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Sorting in Iterated Incumbency Contests

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Abstract This paper analyzes incumbency contests in a large population setting. Incumbents repeatedly face different challengers, holding on to their positions until defeated in a contest. Defeated incumbents turn into challengers until they win a contest against an incumbent, thereby regaining an incumbency position. Individuals are heterogeneous as regards their payoffs from being incumbent. We consider steady-state equilibria and study how and to which extent individuals sort into the incumbency positions depending on their type. In particular, we identify sufficient conditions for positive sorting, meaning that types with higher incumbency payoffs are overrepresented among the incumbents, and show that negative rather than positive sorting may also arise in equilibrium when these conditions are violated. Further results show how incumbency rents, surplus, and sorting are affected by the frequency at which incumbency is contested.

Keywords Contests · Sorting · Incumbency Rents · Steady-State Equilibrium

JEL classification C72 · D72 · D74

1 Introduction

Repeated incumbency contests describe situations in which incumbents face a sequence of challenges to their position, with each challenger seeking to replace the current incumbent in a contest to thereby gain an incumbency rent. Such repeated incumbency contests arise in various contexts and models thus differ in what they assume about the fate of defeated incumbents and challengers. In some models there is a constant inflow of new challengers and defeated contestants never get a second chance to obtain an incumbency position in the future. This is the situation analyzed in Stephan and Ursprung (1998) and Virág (2009). We may think of this as modelling uprisings against medieval kings or modern despots in which only the successful contestant survives. In other models incumbents and challengers simply switch roles after a successful challenge and face each other again in the next period. Such is the case in models of civil wars (Mehlum and Moene, 2006), coups (Acemoglu and Robinson, 2001), struggles between lobbying groups (Polborn, 2006), or electoral competition in two-party systems (Baron, 1996; Azzimonti, 2011; Bai and Lagunoff, 2011; Battaglini, 2014; Forand, 2014).

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In this paper we study a model of repeated incumbency contests in which at any given point in time there are many incumbents and challengers. Defeated contestants return as challengers in the future but when doing so face a different opponent than before. We refer to such a contest as an iterated incumbency contest. In the literature, such iterated incumbency contests have been considered as models of both human and animal territorial conflicts in anarchic environments, in which the transient nature of both victories and losses is salient and individuals move on to different targets when having lost the fight over the control of a given territory. A prominent example of such a model in economics is Hafer (2006); examples from behavioral ecology include Grafen (1987), Eshel and Sansone (1995, 2001), and Kokko et al. (2006). Replacing territories by other contestable resources yields alternative economic interpretations of such models. For instance, one may think of territories as market niches with small-scale entrepreneurs (e. g. mobile app developers or fashion designers) holding on to a market niche until they are dislodged by a competitor and then moving on to try to dislodge a competitor from another niche in turn.[1] Similarly, seats on supervisory boards come with a stream of income and benefits attached to them and are contestable (by lobbying or buying stock in a public company) with defeated contestants typically not coming back to try to obtain a seat on the supervisory board of the same company but rather trying to join the board of another suitable company.

In the examples just described, individuals will typically be heterogeneous in some payoff-relevant dimension. For instance, some entrepreneurs have a higher ability to exploit a market niche than others and some people may enjoy the perks of being on a supervisory board more than others.[2] In such circumstances one might expect that individuals who extract higher intrinsic benefits from being in an incumbency position are better motivated to obtain and defend incumbency position and therefore should be overrepresented among the incumbents. The main purpose of this paper is to test the intuition that such positive sorting into the incumbency positions must arise in the equilibrium of an iterated incumbency contest. More generally, we are interested in the factors determining the extent of sorting in equilibrium and in studying how the iterated nature of the contests affects (relative) incumbency rents. These questions are not only interesting from a descriptive perspective but also from a normative one as positive sorting ensures that those individuals who extract higher benefits from incumbency positions are more likely to hold them.

In our model of an iterated incumbency contest there is a continuum of identical incumbency positions distributed among members of a unit mass population. Each individual can hold at most one incumbency position at a given time. We refer to individuals that do so as incumbents, and individuals that do not as challengers. Time is continuous. Individuals meet other individuals, randomly sampled from the population, at a fixed, exogenous rate. Whenever a meeting is between an incumbent and a challenger, a contest ensues. If the current incumbent wins the contest, then the challenger awaits her next opportunity to contest an incumbent. If the current incumbent loses the contest, then the challenger takes over the position and the incumbent joins the pool of challengers. We study steady states in which the strategies and the utilities of all individuals are time-invariant.

Individuals come in a finite number of types, differing in their commonly known flow payoff obtained while being incumbent. The masses of these types, that we refer to as the population distribution, are exogenous. For any pair of such types, the contest will determine the probability that in a contest between this pair of types the incumbency position changes hands. These probabilities together with the population distribution determine the incumbent distribution, that is, the masses of the different types that hold an incumbency position. The probabilities of winning the contest are in turn determined by the incumbency rents. Crucially, these incumbency rents are endogenous because they not only depend on the flow payoff obtained while being incumbent but also on the future rents lost in defending and the future rents gained by attacking an incumbency positions – with these latter two terms depending on the incumbency rents of other agents and the incumbent distribution.

Regarding the contests, we do not restrict attention to a particular contest model but capture the structure of a variety of complete-information contests by specifying contest payoffs and winning prob-

[1] In reality, rather than dislodging a competitor, entrepreneurs may also innovate, thereby generating a new market niche, and may grow their business by taking over and holding on to more than one niche. Similarly, in the example we discuss next, individuals may join the supervisory board of a newly launched company and may occupy seats on more than one supervisory board. We return to these points in our discussion in Section 6.

[2] It is just as natural to think of heterogeneity between individuals as arising from differences in the costs of fighting for an incumbency position. As we explain in Section 6, our analysis applies equally to this case.

abilities as functions of the incumbency rents of the two contestants. We impose a number of regularity conditions on these functions: (i) both the winning probabilities and the payoffs are continuous in the rents, (ii) payoffs are consistent with the assumptions that either contestant can forego the incumbency position at zero cost and that neither contestant receives more than her rent in case of winning, and (iii) the contestant's winning probabilities are strictly increasing in own rents. This approach is convenient as we can abstract from the details of the contests in our general analysis but still relate to particular contest models when discussing our results. Specifically, our assumptions on the contest outcomes encompass the most familiar complete-information contests, namely the Tullock contest (Tullock, 1980) and the all-pay auction (Baye et al., 1996). We discuss other contests from the literature (surveyed in Konrad, 2009) when introducing the formal model in Section 2.1.

Equilibrium is characterized by an incumbent distribution specifying a time-invariant mass of incumbents for each type and an incumbency rent profile specifying a time-invariant incumbency rent for each type. Equilibrium requires that the incumbent distribution and the rent profile correspond to a steady state and are consistent with optimal behavior in the component contests.

Our first main result (Proposition 1) establishes that there is always an equilibrium with positive sorting in which individuals with higher flow payoffs during incumbency have a higher likelihood of being incumbent. Next, we turn to establish conditions which ensure that *all* equilibria exhibit such positive sorting. Proposition 2 shows this to be the case when meeting are sufficiently infrequent. Proposition 3 identifies a sufficient condition on the structure of the contest ensuring positive sorting for all meeting rates. This condition requires the contest to be monotonic, which entails that the challenger's expected gain from engaging in the contest is increasing in the challenger's rent and, similarly, that the incumbent's expected loss from engaging in the contest is increasing in the incumbent's rent. The all-pay auction and the Tullock contest are such monotonic contests and for these contests we can thus conclude that positive sorting must arise in every equilibrium.

Following Virág (2009), we consider the limit when the meeting process becomes frictionless, so that both incumbents and challengers are continually engaged in contests. With such continual contests the incumbency rents of all types vanish under fairly general conditions (Lemma 3). Somewhat surprisingly, for a broad class of contests, including the all-pay auction and the Tullock contest, positive sorting nevertheless obtains in the limit (Proposition 4) and the surplus generated by the availability of incumbency positions is not fully dissipated (Proposition 5).

For the special case of an iterated all-pay auction with two types we establish uniqueness of equilibrium (Proposition 6) and discuss the comparative statics with respect to the frequency of meetings. In particular, we show that sorting becomes more pronounced with an increase the meeting rate when the incumbency positions are scarce, yet becomes less pronounced when the incumbency positions are abundant (Proposition 7). Finally, adding a fixed cost of attack to the all-pay auction provides an example of a contest that, due to a strong discouragement effect of an increase in the incumbents' rent on the challenger, fails the monotonicity condition on the loss of the incumbent in Proposition 3. This sets the stage for the existence of an equilibrium with negative sorting. Indeed, for a broad range of parameter values exactly one equilibrium with positive sorting and exactly one equilibrium with negative sorting coexist (Proposition 8). In the latter equilibrium individuals with lower flow payoff obtain higher incumbency rents because they face a lower probability of being dislodged from their position.

From a modeling perspective, the most novel aspect of this paper is to use an equilibrium-search model with a large population of heterogeneous agents to model iterated contests. Embedding the individual contests in the dynamic framework of a search model allows us to understand how behavior in the component contests is impacted by the "shadow of the future" while abstracting from the well-known complications arising in a repeated-game framework. Embedding pairwise interactions into a dynamic framework via the use of a search model is, of course, a well-established modeling strategy in other contexts. For surveys of the vast literature concerned with search models of the labor market we refer the reader to Pissarides (1990) or Rogerson et al. (2005). Closer to our concerns is the literature on sorting in search-and-matching models with heterogenous agents, recently surveyed in Chade et al. (2017). Relative to this second strand of literature, the most important distinguishing feature of our analysis arises from the very nature of incumbency contests, namely that meetings do not generate but destroy rents for incumbents and that sorting does not arise from agent's decisions which matches to accept but from the intensity with which agents fight to maintain or change their positions.[3] Models of iterated territorial conflicts in behavioral ecology (Grafen, 1987; Eshel and Sansone, 1995, 2001; Kokko et al., 2006) also embed pairwise conflicts in a search model but differ from our model in that (i) the conflicts between incumbent and intruder are modelled as a Hawk-Dove game in which the contestants simply decide whether to fight or not and (ii) the issue of sorting does not arise because individuals are not heterogeneous.

Closest in spirit to our paper is Virág (2009), who considers a model in which there is only one incumbency position and assumes that defeated incumbents leave the game forever. Contests in Virág (2009) are modelled as an all-pay auction with private values, with values drawn from a continuous distribution. The focus of his analysis is on the question whether sorting must be perfect in the sense that the incumbent distribution is concentrated on the highest possible type. His main finding is that such perfect sorting does not obtain in a stationary equilibrium even in the limit when the arrival rate of challengers goes to infinity. In our framework with perfect information such perfect sorting cannot arise in equilibrium under the assumptions on the contest outcomes we introduce in Section 2. Rather, what is not obvious is whether there is any sorting at all in the limit and it is therefore this question that we address in Proposition 4.

Hafer (2006) considers, as we do, a setup with many incumbency positions and random meetings between pairs of individuals, but her focus is rather different from ours. Instead of focusing on steady states, Hafer (2006) studies the transitional dynamics from an arbitrary initial situation. In her setting contest efforts cease after the transitional period, whereas in our setting costly contests are a recurring phenomenon in steady state. Further important differences are that (i) she, like Virág (2009), considers a setting with incomplete information and (ii) she restricts attention to the war of attrition when modeling the interaction between challenger and incumbent, whereas our main results impose relatively little structure on the incumbency contests.

The question how the anticipation of future incumbency rents determines equilibrium behavior in a sequence of pairwise contests between incumbents and challengers is also the central concern in the models of Stephan and Ursprung (1998), Mehlum and Moene (2006), and Polborn (2006). Stephan and Ursprung (1998) consider a model with a single incumbency position. The incumbent faces a stream of challengers and contestants exit from the game once they have been defeated. Mehlum and Moene (2006) as well as Polborn (2006) consider the repeated interaction between two contestants, who switch roles whenever a successful attack is mounted. The focus in Mehlum and Moene (2006) is on the effect of an incumbency edge on the intensity of fighting; in Polborn (2006) it is (as in Acemoglu and Robinson, 2001) on the optimal timing of an attack. The issue of sorting, which is our central concern, is not studied in any of these contributions.

The remainder of the paper is organized as follows: Section 2 describes our model of iterated incumbency contests and defines equilibrium. Section 3 considers positive sorting. Section 4 analyzes continual contests. Section 5 offers a detailed analysis of the iterated all-pay auction with two types and shows how extending the all-pay auction by adding an an attack cost for the challenger may give rise to equilibria with negative sorting. Section 6 concludes by discussing extensions and open questions. All proofs are in the appendix.

2 The Model

Our model has two main building blocks: a description of equilibrium behavior in a given incumbency contest as a function of the contestants' values of winning or losing the contest and a population framework with heterogenous agents in which incumbency contests arise over time from random meetings between incumbents and challengers. Section 2.1 introduces the first of these building blocks, Section 2.2 the second. Section 2.3 then ties the building blocks together by defining equilibrium. In a nutshell, equilibrium requires that the distributions of agents who hold incumbency positions is in steady state and that all agents' steady-state continuation values of winning or losing an incumbency contest are consistent with equilibrium behavior in the subsequent contests they will face.

[3] The feature that meetings may destroy rents is also present in the otherwise rather different model of marriage and divorce investigated in Cornelius (2003) in which agents may unilaterally decide to leave a current relationship upon encountering a more attractive partner, thereby leaving their current partner worse off.

2.1 Incumbency Contests

We consider two-player contests between an incumbent I and a challenger C. A contest is parameterized by specifying both contestants' values for winning or losing the contest. We denote these values by $(W_I, L_I) \in \mathbb{R}^2_+$ for the incumbent and by $(W_C, L_C) \in \mathbb{R}^2_+$ for the challenger and assume that winning is more attractive than losing: $W_I > L_I$ and $W_C > L_C$. The parameters are assumed to be common knowledge among the contestants. Both contestants are risk neutral. If the challenger wins the contest with probability p and loses it with probability $1-p$, then the challenger's payoff is $pW_C + (1-p)L_C - \kappa_C$ and the incumbent's payoff is $(1-p)W_I + pL_I - \kappa_I$, where $\kappa_C \geq 0$ and $\kappa_I \geq 0$ are the costs the two contestants incur in their fight for the incumbency position.

