# Differential treatment and the winner's effort in contests with incomplete information ${ }^{\text {s }}$ 

Cédric Wasser ${ }^{\text {a,* }}$, Mengxi Zhang ${ }^{\text {b }}$<br>a University of Basel, Faculty of Business and Economics, Peter Merian-Weg 6, 4002 Basel, Switzerland<br>${ }^{\text {b }}$ University of Bonn, Department of Economics, Lennéstr. 37, 53113 Bonn, Germany

## A R T I C L E I N F O

## Article history:

Received 30 July 2021
Available online 22 December 2022

## JEL classification:

D44
D82

## Keywords:

Contests
All-pay auction
Favoritism
Winner's effort
Mechanism design


#### Abstract

We study the design of all-pay contests when the organizer's objective is to maximize the expected winner's effort and contestants have private information about their valuations for the prize. We identify sufficient conditions for every optimal contest to involve differential treatment of ex ante symmetric contestants. Moreover, we provide a complete characterization of optimal contests when valuations are uniformly distributed. Finally, our results for the winner's effort also imply that differential treatment is even more likely to benefit the organizer when her objective is to maximize the expected highest effort.


© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

## 1. Introduction

There are many contest-like situations where the organizer of the contest primarily cares about the winner's performance. For instance, consider an up-or-out promotion contest in which candidates invest in human capital to compete for the opportunity to stay employed. The firm directly benefits from the winner's investment whereas the losers' investments are lost, having served only the purpose of driving up the winner's incentive to invest. ${ }^{1}$ As another example, it has become increasingly popular for organizations and individuals to use online platforms such as Topcoder.com and InnoCentive.com to run crowdsourcing contests to procure innovations or solutions to difficult problems. In such contests, only the quality of the innovation or the solution eventually adopted by the organizer matters. ${ }^{2}$

In this paper, we analyze the design of all-pay contests with a single prize when the goal of the organizer is to maximize the expected winner's effort. We consider a setting where each participating agent privately knows his valuation for the

[^0]prize, with all valuations being independently drawn from the same continuous distribution. ${ }^{3}$ We consider a general class of all-pay contests, which are defined by agent-specific score functions and a tie-breaking rule: In an all-pay contest, each agent simultaneously invests non-refundable effort, which is converted into a score by his score function. The prize is then awarded to (one of) the agent(s) with the highest score, where ties are broken according to the tie-breaking rule. Under complete information, such contests correspond to all-pay contests as studied in Siegel (2009). ${ }^{4}$ Designing a contest consists of choosing among any combination of nondecreasing score functions and tie-breaking rule.

The class of contests we consider is large and contains many well-known examples, such as the standard all-pay auction (which obtains when each score function is the identity function) and modifications thereof with handicaps, head starts, or bid caps. It also allows for a completely random assignment of the prize (when all score functions are constant and the same). A restriction is that random assignment is only possible when agents tie for the highest score, which implies that an agent's winning probability is either zero, or determined by the tie-breaking rule, or one. Hence, contests where the probability of winning is a smooth function of the efforts are excluded, as this would require that an agent can still obtain the prize with a certain probability even if he does not have the highest score. Practical advantages of all-pay contests are that their rules may be easy to communicate to participants and that it is simple to justify to losers that they did not get the prize based on their lower score (which, by contrast, is more difficult when there is a stochastic relation between scores and the assignment of the prize).

For our setting with ex ante symmetric agents, it is well known that symmetric contests are optimal when the goal is to maximize the expected total effort (this follows from Myerson, 1981). Typically, the standard all-pay auction is then optimal and any differential treatment of the agents on average discourages effort. By contrast, we find that for a broad class of environments it is optimal to use differential treatment when the goal is to maximize the expected winner's effort.

As our first main result, we establish conditions under which every optimal contest involves differential treatment. To do so, we consider an optimal symmetric contest and identify two separate asymmetric modifications that each strictly increase the winner's effort. When restricting attention to symmetric contests, the all-pay auction maximizes the expected winner's effort if the valuation distribution is such that an appropriately defined virtual valuation function is strictly increasing. If the virtual valuation function is not monotone, there is an optimal symmetric contest where the (symmetric) score functions for some intervals of efforts are constant and otherwise are equal to the identity function like in the all-pay auction. This leads agents to invest the same effort for a range of valuations, so that with positive probability there is a tie for the highest score, in which case the prize is randomly allocated to ensure symmetry. This gives rise to the first profitable modification: we show that whenever such ties happen at scores above the lowest possible one, there is an asymmetric contest that always allocates the prize deterministically and yields a strictly higher winner's effort. Whenever the virtual valuation function exhibits a corresponding non-monotonicity, optimal contests thus must involve differential treatment. The second modification is profitable whenever the probability density at the highest valuation is greater than under the uniform distribution (e.g., if the distribution function is strictly convex). The expected winner's effort then strictly increases if we modify the optimal symmetric contest by adding appropriately chosen hierarchical "guaranteed-winning options": If an agent invests more than a specific individual cutoff level, then he is guaranteed to win unless some other agent that comes before him in the hierarchy also is above that agent's cutoff. If all agents are below their cutoffs, the contest remains unchanged.

Intuitively, such asymmetric modifications cause some of the contestants to work harder on average and others to work less. When considering total effort, the latter effect always dominates. However, by appropriately designing the differential treatment, the organizer can ensure that the discouragement effect mostly falls on those who are less likely to win, while those who are more likely to win are encouraged. Thus, even though the effect on total effort is always negative, the effect on the winner's effort can be positive. Moreover, as we show, in many environments differential treatment that increases the winner's effort at the same time also increases the contestants' payoffs on average. That is, for the contestants' surplus the reduction in effort costs caused by differential treatment may often outweigh the induced inefficiency in the allocation of the prize.

As our second main result, we fully characterize the set of contests that maximize the expected winner's effort for the case of uniformly distributed valuations. This set of optimal contests is surprisingly large: it contains every contest where the tie-breaking rule is deterministic and where in equilibrium each agent has the same ex ante probability of winning. Hence, both the symmetric all-pay auction and a continuum of asymmetric contests are optimal. This allows for many forms of differential treatment, the only condition being that all agents are ex ante equally likely to win.

As an example, suppose there are two agents with valuations uniformly drawn from $[0,1]$, and the organizer gives the prize to the first agent if his effort is at least $1 / 2$ and the other agent obtains the prize otherwise. Clearly, the first agent exerts effort $1 / 2$ (and wins) if his valuation exceeds $1 / 2$, which happens with probability $1 / 2$, and exerts zero effort otherwise. The other agent always exerts zero effort. As both agents win with the same probability, this is an optimal contest, resulting in expected winner's effort $1 / 4$. Moreover, among all optimal contests, this take-it-or-leave-it offer corresponds, in a sense, to the most efficient production of winner's effort, as no effort is wasted by the loser. By contrast, the all-pay auction always results in a positive loser's effort and minimizes the agents' surplus. Hence, if the organizer primarily cares

[^1]about the winner's effort but secondarily also cares about the agents' surplus or the payoff of one particular agent, she is better off implementing an asymmetric contest. As a further illustration, consider two agents with affine score functions such that the first agent is given both an additive head start, which increases his winning probability, and a multiplicative handicap, which decreases his winning probability. As we show, for any handicap there exists a head start such that the two effects balance out and the corresponding modified all-pay auction with head start and handicap is an optimal contest.

When applied to the example of promotion contests, the above results suggest that employers may benefit from discriminating between candidates even if they are identical from an ex ante perspective. The results for the uniform case, where many forms of differential treatment are optimal but all yield the same ex ante probability of winning, show that the inherent discrimination may not necessarily be recognized as such by an outside observer who relies on aggregate statistics such as promotion rates. Moreover, a manager tasked with organizing the promotion process to maximize the human capital acquired by the winning candidate can achieve this goal and at the same time also favor some candidate based on personal preference. As in the take-it-or-leave-it offer discussed above, the manager could require one candidate to work hard for the promotion and require nothing from the other candidate, who is favored and still gets promoted in half the cases. Accordingly, very different payoffs may result for the two candidates. Importantly, the manager can implement such favoritism without compromising the average quality of promoted candidates.

Another important insight emerges from comparing the goal of maximizing the expected winner's effort to the closely related goal of maximizing the expected highest effort. Whereas in symmetric contests the winner's effort and the highest effort always coincide, under differential treatment the highest effort is greater than or equal to the winner's effort. Thus, in every situation where all optimal contests maximizing the expected winner's effort involve differential treatment, also all optimal contests maximizing the expected highest effort involve differential treatment. Our results for the uniform case enable us to show that, in addition, there are situations where differential treatment strictly increases the expected highest effort but not the expected winner's effort. So, overall, the organizer is even more likely to benefit from differential treatment when maximizing the expected highest effort.

Whether an organizer is interested in the winner's effort or the highest effort may depend on whether she is able to use the product or effort from contestants who do not win the prize. That the organizer is obligated to award the prize to and take the effort from the same contestant is a natural assumption or even necessary in many circumstances. For instance, in an up-or-out promotion contest, the nature of the contest prohibits the organizer from separating the two roles. In many online platforms such as InnoCentive.com, the platform only communicates the winner's contact information to the organizer. One potential reason is to prevent collusion between the organizer and some of the participants. A side effect of such precautions is that the organizer is unable to separate the winner of the contest from the solution provider. This additional restriction is irrelevant when the organizer is restricted to use only symmetric contests. However, as we demonstrate, it can become a binding constraint when asymmetric contest rules are allowed. This finding suggests a possible new approach to increase the organizer's payoff in situations, such as innovation contests, where it is possible to assign the prize to one contestant but implement the submission from a different contestant.

Our paper contributes to the literature on the design of all-pay contests under incomplete information. A number of papers have studied the expected highest effort in symmetric contests, where it is typically equivalent to the expected winner's effort. In particular, Chawla et al. (2019) characterize an optimal symmetric contest that maximizes the expected highest effort when the organizer can withhold the prize, which is closely related to the symmetric benchmark in our analysis. Moldovanu and Sela (2006) and Liu and Lu (2017) study the impact on the expected highest effort of other design parameters, such as the prize structure or splitting the contest into several parallel contests. ${ }^{5}$ For uniformly distributed valuations, Seel and Wasser (2014) show that introducing a head start in the all-pay auction increases the expected highest effort. Combined with our result that the all-pay auction without head start maximizes the expected winner's effort, this also illustrates that the two objectives are not equivalent. Kirkegaard (2012) considers two ex ante asymmetric agents and studies the maximization of total effort when only affine score functions can be used, that is, head starts and handicaps as discussed above. ${ }^{6}$

The winner's effort has also been studied in other types of contest models, typically under complete information. Considering lottery contests, Serena (2017) studies the design of multiplicative biases among heterogeneous contestants and Barbieri and Serena (2021) investigate the design of the temporal structure to maximize the expected winner's effort. Fu and Wu (2022) consider two-stage nested Tullock contests and let the organizer choose whether to disclose contestants' interim status after the preliminary round, whereas Deng et al. (2020) analyze the effect of information policies in a lottery contest with one-sided private information.

The following papers have also identified contest settings where differential treatment of ex ante symmetric contestants can be optimal. For an innovation contest where the quality of a contestant's innovation depends on both his privately known ability and his effort, Pérez-Castrillo and Wettstein (2016) show that the organizer may benefit when the size of the prize depends on the identity of the winner. According to Kawamura and Moreno de Barreda (2014), a head start in an all-pay auction with complete information may increase the probability that the most able participant wins. Drugov and

[^2]Ryvkin (2017) consider a class of symmetric contests where the probability of winning is a smooth function of the efforts, which excludes all-pay contests, and study the effect of introducing a small bias.

The rest of the paper is organized as follows. We present the model and preliminary results in Section 2. In Section 3 we determine sufficient conditions such that all optimal contests involve differential treatment. Section 4 is devoted to characterizing the entire set of optimal contests for the uniform case. We discuss the relation between the winner's effort and the highest effort in Section 5. In Section 6, we show how our results extend to the case where the organizer, in addition to designing the score functions and the tie-breaking rule, can also withhold the prize. We conclude in Section 7. Most proofs are in the Appendix.

## 2. Model and preliminaries

### 2.1. Setup

There are $n \geq 2$ risk-neutral agents who compete for a single indivisible prize. Each agent $i \in N=\{1,2, \ldots, n\}$ has a privately known type $\theta_{i} \in[0, \bar{\theta}]$ with $\bar{\theta}>0$, which represents his valuation for the prize. If agent $i$ exerts effort $b_{i}$, his payoff is $\theta_{i}-b_{i}$ if he obtains the prize and $-b_{i}$ otherwise. It is common knowledge that each agent's type is independently drawn from the same distribution $F$ with continuous density function $f\left(\theta_{i}\right)>0$ for $\theta_{i} \in[0, \bar{\theta}]$.

An (all-pay) contest $(\mathbf{s}, \tau)$ is characterized by score functions $\mathbf{s}=\left(s_{1}(\cdot), \ldots, s_{n}(\cdot)\right)$, that is, a nondecreasing function $s_{i}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$for each $i \in N$, and a tie-breaking rule $\tau$. In a contest, each agent $i \in N$ simultaneously chooses his effort $b_{i} \geq 0$, which results in score $s_{i}\left(b_{i}\right)$, and the prize is given to an agent with the highest score. When several agents tie for the highest score, the prize is assigned according to the tie-breaking rule $\tau$.

To define the tie-breaking rule formally, let $\hat{p}_{i}^{\tau}(M, r)$ denote the probability that agent $i$ wins under tie-breaking rule $\tau$ when the agents in $M \subseteq N$ tie for the highest score and the highest score is $r$. Let $\Pi$ denote the set of all $n$ ! permutations of $N$. For each possible score $r \in \mathbb{R}_{+}, \tau(\cdot, r)$ is a distribution over $\Pi$ (i.e., $\tau(\pi, r) \in[0,1]$ for each $\pi$ and $\sum_{\pi \in \Pi} \tau(\pi, r)=1$ ). If there is a tie at $r$, a permutation $\pi \in \Pi$ is drawn according to $\tau(\cdot, r)$, and then the tie is broken lexicographically according to $\pi$ (i.e., agent $\arg \min _{j \in M} \pi(j)$ wins). Thus

$$
\hat{p}_{i}^{\tau}(M, r)=\sum_{\{\pi \in \Pi: \pi(i)<\pi(j) \text { for all } j \in M \backslash\{i\}\}} \tau(\pi, r)
$$

This formulation includes, for example, random symmetric tie-breaking $\hat{p}_{i}^{\tau}(M, r)=1 /|M|$ if $\tau(\pi, r)=1 / n$ ! for all $\pi$ and deterministic lexicographic tie-breaking if $\tau(\pi, r)=1$ for one $\pi$. Now, for any profile of scores $\mathbf{r}=\left(r_{1}, \ldots, r_{n}\right)$, the probability that agent $i$ obtains the prize under tie-breaking rule $\tau$ is given by

$$
p_{i}^{\tau}(\mathbf{r})= \begin{cases}1 & \text { if } r_{i}>r_{j} \text { for all } j \neq i, \\ \hat{p}_{i}^{\tau}\left(M, r_{i}\right) & \text { if } i \in M=\arg \max _{j \in N} r_{j} \text { and }|M| \geq 2, \\ 0 & \text { if } r_{i}<r_{j} \text { for some } j \neq i\end{cases}
$$

Accordingly, if the agents invest efforts $\mathbf{b}=\left(b_{1}, \ldots, b_{n}\right)$ in a contest $(\mathbf{s}, \tau)$, then each agent $i$ receives score $s_{i}\left(b_{i}\right)$ and wins the contest with probability $p_{i}^{\tau}(\mathbf{s}(\mathbf{b}))$. Let

$$
u_{i}^{\mathbf{s}, \tau}\left(\mathbf{b}, \theta_{i}\right)=p_{i}^{\tau}(\mathbf{s}(\mathbf{b})) \theta_{i}-b_{i}
$$

denote the payoff of agent $i$ with type $\theta_{i}$ in contest $(\mathbf{s}, \tau)$ when efforts $\mathbf{b}$ are invested. We use pure-strategy Bayesian Nash equilibrium as the equilibrium concept. Let $\beta_{i}:[0, \bar{\theta}] \rightarrow \mathbb{R}_{+}$be a strategy of agent $i$, which prescribes an effort $\beta_{i}\left(\theta_{i}\right)$ for each type $\theta_{i}$. A strategy profile $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n}\right)$ is an equilibrium of contest $(\mathbf{s}, \tau)$ if

$$
\beta_{i}\left(\theta_{i}\right) \in \underset{b_{i} \geq 0}{\arg \max } \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[u_{i}^{\mathbf{s}, \tau}\left(\left(\beta_{1}\left(\theta_{1}\right), \ldots, \beta_{i-1}\left(\theta_{i-1}\right), b_{i}, \beta_{i+1}\left(\theta_{i+1}\right), \ldots, \beta_{n}\left(\theta_{n}\right)\right), \theta_{i}\right)\right]
$$

for all $\theta_{i} \in[0, \bar{\theta}]$ and $i \in N$, where $\boldsymbol{\theta}_{-i}=\left(\theta_{1}, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_{n}\right)$.

### 2.2. Implementation

Every equilibrium of every contest induces an effort from each agent and an allocation of the prize as a function of the realized types. We now take a mechanism-design approach to characterize the set of all such efforts and allocations that can be obtained in contests.

