$

It remains to show that there exist $\alpha > 0$ and $K \in \mathbb{N}$ such that $\|w_n\|_{L^6(\mathbb{R})} \geq \alpha$ for every $n \geq K$.

We assume by contradiction that this does not hold, i.e. $\|w_n\|_{L^6(\mathbb{R})} \to 0$ up to subsequences. Therefore by the identity $\partial_x \mathrm{H}(f) = (-\Delta)^{1/2} f$ for $f \in H^1(\mathbb{R})$, Hölder's inequality and the $L^p$-boundedness of the Hilbert transform for $1 < p < +\infty$, we obtain

$$\begin{aligned} \left|\left\langle |w_n|^2, (-\Delta)^{1/2}|w_n|^2 \right\rangle\right| &= \left|\left\langle \partial_x |w_n|^2, \mathrm{H}(|w_n|^2) \right\rangle\right| \\ &\leq 2\|\partial_x w_n\|_{L^2(\mathbb{R})} \|w_n\|_{L^6(\mathbb{R})} \|\mathrm{H}(|w_n|^2)\|_{L^3(\mathbb{R})} \\ &\lesssim \|w_n\|^3_{L^6(\mathbb{R})}. \end{aligned}$$

The last inequality holds since by construction $\|\partial_x w_n\|_{L^2(\mathbb{R})} = 1$. If we consider $\widetilde{E}(w_n)$, keeping the estimate above in mind, we find that $\|w_n\|_{L^6(\mathbb{R})} \to 0$ implies

$$\widetilde{E}(w_n) = \frac{1}{2}\|\partial_x w_n\|^2_{L^2(\mathbb{R})} - \frac{1}{4}\left\langle |w_n|^2, (-\Delta)^{1/2}|w_n|^2 \right\rangle + \frac{1}{24}\|w_n\|^6_{L^6(\mathbb{R})} \to \frac{1}{2},$$

which contradicts (3.9).

**Step 3.** We use Step 2 to conclude in the following way. By Lemma 2.3 the energy $\widetilde{E}$ is invariant under translations and by Lemma 3.3 above weakly lower semicontinuous in $H^1(\mathbb{R})$. Using this and the weak convergence $w_n(\,\cdot + y_n) \rightharpoonup w \neq 0$ in $H^1(\mathbb{R})$ we deduce

$$0 = \lim_{n \to +\infty} \widetilde{E}(w_n) = \lim_{n \to +\infty} \widetilde{E}(w_n(\,\cdot + y_n)) \geq \widetilde{E}(w) \geq 0.$$

Hence $\widetilde{E}(w) = 0$. By Theorem 2.1 we know that every nontrivial vanishing point of $\tilde{E}$ is given by

$$w(x) = e^{i\alpha} \left( \frac{2\lambda}{1 + \lambda^2 (x - x_0)^2} \right)^{1/2}, \quad \text{with arbitrary } \lambda > 0 \text{ and } \alpha, x_0 \in \mathbb{R}.$$

In particular $\|w\|^2_{L^2(\mathbb{R})} = 2\pi$. This is a contradiction to $\|w_n\|^2_{L^2(\mathbb{R})} \leq N < 2\pi$ due to the weak lower semicontinuity of the $L^2$-norm. □

In the previous two lemmas we have seen that for $E$ and $\tilde{E}$ the boundedness of minimizing sequences and weak lower semicontinuity reduce to the same arguments. In contrast, for the existence of a strongly convergent subsequence in $L^2(\mathbb{R})$ of a minimizing sequence, it is crucial to consider sequences in $X$ instead of $H^1(\mathbb{R})$, since the additional condition on the decay leads to compactness of $X$ in $L^2(\mathbb{R})$. This is established in the following proposition.

**Proposition 3.5.** *Let $V$ be as in Assumption **(A)**. Then*

$$X := \left\{ v \in H^1(\mathbb{R}) \mid \sqrt{V} v \in L^2(\mathbb{R}) \right\},$$

*with the corresponding norm*

$$\|v\|_X^2 = \|v\|_{L^2(\mathbb{R})}^2 + \|\partial_x v\|_{L^2(\mathbb{R})}^2 + \|\sqrt{V} v\|_{L^2(\mathbb{R})}^2$$

*is compactly embedded in $L^2(\mathbb{R})$.*

*Proof.* Let $(v_n)$ be a bounded sequence in $X$. Then $(v_n)$ is bounded in $H^1(\mathbb{R})$ as well. By Rellich compactness there exists a subsequence, still denoted by $(v_n)$, that strongly converges in $L^2_{\text{loc}}(\mathbb{R})$. Moreover by the uniform bound $\int_{\mathbb{R}} V|v_n|^2 \, \mathrm{d}x \leq \|v_n\|_X^2 \lesssim 1$ and the assumptions on $V$, stated in Assumption **(A)**, we obtain that $\int_{|x|>R} |v_n|^2 \, \mathrm{d}x$ becomes arbitrary small, independent of $n$, as we choose $R$ sufficiently large. Hence $(v_n)$ actually converges strongly on the whole space $L^2(\mathbb{R})$. □

*Proof of Theorem 3.1.* Let $N \in (0, 2\pi)$ be fixed. Take a minimizing sequence $(v_n)$ of $E$ in $X$ satisfying $\|v_n\|_{L^2(\mathbb{R})}^2 = N$. In particular $(E(v_n))$ is bounded from above and hence by (3.2) there exists $C > 0$ such that

$$\widetilde{E}(v_n) + \frac{1}{2}\|\sqrt{V} v_n\|_{L^2(\mathbb{R})}^2 \leq C.$$

We obtain as a direct consequence that $(v_n)$ obeys the assumptions of Lemma 3.4 and hence $(v_n)$ is bounded in $H^1(\mathbb{R})$. This together with $\tilde{E} \geq 0$ and the inequality above again, $(v_n)$ is actually bounded in $X$ and therefore (up to passing to a subsequence if necessary) weakly converges to an element $v \in X$. Since $X$ is compactly embedded in $L^2(\mathbb{R})$, as we established in Proposition 3.5 above, $v_n \to v$ strongly in $L^2(\mathbb{R})$ (again up to subsequences) and in particular $\|v\|_{L^2(\mathbb{R})}^2 = N$. Therefore we may conclude by the weak lower semicontinuity of the energy $\widetilde{E}$ and the $L^2$ norm, as follows.

$$\begin{aligned} \liminf_{n\to+\infty} E(v_n) &\geq \liminf_{n\to+\infty} \widetilde{E}(v_n) + \liminf_{n\to+\infty} \frac{1}{2}\|\sqrt{V} v_n\|_{L^2(\mathbb{R})}^2 \\ &\geq \widetilde{E}(v) + \frac{1}{2}\|\sqrt{V} v\|_{L^2(\mathbb{R})}^2 = E(v). \end{aligned}$$

This proves that $v$ is a minimizer under the given constraint and concludes our proof. □

## 3.2 Existence of Excited States

In this section we prove the existence of infinitely many excited states of $L^2$-mass $N$, for all $N$ in $(0, 2\pi)$. Moreover we establish an exponential decay rate. If $V$ additionally possesses certain polynomial growth rates, we show in Proposition 3.16 that the excited states actually exhibit superexponential decay.

These existence and decay results are based on well-known concepts adapted to our case.

### Existence of infinitely many excited states

We first establish the main result.

**Theorem 3.6.** *Let $N \in (0, 2\pi)$. Then there exist infinitely many excited states to (CM) with $L^2$-mass $N$. That is to say that the set of critical points*

$$\left\{ v \in X \,\middle|\, E'(v) = 0,\ \|v\|^2_{L^2(\mathbb{R})} = N \right\}$$

*is infinite.*

For the proof we implement the concept of index theory. More specifically we will use the definition of the Krasonelskii genus, which can be understood as a generalization of the dimension of a linear space. For even functionals on a complete symmetric $\mathcal{C}^{1,1}$-manifold in a Banach space, there is a non-linear analog of the Courant-Fischer minimax principle for linear eigenvalue problems. For a more detailed treatment of the Krasonelskii genus see [26] [Chapter II. 5].

For the convenience of the reader, we recall here the main definition and two key results.

**Definition 3.7** ( [26] Definition 5.1)**.** Let $X$ be some Banach space and $A$ be a nonempty, closed, symmetric subset of $X$. Then the Krasonelskii genus is defined by

$$\gamma(A) = \inf\left\{ m \in \mathbb{N} \,\middle|\, \exists h \in \mathcal{C}^0(A;\ \mathbb{R}^m \setminus \{0\}),\ h(-u) = -h(u)\ \forall u \in A \right\},$$

where the infimum of the empty set is defined to be $+\infty$.

**Theorem 3.8** ( [26] Theorem 5.7)**.** *Suppose $E$ is an even $\mathcal{C}^1$ functional on a complete symmetric $\mathcal{C}^{1,1}$-manifold $M \subset X \setminus \{0\}$ in some Banach space $X$. Also suppose $E$ satisfies the Palais-Smale condition and is bounded from below on $M$. Let $\hat{\gamma}(M) = \sup\left\{ \gamma(K) \,\middle|\, K \subset M \text{ compact and symmetric } \right\}$. Then the functional $E$ possesses at least $\hat{\gamma}(M)$ pairs of critical points.*

We will apply this theorem to $M = \mathbb{S}_N := \left\{ v \in X \,\middle|\, \|v\|^2_{L^2(\mathbb{R})} = N \right\}$. Notice that by Lemma 3.2, the energy functional $E$ is even and bounded from below by 0. So Theorem 3.6 follows immediately by the theorem above if we verify that $E$ satisfies the Palais-Smale condition and $\hat{\gamma}(\mathbb{S}_N) = +\infty$.

The last statement is a consequence of the following proposition.

**Proposition 3.9** ( [26] Proposition 5.2)**.** *For any bounded symmetric neighborhood $\Omega$ of the origin in $\mathbb{R}^n$ there holds: $\gamma(\partial\Omega) = n$.*

**Lemma 3.10.** $\hat{\gamma}(\mathbb{S}_N) = +\infty$.

*Proof.* Take $\{\varphi_j\}_{j \in \mathbb{N}}$ a family of $\mathcal{C}_c^\infty(\mathbb{R})$ functions forming an orthonormal system in $L^2(\mathbb{R})$.

For every $n \in \mathbb{N}$ define a compact and symmetric subset of $\mathbb{S}_N$ by

$$K_n = \left\{ \sum_{j=1}^n \lambda_j \varphi_j \,\middle|\, \lambda_j \in \mathbb{R} \text{ and } \sum_{j=1}^n \lambda_j^2 = N \right\}.$$

$K_n$ is homeomorphic to the unit sphere $\mathbb{S}$ in $\mathbb{R}^n$, by a linear homeomorphism. So in particular $\gamma(K_n) = \gamma(\mathbb{S})$ and hence by Proposition 3.9, $\gamma(K_n) = n$.

Since $n \in \mathbb{N}$ was arbitrary $\hat{\gamma}(\mathbb{S}_N) = +\infty$. □

**Lemma 3.11.** *Let $N \in (0, 2\pi)$. Then $E$ satisfies the Palais-Smale condition on $\mathbb{S}_N$.*

*Proof.* Let $(v_n)_n$ be a Palais-Smale sequence for $E$ in $\mathbb{S}_N$, i.e. $(E(v_n))$ is bounded in $\mathbb{R}$ and $E'[v_n] \to 0$ in the dual space of $X$. The aim is to show that $(v_n)$ has a strongly convergent subsequence in $X$.

By the boundedness of $(E(v_n))$ and Lemma 3.4 we already know that $(v_n)$ is bounded in $X$. Hence, by passing to a subsequence if necessary, $v_n \rightharpoonup v$ weakly in $X$. To prove that actually $v_n \to v$ strongly in $X$, we have to verify the convergence of the norms, respectively

$$\begin{aligned} \|v_n\|^2_{L^2(\mathbb{R})} + \|\partial_x v_n\|^2_{L^2(\mathbb{R})} + &\|\sqrt{V} v_n\|^2_{L^2(\mathbb{R})} \\ &\to \|v\|^2_{L^2(\mathbb{R})} + \|\partial_x v\|^2_{L^2(\mathbb{R})} + \|\sqrt{V} v\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.10}$$

First notice that $v_n \to v$ strongly in $L^2(\mathbb{R})$ by the compactness of the embedding $X \subset L^2(\mathbb{R})$ (see Proposition 3.5). Next, consider

$$\begin{aligned} E'[v_n](w) = \Re\big[&\langle \partial_x v_n, \partial_x w\rangle + \langle \sqrt{V} v_n, \sqrt{V} w\rangle \\ &- \big\langle \big((-\Delta)^{1/2}|v_n|^2\big) v_n, w\big\rangle + \tfrac{1}{4}\langle |v_n|^4 v_n, w\rangle\big]. \end{aligned}$$

For $w = v_n$, we obtain

$$\begin{aligned} &\|\partial_x v_n\|^2_{L^2(\mathbb{R})} + \|\sqrt{V} v_n\|^2_{L^2(\mathbb{R})} \\ &\qquad = E'[v_n](v_n) + \Re\big[\big\langle \big((-\Delta)^{1/2}|v_n|^2\big) v_n, v_n\big\rangle - \tfrac{1}{4}\langle |v_n|^4 v_n, v_n\rangle\big] \\ &\qquad = C(n) + D(n) + \Re\big[\langle \partial_x v_n, \partial_x v\rangle + \langle \sqrt{V} v_n, \sqrt{V} v\rangle\big], \end{aligned}$$

where

$$\begin{aligned} C(n) &= E'[v_n](v_n) - E'[v_n](v) \text{ and} \\ D(n) &= \Re\big[\big\langle \big((-\Delta)^{1/2}|v_n|^2\big) v_n, v_n - v\big\rangle - \tfrac{1}{4}\,\langle |v_n|^4 v_n, v_n - v\rangle\big]. \end{aligned}$$

$C(n)$ converges to 0, since $E'[v_n] \to 0$ and $(v_n)$ is a bounded sequence in $X$. Using the boundedness of $(v_n)$ in $H^1(\mathbb{R})$ again, we can easily check that

$$|v_n|^4 v_n \text{ and } \big((-\Delta)^{1/2}|v_n|^2\big) v_n$$

are both bounded in $L^2(\mathbb{R})$. Thus, by the strong convergence of $v_n \to v$ in $L^2(\mathbb{R})$, the quantity $D(n)$ converges to 0 as well. Together with the weak convergence $v_n \rightharpoonup v$ in $X$, we therefore obtain

$$\begin{aligned} \|\partial_x v_n\|^2_{L^2(\mathbb{R})} + \|\sqrt{V} v_n\|^2_{L^2(\mathbb{R})} &= C(n) + D(n) + \Re\big[\langle \partial_x v_n, \partial_x v\rangle + \langle \sqrt{V} v_n, \sqrt{V} v\rangle\big] \\ &\to \|\partial_x v\|^2_{L^2(\mathbb{R})} + \|\sqrt{V} v\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

In combination with the strong convergence $v_n \to v$ in $L^2(\mathbb{R})$ this shows (3.10), which guarantees the strong convergence of $v_n \to v$ in $X$ and therefore concludes the proof. □

*Proof of Theorem 3.6.* The proof directly follows from Theorem 3.8 and Lemmas 3.10 and 3.11. □

### Decay rates

We first establish $L^2$-exponential decay of critical points, for a strictly positive external potential $V$ obeying Assumption **(A)**.

**Lemma 3.12.** *Assume that $V > 0$ satisfies Assumption **(A)**. Let $v \in X$ be a critical point of $E$. Then $ve^{\alpha x} \in L^2(\mathbb{R})$ for every $\alpha > 0$.*

For the proof we refer to [25] [Theorem XIII.70], where exponential decay of eigenfunctions of the operator $H = (-\Delta) + V_1 + V_2$ under suitable conditions is stated. In our case $V_1 := V > 0$ trivially satisfies the assumptions $V_1 \in L^1_{\text{loc}}(\mathbb{R})$ and $V_1(x) \to +\infty$ as $|x| \to \infty$, due to the stronger conditions given in Assumption **(A)**. However the reader should be aware of the fact that the hypothesis $V_2 \in L^{n/2}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ can be replaced by $V_2 \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$, in the case $n = 1$. We will give a few remarks to explain this claim below. In our case we consider $V_2$ to be the self generated term $V_2 = -(-\Delta)^{1/2}(|v|^2) + \frac{1}{4}|v|^4$, which actually belongs to $L^2(\mathbb{R})$.

A key ingredient of the proof is to argue that $H = (-\Delta) + V_1 + V_2$ is an operator with compact resolvent by proving that $V_2$ is $-\Delta$-form bounded with relative bound 0. This is to say that

$$|\langle u, V_2 u\rangle| \leq \varepsilon \langle \partial_x u, \partial_x u\rangle + C_\varepsilon \langle u, u\rangle,$$

for every $\varepsilon > 0$ and corresponding $C_\varepsilon > 0$ independent of $u \in H^1(\mathbb{R})$. For $n = 1$ and $V_2 \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ this can be shown by an application of Hölder, Gagliardo-Nirenberg and Young's inequality.

To establish superexponential decay of critical points we have to strengthen our conditions on $V$ to receive higher regularity. Before we turn to the precise statement we give the following two auxiliary lemmas.

**Lemma 3.13.** *Let $V$ be of class $\mathcal{C}^\infty$ satisfying Assumption **(A)** as well as the following additional conditions for some real numbers $m > 2$ and $R \geq 1$:*

(i) *There exist $0 < D_1 \leq D_2$ such that $D_1\langle x\rangle^m \leq V(x) \leq D_2\langle x\rangle^m$ for every $|x| \geq R$.*

(ii) *For every $\alpha \in \mathbb{N}$ there exists $C_\alpha \geq 0$ such that $|\partial_{x^\alpha} V(x)| \leq C_\alpha \langle x\rangle^{m-\alpha}$ for every $x \in \mathbb{R}$.*

*Then every critical point $v$ of (CM) belongs to $H^2(\mathbb{R})$.*

*Proof.* Lemma 3.13 is a direct consequence of [29] [Lemma 2.4], where it is proven that

$$\|(-\partial_{xx} + V)v\|_{L^2(\mathbb{R})} + \|v\|_{L^2(\mathbb{R})} \simeq \|v\|_{H^2(\mathbb{R})} + \|\langle x\rangle^m v\|_{L^2(\mathbb{R})},$$

for every $v \in \mathcal{S}(\mathbb{R})$. The statement now follows by a density argument and the fact that a critical point $v \in X$ of (CM) satisfies

$$(-\partial_{xx} + V)v = (-\Delta)^{1/2}(|v|^2)v - \frac{1}{4}|v|^4 v + \lambda v \in L^2(\mathbb{R}),$$

where $\lambda$ denotes the corresponding Lagrange multiplier. □

*Remark* 3.14. In the special case of the harmonic potential $V(x) = x^2$, the implication

$$(-\partial_{xx} + x^2)v \in L^2(\mathbb{R}) \quad \Rightarrow \quad \partial_{xx}v \in L^2(\mathbb{R})$$

holds true as well. Here a $L^2$-bound of $\partial_{xx}v$ can be found by using the explicit representation of $\partial_{xx}v$ in terms of Hermite functions.

**Lemma 3.15.** *Let $Q\colon \mathbb{R} \to \mathbb{C}$ be Lipschitz continuous and $f\colon \mathbb{R} \to \mathbb{R}$ be nonnegative and monotone increasing in $|x|$. Then the following implication holds:*

$$\int_{\mathbb{R}} f(x)|Q(x)|^2 \,\mathrm{d}x < +\infty \quad \Rightarrow \quad f|Q|^3 \in L^\infty(\mathbb{R}).$$

*Proof.* Assume that $Q$ is Lipschitz continuous with Lipschitz constant $L > 0$. Further assume $f$ to satisfy the conditions of the lemma.