Let

$$x_I = W_I - L_I > 0 \text{ and } x_C = W_C - L_C > 0 \tag{1}$$

denote the contestants' incumbency rents. We model the outcome of the contest as a function of these rents, thereby gaining the flexibility to accommodate a variety of different contests in our formal analysis. In particular, we suppose that any contest between an incumbent with values (W_I, L_I) and a challenger with values (W_C, L_C) has a unique equilibrium with expected payoff $L_C + \sigma(x_I, x_C)$ and winning probability $\mu(x_I, x_C)$ for the challenger and expected payoff $W_I - \tau(x_I, x_C)$ and winning probability $1 - \mu(x_I, x_C)$ for the incumbent.[4] We are thus expressing equilibrium payoffs in terms of the challenger's expected gain $\sigma(x_I, x_C)$ from participating in the contest (compared to the benchmark of simply obtaining L_C) and the incumbents' expected loss $\tau(x_I, x_C)$ from participating in the contest (compared to the benchmark of simply obtaining W_I).[5] When convenient, we will sometimes refer to $x_I - \tau(x_I, x_C)$ as the incumbents' expected gain (compared to the benchmark of simply obtaining L_I).

Throughout our analysis we impose the following three assumptions on the expected gains, expected losses, and winning probabilities.

Assumption 1. *The functions $\sigma : \mathbb{R}^2_{++} \to \mathbb{R}$, $\tau : \mathbb{R}^2_{++} \to \mathbb{R}$, and $\mu : \mathbb{R}^2_{++} \to [0,1]$ are continuous.*

Assumption 2. *The expected gains $\sigma(x_I, x_C)$ and the expected losses $\tau(x_I, x_C)$ satisfy*

$$0 \leq \sigma(x_I, x_C) \leq x_C, \quad 0 \leq \tau(x_I, x_C) \leq x_I \tag{2}$$

and

$$\sigma(x_I, x_C) - \tau(x_I, x_C) \leq \max\{0, x_C - x_I\} \tag{3}$$

for all $(x_I, x_C) \in \mathbb{R}^2_{++}$.

Assumption 3. *The challenger's winning probability $\mu(x_I, x_C)$ is strictly decreasing in x_I and strictly increasing in x_C.*

Assumption 1 serves to ensure the existence of equilibria in our population model (cf. Proposition 1 in Section 3). It is also required for the limit analysis we conduct in Section 4.

Assumption 2 imposes some minimal consistency requirements on expected gains and losses. The inequalities in (2) state that both participants in the contest are assured to obtain at least their value from losing the contest as their equilibrium payoffs ($\sigma(x_I, x_C) \geq 0, \tau(x_I, x_C) \leq x_I$) and can never expect to gain more than their incumbency rent from participating in the contest ($\sigma(x_I, x_C) \leq x_C$, $\tau(x_I, x_C) \geq 0$). Upon adding $L_C + W_I$ on both sides of it, inequality (3) is easily seen to be equivalent to the statement that the sum of the two contestants equilibrium payoffs is smaller than $\max\{L_C + W_I, L_I + W_C\}$, which is a natural consequence of the fact that there is only one incumbency position to be filled, so that in any outcome of the contest there is one winner and one loser.

Assumption 3 is equivalent to assuming that for both contestants their equilibrium probability of winning the contest is strictly increasing in their own incumbency rents. This assumption excludes the possibility that a contestant's winning probability is constant over some range of rents. In particular, it excludes contests in which one of the contestants enjoys a sufficiently large head-start (Konrad, 2002; Siegel, 2014; Franke et al., 2018): if, say, the incumbent has such a head-start, then the equilibrium of

[4] The presumption that equilibrium in any given contest is unique is restrictive. We impose it to focus on the effects that arise from challengers and incumbents being repeatedly engaged in such contests.

[5] While unconventional, expressing equilibrium payoffs in terms of expected gains for challengers and expected losses for incumbents will greatly simplify the exposition once we embed incumbency contests into a population framework.

the contest will involve a boundary solution in which the challenger has zero probability of winning the contest whenever x_C is sufficiently small relative to x_I. Remark 2 in Section 2.3 explains why we exclude such contests from consideration.

Before we proceed, we note that our formulation of a contest is not confined to simultaneous-move one-shot contests but also covers contests complying with Assumption 1 - 3 in which contestants move sequentially (Baik and Shogren, 1992; Leininger, 1993) or simultaneously sink efforts in multiple periods along the lines of Yildirim (2005). As is the case for contests with head-starts (and for essentially the same reason), contests where the contestants first have to sink a fixed cost in order to participate as in Fu et al. (2015) violate Assumption 3. Section 5.3 illustrates this for an all-pay auction in which the challenger has to pay a fixed cost before attacking the incumbent.

The familiar all-pay auction and the equally familiar Tullock contest will serve as leading examples throughout our subsequent analysis:

Contest 1 (All-Pay Auction). *In the all-pay auction with complete information, the incumbent and the challenger simultaneously sink efforts $e_I, e_C \geq 0$ at cost equal to the chosen effort. A contestant wins for sure if she chooses the higher effort. Ties are broken by a fair coin toss.*

It is well-known (Baye et al., 1996) that this contest has a unique equilibrium in mixed strategies. In this equilibrium, the challenger wins the contest with probability

$$\mu(x_I,x_C) = \begin{cases} \frac{1}{2}\frac{x_C}{x_I} & \text{if } x_C < x_I \\ 1-\frac{1}{2}\frac{x_I}{x_C} & \text{if } x_C \geq x_I \end{cases}. \tag{4}$$

and the equilibrium payoffs are $\max\{x_I - x_C, 0\} + L_I$ *for the incumbent and* $\max\{x_C - x_I, 0\} + L_C$ *for the challenger. In terms of the functions σ and τ we thus have*

$$\sigma(x_I,x_C) = \max\{x_C - x_I, 0\} \tag{5}$$

$$\tau(x_I,x_C) = x_I - \max\{x_I - x_C, 0\}. \tag{6}$$

Contest 2 (Tullock Contest). *In a Tullock contest with complete information, the incumbent and the challenger simultaneously sink efforts $e_I, e_C \geq 0$ at cost equal to the chosen effort. Given such effort choices, the challenger wins with probability*

$$p(e_I,e_C) = \begin{cases} \frac{e_C^r}{e_I^r + e_C^r} & \text{if } e_I + e_C > 0 \\ \frac{1}{2} & \text{if } e_I + e_C = 0 \end{cases}, \tag{7}$$

and I wins with probability $1 - p(e_I, e_C)$.

Under the parameter restriction $0 < r \leq 1$, *which we impose throughout the following, a unique pure strategy equilibrium exists for all* $(x_C, x_I) \in \mathbb{R}^2_{++}$, *in which (Nti, 1999)*

$$\sigma(x_I,x_C) = \frac{x_C^{r+1}}{\left(x_I^r + x_C^r\right)^2}\left[x_C^r + (1-r)x_I^r\right] \tag{8}$$

$$\tau(x_I,x_C) = x_I - \frac{x_I^{r+1}}{\left(x_I^r + x_C^r\right)^2}\left[x_I^r + (1-r)x_C^r\right] \tag{9}$$

and

$$\mu(x_I,x_C) = \frac{x_C^r}{x_I^r + x_C^r}. \tag{10}$$

It is easily verified that the all-pay auction and the Tullock contest satisfy Assumptions 1 - 3. They are also homogeneous, monotonic, and role symmetric in the sense of the following definitions.[6]

[6] For the all-pay auction this claim is immediate from (4)–(6). For the Tullock contest homogeneity and role symmetry are immediate from (8)–(10). The first half of the monotonicity property ($\sigma(x_I, x_C)$ increasing in x_C) is established in Nti (1999). The second half follows similarly by observing that from (9) the derivative of $\tau(x_I, x_C)$ with respect to x_I is positive if and only if

$$x_I^{3r} + (2+r^2)x_I^{2r}x_C^r + (r+1)(1-r)x_I^r x_C^{2r} \leq (x_I^r + x_C^r)^3,$$

which is satisfied for $0 < r \leq 1$ and $(x_I, x_C) \in \mathbb{R}^2_{++}$.

Definition 1 (Homogeneity) A contest is homogenous if $\sigma(x_I, x_C)$ and $\tau(x_I, x_C)$ are both homogenous of degree 1 in (x_I, x_C) and $\mu(x_I, x_C)$ is homogenous of degree 0 in (x_I, x_C).

Definition 2 (Monotonicity) A contest is monotonic if $\sigma(x_I, x_C)$ is increasing in x_C and $\tau(x_I, x_C)$ is increasing in x_I.

Definition 3 (Role Symmetry) A contest is role symmetric if for all $(y,z) \in \mathbb{R}^2_{++}$

$$\mu(y,z) = 1 - \mu(z,y) \tag{11}$$

$$\sigma(y,z) = z - \tau(z,y). \tag{12}$$

These three properties play an important role in our subsequent analysis and we thus offer some discussion here.

It is immediate from Malueg and Yates (2005, Proposition 1) that contests in which the contest success function is homogenous of degree zero in contestants' efforts are also homogenous in the sense of our Definition 1. Many of the contests commonly studied in the literature, including all-pay auctions and Tullock contests with asymmetric bid-effectiveness for the challenger and the incumbent (Leininger, 1993; Franke et al., 2014), are thus homogenous.[7] Such contests also satisfy our Assumption 3 (Malueg and Yates, 2005, Proposition 4) and the first half of our monotonicity property (Malueg and Yates, 2005, Proposition 5), which requires that the challenger's expected gain is increasing in her own rent. The second part of the monotonicity property in Definition 2, requiring that the incumbent's expected loss is increasing in her rent, has, to the best of our knowledge, not been explored in the literature (but does hold, as we have noted above, in the all-pay auction and in the Tullock contest). This requirement is equivalent to the requirement that the incumbents' expected gain $x_I - \tau(x_I, x_C)$ increases at a rate smaller than one in her value of winning the contest. The difficulty in establishing this property is that increasing the incumbent's value of winning the contest may decrease the challenger's equilibrium effort (see Malueg and Yates, 2005, Proposition 3 for the case of homogenous contests). If this discouragement effect is sufficiently strong, then the incumbent's equilibrium payoff might increase by more than the increase in her rent. In Section 5.3 we investigate a contest which fails the monotonicity property for exactly this reason.

Condition (11) in the definition of role symmetry states that a contestant's winning probability does not depend on her role in the contest (i.e., whether she is the incumbent or the challenger) but is solely determined by her own incumbency rent and the incumbency rent of her opponent. Condition (12) makes an analogous statement for contest payoffs: given an incumbency rent z, the expected gains when being a challenger are equal to those when being an incumbent.

2.2 Population Framework

There is a unit mass of risk-neutral and infinitely lived individuals. Time is continuous. All individuals discount future payoffs at rate $\rho > 0$. At each moment in time an individual either holds an incumbency position or not and will be referred to as an incumbent or a challenger accordingly. The mass of incumbency positions is fixed and given by $\theta \in (0,1)$.

Individuals come in $n \geq 2$ different, exogenous types labeled by $i \in N = \{1, \ldots, n\}$. The type of an individual determines the flow payoff $v_i > 0$ received while holding an incumbency position. Types are ordered such that $v_1 < v_2 < \ldots < v_n$. When not holding an incumbency position all individuals receive a flow payoff of zero. The proportion of type i individuals is given by $f_i > 0$. We refer to the vector $f = (f_i, ..., f_n)$, which satisfies $\sum_i^n f_i = 1$, as the population distribution. To avoid a knife-edge case in the proof of Lemma 3, we impose the genericity condition

$$\sum_{j \in J} f_j \neq \theta \text{ for all } J \subset N. \tag{13}$$

We will consider steady states in which the masses of incumbents and challengers of the different types are time-invariant and all incumbency positions are held by some agent, so that at each moment in

[7] Appendix A in Malueg and Yates (2005) provides more examples. Contests in which the contest success function is homogenous of degree zero are also studied in Baik (2004).

time a fraction θ of the individuals are incumbents whereas the remaining fraction $1-\theta$ of the individuals are challengers. We use $g_i \geq 0$ to denote the mass of incumbents of type i in such a steady-state, whereas $f_i - g_i \geq 0$ is the corresponding mass of challengers. Let

$$\mathscr{G} = \{g \in \mathbb{R}^n_+ : g_i \leq f_i, \forall i \in N \text{ and } \sum_{i=1}^{n} g_i = \theta\}. \tag{14}$$

We refer to a vector $g = (g_1, \ldots, g_n) \in \mathscr{G}$ as an incumbent distribution.

Incumbency contests arise when challengers meet incumbents. For simplicity, we suppose that meetings between individuals are random and generated by a quadratic search technology (Diamond and Maskin, 1979) with exogenous meeting rate $m > 0$: Thus, each challenger meets incumbents of type $i \in N$ at rate mg_i and each incumbent meets challengers of type $i \in N$ at rate $m(f_i - g_i)$. Every meeting between an incumbent and a challenger triggers an incumbency contest. If the incumbent wins the contest, both contestants retain their current roles. If the challenger wins the contest, she obtains the incumbency position, whereas the previous incumbent becomes a challenger.

2.3 Equilibrium

As mentioned previously, we restrict attention to steady states. Further, we only consider steady states which are type-symmetric in the sense that any two individuals with the same type have identical continuation values when being in the same role. In line with the notation introduced in Section 2.1, we denote the continuation payoff of an incumbent of type i by W_i and the continuation payoff of a challenger of type i by L_i and let $x_i = W_i - L_i$ denote the incumbency rent of an individual of type i, which we take to be strictly positive.

Payoffs and winning probabilities in every contest are determined as discussed in Section 2.1. In particular, if a challenger of type i encounters an incumbent of type j, then the resulting gain for the challenger is $\sigma(x_j, x_i)$, the resulting loss for the incumbent is $\tau(x_j, x_i)$, and the winning probability for the challenger is $\mu(x_j, x_i)$. In defining equilibrium we can thus focus on the determination of the incumbent distribution $g \in G$ and an incumbency rent profile $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$. Together the incumbent distribution and the incumbency rent profile provide sufficient information to determine all the continuation values (cf. equations (16) and (17) below).