An allocation $\mathbf{q}=\left(q_{1}, \ldots, q_{n}\right)$ is a mapping from type profiles $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{n}\right)$ to winning probabilities: for each $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$ and $i \in N, q_{i}(\boldsymbol{\theta})$ specifies the probability with which agent $i$ obtains the prize. Let

$$
Q_{i}\left(\theta_{i}\right)=\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[q_{i}\left(\theta_{i}, \boldsymbol{\theta}_{-i}\right)\right]
$$

denote agent $i$ 's interim winning probability induced by allocation $\mathbf{q}$ when his type is $\theta_{i}$. We say that a contest (s, $\tau$ ) implements the allocation $\mathbf{q}$ and efforts $\boldsymbol{\beta}$ if $\boldsymbol{\beta}$ is an equilibrium of $(\mathbf{s}, \tau)$ and

$$
\begin{equation*}
q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\mathbf{s}(\boldsymbol{\beta}(\boldsymbol{\theta}))) \quad \text { for all } \boldsymbol{\theta} \in[0, \bar{\theta}]^{n} \text { and } i \in N \tag{1}
\end{equation*}
$$

A pair of allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$ is contest implementable if there is a contest that implements it.
The lemma below characterizes the set of contest implementable allocations and efforts. It makes use of the following definition: a direct score function for agent $i$ is a nondecreasing function $\sigma_{i}:[0, \bar{\theta}] \rightarrow \mathbb{R}_{+}$. Thus, as opposed to score functions $s_{i}$, direct score functions $\sigma_{i}$ are functions of $i$ 's type rather than $i$ 's effort.

Lemma 1. A pair of allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$ is contest implementable if and only if there exist direct score functions $\boldsymbol{\sigma}=$ $\left(\sigma_{1}(\cdot), \ldots, \sigma_{n}(\cdot)\right)$ and a tie-breaking rule $\tau$ such that

$$
\begin{equation*}
q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta})) \quad \text { for all } \boldsymbol{\theta} \in[0, \bar{\theta}]^{n} \text { and } i \in N \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta_{i}\left(\theta_{i}\right)=\theta_{i} Q_{i}\left(\theta_{i}\right)-\int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z \quad \text { for all } \theta_{i} \in[0, \bar{\theta}] \text { and } i \in N \tag{3}
\end{equation*}
$$

Lemma 1 shows that two conditions, (2) and (3), are necessary and sufficient for the existence of a contest (s, $\tau$ ) that implements $(\mathbf{q}, \boldsymbol{\beta})$. Condition (2) characterizes the set of contest implementable allocations $\mathbf{q}$ independently of the efforts $\boldsymbol{\beta}$, and condition (3) pins down the efforts $\boldsymbol{\beta}$ for any given allocation $\mathbf{q}$. An allocation $\mathbf{q}$ is contest implementable if and only if in can be described using direct score functions $\sigma$ and a tie-breaking rule $\tau$ as in (2). As for any contest implementable $\mathbf{q}$ the corresponding efforts $\boldsymbol{\beta}$ are given by (3), Lemma 1 further shows that we can identify any contest implementable pair $(\mathbf{q}, \boldsymbol{\beta})$ with just the allocation $\mathbf{q}$.

In the language of mechanism design, every contest $(\mathbf{s}, \tau)$ is a specific indirect mechanism. By the revelation principle, for each equilibrium of a contest there is an incentive compatible direct mechanism that results in the same allocation and efforts, which corresponds to the implemented $(\mathbf{q}, \boldsymbol{\beta})$. Incentive compatibility is equivalent to each interim winning probability $Q_{i}$ being nondecreasing and the effort of each agent $i$ being fully pinned down by $Q_{i}$ as in (3), which is an instance of the well-known payoff and revenue equivalence result. Hence, if contest ( $\mathbf{s}, \tau$ ) implements ( $\mathbf{q}, \boldsymbol{\beta}$ ), each $Q_{i}$ is nondecreasing and $\boldsymbol{\beta}$ satisfies (3), implying each $\beta_{i}$ is also nondecreasing. Accordingly, we can define the direct score function $\sigma_{i}(\cdot)=s_{i}\left(\beta_{i}(\cdot)\right)$ for each $i$, so that $q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\mathbf{s}(\boldsymbol{\beta}(\boldsymbol{\theta})))=p_{i}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta}))$, which explains why (2) is necessary for implementability. In addition, Lemma 1 establishes that (2) and (3) are also sufficient for implementability. In the proof, we construct for any given $\boldsymbol{\sigma}$ and $\tau$ a contest ( $\mathbf{s}, \tau^{\prime}$ ) that implements allocation and efforts satisfying (2) and (3).

Remark 1. Interpreting the efforts as payments, our contest environment is a special case of the canonical auction environment considered by Myerson (1981). However, as we restrict attention to contests (s, $\tau$ ), the corresponding set of implementable allocations and payments is smaller than in the auction environment. First, in the auction environment any allocation with nondecreasing interim winning probabilities $Q_{i}$ is implementable, whereas any contest implementable allocation satisfies the stronger requirement (2). The latter requires, for example, that $q_{i}$ is nondecreasing in $\theta_{i}$ and nonincreasing in $\theta_{j}$ for all $j \neq i$, and it restricts the situations where $q_{i}$ is neither zero nor one. Second, in the auction environment the condition corresponding to (3) only pins down the interim expected payments for any given allocation, leaving much freedom in designing the ex post payments. By contrast, in contests the interim expected efforts coincide with the ex post efforts since efforts are invested up front and are nonrefundable. Hence, the allocation fully pins down the efforts. Note that this second difference from the auction environment is not specific to the class of contests $(\mathbf{s}, \tau)$ : it arises whenever efforts cannot be conditioned on the allocation of the prize.

### 2.3. The winner's effort and differential treatment

Suppose the contest is held by an organizer who only cares about the winner's effort. If the contest implements allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$, the expected winner's effort is given by

$$
\begin{align*}
W(\mathbf{q}) & =\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \beta_{i}\left(\theta_{i}\right)\right] \\
& =\sum_{i \in N} \int_{0}^{\bar{\theta}} Q_{i}\left(\theta_{i}\right)\left(\theta_{i} Q_{i}\left(\theta_{i}\right)-\int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z\right) f\left(\theta_{i}\right) \mathrm{d} \theta_{i} . \tag{4}
\end{align*}
$$

Note that we define $W$ as a function of only the allocation as for every contest implementable $\mathbf{q}$, the efforts $\boldsymbol{\beta}$ are uniquely pinned down by (3). The organizer's problem of optimally designing the contest corresponds to choosing a contest implementable allocation $\mathbf{q}$ to maximize $W(\mathbf{q})$.

An allocation $\mathbf{q}$ is symmetric if there is a function $Q$ such that $Q_{i}\left(\theta_{i}\right)=Q\left(\theta_{i}\right)$ for all $i \in N$ and almost all $\theta_{i} \in[0, \bar{\theta}]$. We say a contest involves differential treatment if the implemented allocation is not symmetric. The main question we study below is whether it is optimal for the organizer to use differential treatment.

Remark 2. A commonly studied objective for the organizer is maximizing the expected total effort $\sum_{i \in N} \mathbb{E}_{\theta_{i}}\left[\beta_{i}\left(\theta_{i}\right)\right]$ instead of $W$. In this case, the organizer's problem is equivalent to maximizing the revenue in the auction environment considered by Myerson (1981). As agents are ex ante symmetric, Myerson's results imply that a symmetric allocation is always optimal, and if $F$ satisfies Myerson's regularity condition, any differential treatment strictly lowers the expected total effort. The results for the auction environment translate to the contest environment despite the restrictions discussed in Remark 1 because the set of optimal mechanisms in the auction environment does contain a contest. This is no longer true when considering $W$. As the winner's payment equals the total payment in any auction where only the winner pays, the maximal expected winner's payment equals the maximal revenue. Thus, that efforts cannot be conditioned on the allocation of the prize imposes a binding constraint on the problem of maximizing the expected winner's effort in contests.

## 3. When is differential treatment beneficial?

In this section, we derive sufficient conditions on $F$ such that the organizer strictly benefits from differential treatment. To do so, we first determine an optimal symmetric allocation that maximizes the expected winner's effort among all symmetric allocations. Then we identify two ways of introducing asymmetry that improve the expected winner's effort under some condition on $F$.

### 3.1. Benchmark: optimal symmetric contests

Suppose the organizer faces restrictions that only allow her to implement symmetric allocations, that is, $\mathbf{q}$ such that $Q_{i}=Q$ for all $i \in N$. As shown by the following lemma, contest implementable symmetric allocations have the property that the winner is always (one of) the agent(s) with the highest effort.

Lemma 2. Let $\mathbf{q}$ be a contest implementable symmetric allocation and let $\boldsymbol{\beta}$ satisfy (3). Then for almost all $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}, q_{i}(\boldsymbol{\theta})>0$ only if $i \in \arg \max _{j \in N} \beta_{j}\left(\theta_{j}\right)$.

Fix any contest implementable symmetric allocation $\mathbf{q}$. It follows from Lemma 2 that the expected winner's effort, $W(\mathbf{q})$, is the same as the expected highest effort, $\mathbb{E}_{\theta}\left[\max _{i \in N} \beta_{i}\left(\theta_{i}\right)\right]$. Moreover, observe that by Lemma $1, Q_{i}=Q$ implies $\beta_{i}=\beta$ for all agents $i$, where $\beta$ is a nondecreasing function. This further implies that the expected highest effort is the same as the expected effort of the agent with the highest type. We thus have

$$
\begin{equation*}
W(\mathbf{q})=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \beta_{i}\left(\theta_{i}\right)\right]=\mathbb{E}_{\boldsymbol{\theta}}\left[\max _{i \in N} \beta_{i}\left(\theta_{i}\right)\right]=\mathbb{E}_{\boldsymbol{\theta}}\left[\beta\left(\max _{i \in N} \theta_{i}\right)\right] \tag{5}
\end{equation*}
$$

An optimal symmetric allocation $\mathbf{q}$ is a contest implementable symmetric allocation $\mathbf{q}$ such that $W(\mathbf{q}) \geq W(\hat{\mathbf{q}})$ for all contest implementable symmetric allocations $\hat{\mathbf{q}}$. As in Chawla et al. (2019), optimal symmetric allocations can be characterized using a similar approach as that of Myerson (1981). ${ }^{7}$ The key step is to reformulate the objective as a virtual surplus to which Myerson's method can be applied. Using that the distribution of $\max _{i} \theta_{i}$ is $F^{n}$, substituting (3) for $\beta$, and applying integration by parts, we can rewrite (5) as

$$
\begin{aligned}
W(\mathbf{q}) & =\int_{0}^{\bar{\theta}} \beta(\theta) \mathrm{d} F(\theta)^{n} \\
& =\int_{0}^{\bar{\theta}} \theta Q(\theta) n F(\theta)^{n-1} f(\theta) \mathrm{d} \theta-\int_{0}^{\bar{\theta}} \int_{0}^{\theta} Q(z) \mathrm{d} z n F(\theta)^{n-1} f(\theta) \mathrm{d} \theta \\
& =\int_{0}^{\bar{\theta}} \theta Q(\theta) n F(\theta)^{n-1} f(\theta) \mathrm{d} \theta-\int_{0}^{\bar{\theta}} \mathrm{Q}(\theta)\left[1-F(\theta)^{n}\right] \mathrm{d} \theta
\end{aligned}
$$

[^3]$$
=\int_{0}^{\bar{\theta}} n Q(\theta) \psi(\theta) f(\theta) \mathrm{d} \theta
$$
where the virtual valuation function $\psi$ is defined as
\[

$$
\begin{equation*}
\psi(\theta)=\theta F(\theta)^{n-1}-\frac{1-F(\theta)^{n}}{n f(\theta)} \tag{6}
\end{equation*}
$$

\]

The expected winner's effort under any contest implementable symmetric allocation $\mathbf{q}$ can thus be written as

$$
\begin{equation*}
W(\mathbf{q})=\mathbb{E}_{\theta}[n Q(\theta) \psi(\theta)]=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \psi\left(\theta_{i}\right)\right] \tag{7}
\end{equation*}
$$

Now, note that by Lemma $1, Q_{i}$ must be nondecreasing for every contest implementable allocation. If $\psi$ is nondecreasing, (7) is maximized when allocating the prize to the highest type. In general, however, optimal symmetric allocations assign the prize based on ironed virtual valuations. Following Myerson (1981), the ironed virtual valuation function $\bar{\psi}$ is defined by, for all $z \in[0,1]$ and $\theta \in[0, \bar{\theta}]$,

$$
G(z)=\int_{0}^{z} \psi\left(F^{-1}(y)\right) \mathrm{d} y, \quad \bar{G}(z)=\operatorname{conv} G(z), \quad \text { and } \quad \bar{\psi}(\theta)=\bar{G}^{\prime}(F(\theta))
$$

where $\operatorname{conv} G$ denotes the convex hull of $G$ (i.e., $\bar{G}$ is the highest convex function such that $\bar{G}(z) \leq G(z)$ for $z \in[0,1]$ ). The next proposition identifies an optimal symmetric allocation.

Proposition 1. An optimal symmetric allocation $\mathbf{q}$ is given by, for each $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$ and $i \in N$,

$$
q_{i}(\boldsymbol{\theta})= \begin{cases}1 /|M| & \text { if } i \in M=\arg \max _{j \in N} \bar{\psi}\left(\theta_{j}\right)  \tag{8}\\ 0 & \text { otherwise }\end{cases}
$$

(i.e., $q_{i}$ satisfies (2) for $\sigma_{i}\left(\theta_{i}\right)=\bar{\psi}\left(\theta_{i}\right)-\bar{\psi}(0)$ and $\tau$ such that $\hat{p}_{i}^{\tau}(M, r)=1 /|M|$ for all $r$ ).

According to Proposition 1, assigning the prize to an agent with the highest ironed virtual valuation and breaking ties randomly is an optimal symmetric allocation. If $\psi$ is strictly increasing everywhere, the allocation (8) is ex post efficient: the agent with the highest valuation $\theta_{i}$ obtains the prize (ties occur with probability zero). In this case, the all-pay auction is an optimal symmetric contest, which corresponds to a contest $(\mathbf{s}, \tau)$ with score function $s_{i}\left(b_{i}\right)=b_{i}$ for each $i$ and $b_{i}$ and $\hat{p}_{i}^{\tau}(M, r)=1 /|M|$ for all $r$. As $Q(\theta)=F(\theta)^{n-1}$, the implemented efforts (3) are

$$
\begin{equation*}
\beta_{i}(\theta)=\beta^{A}(\theta)=\theta F(\theta)^{n-1}-\int_{0}^{\theta} F(z)^{n-1} \mathrm{~d} z \tag{9}
\end{equation*}
$$

where $\beta^{A}$ is the well-known equilibrium of the all-pay auction (see, e.g., Vojnović, 2016).
If $\psi$ is not strictly increasing everywhere, there are intervals on which $\bar{\psi}$ is constant, and the allocation (8) is not ex post efficient. In this case, an optimal symmetric contest is given by $(\mathbf{s}, \tau)$ with $\tau$ such that $\hat{p}_{i}^{\tau}(M, r)=1 /|M|$ for all $r$ and score function $s_{i}$ defined as follows: for each interval $\left[\theta^{I}, \theta^{I I}\right]$ where $\bar{\psi}$ is constant there is an interval of efforts $\left[b^{I}, b^{I I}\right]$, and $s_{i}\left(b_{i}\right)=\left(b^{I}+b^{I I}\right) / 2$ for $b_{i} \in\left[b^{I}, b^{I I}\right]$ whereas $s_{i}\left(b_{i}\right)=b_{i}$ outside these intervals. Under the slightly stronger condition that $\psi$ is strictly decreasing on some interval (so that $\bar{\psi}$ is different from $\psi$ ), it follows from the proof of Proposition 1 that not only the allocation (8) but every optimal symmetric allocation fails to be ex post efficient.

There is a simple sufficient condition for the virtual valuation function $\psi$ to be monotone: as is easily verified from (6), $\psi\left(\theta_{i}\right)$ is strictly increasing for all $\theta_{i}$ if $f\left(\theta_{i}\right)$ is nondecreasing, that is, if $F$ is convex. On the other hand, if $f$ is differentiable and $f^{\prime}(0)<0$, then $\psi^{\prime}(0)<0$. That is, $\psi$ is strictly decreasing at the lower bound of the support if $F$ is strictly concave. Hence, the all-pay auction is optimal among symmetric contests if $F$ is convex, but not if $F$ is strictly concave. More generally, $\psi$ is strictly decreasing on some interval if there is a sufficiently strong decrease in $f$ on some corresponding interval. As a simple example, let $\bar{\theta}=1$ and consider

$$
f(\theta)= \begin{cases}3 / 2 & \text { if } \theta \leq 7 / 16  \tag{10}\\ 5-8 \theta & \text { if } 7 / 16<\theta<9 / 16 \\ 1 / 2 & \text { if } \theta \geq 9 / 16\end{cases}
$$

Then for $n=2, \psi(\theta)$ is strictly decreasing, approximately, for $\theta \in[0.505,0.563]$ and strictly increasing otherwise. Numerically applying the ironing procedure results in $\bar{\psi}(\theta)$ being constant for $\theta \in[0.460,0.599]$ and strictly increasing otherwise.