If $Q \equiv 0$ there is nothing to prove, so without loss of generality there exists an element $x_0 \in \mathbb{R}$, such that $Q(x_0) \neq 0$. We choose $\varepsilon = |Q(x_0)|/2L$. Then for every $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ the estimate

$$|Q(x_0)| - |Q(x)| \leq L|x_0 - x| \leq L\varepsilon = \frac{1}{2}|Q(x_0)|$$

holds and in particular

$$\frac{1}{2}|Q(x_0)| \leq |Q(x)| \quad \text{for } x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

Let us assume that $x_0 \geq 0$ (if $x_0 < 0$ we consider the integral over $(x_0 - \varepsilon, x_0)$ and proceed in the same way). Then by the inequality above and the definition of $\varepsilon$ we obtain the estimate

$$\begin{aligned} C &:= \int_{\mathbb{R}} f(x)|Q(x)|^2 \,\mathrm{d}x \geq \int_{x_0}^{x_0+\varepsilon} f(x)|Q(x)|^2 \,\mathrm{d}x \\ &\geq \frac{\varepsilon}{4} f(x_0)|Q(x_0)|^2 = \frac{1}{8L} f(x_0)|Q(x_0)|^3, \end{aligned}$$

which proves that $f(x_0)|Q(x_0)|^3 \leq 8CL$. Since this bound does not depend on $x_0$ this concludes the proof. □

**Proposition 3.16.** *Let $V$ satisfy the conditions of Lemma 3.13 for some real number $m > 2$ or let $V(x) = x^2$ be the harmonic potential and $m = 2$. Define $s = m/2 + 1 \geq 2$. Then there exists some $\alpha > 0$, such that every critical point $v$ of (CM) has the following superexponential decay.*

$$|v(x)| \leq Ce^{-\alpha|x|^s} \quad \textit{for every } x \in \mathbb{R},$$

*where $C$ may depend on $v$.*

*Proof.* To show the result we generalize the proof of Theorem 8.1.1 in [8]. Assume that $v$ is a critical point of (CM). By Lemma 3.13, $v$ belongs to $H^2(\mathbb{R})$ and is therefore Lipschitz continuous.

Our aim is to show that

$$\int_{\mathbb{R}} e^{\mu|x|^s}|v(x)|^2\,\mathrm{d}x < +\infty, \tag{3.11}$$

for some $\mu > 0$. If this integrability condition holds true we can apply Lemma 3.15 with $f(x) = e^{\mu|x|^s}$ and conclude that

$$|v(x)| \le Ce^{-\frac{\mu}{3}|x|^s},$$

which is exactly the statement of our proposition.

The strategy behind showing (3.11) is to first approximate $e^{\mu|x|^s}$ by

$$\theta_{\mu,\varepsilon}(x) = e^{\frac{\mu|x|^s}{1+\varepsilon|x|^s}}$$

and then construct a uniform in $\varepsilon$ bound $C = C(v, s, \mu)$, such that

$$\int_{\mathbb{R}} \theta_{\mu,\varepsilon}(x)|v(x)|^2\,\mathrm{d}x \le C \tag{3.12}$$

holds. Letting $\varepsilon \to 0$ we immediately obtain (3.11) by monotone convergence theorem. The remaining part of this proof is to show (3.12).

We consider the equation satisfied by a critical point $v$, i.e.

$$-\partial_{xx}v + Vv - Wv = 0, \quad \text{where} \quad W = (-\Delta)^{1/2}|v|^2 - \frac{1}{4}|v|^4 + \lambda$$

for a Lagrange multiplier $\lambda$. We integrate this against $\overline{v}\,\theta_{\mu,\varepsilon}$ to obtain

$$\begin{aligned} \int_{\mathbb{R}} |\partial_x v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x + \int_{\mathbb{R}} \partial_x v\,\overline{v}\,\partial_x\theta_{\mu,\varepsilon}\,\mathrm{d}x + \int_{\mathbb{R}} V|v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x \\ = \int_{\mathbb{R}} W|v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Recall that $v$ belongs to $H^2(\mathbb{R})$ according to Lemma 3.13, whence it follows by Sobolev embedding theorem that $W$ is an element of $L^\infty(\mathbb{R})$. So the right-hand side is bounded from above by

$$\text{RHS} \le \|W\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x$$

To find a lower bound of the left-hand side of (3.13) we use the estimate $|\partial_x\theta_{\mu,\varepsilon}| \le \theta_{\mu,\varepsilon}\mu s|x|^{s-1}$ and apply Young's inequality in the following way,

$$\begin{aligned} \left|\int_{\mathbb{R}} \partial_x v\,\overline{v}\,\partial_x\theta_{\mu,\varepsilon}\,\mathrm{d}x\right| &\le \int_{\mathbb{R}} 2^{\frac{1}{2}}|\partial_x v|\,\theta_{\mu,\varepsilon}^{\frac{1}{2}}\,\frac{\mu s}{2^{\frac{1}{2}}}|x|^{s-1}|v|\,\theta_{\mu,\varepsilon}^{\frac{1}{2}}\,\mathrm{d}x \\ &\le \int_{\mathbb{R}} |\partial_x v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x + \frac{(\mu s)^2}{4}\int_{\mathbb{R}} |x|^{2s-2}|v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x. \end{aligned}$$

In particular the left-hand side of (3.13) is bounded from below by

$$\text{LHS} \ge \int_{\mathbb{R}} \left(V - \frac{(\mu s)^2}{4}|x|^{2s-2}\right)|v|^2\,\theta_{\mu,\varepsilon}\,\mathrm{d}x$$

By our additional assumption on $V$ there esist $D_1 > 0$ and a radius $R \geq 1$ such that $V(x) \geq D_1|x|^m = D_1|x|^{2s-2}$ for every $|x| \geq R$. Now we fix $\mu > 0$ sufficiently small such that

$$\left(D_1 - \frac{(\mu s)^2}{4}\right) > 0.$$

Notice that $\mu$ only depends on $D_1$ and $s$ and hence on the external potential $V$ but is independent on the critical point $v$. Next we choose $\widetilde{R} \geq R$ accordingly large enough such that

$$\frac{1}{2}\left(D_1 - \frac{(\mu s)^2}{4}\right)\widetilde{R}^{2s-2} > \|W\|_{L^\infty(\mathbb{R})}.$$

Using this and the bound $\theta_{\mu,\varepsilon}(x) \leq e^{\mu|x|^s}$ in the previous estimates, we obtain

$$\begin{aligned}
\text{RHS} &\leq \|W\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x \\
&\leq \|W\|_{L^\infty(\mathbb{R})} \left(\int_{|x|\geq \widetilde{R}} |v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x + e^{\mu \widetilde{R}^s}\|v\|^2_{L^2(\mathbb{R})}\right) \\
&\leq \frac{1}{2}\left(D_1 - \frac{(\mu s)^2}{4}\right)\widetilde{R}^{2s-2} \int_{|x|\geq \widetilde{R}} |v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x + C_1
\end{aligned}$$

and

$$\begin{aligned}
\text{LHS} &\geq \int_{|x|\geq \widetilde{R}} \left(V - \frac{(\mu s)^2}{4}|x|^{2s-2}\right)|v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x \\
&\quad + \int_{|x|\leq \widetilde{R}} \left(V - \frac{(\mu s)^2}{4}|x|^{2s-2}\right)|v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x \\
&\geq \int_{|x|\geq \widetilde{R}} \left(D_1 - \frac{(\mu s)^2}{4}\right)|x|^{2s-2}|v|^2\, \theta_{\mu,\varepsilon}\, \mathrm{d}x \\
&\quad - \frac{(\mu s)^2}{4}\widetilde{R}^{2s-2} e^{\mu\widetilde{R}^s}\|v\|^2_{L^2(\mathbb{R})} \\
&\geq \left(D_1 - \frac{(\mu s)^2}{4}\right)\widetilde{R}^{2s-2}\int_{|x|\geq \widetilde{R}} |v|^2 \theta_{\mu,\varepsilon}\, \mathrm{d}x - C_2.
\end{aligned}$$

By comparing the left- and right-hand side we obtain

$$\int_{|x|\geq \widetilde{R}} |v|^2 \theta_{\mu,\varepsilon}\, \mathrm{d}x \leq \frac{2(C_1 + C_2)}{\left(D_1 - \frac{(\mu s)^2}{4}\right)\widetilde{R}^{2s-2}}.$$

Due to the continuity of $v$ and the bound $\theta_{\mu,\varepsilon}(x) \leq e^{\mu\widetilde{R}^s}$ for $|x| \leq \widetilde{R}$, we thus find the desired uniform in $\varepsilon$ bound which is

$$\int_{\mathbb{R}} |v|^2 \theta_{\mu,\varepsilon}\, \mathrm{d}x \leq C.$$

This is (3.12) and hence the proof is complete. □

# Chapter 4

# Main Results for the Nonlocal Liouville Equation in $\mathbb{R}$

This chapter is a more detailed version of the work published in [3].

We study the one-dimensional nonlocal Liouville equation (L) from the introduction, which reads

$$(-\Delta)^{1/2} w = K e^{w} \quad \text{in } \mathbb{R} \tag{L}$$

subject to the integrability condition

$$\lambda := \int_{\mathbb{R}} K(x) e^{w(x)} \, \mathrm{d}x < +\infty. \tag{4.1}$$

In this chapter we always assume that the $Q$-curvature function $K$ satisfies Assumption **(B)**, which was imposed in the introduction and will be recalled right below. Moreover we remark that we always deal with real-valued functions here.

**Assumption (B).** *We make the following assumptions on $K \colon \mathbb{R} \to \mathbb{R}$.*

(i) *$K$ is strictly positive, even and monotone decreasing in $|x|$.*

(ii) *$K$ is continuously differentiable.*

(iii) *There exist $C > 0$ and $\delta > 0$ such that $K$ satisfies the pointwise bound*

$$\sqrt{K(x)} + |x \partial_x \sqrt{K(x)}| \leq C \langle x \rangle^{-1/2-\delta},$$

*where $\langle x \rangle = \sqrt{1 + x^2}$.*

The aim is to prove existence and uniqueness and establish regularity, radial symmetry and monotonicity of solutions to (L) in $L_{1/2}(\mathbb{R})$.

Our main results are given in the following two theorems below.

**Theorem 4.1.** *Suppose $K$ satisfies Assumption **(B)** and let $w \in L_{1/2}(\mathbb{R})$ be a solution of (L) satisfying (4.1). Then the following properties hold.*

*(i) **Regularity and Universal Bound on $\lambda$:** We have $w \in \mathcal{C}^{1,1/2}_{\mathrm{loc}}(\mathbb{R})$ and $\lambda = \int_{\mathbb{R}} K e^{w} \, \mathrm{d}x$ satisfies $0 < \lambda < 2\pi$.*

*(ii)* ***Symmetry and Monotonicity:*** *$w$ is even and decreasing in $|x|$, i. e., it holds $w(-x) = w(x)$ for all $x \in \mathbb{R}$ and $w(x) \geq w(y)$ whenever $|x| \leq |y|$.*

*(iii)* ***Existence:*** *For every $w_0 \in \mathbb{R}$, there exists a solution $w \in L_{1/2}(\mathbb{R})$ of (L) with $w(0) = w_0$ such that (4.1) holds.*

**Theorem 4.2** (**Uniqueness**). *Suppose $K$ satisfies Assumption **(B)**. If $w, \widetilde{w} \in L_{1/2}(\mathbb{R})$ are solutions of (L) satisfying (4.1), then it holds*

$$\widetilde{w}(0) = w(0) \quad \Rightarrow \quad \widetilde{w} \equiv w.$$

The regularity and asymptotic behaviour of $w$ as well as the universal bound on $\lambda$ follow by well-known arguments adapted to our case. Indeed the proof of the Hölder continuity of $w$ is very strongly inspired by [21], exploiting the fact that entire $s$-harmonic functions are affine, which was proven in the work of M. M. Fall in [14]. The asymptotic behaviour $w(x) = -\frac{\lambda}{\pi} \log|x| + O(1)$ is shown by the methods presented in [21] and uses some technical results given in [24]. Using the continuous differentiability of $w$, by a Pohozaev identity established in [27], we finally achieve the bound on the total $Q$-curvature $\lambda$.

To prove that solutions to (L) are symmetric-decreasing we adapt a moving planes argument for integral equations, which was initiated in the work of [11].

The proofs of existence and uniqueness are the most innovative part of the chapter. We have dedicated a subsection in the introduction to sketching the main arguments, to which we refer here.

## 4.1 Regularity, Asymptotics and Universal Bound

We first collect some results that can be deduced by adapting known arguments. In particular, the results in this section will imply that item (i) in Theorem 4.1 holds true.

Throughout this section, we always assume that $w \in L_{1/2}(\mathbb{R})$ solves (L) subject to (4.1), where $K$ satisfies Assumption **(B)**.

We start by giving some immediate facts about $K$.

**Lemma 4.3.** *It holds that $\sqrt{K}, K \in H^1(\mathbb{R})$ and $K \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

*Proof.* Since $\sqrt{K} \lesssim \langle x \rangle^{-1/2-\delta}$ for some $\delta > 0$, we readily see that $\sqrt{K} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, whence it follows that $K \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Furthermore, by the bound $|x\partial_x \sqrt{K}(x)| \lesssim \langle x \rangle^{-1/2-\delta}$ for some $\delta > 0$ and the fact that $\partial_x \sqrt{K}$ is continuous and hence locally bounded, we deduce that $\partial_x \sqrt{K} \in L^2(\mathbb{R})$ holds. Thus $\partial_x K = 2\sqrt{K}\partial_x\sqrt{K} \in L^2(\mathbb{R})$ since $\sqrt{K} \in L^\infty(\mathbb{R})$. This shows that $\sqrt{K}$ and $K$ both belong to $H^1(\mathbb{R})$. □

Next, we derive the following regularity result for $w$.

**Lemma 4.4.** *It holds that*

$$w(x) = \frac{1}{\pi} \int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{w(y)} \, \mathrm{d}y + C$$

*with some constant $C \in \mathbb{R}$. Moreover $w$ belongs to $\mathcal{C}^0(\mathbb{R})$.*

*Proof.* Define the function $\widetilde{w} := w(x) - \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) K(y)e^{w(y)}\,\mathrm{d}y$. Then $\widetilde{w} \in L_{1/2}(\mathbb{R})$ satisfies $(-\Delta)^{1/2}\widetilde{w} = 0$. Since $w \in L_{1/2}(\mathbb{R})$ it follows by [14] that $\widetilde{w}\colon \mathbb{R} \to \mathbb{R}$ is an affine function and hence constant. This proves the integral representation of $w$.

To conclude that $w$ belongs to $\mathcal{C}^0(\mathbb{R})$ we can adapt the first step of the arguments presented in [21], where regularity for solutions of the equation $(-\Delta)^{n/2}u = |x|^{n\alpha}e^{nu}$ in $\mathbb{R}^n$ with $\alpha > -1$ subject to the integrability condition $\int_{\mathbb{R}^n} |x|^{n\alpha}e^{nu} < +\infty$ is discussed.

For the reader's convenience, we state the necessary modifications for our case. First, we show that $e^w \in L^p_{\mathrm{loc}}(\mathbb{R})$ for any $p \in [1,\infty)$ by an '$\varepsilon$-regularity trick' as follows. Indeed, for any such $p \geq 1$, we can take $0 < \varepsilon < \frac{\pi}{p}$ and we split $Ke^w = f_1 + f_2$ with $f_1, f_2 \geq 0$ such that $f_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\|f_2\|_{L^1(\mathbb{R})} \leq \varepsilon$. Next, we write

$$\widetilde{w} = w_1 + w_2 + w_3$$

with the functions

$$w_i(x) := \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) f_i(y)\,\mathrm{d}y \quad \text{for } i = 1,2, \quad w_3 := w - w_1 - w_2.$$

We have that $w_1 \in \mathcal{C}^0(\mathbb{R})$ and $w_3$ is a constant by the singular integral representation above. For any $R > 0$ be given, we apply Jensen's inequality to find

$$\begin{aligned}
\int_{-R}^{R} e^{pw_2}\,\mathrm{d}x &= \int_{-R}^{R} \exp\left(\int_{\mathbb{R}} \frac{p\|f_2\|_{L^1(\mathbb{R})}}{\pi} \log\left(\frac{1+|y|}{|x-y|}\right) \frac{f_2(y)}{\|f_2\|_{L^1(\mathbb{R})}}\,\mathrm{d}y\right)\mathrm{d}x \\
&\leq \int_{-R}^{R}\int_{\mathbb{R}} \exp\left(\frac{p\|f_2\|_{L^1(\mathbb{R})}}{\pi} \log\left(\frac{1+|y|}{|x-y|}\right)\right) \frac{f_2(y)}{\|f_2\|_{L^1(\mathbb{R})}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{\|f_2\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} f_2(y) \int_{-R}^{R} \left(\frac{1+|y|}{|x-y|}\right)^{\frac{p\|f_2\|_{L^1(\mathbb{R})}}{\pi}} \mathrm{d}x\,\mathrm{d}y < +\infty,
\end{aligned}$$

since $p\|f_2\|_{L^1(\mathbb{R})} \leq p\varepsilon < \pi$ holds. This shows that $e^{w_2} \in L^p_{\mathrm{loc}}(\mathbb{R})$ and hence $e^w \in L^p_{\mathrm{loc}}(\mathbb{R})$ by the regularity of $w_1$ and $w_3$. Notice that since $K \in L^\infty(\mathbb{R})$, this implies that in particular $Ke^w \in L^1(\mathbb{R}) \cap L^2_{\mathrm{loc}}(\mathbb{R})$. Using the integral representation of $w$ again we therefore achieve continuity of $w$. □

The following lemma is devoted to the understanding of the asymptotic behaviour of $w$.

**Lemma 4.5.** *It holds that*

$$\lim_{|x|\to+\infty} \frac{w(x)}{\log|x|} = -\frac{\lambda}{\pi} \quad \text{for } \lambda = \int_{\mathbb{R}} K(y)e^{w(y)}\,\mathrm{d}y.$$

*Proof.* The limit follows from the discussions in [21] (summarized in [Remark 3.2]) and [24] [Lemma 2.4]. For the reader's convenience we give a detailed proof adapted to our problem.

Since the argument contains a lot of technicalities we split it into four steps. Moreover to simplify notation, we assume $C = 0$ in the integral equation of $w$, since this constant does not change the limit.

**Step 1.** By elementary computations we find the lower bound

$$w(x) \geq -\frac{\lambda}{\pi} \log|x| \quad \text{for } |x| \geq 1. \tag{4.2}$$

We just notice that for $|x| \geq 1$, we have $|x-y| \leq |x|+|y| \leq |x|(1+|y|)$ and thus $\log\left(\frac{1+|y|}{|x-y|}\right) \geq \log\left(\frac{1}{|x|}\right) = -\log|x|$. Therefore from the integral representation of $w$ we readily deduce the lower bound.

**Step 2.** We show that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\begin{aligned} w(x) \leq &-\left(\frac{\lambda}{\pi} - \varepsilon\right) \log|x| \\ &+ \frac{1}{\pi}\int_{x-1}^{x+1} \log\left(\frac{1}{|x-y|}\right) K(y) e^{w(y)}\, \mathrm{d}y \quad \text{for every } |x| \geq R. \end{aligned} \tag{4.3}$$

To simplify notation we will always write $f = Ke^w$ in this step. We again consider the integral representation for $w$. For a given $\varepsilon > 0$, we take a radius $R_0 > 3$ large enough such that

$$\frac{1}{\pi}\int_{|y|\geq R_0} f(y)\, \mathrm{d}y < \frac{\varepsilon}{3}. \tag{4.4}$$

For $|x| > 2R_0$ we split $\mathbb{R}$ into the three domains $\mathbb{R} = (-R_0, R_0) \cup A \cup B$, where $A = \{y \in \mathbb{R} \mid |x-y| \leq |x|/2 \text{ and } |y| \geq R_0\} = \{y \in \mathbb{R} \mid |x-y| \leq |x|/2\}$ and $B = \{y \in \mathbb{R} \mid |x-y| > |x|/2 \text{ and } |y| \geq R_0\}$.

Integrating over $(-R_0, R_0)$ we obtain

$$\begin{aligned} \frac{1}{\pi}\int_{-R_0}^{R_0} \log\left(\frac{|x|(1+|y|)}{|x-y|}\right) f(y)\, \mathrm{d}y &\leq \frac{1}{\pi}\int_{-R_0}^{R_0} f(y)\, \mathrm{d}y \log\left(\frac{|x|(1+R_0)}{|x|-R_0}\right) \\ &\leq \frac{\lambda}{\pi} \log\left(\frac{|x|(1+R_0)}{|x|-R_0}\right) \leq \frac{\lambda}{\pi}\log(2(1+R_0)). \end{aligned}$$

In the last step we used the condition $|x| > 2R_0$. According to this uniform bound on the set $|x| > 2R_0$ and by applying (4.4) we derive the following estimate for $|x|$ large enough:

$$\begin{aligned} &\frac{1}{\pi}\int_{-R_0}^{R_0} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\, \mathrm{d}y + \log|x| \frac{1}{\pi}\int_{-R_0}^{R_0} f(y)\, \mathrm{d}y \\ &\qquad \leq \frac{\lambda}{\pi}\log(2(1+R_0)) \leq \left(\frac{\varepsilon}{3} - \frac{1}{\pi}\int_{|y|\geq R_0} f(y)\, \mathrm{d}y\right) \log|x|. \end{aligned}$$

Therefore it holds

$$\frac{1}{\pi}\int_{-R_0}^{R_0} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\, \mathrm{d}y \leq \left(\frac{\varepsilon}{3} - \frac{1}{\pi}\int_{\mathbb{R}} f(y)\, \mathrm{d}y\right) \log|x| = \left(\frac{\varepsilon}{3} - \frac{\lambda}{\pi}\right) \log|x|,$$

for $|x|$ sufficiently large.