In a steady state, the rates at which individuals of different types switch from being challengers to being incumbents and vice versa are jointly determined by the incumbent distribution g and the vector of incumbency rents x, with the former determining the meeting rates and the latter determining the winning probability of a challenger. In particular, the rate at which challengers of type i obtain incumbency positions is given by $m\sum_j g_j \mu(x_j, x_i)$ and the rate at which incumbents of type i lose incumbency positions (and thereby turn into challengers) is given by $m\sum_{j\in N}(f_j - g_j)\mu(x_i, x_j)$. To maintain a steady state, the inflow and the outflow to and from incumbency must balance for each type. Hence, steady state requires the following balance conditions, which are our first equilibrium requirement:

$$(f_i - g_i) \sum_{j\in N} g_j \mu(x_j, x_i) = g_i \sum_{j\in N} (f_j - g_j)\mu(x_i, x_j), \quad \forall i \in N. \tag{15}$$

Our second equilibrium requirement is that (properly discounted) expected gains and losses from future interactions generate continuation values consistent with the incumbency rents presumed in calculating expected gains and losses. Given an incumbency rent profile x and an incumbent distribution g, the continuation value L_i of a type-i challenger is given by

$$L_i = \frac{m}{\rho} \sum_{j\in N} \sigma(x_j, x_i) g_j. \tag{16}$$

This is so because the flow payoff of a challenger is zero, implying that the continuation value of a challenger is nothing but the discounted expected value of the future gains arising from the possibility of contesting an incumbency position. As meetings with a type-j incumbent lead to an expected payoff of $L_i + \sigma(x_j, x_i)$, the expected gain of such a meeting is $\sigma(x_j, x_i)$. Hence, taking all potential meetings and the rates mg_j at which they occur into account, the future expected gain is given by the right side of (16). Similarly, taking the present value of the stream of flow payoff v_i and the expected present value of the future losses arising from being drawn into a contest into account, the continuation value of a type-i incumbent is given by

$$W_i = \frac{v_i}{\rho} - \frac{m}{\rho} \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j). \tag{17}$$

Equations (16) and (17) embody the requirement that continuation values are determined by flow payoffs and expected future gains and losses. To turn this into an equilibrium condition in terms of an incumbency rent profile and incumbent distribution, we recall the identity $x_i = W_i - L_i$ from (1) and subtract (16) from (17) to obtain the value equations

$$\rho x_i = v_i - m \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(x_j, x_i) g_j, \quad \forall i \in N. \tag{18}$$

Together with the the balance conditions (15), the value equations (18) define equilibrium (with the associated equilibrium continuation values given by (16) and (17)):

Definition 4 (Equilibrium) An equilibrium is a tuple $(x, g) = (x_1, \ldots, x_n, g_1, \ldots, g_n) \in \mathbb{R}^n_{++} \times \mathscr{G}$ satisfying the balance conditions (15) and the value equations (18).

From the value equations (18) it is evident that $x_i \leq v_i/\rho$ holds in every equilibrium for all types $i \in N$. That is, the iterated nature of the contests reduces the incumbency rents below the present value of having an incumbency position forever.

Before we proceed to the substance of our analysis, we observe that for any given incumbency rent profile x there is a unique incumbent distribution g solving the balance conditions (15), providing us with a counterpart to Lemma 4 (the "fundamental matching lemma") in Shimer and Smith (2000). For later reference we state this observation as a lemma.

Lemma 1 *For every $x \in \mathbb{R}^n_{++}$, there is a unique $g \in \mathscr{G}$ such that (x, g) solves the balance conditions* (15) *and this g satisfies $0 < g_i < f_i$ for all $i \in N$.*

The proof of Lemma 1 in Appendix A adapts arguments developed in Banaji and Baigent (2008) to establish uniqueness of equilibrium in a model of an electron transfer networks. Besides the continuity of the function $\mu(x_I, x_C)$ (Assumption 1), the key property required for the argument is an implication of Assumption 3, namely that the challenger's winning probability is strictly positive no matter what the types of the challenger and the incumbent are. It is this implication which also yields that any g solving the balance conditions satisfies $0 < g_i < f_i$ for all $i \in N$. These inequalities may be interpreted as the statement that sorting in our model is never perfect, so that every type of agent is represented both in the incumbent and in the challenger population.

Remark 1 It is part of our equilibrium definition that all incumbency rents are strictly positive. Condition (2) in Assumption 2 then ensures that whenever a challenger and an incumbent meet they are at least as well off from participating in the contest rather than avoiding it, thereby justifying our presumption that every such meeting results in a contest. Nevertheless, one may wonder what happens if we had taken the possibility of zero or negative incumbency rents into account. Let us thus suppose that there is some type i for which the continuation value of being a challenger weakly exceeds the continuation value of being an incumbent. When in the role of a challenger a type-i individual would then weakly prefer not to engage in any contests and, presuming that doing so is possible, thus obtain a continuation value of zero. On the other hand, presuming that incumbents can also avoid participating in any contests by abandoning the incumbency position upon meeting a challenger, the continuation payoff of any type-i incumbent must be strictly positive as such an incumbent can simply collect the strictly positive flow payoff v_i until first meeting a challenger. Consequently, the same considerations that motivate Assumption 2 in Section 2.1 preclude the possibility of zero or negative incumbency rents arising in our population framework. Observe, though, that zero incumbency rents arise in the limit case of continual contests that we consider in Section 4.

Remark 2 Suppose that, contrary to what Assumption 3 requires, incumbents have such a strong advantage in the component contest such that for $(x_I, x_C) \in [v_1/\rho, v_n/\rho]^2$ equilibrium in this contest entails $\mu(x_I, x_C) = \sigma(x_I, x_C) = \tau(x_I, x_C) = 0$: the incumbency position is maintained by the current incumbent for sure (which is the violation of Assumption 3) and neither contestant incurs any effort cost. For any

$g \in \mathscr{G}$, the tuple $(v_1/\rho,\dots,v_n/\rho,g_1,\dots,g_n)$ is then an equilibrium because for the incumbency rent profile $x=(v_1/\rho,\dots,v_n/\rho)$ the expected gain and loss terms disappear from the value equations (18) and, further, the balance conditions (15) hold vacuously. Consequently, our model is silent on the determination of the equilibrium incumbent distribution. The import of requiring strict (rather than weak) monotonicity of $\mu(x_I,x_C)$ in its arguments in Assumption 3 is to eliminate this trivial source of equilibrium multiplicity by ensuring that $\mu(x_I,x_C) > 0$ holds for all $(x_I,x_C) \in \mathbb{R}^2_{++}$, thereby implying (see Lemma 1 above) that for any given strictly positive vectors of incumbency rents there is a unique associated incumbent distribution.

3 Positive Sorting

We are interested in determining whether equilibrium induces positive sorting in the sense that the share of individuals of a given type who hold incumbency positions is strictly increasing in type. This is captured in the following definition:

Definition 5 (Positive Sorting) There is positive sorting in equilibrium (x,g) if g_i/f_i is strictly increasing in i.

The following lemma delivers a straightforward necessary and sufficient condition for positive sorting in equilibrium, namely that incumbency rents x_i are increasing in i. The proof is in Appendix B (as are all other proofs for this section).

Lemma 2 *There is positive sorting in equilibrium (x,g) if and only if x_i is strictly increasing in i.*

Recalling that individuals with higher types obtain higher flow payoffs, Lemma 2 simply asserts that positive sorting will obtain whenever higher flow payoffs from holding the incumbency position translate into higher incumbency rents. To obtain this result, the proof demonstrates that every equilibrium features sorting by rents, that is, $x_i > x_k$ implies $g_i/f_i > g_k/f_k$ and $x_i = x_k$ implies $g_i/f_i = g_k/f_k$. This holds because (by Assumption 3) contestants with higher incumbency rents are more likely to win any incumbency contest they engage in and are therefore overrepresented among incumbents.

On the basis of Lemma 2 we can show:

Proposition 1 *An equilibrium (x,g) with positive sorting exists.*

The proof of Proposition 1 follows the same logic as the related existence proof for search-and-matching models with a finite number of types in Lauermann and Nöldeke (2015). It proceeds in two steps. In the first step, artificial bounds - including the requirement that x_i is weakly increasing - are imposed on a fixed-point map obtained from the equilibrium conditions (15) and (18). A standard application of Brouwer's fixed point theorem (where the requisite continuity of the fixed-point map is obtained from Assumption 1) yields the existence of a fixed point. In the second step, Assumption 2 is essential to show that the bounds imposed in the first step are not binding, thereby ensuring that any fixed point of the map considered in the first step is not only an equilibrium but features strictly increasing incumbency rents x_i. The key intuition for the second step is that there cannot be an equilibrium in which two distinct types have identical incumbency rents. This can be seen by noting that the terms describing the expected gains and losses in the value equations (18) would be identical for any two types i and k with $x_i = x_k$ (because these two types would have identical stakes in every contest they encounter and face the same opponents with the same rates). But this implies that the right side of (18) would be higher for the type with the higher flow payoff, thereby contradicting the hypothesis that $x_i = x_k$ holds in equilibrium.

In general, there is no assurance that equilibrium is unique and the result in Proposition 1 therefore does not imply that there is positive sorting in *all* equilibria. Indeed, Section 5.3 presents an example featuring one equilibrium with positive sorting and another with negative sorting (in the obvious sense that g_i/f_i is strictly decreasing in i), indicating that such a stronger result cannot be obtained without imposing additional conditions to capture the intuition that higher flow payoffs should go along with higher incumbency rents. We present two such conditions.

First, the intuition that higher flow payoffs imply higher incumbency rents is correct if the threat of being challenged pales into insignificance for the incumbents, that is, the meeting rate is sufficiently small. This is so because for small m the value equations (18) imply that incumbency rents are approximately proportional to flow payoffs and therefore increasing in type. More formally, we obtain:

Proposition 2 *Let $m \leq \underline{m} = \rho \cdot \min_{i=1,\ldots n-1}\left[\frac{v_{i+1}}{v_i} - 1\right] > 0$. Then there is positive sorting in every equilibrium (x,g).*

When m is large, the sum of the expected future gains and loss terms in the value equations has a significant impact on the determination of incumbency rents, reflecting the importance of the iterated nature of the contests. It is then no longer clear that incumbency rents must be ordered in the same way as flow payoffs. However, a positive link between the flow payoffs v_i and the incumbency gains x_i is ensured from (18) when both the expected gain of a challenger and the expected loss of an incumbent are increasing in own rents. That is, monotonicity of the contest is a sufficient condition for positive sorting in every equilibrium:

Proposition 3 *Suppose the contest is monotonic. Then there is positive sorting in every equilibrium (x,g).*

To see the intuition behind Proposition 3 consider types i and k with $x_i < x_k$. If the contest is monotonic, the inequality $x_i < x_k$ implies that both the expected gain terms and the expected loss terms on the right side of (18) are higher for type k than for type i. Higher future gains mean that type k is in a relatively comfortable position as a challenger, whereas a higher future losses mean that type k's position as an incumbent is more costly to defend. Both of these effects work in the direction of making the incumbency position less advantageous for type k than for type i, implying that the inequality $x_i < x_k$ can only result because type k has a higher flow payoff from holding the incumbency position.

We have already verified (cf. footnote 6) that both the all-pay auction and the Tullock contest are monotonic. Therefore, the following is an immediate corollary to Proposition 3.

Corollary 1 *Suppose the incumbency contest is an all-pay auction or a Tullock contest. Then there is positive sorting in every equilibrium (x,g).*

As we have discussed when defining monotonicity in Section 2.1, the requirement that the incumbent's expected loss $\tau(x_I,x_C)$ is increasing in the incumbent's rent is not innocuous. This raises the question whether a counterpart of Proposition 3 can be obtained without imposing this requirement. The argument in the following remark, considering the case in which the meeting rate is high (relative to the discount rate) and incumbency positions are rare, explains why this is not possible.

Remark 3 Suppose that $\tau(x_I,x_C)$ fails to be increasing in x_I. There then exist $x_1 > x_2$ such that $\tau(x_1,x_3) < \tau(x_2,x_3)$ holds for some $x_3 > 0$ (which we can take to be different from both x_1 and x_2 by continuity of τ). Let us suppose that $\tau(x_2,x_3) < \tau(x_3,x_3)$ holds.[8] Let $n = 3$ and let (v_1,v_2,v_3) denote the vector of flow payoffs. Now observe that in the limit as $\theta \to 0$ we must have $g_i \to 0$ for all i, so that for small θ the value equations yield

$$v_i \approx \rho x_i + m \sum_{j=1,2,3} \tau(x_i,x_j) f_j.$$

Suppose, further, that the population proportion f_3 of type 3 is close to 1 (implying that the population proportions of types 1 and 2 are close to zero) and that ρ is very small. We then obtain the approximation

$$v_i \approx m\tau(x_i,x_3).$$

By construction, $\tau(x_i,x_3)$ is strictly increasing in i, so that whenever the above approximations are sufficiently good, we will be able to find values $v_1 < v_2 < v_3$ such that the given incumbency rent profile (x_1,x_2,x_3) together with the (from Lemma 1) uniquely determined incumbent distribution (g_1,g_2,g_3) solve the value equations as well as the balance conditions. As $x_1 > x_2$ holds, such an equilibrium does not feature positive sorting.[9]

[8] Up to a relabelling of types the following argument goes through if this last inequality is reversed.

[9] Note that a similar argument, considering the case of large rather than small θ, can be used to show that the the first half of the monotonicity property ($\sigma(x_I,x_C)$ increasing in x_C) is also necessary to ensure that for all feasible specifications of the parameters $(v_1,\ldots v_n)$, $(f_1,\ldots f_n)$, θ, m, and ρ there is positive sorting in every equilibrium.

4 Continual Contests

Proposition 2 in the previous section has considered the case in which contests for the incumbency position occur only rarely because the meeting rate m is low. We have seen that for such sporadic contests positive sorting always obtains in equilibrium. In particular, in the limit as $m \to 0$ incumbency rents converge to $x_i = v_i/\rho$ and the equilibrium sorting pattern is determined by the solution - which we have shown to be unique in Lemma 1 - to the balance conditions (15) given these incumbency rents. Here we consider the opposite extreme of continual contests in the limit as $m \to \infty$. Formally, we study limit equilibria in the sense of the following definition.