### 3.2. Improvement 1: deterministic tie-breaking

When two or more agents tie for the highest ironed virtual valuation, the optimal symmetric allocation identified in Proposition 1 randomly assigns the prize among those agents. According to the next result, given any contest implementable allocation that is specified with a random tie-breaking rule, this tie-breaking rule can be replaced by a deterministic one to obtain a contest implementable allocation that generates at least the same and sometimes a strictly higher expected winner's effort. A tie-breaking rule $\tau$ is deterministic if $\hat{p}_{i}^{\tau}(M, r) \in\{0,1\}$ for all $M$ and $r$.

Proposition 2. Consider any direct score functions $\boldsymbol{\sigma}$ and tie-breaking rule $\tau$. Denote by $\mathbf{q}^{\boldsymbol{\sigma}, \tau}=\left(p_{1}^{\tau}(\boldsymbol{\sigma}(\cdot)), \ldots, p_{n}^{\tau}(\boldsymbol{\sigma}(\cdot))\right)$ the contest implementable allocation defined by $\sigma$ and $\tau$.
(i) There is a deterministic tie-breaking rule $\tau^{*}$ such that $W\left(\mathbf{q}^{\sigma, \tau^{*}}\right) \geq W\left(\mathbf{q}^{\sigma, \tau}\right)$.
(ii) There is a deterministic tie-breaking rule $\tau^{*}$ such that $W\left(\mathbf{q}^{\sigma, \tau^{*}}\right)>W\left(\mathbf{q}^{\sigma, \tau}\right)$ if there is a score $r>0$ and a set of agents $M$ with $|M| \geq 2$ such that $\mathbb{P}\left[\max _{j \in N} \sigma_{j}\left(\theta_{j}\right)=r\right]>0$ and for all $i \in M, \hat{p}_{i}^{\tau}(M, r) \in(0,1), \mathbb{P}\left[\sigma_{i}\left(\theta_{i}\right)=r\right]>0$, and $\mathbb{P}\left[\sigma_{i}\left(\theta_{i}\right)<r\right]>0$.

Any combination of direct score functions $\boldsymbol{\sigma}$ and tie-breaking rule $\tau$ define a contest implementable allocation. Holding $\sigma$ fixed, part (i) of Proposition 2 assures there is always a deterministic tie-breaking rule such that the defined allocation yields a weakly higher winner's effort $W$ than the allocation defined by any other tie-breaking rule. If ties occur with probability zero, $W$ is, of course, constant across all tie-breaking rules. If ties occur with positive probability, there is a deterministic tie-breaking rule that yields either the same or even a strictly higher $W$ than any random tie-breaking rule. Part (ii) of Proposition 2 identifies a sufficient condition under which randomly breaking relevant ties yields a strictly lower $W$ : there is a score $r$ such that at least two agents tie for the highest score at $r$ with positive probability and those agents also obtain a score less than $r$ with positive probability (i.e., their direct score functions assign a lower score than $r$ to an interval of types).

To illustrate the proof of Proposition 2, let $n=2$ and fix direct score functions $\sigma$ and a score $r$ such that for each agent $i, \sigma_{i}\left(\theta_{i}\right)=r$ for $\theta_{i} \in\left[x_{i}, y_{i}\right]$ with $x_{i}<y_{i}$. Then there is a tie at $r$ with positive probability, and the contest implementable allocation defined by any tie-breaking rule $\tau$ is such that $Q_{i}^{\sigma, \tau}\left(\theta_{i}\right)=K_{i}$ for $\theta_{i} \in\left[x_{i}, y_{i}\right]$ for some constant $K_{i}$. By Lemma 1 , the effort $\beta_{i}\left(\theta_{i}\right)$ implemented under this allocation is also constant for $\theta_{i} \in\left[x_{i}, y_{i}\right]$. Now, suppose the tie-breaking rule is changed in favor of agent $i$, which implies that $K_{i}$ increases. This change has two effects on agent $i$ : First, for any type $\theta_{i} \in\left[x_{i}, y_{i}\right]$, it increases both the probability that $i$ is the winner and $i$ 's effort $\beta_{i}\left(\theta_{i}\right)$, as achieving score $r$ becomes more attractive for $i$. Second, for any type $\theta_{i}>y_{i}$, it leaves the probability that $i$ is the winner unchanged but it decreases $\beta_{i}\left(\theta_{i}\right)$, as $i$ 's benefit from scores above $r$ becomes lower. Because of the first effect the expected winner's effort quadratically increases in $K_{i}$ and because of the second effect it linearly decreases in $K_{i}$. As a result, $W$ is convex in $K_{1}$ and $K_{2}$. Therefore $W$ is maximized at a tie-breaking rule that is as asymmetric as possible, that is, deterministic. The proof of part (i) generalizes this insight to $n \geq 2$ agents, and part (ii) follows because $W$ is strictly convex in that case.

Under the optimal symmetric allocation identified in Proposition 1, the prize is randomly assigned whenever there is a tie for the highest ironed virtual valuation. By part (i) of Proposition 2, using a deterministic tie-breaking rule weakly increases the expected winner's effort. Unless $\psi$ is strictly increasing everywhere, ties occur with positive probability and deterministic tie-breaking results in differential treatment, which implies that some form of differential treatment is optimal for organizer. If $\bar{\psi}$ is constant on some interval at a level greater than $\bar{\psi}(0)$, part (ii) of Proposition 2 implies that deterministic tie-breaking strictly increases $W$, that is, the organizer strictly benefits from differential treatment. ${ }^{8}$

### 3.3. Improvement 2: differential treatment at the top

We now present a second channel through which differential treatment is beneficial when $f(\bar{\theta})>1 / \bar{\theta}$, that is, when high types are sufficiently likely. Specifically, there is a threshold $x \in(0, \bar{\theta})$ such that the expected winner's effort increases if the prize is allocated lexicographically rather than efficiently among agents with types above $x$.

Consider an allocation that is efficient at the top: let $\theta^{T} \in[0, \bar{\theta})$ and let $\mathbf{q}^{T}$ be a contest implementable symmetric allocation such that $q_{i}^{T}(\cdot)=p_{i}^{\tau}(\boldsymbol{\sigma}(\cdot))$ for some direct score functions $\boldsymbol{\sigma}$ and tie-breaking rule $\tau$ where $\sigma_{i}\left(\theta_{i}\right)=\theta_{i}$ for all $\theta_{i} \in\left[\theta^{T}, \bar{\theta}\right]$ and $i \in N$. Hence, the corresponding interim winning probabilities satisfy $Q^{T}\left(\theta_{i}\right)=F\left(\theta_{i}\right)^{n-1}$ for all $\theta_{i} \in\left(\theta^{T}, \bar{\theta}\right]$. Now, for a given threshold $x \in\left(\theta^{T}, \bar{\theta}\right]$, let the contest implementable allocation $\mathbf{q}^{\chi}$ be such that $q_{i}^{\chi}(\cdot)=p_{i}^{\tau}\left(\boldsymbol{\sigma}^{\chi}(\cdot)\right)$, where for all $i \in N$,

$$
\sigma_{i}^{\chi}\left(\theta_{i}\right)= \begin{cases}\sigma_{i}\left(\theta_{i}\right) & \text { if } \theta_{i} \leq x \\ x+\frac{n-i}{n-1}(1-x) & \text { if } \theta_{i}>x\end{cases}
$$

[^4]The corresponding interim winning probabilities are

$$
Q_{i}^{x}\left(\theta_{i}\right)= \begin{cases}Q^{T}\left(\theta_{i}\right) & \text { if } \theta_{i} \leq x \\ F(x)^{i-1} & \text { if } \theta_{i}>x\end{cases}
$$

The allocation $\mathbf{q}^{x}$ coincides with $\mathbf{q}^{T}$ except when two or more agents have types greater than $x$. In this case, as $\sigma_{i}^{x}\left(\theta_{i}\right)$ is constant in $\theta_{i}$ and decreasing in $i, \mathbf{q}^{x}$ assigns the prize to agent $\min \left\{i \in N \mid \theta_{i}>x\right\}$ whereas $\mathbf{q}^{T}$ allocates it efficiently. The following proposition shows that there exists a threshold $x$ such that the expected winner's effort is greater under $\mathbf{q}^{x}$.

Proposition 3. Suppose $f(\bar{\theta})>1 / \bar{\theta}$. Then $W\left(\mathbf{q}^{\chi}\right)>W\left(\mathbf{q}^{T}\right)$ for some $x \in\left(\theta^{T}, \bar{\theta}\right)$.
Note that the optimal symmetric allocation given in Proposition 1 is always an efficient-at-the-top allocation $\mathbf{q}^{T}$. This is because $\psi(\bar{\theta})=\bar{\theta}>\psi\left(\theta_{i}\right)$ for all $\theta_{i}<\bar{\theta}$, which implies that $\bar{\psi}$ is strictly increasing for types close to $\bar{\theta}$. Consequently, Proposition 3 shows that if $f(\bar{\theta})>1 / \bar{\theta}$, differential treatment in the form of allocation $\mathbf{q}^{\chi}$ yields a strictly higher expected winner's effort than any contest implementable symmetric allocation.

Suppose $\theta^{T}=0$. Then $\mathbf{q}^{T}$ is the ex post efficient allocation, as is implemented by the all-pay auction (and $\mathbf{q}^{T}$ coincides with (8) if $\psi$ is strictly increasing). The implemented efforts are $\beta^{A}$ as defined in (9). On the other hand, the implemented efforts under the modified allocation $\mathbf{q}^{x}$ are $\beta_{i}\left(\theta_{i}\right)=\beta^{A}\left(\theta_{i}\right)$ for $\theta_{i} \leq x$ and $\beta_{i}\left(\theta_{i}\right)=\beta^{A}(x)+x F(x)^{i-1}\left(1-F(x)^{n-i}\right)$ for $\theta_{i} \geq$ $x$. The allocation $\mathbf{q}^{x}$ can be implemented using a modified all-pay auction where each agent $i$ is offered a hierarchical guaranteed-winning option $b_{i}^{G W}$ : if agent $i$ invests at least effort $b_{i}^{G W}$, he is guaranteed to win against all agents $j>i$. This corresponds to a contest $(\mathbf{s}, \tau)$ where for all $i \in N, s_{i}\left(b_{i}\right)=b_{i}$ if $b_{i}<b_{i}^{G W}$ and $s_{i}\left(b_{i}\right)=n+1-i$ if $b_{i} \geq b_{i}^{G W}$. The allocation $\mathbf{q}^{x}$ is implemented if $b_{i}^{G W}=\beta^{A}(x)+x F(x)^{i-1}\left(1-F(x)^{n-i}\right)$.

In response to this modification of the all-pay auction, each agent $i$ with type above $x$ changes his effort to $b_{i}^{G W}$, which is decreasing in $i$. Thus, agents with high priorities invest more and agents with low priorities invest less. Moreover, agents with high priorities are more likely to win, making their efforts more valuable to the organizer relative to the efforts of agents with low priorities. Proposition 3 shows that it is possible to find a suitable threshold $x$ such that the overall effect on $W$ is positive if $f(\bar{\theta})>1 / \bar{\theta}$.

As we had seen above, the all-pay auction is not optimal if $F$ is strictly concave, as there are other symmetric contests that generate a greater expected winner's effort. By Proposition 3, the all-pay auction is also not optimal if $F$ is strictly convex (which implies $f(\bar{\theta})>1 / \bar{\theta}$ ), as it is outperformed by some asymmetric contests. Interestingly, as we will find in Section 4, the all-pay auction is optimal if $F$ is uniform (i.e., both concave and convex), while many asymmetric contests are also optimal in that case.

### 3.4. The main result

We are ready to present the main result of this section, which summarizes our insights obtained above. We have identified two situations where differential treatment improves upon the optimal symmetric allocation in Proposition 1. First, part (ii) of Proposition 2 implies that if $\bar{\psi}$ is constant at a value greater than $\bar{\psi}(0)$, deterministic tie-breaking improves upon the random tie-breaking involved in the optimal symmetric allocation. Second, Proposition 3 shows that if $f(\bar{\theta})>1 / \bar{\theta}$, differential treatment when types are close to $\bar{\theta}$ improves upon the optimal symmetric allocation, which is efficient at the top. We obtain the following theorem.

Theorem 1. Suppose at least one of the following two conditions holds:
(i) There are $\theta^{I}, \theta^{I I}$ such that $0<\theta^{I}<\theta^{I I}$ and $\bar{\psi}(0)<\bar{\psi}\left(\theta^{I}\right)=\bar{\psi}\left(\theta^{I I}\right)$.
(ii) $f(\bar{\theta})>1 / \bar{\theta}$.

Then, there is a contest implementable allocation $\mathbf{q}$ such that $W(\mathbf{q})>W(\hat{\mathbf{q}})$ for every contest implementable symmetric allocation $\hat{\mathbf{q}}$. Hence, differential treatment occurs under every contest that maximizes the expected winner's effort.

The above theorem presents two separate conditions on the type distribution $F$, and it establishes that the organizer strictly benefits from differential treatment if $F$ satisfies one (or both) of them. Condition (i) requires that the virtual valuation function $\psi$ is nonincreasing on some interval in such a way that the ironed counterpart $\bar{\psi}$ is constant on an interval where its value is higher than at the lowest type. This is satisfied if there is a sufficiently strong decrease in the density $f$ on an interval of sufficiently high types. A simple example of such a distribution is given in (10). Condition (ii) requires that the (continuous) density $f$ is greater than that of the uniform distribution at the upper bound of the support. Being a local property of $F$, one can easily construct examples that satisfy (ii) in addition to (i). But (ii) is also satisfied by many distributions that do not satisfy (i). Specifically, as discussed in Subsection 3.1, if $F$ is convex, $\psi$ is strictly increasing and thus condition (i) does not hold. However, except for the uniform distribution every convex $F$ satisfies condition (ii). On the other hand, concave $F$ do not satisfy (ii), but they may satisfy (i).

Theorem 1 identifies a broad class of situations where an organizer who aims at maximizing the winner's effort is strictly better off when she is not restricted to use only contests that implement a symmetric allocation. We next demonstrate that in many cases also the agents are strictly better off when the organizer does not face such a restriction. Using Lemma 1 , the agents' surplus, that is, the sum of their ex ante expected payoffs can be written as

$$
\begin{align*}
\sum_{i \in N} \mathbb{E}_{\theta_{i}}\left[Q_{i}\left(\theta_{i}\right) \theta_{i}-\beta_{i}\left(\theta_{i}\right)\right] & =\sum_{i \in N} \int_{0}^{\bar{\theta}} \int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z f\left(\theta_{i}\right) \mathrm{d} \theta_{i} \\
& =\sum_{i \in N} \int_{0}^{\bar{\theta}} Q_{i}(z)(1-F(z)) \mathrm{d} z \\
& =\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \frac{1-F\left(\theta_{i}\right)}{f\left(\theta_{i}\right)}\right] \tag{11}
\end{align*}
$$

Now, suppose $F$ is convex. Then $\psi$ is strictly increasing and the ex post efficient allocation is the unique optimal symmetric allocation (it is optimal according to Proposition 1 and the uniqueness can be easily verified using (7)). Moreover, ( $1-$ $\left.F\left(\theta_{i}\right)\right) / f\left(\theta_{i}\right)$ is strictly decreasing in $\theta_{i}$, so that the ex post efficient allocation uniquely minimizes the agents' surplus (11) among all contest implementable allocations. If, in addition, condition (ii) of Theorem 1 holds, which only excludes the uniform distribution, allocations that maximize the winner's effort are not ex post efficient and thus yield a higher agents' surplus.

Corollary 1. Suppose $F$ is convex and $f(\bar{\theta})>1 / \bar{\theta}$. Then every contest that maximizes the expected winner's effort involves differential treatment and yields a strictly higher agents' surplus than the optimal symmetric allocation.

So far, we have focused on situations where every optimal contest uses differential treatment. In the next section, we present a situation where many optimal contests use differential treatment, but also one symmetric allocation is optimal.

## 4. Optimal contests in the uniform case

In this section, we will obtain a full characterization of all contest implementable allocations that maximize the expected winner's effort when types are uniformly distributed. We start with some preliminaries and then focus on the uniform case.

### 4.1. Preliminaries

To identify the contests that maximize the expected winner's effort, we consider the problem of maximizing $W(\mathbf{q})$ over all contest implementable allocations $\mathbf{q}$. It will be useful to separate the expected winner's effort as given in (4) into two parts:

$$
W(\mathbf{q})=W^{I}(\mathbf{q})-W^{I I}(\mathbf{q})
$$

where

$$
W^{I}(\mathbf{q})=\int_{0}^{\bar{\theta}} \sum_{i \in N} Q_{i}\left(\theta_{i}\right)^{2} \theta_{i} f\left(\theta_{i}\right) \mathrm{d} \theta_{i} \quad \text { and } \quad W^{I I}(\mathbf{q})=\sum_{i \in N} \int_{0}^{\bar{\theta}} Q_{i}\left(\theta_{i}\right) \int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z f\left(\theta_{i}\right) \mathrm{d} \theta_{i}
$$

In the mechanism-design literature, optimal allocations are usually studied for objectives that take the form

$$
\sum_{i \in N} \int_{0}^{\bar{\theta}} \Gamma\left(Q_{i}(\theta), \theta\right) f(\theta) \mathrm{d} \theta
$$

for some function $\Gamma$. When restricting attention to symmetric allocations in Subsection 3.1, we used the observation that the winner's effort equals the highest type's effort to rewrite the objective $W(\mathbf{q})$ in this form with $\Gamma(Q, \theta)=Q \psi(\theta)$, enabling us to apply Myerson's (1981) approach because $\Gamma$ is linear in $Q$. This is not possible when we allow for asymmetric allocations. Considered in isolation, part $W^{I}$ can be written using $\Gamma(Q, \theta)=Q^{2} \theta$, which belongs to a class of objectives for which Gershkov et al. (2021) recently identified the conditions under which symmetric allocations are optimal when $n=2$.