On $A$ we use that $1+|y| \le 1+3/2|x| \le 2|x|$ to find the estimates below.

$$\begin{aligned}\frac{1}{\pi}\int_A \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\,\mathrm{d}y \\ &\le \frac{1}{\pi}\int_{x-1}^{x+1} \log\left(\frac{1}{|x-y|}\right) f(y)\,\mathrm{d}y + \log(2|x|)\frac{1}{\pi}\int_A f(y)\,\mathrm{d}y \\ &\le \frac{1}{\pi}\int_{x-1}^{x+1} \log\left(\frac{1}{|x-y|}\right) f(y)\,\mathrm{d}y + \frac{\varepsilon}{3}\left(\log(2)+\log|x|\right).\end{aligned}$$

Here we again applied (4.4).

For $y \in B$ we always have $4|x-y| \ge |y|+1$. For $|y|+1 \le 2|x|$ we immediately see

$$4|x-y| \ge 2|x| \ge |y|+1,$$

whereas for $|y|+1 \ge 2|x|$ this can be shown by

$$4|x-y| \ge 4(|y|-|x|) \ge 4(|y|-(|y|+1)/2) = 2|y|-2 \ge |y|+1,$$

where the last step follows by $|y| \ge R_0 > 3$. So

$$\frac{1}{\pi}\int_B \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\,\mathrm{d}y \le \log(4)\frac{\varepsilon}{3}.$$

Combining the last three integral estimates we immediately find (4.3) for $R$ sufficiently large.

**Step 3.** We show that for any $q \in [1,+\infty)$ there exists some constant $C > 0$ such that

$$\int_{x-1}^{x+1} e^{qw(y)}\,\mathrm{d}y \le C \quad \text{for every } x \in \mathbb{R}. \tag{4.5}$$

To prove this estimate we use (4.3) from Step 2 with $\varepsilon = -\lambda/\pi$. So there exists $R > 0$ such that for every $|z| \ge R+1$

$$\begin{aligned} w(z) &\le \frac{1}{\pi}\int_{z-1}^{z+1} \log\left(\frac{1}{|z-y|}\right) K(y)e^{w(y)}\,\mathrm{d}y \\ &= \frac{1}{\pi}\int_{|y|\ge R} \mathbb{1}_{\{|y-z|\le 1\}}(y) \log\left(\frac{1}{|z-y|}\right) K(y)e^{w(y)}\,\mathrm{d}y \end{aligned}$$

holds. Since $Ke^w \in L^1(\mathbb{R})$, by enlarging $R > 0$ if necessary, we may additionally assume

$$\alpha := \int_{|y|\ge R} K(y)e^{w(y)}\,\mathrm{d}y < \pi/q.$$

Next, we use the estimate for $w(z)$ above and apply Jensen's inequality with

$\mathrm{d}\mu(y) = \frac{K(y)e^{w(y)}}{\alpha}\mathrm{d}y$ to obtain for $|x| > R+2$

$$\begin{aligned}
\int_{x-1}^{x+1} e^{qw(z)}\,\mathrm{d}z &\leq \int_{x-1}^{x+1} \exp\left(\frac{\alpha q}{\pi}\int_{|y|\geq R} \mathbb{1}_{\{|y-z|\leq 1\}}(y)\log\left(\frac{1}{|z-y|}\right)\,\mathrm{d}\mu(y)\right)\,\mathrm{d}z \\
&\leq \int_{x-1}^{x+1}\int_{|y|\geq R} \exp\left(\frac{\alpha q}{\pi}\mathbb{1}_{\{|y-z|\leq 1\}}(y)\log\left(\frac{1}{|z-y|}\right)\right)\,\mathrm{d}\mu(y)\,\mathrm{d}z \\
&\leq \int_{x-1}^{x+1}\int_{|y|\geq R} \left(1+\left(\tfrac{1}{|z-y|}\right)^{\frac{\alpha q}{\pi}}\right)\,\mathrm{d}\mu(y)\,\mathrm{d}z \\
&\leq \int_{|y|\geq R}\int_{x-1}^{x+1} \left(1+\left(\tfrac{1}{|z-x|}\right)^{\frac{\alpha q}{\pi}}\right)\,\mathrm{d}z\,\mathrm{d}\mu(y) \\
&= \int_{|y|\geq R}\left(2+\left(\tfrac{2}{1-\frac{\alpha q}{\pi}}\right)\right)\,\mathrm{d}\mu(y) = 2+\left(\tfrac{2}{1-\frac{\alpha q}{\pi}}\right).
\end{aligned}$$

Here in the last two steps we used that $\frac{\alpha q}{\pi} < 1$ and $\int_{|y|\geq R} 1\,\mathrm{d}\mu(y) = 1$. For $|x| \leq R+2$ we use the continuity of $w$ to obtain the uniform bound

$$\int_{x-1}^{x+1} e^{qw(z)}\,\mathrm{d}z \leq \int_{|z|\leq R+3} e^{qw(z)}\,\mathrm{d}z < +\infty.$$

Now (4.5) directly follows from the estimates above.

**Step 4.** We combine the previous steps to conclude. First we notice that by Hölder's inequality with $p, q \in (1, +\infty)$, $1/p + 1/q = 1$ and inequality (4.5) above, we find some $C > 0$ such that

$$\begin{aligned}
&\int_{x-1}^{x+1} \log\left(\frac{1}{|x-y|}\right) K(y)e^{w(y)}\,\mathrm{d}y \\
&\quad\leq \|K\|_{L^\infty(\mathbb{R})}\left(\int_{x-1}^{x+1}\log\left(\frac{1}{|x-y|}\right)^p\,\mathrm{d}y\right)^{1/p}\left(\int_{x-1}^{x+1} e^{qw(y)}\,\mathrm{d}y\right)^{1/q} \leq C
\end{aligned}$$

for every $x \in \mathbb{R}$.
Applying this to the upper bound (4.3) and recalling the lower bound (4.2) we conclude by taking the limit $|x| \to +\infty$. □

In the next lemma we establish the Hölder regularity stated in item (i) in Theorem 4.1.

**Lemma 4.6.** *We have that* $w_+ = \max\{w, 0\} \in L^\infty(\mathbb{R})$ *and* $Ke^w \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*. Finally, it holds that* $w \in C^{1,1/2}_{\mathrm{loc}}(\mathbb{R})$*.*

*Proof.* Clearly, the limit given in Lemma 4.5 implies that $w(x) < 0$ for $|x| \geq R$ with $R > 0$ sufficiently large. Since $w \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ by Lemma 4.4, we thus find $w^+ = \max\{w, 0\} \in L^\infty(\mathbb{R})$ and hence $e^w \in L^\infty(\mathbb{R})$. By our assumptions on $K$, this implies

$$0 < K(x)e^{w(x)} \leq C\langle x\rangle^{-1-2\delta} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

for some $\delta > 0$.

According to Lemma A.2, $w$ is weakly differentiable and $\partial_x w = -\mathrm{H}(Ke^w) \in L^2(\mathbb{R})$. We obtain $\partial_x(Ke^w) = \partial_x K e^w + Ke^w \partial_x w \in L^2(\mathbb{R})$ as a direct consequence. In particular $Ke^w \in H^1(\mathbb{R})$ which yields $\partial_{xx} w = -\partial_x \mathrm{H}(Ke^w) = -\mathrm{H}(\partial_x(Ke^w)) \in L^2(\mathbb{R})$ by Lemma A.2 (iv). In consideration of the fact that $w$ is continuous we thus obtain $w \in H^2_{\mathrm{loc}}(\mathbb{R}) \subset \mathcal{C}^{1,1/2}_{\mathrm{loc}}(\mathbb{R})$. □

The next lemma shows that the asymptotics of Lemma 4.5 can be sharpened.

**Lemma 4.7.** *The asymptotics*

$$w(x) = -\frac{\lambda}{\pi} \log|x| + O(1) \quad \text{as} \quad |x| \to +\infty$$

*holds, where* $\lambda = \int_{\mathbb{R}} Ke^w \,\mathrm{d}x > 0$. *Finally, we have that*

$$\int_{\mathbb{R}} \log(1+|x|) K(x) e^{w(x)} \,\mathrm{d}x < +\infty$$

*and in particular* $w$ *is given by the singular integral representation*

$$w(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \log|x-y| K(y) e^{w(y)} \,\mathrm{d}y + C.$$

*Proof.* We recall the estimate in the proof of Lemma 4.6

$$0 < K(x) e^{w(x)} \leq C \langle x \rangle^{-1-2\delta} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

for some $\delta > 0$. Thus the function $f := Ke^w$ satisfies $\log(1+|\cdot|) f \in L^1(\mathbb{R})$. The integral representation of $w$ now is an immediate consequence of Lemma 4.4. In particular we find

$$w(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \log|x-y| f(y) \,\mathrm{d}y + C = -\frac{\lambda}{\pi} \log|x| - \frac{1}{\pi} \int_{\mathbb{R}} \log\left|\frac{x-y}{x}\right| f(y) \,\mathrm{d}y + C$$

with some constant $C \in \mathbb{R}$. The asymptotic formula for $w$ now follows from

$$\lim_{|x| \to +\infty} \int_{\mathbb{R}} \log\left|\frac{x-y}{x}\right| f(y) \,\mathrm{d}y = 0. \tag{4.6}$$

Indeed, this can be seen by splitting the integration into the sets $\{|x-y| \geq |x|/2\}$ and $\{|x-y| \leq |x|/2\}$ and by using that $f \in L^1(\mathbb{R}; (1+\log(1+|x|))\mathrm{d}x) \cap L^2(\mathbb{R})$ and dominated convergence. For the details see Lemma A.7. □

In the following lemma we give an upper bound to $\lambda$ using a Pohozaev-type argument for Liouville equations; see, e.g., [12,27]. For the reader's convenience, we state the proof adapted to our case.

**Lemma 4.8.** *The total Q-curvature satisfies* $0 < \lambda < 2\pi$.

*Proof.* Since $K$ is strictly positive by Assumption **(B)**, we readily see that $\lambda > 0$ holds. To find the upper bound we first differentiate the integral equation for $w$, given in Lemma 4.7, to obtain

$$\partial_x w(x) = -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{1}{x-y} K(y) e^{w(y)} \,\mathrm{d}y. \tag{4.7}$$

Multiplication with $xK(x)e^{w(x)}$ and integration of the right-hand side over $[-R, R]$ yields

$$\begin{aligned} \mathrm{I} := & -\frac{1}{\pi}\int_{-R}^{R} PV\int_{\mathbb{R}} \frac{xK(x)}{x-y} e^{w(x)} K(y) e^{w(y)}\, \mathrm{d}y\, \mathrm{d}x \\ = & -\frac{1}{2\pi}\int_{-R}^{R}\int_{\mathbb{R}} K(x)e^{w(x)}K(y)e^{w(y)}\, \mathrm{d}y\, \mathrm{d}x \\ & -\frac{1}{2\pi}\int_{-R}^{R} PV\int_{\mathbb{R}} \frac{x+y}{x-y} K(x)e^{w(x)}K(y)e^{w(y)}\, \mathrm{d}y\, \mathrm{d}x \\ \to & -\frac{\Lambda^2}{2\pi} + 0 \quad \text{as} \quad R \to +\infty. \end{aligned}$$

On the other hand, if we use the left-hand side in (4.7) we deduce

$$\begin{aligned} \mathrm{II} := & \int_{-R}^{R} xK(x)e^{w(x)}\partial_x w(x)\, \mathrm{d}x = \int_{-R}^{R} xK(x)\partial_x e^{w(x)}\, \mathrm{d}x \\ = & \, xK(x)e^{w(x)}\Big|_{x=-R}^{R} - \int_{-R}^{R} \partial_x(xK(x))e^{w(x)}\, \mathrm{d}x \\ \to & -\lambda - \int_{\mathbb{R}} x\partial_x K(x)e^{w(x)}\, \mathrm{d}x \quad \text{as} \quad R \to +\infty. \end{aligned}$$

Note that $xK(x)e^{w(x)}\Big|_{x=-R}^{R} \to 0$ as $R \to +\infty$ since $|xK(x)e^{w(x)}| \le C\langle x\rangle^{-2\delta}$ for some $\delta > 0$ in view of $e^w \in L^\infty(\mathbb{R})$ and our assumptions on $K$. Furthermore, we notice that $x\partial_x K e^w \in L^1(\mathbb{R})$ again by the Assumption **(B)**. Since I = II, we deduce that

$$\frac{\lambda}{2\pi}(2\pi - \lambda) = -\int_{\mathbb{R}} x\partial_x K(x)e^{w(x)}\, \mathrm{d}x.$$

We see that $\int_{\mathbb{R}} x\partial_x K(x)e^{w(x)}\, \mathrm{d}x < 0$, because $K$ is monotone decreasing in $|x|$ and non-constant. This implies that $\lambda < 2\pi$ must hold. □

## 4.2 Radial Symmetry and Monotonicity

This section is devoted to the proof of item (ii) in Theorem 4.1. We implement the method of moving planes; actually, it is a 'moving point' argument since we are in one space dimension. Because of the nonlocal nature of the problem, it is expedient to work with the equation for $w(x)$ written in integral form. We then adapt the moving plane method generalized to integral equations, which was initiated in the work of [11].

From Lemma 4.7 we recall the integral representation

$$w(x) = \int_{\mathbb{R}} G(x-y)K(y)e^{w(y)}\, \mathrm{d}y + C, \quad \text{where } G(x) = -\frac{1}{\pi}\log|x| \tag{4.8}$$

for a suitable $C \in \mathbb{R}$.

There is a variety of different notations for moving planes arguments. We will use the following: For $\lambda > 0$ and $x \in \mathbb{R}$, we set

$$\Sigma_\lambda := [\lambda, +\infty),\ x_\lambda := 2\lambda - x,\ w_\lambda(x) := w(x_\lambda),\ \Sigma_\lambda^w := \{x \ge \lambda \mid w(x) > w_\lambda(x)\}.$$

Our aim is to prove that for every solution $w$ to (L), $\Sigma_\lambda^w$ is empty for all $\lambda > 0$. We first assume this to be true and explain how this implies that $w$ is radial symmetric-decreasing, which is statement (ii) in Theorem 4.1.

*Proof of Theorem 4.1 (ii).* First notice that by the symmetry of $K$, we have the following symmetry condition: Every solution $w$ to (L) gives rise to another solution $\widetilde{w}$ defined by $\widetilde{w}(x) = w(-x)$. So everything we prove for $w$ will also hold true for $\widetilde{w}$.

Next, for every $x > 0$ and $y \in (-x, x)$ we choose $\lambda = \frac{x+y}{2}$, which is positive. Since $x_\lambda = 2\lambda - x = y$ and $\Sigma_\lambda^w$ is empty by assumption, the inequality $w(x) \leq w_\lambda(x) = w(y)$ must hold. This immediately leads to two properties.

(i) $w$ is monotonically decreasing on $(0, +\infty)$ and

(ii) by continuity of $w$ and letting $y \to -x$ we find that $w(x) \leq w(-x)$.

Since our argument will also hold if we replace $w$ by $\widetilde{w}$, the second property proves that $w$ is symmetric. □

To show that $\Sigma_\lambda^w = \emptyset$ for $\lambda > 0$, we use a continuation method (open/closed argument) specified in the next lemma.

**Lemma 4.9 (Continuation Method).** *Assume the following properties hold true*

(i) *There exists $\lambda_0 > 0$ such that $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0$.*

(ii) *If $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0$, then there exists $\varepsilon > 0$ such that $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0 - \varepsilon$.*

(iii) *If $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_n$ and $\lim_{n\to+\infty} \lambda_n = \lambda_0$, then $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0$.*

*Then $\Sigma_\lambda^w = \emptyset$ for all $\lambda > 0$*

*Proof.* Consider the set $I = \{\lambda_0 > 0 \mid \Sigma_\lambda^w = \emptyset \text{ for all } \lambda > \lambda_0\}$. This set is open and closed in $(0, +\infty)$ by (ii) and (iii), respectively. By (i), we have $I \neq \emptyset$. Since $(0, +\infty)$ is connected, it follows that $I = (0, +\infty)$. □

The following lemma proves the initialisation (condition (i) of Lemma 4.9) of our moving planes argument.

**Lemma 4.10.** *There exists $\lambda_0 > 0$ such that $\Sigma_\lambda^w = \emptyset$ for every $\lambda > \lambda_0$.*

*Proof.* By using the integral representation (4.8) of $w$ and the symmetry of $G$, we obtain

$$\begin{aligned} w(x) &= \int_\lambda^\infty G(x-y)K(y)e^{w(y)}\,\mathrm{d}y + \int_{-\infty}^\lambda G(x-y)K(y)e^{w(y)}\,\mathrm{d}y + C \\ &= \int_\lambda^\infty G(x-y)K(y)e^{w(y)}\,\mathrm{d}y + \int_\lambda^\infty G(x_\lambda - y)K(y_\lambda)e^{w_\lambda(y)}\,\mathrm{d}y + C. \end{aligned}$$

Replacing $x$ by $x_\lambda$ this yields

$$w_\lambda(x) = w(x_\lambda) = \int_\lambda^\infty G(x_\lambda - y)K(y)e^{w(y)}\, \mathrm{d}y + \int_\lambda^\infty G(x-y)K(y_\lambda)e^{w_\lambda(y)}\, \mathrm{d}y + C.$$

Combining these two equations we obtain the following formula for the difference

$$w(x) - w_\lambda(x) = \int_\lambda^\infty (G(x-y) - G(x_\lambda - y)) \left(K(y)e^{w(y)} - K(y_\lambda)e^{w_\lambda(y)}\right) \mathrm{d}y.$$

Since $G$ and $K$ are monotone decreasing in $|x|$, we find

$$G(x-y) \geq G(x_\lambda - y) \quad \text{and} \quad K(y_\lambda) \geq K(y) > 0 \quad \text{for every } x, y \in \Sigma_\lambda.$$

Hence for any $x \in \Sigma_\lambda^w \subset \Sigma_\lambda$, we derive the estimate

$$\begin{aligned}
w(x) - w_\lambda(x) &= \int_\lambda^\infty (G(x-y) - G(x_\lambda - y)) \left(K(y)e^{w(y)} - K(y_\lambda)e^{w_\lambda(y)}\right) \mathrm{d}y \\
&\leq \int_\lambda^\infty (G(x-y) - G(x_\lambda - y))\, K(y) \left(e^{w(y)} - e^{w_\lambda(y)}\right) \mathrm{d}y \\
&\leq \int_{\Sigma_\lambda^w} (G(x-y) - G(x_\lambda - y))\, K(y) \left(e^{w(y)} - e^{w_\lambda(y)}\right) \mathrm{d}y \\
&\leq \int_{\Sigma_\lambda^w} (G(x-y) - G(x_\lambda - y))\, K(y)e^{w(y)} \left(w(y) - w_\lambda(y)\right) \mathrm{d}y \\
&= \int_{\Sigma_\lambda^w} (G(x-y) - G(x_\lambda - y))\, F_\lambda(y)\, \mathrm{d}y,
\end{aligned}$$

where we denote $F_\lambda(y) = K(y)e^{w(y)} (w(y) - w_\lambda(y)) > 0$ on $\Sigma_\lambda^w$.