Definition 6 (Limit Equilibrium) A tuple $(x,g) \in \mathbb{R}^n_+ \times \mathscr{G}$ is a limit equilibrium if there exists a sequence $(m^k, x^k, g^k)_{k=1}^{\infty}$ in $\mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathscr{G}$ such that (i) for all k the tuple (x^k, g^k) is an equilibrium given the meeting rate m^k, (ii) the sequence of meeting rates diverges to infinity, and (iii) the sequence $(x^k, g^k)_{k=1}^{\infty}$ converges to (x,g).

Note that the definition of a limit equilibrium does not require the limit incumbency rent profile x to be strictly positive. Indeed, our first result is that quite generally incumbency rents vanish as the meeting rate goes to infinity.

Lemma 3 *Suppose that (i) $\sigma(y,y) > 0$ holds for all $y > 0$ or (ii) the contest is role-symmetric. Then every limit equilibrium (x,g) satisfies $x = 0$.*

The proof for Lemma 3 (and all other results in this section) is in Appendix C. The conclusion obtained in this lemma is intuitive: as the meeting rate becomes large, individuals switch roles very frequently, so that the difference between the continuation values of incumbents and challengers should become small. Alas, this argument might fail if there were some type who never has a chance of obtaining a strictly positive gain when in the role of a challenger and, simultaneously, is effectively insulated from any competitive pressure when holding the incumbency position. Condition (i) in the statement of Lemma 3 eliminates this possibility by ensuring that a challenger obtains a strictly positive gain when contesting an incumbent with the same, strictly positive incumbency rent. As indicated by condition (ii) this requirement can be dispensed with when the contest is role-symmetric. The reason is that if $\sigma(y,y) = 0$ holds in a role symmetric contest (as is the case for all $y > 0$ in an all-pay auction, cf. equation (5)), then it is the case that there is full rent-dissipation if two contestants with incumbency rents y are matched with each other.[10] In conjunction with the genericity requirement (13) this suffices to imply the desired result.

The result that incumbency rents vanish in continual contests, suggests two conjectures. First, even if positive sorting holds for all finite meeting rates, it vanishes in the limit as m converges to infinity. That is, the ratios g_i/f_i all converge to θ. Second, the aggregate flow payoff $\sum_{i \in N} g_i v_i$ accruing to the holders of the incumbency positions is fully dissipated in the same limit. That is, the aggregate surplus

$$S = \sum_{i \in N} [(f_i - g_i)L_i + g_i W_i] = \sum_{i \in N} [f_i L_i + g_i x_i] \tag{19}$$

converges to zero.[11] In the following we show that under fairly mild additional assumptions both of these conjectures are false for homogenous contests.

Proposition 4 *Suppose the contest is homogenous, monotonic, and satisfies condition (i) or (ii) from Lemma 3. Then there is positive sorting in every limit equilibrium (x,g).*

The proof of Proposition 4 makes essential use of homogeneity to express the equilibrium conditions in terms of ratios of incumbency rents. The role of the additional conditions is to ensure that there is positive sorting along the equilibrium sequence converging to the limit equilibrium and that the arguments proving Lemma 3 are applicable to make the appropriate inferences for the convergence of the incumbency rents. As the proof makes clear, the reason why positive sorting persists in the limit is that all incumbency rents converge to zero at the same rate and in such a way that the limits of the ratios of incumbency rents x_j/x_i remain strictly increasing in j. As for homogeneous contests the ratios x_j/x_i

[10] To see this, observe that from (12) in Definition 3 we have $\sigma(y,y) + \tau(y,y) = y$ in every role-symmetric contest. Therefore $\sigma(y,y) = 0$ implies $\tau(y,y) = y$, so that the challenger's equilibrium payoff is L_C and the incumbent's equilibrium payoff is L_I.

[11] The second equality in (19) follows upon substituting the definition $x_i = W_i - L_i$ into the sum appearing before the equal sign.

uniquely determine the sorting pattern arising from the balance conditions, this suffices to deliver the result.

To state our result about surplus dissipation for continual contests, it will be convenient to say that there is incomplete surplus dissipation in a limit equilibrium (x,g) if for any sequence of equilibria $(x^k,g^k)_{k=1}^{\infty}$ converging to (x,g) in the sense of Definition 6 every limit point of the associated sequence of equilibrium surpluses $(S^k)_{k=1}^{\infty}$ is strictly positive. Formally, this associated sequence of equilibrium surpluses is defined by

$$S^k = \sum_{i\in N}\left[f_i\frac{m^k}{\rho}\sum_{j\in N}\sigma(x_j^k,x_i^k)g_j^k + g_i^k x_i^k\right], \tag{20}$$

where we have substituted (16) into the rightmost expression from (19) to obtain an expression depending only on (m^k,x^k,g^k). With this terminology in place, we can state the following result:

Proposition 5 *Suppose the conditions from Proposition 4 are satisfied and $\sigma(x_I,x_C) > 0$ holds for $x_I < x_C$. Then there is incomplete surplus dissipation in every limit equilibrium (x,g).*

While the incumbency rents x_i^k (by Lemma 3) and also the expected gains $\sigma(x_j^k,x_i^k)$ (by Assumption 2) in equation (20) all converge to zero, the proof of Proposition 5 shows that the additional condition $\sigma(x_I,x_C) > 0$ for $x_I < x_C$ ensures that challengers with the highest type nevertheless have strictly positive continuation values in the limit. The intuition for this result is that incumbency rents for the highest type disappear not because incumbents of this type face continual challenges to their position but because challengers of the highest type find it very easy to obtain incumbency positions when meetings are frequent.[12]

Both the all-pay auction and the Tullock contest are homogenous, monotonic, and role symmetric, so that Proposition 4 is applicable to them. Further, it is immediate from (5) and (8) that the additional condition appearing in Proposition 5 holds for both of these contests, too. Hence, we may state without further proof:

Corollary 2 *Suppose the contest is an all-pay auction or a Tullock contest. Then there is positive sorting and incomplete surplus dissipation in every limit equilibrium (x,g).*

5 The All-Pay Auction with Two Types

In this section we conduct a more detailed study of the equilibria in the all-pay auction with two types. For this special case we can provide insights into the structure of equilibria of iterated contests that go beyond the sufficient conditions for positive sorting and the limit results presented in the previous sections. In particular, we establish uniqueness of equilibrium and derive the comparative statics of this unique equilibrium with respect to the meeting rate m. In addition, the all-pay auction with two-types provides a convenient building block for the construction of an iterated contest that illustrates the possibility of equilibrium multiplicity and negative sorting. All proofs for this section are in Appendix D.

5.1 Uniqueness of Equilibrium

From Corollary 1 we know that there is positive sorting in all equilibria (x_1,x_2,g_1,g_2) of the all-pay auction with two-types. From Lemmas 1 and 2 every equilibrium thus satisfies

$$x_2 > x_1 > 0,\;\; f_1 > g_1 > 0,\;\; f_2 > g_2 > 0. \tag{21}$$

Using the winning probabilities for the all-pay auction (4) and the inequality $x_2 > x_1$, the balance conditions (15) reduce to[13]

$$g_2(f_1 - g_1) = \left[2\frac{x_2}{x_1} - 1\right] g_1(f_2 - g_2). \tag{22}$$

[12] This conclusion hinges on the fact that, as we have assumed, there are at least two different types of individuals. The all-pay auction without heterogeneity of types provides a simple example of a contest satisfying all the conditions of Proposition 5 but leading to complete surplus dissipation in every limit equilibrium.

[13] With only two types the balance conditions for both types are equivalent to $g_2(f_1 - g_1)\mu(x_2,x_1) = g_1(f_2 - g_2)\mu(x_1,x_2)$. Inserting the winning probabilities for the all-pay auction yields (22).

Similarly, substituting the expected gains (5) and losses (6) for the all-pay auction into the value equations (18) and using $x_2 > x_1$, the value equations reduce to

$$\rho x_1 = v_1 - m(1-\theta)x_1 \tag{23}$$

$$\rho x_2 = v_2 - m(1-\theta)x_2 - m(2g_1 - f_1)(x_2 - x_1), \tag{24}$$

where we have used $f_1 + f_2 = 1$ and

$$g_1 + g_2 = \theta \tag{25}$$

to simplify the resulting expressions. A tuple (x_1, x_2, g_1, g_2) thus is an equilibrium of the iterated all-pay auction with two types if and only if it satisfies the inequalities in (21) and solves equations (22) - (25).

Note that the value equation (23) for type 1 is independent of the incumbency rent x_2 and can be solved for

$$x_1 = \frac{v_1}{\rho + m(1-\theta)}. \tag{26}$$

This is due to the special structure of the all-pay auction: because they have the lower incumbency rent, the expected gain for type-1 challengers is zero and the expected loss of type-1 incumbents is equal to their incumbency rent. Therefore, the incumbency rent for type 1 is simply the present value $v_1/(\rho + m(1-\theta))$ of the flow payoff v_1, where the discount rate is incremented by $m(1-\theta)$ to take the complete loss of the incumbency rent into account that occurs whenever an incumbent of type 1 meets a challenger.

In contrast, the value equation (24) for type 2 indicates that the incumbency rent of type 2 depends on x_1 unless $2g_1 - f_1 = 0$ holds. To understand this, observe that the situation of type-2 individuals only differs from the one just described for type 1 individuals – complete loss of rent when meeting a challenger, no gain when meeting an incumbent – whenever they meet a type-1 individual. In such a meeting they suffer the expected loss x_1 as an incumbent and obtain the expected gain $x_2 - x_1$ as a challenger. In both cases, type-2 individuals are thus better off by $x_2 - x_1$ when meeting a type-1 individual rather than a type-2 individual. Now, observe that being better off in the role of an incumbent increases the incumbency rent of an individual, whereas being better off in the role of a challenger decreases the incumbency rent. Therefore, how the difference $x_2 - x_1$ affects the incumbency rent x_2 depends on whether type-2 individuals are more likely to meet type-1 individuals as incumbents or as challengers. In particular, if there are just as many type-1 individuals among the incumbents as there are among the challengers, that is, $g_1 = f_1/2$ holds, there is no effect. In this special case, the incumbency rent of type-2 individuals is thus determined as if all meetings were with type-2 individuals and given by $x_2 = v_2/(\rho + m(1-\theta))$. Consequently, with $g_1 = f_1/2$ the ratio of incumbency rents x_2/x_1, which is all that matters for determining the solution to the balance condition (22), is equal to the ratio of flow payoffs: $x_2/x_1 = v_2/v_1$.

More generally, the intuition provided above suggests (and substituting the value of x_1 given in (26) into (24) and solving for x_2 shows, so that no proof is required) the following result.

Lemma 4 *For all $g_1 \in (0, f_1)$ the value equations* (23) *and* (24) *have a unique solution (x_1, x_2). In this solution x_1 is given by* (26) *and both x_2 and x_2/x_1 are strictly decreasing in g_1. Further, $x_2/x_1 = v_2/v_1$ holds if and only if $g_1 = f_1/2$.*

That we can solve for the incumbency rents (x_1, x_2) from the value equations for given $g_1 > 0$ does not, in itself, imply that equilibrium is unique. The issue is that, as indicated by (22), the equilibrium incumbent distribution does in turn depend on the ratio x_2/x_1. Indeed, it follows from (22) that an increase in the ratio of incumbency rents x_2/x_1 increases the extent of positive sorting (that is, increases g_2 relative to g_1) and thereby decreases g_1. From Lemma 4, a decrease in g_1 increases x_2/x_1. Therefore, the function defined by mapping x_2/x_1 into g_1 via the balance condition and then back into x_2/x_1 via the value equations is strictly increasing and thus has the potential for multiple fixed points, corresponding to multiple equilibria. Nevertheless, we can prove uniqueness of equilibrium by verifying directly that equations (22) – (25) only have one solution satisfying the inequalities in (21):

Proposition 6 *Let the contest be the all-pay auction and let $n = 2$. Then there exists a unique equilibrium (x, g).*

5.2 Comparative Statics

Here we investigate whether a higher frequency of meetings promotes or impedes positive sorting. To do so, we measure the extent of sorting by the ratio g_2/g_1. For given θ, f_1, and f_2 this seems a natural measure of sorting in the two-type model we consider here because an increase in this ratio indicates that the fraction of type 2 individuals who hold an incumbency position has increased, whereas the fraction of type 1 individuals who hold an incumbency position has decreased.

As we have noted in the paragraph preceding the statement of Proposition 6, the balance condition (22) dictates that higher values of the ratio x_2/x_1 are associated with a higher extent of positive sorting. Identifying conditions under which the extent of sorting is increasing (resp. decreasing) in the meeting rate m is thus tantamount to identifying conditions under which the ratio x_2/x_1 is increasing (resp. decreasing) in m in the unique equilibrium associated with the meeting rate m. How the equilibrium incumbency rent ratio x_2/x_1 depends on m is in turn determined by the scarcity of the incumbency conditions. To see why, it is convenient to begin by considering a special case.

As we have noted in Lemma 4, the equilibrium ratio of incumbency rents x_2/x_1 is equal to the ratio of flow payoffs v_2/v_1 if and only if the rate at which type-2 individuals experience conflicts with type-1 individuals does not depend on the current role of the type-2 individuals, that is, $g_1 = f_1/2$ holds. Hence, whether or not the values $g_1 = f_1/2$ and $x_2/x_1 = v_2/v_1$ (together with the value of g_2 and x_1 given by (25) and (26)) constitute an equilibrium is entirely determined by the balance condition (22). Substituting $g_1 = f_1/2$ and $x_2/x_1 = v_2/v_1$ into the balance equation (22) and solving for θ it follows that such an equilibrium exists if and only if

$$\theta = \theta^* := \frac{1}{2} + \frac{1}{2}\left(1 - \frac{v_1}{v_2}\right)(1 - f_1) \in (0,1). \tag{27}$$

Noting that – because the balance condition is independent of m - this critical value θ^* is independent of m, it follows that a change in meeting rate has no effect on the extent of positive sorting if $\theta = \theta^*$ holds.