Part $W^{I I}$, however, cannot be written using $\Gamma$ and is much less tractable. ${ }^{9}$ And, of course, also the interaction between $W^{I}$ and $W^{I I}$ must be dealt with. ${ }^{10}$

The optimization problem becomes significantly more tractable for uniform $F$. As we show below, $W^{I I}$ then takes a simple form and $W^{I}$ is constant for all deterministic allocations. For the latter result, we will make use of the following lemma, which holds for general $F$. We say an allocation $\mathbf{q}$ is deterministic if $q_{i}(\boldsymbol{\theta}) \in\{0,1\}$ for almost all $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$ and all $i \in N$. The lemma shows that for every contest implementable deterministic allocation the interim winning probabilities can be represented by some direct score functions alone.

Lemma 3. Let $\mathbf{q}$ be a contest implementable deterministic allocation. There exist direct score functions $\boldsymbol{\sigma}$ such that $Q_{i}\left(\theta_{i}\right)=$ $\prod_{j \neq i} F\left(\sigma_{j}^{-1}\left(\sigma_{i}\left(\theta_{i}\right)\right)\right)$ for all $i \in N$ and almost all $\theta_{i} \in[0, \bar{\theta}]$, where $\sigma_{i}^{-1}(r)=\inf \left\{\theta_{i} \in[0, \bar{\theta}]: \sigma_{i}\left(\theta_{i}\right) \geq r\right\}$.

### 4.2. The uniform case

We now assume types are uniformly distributed, that is, $f\left(\theta_{i}\right)=1 / \bar{\theta}$ for all $\theta_{i} \in[0, \bar{\theta}]$. We first focus on part $W^{I}$ of the objective. To illustrate our approach, let $n=2$ and $\bar{\theta}=1$. Consider any contest implementable deterministic allocation such that $Q_{1}$ is strictly increasing, is differentiable, and satisfies $Q_{1}(0)=0$ and $Q_{1}(1)=1$. By Lemma 3, we have $Q_{2}(\theta)=$ $\sigma_{1}^{-1}\left(\sigma_{2}(\theta)\right)=Q_{1}^{-1}(\theta)$ for $\theta \in[0,1]$. Using

$$
\int_{0}^{1} Q_{2}(\theta)^{2} \theta \mathrm{~d} \theta=\int_{0}^{1} Q_{1}^{-1}(\theta)^{2} \theta \mathrm{~d} \theta=\int_{0}^{1} z^{2} Q_{1}(z) Q_{1}^{\prime}(z) \mathrm{d} z
$$

where the second equality obtains by the change of variable $\theta=Q_{1}(z)$, we have

$$
W^{I}(\mathbf{q})=\int_{0}^{1}\left(Q_{1}(\theta)^{2} \theta+\theta^{2} Q_{1}(\theta) Q_{1}^{\prime}(\theta)\right) \mathrm{d} \theta=\left[\frac{1}{2} Q_{1}(\theta)^{2} \theta^{2}\right]_{0}^{1}=\frac{1}{2}
$$

Hence, $W^{I}$ is the same for all such allocations, and it equals $\mathbb{E}\left[\theta_{i}\right]$. The next lemma generalizes this result to all contest implementable deterministic allocations, arbitrary $n$, and arbitrary $\bar{\theta}$. The proof is based on a generalization of Young's inequality to $n \geq 2$ nondecreasing functions.

Lemma 4. Let $F$ be uniform. Then, $W^{I}(\mathbf{q})=\bar{\theta} / 2$ for every contest implementable deterministic allocation $\mathbf{q}$.
Turning to part $W^{I I}$ of the objective, define $R_{i}\left(\theta_{i}\right)=\int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z$. Then we can write $W^{I I}$ for every allocation $\mathbf{q}$ as

$$
\begin{equation*}
W^{I I}(\mathbf{q})=\sum_{i \in N} \int_{0}^{\bar{\theta}} R_{i}^{\prime}\left(\theta_{i}\right) R_{i}\left(\theta_{i}\right) \frac{1}{\bar{\theta}} \mathrm{~d} \theta_{i}=\sum_{i \in N} \frac{1}{2 \bar{\theta}} R_{i}(\bar{\theta})^{2}=\frac{\bar{\theta}}{2} \sum_{i \in N}\left(\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]\right)^{2} \tag{12}
\end{equation*}
$$

Thus $W^{I I}$ is independent of the details of the allocation and is only a function of the ex ante winning probability $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]$ of each agent.

Combining Lemma 4 and (12) we obtain the expected winner's effort for every contest implementable deterministic allocation $\mathbf{q}$ as

$$
\begin{equation*}
W(\mathbf{q})=\frac{\bar{\theta}}{2}-\frac{\bar{\theta}}{2} \sum_{i \in N}\left(\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]\right)^{2} \tag{13}
\end{equation*}
$$

Accordingly, the problem of identifying the contest implementable deterministic allocations that maximize $W$ reduces to maximizing (13) with respect to the ex ante winning probability of each agent. Note that (13) is strictly concave in $\mathbb{E}\left[Q_{1}\left(\theta_{1}\right)\right], \ldots, \mathbb{E}\left[Q_{n}\left(\theta_{n}\right)\right]$. Hence, as every allocation satisfies $\sum_{i=1}^{n} \mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1$, (13) is maximized if and only if $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1 / n$ for all $i \in N$. This condition characterizes all optimal contest implementable deterministic allocations. By Proposition 2 the expected winner's effort cannot be higher under non-deterministic allocations. In the proof of the following theorem, we further show that any non-deterministic allocation generates a strictly lower expected winner's effort.

[^5]Theorem 2. Let $F$ be uniform. The contest implementable allocation $\mathbf{q}$ maximizes the expected winner's effort $W(\mathbf{q})$ if and only if $\mathbf{q}$ is deterministic and $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1 / n$ for all $i \in N$.

Theorem 2 shows that the set of allocations that maximize the expected winner's effort is large. The only requirements are that the allocation is deterministic and that every agent has the same ex ante probability of winning. This leaves a lot of flexibility at the interim stage, so that many forms of differential treatment (i.e., $Q_{i}(\theta) \neq Q_{j}(\theta)$ for a set of types $\theta$ with positive measure) are optimal. On the other hand, a unique symmetric allocation is optimal: the ex post efficient allocation that is implemented by the all-pay auction. By (13), $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1 / n$ implies that the maximized expected winner's effort equals $\bar{\theta}(n-1) /(2 n)$.

As an illustration, suppose there are only two agents and $\bar{\theta}=1$. In this case, every nondecreasing function $a:[0,1] \rightarrow$ $[0,1]$ such that $\int_{0}^{1} a(z) \mathrm{d} z=1 / 2$ gives rise to an optimal contest implementable deterministic allocation $\mathbf{q}$ where $q_{1}(\boldsymbol{\theta})=1$ if and only if $\theta_{2}<a\left(\theta_{1}\right)$.

For example, consider a contest $(\mathbf{s}, \tau)$ where $s_{1}\left(b_{1}\right)=h+c b_{1}$ and $s_{2}\left(b_{2}\right)=b_{2}$ for some $h \geq 0$ and $c \in(0,1]$ and where $\hat{p}_{1}^{\tau}(M, r)=0$ for all $r$. This corresponds to the biased all-pay auction with head start $h$ and handicap $c$ studied by Kirkegaard (2012). If types are uniformly distributed on [0, 1], Kirkegaard (2012, Proposition 1) implies that the implemented efforts are

$$
\beta_{1}\left(\theta_{1}\right)=\left\{\begin{array}{ll}
0 & \text { if } \theta_{1}<h^{\frac{c}{1+c}} \\
\frac{1}{1+c} \theta_{1}^{\frac{1+c}{c}}-\frac{1}{1+c} h & \text { if } \theta_{1} \geq h^{\frac{c}{1+c}}
\end{array} \quad \beta_{2}\left(\theta_{2}\right)= \begin{cases}0 & \text { if } \theta_{2}<h^{\frac{1}{1+c}} \\
\frac{c}{1+c} \theta_{2}^{1+c}+\frac{1}{1+c} h & \text { if } \theta_{2} \geq h^{\frac{1}{1+c}}\end{cases}\right.
$$

Moreover, the deterministic allocation $\mathbf{q}$ is implemented where $q_{1}(\boldsymbol{\theta})=1$ if and only if $\theta_{2}<a\left(\theta_{1}\right)=\max \left\{h^{1 /(1+c)}, \theta_{1}^{1 / c}\right\}$. It follows that $\mathbb{E}\left[Q_{1}\left(\theta_{1}\right)\right]=\min \{(h+c) /(1+c), 1\}$. Consequently, any combination of head start $h$ and handicap $c$ such that $(h+c) /(1+c)=1 / 2$ maximizes the expected winner's effort. Special cases include $h=0$ and $c=1$, which corresponds to the all-pay auction, as well as $h=1 / 2$ and $c=0$, which corresponds to a take-it-or-leave-it offer to agent 2 to win the prize for effort $1 / 2$.

Whereas the organizer is indifferent between all contest implementable deterministic allocations where $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1 / n$ for all $i$, the agents' payoffs differ across these allocations. With uniform $F$ the agents' surplus (i.e., the sum of their payoffs) as given in (11) becomes $\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta})\left(\bar{\theta}-\theta_{i}\right)\right]$, which is (uniquely) minimized by the ex post efficient allocation. Hence, among all optimal allocations identified in Theorem 2, the symmetric one minimizes the agents' surplus. For example, for $n=2$ and $\bar{\theta}=1$, the all-pay auction yields sum of payoffs $1 / 3$ whereas the take-it-or-leave-it offer to agent 2 yields sum of payoffs $3 / 8$.

Even though the agents' surplus when the organizer uses differential treatment is always higher than in the all-pay auction, not necessarily all agents benefit from differential treatment. For instance, with $n=2$ and $\bar{\theta}=1$ the take-it-or-leave-it offer to agent 2 results in payoff $1 / 4$ for agent 1 and payoff $1 / 8$ for agent 2 , while the all-pay auction results in payoff $1 / 6$ for each agent. In this case, differential treatment makes agent 1 better off and agent 2 worse off, and one can show that this is also true for all other combinations of head start $h$ and handicap $c$ that maximize the winner's effort. However, with other forms of differential treatment, it is possible that both agents benefit. For example, consider the allocation where $q_{1}(\boldsymbol{\theta})=1$ if and only if $\theta_{1}>1-1 / \sqrt{2}$ and $\theta_{2}<1 / \sqrt{2}$, which belongs to the set of allocations that maximize the winner's effort. As can be easily verified, under this allocation each agent obtains payoff $\sqrt{2} / 8>1 / 6$, that is, both agents are better off than in the all-pay auction.

## 5. Winner's effort vs. highest effort

In this section, we compare the expected winner's effort to the expected highest effort, another possible objective for the organizer. Although the two objectives are closely related, we show that the corresponding maximization problems differ, and we discuss implications for the role of institutional constraints an organizer may face in applications.

In a contest that implements allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$, the expected highest effort is

$$
H(\mathbf{q})=\mathbb{E}_{\boldsymbol{\theta}}\left[\max _{i \in N} \beta_{i}\left(\theta_{i}\right)\right] .
$$

Clearly, we have $H(\mathbf{q}) \geq W(\mathbf{q})$ for every contest implementable allocation $\mathbf{q}$, as the winner is not necessarily the agent with the highest effort.

By Lemma 2, $H(\mathbf{q})=W(\mathbf{q})$ for every contest implementable symmetric allocation $\mathbf{q}$. As a consequence, if differential treatment improves the winner's effort, it also improves the highest effort. Specifically, under the conditions given in Theorem 1, there is a contest implementable allocation $\mathbf{q}$ such that

$$
H(\mathbf{q}) \geq W(\mathbf{q})>W(\hat{\mathbf{q}})=H(\hat{\mathbf{q}})
$$

for every contest implementable symmetric allocation $\hat{\mathbf{q}}$, that is, differential treatment occurs in every contest that maximizes the expected highest effort.

Now, suppose types are uniformly distributed, as in the preceding section. Let $\bar{\theta}=1$ and recall the two-agents example with the biased all-pay auction with head start $h$ and handicap $c$, which maximizes the expected winner's effort if and only if $(h+c) /(1+c)=1 / 2$. Observe that in case of the all-pay auction $(h=0, c=1)$ and the take-it-or-leave-it offer to agent $2(h=1 / 2, c=0)$, the winner also invests the highest effort, so that $H$ and $W$ coincide. However, the equivalence fails for some less extreme choices of $h$ and $c$. For example, let $h=1 / 4$ and $c=1 / 2$. Then agent 1 wins if and only if $\theta_{2}<\max \left\{(1 / 4)^{2 / 3},\left(\theta_{1}\right)^{2}\right\}$, and the implemented efforts are

$$
\beta_{1}\left(\theta_{1}\right)=\left\{\begin{array}{ll}
0 & \text { if } \theta_{1}<(1 / 4)^{1 / 3} \\
\frac{2}{3}\left(\theta_{1}\right)^{3}-\frac{1}{6} & \text { if } \theta_{1} \geq(1 / 4)^{1 / 3},
\end{array} \quad \beta_{2}\left(\theta_{2}\right)= \begin{cases}0 & \text { if } \theta_{2}<(1 / 4)^{2 / 3} \\
\frac{1}{3}\left(\theta_{2}\right)^{3 / 2}+\frac{1}{6} & \text { if } \theta_{2} \geq(1 / 4)^{2 / 3}\end{cases}\right.
$$

Observe that for type profiles $\boldsymbol{\theta}$ such that $\theta_{1} \in\left((1 / 4)^{1 / 3},(5 / 8)^{1 / 3}\right)$ and $\theta_{2} \in\left((1 / 4)^{2 / 3},\left(\theta_{1}\right)^{2}\right)$, agent 1 wins but $\beta_{1}\left(\theta_{1}\right)<$ $1 / 4<\beta_{2}\left(\theta_{2}\right)$. Hence, the expected highest effort exceeds the maximum expected winner's effort.

The example discussed above has important implications. Combined with Theorem 2, it shows that if types are uniformly distributed, there exists a contest implementable allocation $\mathbf{q}$ such that $H(\mathbf{q})>W(\mathbf{q}) \geq W(\hat{\mathbf{q}})=H(\hat{\mathbf{q}})$ for all contest implementable symmetric allocations $\hat{\mathbf{q}}$, whereas $W$ is maximized by a contest implementable symmetric allocation. Thus, differential treatment also strictly benefits the organizer in the uniform case if her payoff is $H(\mathbf{q})$ instead of $W(\mathbf{q})$. This demonstrates that the set of environments where the winner's effort can be increased with differential treatment is a proper subset of the set of environments where the highest effort can be increased with differential treatment.

The organizer of an innovation or crowdsourcing contest ideally wants to implement the best submitted solution, that is, the highest effort. However, for the reasons outlined in Section 1, the organizer may face the constraint that she is only allowed to implement the winning solution. Thus she can only choose the allocation to maximize the winner's effort. When restricting attention to symmetric allocations, the two objectives are equivalent and such a constraint is not binding. However, as shown above, the equivalence fails for asymmetric allocations, and thus the constraint becomes binding. This creates an incentive for the organizer to invest in a commitment device, so that she can credibly separate the winner from the contestant with the best submission.

## 6. Extension: withholding the prize

In every contest $(\mathbf{s}, \tau)$, the prize is always allocated to one of the agents. Formally, every contest implementable allocation satisfies $\sum_{i \in N} q_{i}(\boldsymbol{\theta})=1$ for all $\boldsymbol{\theta}$. We now extend our definition of contests to allow for the possibility that the organizer withholds the prize. This means that also some allocations where $\sum_{i \in N} q_{i}(\boldsymbol{\theta})<1$ for some $\boldsymbol{\theta}$ become implementable. We define such contests as follows.

A reserve contest $\left(\mathbf{s}, \tau, s^{*}\right)$ is characterized by score functions $\mathbf{s}$, a tie-breaking rule $\tau$, and a reserve score $s^{*} \in \mathbb{R}_{+}$. Given profile of efforts $\mathbf{b}$, if $s_{i}\left(b_{i}\right)<s^{*}$ for all $i \in N$, the prize is withheld (i.e., there is no winner), and otherwise the winner is determined as in the contest $(\mathbf{s}, \tau)$. Note that the additional design parameter $s^{*}$ together with the score functions enable the organizer to set for each agent $i$ an individual minimum level of effort below which $i$ cannot win the prize. This is similar to setting individual reserve prices in an auction.