Next, we establish the upper bounds

$$\begin{aligned}
-G(x_\lambda - y) &= \frac{1}{\pi} \log(x + y - 2\lambda) \leq \frac{1}{\pi} \log(x+y) \\
&\leq \frac{1}{\pi} \log(2\max\{x, y\}) = \frac{1}{\pi} (\log(2) + \max\{\log(x), \log(y)\}) \\
&\leq \frac{1}{\pi} (\log(2) + |\log(x)| + |\log(y)|) \quad \text{for } x, y \in \Sigma_\lambda
\end{aligned}$$

and

$$G(x-y) = -\frac{1}{\pi} \log|x-y| \leq 0 \quad \text{for } |x-y| \geq 1.$$

So for every $x \in \Sigma_\lambda^w$, it holds

$$\begin{aligned}
0 < w(x) - w_\lambda(x) &\leq \int_{\Sigma_\lambda^w} (G(x-y) - G(x_\lambda - y))\, F_\lambda(y)\, \mathrm{d}y \\
&\leq \int_{\Sigma_\lambda^w \cap \{|x-y| \leq 1\}} G(x-y) F_\lambda(y)\, \mathrm{d}y \\
&\quad + \frac{1}{\pi} \int_{\Sigma_\lambda^w} (\log(2) + |\log(x)| + |\log(y)|)\, F_\lambda(y)\, \mathrm{d}y.
\end{aligned}$$

We integrate this inequality against $\langle x\rangle^{-\alpha}$, where $\alpha := 1+\delta$ is taken from Assumption **(B)**, and obtain

$$\begin{aligned} \|\langle x\rangle^{-\alpha}(w-w_\lambda)\|_{L^1(\Sigma_\lambda^w)} &\le \int_{\Sigma_\lambda^w}\left(\int_{\Sigma_\lambda^w\cap\{|x-y|\le 1\}}\langle x\rangle^{-\alpha}G(x-y)F_\lambda(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &\quad + \int_{\Sigma_\lambda^w}\left(\frac{1}{\pi}\int_{\Sigma_\lambda^w}\langle x\rangle^{-\alpha}\left(\log(2)+|\log(x)|+|\log(y)|\right)F_\lambda(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\Sigma_\lambda^w}\left(C_{1,\Sigma_\lambda^w}(y)+C_{2,\Sigma_\lambda^w}(y)\right)F_\lambda(y)\,\mathrm{d}y, \end{aligned} \tag{4.9}$$

where we denote

$$\begin{aligned} C_{1,\Sigma_\lambda^w}(y) &:= \int_{\Sigma_\lambda^w\cap\{|x-y|\le 1\}}\langle x\rangle^{-\alpha}G(x-y)\,\mathrm{d}x \\ &\le \int_{y-1}^{y+1}G(x-y)\,\mathrm{d}x = -\frac{1}{\pi}\int_{-1}^{1}\log|x|\,\mathrm{d}x = \frac{2}{\pi} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} C_{2,\Sigma_\lambda^w}(y) &:= \frac{1}{\pi}\int_{\Sigma_\lambda^w}\langle x\rangle^{-\alpha}\left(\log(2)+|\log(x)|+|\log(y)|\right)\,\mathrm{d}x \\ &\le \frac{1}{\pi}\left(\log(2)+|\log(y)|\right)\int_0^\infty\langle x\rangle^{-\alpha}\,\mathrm{d}x + \frac{1}{\pi}\int_0^\infty\langle x\rangle^{-\alpha}|\log(x)|\,\mathrm{d}x \\ &= C(\alpha)\left(1+|\log(y)|\right), \end{aligned} \tag{4.11}$$

where $C(\alpha)>0$ is a constant which does not depend on $\lambda>0$. We plug these two estimates into inequality (4.9). So by recalling the definition $F_\lambda(y) = K(y)e^{w(y)}\left(w(y)-w_\lambda(y)\right)>0$ on $\Sigma_\lambda^w$ , we have found a constant $C(\alpha)$, independent of $\lambda$, such that

$$\begin{aligned} &\|\langle x\rangle^{-\alpha}(w-w_\lambda)\|_{L^1(\Sigma_\lambda^w)} \\ &\le C(\alpha)\int_{\Sigma_\lambda^w}\left(1+|\log(y)|\right)K(y)e^{w(y)}\left(w(y)-w_\lambda(y)\right)\,\mathrm{d}y \\ &\le C(\alpha)\sup_{y\ge\lambda}\left(\left(1+|\log(y)|\right)K(y)e^{w(y)}\langle y\rangle^{\alpha}\right)\|\langle x\rangle^{-\alpha}(w-w_\lambda)\|_{L^1(\Sigma_\lambda^w)}. \end{aligned} \tag{4.12}$$

Recall that $w$ is bounded from above on account of Lemma 4.6. Therefore we know that $e^w\in L^\infty(\mathbb{R})$. Moreover by Assumption **(B)**, $K(y)\lesssim\langle y\rangle^{-1-2\delta}$. Therefore by our choice $\alpha=1+\delta$

$$\sup_{y\ge\lambda}\left(\left(1+|\log(y)|\right)K(y)e^{w(y)}\langle y\rangle^{\alpha}\right)\lesssim\sup_{y\ge\lambda}\left(\left(1+|\log(y)|\right)\langle y\rangle^{-\delta}\right)$$

becomes arbitrary small, for $\lambda$ sufficiently large. In particular there exists $\lambda_0>0$ such that for every $\lambda>\lambda_0$

$$\|\langle x\rangle^{-\alpha}(w-w_\lambda)\|_{L^1(\Sigma_\lambda^w)}\le\frac{1}{2}\|\langle x\rangle^{-\alpha}(w-w_\lambda)\|_{L^1(\Sigma_\lambda^w)},$$

and hence the set $\Sigma_\lambda^w$ has measure zero for $\lambda>\lambda_0$. By the continuity of $w-w_\lambda$ this implies that $\Sigma_\lambda^w=\emptyset$. □

In the next lemma we establish the continuation result, given by condition (ii) in Lemma 4.9.

**Lemma 4.11.** *If $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0$, then there exists $\varepsilon > 0$ such that $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0 - \varepsilon$.*

*Proof.* We split the proof into the following two steps:

**Step 1.** We show that

$$w(x) < w_{\lambda_0}(x) \quad \text{for every } x > \lambda_0. \tag{4.13}$$

By assumption $w(x) \le w_\lambda(x)$ for every $x \ge \lambda$ and for every $\lambda > \lambda_0$. By continuity of $w$ we conclude that $w(x) \le w_{\lambda_0}(x)$ for every $x \ge \lambda_0$.

Now we argue by contradiction. Assume that there exists $x > \lambda_0$ such that $w(x) = w_{\lambda_0}(x)$. On the one hand we find that

$$\begin{aligned}(-\Delta)^{1/2}(w_{\lambda_0} - w)(x) &= K(x_{\lambda_0})e^{w_{\lambda_0}(x)} - K(x)e^{w(x)} \\ &= \left(K(x_{\lambda_0}) - K(x)\right) e^{w(x)} \ge 0,\end{aligned}$$

since $K$ is monotone decreasing in $|x|$. On the other hand, by using the singular integral expression for $(-\Delta)^{1/2}$, we conclude

$$\begin{aligned}(-\Delta)^{1/2}(w_{\lambda_0} - w)(x) &= \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(w_{\lambda_0} - w)(x) - (w_{\lambda_0} - w)(y)}{(x-y)^2}\, \mathrm{d}y \\ &= -\frac{1}{\pi} PV \int_{\mathbb{R}} \frac{(w_{\lambda_0} - w)(y)}{(x-y)^2}\, \mathrm{d}y \\ &= -\frac{1}{\pi} PV \int_{\lambda_0}^{\infty} \left(\frac{1}{(x-y)^2} - \frac{1}{(x-y_{\lambda_0})^2}\right)(w_{\lambda_0} - w)(y)\, \mathrm{d}y \\ &\le 0.\end{aligned}$$

So we must have equality. Since $(x-y)^{-2} - (x-y_{\lambda_0})^{-2} > 0$ for $x, y > \lambda_0$, we deduce that $w_{\lambda_0} \equiv w$ on $\Sigma_{\lambda_0}$. Therefore for every $y \in \Sigma_{\lambda_0}$,

$$0 = (-\Delta)^{1/2}(w_{\lambda_0} - w)(y) = \left(K(y_{\lambda_0}) - K(y)\right) e^{w(y)}.$$

Thus $K(y) = K(y_{\lambda_0})$ for every $y \in \Sigma_{\lambda_0}$, which means that $K\colon \mathbb{R} \to \mathbb{R}$ is symmetric with respect to the reflection at $\{y = \lambda_0\}$. Since by Assumption **(B)** (i) $K$ is also symmetric with respect to the origin and monotone decreasing in $|x|$, we conclude that $K$ is constant, which contradicts Assumption **(B)** (iii). This completes the proof of the strict inequality (4.13).

**Step 2.** We will argue, by using dominated convergence theorem, that there exists $C(\lambda) \searrow 0$ as $\lambda \nearrow \lambda_0$, satisfying

$$\begin{aligned}\|\langle x\rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma_\lambda^w)} \le{}& C(\lambda) \sup_{y \ge \lambda} \Big( (1 + |\log(y)|)\, K(y) e^{w(y)} \langle y\rangle^{\alpha} \Big) \\ &\times \|\langle x\rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma_\lambda^w)}.\end{aligned} \tag{4.14}$$

In particular for $\varepsilon > 0$ sufficiently small, we find that

$$\|\langle x\rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma_\lambda^w)} \le \frac{1}{2} \|\langle x\rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma_\lambda^w)}, \quad \text{for every } \lambda > \lambda_0 - \varepsilon,$$

which implies that $\Sigma_\lambda^w$ has Lebesque measure zero and hence is empty for $\lambda > \lambda_0 - \varepsilon$ by the continuity of $w - w_\lambda$. So to conclude the proof we have to show that (4.14) holds true.

First notice that by (4.13) and the continuity of $w$

$$\lim_{\lambda \nearrow \lambda_0} \mathbb{1}_{\Sigma_\lambda^w}(x) = 0 \quad \text{for every } x \neq \lambda_0. \tag{4.15}$$

From the proof of Lemma 4.10 we recall the estimate (4.9)

$$\begin{aligned} &\|\langle x\rangle^{-\alpha}(w - w_\lambda)\|_{L^1(\Sigma_\lambda^w)} \\ &\qquad \le \int_{\Sigma_\lambda^w} \left(C_{1,\Sigma_\lambda^w}(y) + C_{2,\Sigma_\lambda^w}(y)\right) K(y) e^{w(y)} \left(w(y) - w_\lambda(y)\right) \,\mathrm{d}y. \end{aligned}$$

Similarly to (4.10), we find the estimate

$$C_{1,\Sigma_\lambda^w}(y) \le \left(\int_{\Sigma_\lambda^w} \langle x\rangle^{-2\alpha} \,\mathrm{d}x\right)^{1/2} \left(\frac{1}{\pi}\int_{-1}^{1} (\log|x|)^2 \,\mathrm{d}x\right)^{1/2}.$$

The second integral is just a constant. The first integral converges to 0 as $\lambda \nearrow \lambda_0$ by (4.15) and dominated convergence theorem. For $C_{2,\Sigma_\lambda^w}$ we argue in the same way. Summing this up there exists $C(\lambda)$, which converges to 0 as $\lambda \nearrow \lambda_0$, such that $C_{1,\Sigma_\lambda^w}(y) + C_{2,\Sigma_\lambda^w}(y) \le C(\lambda)\,(1 + |\log(y)|)$, which directly yields (4.14) and therefore concludes the proof. □

Finally, we show the closedness, i.e. condition (iii) of Lemma 4.9.

**Lemma 4.12.** *If $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_n$ and $\lim_{n\to+\infty} \lambda_n = \lambda_0$, then $\Sigma_\lambda^w = \emptyset$ for all $\lambda > \lambda_0$.*

*Proof.* Let $\lambda > \lambda_0$ be arbitrary. Since $\lim_{n\to+\infty} \lambda_n = \lambda_0$ there exists $n \in \mathbb{N}$ such that $\lambda > \lambda_n$ and hence $\Sigma_\lambda^w = \emptyset$. □

## 4.3 Compactness, A-Priori Estimates and Existence

In this section, we derive results which will be used to prove Theorem 4.1 (iii) about existence of solutions. Some estimates will also be later needed in the proof of Theorem 4.2.

We organize the proof of our existence result as follows. We first put the nonlocal Liouville equation (L) down to a fixed-point equation on the set of radial symmetric-decreasing functions in $H^1(\mathbb{R})$, satisfying some condition on the decay. Next, we establish a local Lipschitz estimate to achieve compactness and recall the Pohozaev-type estimate of Lemma 4.8 in order to derive suitable a-priori bounds on fixed-points. Combining these results, we are in a position to use a version of Schauder's fixed-point theorem applicable to our case.

### Fixed-point equation and functional Setup

We start by recasting the nonlocal Liouville equation (L) into an integral equation by the following proposition.

**Proposition 4.13.** *The following two are equivalent:*

(i) $w \in L_{1/2}(\mathbb{R})$ *is a solution to (L) with finiteness condition* $Ke^w \in L^1(\mathbb{R})$.

(ii) $v = \sqrt{Ke^w} \in H^1(\mathbb{R})$ *solves*

$$v(x) = v(0)\sqrt{\frac{K(x)}{K(0)}}e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y}. \tag{4.16}$$

*If one of these conditions is satisfied it holds*

$$\int_{\mathbb{R}} \log(1+|x|)K(x)e^{w(x)}\,\mathrm{d}x = \int_{\mathbb{R}} \log(1+|x|)v(x)^2\,\mathrm{d}x < +\infty.$$

*Finally, every solution* $v \in H^1(\mathbb{R})$ *of (4.16) must be symmetric-decreasing, i.e. it holds that* $v = v^\star$.

*Remark.* By differentiating, we readily check that $v$ solves the nonlinear equation

$$\partial_x v = (\partial_x \log\sqrt{K})v - \frac{1}{2}\mathrm{H}(v^2)v. \tag{4.17}$$

We remark that, for the special case $K(x) = e^{-x^2}$, we retrieve the ground state soliton equation $\partial_x v + xv + \frac{1}{2}\mathrm{H}(|v|^2)v = 0$ for the harmonic Calogero–Moser DNLS. Of course, the following analysis will allow for more general $K(x)$ that satisfy Assumption **(B)**.

*Proof.*
(i)⇒(ii):
In Lemma 4.6 we have seen that $Ke^w$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. According to Lemma A.2, we thus find

$$\partial_x w = -\mathrm{H}(Ke^w).$$

Since $w$ is $\mathcal{C}^1$, again by Lemma 4.6, this is equivalent to the integral equation

$$w(x) = w(0) - \int_0^x \mathrm{H}(Ke^w)(y)\,\mathrm{d}y.$$

Using that $K(x) > 0$, we can rewrite this as

$$K(x)e^{w(x)} = K(0)e^{w(0)}\frac{K(x)}{K(0)}e^{-\int_0^x \mathrm{H}(Ke^w)(y)\,\mathrm{d}y}.$$

Finally, using the definition of $v$ and taking the square root in the previous equation, we obtain (4.16).

By the finiteness condition (4.1) the function $v = \sqrt{Ke^w}$ belongs to $L^2(\mathbb{R})$. To verify that $v$ is an element of $H^1(\mathbb{R})$, we differentiate (4.16) to obtain

$$\partial_x v = \frac{v(0)}{\sqrt{K(0)}} \left( \left(\partial_x \sqrt{K}\right) - \sqrt{K}\frac{1}{2}\mathrm{H}(v^2) \right) e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y}.$$

In view of Theorem 4.1 (ii) $w$ is even and decreasing in $|x|$. Thus the positive function $v = \sqrt{Ke^w}$ satisfies $v = v^\star$ due to the symmetric-decreasing property of $K$. In particular by Lemma A.3, we derive $e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y} \leq 1$ for every $x \in \mathbb{R}$. Using the asymptotic behaviour of $K$, given in Assumption **(B)** (iii), this proves that

$$\begin{aligned} \|\partial_x v\|_{L^2(\mathbb{R})} &\leq \frac{v(0)}{\sqrt{K(0)}} \left( \|\partial_x \sqrt{K}\|_{L^2(\mathbb{R})} + \|\sqrt{K}\|_{L^\infty(\mathbb{R})} \|\mathrm{H}(v^2)\|_{L^2(\mathbb{R})} \right) \\ &\lesssim v(0) \left(1 + \|v\|^2_{L^4(\mathbb{R})}\right) < +\infty, \end{aligned} \tag{4.18}$$

since $v = \sqrt{Ke^w} \in L^4(\mathbb{R})$ in view of the fact that $Ke^w \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We conclude that $v \in H^1(\mathbb{R})$ holds. Moreover, by Lemma 4.7, we obtain

$$\int_{\mathbb{R}} \log(1+|x|) v(x)^2 \,\mathrm{d}x < +\infty.$$

(ii) $\Rightarrow$ (i):
Let $v \in H^1(\mathbb{R})$ with $v(0) > 0$ be a solution of (4.16). Then the function $w = \log(K^{-1}v^2)$ belongs to $L_{1/2}(\mathbb{R})$ since by Hölder's inequality

$$\begin{aligned} |\log(K^{-1}v^2)| &= \left| \log\left(v(0)^2 K(0)^{-1}\right) - \int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y \right| \\ &\leq C + |x|^{1/2} \|v\|^2_{L^4(\mathbb{R})} \in L_{1/2}(\mathbb{R}). \end{aligned}$$

Moreover trivially the finiteness condition $Ke^w = v^2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is satisfied. Differentiating $w = \log(K^{-1}v^2)$ while taking into account that $v$ solves (4.17), we obtain

$$\partial_x w = -\mathrm{H}(Ke^w) \in L^2(\mathbb{R}).$$

Thus we can apply Lemma A.2 to show that $w$ is a solution of (L). □

Based on this proposition, we give the following definitions. We let $X$ denote the real Hilbert space given by

$$X := \left\{ u\colon \mathbb{R} \to \mathbb{R} \;\middle|\; \|u\|_X < +\infty \right\}, \tag{4.19}$$

where we define the norm via

$$\|u\|_X^2 = \|u\|^2_{H^1(\mathbb{R})} + \int_{\mathbb{R}} \log(1+|x|) u(x)^2 \,\mathrm{d}x.$$

Notice that this is the same definition of the space $X$ that we used in Chapter 3 in the special case $V(x) = \log(1+|x|)$, which trivially satisfies Assumption **(A)**. In particular, by Proposition 3.5, the embedding $X \subset L^2(\mathbb{R})$ is compact.

Moreover, we define

$$X^\star := \left\{ u \in X \;\middle|\; u = u^\star \right\},$$

which is the set of radial symmetric-decreasing functions that belong to the space $X$. Notice that $X^\star$ is a closed and convex subset of $X$.

For $\lambda > 0$ and $u \in X^\star$ given, we set

$$T_\lambda(u) = \lambda \sqrt{K(x)} e^{-\frac{1}{2}\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y}. \tag{4.20}$$

In view of Proposition 4.13, we note that for $v \in X^\star$ and $\lambda := v(0)/\sqrt{K(0)}$ we obtain the equivalence

$$T_\lambda(v) = v \iff w = \log(K^{-1}v^2) \text{ solves (L) with } \int_{\mathbb{R}} K e^w = \int_{\mathbb{R}} v^2, \tag{4.21}$$

where, of course, we always assume that $K$ satisfies Assumption **(B)**.

### Compactness and A-Priori Bounds

In order to prepare the proof of Theorem 4.1 (iii), we establish some properties of the functional $T_\lambda \colon X^\star \to X^\star$.

We begin by recording the following fact.

**Lemma 4.14.** *For every $\lambda > 0$ the map $T_\lambda \colon X^\star \to X^\star$ is well-defined.*

*Proof.* We show that $T_\lambda$ maps $X^\star$ into itself. Indeed, let $\lambda > 0$ and $u \in X^\star$ be given. By Lemma A.1 (v) and Lemma A.3

$$\psi_u(x) = e^{-\frac{1}{2}\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y}$$

is a symmetric-decreasing function with $0 < \psi_u(x) \leq 1$ for every $x \in \mathbb{R}$. Using Assumption **(B)** (i), we find that $T_\lambda(u) = \lambda\sqrt{K}\psi_u$ is symmetric-decreasing as well. By the decay properties of $K$ given in Assumption **(B)** (iii) and the boundedness of $\psi_u$ we see that $T_\lambda(u)$ and $(\log(1+|x|))^{1/2}T_\lambda(u)$ both belong to $L^2(\mathbb{R})$. Finally, we recall estimate (4.18) to find $\partial_x T_\lambda(u) \in L^2(\mathbb{R})$, whence it follows that $T_\lambda(u) \in X^\star$. □

Next, we establish the following local Lipschitz estimate.