When incumbency positions are scarce ($\theta < \theta^*$), then one would expect the equilibrium incumbent distribution to feature a lower mass of type-1 individuals than when $\theta = \theta^*$ holds, so that $g_1 < f_1/2$ holds. By Lemma 4 the inequality $g_1 < f_1/2$ implies that for any $m > 0$ the equilibrium ratio x_2/x_1 is strictly larger than the ratio of the flow payoffs v_2/v_1. That is, when incumbency positions are scarce, the iterated nature of the contests has a smaller impact (when measured relative to the flow payoffs) on type-2 individuals than on type-1 individuals. It is then natural that this effect should be more pronounced when the meeting rate is high rather than low, implying that the extent of sorting is increasing in m when incumbency positions are scarce. Similarly, when incumbency positions are abundant ($\theta > \theta^*$), one would expect the incumbency rent ratio x_2/x_1 to be lower than v_2/v_1 and to be decreasing in m, implying that the extent of sorting is decreasing in m. The proof of the following proposition verifies that this intuition is correct.

Proposition 7 *Let the contest be the all-pay auction, let $n = 2$, and let θ^* be as given in* (27).

(i) If $\theta < \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is strictly increasing in m.
(ii) If $\theta = \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is constant in m.
(iii) if $\theta > \theta^*$, then the ratio of rents x_2/x_1 and the extent of sorting in equilibrium is strictly decreasing in m.

5.3 Negative Sorting and Multiple Equilibria in the All-Pay Auction with Attack Costs

We have indicated in Section 2.1 that a sufficiently strong discouragement effect on challengers might lead to a failure of the monotonicity requirement that the expected loss $\tau(x_I, x_C)$ is increasing in x_I. We have also indicated that we are not aware of any simultaneous-move contest featuring such a strong discouragement effect, precluding us from illustrating failures of positive sorting with such a contest. On the other hand, strong discouragement effects arise naturally in contests in which the challenger must first make a costly decision to attack before engaging the incumbent in a contest: in such a contest sufficiently weak incumbents will be attacked and thus suffer a strictly positive expected loss, whereas sufficiently strong incumbents will be protected from challengers and thus suffer no loss. Here we illustrate how this may lead to the existence of an equilibrium with negative sorting (i. e. an equilibrium (x,g) in which g_i/f_i is strictly decreasing in i) by considering an all-pay auction with attack costs (while maintaining the assumption that there are two types of agents).

Contest 3 (All-Pay Auction with Deterministic Attack Costs)**.** *The contest is given by the following sequential game with perfect information: The challenger first decides whether to attack the incumbent or not. If the challenger decides not to attack, then no further interaction takes place, neither player incurs any cost and both individuals retain their current roles. If the challenger attacks, then the all-pay auction as described in Section 2.1 is played with the challenger incurring an additional cost* $\bar{c} > 0$.

Imposing the refinement that the challenger attacks in case of indifference, this game has a unique subgame-perfect equilibrium with

$$\sigma(x_I,x_C) = \begin{cases} 0 & \text{if } x_C < x_I + \bar{c} \\ x_C - x_I - \bar{c} & \text{if } x_C \geq x_I + \bar{c} \end{cases} \tag{28}$$

$$\tau(x_I,x_C) = \begin{cases} 0 & \text{if } x_C < x_I + \bar{c} \\ x_I & \text{if } x_C \leq x_I \end{cases} \tag{29}$$

$$\mu(x_I,x_C) = \begin{cases} 0 & \text{if } x_C < x_I + \bar{c} \\ 1 - \dfrac{1}{2}\dfrac{x_I}{x_C} & \text{if } x_C \geq x_I + \bar{c} \end{cases} \tag{30}$$

Note that for $y < x_C - \bar{c} < z$ we have $\tau(y,x_C) > \tau(z,x_C) = 0$. That is, an increase in x_I from below the threshold $x_C - \bar{c}$ (which makes the challenger just indifferent to attack) to above this threshold discourages the challenger from attacking and thereby reduces the expected loss of the incumbent to zero. Hence, the all-pay auction with deterministic attack costs is indeed not monotonic.

While the all pay-auction with deterministic attack costs satisfies Assumption 2, it violates Assumptions 1 (as neither $\tau(x_I,x_C)$ nor $\mu(x_I,x_C)$ is continuous) and 3 (as $\mu(x_I,x_C)$ is equal to zero unless it is worthwhile for the challenger to attack). None of the following arguments and conclusions hinge on these violations of Assumptions 1 and 3; as we show in an earlier version of this paper (Häfner and Nöldeke, 2016), the analysis carries over to a contest with stochastic (rather than deterministic) attack costs that complies with Assumptions 1 to 3. Therefore, nothing of substance is lost by focusing on the much simpler contest with deterministic attack costs.

In the presence of a strictly positive attack cost, it is clear that whenever a challenger and incumbent of the same type $i = 1,2$ meet, the challenger's winning probability $\mu(x_i,x_i)$, as well as the expected gains $\sigma(x_i,x_i)$ and the expected losses $\tau(x_i,x_i)$ are equal to zero. We may thus rewrite the balance conditions (15) as

$$(f_1 - g_1)g_2\mu(x_2,x_1) = g_1(f_2 - g_2)\mu(x_1,x_2) \tag{31}$$

and the value equations (18) as

$$\rho x_1 = v_1 - m\tau(x_1,x_2)(f_2 - g_2) - m\sigma(x_2,x_1)g_2 \tag{32}$$

$$\rho x_2 = v_2 - m\tau(x_2,x_1)(f_1 - g_1) - m\sigma(x_1,x_2)g_1. \tag{33}$$

We have noted in Remark 2 that violations of Assumption 3 may lead to a multitude of trivial equilibria in which the incumbent distribution is arbitrary, incumbency rents are given by the present value of the flow payoffs ($x_i = v_i/\rho$), and incumbents maintain their positions without investing any effort in fending of challengers. To avoid such trivial equilibria here, it suffices to assume

$$v_2 - v_1 > \rho\bar{c}. \tag{34}$$

This condition ensures that type-2 challengers with incumbency rent $x_2 = v_2/\rho$ will wish to attack type-1 incumbents with incumbency rent $x_1 = v_1/\rho$ and by doing so will gain the incumbency position with strictly positive probability. In addition, we cut down on the number of cases to be considered by supposing

$$f_1 > \theta > f_2, \text{ and } v_1 > \rho\bar{c}. \tag{35}$$

The first part of this condition means that type-1 agents are abundant and type-2 agents are scarce relative to the mass of incumbency positions, whereas the second part means that type-1 challengers are willing to attack in the most favorable of all circumstances in which the present value of their incumbency rents takes the maximal possible value v_1/ρ and they face an incumbent with rent arbitrarily close to zero.

Finally, let us define

$$\bar{m} = \frac{v_2 - v_1 + \rho\bar{c}}{(f_1 - \theta)\,(v_1/\rho - \bar{c})} > 0, \tag{36}$$

where the inequality follows from (34)–(35). We then have:

Proposition 8 *Let $n = 2$ and let* (34) *and* (35) *hold. Then the all-pay auction with deterministic attack costs has a unique equilibrium if $m < \bar{m}$ holds and in this equilibrium there is positive sorting. If $m \geq \bar{m}$ holds, then the all-pay auction with deterministic attack costs has exactly two equilibria and there is positive sorting in one of these equilibria and negative sorting in the other.*

The proof of Propositon 8 is constructive in that it explicitly identifies the two only candidates for equilibria and then shows that the first of these, featuring positive sorting, is always an equilibrium whereas the second candidate, featuring negative sorting, is an equilibrium if and only if $m \geq \bar{m}$ holds.

In light of Proposition 1 it is not surprising that an equilibrium with positive sorting exists. Indeed, even though Assumptions 1 and 3 are violated, this equilibrium also features sorting by rents and the natural inequality $x_2 > x_1$ identified in Lemma 2 as necessary for positive sorting. What may be more surprising is that under the relatively mild parameter restrictions embodied in (34) and (35) the only additional condition for the existence of an equilibrium with negative sorting is that the meeting rate is sufficiently high. (Recall that Proposition 2 precludes the existence of an equilibrium with negative sorting for low meeting rates.) What happens in this equilibrium is that type-1 individuals occupy all the incumbency position and are never attacked. Thus type-1 individuals bear no future losses as incumbents and – because they never attack a type-1 incumbent – also have no future gains as challengers. This implies that the incumbency rent for type-1 individuals is the present value of their flow payoff, $x_1 = v_1/\rho$. Type-2 agents, on the other hand, are deterred from attacking incumbents because they anticipate being attacked by type-1 challengers once they have acquired an incumbency position. This is self-confirming for high meeting rates because the anticipation of future losses makes the incumbency rent for type-2 individuals sufficiently low as to encourage type-1 challengers to mount an attack whenever they meet a type-2 incumbent.

6 Conclusion

We have developed a model of repeated contests over incumbency positions among a population of players obtaining heterogeneous but commonly known flow payoffs during incumbency: Incumbents recurrently defend their positions against challengers, unsuccessful challengers continue searching for incumbency positions to contest, and defeated incumbents may regain incumbency positions again in the future by mounting successful challenges. We have identified conditions on the structure of the contests which ensure positive sorting in all equilibria and have shown that these conditions hold in the two standard complete-information contests, namely the all-pay auction and the Tullock contests. We have also established that under fairly general conditions positive sorting and incomplete surplus dissipation arise in the limit case of continual conflicts. For the all-pay auction with two types we have shown uniqueness of equilibrium and have discussed how the frequency at which incumbency is contested affects the extent of sorting in this equilibrium. Finally, we have provided a non-trivial example of a contest which gives rise to an equilibrium with negative sorting in which individuals with lower flow payoffs from holding the incumbency position have higher incumbency rents in equilibrium.

To obtain these results we have made many simplifying assumptions, including (i) heterogeneity among individuals stems from differences in the intrinsic benefits of holding the incumbency position rather than from different marginal costs of effort or, equivalently, differences in ability (measured as a factor multiplying efforts in the contest success function), (ii) search for incumbency positions is random rather than directed, and (iii) incumbency positions can only be obtained by defeating a current incumbent and no individual can hold more than one incumbency position at any given time. These assumptions call for some remarks.

First, when the contestants' payoffs are separable in the value of the prize and the linear costs of effort (as in our model), then (as Baye et al. (1996) explain in the context of an all-pay auction) there is a strategic equivalence between one-shot contests in which the contestants have heterogeneous valuations and one-shot contests in which the players have different marginal cost of providing effort. This equivalence carries over to the iterated contests we consider: Suppose (x,g) is an equilibrium in our model with n types having valuations $v_1 < ... < v_n$ and consider the alternative model in which all types derive the same flow payoff 1 from the incumbency positions but have marginal costs $c_i = 1/v_i$ for providing effort. Then $(\hat{x},g)$ with $\hat{x}_i = x_i/v_i$ is an equilibrium in this alternative model and, vice versa, if $(\hat{x},g)$ is an equilibrium in the alternative model then (x,g) with $x_i = v_i\hat{x}_i$ is an equilibrium in our model.[14] In particular, because the equilibrium $(\hat{x},g)$ features the same incumbent distribution as the equilibrium (x,g), the equilibrium sorting pattern does not depend on how heterogeneity is modeled. Observe that because the order of the cost parameters $c_i = 1/v_i$ is the reverse of the order of the flow-payoff parameters v_i our results about positive sorting in equilibrium translate into the statement that in equilibrium types with lower marginal cost (higher ability) have a higher likelihood of being in the incumbency condition.

Second, while a formal investigation of a model with directed search is beyond the scope of this paper, we can offer some conjectures. Specifically, suppose that the challenger's expected gain $\sigma(x_I,x_C)$ is decreasing in x_I.[15] It will then be the case that challengers prefer to seek out incumbents with lower incumbency rents and when meeting such opponents have a higher probability to take over their position. Hence, if challengers have some ability to direct their search towards such opponents, individuals with low incumbency rents will face more frequent challenges and will also be dislodged from their positions more frequently. As a consequence, it seems likely that in an equilibrium with positive sorting both the incumbency rent profile x and the incumbent distribution g will be more spread out than in our model. At the same time, endogenizing the meetings exacerbates the potential for equilibrium multiplicity and negative sorting: if, say, there are only two types and all individuals focus their attacks on the high type of incumbent, then this will tend to depress the incumbency rent for the high type and increase the incumbency rent for the low type (whose incumbency positions are secure). If these effects are strong enough, then the incumbency rent for the high type may be lower than the one for the low type – which in turn would justify the hypothesis that it is attractive for challengers to seek out high type incumbents.

Third, it is, in principle, straightforward to extend our model to allow challengers to discover new, unoccupied incumbency positions while searching (as would be natural in our example of entrepreneurs and market niches from the Introduction). Doing so, however, will make the fraction θ of the population holding an incumbency position endogenous and requires other changes, such as allowing for the exogenous destruction of incumbency positions, to maintain a steady state in the face of a constant inflow into incumbency positions. This would add another layer of complexity to our equilibrium analysis and we therefore believe that such an extension should be first investigated in the absence of heterogeneity and the attendant sorting effects. Allowing individuals to accumulate multiple incumbency conditions would be a very interesting extension because it will lead to an endogenously determined resource distribution. Again, we believe that this issue might best be tackled in a model without heterogeneity first.

Our analysis has revealed that the properties of the expected loss $\tau(x_I,x_C)$ play a crucial role in determining whether there must be positive sorting in equilibrium. As mentioned in our discussion of the definition of a monotonic contest in Section 2.1, conditions on the contest technology ensuring monotonicity of a contest have not yet been explored in the literature. Identifying such conditions would not only complement our analysis but would also be of relevance in models of repeated contests in which defeated contestants leave the game (as, for instance, in Virág, 2009) in which continuation values are solely dependent on future expected losses (as they are in the limit case $\theta \to 0$ of our model, cf. Remark 3).

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[14] The result follows directly from the observation – which is straightforward to verify – that the equilibrium winning probability $\hat{\mu}$ in the heterogeneous costs setting satisfies $\hat{\mu}(\hat{x}_i,\hat{x}_j) = \mu(x_i,x_j)$ for all i, j, so that the balance conditions are satisfied for $(\hat{x},g)$ if and only if they are satisfied for (x,g). Further, the gains and losses, $\hat{\sigma}$ and $\hat{\tau}$, in the heterogeneous costs setting satisfy $\hat{\tau}(\hat{x}_i,\hat{x}_j) = \tau(x_i,x_j)/v_i$ and $\hat{\sigma}(\hat{x}_j,\hat{x}_i) = \sigma(x_j,x_i)/v_i$ for all i, j, so that the value equations are satisfied for $(\hat{x},g)$ if and only if they are satisfied for (x,g).