We say that a pair of allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$ is reserve-contest implementable if there is a reserve contest that implements it. That is, $\boldsymbol{\beta}$ is an equilibrium of some reserve contest $\left(\mathbf{s}, \tau, s^{*}\right)$ and for each $i \in N$ and $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$, if $s_{i}\left(\beta_{i}\left(\theta_{i}\right)\right)<$ $s^{*}$ then $q_{i}(\boldsymbol{\theta})=0$ and otherwise $q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\mathbf{s}(\boldsymbol{\beta}(\boldsymbol{\theta})))$.

From the perspective of the agents, competing in a reserve contest with $n$ agents and reserve score $s^{*}$ is equivalent to competing in a contest with one additional agent whose score function satisfies $s_{n+1}\left(b_{n+1}\right)=s^{*}$ for all $b_{n+1} \in \mathbb{R}_{+}$and who never wins in case of a tie. Indeed, in such a contest with $n+1$ agents, the effort chosen by agent $n+1$ is always zero and the efforts of the other agents correspond to an equilibrium if and only if they do so in the reserve contest with $n$ agents. Hence, the same allocation is implemented in both cases, except that agent $n+1$ obtains the prize in the contest whenever it is withheld in the reserve contest. This observation immediately yields the following characterization of reserve-contest implementable allocations and efforts.

Lemma 5. Allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$ for $n$ agents are reserve-contest implementable if and only if allocation and efforts $(\breve{\mathbf{q}}, \breve{\boldsymbol{\beta}})$ for $n+1$ agents are contest implementable, where

$$
\begin{aligned}
& \breve{q}_{i}\left(\theta_{1}, \ldots, \theta_{n+1}\right)= \begin{cases}q_{i}\left(\theta_{1}, \ldots, \theta_{n}\right) & \text { if } i \leq n \\
1-\sum_{j=1}^{n} q_{j}\left(\theta_{1}, \ldots, \theta_{n}\right) & \text { if } i=n+1,\end{cases} \\
& \breve{\beta}_{i}\left(\theta_{i}\right)= \begin{cases}\beta_{i}\left(\theta_{i}\right) & \text { if } i \leq n \\
0 & \text { if } i=n+1\end{cases}
\end{aligned}
$$

for all $\left(\theta_{1}, \ldots, \theta_{n+1}\right) \in[0, \bar{\theta}]^{n+1}$. Moreover, $W(\mathbf{q})=W(\breve{\mathbf{q}})$.

Using Lemma 5, many results we obtained for contest implementable allocations in the preceding sections can easily be extended to reserve-contest implementable allocations.

As in Section 3, we next identify situations where differential treatment increases the winner's effort. It is straightforward to adapt Proposition 1 and its proof to reserve-contest implementable allocations, the only difference being that the prize is withheld if the highest ironed virtual valuation is negative ${ }^{11}$ : an optimal reserve-contest implementable symmetric allocation $\mathbf{q}$ is given by, for each $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$ and $i \in N$,

$$
q_{i}(\boldsymbol{\theta})= \begin{cases}1 /|M| & \text { if } \bar{\psi}\left(\theta_{i}\right) \geq 0 \text { and } i \in M=\arg \max _{j \in N} \bar{\psi}\left(\theta_{j}\right)  \tag{14}\\ 0 & \text { otherwise }\end{cases}
$$

If $\bar{\psi}\left(\theta^{I}\right)=\bar{\psi}\left(\theta^{I I}\right) \geq 0$ for some $\theta^{I}<\theta^{I I}$, this optimal symmetric $\mathbf{q}$ assigns the prize randomly with positive probability. Then, applying part (ii) of Proposition 2 to the corresponding contest implementable allocation with $n+1$ agents (i.e., $\breve{\mathbf{q}}$ in Lemma 5), one can show that there is a reserve-contest implementable deterministic allocation that generates a strictly higher expected winner's effort. ${ }^{12}$ Moreover, note that the optimal symmetric allocation (14) is an efficient-at-the-top allocation. As the proof of Proposition 3 continues to hold if $\sum_{i \in N} q_{i}^{T}(\boldsymbol{\theta})<1$ for some $\boldsymbol{\theta}$, Proposition 3 also applies here, showing that the expected winner's effort can be increased if $f(\bar{\theta})>1 / \bar{\theta}$. Consequently, we obtain the following counterpart to Theorem 1.

Theorem 1R. Suppose at least one of the following two conditions holds:
(i) There are $\theta^{I}, \theta^{I I}$ such that $0<\theta^{I}<\theta^{I I}$ and $0 \leq \bar{\psi}\left(\theta^{I}\right)=\bar{\psi}\left(\theta^{I I}\right)$.
(ii) $f(\bar{\theta})>1 / \bar{\theta}$.

Then, there is a reserve-contest implementable allocation $\mathbf{q}$ such that $W(\mathbf{q})>W(\hat{\mathbf{q}})$ for every reserve-contest implementable symmetric allocation $\hat{\mathbf{q}}$. Hence, differential treatment occurs under every reserve contest that maximizes the expected winner's effort.

Theorem 1R identifies sufficient conditions on $F$ such that the organizer strictly benefits from differential treatment even if she can withhold the prize. While condition (i) is satisfied by a smaller set of $F$ than if the prize cannot be withheld, condition (ii) remains unaffected. ${ }^{13}$

As in Section 4, suppose now that types are uniformly distributed. Let $\mathbf{q}$ be a reserve-contest implementable allocation for $n$ agents and let $\breve{\mathbf{q}}$ be the corresponding contest implementable allocation with $n+1$ agents as defined in Lemma 5 . As a (non-)deterministic $\mathbf{q}$ corresponds to a (non-)deterministic $\breve{\mathbf{q}}$, Proposition 2 implies that for every reserve-contest implementable non-deterministic allocation, there is a reserve-contest implementable deterministic allocation that generates a weakly higher expected winner's effort. Using (13), we obtain that for every reserve-contest implementable deterministic allocation $\mathbf{q}$,

$$
\begin{align*}
W(\mathbf{q})=W(\breve{\mathbf{q}}) & =\frac{\bar{\theta}}{2}-\frac{\bar{\theta}}{2} \sum_{i=1}^{n}\left(\mathbb{E}\left[\breve{Q}_{i}\left(\theta_{i}\right)\right]\right)^{2}-\frac{\bar{\theta}}{2}\left(\mathbb{E}\left[\breve{Q}_{n+1}\left(\theta_{n+1}\right)\right]\right)^{2} \\
& =\frac{\bar{\theta}}{2}-\frac{\bar{\theta}}{2} \sum_{i=1}^{n}\left(\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]\right)^{2}-\frac{\bar{\theta}}{2}\left(1-\sum_{i=1}^{n} \mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]\right)^{2} \tag{15}
\end{align*}
$$

Maximizing (15) and showing that any non-deterministic allocation generates a strictly lower $W$, we obtain the following counterpart to Theorem 2.

Theorem 2R. Let $F$ be uniform. The reserve-contest implementable allocation $\mathbf{q}$ maximizes the expected winner's effort $W(\mathbf{q})$ if and only if $\mathbf{q}$ is deterministic and $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=1 /(n+1)$ for all $i \in N$.

Theorem 2R characterizes the set of allocations that maximize the expected winner's effort when the prize can be withheld. Optimality only requires that the allocation is deterministic and leads to the same ex ante probability of winning for each of the $n$ agents, namely $1 /(n+1)$. Accordingly, the ex ante probability that the prize is withheld also equals $1 /(n+1)$. Using (15), we find that the maximized expected winner's effort equals $\bar{\theta} n /(2 n+2)$.

Again, like when the prize cannot be withheld, many forms of differential treatment are optimal, and there is also a unique optimal symmetric allocation. For $\bar{\theta}=1$, the optimal symmetric allocation is implemented by a modified all-pay

[^6]auction where the prize is withheld if the highest effort is below the minimum level of effort $b^{*}=(n+1)^{-(n+1) / n}$. By contrast, one can verify that the following contest with differential treatment is also optimal: agent $i \in N$ wins the prize if and only if his effort is greater than or equal to $i /(n+1)$ and the effort of each agent $j>i$ is less than $j /(n+1)$.

As in Section 5, we use an example to demonstrate that maximizing the winner's effort is different from maximizing the highest effort. Suppose there are two agents, $F$ is uniform, and $\bar{\theta}=1$. Consider the reserve-contest implementable deterministic allocation $\mathbf{q}$ such that $q_{1}(\boldsymbol{\theta})=1$ if and only if $\theta_{1} \geq \max \left\{3 / 5,\left(6 \theta_{2}-1\right) / 5\right\}$ whereas $q_{2}(\boldsymbol{\theta})=1$ if and only if $\theta_{2} \geq 5 / 9$ and $\theta_{1}<\max \left\{3 / 5,\left(6 \theta_{2}-1\right) / 5\right\}$. As is easily verified, $\mathbb{E}\left[Q_{1}\left(\theta_{1}\right)\right]=\mathbb{E}\left[Q_{2}\left(\theta_{2}\right)\right]=1 / 3$, and thus $\mathbf{q}$ maximizes the expected winner's effort (Theorem 2R). The implemented efforts are

$$
\beta_{1}\left(\theta_{1}\right)=\left\{\begin{array}{ll}
0 & \text { if } \theta_{1}<3 / 5, \\
\frac{5}{12}\left(\theta_{1}\right)^{2}+\frac{1}{4} & \text { if } \theta_{1} \geq 3 / 5,
\end{array} \quad \beta_{2}\left(\theta_{2}\right)= \begin{cases}0 & \text { if } \theta_{2}<5 / 9 \\
\frac{1}{3} & \text { if } 5 / 9 \leq \theta_{2} \leq 2 / 3 \\
\frac{3}{5}\left(\theta_{2}\right)^{2}+\frac{1}{15} & \text { if } \theta_{2}>2 / 3\end{cases}\right.
$$

Now, if $\theta_{1} \in[3 / 5,16 / 25]$ and $\theta_{2} \in(7 / 10, \sqrt{5} / 3)$, then $q_{2}(\boldsymbol{\theta})=1$ but $\beta_{2}\left(\theta_{2}\right)<2 / 5 \leq \beta_{1}\left(\theta_{1}\right)$. That is, with positive probability a type profile realizes such that agent 2 is the winner although agent 1 's effort is higher than agent 2's. Consequently, the expected highest effort exceeds the highest possible expected winner's effort. So, also when she can withhold the prize, the organizer strictly benefits from differential treatment more often when her objective is the highest effort than when it is the winner's effort.

## 7. Conclusion

A natural objective for the organizer of a contest is to maximize the expected winner's effort. We find that for a broad class of incomplete information environments, the organizer benefits from appropriately designed differential treatment of contestants even if they are identical from the ex ante perspective. This result stands in contrast to the case where the organizer's goal is to maximize the total effort and symmetric treatment is always optimal. Optimality for maximizing the winner's effort may thus offer an alternative explanation for existing discrimination in contest-like situations. For the case of uniformly distributed valuations, we find that every contest where each participant is ex ante equally like to win maximizes the expected winner's effort. Accordingly, symmetric treatment and many forms of differential treatment can be optimal at the same time, which allows the organizer to also pursue secondary goals without affecting her primary goal. Specifically, optimal differential treatment can increase the contestants' welfare on average, but may make some of the contestants worse off. We also find that when the organizer can use differential treatment, the problem of maximizing the expected highest effort no longer coincides with that of maximizing the expected winner's effort. In particular, the maximum of the highest effort may be strictly greater. Applied to innovation contests, for example, this insight suggests that the flexibility to separate the winner of the prize from the contestant whose innovation is adopted can be of great value to the organizer.

## Declaration of competing interest

None.

## Data availability

No data was used for the research described in the article.

## Appendix A. Proofs

Proof of Lemma 1. First, we show that (2) and (3) are necessary for implementability. Suppose allocation and efforts ( $\mathbf{q}, \boldsymbol{\beta}$ ) are contest implementable: there is a contest $(\mathbf{s}, \tau)$ such that $\boldsymbol{\beta}$ is an equilibrium of $(\mathbf{s}, \tau)$ and $\mathbf{q}$ satisfies (1). Consider the following direct mechanism where each agent reports his type instead of choosing his effort: if types $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$ are reported, each agent $i$ receives the prize with probability $q_{i}(\boldsymbol{\theta})$ and must exert effort $\beta_{i}\left(\theta_{i}\right)$. By the revelation principle, this direct mechanism is Bayesian incentive compatible, that is,

$$
\begin{equation*}
U_{i}\left(\theta_{i}\right)=\theta_{i} Q_{i}\left(\theta_{i}\right)-\beta_{i}\left(\theta_{i}\right) \geq \theta_{i} Q_{i}\left(\theta_{i}^{\prime}\right)-\beta_{i}\left(\theta_{i}^{\prime}\right) \quad \text { for all } \theta_{i}, \theta_{i}^{\prime} \in[0, \bar{\theta}] \text { and } i \in N \tag{A.1}
\end{equation*}
$$

By standard arguments (see, e.g., Myerson, 1981), (A.1) holds if and only if

$$
\begin{equation*}
Q_{i}\left(\theta_{i}\right) \text { is nondecreasing in } \theta_{i} \text { and } U_{i}\left(\theta_{i}\right)=U_{i}(0)+\int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z \tag{A.2}
\end{equation*}
$$

for all $\theta_{i} \in[0, \bar{\theta}]$ and $i \in N$. Note that $U_{i}(0)=0$ because in any equilibrium of $(\mathbf{s}, \tau)$, agent $i$ with valuation $\theta_{i}=0$ invests zero effort and obtains payoff zero. The definition of $U_{i}$ and (A.2) then imply (3). Moreover, $\beta_{i}$ is nondecreasing because $Q_{i}$ is nondecreasing. We can thus define for each $i \in N$ the direct score function $\sigma_{i}(\cdot)=s_{i}\left(\beta_{i}(\cdot)\right)$. Then, (1) is equivalent to $q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta}))$, which implies (2).

Now, we show that (2) and (3) are sufficient for implementability. To do so, we will need the following auxiliary result.
Claim A.1. Suppose allocation and efforts ( $\mathbf{q}, \boldsymbol{\beta}$ ) satisfy (2) and (3) for some direct score functions $\boldsymbol{\sigma}$ and tie-breaking rule $\tau$. Then (q, $\boldsymbol{\beta}$ ) also satisfy (2) and (3) for some $\hat{\boldsymbol{\sigma}}$ and $\hat{\tau}$ such that for all $i \in N$ and $0 \leq x_{i} \leq y_{i} \leq 1$,

$$
\begin{align*}
\beta_{i}\left(x_{i}\right)=\beta_{i}\left(y_{i}\right) & \Longrightarrow \hat{\sigma}_{i}\left(x_{i}\right)=\hat{\sigma}_{i}\left(y_{i}\right),  \tag{A.3}\\
\beta_{i}\left(x_{i}-\right)<\beta_{i}\left(x_{i}+\right) & \Longrightarrow \hat{\sigma}_{i}\left(x_{i}-\right)<\hat{\sigma}_{i}\left(x_{i}+\right) \tag{A.4}
\end{align*}
$$

where $\beta_{i}\left(x_{i}-\right)=\lim _{\theta_{i} \uparrow x_{i}} \beta_{i}\left(\theta_{i}\right)$ and $\beta_{i}\left(x_{i}+\right)=\lim _{\theta_{i} \downarrow x_{i}} \beta_{i}\left(\theta_{i}\right)$ denote the one-sided limits.
Proof. Consider any $i \in N$ and $x_{i}<y_{i}$ such that $\beta_{i}\left(x_{i}\right)=\beta_{i}\left(y_{i}\right)$ but $\sigma_{i}\left(x_{i}\right)<\sigma_{i}\left(y_{i}\right)$. By (3), $\beta_{i}\left(y_{i}\right)-\beta_{i}\left(x_{i}\right)=x_{i}\left[Q_{i}\left(y_{i}\right)-\right.$ $\left.Q_{i}\left(x_{i}\right)\right]+\int_{x_{i}}^{y_{i}}\left[Q_{i}\left(y_{i}\right)-Q_{i}(z)\right] \mathrm{d} z$. Hence, $\beta_{i}\left(y_{i}\right)=\beta_{i}\left(x_{i}\right)$ if and only if $Q_{i}\left(y_{i}\right)=Q_{i}\left(x_{i}\right)$. By (2), $q_{i}\left(\theta_{i}, \boldsymbol{\theta}_{-i}\right)$ is nondecreasing in $\theta_{i}$, so that $Q_{i}\left(y_{i}\right)=Q_{i}\left(x_{i}\right)$ implies $q_{i}\left(x_{i}, \boldsymbol{\theta}_{-i}\right)=q_{i}\left(y_{i}, \boldsymbol{\theta}_{-i}\right)$ for almost all $\boldsymbol{\theta}_{-i}$. Note that this means $p_{i}^{\tau}\left(\sigma_{i}\left(\theta_{i}\right), \boldsymbol{\sigma}_{-i}\left(\boldsymbol{\theta}_{-i}\right)\right)=$ $p_{i}^{\tau}\left(\sigma_{i}\left(x_{i}\right), \boldsymbol{\sigma}_{-i}\left(\boldsymbol{\theta}_{-i}\right)\right)$ for all $\theta_{i} \in\left[x_{i}, y_{i}\right]$, that is, the scores in $\left[\sigma_{i}\left(x_{i}\right), \sigma_{i}\left(y_{i}\right)\right]$ all result in the same winning probability for agent $i$. Define $\tilde{\boldsymbol{\sigma}}$ such that $\tilde{\sigma}_{i}\left(\theta_{i}\right)=\sigma_{i}\left(x_{i}\right)$ for all $\theta_{i} \in\left[x_{i}, y_{i}\right]$ whereas $\tilde{\boldsymbol{\sigma}}$ coincides with $\boldsymbol{\sigma}$ otherwise. Clearly, ( $\left.\mathbf{q}, \boldsymbol{\beta}\right)$ also satisfies (2) and (3) for $\tilde{\boldsymbol{\sigma}}$ and $\tau$. Repeating this procedure for any $i \in N$ and $x_{i}<y_{i}$ such that $\beta_{i}\left(x_{i}\right)=\beta_{i}\left(y_{i}\right)$ but $\tilde{\sigma}_{i}\left(x_{i}\right)<$ $\tilde{\sigma}_{i}\left(y_{i}\right)$, we eventually obtain $\hat{\boldsymbol{\sigma}}$ that satisfies (A.3).