**Lemma 4.15.** *The map $T_\lambda \colon X^\star \to X^\star$ satisfies the estimate*

$$\|T_\lambda(v) - T_\lambda(u)\|_X \lesssim \lambda(\|u\|_X + \|v\|_X)(1 + \|u\|_X^2 + \|v\|_X^2)\|u - v\|_{L^2(\mathbb{R})}.$$

*In particular $T_\lambda \colon X^\star \to X^\star$ is continuous.*

*Proof.* We first establish some auxiliary estimates as follows. For convenience we again use the notation

$$\psi_u(x) = e^{-\frac{1}{2}\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y}.$$

Recall that by Lemma A.3, $0 < \psi_u(x) \leq 1$ for any $u \in X^\star$. Furthermore, by using the boundedness of the Hilbert transform H on $L^p(\mathbb{R})$ for any $p \in (1, +\infty)$ and applying Hölder's inequality and Sobolev embedding theorem, we deduce, for any $x \in \mathbb{R}$, the pointwise bound

$$\begin{aligned}
|\psi_u(x) - \psi_v(x)| &\leq \frac{1}{2}\left|\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y - \int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y\right| \\
&\leq \frac{1}{2}\int_0^{|x|} \left|\mathrm{H}(u^2 - v^2)(y)\right|\,\mathrm{d}y \leq \frac{1}{2}|x|^{1/q}\|\mathrm{H}(u^2 - v^2)\|_{L^p(\mathbb{R})} \\
&\leq C|x|^{1/q}\|u^2 - v^2\|_{L^p(\mathbb{R})} \leq C|x|^{1/q}\|u + v\|_{L^r(\mathbb{R})}\|u - v\|_{L^2(\mathbb{R})} \\
&\leq C|x|^{1/q}(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})},
\end{aligned}$$

with $C > 0$ only depending on $p$ and $1/q + 1/p = 1$, $1 < p \leq 2$, $1/2 + 1/r = 1/p$ and hence $2 < r \leq +\infty$. Since $|x|^{1/q}\sqrt{K(x)} \leq C|x|^{1/q}\langle x\rangle^{-1/2-\delta} \in L^2(\mathbb{R})$ for $q \geq 2$ sufficiently large (i.e. $q > 1/\delta$), we obtain

$$\begin{aligned}
\|T_\lambda(u) - T_\lambda(v)\|_{L^2(\mathbb{R})} &\leq C\lambda\||x|^{1/q}\sqrt{K}\|_{L^2(\mathbb{R})}(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})} \\
&\leq C\lambda(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})},
\end{aligned}$$

where $C > 0$ is depending on $K$. Likewise we use $\sqrt{\log(1 + |x|)}|x|^{1/q}\sqrt{K(x)} \in L^2(\mathbb{R})$ for $q \geq 2$ as above, to conclude

$$\|\sqrt{\log(1 + |x|)}\,(T_\lambda(u) - T_\lambda(v))\,\|_{L^2(\mathbb{R})} \leq C\lambda(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})}.$$

Next, we notice that

$$\partial_x T_\lambda(u) = \lambda\left((\partial_x\sqrt{K})\psi_u - \frac{1}{2}\sqrt{K}\psi_u\mathrm{H}(u^2)\right).$$

Using that $|x|^{1/q}\partial_x\sqrt{K} \in L^2(\mathbb{R})$, $|x|^{1/q}\sqrt{K},\ \sqrt{K} \in L^\infty(\mathbb{R})$ and $0 < \psi_u, \psi_v \leq 1$, we find

$$\begin{aligned}
&\|\partial_x(T_\lambda(u) - T_\lambda(v))\|_{L^2(\mathbb{R})} \\
&\quad\leq \lambda\|(\partial_x\sqrt{K})\,(\psi_u - \psi_v)\,\|_{L^2(\mathbb{R})} + \tfrac{\lambda}{2}\|\sqrt{K}\,(\psi_u - \psi_v)\,\mathrm{H}(u^2)\|_{L^2(\mathbb{R})} \\
&\qquad+ \tfrac{\lambda}{2}\|\sqrt{K}\psi_v\mathrm{H}(u^2 - v^2)\|_{L^2(\mathbb{R})} \\
&\quad\leq C\lambda\||x|^{1/q}\partial_x\sqrt{K}\|_{L^2(\mathbb{R})}(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})} \\
&\qquad+ C\lambda\||x|^{1/q}\sqrt{K}\|_{L^\infty(\mathbb{R})}\|\mathrm{H}(u^2)\|_{L^2(\mathbb{R})}(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})} \\
&\qquad+ \tfrac{\lambda}{2}\|\sqrt{K}\|_{L^\infty(\mathbb{R})}\|\,\mathrm{H}(u^2 - v^2)\|_{L^2(\mathbb{R})} \\
&\quad\leq C\lambda\left(1 + \|u\|_X^2\right)(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Here, we also used that by Sobolev embedding

$$\|\mathrm{H}(u^2)\|_{L^2(\mathbb{R})} = \|u^2\|_{L^2(\mathbb{R})} = \|u\|_{L^4(\mathbb{R})}^2 \leq C\|u\|_X^2$$

as well as

$$\|\mathrm{H}(u^2 - v^2)\|_{L^2(\mathbb{R})} = \|u^2 - v^2\|_{L^2(\mathbb{R})} \leq C(\|u\|_X + \|v\|_X)\|u - v\|_{L^2(\mathbb{R})}.$$

In view of the estimates above, we complete the proof. □

Next, we establish the following result.

**Lemma 4.16.** *The map $T_\lambda \colon X^\star \to X^\star$ is compact.*

*Proof.* Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $X^\star$. By Proposition 3.5 up to passing to a subsequence $(u_k)_{k\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{R})$ and thus forms a Cauchy sequence in $L^2(\mathbb{R})$. From the estimate in Lemma 4.15, we deduce that $(T_\lambda(u_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $X$, which converges to some element in $X^\star$, due to the closedness of this subset. □

As a next step, we show the following a-priori bound for fixed-points of the compact map $T_\lambda \colon X^\star \to X^\star$.

**Lemma 4.17.** *There exists $C > 0$ only depending on $K$, such that for any $v \in X^\star$ with $v = T_\lambda(v)$, it holds that*

$$\|v\|_X = \|T_\lambda(v)\|_X \le C\left(\lambda^2 + \lambda\right).$$

*Proof.* Suppose that $v \in X^\star$ is a fixed-point of $T_\lambda$. By the Pohozaev-type result in Lemma 4.8, we deduce the a-priori bound

$$\|v\|^2_{L^2(\mathbb{R})} < 2\pi \lesssim 1.$$

Next, recall that $e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y} \le 1$ for every $x \in \mathbb{R}$ by an application of Lemma A.3. Hence we directly obtain the pointwise bound

$$0 < v(x) = \lambda\sqrt{K(x)}e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y} \lesssim \lambda\langle x\rangle^{-1/2-\delta}, \tag{4.22}$$

for some $\delta > 0$. From this we readily deduce the estimates

$$\|v\|_{L^2(\mathbb{R})} \lesssim \lambda \quad \text{and} \quad \|\sqrt{\log(1+|\cdot|)}v\|_{L^2(\mathbb{R})} \lesssim \lambda.$$

To bound $\|\partial_x v\|_{L^2(\mathbb{R})}$, we use the inequality $e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y} \le 1$ once again together with the fact that $\mathrm{H}$ is an isometry on $L^2(\mathbb{R})$ and the estimate $\|v\|_{L^\infty(\mathbb{R})} \lesssim \lambda$ which is a direct consequence of (4.22). In the last two steps we apply the Gagliardo-Nirenberg interpolation estimate combined with the bound $\|v\|_{L^2(\mathbb{R})} \lesssim 1$. In summary we find

$$\begin{aligned}
\|\partial_x v\|_{L^2(\mathbb{R})} &= \|\partial_x T_\lambda(v)\|_{L^2(\mathbb{R})} \\
&\le \lambda\|(\partial_x\sqrt{K})e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y}\|_{L^2(\mathbb{R})} + \tfrac{1}{2}\|v\mathrm{H}(v^2)\|_{L^2(\mathbb{R})} \\
&\le \lambda\|(\partial_x\sqrt{K})\|_{L^2(\mathbb{R})} + \tfrac{1}{2}\|v\|_{L^\infty(\mathbb{R})}\|v^2\|_{L^2(\mathbb{R})} \\
&\lesssim \lambda\left(1 + \|v\|^2_{L^4(\mathbb{R})}\right) \lesssim \lambda\left(1 + \|v\|^{3/2}_{L^2(\mathbb{R})}\|\partial_x v\|^{1/2}_{L^2(\mathbb{R})}\right) \\
&\lesssim \lambda\left(1 + \|\partial_x v\|^{1/2}_{L^2(\mathbb{R})}\right).
\end{aligned}$$

An elementary argument now yields $\|\partial_x v\|_{L^2(\mathbb{R})} \lesssim \lambda^2 + \lambda$. By recalling the definition of $\|v\|_X$ and the bounds found above, we deduce

$$\|v\|_X \lesssim \lambda^2 + \lambda,$$

which completes the proof. □

### Proof of Theorem 4.1 (iii)

We are now in a position to show the following existence result.

**Proposition 4.18.** *For any $\lambda > 0$, the map $T_\lambda : X^\star \to X^\star$ has a fixed-point. Consequently, for any $v_0 > 0$ there exists a solution $v \in X^\star$ of equation (4.16) with $v(0) = v_0 = \lambda\sqrt{K(0)}$.*

*Likewise, for any $w_0 \in \mathbb{R}$, there exists a solution $w \in L_{1/2}(\mathbb{R})$ of (L) with $Ke^w \in L^1(\mathbb{R})$ and $w(0) = w_0$.*

*Proof.* Let $\lambda > 0$ be given. Suppose there exist $v \in X^\star$ and $\sigma \in (0,1]$ such that $v = \sigma T_\lambda(v)$, which is the same as to say that $v = T_{\sigma\lambda}(v)$. Then by Lemma 4.17, there exists some constant $C > 0$ such that $\|v\|_X \le C\left(\sigma^2\lambda^2 + \sigma\lambda\right) \le C\left(\lambda^2 + \lambda\right)$. Hence for $M > C\left(\lambda^2 + \lambda\right)$ the following implication holds true:

$$\exists v \in X^\star \text{ and } \exists \sigma \in (0,1] \text{ with } v = \sigma T_\lambda(v) \quad \Rightarrow \quad \|v\|_X < M. \tag{4.23}$$

Next, we consider the set

$$S := \left\{u \in X \;\middle|\; \|u\|_X \le M\right\} \cap X^\star,$$

which is closed and convex as the intersection of closed convex sets. On $S$ we define the map $T^\star : S \to S$ via

$$T^\star(v) := \begin{cases} T_\lambda(v) & \text{if } \|T_\lambda(v)\|_X \le M, \\ M\frac{T_\lambda(v)}{\|T_\lambda(v)\|_X} & \text{if } \|T_\lambda(v)\|_X \ge M. \end{cases}$$

Clearly, the map $T^\star : S \to S$ is well-defined and continuous. Furthermore, by the compactness of $T_\lambda : X^\star \to X^\star$, the map $T^\star : S \to S$ is compact as well. Thus, due to the boundedness of $S \subset X$, the image $T^\star(S)$ is precompact in the Banach space $X$. We now apply a suitable version of Schauder's fixed-point theorem (see for instance [18] [Corollary 11.2]), namely every continuous function $\mathfrak{T} : \mathfrak{S} \to \mathfrak{S}$, where $\mathfrak{S}$ is a closed convex set in a Banach space and its image $\mathfrak{T}(\mathfrak{S})$ is precompact, has a fixed-point.

We conclude the map $T^\star : S \to S$ has a fixed-point $v \in S$. We claim that $v \in S$ is a fixed-point of $T_\lambda$ as well. To show this, let us suppose that $\|T_\lambda(v)\|_X \ge M$. Then $\|v\|_X = \|T^\star(v)\|_X = M$ and $v = T^\star(v) = \sigma T_\lambda(v)$ with $\sigma = M/\|T_\lambda(v)\|_X \le 1$. But this contradicts (4.23).

Thus we have proven that there exists $v \in S \subset X^\star$ such that $T_\lambda(v) = v$, whence $v$ solves (4.16) with $v(0) = \lambda\sqrt{K(0)}$. Finally, by (4.21), the existence of $v$ with $v(0) = v_0 > 0$ given is equivalent to the fact that $w = \log(K^{-1}v^2) \in L_{1/2}(\mathbb{R})$ is a solution of (L) with $w(0) = w_0 = \log(K^{-1}(0)v(0)^2) \in \mathbb{R}$ and $\int_{\mathbb{R}} Ke^w \,\mathrm{d}x = \int_{\mathbb{R}} v^2 \,\mathrm{d}x < +\infty$. □

## 4.4 Uniqueness

In this section, we prove the uniqueness result stated in Theorem 4.2. This will be the main result of this chapter.

Our aim is to show global uniqueness (with respect to a given initial datum $v_0$) of the corresponding integral equation, introduced in (4.16). This, due to Proposition 4.13 proves Theorem 4.2.

The main part of our proof is to establish local uniqueness. We will use the implicit function theorem to argue that in a small neighbourhood of a given solution there exists a unique $\mathcal{C}^1$-branch of solutions parametrized by the initial value $v_0$. The difficulty here is to prove the invertibility of the Fréchet derivative of our functional, which turns out to be of the form $\mathcal{K}-\mathbb{1}$ for a compact operator $\mathcal{K}$ on the space of even functions in $L^2(\mathbb{R})$. Thus, according to the Fredholm alternative, proving the existence of a bounded inverse is essentially reduced to verifying that the kernel of $\mathcal{K} - \mathbb{1}$ is trivial. This is where the monotonicity formula established in Lemma 4.21 for even functions is of great importance.

Once we have proven local uniqueness, we will see that every branch of solutions can be extended to 0. Thus, we obtain global uniqueness as a direct consequence of the fact that solutions are unique for small initial data, which holds true by the a-priori bounds stated in Lemma 4.17.

First we introduce the notation and establish the functional setup. Recall the definition of the Hilbert space $X$ above, given in (4.19). In the following we will always consider

$$X_{\text{even}} := \big\{u \in X \;\big|\; u(x) = u(-x) \text{ for } x \in \mathbb{R}\big\},$$

which is the set of symmetric functions that belong to $X$. We define the map $F\colon X_{\text{even}} \times (0, +\infty) \to X_{\text{even}}$ by setting

$$F(u,\lambda)(x) := \lambda\sqrt{K(x)}e^{-\frac{1}{2}\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y} - u(x) = \left(T_\lambda(u) - u\right)(x), \tag{4.24}$$

in terms of the map $T_\lambda$ introduced in (4.20). However, the reader should be aware of the fact that we extend the map $T_\lambda$ from $X^\star$ to $X_{\text{even}}$ here. By standard estimates, it is straightforward to check that the mapping $F : X_{\text{even}} \times (0, +\infty) \to X_{\text{even}}$ is indeed well-defined and of class $\mathcal{C}^1$; see Lemma A.8 below. By construction, we have the equivalence

$$F(v,\lambda) = 0 \quad \Longleftrightarrow \quad v \in X \text{ solves (4.16) with } \lambda = \frac{v(0)}{\sqrt{K(0)}}. \tag{4.25}$$

## Nondegeneracy and Local Uniqueness

Our next goal is to apply the implicit function theorem in order to construct a locally unique $C^1$-branch $\lambda \mapsto v_\lambda$ around a given solution $(v, \lambda)$ satisfying $F(v, \lambda) = 0$. As a key result, we shall need to prove that the Fréchet derivative $\partial_v F$ has a bounded inverse on $X_{\text{even}}$. Indeed, we notice that

$$\partial_v F(v, \lambda) = \mathcal{K} - \mathbb{1},$$

where $\mathcal{K} : X_{\text{even}} \to X_{\text{even}}$ denotes the bounded linear operator given by

$$(\mathcal{K}f)(x) = -v(x) \int_0^x \mathrm{H}(vf)(y)\,\mathrm{d}y.$$

We record the following basic fact.

**Lemma 4.19.** *Suppose that $F(v,\lambda)=0$. Then the linear operator $\mathcal{K}$ extends to a bounded map from $L^2_{even}(\mathbb{R})$ into $X_{even}$. As a consequence, the linear operator $\mathcal{K}: L^2_{even}(\mathbb{R}) \to L^2_{even}(\mathbb{R})$ is compact.*

*Proof.* We show that the linear operator $\mathcal{K}$ extends to a bounded map from $L^2_{\mathrm{even}}(\mathbb{R})$ into $X_{\mathrm{even}}$ as follows. Let $f \in L^2_{\mathrm{even}}(\mathbb{R})$ be given. Using that $v \in H^1(\mathbb{R})$ and by Sobolev embeddings, we deduce from Hölder's inequality together with the boundedness of H on $L^p(\mathbb{R})$ when $p \in (1,+\infty)$ that we have the pointwise bound

$$\begin{aligned}\left| \int_0^x \mathrm{H}(vf)(y)\, \mathrm{d}y \right| &\leq |x|^{1/q} \|\mathrm{H}(vf)\|_{L^p(\mathbb{R})} \leq C|x|^{1/q}\|vf\|_{L^p(\mathbb{R})} \\ &\leq C|x|^{1/q}\|v\|_{H^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})} \leq C|x|^{1/q}\|f\|_{L^2(\mathbb{R})}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\leq 2$, and with some constant $C=C(p,v)>0$. Next, we use that $F(v,\lambda)=0$ holds and thus $v=v^\star \in X^\star$ is symmetric-decreasing (see Proposition 4.13). By Lemma A.3, this implies that $e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)\,\mathrm{d}y} \leq 1$ for all $x \in \mathbb{R}$. In particular, this shows that $0 < v(x) \leq \lambda\sqrt{K(x)}$, which implies the pointwise bound

$$|(\mathcal{K}f)(x)| \leq C\|f\|_{L^2}\sqrt{K(x)}|x|^{1/q} \leq C\|f\|_{L^2}\langle x\rangle^{-\frac{1}{2}-\varepsilon}, \tag{4.26}$$

for some $\varepsilon>0$, where the last inequality follows from the assumed bound for $K$ and by taking $q \gg 1$ sufficiently large (and thus $p>1$ sufficiently close to 1). Clearly, the bound (4.26) shows that

$$\int_{\mathbb{R}} |\mathcal{K}f(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}} \log(1+|x|)|\mathcal{K}f(x)|^2\,\mathrm{d}x \leq C\|f\|^2_{L^2(\mathbb{R})}$$

with some constant $C>0$ independent of $f$.

Next, by differentiating and using the equation satisfied by $v$, we observe that

$$\begin{aligned}\|\partial_x\mathcal{K}f\|_{L^2(\mathbb{R})} &\leq \|(\partial_x v)\int_0^x \mathrm{H}(vf)(y)\,\mathrm{d}y\|_{L^2} + \|v\mathrm{H}(vf)\|_{L^2(\mathbb{R})} \\ &\leq C\|(\partial_x v)|x|^{1/q}\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})}\|vf\|_{L^2(\mathbb{R})} \\ &\leq C\big(\|(\partial_x\sqrt{K})|x|^{1/q}\|_{L^2(\mathbb{R})} \\ &\quad + \|\sqrt{K}|x|^{1/q}\|_{L^2(\mathbb{R})}\|\mathrm{H}(v^2)\|_{L^\infty(\mathbb{R})} + \|v\|^2_{L^\infty(\mathbb{R})}\big)\|f\|_{L^2(\mathbb{R})} \\ &\leq C\|f\|_{L^2(\mathbb{R})}\end{aligned}$$

with some constant $C>0$ independent of $f$. Again, we have chosen $q \gg 1$ sufficiently large and we have used the pointwise bounds for $\sqrt{K}$ and $\partial_x\sqrt{K}$. Moreover we used $\|\mathrm{H}(v^2)\|_{L^\infty(\mathbb{R})} \lesssim \|\mathrm{H}(v^2)\|_{H^1(\mathbb{R})} = \|v^2\|_{H^1(\mathbb{R})} \lesssim \|v\|^2_{H^1(\mathbb{R})}$. In summary, we have shown that

$$\|\mathcal{K}f\|_X \leq C\|f\|_{L^2(\mathbb{R})}$$

with some constant $C>0$ independent of $f$. Since $v$ and $f$ are even functions, we readily check that $\mathcal{K}f$ is even as well. Hence we have proven that the linear map $\mathcal{K}: L^2_{\mathrm{even}}(\mathbb{R}) \to X_{\mathrm{even}}$ is bounded.

Finally, we note that the map $\mathcal{K} : L^2_{\text{even}}(\mathbb{R}) \to L^2_{\text{even}}(\mathbb{R})$ is compact due to fact that the embedding $X \subset L^2(\mathbb{R})$ is compact; see Proposition 3.5 above, with $V(x) = \log(1 + |x|)$ . □

Next, we establish the following key result.