[15] This additional assumption is satisfied in the homogenous contests considered in (Malueg and Yates, 2005); see their Propositions 4 and 5.

A Proof of Lemma 1

Fix $x \in \mathbb{R}^n_{++}$. From Assumption 3 we have $\mu(x_i, x_j) > 0$ for all $i, j \in N$ (as otherwise there exists $(x_I, x_C) \in \mathbb{R}^2_{++}$ such that $\mu(x_I, x_C) = 0$ holds, with Assumption 3 then implying that μ takes on strictly negative values on its domain, which is impossible). Observing that $g_i = 0$ implies that the right side of the balance condition (15) for type i is equal to zero whereas the left side is strictly positive (as $f_i - g_i = f_i > 0$ and $g \in \mathscr{G}$ implies $g_j > 0$ for some $j \in N$), it follows that if $(x, g) \in \mathbb{R}^n_{++} \times \mathscr{G}$ solves (15), then $g_i > 0$ holds for all $i \in N$. An analogous argument shows that $f_i - g_i > 0$ must also hold for all $i \in N$, delivering the inequalities $0 < g_i < f_i$ for all $i \in N$.

Let $\mathscr{D} = \prod_{i \in N}[0, f_i]$ and let $\mathscr{D}_{-k} = \prod_{i \neq k \in N}[0, f_i]$ for $k \in N$. Define the map $F : \mathscr{D} \to \mathbb{R}^n$ by letting its i-th coordinate be given by the net outflow of type i from the incumbency position, that is,

$$F_i(g) = \sum_{j \in N} \left[g_i (f_j - g_j) \mu(x_i, x_j) - (f_i - g_i) g_j \mu(x_j, x_i) \right], \tag{37}$$

for all $i \in N$. By construction, $g \in \mathscr{D}$ then solves the balance conditions (15) if and only if $F(g) = 0$ holds. A straightforward application of Brouwer's fixed point theorem (considering the function G, defined by (41) in the proof of Proposition 1 below, for the given value of x) shows that for every $\theta \in (0,1)$ there exists a solution to the equation $F(g) = 0$ satisfying $\sum_{j \in N} g_j = \theta$, so that $g \in \mathscr{G}$ holds. Similarly, for any given $\bar{g}_k \in [0, f_k]$ there exists a vector $g \in \mathscr{D}$ satisfying $g_k = \bar{g}_k$ and solving $F(g) = 0$. It remains to show that for every $\theta \in (0,1)$ there is exactly one solution to the equation $F(g) = 0$ satisfying $\sum_{j \in N} g_j = \theta$.

Towards this end, we begin by observing that the Jacobian matrix $J(g)$ of the mapping F is defined for all $g \in \mathscr{D}$ and is continuous in g. Let $J_{-k}(g)$ denote the Jacobian matrix with the k-th row and column deleted. Straightforward calculation shows that - because $\mu(x_i, x_j) > 0$ holds for all $i, j \in N$ - the matrix $J_{-k}(g)$ has a positive strictly (column) dominant diagonal and negative non-diagonal entries. Hence, $J_{-k}(g)$ is a P-matrix and a Leontief matrix (Gale and Nikaido, 1965). As the domain $\mathscr{D}_{-k}$ is rectangular, it follows (Gale and Nikaido, 1965) that for any given $\bar{g}_k \in [0, f_k]$ there is exactly one solution $g \in \mathscr{D}$ to the equation $F(g) = 0$ satisfying $g_k = \bar{g}_k$. Using $H : [0, f_k] \to \mathscr{D}_{-k}$ to denote the function mapping $\bar{g}_k$ into the remaining coordinates of the vector g solving $F(g) = 0$, it is easy to see that $\sum_{j \neq k} H_j(0) = 0$ and $\sum_{j \neq k} H_j(f_k) = \sum_j f_j = 1 - f_k$ holds. Further, the implicit function theorem ensures that H is continuous and - because $J_{-k}(g)$ is a Leontief matrix, so that all of the elements of its inverse are positive (Gale and Nikaido, 1965) - increasing. It follows that the function $Z : [0, f_k] \to \mathbb{R}_+$ defined by $Z(\bar{g}_k) = \sum_j H_j(\bar{g}_k) + \bar{g}_k$ is continuous, strictly increasing, and satisfies $Z(0) = 0$ and $Z(1) = 1$. Consequently, for every $\theta \in (0,1)$ all solutions to the equation $F(g) = 0$ satisfying $\sum_j g_j = \theta$ have the property that g_k is given by the unique solution to the condition $Z(g_k) = \theta$. As this argument applies for all $k \in N$, the desired uniqueness result follows.

B Proofs for Section 3

Proof of Lemma 2. Let (x, g) be an equilibrium. As noted at the beginning of the proof of Lemma 1, Assumption 3 implies $\mu(x_i, x_j) > 0$ for all $i, j \in N$. From Lemma 1 we also have $g_j > 0$ and $f_j - g_j > 0$ for all $j \in N$. Therefore, we may rearrange the flow balance conditions (15) as

$$\frac{g_i}{f_i - g_i} = \frac{\sum_{j \in N} g_j \mu(x_j, x_i)}{\sum_{j \in N} (f_j - g_j) \mu(x_i, x_j)} \tag{38}$$

for all $i \in N$. By Assumption 3, $g_j > 0$, and $f_j - g_j > 0$, each summand in the numerator of the right hand expression in (38) is strictly increasing in x_i and each summand in the denominator is strictly decreasing in x_i. Hence, we obtain $x_i > x_k \Rightarrow g_i/(f_i - g_i) > g_k/(f_k - g_k) \Leftrightarrow g_i/f_i > g_k/f_k$. Further, for any two types i and k such that $x_i = x_k$ holds, the right hand side of (38) is identical, implying $g_i/(f_i - g_i) = g_k/(f_k - g_k)$ and therefore $g_i/f_i = g_k/f_k$. It is then immediate that x_i strictly increasing in i is necessary and sufficient for positive sorting. □

Proof of Proposition 1. Fix x_0 satisfying $0 < x_0 < v_1/(\rho + m)$. Let $\mathscr{X} = \{x \in \mathbb{R}^n_+ : x_{i-1} \leq x_i \leq v_i/\rho, \forall i \in N\}$ and recall $\mathscr{G} = \{g \in \mathbb{R}^n_+ : g_i \leq f_i, \forall i \in N \text{ and } \sum_{i=1}^n g_i = \theta\}$. From Lemma 2 it suffices to establish the existence of an equilibrium (x, g) with x_i strictly increasing. To do so, we proceed in two steps.
STEP 1: Define the vector-valued map $V : \mathscr{X} \times \mathscr{G} \to \mathscr{X}$ iteratively by setting

$$V_1(x, g) = \max \left\{ \frac{v_1 - m \sum_{j \in N} \tau(x_1, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(x_j, x_1) g_j}{\rho}, x_0 \right\} \tag{39}$$

$$V_i(x, g) = \max \left\{ \frac{v_i - m \sum_{j \in N} \tau(x_i, x_j)(f_j - g_j) - m \sum_{j \in N} \sigma(x_j, x_i) g_j}{\rho}, V_{i-1}(x, g) \right\}, \text{ for } i = 2, \ldots, n. \tag{40}$$

and let $G : \mathscr{X} \times \mathscr{G} \to \mathscr{G}$ be given by

$$G_i(x, g) = g_i + (f_i - g_i) \sum_{j \in N} g_j \mu(x_j, x_i) - g_i \sum_{j \in N} (f_j - g_j) \mu(x_i, x_j), \text{ for all } i \in N. \tag{41}$$

Because $\sigma(x_j, x_i) \geq 0$ and $\tau(x_i, x_j) \geq 0$ holds for all $x \in \mathscr{X}$ (Assumption 2), it is immediate that V does indeed map into $\mathscr{X}$. To verify that G maps into $\mathscr{G}$ observe that the inequalities $0 \leq \mu(y, z) \leq 1$, which hold for all $(y, z) \in \mathbb{R}^2_{++}$, imply

$$G_i(x, g) \geq g_i - g_i(1 - \theta) = g_i \theta \geq 0$$

and

$$G_i(x, g) \leq g_i + (f_i - g_i)\theta = \theta f_i + (1 - \theta) g_i \leq f_i,$$

for all $(x, g) \in \mathscr{X} \times \mathscr{G}$. Further, adding (41) over all i, gives $\sum_{i=1}^n G_i(x, g) = \sum_{i=1}^n g_i = \theta$.

The sets $\mathscr{X}$ and $\mathscr{G}$ are both non-empty, compact and convex. By continuity of σ, τ, and μ (Assumption 1), the functions V and G are both continuous. Hence, Brouwer's fixed point theorem implies that the mapping $(V,G) : \mathscr{X} \times \mathscr{G} \to \mathscr{X} \times \mathscr{G}$ has a fixed point.

STEP 2: Comparing (39) and (40) with (18) and (41) with (15) it is immediate from Lemma 2 that any fixed point (x^*, g^*) of the mapping (V,G) is an equilibrium with positive sorting if the inequalities

$$x_1^* > x_0 \tag{42}$$

$$x_i^* > x_{1-1}^*, \text{ for } i = 2, \dots, n \tag{43}$$

hold. To finish the proof, it thus suffices to show that any fixed point of the mapping (V,G) satisfies these inequalities.

First, suppose (x^*, g^*) is a fixed point of (V,G) violating (42). As $x^* \in \mathscr{X}$ holds, we then have $x_1^* = x_0$. From the definition of V_1 in (39), we obtain

$$\rho x_0 \geq v_1 - m \sum_{j \in N} \tau(x_1^*, x_j^*)(f_j - g_j^*) - m \sum_{j \in N} \sigma(x_j^*, x_1^*) g_j^*. \tag{44}$$

Using the upper bounds from (2) in Assumption 2 we also have $\tau(x_1^*, x_j^*) \leq x_1^* = x_0$ and $\sigma(x_j^*, x_1^*) \leq x_0$, yielding

$$v_1 - m \sum_{j \in N} \tau(x_1^*, x_j^*)(f_j - g_j^*) - m \sum_{j \in N} \sigma(x_j^*, x_1^*) g_j^* \geq v_1 - m x_0. \tag{45}$$

Combining (44) and (45), we obtain

$$\rho x_0 \geq v_1 - m x_0 \Rightarrow x_0 \geq \frac{v_1}{m + \rho}.$$

As x_0 had been chosen to satisfy $x_0 < v_1/(m+\rho)$, this is a contradiction. Consequently, every fixed point (x^*, g^*) of (V,G) satisfies (42).

Second, suppose (x^*, g^*) is a fixed point of (V,G) violating (43). As $x^* \in \mathscr{X}$ holds, there then exists $i \geq 2$ such that $x_i^* = x_{i-1}^*$ holds. Consider the smallest such i, ensuring that

$$\rho x_{i-1}^* = v_{i-1} - m \sum_{j \in N} \tau(x_{i-1}^*, x_j^*)(f_j - g_j^*) - m \sum_{j \in N} \sigma(x_j^*, x_{i-1}^*) g_j^*$$

holds. (In case $i = 2$, the preceding equality obtains because we have already shown $x_1^* > x_0$.) Using $v_i > v_{i-1}$ and the hypothesis $x_i^* = x_{i-1}^*$ on the right hand side of this equation yields

$$\rho x_{i-1}^* < v_i - m \sum_{j \in N} \tau(x_i^*, x_j^*)(f_j - g_j^*) - m \sum_{j \in N} \sigma(x_j^*, x_i^*) g_j^*.$$

Using the fixed-point condition $V_{i-1}(x^*, g^*) = x_{i-1}^*$ and the definition of V_i in (40) we thus obtain

$$V_i(x^*, g^*) > x_{i-1}^*.$$

From the fixed-point condition $V_i(x^*, g^*) = x_i^*$, this inequality contradicts the hypothesis $x_i^* = x_{i-1}^*$. Consequently, every fixed point (x^*, g^*) of (V,G) satisfies (43). □

Proof of Proposition 2. Let (x,g) be an equilibrium. From the bounds (2) in Assumption 2 and the value equations in (18) we obtain

$$v_i \geq \rho x_i \geq v_i - m x_i \Leftrightarrow x_i \in \left[\frac{v_i}{\rho + m}, \frac{v_i}{\rho}\right].$$

Hence, $x_{i+1} > x_i$ is implied by

$$\frac{v_i}{\rho} < \frac{v_{i+1}}{\rho + m} \Leftrightarrow m < \rho \left[\frac{v_{i+1}}{v_i} - 1\right].$$

By definition of $\underline{m}$, the latter inequality holds for $i = 1, \dots n-1$ whenever $m < \underline{m}$. The result then follows from Lemma 2. □

Proof of Proposition 3. Deducting the value equations (18) for two adjacent types $i, i+1 \in N$ from each other, we obtain

$$m \sum_{j \in N} \left[\left[\tau(x_{i+1}, x_j) - \tau(x_i, x_j) \right] (f_j - g_j) + \left[\sigma(x_j, x_{i+1}) - \sigma(x_j, x_i) \right] g_j \right] + \rho \left[x_{i+1} - x_i \right] = v_{i+1} - v_i.$$

The right side of this expression is strictly positive, whereas for $x_i \geq x_{i+1}$ monotonicity of the contest implies that the left side is negative. Hence, in every equilibrium $x_{i+1} > x_i$ must hold for all $i, i+1 \in N$, so that the result follows from Lemma 2. □

C Proofs for Section 4

Proof of Lemma 3. Let (x,g) be a limit equilibrium and let $(m^k, x^k, g^k)_{k=1}^{\infty}$ be the associated sequence with the properties indicated in Definition 6.

From the value equations (18) and the requirement that (x^k, g^k) is an equilibrium when the meeting rate is m^k we have

$$\rho x_i^k + m^k \left[\sum_j \tau(x_i^k, x_j^k)(f_j - g_j^k) + \sigma(x_j^k, x_i^k) g_j^k \right] = v_i \tag{46}$$

for all $i \in N$. We now consider the the two cases appearing in the statement of the lemma.