Consider any $i \in N$ and $x_{i} \in(0,1)$ such that $\beta_{i}\left(x_{i}-\right)<\beta_{i}\left(x_{i}+\right)$ but $\sigma_{i}\left(x_{i}-\right)=\sigma_{i}\left(x_{i}+\right)$. Fix $\varepsilon>0$. We construct new direct score functions $\tilde{\boldsymbol{\sigma}}$ and a tie-breaking rule $\tilde{\tau}$ as follows. For every $j \in N, \tilde{\sigma}_{j}\left(\theta_{j}\right)=\sigma_{j}\left(\theta_{j}\right)$ if $\sigma_{j}\left(\theta_{j}\right)<\sigma_{i}\left(x_{i}\right)$ and $\tilde{\sigma}_{j}\left(\theta_{j}\right)=$ $\sigma_{j}\left(\theta_{j}\right)+\varepsilon$ if $\sigma_{j}\left(\theta_{j}\right) \geq \sigma_{i}\left(x_{i}\right)$. Moreover, $\tilde{\tau}(\cdot, r)=\tau(\cdot, r)$ if $r<\sigma_{i}\left(x_{i}\right)$ and $\tilde{\tau}(\cdot, r)=\tau(\cdot, r-\varepsilon)$ if $r \geq \sigma_{i}\left(x_{i}\right)+\varepsilon$. Then $\tilde{\sigma}_{i}\left(x_{i}-\right)<$ $\tilde{\sigma}_{i}\left(x_{i}+\right)$ and $p_{j}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta}))=p_{j}^{\tilde{\tau}}(\tilde{\boldsymbol{\sigma}}(\boldsymbol{\theta}))$ for all $j \in N$ and $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}$, so that $(\mathbf{q}, \boldsymbol{\beta})$ also satisfies (2) and (3) for $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\tau}$. Repeating this procedure for any $i$ and $x_{i}$ such that $\beta_{i}\left(x_{i}-\right)<\beta_{i}\left(x_{i}+\right)$ but $\sigma_{i}\left(x_{i}-\right)=\sigma_{i}\left(x_{i}+\right)$, we eventually obtain $\hat{\boldsymbol{\sigma}}$ that satisfies (A.4) and the corresponding $\hat{\tau}$.

Suppose allocation and efforts $(\mathbf{q}, \boldsymbol{\beta})$ satisfy (2) and (3) for some direct score functions $\boldsymbol{\sigma}$ and tie-breaking rule $\tau$. By Claim A.1, there exist direct score functions $\hat{\boldsymbol{\sigma}}$ and a tie-breaking rule $\hat{\tau}$ that also satisfies (2) and (3) and, in addition, (A.3) and (A.4). Conditions (2) and (3) imply (A.2) and, thus, that the direct mechanism corresponding to (q, $\boldsymbol{\beta}$ ) is Bayesian incentive compatible. Define the score functions $\mathbf{s}$ such that

$$
s_{i}\left(b_{i}\right)=\sup \left\{\hat{\sigma}_{i}\left(\theta_{i}\right) \mid \theta_{i} \in[0, \bar{\theta}] \text { and } \beta_{i}\left(\theta_{i}\right) \leq b_{i}\right\} \quad \text { for each } i \in N
$$

As $\beta_{i}(0)=0, s_{i}\left(b_{i}\right)$ is defined for all $b_{i} \geq 0$. To complete the proof, we show that the contest $(\mathbf{s}, \hat{\tau})$ implements $(\mathbf{q}, \boldsymbol{\beta})$. Suppose agent $i$ invests effort $b$. If $b=\beta_{i}\left(\theta_{i}\right)$ for some $\theta_{i}$, then by (A.3) we have $s_{i}(b)=\hat{\sigma}_{i}\left(\theta_{i}\right)$, that is, agent $i$ obtains the same score as when reporting $\theta_{i}$ in the direct mechanism. If $b$ is not in the image of $\beta_{i}$, then $b$ is dominated for $i$ by some lower effort: Efforts $b>\beta_{i}(\bar{\theta})$ yield score $\hat{\sigma}_{i}(\bar{\theta})$ and are dominated by effort $\beta_{i}(\bar{\theta})$. Efforts $b \in\left(\beta_{i}\left(x_{i}-\right), \beta_{i}\left(x_{i}+\right)\right)$ for some $x_{i}$ yield score $\hat{\sigma}_{i}\left(x_{i}-\right)$ and are dominated by effort $\beta_{i}\left(x_{i}-\right)$ (and because of (A.4), effort $b=\beta_{i}\left(x_{i}+\right)>\beta_{i}\left(x_{i}-\right)$ is not dominated as it yields score $\hat{\sigma}_{i}\left(x_{i}+\right)>\hat{\sigma}_{i}\left(x_{i}-\right)$ ). Consequently, each agent $i$ investing effort $\beta_{i}\left(\theta_{i}\right)$ if his type is $\theta_{i}$ is a pure-strategy Bayesian Nash equilibrium of contest ( $\mathbf{s}, \hat{\tau}$ ): for each $i$, efforts outside the image of $\beta_{i}$ are dominated and choosing an effort within the image of $\beta_{i}$ is equivalent to reporting a type in the direct mechanism, where truthful reporting is a pure-strategy Bayesian Nash equilibrium as it is Bayesian incentive compatible.

Proof of Lemma 2. Consider a contest implementable symmetric allocation $\mathbf{q}$, and recall this means that for almost all type profiles $\boldsymbol{\theta} \in[0, \bar{\theta}]^{n}, Q_{i}\left(\theta_{i}\right)=Q\left(\theta_{i}\right)$ for all $i \in N$ for some function $Q$. We will show that the lemma holds for any such type profile. The proof is by contradiction. So suppose there is such a type profile $\hat{\boldsymbol{\theta}}$ such that $q_{1}(\hat{\boldsymbol{\theta}})=K>0$ but $\beta_{1}\left(\hat{\theta}_{1}\right)<\beta_{2}\left(\hat{\theta}_{2}\right)$ (where the focus on agents 1 and 2 is without loss of generality). As $\mathbf{q}$ is symmetric, $\beta_{i}\left(\theta_{i}\right)=\beta\left(\theta_{i}\right)$ for almost all $\theta_{i}$ and all $i \in N$. Hence, $\beta_{1}\left(\hat{\theta}_{1}\right)<\beta_{2}\left(\hat{\theta}_{2}\right)$ implies $\hat{\theta}_{1}<\hat{\theta}_{2}$ and $Q\left(\hat{\theta}_{1}\right)<Q\left(\hat{\theta}_{2}\right)\left(\right.$ since $\beta\left(\hat{\theta}_{1}\right)=\beta\left(\hat{\theta}_{2}\right)$ if $Q\left(\hat{\theta}_{1}\right)=Q\left(\hat{\theta}_{2}\right)$ ). As $\mathbf{q}$ is contest implementable, it satisfies (2) for some direct score functions $\boldsymbol{\sigma}$. As $K>0$ and $Q\left(\hat{\theta}_{1}\right)<Q\left(\hat{\theta}_{2}\right)$, we have $\sigma_{2}\left(\hat{\theta}_{2}\right) \leq \sigma_{1}\left(\hat{\theta}_{1}\right)<\sigma_{1}\left(\hat{\theta}_{2}\right)$, and therefore $q_{1}\left(\hat{\theta}_{2}, \hat{\theta}_{2}, \hat{\boldsymbol{\theta}}_{-1,2}\right)=1$ (if $K<1$ because of a tie at $\hat{\boldsymbol{\theta}}$, there can be no longer any tie when agent 1 's score increases to $\sigma_{1}\left(\hat{\theta}_{2}\right)$ ). Now, as $q_{1}(\boldsymbol{\theta})$ is nondecreasing in $\theta_{1}$ and nonincreasing in $\theta_{j}$ for $j>1$, we have for all $\theta_{j} \leq \hat{\theta}_{j}$ with $j>1$,

$$
q_{1}(\boldsymbol{\theta})\left\{\begin{array} { l l } 
{ \geq K } & { \text { for } \theta _ { 1 } \in ( \hat { \theta } _ { 1 } , \hat { \theta } _ { 2 } ) , } \\
{ = 1 } & { \text { for } \theta _ { 1 } \geq \hat { \theta } _ { 2 } }
\end{array} \quad \text { and thus } \quad q _ { 2 } ( \boldsymbol { \theta } ) \left\{\begin{array}{ll}
\leq 1-K & \text { for } \theta_{1} \in\left(\hat{\theta}_{1}, \hat{\theta}_{2}\right) \\
=0 & \text { for } \theta_{1} \geq \hat{\theta}_{2}
\end{array}\right.\right.
$$

Consequently, letting $P=\mathbb{P}_{\boldsymbol{\theta}_{-1,2}}\left[\sigma_{j}\left(\theta_{j}\right) \leq \sigma_{2}\left(\hat{\theta}_{2}\right)\right.$ for all $\left.j \notin\{1,2\}\right]$,

$$
Q_{1}\left(\hat{\theta}_{2}\right) \geq F\left(\hat{\theta}_{2}\right) P \quad \text { and } \quad Q_{2}\left(\hat{\theta}_{2}\right) \leq\left[(1-K)\left[F\left(\hat{\theta}_{2}\right)-F\left(\hat{\theta}_{1}\right)\right]+F\left(\hat{\theta}_{1}\right)\right] P
$$

which contradicts $Q_{1}\left(\hat{\theta}_{2}\right)=Q_{2}\left(\hat{\theta}_{2}\right)=Q\left(\hat{\theta}_{2}\right)$.
Proof of Proposition 1. Note that for all nondecreasing $Q$

$$
\begin{aligned}
\int_{0}^{\bar{\theta}} Q(\theta)[\bar{\psi}(\theta)-\psi(\theta)] f(\theta) \mathrm{d} \theta & =\int_{0}^{\bar{\theta}} Q(\theta)\left[\bar{G}^{\prime}(F(\theta))-G^{\prime}(F(\theta))\right] f(\theta) \mathrm{d} \theta \\
& =\int_{0}^{\bar{\theta}}[G(F(\theta))-\bar{G}(F(\theta))] \mathrm{d} Q(\theta) \geq 0
\end{aligned}
$$

where the second line uses integration by parts and that $\bar{G}(0)=G(0), \bar{G}(1)=G(1)$, and $\bar{G}(z) \leq G(z)$ for $z \in(0,1)$ because $\bar{G}=\operatorname{conv} G$. Moreover, the inequality holds with equality if $Q$ is constant on any interval $[x, y]$ where $G(F(\theta))>\bar{G}(F(\theta))$ for all $\theta \in[x, y]$. Hence,

$$
\begin{equation*}
W(\mathbf{q})=\int_{0}^{\bar{\theta}} n Q(\theta) \psi(\theta) f(\theta) \mathrm{d} \theta \leq \int_{0}^{\bar{\theta}} n Q(\theta) \bar{\psi}(\theta) f(\theta) \mathrm{d} \theta=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \bar{\psi}\left(\theta_{i}\right)\right] \tag{A.5}
\end{equation*}
$$

with equality if $Q$ is constant on any interval of $\theta$ where $G(F(\theta))>\bar{G}(F(\theta))$.
Consider the allocation $\mathbf{q}$ given in (8). Clearly, $\mathbf{q}$ maximizes $\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in N} q_{i}(\boldsymbol{\theta}) \bar{\psi}\left(\theta_{i}\right)\right]$. Moreover, because $\bar{G}=\operatorname{conv} G, \bar{\psi}$ is constant on any interval of $\theta$ where $G(F(\theta))>\bar{G}(F(\theta))$. Hence, also $Q$ is constant on any such interval, implying that (A.5) holds with equality. Thus $\mathbf{q}$ also maximizes $W(\mathbf{q})$.

Proof of Proposition 2. The winner's effort under contest implementable allocation $\mathbf{q}^{\boldsymbol{\sigma}, \tau}$ can be written as $W\left(\mathbf{q}^{\boldsymbol{\sigma}, \tau}\right)=$ $\sum_{i=1}^{n} \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right]$ where the contribution of type $\theta_{i}$ of agent $i$ is

$$
B_{i}\left(\theta_{i}\right)=Q_{i}\left(\theta_{i}\right) \beta_{i}\left(\theta_{i}\right)=Q_{i}\left(\theta_{i}\right)^{2} \theta_{i}-Q_{i}\left(\theta_{i}\right) \int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z
$$

and $Q_{i}\left(\theta_{i}\right)=\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[q_{i}^{\boldsymbol{\sigma}, \tau}\left(\theta_{i}, \boldsymbol{\theta}_{-i}\right)\right]$ (i.e., we drop the superscript $\boldsymbol{\sigma}, \tau$ to simplify the notation).
First note that if ties happen only for a set of type profiles $\boldsymbol{\theta}$ with measure zero (e.g., if each $\sigma_{i}$ is strictly increasing), $W\left(\mathbf{q}^{\boldsymbol{\sigma}, \tau}\right)$ is constant across all tie-breaking rules $\tau$. In this case, part (i) clearly holds.

Now, suppose there is a score $r \geq 0$ and at least two agents $i$ such that $\mathbb{P}\left[\sigma_{i}\left(\theta_{i}\right)=r\right]>0$. This means that for some $0 \leq x_{i}<y_{i} \leq \bar{\theta}, \sigma_{i}\left(\theta_{i}\right)<r$ if $\theta_{i}<x_{i}, \sigma_{i}\left(\theta_{i}\right)=r$ if $\theta_{i} \in\left(x_{i}, y_{i}\right)$, and $\sigma_{i}\left(\theta_{i}\right)>r$ if $\theta_{i}>y_{i}$. Accordingly, $Q_{i}\left(\theta_{i}\right)=K_{i}$ for all $\theta_{i} \in\left(x_{i}, y_{i}\right)$ for some constant $K_{i}$ that depends on tie-breaking at $r$, that is, $\tau(\cdot, r)$. Type $\theta_{i}$ 's contribution to $W\left(\mathbf{q}^{\sigma, \tau}\right)$ can then be written as

$$
B_{i}\left(\theta_{i}\right)= \begin{cases}Q_{i}\left(\theta_{i}\right)^{2} \theta_{i}-Q_{i}\left(\theta_{i}\right) \int_{0}^{\theta_{i}} Q_{i}(z) \mathrm{d} z & \text { for } \theta_{i}<x_{i} \\ K_{i}^{2} x_{i}-K_{i} \int_{0}^{x_{i}} Q_{i}(z) \mathrm{d} z & \text { for } \theta_{i} \in\left(x_{i}, y_{i}\right) \\ Q_{i}\left(\theta_{i}\right)^{2} \theta_{i}-Q_{i}\left(\theta_{i}\right)\left(\int_{0}^{x_{i}} Q_{i}(z) \mathrm{d} z+K_{i}\left(y_{i}-x_{i}\right)+\int_{y_{i}}^{\theta_{i}} Q_{i}(z) \mathrm{d} z\right) & \text { for } \theta_{i}>y_{i}\end{cases}
$$

For agent $i$ 's expected contribution follows

$$
\mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right]=\omega_{i}+\int_{x_{i}}^{y_{i}}\left(K_{i}^{2} x_{i}-K_{i} \int_{0}^{x_{i}} Q_{i}(z) \mathrm{d} z\right) \mathrm{d} F\left(\theta_{i}\right)-\int_{y_{i}}^{\bar{\theta}} Q_{i}\left(\theta_{i}\right) K_{i}\left(y_{i}-x_{i}\right) \mathrm{d} F\left(\theta_{i}\right)
$$

where $\omega_{i}$ represents terms that are independent of $K_{i}$. Note that

$$
\begin{aligned}
\frac{\partial \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right]}{\partial K_{i}} & =\int_{x_{i}}^{y_{i}}\left(2 K_{i} x_{i}-\int_{0}^{x_{i}} Q_{i}(z) \mathrm{d} z\right) \mathrm{d} F\left(\theta_{i}\right)-\int_{y_{i}}^{\bar{\theta}} Q_{i}\left(\theta_{i}\right)\left(y_{i}-x_{i}\right) \mathrm{d} F\left(\theta_{i}\right) \\
\frac{\partial^{2} \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right]}{\partial K_{i}^{2}} & =\int_{x_{i}}^{y_{i}} 2 x_{i} \mathrm{~d} F\left(\theta_{i}\right)
\end{aligned}
$$

Let $M$ denote the set of agents that with positive probability tie at score $r$. Since $\partial^{2} \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right] / \partial K_{i}^{2} \geq 0$ for $i \in M$, $W\left(\mathbf{q}^{\boldsymbol{\sigma}, \tau}\right)=\sum_{i=1}^{n} \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right]$ is a convex function of $\left(K_{i}\right)_{i \in M}$.