**Lemma 4.20.** *Let $F(v, \lambda) = 0$ hold. Then the Fréchet derivative $\partial_v F = \mathcal{K} - \mathbb{1}$ is invertible on $X_{even}$ with bounded inverse.*

*Proof.* Since $\mathcal{K}$ maps $L^2_{\text{even}}(\mathbb{R})$ into $X_{\text{even}}$, it suffices to show $\mathcal{K} - \mathbb{1}$ is invertible on $L^2_{\text{even}}(\mathbb{R})$. By the compactness of $\mathcal{K}$ on $L^2_{\text{even}}(\mathbb{R})$ and the Fredholm alternative, this amounts to showing the implication

$$\mathcal{K}f = f \quad \text{and} \quad f \in L^2_{\text{even}}(\mathbb{R}) \quad \Rightarrow \quad f = 0. \tag{4.27}$$

Indeed, let us assume that $f \in L^2_{\text{even}}(\mathbb{R})$ solves $\mathcal{K}f = f$. Since $f \in \text{ran}\,(\mathcal{K})$, we obtain that $f \in X_{\text{even}}$ and in particular the function $f$ is continuous. Next, we note that the equation $\mathcal{K}f = f$ can be written as

$$v\psi = f,$$

where we define the even and continuous differentiable function $\psi : \mathbb{R} \to \mathbb{R}$ by setting

$$\psi(x) := -\int_0^x \mathrm{H}(vf)(y)\mathrm{d}y.$$

We see that $\partial_x \psi = -\mathrm{H}(vf) \in L^2(\mathbb{R})$. Therefore, according to Lemma A.2, we find that $\psi \in \dot{H}^1_{\text{even}}(\mathbb{R})$ solves the equation

$$(-\Delta)^{1/2}\psi - v^2\psi = 0 \quad \text{in } \mathbb{R}.$$

Notice that $W = -v^2$ is $\mathcal{C}^1$ and monotone increasing on $[0, +\infty)$ and furthermore satisfies $W(x)\psi(x)^2 = -(\mathcal{K}f(x))^2 \to 0$ as $x \to +\infty$ by the pointwise bound in (4.26). So by using that $\psi \in \dot{H}^1_{\text{even}}(\mathbb{R})$ with $\psi(0) = 0$, we obtain that $\psi \equiv 0$ by Lemma 4.21 below. Thus $f = v\psi = 0$ is the zero function. This shows (4.27).

In summary, we have shown that the bounded linear operator $\partial_v F = \mathcal{K} - \mathbb{1}$ is invertible on $X_{\text{even}}$. By bounded inverse theorem, its inverse $(\partial_v F)^{-1} : X_{\text{even}} \to X_{\text{even}}$ is bounded as well. □

**Lemma 4.21 (Key Lemma).** *Let $W : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-function with $W'(x) \geq 0$ for $x \geq 0$. Assume that $\psi \in \dot{H}^1_{\text{even}}(\mathbb{R})$ solves*

$$(-\Delta)^{1/2}\psi + W\psi = 0 \quad \text{in } \mathbb{R}$$

*with $W(x)\psi(x)^2 \to 0$ as $x \to +\infty$. Then $\psi(0) = 0$ implies that $\psi \equiv 0$.*

*Remark.* The assumption above that $\psi$ is an even function is essential. For example, the odd function $\psi(x) = \frac{2x}{x^2+1} \in H^1(\mathbb{R})$ (and hence $\psi(0) = 0$) solves the equation

$$(-\Delta)^{1/2}\psi - \frac{2}{1 + x^2}\psi = 0 \quad \text{in } \mathbb{R}.$$

*Proof.* By integrating the equation on $[0, R)$ against $\partial_x \psi \in L^2(\mathbb{R})$, we find

$$\mathrm{I}_R + \mathrm{II}_R := \int_0^R ((-\Delta)^{1/2}\psi)(x)\partial_x\psi(x)\,\mathrm{d}x + \int_0^R W(x)\psi(x)\partial_x\psi(x)\,\mathrm{d}x = 0$$

for every $R > 0$. We remark that $(-\Delta)^{1/2}\psi \in L^2(\mathbb{R})$ and $W\psi \in L^2_{\mathrm{loc}}(\mathbb{R})$ holds and hence the integrals above are absolutely convergent. Since the limit $\mathrm{I} := \lim_{\mathbb{R}\to+\infty} \mathrm{I}_R$ exists, also $\mathrm{II} := \lim_{R\to+\infty} \mathrm{II}_R = -\lim_{\mathbb{R}\to+\infty} \mathrm{I}_R = -\mathrm{I}$ is well-defined. To analyze the first term, we notice that

$$\begin{aligned}\mathrm{I} &= \int_0^{+\infty} (\mathrm{H}\partial_x\psi)(x)\partial_x\psi(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{+\infty}\left(PV\int_{-\infty}^{+\infty}\frac{\partial_y\psi(y)}{x-y}\,\mathrm{d}y\right)\partial_x\psi(x)\mathrm{d}x \\ &= -\frac{1}{\pi}\int_0^{+\infty}\int_0^{+\infty}\frac{\partial_x\psi(x)\partial_y\psi(y)}{x+y}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where the last step follows by using the anti-symmetry $\partial_y\psi(-y) = -\partial_y\psi(y)$. Next, we recall the known formula (allowing also for complex-valued functions for the moment):

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\overline{\phi}(t)\phi(s)}{t+s}\,\mathrm{d}s\,\mathrm{d}t = \int_{-\infty}^{+\infty}|(\mathsf{L}\phi)(\lambda)|^2\,\mathrm{d}\lambda \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}_+),$$

where $(\mathsf{L}\phi)(\lambda) = \int_0^{+\infty} e^{\lambda t}\phi(t)\,dt$ denotes the one-side Laplace transform; see, e. g., [28]. From this formula, which extends to $L^2(\mathbb{R}_+)$ by density, we readily deduce the classical fact the the Carleman–Hankel operator with the kernel $(x+y)^{-1}$ on $L^2(\mathbb{R}_+)$ is positive definite. Hence, if we go back to the expression for I, we deduce that

$$\mathrm{I} \le 0 \quad \text{with } \mathrm{I} = 0 \text{ if and only if } \partial_x\psi \equiv 0.$$

On the other hand, by integration by parts and using that $\psi(0) = 0$ and $W(x)\psi(x)^2 \to 0$ as $x \to +\infty$, we obtain from $W'(x) \ge 0$ for $x \ge 0$ that

$$\mathrm{II} = \int_0^{+\infty} W(x)\psi(x)\partial_x\psi(x)\,\mathrm{d}x = -\frac{1}{2}\int_0^{+\infty} W'(x)\psi(x)^2\,\mathrm{d}x \le 0.$$

Because of $\mathrm{I} + \mathrm{II} = 0$, we conclude that $\mathrm{I} = 0$ must hold and thus $\partial_x\psi \equiv 0$. Hence $\psi$ is a constant function. Since $\psi(0) = 0$ by assumption, this implies that $\psi \equiv 0$. □

By applying the implicit function theorem together with Lemma 4.20, we can construct a unique local branch around any given solution of the equation $F(v, \lambda) = 0$ as follows.

**Proposition 4.22.** *Let $(v, \lambda) \in X_{even} \times (0, +\infty)$ solve $F(v, \lambda) = 0$. Then there exists an open interval $I = (\lambda - \varepsilon, \lambda + \varepsilon) \cap (0, +\infty)$ with some $\varepsilon > 0$ and a $\mathcal{C}^1$-map*

$$I \to X_{even}, \quad t \mapsto v_t$$

*such that $v_{t=\lambda} = v$ and $F(v_t, t) = 0$ for all $t \in I$. Moreover, there exists a neighborhood $N \subset X_{even}$ around $v$ such that $F(u, t) = 0$ with $(u, t) \in N \times I$ implies that $u = v_t$.*

### Global Uniqueness

We will see that we can extend every local branch $t \mapsto v_t$, given in Proposition 4.22, to all of $t \in (0, \lambda]$ thanks to a-priori bounds. First, we notice that we must have global uniqueness of solutions of $F(v, \lambda) = 0$ for sufficiently small $\lambda > 0$.

**Proposition 4.23.** *There exists $\lambda_\star > 0$ such that the solution $v \in X_{even}$ of $F(v, \lambda) = 0$ is unique for $0 < \lambda \leq \lambda_\star$.*

*Proof.* Let $u, v \in X_{\text{even}}$ both solve $F(u, \lambda) = F(v, \lambda) = 0$. Since $u, v \in X_{\text{even}}$ are symmetric-decreasing by Proposition 4.13, we see that $u$ and $v$ are fixed-points of the map $T_\lambda : X^\star \to X^\star$. If we combine the a-priori bounds in Lemma 4.17 with Lipschitz estimates in Lemma 4.15, we deduce that

$$\|u - v\|_{L^2(\mathbb{R})} \lesssim \lambda(\lambda^6 + \lambda)\|u - v\|_{L^2(\mathbb{R})}.$$

Thus there exists $\lambda_\star > 0$ sufficiently small such that $0 < \lambda \leq \lambda_\star$ implies $\|u - v\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\|u - v\|_{L^2(\mathbb{R})}$ and hence $u \equiv v$. $\square$

**Proposition 4.24.** *Let $(v, \lambda) \in X_{even} \times (0, +\infty)$ solve $F(v, \lambda) = 0$. Then there exists a $\mathcal{C}^1$-map*

$$(0, \lambda] \to X_{even}, \quad t \mapsto v_t$$

*such that $v_{t=\lambda} = v$ and $F(v_t, t) = 0$ for all $t \in (0, \lambda]$.*

*Proof.* Let $v \in X_{\text{even}}$ solve $F(v, \lambda) = 0$. First we define

$$t_\star \coloneqq \inf\Big\{t > 0 \,\Big|\, \text{there exists a } \mathcal{C}^1\text{-map } (t, \lambda] \to X_{\text{even}},\ s \mapsto v_s, \\ \text{such that } v_\lambda = v \text{ and } F(v_s, s) = 0 \text{ for all } s \in (t, \lambda]\Big\}$$

We have to show that $t_\star = 0$ to finish the proof.

By contradiction we assume that $t_\star > 0$. Let $(t_n)_n$ be a sequence in $(t_\star, \lambda)$ such that $t_n \to t_\star$. Then by definition of $t_\star$ we obtain $v_{t_n} = T_{t_n}(v_{t_n})$ and hence $v_{t_n}$ belongs to $X^\star$ by Proposition 4.13. Thanks to the a-priori bounds in Lemma 4.17 we immediately find the uniform bound

$$\|v_{t_n}\|_X \leq C(t_n^2 + t_n) \leq C(\lambda^2 + \lambda).$$

Therefore by applying Banach-Alaoglu and the compactness of the embedding $X \subset L^2(\mathbb{R})$, there exists a function $v_\star \in X^\star$ such that $v_{t_n} \rightharpoonup v_\star$ weakly converges in $X$ and $v_{t_n} \to v_\star$ strongly converges in $L^2(\mathbb{R})$ up to subsequences. By the local Lipschitz estimate in Lemma 4.15 and the uniform bound above, this yields

$$\begin{aligned}\|v_{t_n} - T_{t_\star}(v_\star)\|_X &= \|T_{t_n}(v_{t_n}) - T_{t_\star}(v_\star)\|_X \\ &\leq \|T_{t_n}(v_{t_n}) - T_{t_n}(v_\star)\|_X + \|T_{t_n}(v_\star) - T_{t_\star}(v_\star)\|_X \\ &\lesssim t_n(\lambda^6 + \lambda)\|v_{t_n} - v_\star\|_{L^2(\mathbb{R})} + |t_n - t_\star|\|T_1(v_\star)\|_X \to 0.\end{aligned}$$

In particular $v_{t_n} \to T_{t_\star}(v_\star)$ in $X$ whence it follows that $v_\star = T_{t_\star}(v_\star)$. So we are in the setting to apply Proposition 4.22 and extend the map $s \mapsto v_s$ to achieve a contradiction to the definition of $t_\star$. $\square$

We are now ready to prove global uniqueness.

**Theorem 4.25.** *Suppose $K$ satisfies Assumption **(B)**. If $v, \tilde{v} \in X^\star$ are solutions of (4.16), then it holds*

$$\widetilde{v}(0) = v(0) \quad \Rightarrow \quad \widetilde{v} \equiv v.$$

*Proof.* Let $v, \widetilde{v} \in X^\star$ be two solutions of (4.16). By the previous discussion, we have $F(v, \lambda) = F(\widetilde{v}, \lambda) = 0$ with $\lambda = v(0)/\sqrt{K(0)} > 0$. By Proposition 4.24, we can construct two branches

$$t \mapsto v_t \quad \text{and} \quad t \mapsto \widetilde{v}_t \quad \text{for all } t \in (0, \lambda]$$

such that $F(v_t, t) = F(\widetilde{v}_t, t) = 0$ for $t \in (0, \lambda]$ and $v_{t=\lambda} = v$ and $\widetilde{v}_{t=\lambda} = \widetilde{v}$. Suppose now $v \neq \widetilde{v}$. By the local uniqueness property in Proposition 4.22, the branches can never intersect, i. e., we have $v_t \neq \widetilde{v}_t$ for all $t \in (0, \lambda]$. But this contradicts the uniqueness result in Proposition 4.23 whenever $0 < t \ll 1$ is sufficiently small. Therefore, we conclude that $v = \widetilde{v}$. □

# Chapter 5

# Harmonic CM DNLS

The core of this chapter is to use the results presented in Chapter 3 and 4 with the objective to explicitly compute the ground state energy of the Calogero-Moser derivative NLS with external harmonic potential $V(x) = x^2$. Furthermore we will give a small outlook on local well-posedness of the time evolution. The equation we consider reads

$$i\partial_t \psi = -\partial_{xx}\psi + x^2\psi - \left((-\Delta)^{1/2}|\psi|^2\right)\psi + \frac{1}{4}|\psi|^4\psi \tag{5.1}$$

for a complex-valued field $\psi\colon [0,+\infty) \times \mathbb{R} \to \mathbb{C}$, where $\psi \in \mathcal{C}^0([0,T]; X)$ with $X := \left\{ v \in H^1(\mathbb{R}) \;\middle|\; xv \in L^2(\mathbb{R}) \right\}$.

## 5.1 Ground state energy of the harmonic CM DNLS

It is clear that the external potential $V(x) = x^2$ satisfies Assumption **(A)**. Therefore we can apply Theorem 3.1 to obtain the existence of ground states in $X$ under the constraint that the $L^2$-mass $N$ belongs to $(0, 2\pi)$. Recalling equation (3.3)

$$E(v) = \frac{1}{2}\|\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v\|^2_{L^2(\mathbb{R})} + \frac{1}{4\pi}N\,(2\pi - N)$$

in Lemma 3.2, we obtain that the energy is bounded from below by $\frac{1}{4\pi}N\,(2\pi - N)$. Our aim is to prove that for every $N \in (0, 2\pi)$ the ground state energy is actually given by this lower bound. Obviously this holds true if there exists $v \in X$ obeying $\|v\|^2_{L^2(\mathbb{R})} = N$, which solves the first order differential equation

$$\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v = 0. \tag{5.2}$$

By Proposition 4.18 we directly see that for every initial value $v_0 > 0$ there exists a solution $v \in X$ to the corresponding integral equation (and hence for (5.2)) satisfying $v(0) = v_0$. The question we want to answer is, whether we can also prove the existence of a solution to (5.2) for any given $L^2$-mass $N \in (0, 2\pi)$. That this is actually the case is stated in the next theorem.

**Theorem 5.1.** *For every $L^2$-mass $N \in (0, 2\pi)$ there exists $v \in X$ satisfying*

$$\partial_x v + xv + \tfrac{1}{2}\mathrm{H}(|v|^2)v = 0 \quad \textit{and} \quad \|v\|^2_{L^2(\mathbb{R})} = N.$$

Notice that a solution $v \in X$ to (5.2) solves the corresponding fixed-point equation

$$v = T_\lambda(v) := \lambda e^{-\frac{x^2}{2}} e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y}, \tag{5.3}$$

for $\lambda = v(0)$. According to Proposition 4.13, the function $v$ is radial-symmetric decreasing (up to a constant phase $e^{i\alpha}$).

By means of this we obtain the following corollary of Theorem 5.1.

**Corollary 5.2.** *For every $L^2$-mass $N \in (0, 2\pi)$ the ground state energy of (5.1) is*

$$I(N) := \inf\left\{E(v) \;\middle|\; v \in X,\ \|v\|^2_{L^2(\mathbb{R})} = N\right\} = \frac{1}{4\pi} N\,(2\pi - N)\,.$$

*Moreover every corresponding ground state soliton $v \in X$ of (1.3) is a radial-symmetric decreasing function (up to a phase $e^{i\alpha}$).*

*Proof. (Theorem 5.1)* We define the functional $G_N \colon X \to X$ by setting

$$G_N(v) = \sqrt{N}\frac{T_1(v)}{\|T_1(v)\|_{L^2(\mathbb{R})}}, \tag{5.4}$$

where $T_\lambda \colon X \to X$ is the functional defined in (5.3), which was previously introduced in Section 4.3 in a more general setting. Note that $G_N$ is well-defined because we have $T_1(v) \not\equiv 0$ for any $v \in X$. Moreover every fixed-point $v \in X$ of $G_N$ solves (5.2) with $\|v\|^2_{L^2(\mathbb{R})} = N$.

In the proof we apply Leray-Schauder fixed-point theorem (see [18] [Theorem 11.3]) to $G_N$. So we have to verify that $G_N \colon X \to X$ is a continuous, compact mapping and there exists a suitable $M > 0$ such that the following implication holds true:

$$\exists v \in X \text{ and } \exists \sigma \in (0,1] \text{ with } v = \sigma G_N(v) \quad \Rightarrow \quad \|v\|_X < M. \tag{5.5}$$

In the spirit of Lemma 4.15 and 4.16 we can argue that $T_1 : X \to X$ is locally Lipschitz continuous and compact. However we have to take care of the fact that in these lemmas we made the stronger assumption that the map is defined on $X^\star$, the set of radial symmetric-decreasing functions in $X$. Whereas in the proof of Lemma 4.15, we used this assumption to derive the uniform bound $e^{-\frac{1}{2}\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y} \leq 1$, in our particular case this is not necessary. Instead it suffices to bound the exponential by

$$\begin{aligned}\left|\int_0^x \mathrm{H}(v^2)(y)\,\mathrm{d}y\right| &\leq |x|^{1/2}\|\mathrm{H}(v^2)\|_{L^2(\mathbb{R})} \\ &= |x|^{1/2}\|v^2\|_{L^2(\mathbb{R})} \leq C|x|^{1/2}\|v\|^2_X,\end{aligned} \tag{5.6}$$

because of the quadratic exponential decay of the factor $e^{-\frac{x^2}{2}}$. We omit the details.

The continuity of $G_N$ directly follows from the continuity of $T_1$. The compactness can be proved as follows:

Let $(v_n)$ be a bounded sequence in $X$. For convenience we write $D := \sup_{k\in\mathbb{N}}(\|v_k\|_X^2) < +\infty$. Since $T_1\colon X \to X$ is compact, after passing to a subsequence if necessary, there exists a function $f \in X$ such that $(T_1(v_n))$ strongly converges to $f$ in $X$ and in particular in $L^2(\mathbb{R})$. We apply estimate (5.6) to obtain the lower bound

$$T_1(v_n) \geq e^{-\frac{x^2}{2}} e^{-C|x|^{1/2}D}.$$

This yields

$$\|T_1(v_n)\|_{L^2(\mathbb{R})} \geq C > 0,$$

for $C$ depending only on $D$. So $T_1(v_n) \to f \neq 0$ in $L^2(\mathbb{R})$. We obtain as a direct consequence that

$$G_N(v_n) = \sqrt{N}\frac{T_1(v_n)}{\|T_1(v_n)\|_{L^2(\mathbb{R})}} \to \sqrt{N}\frac{f}{\|f\|_{L^2(\mathbb{R})}} \quad \text{in } X, \quad \text{as } n \to +\infty$$

which concludes the proof of compactness.

It remains to verify implication (5.6). So we consider solutions $v = \sigma G_N(v)$ for $\sigma \in (0,1]$. We see that $v = \sigma G_N(v) = G_{\sigma^2 N}(v)$ and hence $v$ solves (5.2). Thus

$$E(v) = \frac{1}{4\pi}\sigma^2 N\left(2\pi - \sigma^2 N\right) \leq \frac{\pi}{4} \quad \text{and} \quad \|v\|_{L^2(\mathbb{R})}^2 = \sigma^2 N \leq N < 2\pi.$$

According to Lemma 3.4 we deduce the a-priori bound $\|v\|_X \leq C$.