(i) Equation (46) implies

$$\rho x_i^k + m^k \left[\tau(x_i^k, x_i^k)(f_i - g_i^k) + \sigma(x_i^k, x_i^k) g_i^k)\right] \leq v_i.$$

Using the inequality $\tau(y,y) \geq \sigma(y,y)$ implied by (3) in Assumption 2, this implies

$$\rho x_i^k + m^k \sigma(x_i^k, x_i^k) f_i \leq v_i. \tag{47}$$

Suppose there exists some $i \in N$ such that the sequence $(x_i^k)_{k=1}^\infty$ does not converge to zero, so that it converges to some $x_i^* > 0$.

As $\sigma(x_i^*, x_i^*) > 0$ holds by assumption and $(m^k)_{k=1}^\infty$ diverges to infinity, it then follows that the left side of (47) diverges to infinity, yielding a contradiction. Therefore $\lim_{k\to\infty} x_i^k = 0$ holds for all $i \in N$.

(ii) Define

$$\zeta = \min_{g \in \mathscr{G}} Z(g),$$

where

$$Z(g) = \max_{i \in N} \min\{g_i, f_i - g_i\}.$$

The function $Z : \mathscr{G} \to \mathbb{R}_+$ is clearly continuous and, using the genericity condition (13), strictly positive on its domain. As $\mathscr{G}$ is compact, it follows that ζ is not only well-defined, but satisfies $\zeta > 0$.

Considering that all the summands multiplying m^k in (46) are positive and that $Z(g^k) \geq \zeta$ holds, we obtain the existence of j^k such that

$$\rho x_i^k + m^k \left[\tau(x_i^k, x_{j^k}^k) + \sigma(x_{j^k}^k, x_i^k)\right] \zeta \leq v_i \tag{48}$$

holds for all i. Using the assumption of role symmetry, the term in square brackets in (48) is identical to x_i^k by (12). This delivers the inequality

$$[\rho + m^k \zeta] x_i^k \leq v_i$$

for all i and k. As $\zeta > 0$ holds and the sequence $(m^k)_{k=1}^\infty$ diverges to infinity, the desired conclusion $\lim_{k\to\infty} x_i^k = 0$ for all i is then immediate. □

Proof of Proposition 4. Let (x,g) be a limit equilibrium and let $(m^k, x^k, g^k)_{k=1}^\infty$ be the associated sequence with the properties indicated in Definition 6. As the conditions of Lemma 3 are satisfied, we have $x = 0$.

Using homogeneity of the contest, we may rewrite the balance conditions as

$$(f_i - g_i^k) \sum_{j \in N} g_j^k \mu(x_j^k / x_i^k, 1) = g_i^k \sum_{j \in N} (f_j - g_j^k) \mu(1, x_j^k / x_i^k), \ \forall i \in N. \tag{49}$$

Similarly, the value equations can be rewritten as

$$x_i^k \left[\rho + m^k Z_i(x^k)\right] = v_i, \tag{50}$$

where

$$Z_i(x^k) = \sum_{j \in N} \left[\tau(1, \frac{x_j^k}{x_i^k})(f_j - g_j^k) + \sigma(\frac{x_j^k}{x_i^k}, 1) g_j^k\right], \tag{51}$$

Observe that (50) implies

$$\frac{x_j^k}{x_i^k} = \left(\frac{v_j}{v_i}\right) \left(\frac{\rho/m^k + Z_i(x^k)}{\rho/m^k + Z_j(x^k)}\right). \tag{52}$$

From the bounds (2) in Assumption 2 we have that $Z_i(x^k)$ as defined in (51) is bounded above by 1. Further (cf. the proof of Lemma 3), under condition (i) of Lemma 3 $Z_i(x^k)$ is bounded below by $\sigma(1,1) f_i > 0$ and under condition (ii) of Lemma 3 it is bounded below by $\zeta > 0$. No matter which of these two lower bounds is applicable, we thus obtain from (52) that there exists $\underline{\alpha} > 0$ and $\overline{\alpha} > \underline{\alpha}$ such that

$$\alpha_{ij}^k = \frac{x_j^k}{x_i^k} \in [\underline{\alpha}, \overline{\alpha}]$$

holds for all $i, j \in N$ and k. Consequently, the sequence $(m^k, x^k, g^k)_{k=1}^\infty$ has a subsequence for which all the ratios x_j^k / x_i^k converge to some finite limit $\alpha_{ij} > 0$. Taking limits along this subsequence and using (49), we then obtain that the limit equilibrium (x,g) satisfies

$$(f_i - g_i) \sum_{j \in N} g_j \mu(\alpha_{ij}, 1) = g_i \sum_{j \in N} (f_j - g_j) \mu(1, \alpha_{ij}), \ \forall i \in N. \tag{53}$$

Provided that α_{ij} is strictly increasing in j, the desired result then follows by applying the same argument as in the proof of Lemma 2. It therefore remains to show that α_{ij} is strictly increasing in j.

As the contest under consideration is monotonic, Proposition 3 implies that $j > i$ implies $x_j^k > x_i^k$ for all k. Consequently, we have $\alpha_{ij} \geq 1$ for all $j > i$. As $\alpha_{ij} > 1$ for all $j > i$ implies that α_{ij} is strictly increasing in j (because for $\ell > j > i$ we have $\alpha_{i\ell} = \alpha_{ij}\alpha_{j\ell}$), it thus remains to exclude the possibility that $\alpha_{ij} = 1$ holds for some $i \neq j$. Suppose to the contrary that such a pair of types exists. As $\alpha_{ij} = 1$ implies $\alpha_{i\ell} = \alpha_{j\ell}$ for all $\ell \in N$, (51) then implies that along the relevant subsequence $Z_i(x^k)$ and $Z_j(x^k)$ converges to the same limit. Taking limits in (52) along the same subsequence, we then obtain $\alpha_{ij} = v_j / v_i$. As $v_j \neq v_i$ holds for $i \neq j$, this contradicts the hypothesis $\alpha_{ij} = 1$, finishing the proof. □

Proof of Proposition 5. Let (x,g) be a limit equilibrium and let $(m^k,x^k,g^k)_{k=1}^{\infty}$ be the associated sequence with the properties indicated in Definition 6. As the conditions of Lemma 3 are satisfied, we have $x=0$. As in the proof of Proposition 4, we may suppose that the sequences $(\alpha_{ij}^k)_{k=1}^{\infty}$ defined by $\alpha_{ij}^k = x_j^k/x_i^k$ converge to finite, strictly positive limits α_{ij} that satisfy $\alpha_{nj}<1$ for $j<n$. As the limit equilibrium satisfies (53), $g_j>0$ holds for all $j \in N$ (cf. the initial paragraph in the proof of Lemma 1).

Let $(S^k)_{k=1}^{\infty}$ be the sequence of surpluses defined by (20) and suppose that there exists a subsequence, which we may take to be the sequence $(m^k,x^k,g^k)_{k=1}^{\infty}$ itself, such that $(S^k)_{k=1}^{\infty}$ converges to zero. In the following we argue that this results in a contradiction, thereby establishing the proposition.

Fix $j<n$. If $(S^k)_{k=1}^{\infty}$ converges to zero, then clearly the same is true for $(m^k\sigma(x_j^k,x_n^k)g_j^k)_{k=1}^{\infty}$. Using homogeneity to rewrite $\sigma(x_j^k,x_n^k)$, we thus obtain

$$\lim_{k\to\infty} m^k x_n^k \sum_{j\in N} \sigma(\alpha_{nj}^k,1)g_j^k = 0. \tag{54}$$

We have $\lim_{k\to\infty}\sigma(\alpha_{nj}^k,1) = \sigma(\alpha_{nj},1) > 0$, where the inequality follows from $\alpha_{nj}<1$ and the assumption that $\sigma(x_I,x_C)>0$ holds for $x_I<x_C$. As we also have $\lim_{k\to\infty} g_j^k = g_j > 0$, (54) can only hold if $\lim_{k\to\infty} m^k x_n^k = 0$. Noting that, as established in the proof of Proposition 4, the expression $Z_n(x^k)$ defined in equation (51) is bounded above by 1, this in turn implies

$$\lim_{k\to\infty} x_n^k \left[\rho + m^k Z_n(x^k)\right] = 0,$$

contradicting equation (50). □

D Proofs for Section 5

Proof of Proposition 6. Let (x_1,x_2) satisfying $x_2>x_1>0$ solve (23) – (24) for given $g_1>0$. To simplify notation let $\alpha = x_2/x_1$.

It is readily verified that (23) – (24) imply

$$[\alpha-1][\rho+m(1-\theta)+m(2g_1-f_1)]v_1 = [v_2-v_1][\rho+m(1-\theta)]. \tag{55}$$

Substituting $g_2=\theta-g_1$ from (25) into (22) yields

$$(\theta-g_1)(f_1-g_1) = [2\alpha-1]g_1(f_2-\theta+g_1),$$

which is equivalent to

$$f_1\theta - g_1 = 2[\alpha-1][g_1(f_2-\theta)+g_1^2]. \tag{56}$$

Using (55) to substitute for $\alpha-1$ into (56) yields an equilibrium condition that only depends on g_1, namely

$$[f_1\theta-g_1][\rho+m(1-\theta)+m(2g_1-f_1)]v_1 = 2[v_2-v_1][\rho+m(1-\theta)][g_1(f_2-\theta)+g_1^2]. \tag{57}$$

As existence of an equilibrium is assured by Proposition 1 and (x_2,x_1) are uniquely determined by g_1 (Lemma 4), it suffices to establish that (57) cannot have more than one solution in the interval $\mathscr{G}_1 = [\max\{\theta-f_2,0\},\min\{f_1,\theta\}]$. To do so we distinguish two cases: (i) $f_2-\theta\geq 0$ and (ii) $f_2-\theta<0$.

We begin with case (i). Because $f_2-\theta\geq 0$, we have $\mathscr{G}_1=[0,\min\{f_1,\theta\}]$. At $g_1=0$ the right side of (57) is zero, and the left side is $f_1\theta[\rho+m[f_2-\theta]]>0$. Furthermore, because the left side of (57) is concave in g_1 and the right side is convex g_1, there can only be one $g_1\geq 0$ solving (57).

We turn to case (ii). Because $f_2-\theta<0$, we have $\mathscr{G}_1=[\theta-f_2,\min\{f_1,\theta\}]$. Observe that the right side of (57) has one root at $g_1=0$ and a second one at $g_1=\theta-f_2>0$. The right side of (57) is strictly increasing at $g_1=\theta-f_2$ and hence, because it is convex in g_1, it is strictly increasing at all $g_1>\theta-f_2$. Next, we note that the left side of (57) has one root at $g_1=[\theta-f_2-\rho/m]/2<\theta-f_2$ and one root at $g_1=f_1\theta>\theta-f_2$. Because the left side of (57) is concave and strictly decreasing at the root $g_1=f_1\theta$, it follows that there is a unique g_1 satisfying $\theta-f_2<g_1<f_1\theta<\min\{f_1,\theta\}$ that solves (57). □

Proof of Proposition 7. The result for case (ii) has already been established in the text. We consider case (i) here; the argument for case (iii) is analogous.

Let $\theta<\theta^*$. As in the proof of Proposition 6 we use (25) to rewrite the balance condition (22) as

$$(\theta-g_1)(f_1-g_1) = [2\alpha-1]g_1(f_2-\theta+g_1). \tag{58}$$

for $\alpha\geq 1$. By Lemma 1, this equation has a unique solution $g_1(\alpha)\in\mathscr{G}_1=[\max\{\theta-f_2,0\},\min\{f_1,\theta\}]$.

Next, let

$$F(\alpha,m) = \frac{v_2}{v_1}\left(\frac{\rho+m(1-\theta)}{\rho+m(1-\theta)+m(1-1/\alpha)(2g_1(\alpha)-f_1)}\right)-\alpha, \tag{59}$$

As (23) and (24) imply

$$\frac{x_2}{x_1} = \frac{v_2}{v_1}\left(\frac{\rho+m(1-\theta)}{\rho+m(1-\theta)+m(1-x_1/x_2)(2g_1-f_1)}\right),$$

it is easily verified that if (x_1,x_2,g_1,g_2) is an equilibrium for meeting rate $m>0$, then $\alpha=x_2/x_1$ solves

$$F(\alpha,m)=0. \tag{60}$$

Vice versa, if α solves (60), then letting x_1 be given by (26), setting $x_2=\alpha x_1$, $g_1=g_1(\alpha)$ and $g_2=\theta-g_1(\alpha)$ provides a solution to the equilibrium conditions. Hence, from the uniqueness result in Proposition 6, equation (60) has a unique solution $\alpha(m)$ satisfying

$\alpha(m) > 1$. As $F(1,m) > 0$ holds, this solution must occur at a point where $F(\alpha,m)$ intersects 0 from above. Inspection of (59) reveals that for $\alpha > 1$ the partial derivative of F with respect to m satisfies

$$F_m(\alpha,m)\begin{cases} >0 & \text{if } 2g_1(\alpha) < f_1 \\ =0 & \text{if } 2g_1(\alpha) = f_1 \\ <0 & \text{if } 2g_1(\alpha) > f_1 \end{cases}.$$

It follows that $\alpha(m)$ is strictly increasing in m if $2g_1(\alpha(m)) < f_1$ holds. Further, as observed in the text, $g_1(\alpha)$ is clearly strictly decreasing in α and $g_2(\alpha)$ is strictly increasing in α. Therefore, to finish our argument, it suffices to show that $2g_1(\alpha(m)) < f_1$ holds for all $m > 0$. In fact, as $g_1(\alpha(m))$ is strictly decreasing in m whenever $2g_1(\alpha(m)) < f_1$ holds, it suffices to show this for m sufficiently small.