Recall that a tie-breaking rule $\tau$ lexicographically breaks the tie according to $\pi$ that is drawn from $\tau(\cdot, r)$. Hence, each $K_{i}$ is a convex combination of the winning probabilities under all possible lexicographical tie-breaking rules:

$$
K_{i}=\sum_{\pi \in \Pi} \tau(\pi, r) \prod_{\{j: \pi(j)<\pi(i)\}} F\left(y_{j}\right) \prod_{\{j: \pi(j)>\pi(i)\}} F\left(x_{j}\right)
$$

(where $x_{j}=y_{j} \in[0, \bar{\theta}]$ for $j \notin M$ ). Accordingly, choosing the tie-breaking rule at $r$ corresponds to choosing $\left(K_{i}\right)_{i \in M}$ from a convex set. Hence, the convex function $W\left(\mathbf{q}^{\sigma, \tau}\right)$ is maximized at an extreme point of this convex set, that is, at a $\left(K_{i}\right)_{i \in M}$ that corresponds to deterministic tie-breaking at $r$ (i.e., $\tau(\pi, r)=1$ for one $\pi$ ). As this argument holds for any score $r$ at which ties happen with positive probability, $W\left(\mathbf{q}^{\sigma, \tau}\right)$ is maximized at a deterministic tie-breaking rule $\tau$. This completes the proof of part (i).

In part (ii), there is a positive probability that the winner is determined by breaking a tie at score $r$ among agents $i \in M$ and, in addition, $\mathbb{P}\left[\sigma_{i}\left(\theta_{i}\right)<r\right]>0$ for all $i \in M$. The latter implies $x_{i}>0$ and thus $\partial^{2} \mathbb{E}_{\theta_{i}}\left[B_{i}\left(\theta_{i}\right)\right] / \partial K_{i}^{2}>0$ for $i \in M$. Hence, $W\left(\mathbf{q}^{\boldsymbol{\sigma}, \tau}\right)$ is a strictly convex function of $\left(K_{i}\right)_{i \in M}$ and is strictly maximized at an extreme point of the set of possible $\left(K_{i}\right)_{i \in M}$. Any tie-breaking rule $\tau$ such that $\hat{p}_{i}^{\tau}(M, r) \in(0,1)$ for $i \in M$ does not correspond to this extreme point and results in a strictly lower $W\left(\mathbf{q}^{\sigma, \tau}\right)$.

Proof of Proposition 3. Consider the allocation $\mathbf{q}^{x}$ for some $x \in\left(\theta^{T}, \bar{\theta}\right]$. With $Q_{i}^{x}(\theta)=Q^{T}(\theta)$ if $\theta \leq x$ and $Q_{i}^{x}(\theta)=F(x)^{i-1}$ if $\theta>x$, the sum of the agents' contributions to the winner's effort for a common type $\theta$ is

$$
\sum_{i \in N} Q_{i}^{x}(\theta) \beta_{i}(\theta)= \begin{cases}n Q^{T}(\theta)^{2} \theta-n Q^{T}(\theta) \int_{0}^{\theta} Q^{T}(z) \mathrm{d} z & \text { if } \theta \leq x \\ x \sum_{i=1}^{n} F(x)^{2 i-2}-\left(\sum_{i=1}^{n} F(x)^{i-1}\right) \int_{0}^{x} Q^{T}(z) \mathrm{d} z & \text { if } \theta>x\end{cases}
$$

By the formula for the sum of the first $n$ terms of a geometric series,

$$
\sum_{i=1}^{n} F(x)^{i-1}=\frac{1-F(x)^{n}}{1-F(x)} \quad \text { and } \quad \sum_{i=1}^{n} F(x)^{2 i-2}=\frac{1-F(x)^{2 n}}{1-F(x)^{2}}
$$

Accordingly, the expected winner's effort $W\left(\mathbf{q}^{x}\right)$ as a function of $x$ amounts to

$$
\begin{aligned}
\hat{W}(x)=\mathbb{E}_{\theta}\left[\sum_{i \in N} Q_{i}^{x}(\theta) \beta_{i}(\theta)\right]= & \int_{0}^{x}\left(n Q^{T}(\theta)^{2} \theta-n Q^{T}(\theta) \int_{0}^{\theta} Q^{T}(z) \mathrm{d} z\right) \mathrm{d} F(\theta) \\
& +[1-F(x)]\left(x \frac{1-F(x)^{2 n}}{1-F(x)^{2}}-\frac{1-F(x)^{n}}{1-F(x)} \int_{0}^{x} Q^{T}(z) \mathrm{d} z\right)
\end{aligned}
$$

Taking the first derivative with respect to $x$, we obtain

$$
\begin{aligned}
\hat{W}^{\prime}(x)= & n Q^{T}(x)^{2} x f(x)+n\left[F(x)^{n-1}-Q^{T}(x)\right] f(x) \int_{0}^{x} Q^{T}(z) \mathrm{d} z \\
& +\frac{1-F(x)^{2 n}}{1+F(x)}-x f(x) \frac{2 n F(x)^{2 n-1}+(2 n-1) F(x)^{2 n}+1}{[1+F(x)]^{2}}-\left[1-F(x)^{n}\right] Q^{T}(x)
\end{aligned}
$$

Using $Q^{T}(x)=F(x)^{n-1}$ (since $x>\theta^{T}$ ), this simplifies to

$$
\hat{W}^{\prime}(x)=-\frac{x f(x)}{[1+F(x)]^{2}} v(x)+\frac{1-F(x)^{2 n}}{1+F(x)}-\left[1-F(x)^{n}\right] F(x)^{n-1}
$$

where $v(x)=1-n F(x)^{2 n-2}+(n-1) F(x)^{2 n}$. Note that $v(x)>0$ for all $x<\bar{\theta}$ since $v(\bar{\theta})=0$ and $v^{\prime}(x)=-2 n(n-$ 1) $F(x)^{2 n-3}\left[1-F(x)^{2}\right] f(x)<0$. Hence,

$$
\hat{W}^{\prime}(x)<0 \Longleftrightarrow x f(x)>\frac{\mu(x)}{v(x)}
$$

where $\mu(x)=1+F(x)-F(x)^{n-1}-2 F(x)^{n}-F(x)^{n+1}+F(x)^{2 n-1}+F(x)^{2 n}$. It is straightforward to verify that $\nu(\bar{\theta})=v^{\prime}(\bar{\theta})=$ $\mu(\bar{\theta})=\mu^{\prime}(\bar{\theta})=0$. Applying l'Hospital's rule twice yields

$$
\begin{aligned}
\lim _{x \rightarrow \bar{\theta}} \frac{\mu(x)}{\nu(x)} & =\lim _{x \rightarrow \bar{\theta}} \frac{1-(n-1) F(x)^{n-2}-2 n F(x)^{n-1}-(n+1) F(x)^{n}+(2 n-1) F(x)^{2 n-2}+2 n F(x)^{2 n-1}}{-2 n(n-1) F(x)^{2 n-3}+2 n(n-1) F(x)^{2 n-1}} \\
& =\lim _{x \rightarrow \bar{\theta}} \frac{\xi(x)}{-2 n(n-1)(2 n-3) F(x)^{2 n-4}+2 n(n-1)(2 n-1) F(x)^{2 n-2}} \\
& =1
\end{aligned}
$$

where $\underline{\xi}(x)=-(n-1)(n-2) F(x)^{n-3}-2 n(n-1) F(x)^{n-2}-(n+1) n F(x)^{n-1}+(2 n-1)(2 n-2) F(x)^{2 n-3}+2 n(2 n-1) F(x)^{2 n-2}$ and $\xi(\bar{\theta})=4 n(n-1)$. Since $f(\bar{\theta})>1 / \bar{\theta}$ and because of the continuity of $f(x) x$ and $\mu(x) / \nu(x)$, there is thus an $\hat{x}<\bar{\theta}$ such that $\hat{W}^{\prime}(x)<0$ for all $x \in(\hat{x}, \bar{\theta})$. Consequently, for $x \in[\hat{x}, \bar{\theta}), W\left(\mathbf{q}^{x}\right)=\hat{W}(x)>\hat{W}(\bar{\theta})=W\left(\mathbf{q}^{T}\right)$.

Proof of Theorem 1. In the main text.
Proof of Corollary 1. In the main text.

Proof of Lemma 3. Consider a contest implementable deterministic allocation $\mathbf{q}$. By Lemma 1, there are direct score functions $\boldsymbol{\sigma}$ and a tie-breaking rule $\tau$ such that $q_{i}(\boldsymbol{\theta})=p_{i}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta}))$ for all $\boldsymbol{\theta}$ and $i$. As the allocation is deterministic, there is a deterministic $\tau$ with this property, and this $\tau$ is thus lexicographic at each score. Note that $\tau$ affects $\mathbf{q}$ on a set of type profiles with positive measure only if the are two agents $i$ and $j$ such that $\mathbb{P}_{\theta_{i}, \theta_{j}}\left[\sigma_{i}\left(\theta_{i}\right)=\sigma_{j}\left(\theta_{j}\right)\right]>0$, that is, there is a positive probability for a tie between $i$ and $j$. We will now show that if ties occur with positive probabilities under $\sigma$, then the same allocation can also be represented with modified direct score functions $\hat{\boldsymbol{\sigma}}$ under which ties occur with probability zero.

Suppose the agents in $M \subseteq N$ with $|M| \geq 2$ tie with positive probability at score $r$, and $\tau(\pi, r)=1$ for permutation $\pi$, that is, $i$ beats $j$ in a tie at $r$ if $\pi(i)<\pi(j)$. As score $r$ has positive probability for these agents, there are $x_{i}<y_{i}$ for each $i \in M$ such that $\sigma_{i}\left(\theta_{i}\right)=r$ for $\theta_{i} \in\left(x_{i}, y_{i}\right)$ and $\sigma_{i}\left(\theta_{i}\right) \neq r$ for $\theta_{i} \notin\left[x_{i}, y_{i}\right]$. Let $\varepsilon>0$ and consider the modified direct score functions $\hat{\boldsymbol{\sigma}}$ where

$$
\hat{\sigma}_{i}\left(\theta_{i}\right)= \begin{cases}\sigma_{i}\left(\theta_{i}\right) & \text { if } \theta_{i}<x_{i}, \\
\sigma_{i}\left(\theta_{i}\right)+(n-\pi(i)) \varepsilon & \text { if } \theta_{i} \in\left[x_{i}, y_{i}\right], \quad \text { and } \quad \hat{\sigma}_{j}\left(\theta_{j}\right)=\left\{\begin{array}{ll}
\sigma_{j}\left(\theta_{j}\right) & \text { if } \sigma_{j}\left(\theta_{j}\right)<r \\
\sigma_{i}\left(\theta_{i}\right)+n \varepsilon & \text { if } \theta_{i}>y_{i}
\end{array} \sigma_{j}\left(\theta_{j}\right)+n \varepsilon\right. \\
\text { if } \sigma_{j}\left(\theta_{j}\right) \geq r\end{cases}
$$

for all $i \in M$ and all $j \notin M$. Let the tie-breaking rule $\hat{\tau}$ be such that $\hat{\tau}(\cdot, s)=\tau(\cdot, s)$ for $s<r, \hat{\tau}(\cdot, s)=\tau(\cdot, r)$ for $s \in[r, r+n \varepsilon]$, and $\hat{\tau}(\cdot, s)=\tau(\cdot, s-n \varepsilon)$ for $s>r+n \varepsilon$. Observe that $p_{i}^{\tau}(\boldsymbol{\sigma}(\boldsymbol{\theta}))=p_{i}^{\hat{\tau}}(\hat{\boldsymbol{\sigma}}(\boldsymbol{\theta}))$ for almost all $\boldsymbol{\theta}$, that is, the allocation represented by $\hat{\boldsymbol{\sigma}}$ and $\hat{\tau}$ differs from $\mathbf{q}$ on a set of type profiles of measure zero. Moreover, the probability of a tie between agents with types $\theta_{i} \in\left[x_{i}, y_{i}\right]$ is zero under $\hat{\boldsymbol{\sigma}}$.

Iteratively modifying the direct score functions as in the preceding paragraph at every score $r$ where ties occur with positive probability, we obtain direct score functions $\hat{\boldsymbol{\sigma}}$ that represent an allocation almost identical to $\mathbf{q}$ and that result in ties occurring with probability zero. Accordingly, the interim winning probabilities $Q_{1}, \ldots, Q_{n}$ can be represented with $\hat{\boldsymbol{\sigma}}$ and are independent of the tie-breaking rule almost everywhere: for each $i \in N, Q_{i}\left(\theta_{i}\right)=\mathbb{P}_{\boldsymbol{\theta}_{-i}}\left[\max _{j \neq i} \hat{\sigma}_{j}\left(\theta_{j}\right)<\hat{\sigma}_{i}\left(\theta_{i}\right)\right]=$ $\prod_{j \neq i} F\left(\hat{\sigma}_{j}^{-1}\left(\hat{\sigma}_{i}\left(\theta_{i}\right)\right)\right)$ for almost all $\theta_{i}$, where $\hat{\sigma}_{j}^{-1}$ is a generalized inverse of $\hat{\sigma}_{j}$.

Proof of Lemma 4. We first establish a technical result that we will use to prove Lemma 4. The following lemma states a generalization of Young's inequality. It is a higher dimensional analogue of a result by Mitroi and Niculescu (2011) on Young's inequality for nondecreasing functions. A related higher dimensional analogue is stated without proof in Mitroi and Niculescu (2011, Remark 2.6). Given a nondecreasing function $\phi_{i}$, let its generalized inverse be defined as $\phi_{i}^{-1}\left(s_{i}\right)=\inf \left\{t_{i}\right.$ : $\left.\phi_{i}\left(t_{i}\right) \geq s_{i}\right\}$.

Lemma A. 1 (Young's inequality). Let $\phi_{i}:[0, \infty) \rightarrow[0, \infty)$ for $i \in\{1, \ldots, n\}$ be nondecreasing functions such that $\phi_{i}(0)=0$, and let $K:[0, \infty)^{n} \rightarrow[0, \infty)$ be a locally absolutely continuous function that is strictly positive almost everywhere. Then, for any $t_{1}, \ldots, t_{n}>$ 0

$$
\begin{equation*}
\left.\int_{0}^{\phi_{1}\left(t_{1}\right)} \int_{0}^{\phi_{2}\left(t_{2}\right)} \cdots \int_{0}^{\phi_{n}\left(t_{n}\right)} K(\mathbf{s}) \mathrm{d} s_{n} \ldots \mathrm{~d} s_{2} \mathrm{~d} s_{1} \leq \sum_{i=1}^{n} \int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \int_{0}^{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i}\right) \mathrm{d} s_{i} \tag{A.6}
\end{equation*}
$$

where $\mathrm{d} \mathbf{s}_{-i}$ denotes $\mathrm{d} s_{n} \ldots \mathrm{~d} s_{i+1} \mathrm{~d} s_{i-1} \ldots \mathrm{~d} s_{1}$. Inequality (A.6) holds with equality if and only if $t_{1}=t_{i}$ for all $i>1$.
Proof. Consider first $n=2$. By Mitroi and Niculescu (2011, Theorem 2.3), we have for every nondecreasing function $g:[0, \infty) \rightarrow[0, \infty)$ such that $g(0)=0$ and $\lim _{x \rightarrow \infty} g(x)=\infty$ and any $t>0, c \geq 0$

$$
\begin{equation*}
\int_{0}^{t} \int_{0}^{c} K(x, y) \mathrm{d} y \mathrm{~d} x \leq \int_{0}^{t} \int_{0}^{g(x)} K(x, y) \mathrm{d} y \mathrm{~d} x+\int_{0}^{c} \int_{0}^{g^{-1}(y)} K(x, y) \mathrm{d} x \mathrm{~d} y \tag{A.7}
\end{equation*}
$$

with equality if and only if $c \in[g(t-), g(t+)]$. Given any nondecreasing functions $\phi_{1}, \phi_{2}$ and $t_{1}, t_{2}>0$, we can set $g(x)=$ $\phi_{2}\left(\phi_{1}^{-1}(x)\right), t=\phi_{1}\left(t_{1}\right)$, and $c=\phi_{2}\left(t_{2}\right)$. Hence, (A.7) yields (A.6) for $n=2$, with equality if and only if $t_{1}=t_{2}$.