In summary, the conditions of the Leray-Schauder fixed-point Theorem are satisfied and hence $G_N$ contains a fixed-point $v \in X$, which concludes the proof. □

## 5.2 Lens transform

In this section we will illustrate the connection between the harmonic Calogero-Moser NLS and the equation without external potential, which reads

$$i\partial_t\varphi = -\partial_{xx}\varphi - \left((-\Delta)^{1/2}|\varphi|^2\right)\varphi + \frac{1}{4}|\varphi|^4\varphi. \tag{5.7}$$

This might be of high interest, since we have a much deeper understanding of equation (5.7). We will use this connection to give a small outlook on the question of local well-posedness. A direct approach can be used to establish well-posedness on a small, not dense subset.

The following segment is based on the application of the Lens transform (see for example [7]). We start by explaining how solutions to (5.1) give rise to solutions of (5.7).

**Lemma 5.3.** *Let $\psi \in \mathcal{C}^0([0,\widetilde{T}];X)$ be a solution to (5.1). Define the Lens transform via*

$$\varphi(t,x) = \mathcal{L}(\psi)(t,x) = \frac{1}{(1+4t^2)^{1/4}} e^{\frac{itx^2}{(1+4t^2)}} \psi\left(\frac{\arctan(2t)}{2}, \frac{x}{(1+4t^2)^{1/2}}\right).$$

*Then* $\varphi \in \mathcal{C}^0([0,T];H^1(\mathbb{R}))$ *is a solution to (5.7). Here* $T = \tan(2\widetilde{T})/2$ *if* $\widetilde{T} < \pi/4$ *and any* $T > 0$ *can be chosen if* $\widetilde{T} \geq \pi/4$.

Before we turn to the proof we refer to the following remark, which will be used later on.

*Remark* 5.4. For $\tilde{t} < \pi/4$ the inverse of the Lens transform is given by

$$\mathcal{L}^{-1}(\varphi)(\tilde{t},\tilde{x}) = \frac{1}{\cos(2\tilde{t})^{\frac{1}{2}}} e^{-i\tilde{x}^2 \frac{\tan(2\tilde{t})}{2}} \varphi\left(\frac{\tan(2\tilde{t})}{2}, \frac{\tilde{x}}{\cos(2\tilde{t})}\right).$$

Whereas we only obtain $\psi(\tilde{t},\,\cdot\,) := \mathcal{L}^{-1}(\varphi)(\tilde{t},\,\cdot\,) \in L^2(\mathbb{R})$ for $\varphi(t,\,\cdot\,) \in H^1(\mathbb{R})$, it is not hard to show that the Lens transform actually defines a homeomorphism $\mathcal{C}^0([0,\tilde{T}];X) \to \mathcal{C}^0([0,T];X)$, for every $\tilde{T} < \pi/4$ and $T = \tan(2\tilde{T})/2$, when $X := \left\{v \in H^1(\mathbb{R}) \;\middle|\; xv \in L^2(\mathbb{R})\right\}$.

*Proof. (Lemma 5.3)* To simplify notation we write

$$\tilde{t} = \frac{\arctan(2t)}{2} \quad \text{and} \quad \tilde{x} = \frac{x}{(1+4t^2)^{1/2}}.$$

So

$$\varphi(t,x) = \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{1/4}}\psi(\tilde{t},\tilde{x}).$$

First, we notice that the $L^2$-norm is preserved under this transformation, which means that

$$\|\varphi(t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} = \|\psi(\tilde{t},\,\cdot\,)\|^2_{L^2(\mathbb{R})}$$

for every $t \geq 0$. Whereas to guarantee that $\varphi(t,\,\cdot\,)$ belongs to $H^1(\mathbb{R})$ it is indispensible to use the assumption that $\psi(\tilde{t},\,\cdot\,)$ is an element of $X$. This can be directly seen by the following computations.

$$\partial_x\varphi(t,x) = \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{1/4}}\left[\frac{i2t\tilde{x}}{(1+4t^2)^{1/2}}\psi(\tilde{t},\tilde{x}) + \frac{1}{(1+4t^2)^{1/2}}(\partial_x\psi)(\tilde{t},\tilde{x})\right]$$

and hence

$$\begin{aligned}
\|\partial_x\varphi(t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} &\leq \frac{2}{(1+4t^2)}\int_{\mathbb{R}}\left(4t^2\tilde{x}^2|\psi(\tilde{t},\tilde{x})|^2 + |(\partial_x\psi)(\tilde{t},\tilde{x})|^2\right)\frac{\mathrm{d}x}{(1+4t^2)^{1/2}} \\
&= \frac{2}{(1+4t^2)}\left(4t^2\|\tilde{x}\psi(\tilde{t},\,\cdot\,)\|^2_{L^2(\mathbb{R})} + \|(\partial_x\psi)(\tilde{t},\,\cdot\,)\|^2_{L^2(\mathbb{R})}\right) \\
&\leq 2\left(\|\tilde{x}\psi(\tilde{t},\,\cdot\,)\|^2_{L^2(\mathbb{R})} + \|(\partial_x\psi)(\tilde{t},\,\cdot\,)\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}$$

Therefore by our assumption on $\psi(\tilde{t},\,\cdot\,)$, we obtain that $\varphi(t,\,\cdot\,)$ is indeed an element of $H^1(\mathbb{R})$.

Some further computations, mainly using triangle inequality, yield that $\varphi \in \mathcal{C}^0([0,T];H^1(\mathbb{R}))$.

Next, we prove that $\varphi$ solves (5.7). By a density argument, we can always assume $\varphi$ to be regular enough. First notice that

$$\partial_t\tilde{t} = \frac{1}{(1+4t^2)} \quad \text{and} \quad \partial_t\tilde{x} = -\frac{4t\tilde{x}}{1+4t^2}.$$

So

$$i\partial_t\varphi(t,x) = \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{5/4}}\big[\left(-i2t-\tilde{x}^2+4t^2\tilde{x}^2\right)\psi(\tilde{t},\tilde{x}) \\ + i(\partial_t\psi)(\tilde{t},\tilde{x}) - i4t\tilde{x}(\partial_x\psi)(\tilde{t},\tilde{x})\big].$$

Moreover

$$\partial_{xx}\varphi(t,x) \\ = \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{5/4}}\left[\left(i2t-4t^2\tilde{x}^2\right)\psi(\tilde{t},\tilde{x}) + i4t\tilde{x}(\partial_x\psi)(\tilde{t},\tilde{x}) + (\partial_{xx}\psi)(\tilde{t},\tilde{x})\right].$$

To express the fractional Laplace of $|\varphi|^2$ in terms of $\psi$ we make the following computation using some basic facts about the Fourier transform. For convenience we write $a = (1+4t^2)^{1/2}$.

$$\begin{aligned}\left((-\Delta)^{1/2}|\varphi(t,\,\cdot\,)|^2\right)(x) &= \frac{1}{a}\mathcal{F}^{-1}\left(|\xi|\mathcal{F}(|\psi(\tilde{t},\tfrac{\cdot}{a})|^2)(\xi)\right)(x)\\ &= \frac{1}{a}\mathcal{F}^{-1}\left(|a\xi|\mathcal{F}(|\psi(\tilde{t},\,\cdot\,)|^2)(a\xi)\right)(x)\\ &= \frac{1}{a^2}\mathcal{F}^{-1}\left(|\xi|\mathcal{F}(|\psi(\tilde{t},\,\cdot\,)|^2)(\xi)\right)(x/a)\\ &= \frac{1}{a^2}\left((-\Delta)^{1/2}|\psi(\tilde{t},\,\cdot\,)|^2\right)(x/a)\\ &= \frac{1}{1+4t^2}\left((-\Delta)^{1/2}|\psi(\tilde{t},\,\cdot\,)|^2\right)(\tilde{x}).\end{aligned}$$

Combining the previous computations, we obtain

$$\begin{aligned}&(i\partial_t\varphi + \partial_{xx}\varphi)(t,x)\\ &\quad= \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{5/4}}\left[-\tilde{x}^2\psi(\tilde{t},\tilde{x}) + i(\partial_t\psi)(\tilde{t},\tilde{x}) + (\partial_{xx}\psi)(\tilde{t},\tilde{x})\right]\\ &\quad= \frac{e^{it\tilde{x}^2}}{(1+4t^2)^{5/4}}\left[-\left((-\Delta)^{1/2}|\psi(\tilde{t},\,\cdot\,)|^2\right)(\tilde{x})\psi(\tilde{t},\tilde{x}) + \frac{1}{4}\left(|\psi|^4\psi\right)(\tilde{t},\tilde{x})\right]\\ &\quad= -\left((-\Delta)^{1/2}|\varphi(t,\,\cdot\,)|^2\right)(x)\varphi(t,x) + \frac{1}{4}\left(|\varphi|^4\varphi\right)(t,x),\end{aligned}$$

which is the desired result. □

We mention that we have a much better understanding of (5.7) than of the case where a harmonic potential is added. This makes it worth considering the question under which conditions a function $\varphi$ satisfying (5.7) gives rise to a solution $\psi = \mathcal{L}^{-1}(\varphi)$ of (5.1).

However, two problems arise. A solution $\varphi$ of (5.7) does not necessarily satisfy the decay condition $x\varphi \in L^2(\mathbb{R})$. Though for $\varphi(t,\,\cdot\,) \in H^1(\mathbb{R})$ we only obtain $\mathcal{L}^{-1}(\varphi)(\tilde{t},\,\cdot\,) \in L^2(\mathbb{R})$, so in general we cannot expect $\psi$ to be a solution of (5.1). That this is not a hypothetical issue which mostly affects pathological cases becomes apparent by the following important example.

Whereas $\varphi := \sqrt{2/(1+x^2)} \in H^1(\mathbb{R})$ is a stationary solution to (5.7), its inverse Lens transform $\mathcal{L}^{-1}(\varphi)$ does not even belong to $H^1(\mathbb{R})$. In addition although in the case that $\varphi$ is a global-in-time solution, $\psi$ may only be defined for $\tilde{t} \in (-\pi/4, \pi/4)$.

Nevertheless, we can give an implicit statement. We start by introducing the required notation. Let $H^s_+(\mathbb{R}) := \{u \in H^s(\mathbb{R}) \mid \operatorname{supp}(\hat{u}) \subset [0,\infty)\}$ and let $\Pi_+ \colon L^2(\mathbb{R}) \to L^2_+(\mathbb{R}) = H^0_+(\mathbb{R})$ denote the Cauchy–Szegő orthogonal projector. We consider the Calogero-Szegő NLS, which reads as

$$i\partial_t \rho = -\partial_{xx}\rho + 2i\partial_x \Pi_+(|\rho|^2)\rho. \tag{5.8}$$

In [17] local well-posedness of (5.8) is established on $H^s_+(\mathbb{R})$, as $s > 1/2$. We are only interested in the case $s = 1$ here.

Next we apply the Gauge transform $\varphi(t,x) = \rho(t,x)e^{-\frac{i}{2}\int_{-\infty}^x |\rho(t,y)|^2 \,\mathrm{d}y}$, which is a homeomorphism on $H^s(\mathbb{R})$ for $s \geq 0$ and then a homeomorphism on $X$ too. It turns out that $\varphi$ solving (5.7) is equivalent to $\rho$ solving (5.8), see Lemma A.9. So we achieve local well-posedness of (5.7) at least in the energy space $\{ue^{-\frac{i}{2}\int_{-\infty}^x |u|^2 \,\mathrm{d}y} \mid u \in H^1_+(\mathbb{R})\}$.

Notice that for an initial value $\varphi_0 \in X \cap \{ue^{-\frac{i}{2}\int_{-\infty}^x |u|^2 \,\mathrm{d}y} \mid u \in H^1_+(\mathbb{R})\}$ the variance $\|x\varphi\|^2_{L^2(\mathbb{R})}$ of the locally unique solution $\varphi$ to (5.7) does not blow-up on a time inverval $[0,T]$, where the quantity $T > 0$ is only depending on $\|\varphi_0\|_{H^1(\mathbb{R})}$, see Lemma A.10. According to Remark 5.4 on the Lens transform we thus have established local well-posedness of the harmonic Calogero-Moser NLS on $X \cap \mathcal{L}^{-1}\left(\{ue^{-\frac{i}{2}\int_{-\infty}^x |u|^2 \,\mathrm{d}y} \mid u \in H^1_+(\mathbb{R})\}\right)$.

# Appendix A

# Some Technical Facts

## A.1 Properties of the Hilbert Transform and the Fractional Laplacian

First, we collect some basic properties of the Hilbert transform.

**Lemma A.1.** *Let* $\mathrm{H}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *be the Hilbert transform defined by* $\widehat{\mathrm{H}(f)}(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi)$. *Then* H *satisfies the following properties.*

(i) *If* $f$ *is real-valued then* $\mathrm{H}(f)$ *is real-valued too.*

(ii) H *is an anti-involution, that is to say that* $\mathrm{H}(\mathrm{H}(f)) = -f$.

(iii) H *is an anti self-adjoint operator on* $L^2(\mathbb{R})$, *i.e.* $\mathrm{H}^\star = -\mathrm{H}$.

(iv) $\mathrm{H}\colon H^1(\mathbb{R}) \to H^1(\mathbb{R})$ *is an isometry and* $\partial_x \mathrm{H}(f) = \mathrm{H}(\partial_x f) = (-\Delta)^{1/2} f$ *holds.*

(v) H *anticommutes with the reflection, i.e. for* $Rf(x) = f(-x)$ *we obtain* $\mathrm{H}(Rf)(x) = -\mathrm{H}(f)(-x)$. *In particular* $\mathrm{H}(f)$ *is odd for any even function* $f$.

**Lemma A.2.** *Let* $f \in L^2(\mathbb{R})$ *and* $w \in L_{1/2}(\mathbb{R})$. *The following equivalence holds true:*

$$(-\Delta)^{1/2} w = f \iff w \text{ belongs to } \dot{H}^1(\mathbb{R}) \text{ and solves } \partial_x w = -\mathrm{H}(f).$$

**Lemma A.3.** *Lef* $f \in L^2(\mathbb{R})$ *satisfy* $f = f^\star$. *Then*

$$\mathrm{H}(f)(x) \geq 0 \quad \text{for } x \geq 0 \quad \text{and} \quad \mathrm{H}(f)(x) \leq 0 \quad \text{for } x \leq 0.$$

*Proof.* Since $f = f^\star$ is an even function, we note that by Lemma A.1 (v) above, its Hilbert transform is an odd function. Thus it suffices to show that $\mathrm{H}(f)(x) \geq 0$ for $x \geq 0$. Next, we consider the singular integral representation of the Hilbert transformation

$$\mathrm{H}(f)(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{+\infty} \frac{f(x-y) - f(x+y)}{y} \, \mathrm{d}y.$$

Since $|x-y| \leq |x|+|y| = |x+y|$ for $x, y \geq 0$ and $f = f^*$ is symmetric-decreasing, we see that $f(x-y) \geq f(x+y)$ for all $x, y \geq 0$. Thus the claim directly follows from the integral expression for $\mathrm{H}(f)(x)$. □

**Lemma A.4.** *Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then $\varphi$ satisfies the decay bound*

$$|(-\Delta)^{1/2}\varphi(x)| \leq \frac{C}{x^2},$$

*for a suitable $C = C(\varphi) > 0$.*

This inequality is a special case of a statement established in [20], where $|(-\Delta)^s\varphi| \lesssim |x|^{-n-2s}$ for any $s > 0$ is proven.

*Proof.* For $x \neq 0$, we write $\mathbb{R}$ as a union of the sets

$$A_1 = (-|x|/2, |x|/2) \quad \text{and} \quad A_2 = \mathbb{R} \setminus A_1.$$

By the singular integral representation we obtain

$$\begin{aligned}(-\Delta)^{1/2}\varphi(x) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{(x-y)^2}\,\mathrm{d}y \\ &= -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{y^2}\,\mathrm{d}y.\end{aligned}$$

We thus derive

$$|(-\Delta)^{1/2}\varphi(x)| \leq \frac{1}{2\pi}(I_1 + I_2),$$

where

$$I_j = \left| \int_{A_j} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{y^2}\,\mathrm{d}y \right|.$$

On $A_1$ we find

$$\varphi(x+y)+\varphi(x-y)-2\varphi(x) = \frac{1}{2}(\varphi''(\xi_y)+\varphi''(\xi_{-y}))y^2,$$

where $\xi_y$ and $\xi_{-y}$ belong to $(x-|x|/2, x+|x|/2)$ and thus $|\xi_y|, |\xi_{-y}| \geq |x|/2$. Since $\varphi$ is a Schwartz function we derive $|\varphi''(z)| \leq \frac{C}{|z|^3}$ for a suitable $C = C(\varphi)$, whence it follows that

$$I_1 \leq \frac{1}{2}\int_{A_1} (|\varphi''(\xi_y)| + |\varphi''(\xi_{-y})|)\,\mathrm{d}y \lesssim \frac{1}{|x|^3}\int_{A_1} \mathrm{d}y = \frac{1}{|x|^2}.$$

On $A_2$ we immediately find

$$\begin{aligned} I_2 &\lesssim \frac{1}{|x|^2}\int_{A_2} |\varphi(x+y)+\varphi(x-y)|\,\mathrm{d}y + |\varphi(x)|\int_{A_2}\frac{1}{y^2}\,\mathrm{d}y \\ &\lesssim \frac{\|\varphi\|_{L^1(\mathbb{R})}}{|x|^2} + \frac{|\varphi(x)|}{|x|} \lesssim \frac{1}{|x|^2},\end{aligned}$$

where in the last step we use $|\varphi(x)| \leq \frac{C}{|x|}$ for a $C > 0$ depending only on $\varphi$. □

## A.2 Functional Tools

**Theorem A.5** ( [22] Theorem 8.10)**.** *Let $1 < p < +\infty$ and let $(f_j)$ be a bounded sequence of functions in $W^{1,p}(\mathbb{R}^n)$. Suppose that for some $\varepsilon > 0$ the set $E^j := \{x \mid |f_j(x)| > \varepsilon\}$ has a measure $|E^j| > \delta > 0$ for some $\delta$ and all $j$. Then there exists a sequence of vectors $y_j \in \mathbb{R}^n$ such that the translated sequence $f_j(\,\cdot\, + y_j)$ has a subsequence that converges weakly in $W^{1,p}(\mathbb{R}^n)$ to a nonzero function.*

**Theorem A.6** ( [22] Chapter II Exercise 22)**.** *Suppose that $1 \leq p < q < r \leq +\infty$ and that $f$ is a function in $L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $\|f\|_{L^p(\mathbb{R}^n)} \leq C_p < +\infty$, $\|f\|_{L^r(\mathbb{R}^n)} \leq C_r < +\infty$ and $\|f\|_{L^q(\mathbb{R}^n)} \geq C_q > 0$. Then there exist constants $\varepsilon > 0$ and $M > 0$, depending only on $p, q, r, C_p, C_q, C_r$ such that $|\{x \mid |f(x)| > \varepsilon\}| > M$.*

**Lemma A.7.** *Let $f \in L^1(\mathbb{R}; (1 + \log(1 + |x|))\mathrm{d}x) \cap L^2(\mathbb{R})$. Then*

$$\lim_{|x|\to+\infty} \int_{\mathbb{R}} \log\left|\frac{x-y}{x}\right| f(y)\,\mathrm{d}y = 0. \tag{A.1}$$

*Proof.* For convenience we use the notation $k(x,y) = \log\left|\frac{x-y}{x}\right|$ for the kernel of the singular integral. Moreover we always assume $|x| \geq 2$. We split $\mathbb{R}$ into the two sets $\{|x-y| \geq |x|/2\}$ and $\{|x-y| \leq |x|/2\}$.