Let $\alpha^* = \lim_{m\to 0}\alpha(m)$. Because $g(\alpha) \in \mathscr{G}_1$ holds for all $m > 0$, it is easy to see from (23) and (24) that this limit exists and satisfies $\alpha^* = v_2/v_1$. Now, by definition of $g_1(\alpha)$, we have that $g_1(\alpha^*)$ satisfies (58), so that we obtain

$$(\theta - g_1(\alpha^*))(f_1 - g_1(\alpha^*)) = [2\alpha^* - 1]g_1(\alpha^*)(f_2 - \theta + g_1(\alpha^*)),$$

which we may rewrite as

$$\frac{\theta - g_1(\alpha^*)}{1 - f_1 - \theta + g_1(\alpha^*)} = [2\alpha^* - 1]\frac{g_1(\alpha^*)}{f_1 - g_1(\alpha^*)}.$$

From this equality we see that

$$2g_1(\alpha^*) < f_1 \Leftrightarrow \frac{\theta - f_1/2}{1 - \theta - f_1/2} < [2\alpha^* - 1] \Leftrightarrow \theta < \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{\alpha^*}\right)(1 - f_1),$$

and, consequently,

$$2g_1(\alpha^*) < f_1 \Leftrightarrow \theta < \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{\alpha^*}\right)(1 - f_1) = \theta^*,$$

where the last equality follows from the definition of θ^* in (27) and $\alpha^* = v_2/v_1$. Therefore $\theta < \theta^*$ implies that $2g_1(\alpha(m)) < f_1$ holds for all sufficiently small $m > 0$. □

Proof of Proposition 8. We consider five cases, thereby exhausting all possible equilibrium configurations and proving the result.

CASE 1: There is no equilibrium (x,g) with $x_1 = x_2$: from (28)–(29) $x_1 = x_2$ implies $\sigma(x_1,x_2) = \sigma(x_2,x_1) = \tau(x_2,x_1) = \tau(x_1,x_2) = 0$, so that (32)–(33) imply $x_1 = v_1/\rho < x_2 = v_2/\rho$, yielding a contradiction.

CASE 2: There is no equilibrium (x,g) with $x_1 + \bar{c} > x_2 > x_1$: From (28)–(29) the inequality $x_2 > x_1$ implies $\sigma(x_2,x_1) = \tau(x_2,x_1) = 0$. Similarly, the inequality $x_1 + \bar{c} > x_2$ implies $\sigma(x_1,x_2) = \tau(x_1,x_2) = 0$. Therefore, (32) implies $x_1 = v_1/\rho$ and (33) implies $x_2 = v_2/\rho$. Inequality (34) then implies $x_2 - x_1 > \bar{c}$, contradicting the inequality $x_1 + \bar{c} > x_2$.

CASE 3: There is no equilibrium (x,g) with $x_2 + \bar{c} > x_1 > x_2$: The argument is analogous to the one for the previous case.

CASE 4: There exists exactly one equilibrium (x,g) satisfying $x_2 \geq x_1 + \bar{c}$. This equilibrium is given by

$$x_1^* = \frac{v_1}{\rho},\ \ x_2^* = \frac{v_2 + m[v_1/\rho + \bar{c}][\theta - f_2]}{\rho + m[\theta - f_2]},\ \ g_1^* = \theta - f_2,\ \ g_2^* = f_2, \tag{61}$$

so that $g_2^*/f_2 = 1 > g_1^*/f_1$ holds and there is positive sorting in this equilibrium.

To prove this claim, we begin by supposing that (x,g) is an equilibrium satisfying $x_2 \geq x_1 + \bar{c}$ and identify (x^*,g^*) as given in (61) as the unique candidate for such an equilibrium. Next, we argue that (x^*,g^*) is indeed an equilibrium by showing that it satisfies $x_2^* \geq x_1^* + \bar{c}$.

From (28)–(30) we have that $x_2 \geq x_1 + \bar{c} > x_1$ implies $\sigma(x_2,x_1) = \tau(x_2,x_1) = \mu(x_2,x_1) = 0$ and $\mu(x_1,x_2) > 0$. From $\mu(x_2,x_1) = 0$ and $\mu(x_1,x_2) > 0$, the balance condition (31) implies $g_1(f_2 - g_2) = 0$. From the inequality $\theta > f_2$ in condition (35) and $g \in \mathscr{G}$, we can infer $g_1 > 0$, so that $g_2 = f_2$ follows. As we also have $\sigma(x_2,x_1) = 0$, the value equation (32) for type 1 then implies $x_1 = v_1/\rho$. Further, from $g \in \mathscr{G}$ and $g_2 = f_2$ we have $g_1 = \theta - f_2 > 0$. Using $\tau(x_2,x_1) = 0$ and $g_1 = \theta - f_2$ the value equation (33) for type 2 becomes

$$\rho x_2 = v_2 - m\sigma(x_1,x_2)(\theta - f_2).$$

Using $x_1 = v_1/\rho$ and (28) to infer $\sigma(x_1,x_2) = x_2 - x_1 - \bar{c}$, this yields

$$x_2 = \frac{v_2 + m[v_1/\rho + \bar{c}][\theta - f_2]}{\rho + m[\theta - f_2]},$$

showing that the conditions in (61) identify the unique candidate for an equilibrium satisfying $x_2 \geq x_1 + \bar{c}$.

The above arguments already show that (x^*,g^*) satisfies (31)–(33) – and therefore is an equilibrium – provided that the inequality $x_2^* \geq x_1^* + \bar{c}$ does indeed hold. Observing that (61) implies

$$x_2^* - x_1^* = \frac{v_2 - v_1 + m\bar{c}(\theta - f_2)}{\rho + m(\theta - f_2)},$$

it follows that $x_2^* - x_1^* > \bar{c}$ is implied by $v_2 - v_1 > \rho\bar{c}$, which we have assumed in (34).

CASE 5: There exists no equilibrium (x,g) satisfying $x_1 \geq x_2 + \bar{c}$ if $m < \bar{m}$ and there exists exactly one equilibrium (x,g) satisfying $x_1 \geq x_2 + \bar{c}$ if $m \geq \bar{m}$. This equilibrium is then given by

$$\hat{x}_1 = \frac{v_1}{\rho}, \ \hat{x}_2 = \frac{v_2}{\rho + m(f_1 - \theta)}, \ \hat{g}_1 = \theta, \ \hat{g}_2 = 0, \tag{62}$$

so that $\hat{g}_1/f_1 > 0 = \hat{g}_2/f_1$ holds and there is negative sorting in this equilibrium.

To prove this claim, we proceed as in the previous case: We begin by supposing that (x,g) is an equilibrium satisfying $x_1 \geq x_2 + \bar{c}$ and identify $(\hat{x},\hat{g})$ as given in (62) as the unique candidate for such an equilibrium. Next we argue that $(\hat{x},\hat{g})$ is indeed such an equilibrium if and only if $m \geq \bar{m}$ holds.

From (28)–(30) we have that $x_1 \geq x_2 + \bar{c} > x_2$ implies $\sigma(x_1,x_2) = \tau(x_1,x_2) = \mu(x_1,x_2) = 0$ and $\mu(x_2,x_1) > 0$. From $\mu(x_1,x_2) = 0$ and $\mu(x_2,x_1) > 0$, the balance condition (31) then implies $g_2(f_1 - g_1) = 0$. From the inequality $f_1 > \theta$ in condition (35) and $g \in \mathscr{G}$, we can infer $f_1 - g_1 > 0$, so that $g_2 = 0$ follows. As we also have $\tau(x_1,x_2) = 0$, the value equation (32) for type 1 then implies $x_1 = v_1/\rho$. Further, from $g \in \mathscr{G}$ and $g_2 = 0$ we have $g_1 = \theta > 0$. Using $\sigma(x_1,x_2) = 0$ and $f_1 - g_1 = f_1 - \theta$ the value equation (33) for type 2 becomes $\rho x_2 = v_2 - m(f_1 - \theta)x_2$, which is the second equality in (62). This shows that the conditions in (62) identify the unique candidate for an equilibrium satisfying $\hat{x}_1 \geq \hat{x}_2 + \bar{c}$.

The above arguments already show that $(\hat{x},\hat{g})$ satisfies (31)–(33) – and therefore is an equilibrium – provided that the inequality $\hat{x}_1 \geq \hat{x}_2 + \bar{c}$ does indeed hold. Upon observing that (62) implies

$$\hat{x}_1 - \hat{x}_2 = \frac{v_1}{\rho} - \frac{v_2}{\rho + m(f_1 - \theta)},$$

and that (using the fact that both terms in its denominator on the right side of (36) are strictly positive by (35)) the inequality $m \geq \bar{m}$ can be rearranged to read

$$\frac{v_1}{\rho} - \frac{v_2}{\rho + m(f_1 - \theta)} \geq \bar{c},$$

from which the desired result follows. □

References

ACEMOGLU, D. AND J. A. ROBINSON (2001): "A Theory of Political Transitions," *American Economic Review*, 938–963.

AZZIMONTI, M. (2011): "Barriers to Investment in Polarized Societies," *American Economic Review*, 101, 2182–2204.

BAI, J. H. AND R. LAGUNOFF (2011): "On the Faustian Dynamics of Policy and Political Power," *Review of Economic Studies*, 78, 17–48.

BAIK, K. H. (2004): "Two-Player Asymmetric Contests with Ratio-Form Contest Success Functions," *Economic Inquiry*, 42, 679–689.

BAIK, K. H. AND J. F. SHOGREN (1992): "Strategic Behavior in Contests: Comment," *American Economic Review*, 82, 359–362.

BANAJI, M. AND S. BAIGENT (2008): "Electron Transfer Networks," *Journal of Mathematical Chemistry*, 43, 1355–1370.

BARON, D. P. (1996): "A Dynamic Theory of Collective Goods Programs," *American Political Science Review*, 90, 316–330.

BATTAGLINI, M. (2014): "A Dynamic Theory of Electoral Competition," *Theoretical Economics*, 9, 515–554.

BAYE, M. R., D. KOVENOCK, AND C. G. D. VRIES (1996): "The All-Pay Auction with Complete Information," *Economic Theory*, 8, 291–305.

CHADE, H., J. EECKHOUT, AND L. SMITH (2017): "Sorting Through Search and Matching Models in Economics," *Journal of Economic Literature*, 55, 493–544.

CORNELIUS, T. (2003): "A Search Model of Marriage and Divorce," *Review of Economic Dynamics*, 6, 135–155.

DIAMOND, P. A. AND E. MASKIN (1979): "An Equilibrium Analysis of Search and Breach of Contract I: Steady States," *Bell Journal of Economics*, 10, 282–316.

ESHEL, I. AND E. SANSONE (1995): "Owner-Intruder conflict, Grafen Effect and Self-assessment. The Bourgeois Principle Re-examined," *Journal of Theoretical Biology*, 177, 341–356.

——— (2001): "Multiple Asymmetry and Concord Resolutions of a Conflict," *Journal of Theoretical Biology*, 213, 209–222.

FORAND, J. G. (2014): "Two-Party Competition with Persistent Policies," *Journal of Economic Theory*, 152, 64–91.

FRANKE, J., C. KANZOW, W. LEININGER, AND A. SCHWARTZ (2014): "Lottery versus All-Pay Auction Contests: A Revenue Dominance Theorem," *Games and Economic Behavior*, 83, 116 – 126.

FRANKE, J., W. LEININGER, AND C. WASSER (2018): "Optimal Favoritism in All-Pay Auctions and Lottery Contests," *European Economic Review*, 104, 22 – 37.

FU, Q., Q. JIAO, AND J. LU (2015): "Contests with Endogenous Entry," *International Journal of Game Theory*, 44, 387–424.

GALE, D. AND H. NIKAIDO (1965): "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, 159, 81–93.

GRAFEN, A. (1987): "The Logic of Divisively Asymmetric Contests: Respect for Ownership and the Desperado Effect," *Animal Behaviour*, 35, 462–467.

HAFER, C. (2006): "On the Origins of Property Rights: Conflict and Production in the State of Nature," *Review of Economic Studies*, 73, 119–143.

HÄFNER, S. AND G. NÖLDEKE (2016): "Sorting in Iterated Incumbency Contests," Working papers 2016/02, Faculty of Business and Economics - University of Basel.

KOKKO, H., A. LOPEZ-SEPULCRE, AND L. J. MORELL (2006): "From Hawk and Doves to Self-Consistent Games of Territorial Behavior," *American Naturalist*, 167, 901–912.

KONRAD, K. (2009): *Strategy and Dynamics in Contests*, Oxford University Press.

KONRAD, K. A. (2002): "Investment in the Absence of Property Rights; the Role of Incumbency Advantages," *European Economic Review*, 46, 1521–1537.

LAUERMANN, S. AND G. NÖLDEKE (2015): "Existence of Steady-State Equilibria in Matching Models with Search Frictions," *Economics Letters*, 131, 1 – 4.

LEININGER, W. (1993): “More Efficient Rent Seeking – a Münchhausen Solution,” *Public Choice*, 75, 43–62.

MALUEG, D. A. AND A. J. YATES (2005): “Equilibria and Comparative Statics in Two-Player Contests,” *European Journal of Political Economy*, 21, 738–752.

MEHLUM, H. AND K. MOENE (2006): “Fighting Against the Odds,” *Economics of Governance*, 7, 75–87.

NTI, K. O. (1999): “Rent-Seeking with Asymmetric Valuations,” *Public Choice*, 98, 415–430.

PISSARIDES, C. A. (1990): *Equilibrium Unemployment Theory*, MIT press.

POLBORN, M. (2006): “Investment under Uncertainty in Dynamic Conflicts,” *Review of Economic Studies*, 73, 505–529.

ROGERSON, R., R. SHIMER, AND R. WRIGHT (2005): “Search-Theoretic Models of the Labor Market: A Survey,” *Journal of Economic Literature*, 43, 959–988.

SHIMER, R. AND L. SMITH (2000): “Assortative Matching and Search,” *Econometrica*, 68, 343–369.

SIEGEL, R. (2014): “Asymmetric Contests with Head Starts and Nonmonotonic Costs,” *American Economic Journal: Microeconomics*, 6, 59–105.

STEPHAN, J. AND H. W. URSPRUNG (1998): “The Social Cost of Rent Seeking When Victories are Potentially Transient and Losses Final,” in *Trade, Growth, and Economic Policy in Open Economies: Essays in Honour of Hans-Jürgen Vosgerau*, ed. by K.-J. Koch and K. Jaeger, Springer, 369–380.

TULLOCK, G. (1980): “Efficient Rent Seeking,” in *Toward a Theory of the Rent-Seeking Society*, ed. by J. M. Buchanan, R. D. Tollison, and G. Tullock, Texas A&M University Press.

VIRÁG, G. (2009): “Efficiency and Competition in the Long Run: The Survival of the Unfit,” *Games and Economic Behavior*, 67, 315–330.

YILDIRIM, H. (2005): “Contests with Multiple Rounds,” *Games and Economic Behavior*, 51, 213–227.