To integrate on $\{|x-y| \geq |x|/2\}$, we notice that

$$\log\left(\frac{1}{2}\right) \leq k(x,y)\mathbb{1}_{\{|x-y|\geq|x|/2\}} \leq \log\left(\frac{|x|+|y|}{|x|}\right) \leq \log\left(1+|y|\right).$$

So

$$|k(x,y)\mathbb{1}_{\{|x-y|\geq|x|/2\}} f(y)| \leq (\log(2) + \log\left(1+|y|\right))\,|f(y)|,$$

where the right-hand side belongs to $L^1(\mathbb{R})$ by assumption. Since in addition $\lim_{|x|\to+\infty} k(x,y) = 0$ we can argue by dominated convergence theorem that

$$\lim_{|x|\to+\infty} \int_{\{|x-y|\geq|x|/2\}} k(x,y)f(y)\,\mathrm{d}y = 0.$$

On $\{|x-y| \leq |x|/2\}$, we treat the following three integrals separately.

$$\begin{aligned}\int_{\{|x-y|\leq|x|/2\}} k(x,y)f(y)\,\mathrm{d}y = &-\log|x| \int_{\{|x-y|\leq|x|/2\}} f(y)\,\mathrm{d}y \\ &+ \int_{\{1<|x-y|\leq|x|/2\}} \log|x-y| f(y)\,\mathrm{d}y \\ &+ \int_{\{|x-y|\leq 1\}} \log|x-y| f(y)\,\mathrm{d}y.\end{aligned}$$

First notice that for $|x-y| \leq |x|/2$ it holds $|y| \geq |x| - |x-y| \geq |x|/2$. Thus by recalling that we assume $|x| \geq 2$, we obtain

$$0 \leq \log|x| \leq \log(2|y|) = \log(2) + \log|y| \leq \log(2) + \log(1+|y|).$$

Therefore we derive

$$\begin{aligned}0 &\leq \log|x| \int_{\{|x-y|\leq |x|/2\}} |f(y)|\,\mathrm{d}y \\ &\leq \log(2) \int_{\{|y|\geq |x|/2\}} |f(y)|\,\mathrm{d}y + \int_{\{|y|\geq |x|/2\}} \log(1+|y|)|f(y)|\,\mathrm{d}y,\end{aligned}$$

which clearly converges to 0 as $|x| \to +\infty$ due to the assumption that $f$ and $\log(1+|y|)f$ both belong to $L^1(\mathbb{R})$.

For the second integral we again use that $|y| \geq |x| - |x-y| \geq |x|/2$ for $|x-y| \leq |x|/2$ to obtain

$$\begin{aligned}0 &\leq \int_{\{1<|x-y|\leq |x|/2\}} \log|x-y||f(y)|\,\mathrm{d}y \\ &\leq \int_{\{1<|x-y|\leq |x|/2\}} \log(|x|/2)|f(y)|\,\mathrm{d}y \leq \int_{\{|y|\geq |x|/2\}} \log(1+|y|)|f(y)|\,\mathrm{d}y.\end{aligned}$$

As seen above this converges to 0 as $|x| \to +\infty$.

For the last integral we simply use Hölder's inequality to derive

$$\int_{\{|x-y|\leq 1\}} \big|\log|x-y|f(y)\big|\,\mathrm{d}y \leq \left(\int_{-1}^{1} \log(z)^2\,\mathrm{d}z\right)^{\frac{1}{2}} \left(\int_{\{|x-y|\leq 1\}} f(y)^2\,\mathrm{d}y\right)^{\frac{1}{2}},$$

which tends to 0 since $f \in L^2(\mathbb{R})$. □

**Lemma A.8.** *The map $F\colon X_{even} \times (0,+\infty) \to X_{even}$ given in (4.24) is well-defined and of class $\mathcal{C}^1$.*

*Proof.* Let $(u,\lambda) \in X_{\mathrm{even}} \times (0,+\infty)$ be given. First we show that $F(u,\lambda) \in X_{\mathrm{even}}$ as follows. We set

$$h_u(x) := -\int_0^x \mathrm{H}(u^2)(y)\,\mathrm{d}y.$$

We claim that

$$h_u(x) \leq A\|u\|_X^2 \quad \text{for all } x \in \mathbb{R}, \tag{A.2}$$

with some constant $A > 0$ independent of $u$ and $x$. Indeed, we note that $\partial_x h_u = -\mathrm{H}(u^2)$ and hence we obtain $(-\Delta)^{1/2} h_u = u^2$, according to Lemma A.2. By adapting the arguments in the proof of Lemma 4.5 and using that $h_u \in L_{1/2}(\mathbb{R})$ as well as $\log(1+|\cdot|)|u|^2 \in L^1(\mathbb{R})$, we deduce that

$$h_u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) u(y)^2\,\mathrm{d}y + C = -\frac{1}{\pi}\int_{\mathbb{R}} \log|x-y|u(y)^2\,\mathrm{d}y + C_0$$

with some constant $C_0 \in \mathbb{R}$. Since $h_u(0) = 0$, we find the upper bound

$$C_0 = \frac{1}{\pi}\int_{\mathbb{R}} \log|y|u(y)^2\,\mathrm{d}y \leq \frac{1}{\pi}\int_{\mathbb{R}} \log(1+|y|)u(y)^2\,\mathrm{d}y \leq \frac{1}{\pi}\|u\|_X^2.$$

Furthermore, we estimate

$$
\begin{aligned}
-\frac{1}{\pi}\int_{\mathbb{R}} \log|x-y| u(y)^2 \,\mathrm{d}y &\le -\frac{1}{\pi}\int_{x-1}^{x+1} \log|x-y| u(y)^2 \,\mathrm{d}y \\
&\le \frac{1}{\pi}\,\|\log|\cdot|\|_{L^2((-1,1))}\,\|u^2\|_{L^2(\mathbb{R})} \le B\|u\|_X^2
\end{aligned}
$$

with some constant $B > 0$. This completes the proof of (A.2).

Using (A.2), we see that $e^{\frac{1}{2}h_u} \in L^\infty(\mathbb{R})$ and thus, by our assumptions on $K$, we deduce that $F(u,\lambda) = \lambda\sqrt{K}e^{\frac{1}{2}h_u} - u$ satisfies

$$
\int_{\mathbb{R}} |F(u,\lambda)(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}} \log(1+|x|)|F(u,\lambda)(x)|^2 \,\mathrm{d}x < +\infty.
$$

Similarly, we show that $\partial_x F(u,\lambda) = \lambda e^{\frac{1}{2}h_u}\left(\partial_x\sqrt{K} - \frac{1}{2}\sqrt{K}\mathrm{H}(u^2)\right) \in L^2(\mathbb{R})$. Finally, it is easy to see that $F(u,\lambda)(-x) = F(u,\lambda)(x)$ using that $u(x) = u(-x)$ holds for $u \in X_{\text{even}}$. This proves that $F(u,\lambda) \in X_{\text{even}}$.

The fact that the map $F\colon X_{\text{even}} \times (0,+\infty) \to X_{\text{even}}$ is continuous follows by using dominated convergence together with previous bounds, standard estimates, and the fact that $u_n \to u$ in $X_{\text{even}}$ implies that $h_{u_n} \to h_u$ pointwise. This pointwise convergence can be easily shown by the estimates

$$
\begin{aligned}
|h_{u_n}(x) - h_u(x)| &\le \int_0^{|x|} |\mathrm{H}(u_n^2 - u^2)(y)| \,\mathrm{d}y \\
&\le |x|^{1/2}\|u_n^2 - u^2\|_{L^2(\mathbb{R})} \le |x|^{1/2}\|u_n + u\|_X\|u_n - u\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Furthermore, again by dominated convergence it is straightforward to verify that $F$ is of class $\mathcal{C}^1$. We omit the details. □

## A.3 Properties of Equations

**Lemma A.9.** *Consider the Gauge transform* $\varphi(t,x) = \rho(t,x)e^{-\frac{i}{2}\int_{-\infty}^x |\rho(t,y)|^2 \,\mathrm{d}y}$. *The following two are equivalent.*

(i) $\rho \in \mathcal{C}^0([0,T]; H^1(\mathbb{R}))$ *solves (5.8).*

(ii) $\varphi \in \mathcal{C}^0([0,T]; H^1(\mathbb{R}))$ *solves (5.7).*

*Proof.* First, notice that the Gauge transform is a homeomorphism on $H^1(\mathbb{R})$.

Next, we assume that $\rho \in H^1(\mathbb{R})$ solves (5.8) and prove that then $\varphi(t,x) = \rho(t,x)e^{-\frac{i}{2}\int_{-\infty}^x |\rho(t,y)|^2 \,\mathrm{d}y}$ is a solution of (5.7). To show the opposite direction one can argue in the exact same way. In the following every computation is done on a formal level. We write $\mu(t,x) = e^{-\frac{i}{2}\int_{-\infty}^x |\rho(t,y)|^2 \,\mathrm{d}y}$ to simplify the notation. We start by computing the following derivative.

$$
i\partial_t\varphi = \mu\left(\int_{-\infty}^x \Re(\bar{\rho}\,\partial_t\rho)\,\mathrm{d}y\,\rho + i\partial_t\rho\right) = \mu\left(\Im\int_{-\infty}^x \bar{\rho}\,i\partial_t\rho\,\mathrm{d}y\,\rho + i\partial_t\rho\right). \tag{A.3}
$$

Using that $\rho$ is a solution to (5.8) and $\Pi_+ = \frac{1}{2}(1 + i\mathrm{H})$, we obtain

$$\begin{aligned}\Im \int_{-\infty}^{x} \bar{\rho}\, i\partial_t \rho \,\mathrm{d}y &= \Im \int_{-\infty}^{x} \bar{\rho}\left(-\partial_{xx}\rho + 2i\partial_x \Pi_+(|\rho|^2)\rho\right) \,\mathrm{d}y \\ &= \Im \int_{-\infty}^{x} 2i\partial_x \Pi_+(|\rho|^2)|\rho|^2 \,\mathrm{d}y - \Im\left(\bar{\rho}\,\partial_x \rho\right) \\ &= \Im \int_{-\infty}^{x} \partial_x (i - \mathrm{H})(|\rho|^2)|\rho|^2 \,\mathrm{d}y + \frac{i}{2}\left(\bar{\rho}\,\partial_x \rho - \rho\,\partial_x \bar{\rho}\right).\end{aligned}$$

We recall that by item (i) of Lemma A.1, $\mathrm{H}(|\rho|^2)$ is real-valued, whence it follows that

$$\Im \int_{-\infty}^{x} \partial_x (i - \mathrm{H})(|\rho|^2)|\rho|^2 \,\mathrm{d}y = \int_{-\infty}^{x} \left(\partial_x |\rho|^2\right)|\rho|^2 \,\mathrm{d}y = \frac{1}{2}\int_{-\infty}^{x} \partial_x |\rho|^4 \,\mathrm{d}y = \frac{1}{2}|\rho|^4.$$

Inserting this result into (A.3) and applying (5.8) with $i\partial_x \Pi_+ = \frac{1}{2}(i\partial_x - (-\Delta)^{1/2})$, we arrive at

$$\begin{aligned} i\partial_t \varphi &= \mu\left(\frac{1}{2}|\rho|^4\rho + \frac{i}{2}\left(\bar{\rho}\,\partial_x\rho - \rho\,\partial_x\bar{\rho}\right)\rho + i\partial_t\rho\right) \\ &= \mu\left(\frac{1}{2}|\rho|^4\rho + \frac{i}{2}\left(\bar{\rho}\,\partial_x\rho - \rho\,\partial_x\bar{\rho}\right)\rho - \partial_{xx}\rho + i(\partial_x|\rho|^2)\rho - (-\Delta)^{1/2}(|\rho|^2)\rho\right) \\ &= \mu\left(\frac{1}{2}|\rho|^4\rho + i|\rho|^2\partial_x\rho + \frac{i}{2}(\partial_x|\rho|^2)\rho - \partial_{xx}\rho - (-\Delta)^{1/2}(|\rho|^2)\rho\right).\end{aligned}$$

In the last step we used the identity $\frac{1}{2}\left(\bar{\rho}\,\partial_x\rho - \rho\,\partial_x\bar{\rho} + (\partial_x|\rho|^2)\right)\rho = |\rho|^2\partial_x\rho$. Next, we compute

$$\partial_{xx}\varphi = \mu\left(-\frac{1}{4}|\rho|^4\rho - i|\rho|^2\partial_x\rho - \frac{i}{2}(\partial_x|\rho|^2)\rho + \partial_{xx}\rho\right).$$

Combining these two equations yields

$$i\partial_t\varphi + \partial_{xx}\varphi = \mu\left(\frac{1}{4}|\rho|^4\rho - (-\Delta)^{1/2}(|\rho|^2)\rho\right) = \frac{1}{4}|\varphi|^4\varphi - (-\Delta)^{1/2}(|\varphi|^2)\varphi,$$

which is exactly (5.7). □

**Lemma A.10.** *Let $\varphi \in \mathcal{C}^0([0,T]; H^1(\mathbb{R}))$ be a solution to*

$$i\partial_t\varphi = -\partial_{xx}\varphi - \left((-\Delta)^{1/2}|\varphi|^2\right)\varphi + \frac{1}{4}|\varphi|^4\varphi,$$

*with initial datum $\varphi_0 \in X \coloneqq \left\{v \in H^1(\mathbb{R}) \;\middle|\; xv \in L^2(\mathbb{R})\right\}$. Then the variance $\|x\varphi\|^2_{L^2(\mathbb{R})}$ has no blow-up on the time interval $[0,T]$ and thus $\varphi$ belongs to $\mathcal{C}^0([0,T]; X)$.*

*Proof.* We consider the following estimate on the derivative of the variance

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x\varphi\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} x^2 2\Re(\bar\varphi\,\partial_t\varphi)\,\mathrm{d}x = 2\Im\int_{\mathbb{R}} x^2\bar\varphi\, i\partial_t\varphi\,\mathrm{d}x \\
&= 2\Im\int_{\mathbb{R}} x^2\bar\varphi\left(-\partial_{xx}\varphi - \left((-\Delta)^{1/2}|\varphi|^2\right)\varphi + \frac{1}{4}|\varphi|^4\varphi\right)\,\mathrm{d}x \\
&= 2\Im\int_{\mathbb{R}} x^2\bar\varphi\left(-\partial_{xx}\varphi\right)\,\mathrm{d}x \\
&= 2\Im\left(\int_{\mathbb{R}} 2x\bar\varphi\,\partial_x\varphi\,\mathrm{d}x + \int_{\mathbb{R}} x^2|\partial_x\varphi|^2\,\mathrm{d}x\right) \\
&\le 4\|x\varphi\|_{L^2(\mathbb{R})}\|\partial_x\varphi\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

On the other hand we may use the expression

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|x\varphi\|^2_{L^2(\mathbb{R})} = 2\|x\varphi\|_{L^2(\mathbb{R})}\frac{\mathrm{d}}{\mathrm{d}t}\|x\varphi\|_{L^2(\mathbb{R})},
$$

whence it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|x\varphi\|_{L^2(\mathbb{R})} \le 2\|\partial_x\varphi\|_{L^2(\mathbb{R})}.
$$

In view of the fact that $\varphi \in \mathcal{C}^0([0,T];H^1(\mathbb{R}))$, the right hand side is bounded by a constant $C > 0$, which directly yields

$$
\|x\varphi\|_{L^2(\mathbb{R})} \le CT + \|x\varphi_0\|_{L^2(\mathbb{R})},
$$

which proves the desired result. □

# List of Notations

| | |
|---|---|
| $\mathcal{C}^0(X;Y)$ | space of continuous functions $X \to Y$ |
| $\mathcal{C}^k(X;Y)$ | space of $k$-times continuously differentiable functions $X \to Y$ |
| $\mathcal{C}^k(U)$ | space of $k$-times continuously differentiable functions $U \subset \mathbb{R} \to \mathbb{C}$ (or $\mathbb{R}$) |
| $\mathcal{C}_c^\infty(\mathbb{R}^n)$ | space of infinitely many-times differentiable functions $\mathbb{R}^n \to \mathbb{C}$ (or $\mathbb{R}$) with compact support |
| $\mathcal{C}^{k,\alpha}(U)$ | $= \left\{ u \in \mathcal{C}^k(U) \,\middle|\, \lVert u \rVert_{\mathcal{C}^{k,\alpha}(U)} := \sum_{j=1}^k \lVert \partial_{x^j} u \rVert_{L^\infty(U)} + [u]_{\mathcal{C}^{k,\alpha}(U)} < +\infty \right\}$, where $U \subset \mathbb{R}$ and $[u]_{\mathcal{C}^{k,\alpha}(U)} := \sup\left\{ \frac{\lvert \partial_{x^j} u(x) - \partial_{x^j} u(y) \rvert}{\lvert x-y \rvert^\alpha} \,\middle|\, x, y \in U,\ x \neq y \right\}$ |
| $L^p(U)$ | $= \left\{ u \colon U \to \mathbb{C} \text{ (or } \mathbb{R}) \text{ Lebesgue measurable} \,\middle|\, \lVert u \rVert_{L^p(\mathbb{R})} < +\infty \right\}$, where $U \subset \mathbb{R}^n$ and $\lVert u \rVert_{L^p(U)} := \begin{cases} \left( \int_U \lvert u(x) \rvert^p \,\mathrm{d}x \right)^{1/p} & \text{if } p \in [1, +\infty) \\ \operatorname{ess\,sup}\{ \lvert u(x) \rvert \mid x \in U \} & \text{if } p = +\infty \end{cases}$ |
| $L^2_{\text{even}}(\mathbb{R})$ | $= \left\{ u \in L^2(\mathbb{R}) \,\middle|\, u(x) = u(-x) \right\}$, |
| $H^s(\mathbb{R})$ | $= \left\{ u \in \mathcal{S}'(\mathbb{R}) \,\middle|\, \lVert u \rVert_{H^s(\mathbb{R})} = \lVert (1 + \lvert \xi \rvert^2)^{s/2} \hat{u} \rVert_{L^2(\mathbb{R})} < +\infty \right\}$ |
| $\dot{H}^1(\mathbb{R})$ | $= \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) \,\middle|\, \lVert \partial_x u \rVert_{L^2(\mathbb{R})} < +\infty \right\}$ |
| $\dot{H}^1_{\text{even}}(\mathbb{R})$ | $= \left\{ u \in \dot{H}^1(\mathbb{R}) \,\middle|\, u(x) = u(-x) \right\}$, |
| $L_{1/2}(\mathbb{R})$ | $= \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} \frac{\lvert u(x) \rvert}{1+x^2} \,\mathrm{d}x < +\infty \right\}$ |
| $\mathcal{S}(\mathbb{R})$ | Schwartz space |
| $\mathcal{S}'(\mathbb{R})$ | space of tempered distributions |
| $\mathcal{F}(u)(\xi)$ | $= \hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix\xi} u(x) \,\mathrm{d}x$, the Fourier transform |

| | |
|---|---|
| $\mathcal{F}^{-1}(u)(x)$ | $= \check{u}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix\xi} u(\xi)\, \mathrm{d}\xi$, the inverse Fourier transform |
| $(-\Delta)^s u$ | $= \mathcal{F}^{-1}(\lvert\xi\rvert^{2s}\hat{u})$, the fractional-laplacian |
| $\mathrm{H}(u)$ | $= \mathcal{F}^{-1}(-i\mathrm{sgn}(\xi)\hat{u})$, the Hilbert transform |
| $\langle u, v\rangle$ | $= \int_{\mathbb{R}} \bar{u}v\, \mathrm{d}x$, the complex inner product on $L^2(\mathbb{R})$ |
| $v^\star$ | symmetric-decreasing rearrangement of $v$ |
| $\langle x\rangle$ | $= \sqrt{1+x^2}$ |
| $\Re$ | real part |
| $\Im$ | imaginary part |
| $PV$ | principle value |

# Basic Inequalities

The following inequalities are frequently used throughout the work.

$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ — Young's inequality

where $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$\|uv\|_{L^r(\mathbb{R})} \leq \|u\|_{L^p(\mathbb{R})} \|v\|_{L^q(\mathbb{R})}$, — Hölder's inequality

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $1 \leq p, q, r \leq +\infty$.

$\|u\|_{L^r(\mathbb{R})} \leq \|u\|^{\vartheta}_{L^p(\mathbb{R})} \|u\|^{1-\vartheta}_{L^q(\mathbb{R})}$, — Interpolation inequality

where $\frac{\vartheta}{p} + \frac{1-\vartheta}{q} = \frac{1}{r}$ and $1 \leq p \leq r \leq q \leq +\infty$.

$\|u\|_{L^r(\mathbb{R})} \leq C_{p,q,r} \|\partial_x u\|^{\vartheta}_{L^p(\mathbb{R})} \|u\|^{1-\vartheta}_{L^q(\mathbb{R})}$, — Gagliardo-Nirenberg inequality

where $\vartheta \left(\frac{1}{p} - 1\right) + \frac{1-\vartheta}{q} = \frac{1}{r}$,

$0 \leq \vartheta \leq 1$ and $1 \leq p, q, r < +\infty$.

$\|u\|_{\mathcal{C}^{k-1,\frac{1}{2}}(\mathbb{R})} \leq C_k \|u\|_{H^k(\mathbb{R})}$, — Sobolev inequality

where $k$ is a positive integer.

$\|\mathrm{H}u\|_{L^p(\mathbb{R})} \leq C_p \|u\|_{L^p(\mathbb{R})}$, — Boundedness of the Hilbert transform

where $1 < p < +\infty$.

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# Curriculum Vitae

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