We now prove inequality (A.6) for $n>2$ by induction. Suppose (A.6) holds for $\hat{n}=n-1$. Let $\hat{K}\left(s_{1}, \ldots, s_{n-1}\right)=$ $\int_{0}^{\phi_{n}\left(t_{n}\right)} K\left(s_{1}, \ldots, s_{n}\right) \mathrm{d} s_{n}$. Then (A.6) for $\hat{n}=n-1$ and $\hat{K}$ implies

$$
\begin{align*}
& \int_{0}^{\phi_{1}\left(t_{1}\right)} \int_{0}^{\phi_{2}\left(t_{2}\right)} \cdots \int_{0}^{\phi_{n}\left(t_{n}\right)} K(\mathbf{s}) \mathrm{d} s_{n} \ldots \mathrm{~d} s_{2} \mathrm{~d} s_{1} \\
& \left.\leq \sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}\left[\int_{0}^{\phi_{n}\left(t_{n}\right)} K(\mathbf{s}) \mathrm{d} s_{n}\right] \mathrm{d} s_{-i, n}\right) \mathrm{d} s_{i} \\
& =\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)}\left(\int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}\left[\int_{0}^{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} s_{n}\right] \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{i} \\
& \left.+\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}\left[\int_{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}^{\phi_{n}\left(t_{n}\right)} K(\mathbf{s}) \mathrm{d} s_{n}\right] \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{i} \\
& =\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)}\left(\int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i}\right) \mathrm{d} s_{i} \\
& +\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}^{\phi_{n}\left(t_{n}\right)}\left(\int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{n} \mathrm{~d} s_{i} . \tag{A.8}
\end{align*}
$$

Changing the order of integration in the summands on the last line gives

$$
\begin{aligned}
& \left.\int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}^{\phi_{n}\left(t_{n}\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{n} \mathrm{~d} s_{i} \\
& \left.=\int_{0}^{\phi_{n}\left(t_{n}\right)} \int_{0}^{\phi_{i}\left(\phi_{n}^{-1}\left(s_{n}\right)\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{i} \mathrm{~d} s_{n}
\end{aligned}
$$

and therefore the sum becomes

$$
\begin{aligned}
& \left.\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(t_{i}\right)} \int_{\phi_{n}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)}^{\phi_{n}\left(t_{n}\right)} \int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{n} \mathrm{~d} s_{i} \\
& =\int_{0}^{\phi_{n}\left(t_{n}\right)}\left[\sum_{i=1}^{n-1} \int_{0}^{\phi_{i}\left(\phi_{n}^{-1}\left(s_{n}\right)\right)}\left(\int_{0}^{\phi_{1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-i, n}\right) \mathrm{d} s_{i}\right] \mathrm{d} s_{n}
\end{aligned}
$$

Now, applying (A.6) for $\hat{n}=n-1$ with equality (as $t_{1}=\cdots=t_{n-1}=\phi_{n}^{-1}\left(s_{n}\right)$ ), we get

$$
[\cdots]=\int_{0}^{\phi_{1}\left(\phi_{n}^{-1}\left(s_{n}\right)\right)} \cdots \int_{0}^{\phi_{n-1}\left(\phi_{n}^{-1}\left(s_{n}\right)\right)} K(\mathbf{s}) \mathrm{d} \mathbf{s}_{-n}
$$

Consequently, (A.8) is equal to the right-hand side of (A.6) for $n$. We thus have shown that (A.6) holds for $n$ if it holds for $\hat{n}=n-1$.

We are now ready to prove Lemma 4. According to Lemma 3, for every contest implementable deterministic allocation $\mathbf{q}$, there are nondecreasing functions $\sigma_{1}, \ldots, \sigma_{n}$ such that $Q_{i}\left(\theta_{i}\right)=\prod_{j \neq i} F\left(\sigma_{j}^{-1}\left(\sigma_{i}\left(\theta_{i}\right)\right)\right.$ ) for all $i \in N$ and almost all $\theta_{i} \in[0, \bar{\theta}]$. Moreover, we can assume without loss of generality that $\sigma_{i}(0)=0$ and $\sigma_{i}(\bar{\theta})=\bar{\sigma}$ for some $\bar{\sigma}>0$ for all $i \in N$, as this affects $\mathbf{q}$ only on a set of type profiles with measure zero. Then, $W^{I}$ can be written as

$$
\begin{equation*}
W^{I}(\mathbf{q})=\sum_{i=1}^{n} \int_{0}^{\bar{\theta}} Q_{i}\left(\theta_{i}\right)^{2} \theta_{i} \frac{1}{\bar{\theta}} \mathrm{~d} \theta_{i}=\sum_{i=1}^{n} \int_{0}^{\bar{\theta}} \theta_{i}\left(\prod_{j \neq i} \frac{\sigma_{j}^{-1}\left(\sigma_{i}\left(\theta_{i}\right)\right)}{\bar{\theta}}\right)^{2} \frac{1}{\bar{\theta}} \mathrm{~d} \theta_{i} \tag{A.9}
\end{equation*}
$$

where $\sigma_{j}^{-1}\left(\sigma_{i}\left(\theta_{i}\right)\right) \in[0, \bar{\theta}]$ is well-defined since $\sigma_{i}(0)=0$ and $\sigma_{i}(\bar{\theta})=\bar{\sigma}$ for each $i \in N$. We now apply Young's inequality. For $K(\mathbf{s})=2^{n-1} \prod_{i=1}^{n} s_{i}$, Lemma A. 1 yields

$$
\begin{equation*}
\frac{1}{2} \prod_{i=1}^{n} \phi_{i}\left(t_{i}\right)^{2} \leq \sum_{i=1}^{n} \int_{0}^{\phi_{i}\left(t_{i}\right)} s_{i}\left(\prod_{j \neq i} \phi_{j}\left(\phi_{i}^{-1}\left(s_{i}\right)\right)^{2}\right) \mathrm{d} s_{i} \tag{A.10}
\end{equation*}
$$

with equality if $t_{1}=t_{i}$ for all $i>1$. Let $\phi_{i}=\sigma_{i}^{-1}$ and $t_{i}=\bar{\sigma}$ for each $i \in N$. Thus, (A.10) holds with equality and $\phi_{i}\left(t_{i}\right)=$ $\sigma_{i}^{-1}(\bar{\sigma})=\bar{\theta}$, resulting in

$$
\begin{equation*}
\frac{1}{2} \bar{\theta}^{2 n}=\sum_{i=1}^{n} \int_{0}^{\bar{\theta}} \theta_{i} \prod_{j \neq i} \sigma_{j}^{-1}\left(\sigma_{i}\left(\theta_{i}\right)\right)^{2} \mathrm{~d} \theta_{i} \tag{A.11}
\end{equation*}
$$

Using (A.11) in (A.9) yields $W^{I}(\mathbf{q})=\bar{\theta} / 2$.
Proof of Theorem 2. That a contest implementable deterministic allocation $\mathbf{q}$ maximizes $W(\mathbf{q})$ if and only if $\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]=$ $1 / n$ is shown in the main text. We are left to show that non-deterministic allocations $\mathbf{q}$ do not maximize $W(\mathbf{q})$. By contradiction, suppose the contest implementable non-deterministic allocation $\mathbf{q}^{\sigma, \tau}$ maximizes $W$, where we use the notation introduced in Proposition 2. As $\mathbf{q}^{\sigma, \tau}$ is non-deterministic, there must be a score $r \geq 0$ where with positive probability the set of agents $M \subseteq N$ with $|M| \geq 2$ tie at $r$ for the highest score and the tie-breaking rule $\tau$ is not deterministic at $r$. As in the proof of Proposition 2, for $i \in M$, let $x_{i}<y_{i}$ be such that $\sigma_{i}\left(\theta_{i}\right)=r$ if $\theta_{i} \in\left[x_{i}, y_{i}\right]$. Let $Q_{i}\left(\theta_{i}\right)=K_{i}$ for $\theta_{i} \in\left[x_{i}, y_{i}\right]$.

As shown in the proof of Proposition 2, $W\left(\mathbf{q}^{\sigma, \tau}\right)$ is a convex function of $\left(K_{i}\right)_{i \in M}$, and choosing the tie-breaking rule at $r$ is equivalent to choosing $\left(K_{i}\right)_{i \in M}$ from a convex set whose extreme points correspond to deterministic tie-breaking. Now, since $\tau$ is not deterministic at $r,\left(K_{i}\right)_{i \in M}$ corresponds to a convex combination of at least two extreme points. Since $\mathbf{q}^{\sigma, \tau}$ is optimal, these extreme points as well as all convex combinations of them maximize the expected winner's effort. Let $\left(K_{i}^{\prime}\right)_{i \in M}$ and $\left(K_{i}^{\prime \prime}\right)_{i \in M}$ be two such extreme points, and let $\tau^{\prime}$ and $\tau^{\prime \prime}$ denote corresponding deterministic tie-breaking rules. (If $\tau$ is not deterministic at scores other than $r$, let at all those scores $\tau^{\prime}$ and $\tau^{\prime \prime}$ be equal to each other, deterministic, and such that they generate the same $W$ as $\tau$, which is possible by Proposition 2.) As $\mathbf{q}^{\sigma, \tau^{\prime}}$ are $\mathbf{q}^{\sigma, \tau^{\prime \prime}}$ are optimal deterministic allocations, $\mathbb{E}\left[Q_{i}^{\sigma, \tau^{\prime}}\left(\theta_{i}\right)\right]=\mathbb{E}\left[Q_{i}^{\sigma, \tau^{\prime \prime}}\left(\theta_{i}\right)\right]=1 / n$ for all $i \in N$. Clearly, there is an agent $i \in M$ such that $K_{i}^{\prime}>K_{i}^{\prime \prime}$. But then $\mathbb{E}\left[Q_{i}^{\sigma, \tau^{\prime}}\left(\theta_{i}\right)\right]=\mathbb{E}\left[Q_{i}^{\sigma, \tau^{\prime \prime}}\left(\theta_{i}\right)\right]+\left[F\left(y_{i}\right)-F\left(x_{i}\right)\right]\left(K_{i}^{\prime}-K_{i}^{\prime \prime}\right)>\mathbb{E}\left[Q_{i}^{\sigma, \tau^{\prime \prime}}\left(\theta_{i}\right)\right]$, which is a contradiction.

Proof of Lemma 5. In the main text.
Proof of Theorem 1R. In the main text.
Proof of Theorem 2R. A reserve-contest implementable allocation $\mathbf{q}$ maximizes $W(\mathbf{q})$ if it is deterministic and maximizes (15). Let $\bar{Q}_{i}=\mathbb{E}\left[Q_{i}\left(\theta_{i}\right)\right]$ for each $i$, so that (15) divided by $\bar{\theta}$ is equal to

$$
\bar{W}\left(\bar{Q}_{1}, \ldots, \bar{Q}_{n}\right)=\frac{1}{2}-\frac{1}{2} \sum_{i=1}^{n}\left(\bar{Q}_{i}\right)^{2}-\frac{1}{2}\left(1-\sum_{i=1}^{n} \bar{Q}_{i}\right)^{2}
$$

Taking derivatives, we have for all $j \in N$,

$$
\frac{\partial \bar{W}}{\partial \bar{Q}_{j}}=1-\bar{Q}_{j}-\sum_{i=1}^{n} \bar{Q}_{i}, \quad \frac{\partial^{2} \bar{W}}{\partial \bar{Q}_{j}^{2}}=-2, \quad \frac{\partial^{2} \bar{W}}{\partial \bar{Q}_{j} \partial \bar{Q}_{k}}=-1 \text { for } j \neq k
$$

It is straightforward to verify that $\bar{W}$ is strictly concave (the Hessian is negative definite). Hence, $\bar{W}$ and thus (15) is maximized if and only if $\partial \bar{W} / \partial \bar{Q}_{j}=0$ for all $j$, which has the unique solution $\bar{Q}_{j}=1 /(n+1)$ for all $j$.

We are left to show that reserve-contest implementable non-deterministic allocations do not also maximize $W$. By contradiction, suppose the reserve-contest implementable non-deterministic allocation $\mathbf{q}$ maximizes $W$, and let $\breve{\mathbf{q}}$ be the corresponding contest implementable allocation with $n+1$ agents as defined in Lemma 5 . Let the direct score functions $\sigma$ and the tie-breaking rule $\tau$ be such that $\breve{q}_{i}(\cdot)=p_{i}^{\tau}(\sigma(\cdot))$ for all $i$, and let $\sigma^{*}=\sigma_{n+1}\left(\theta_{n+1}\right)$ for all $\theta_{n+1}$ denote the constant score of agent $n+1$. Because $\mathbf{q}$ is non-deterministic, there must be a score $r \geq \sigma^{*}$ where ties occur with positive probability and are randomly broken under $\tau$. Now, note that the prize is withheld with positive probability under $\mathbf{q}$ because nondeterministic contest implementable allocations are not optimal by Theorem 2. Hence, for the agents involved in a tie at $r$, $r$ cannot be their lowest possible score. But then, according to part (ii) of Proposition 2, replacing $\tau$ with a deterministic tie-breaking rule strictly increases $W$, which contradicts that q maximizes $W$.

## References

Barbieri, S., Serena, M., 2021. Winner's effort maximization in large contests. J. Math. Econ. 96, 102512.
Boudreau, K.J., Lakhani, K.R., Menietti, M., 2016. Performance responses to competition across skill levels in rank-order tournaments: field evidence and implications for tournament design. Rand J. Econ. 47, 140-165.
Chawla, S., Hartline, J.D., Sivan, B., 2019. Optimal crowdsourcing contests. Games Econ. Behav. 113, 80-96.
Deng, S., Fang, H., Fu, Q., Wu, Z., 2020. Confidence management in tournaments. Mimeo.
Drugov, M., Ryvkin, D., 2017. Biased contests for symmetric players. Games Econ. Behav. 103, 116-144.
Franke, J., Leininger, W., Wasser, C., 2018. Optimal favoritism in all-pay auctions and lottery contests. Eur. Econ. Rev. 104, 22-37.
$\mathrm{Fu}, \mathrm{Q} ., \mathrm{Wu}, \mathrm{Z} .$, 2022. Disclosure and favoritism in sequential elimination contests. Am. Econ. J. Microecon. 14, 78-121.
Gershkov, A., Moldovanu, B., Strack, P., Zhang, M., 2021. A theory of auctions with endogenous valuations. J. Polit. Econ. 129, 1011-1051.
Kawamura, K., Moreno de Barreda, I., 2014. Biasing selection contests with ex-ante identical agents. Econ. Lett. 123, 240-243.
Kirkegaard, R., 2012. Favoritism in asymmetric contests: head starts and handicaps. Games Econ. Behav. 76, 226-248.
Liu, X., Lu, J., 2017. Optimal prize-rationing strategy in all-pay contests with incomplete information. Int. J. Ind. Organ. 50, 57-90.
Mitroi, F.-C., Niculescu, C.P., 2011. An extension of Young's inequality. Abstr. Appl. Anal. 2011, 1-18.
Moldovanu, B., Sela, A., 2001. The optimal allocation of prizes in contests. Am. Econ. Rev. 91, 542-558.
Moldovanu, B., Sela, A., 2006. Contest architecture. J. Econ. Theory 126, 70-96.
Myerson, R.B., 1981. Optimal auction design. Math. Oper. Res. 6, 58-73.
Pérez-Castrillo, D., Wettstein, D., 2016. Discrimination in a model of contests with incomplete information about ability. Int. Econ. Rev. 57, 881-914.
Seel, C., Wasser, C., 2014. On optimal head starts in all-pay auctions. Econ. Lett. 124, 211-214.
Serena, M., 2017. Quality contests. Eur. J. Polit. Econ. 46, 15-25.
Siegel, R., 2009. All-pay contests. Econometrica 77, 71-92.
Vojnović, M., 2016. Contest Theory: Incentive Mechanisms and Ranking Methods. Cambridge University Press.


[^0]:    We are grateful to two anonymous referees and an anonymous advisory editor for many valuable comments and suggestions. We also thank participants at the SAET Conference 2021, the 8th Annual Conference on Contests: Theory and Evidence, the 2022 Conference on Mechanism and Institution Design, the EARIE 2022, and the VfS Annual Conference 2022 for helpful comments. Zhang gratefully acknowledges financial support from the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) via Germany’s Excellence Strategy - EXC 2047/1-390685813 and CRC TR-224 (project B01).

    * Corresponding author.

    E-mail addresses: c.wasser@unibas.ch (C. Wasser), mzhang@uni-bonn.de (M. Zhang).
    1 The accumulated human capital may not be entirely firm specific and thus may still have some value outside the firm. But as long as candidates take this external benefit into account when assessing their investment cost, the situation still resembles a contest.
    2 See Serena (2017) for more examples where the organizer wishes to maximize the winner's effort.

[^1]:    ${ }^{3}$ An alternative interpretation of our model is that agents have private information about their ability or their constant marginal cost of effort.
    ${ }^{4}$ Specifically, they correspond to the subclass called "separable contests" in Siegel (2009). In Siegel's formulation the agents directly choose their scores and have individual costs for reaching a certain score. Designing score functions in our formulation is equivalent to designing cost functions in Siegel's formulation.

[^2]:    5 These papers build on the model introduced by Moldovanu and Sela (2001). Analyzing data from Topcoder.com, Boudreau et al. (2016) find this model to be consistent with the observed behavior of contestants.
    ${ }^{6}$ For analyses of optimal head starts and handicaps under complete information, see Franke et al. (2018) and the references therein.

[^3]:    ${ }^{7}$ Chawla et al. (2019) identify a symmetric allocation that maximizes the expected highest effort when the organizer can withhold the prize. In our setting the organizer must always allocate the prize to some agent. In Section 6 , we extend our analysis to the case where the organizer can withhold the prize.

[^4]:    ${ }^{8}$ If $\bar{\psi}$ is constant on some interval $[0, y]$ and strictly increasing otherwise, replacing the random tie-breaking rule with a deterministic one leaves $W$ unchanged. The corresponding allocation is, e.g., implemented by a biased all-pay auction where one agent has a head start (i.e., $s_{i}(b)=b+h$ for one $i$ and $s_{j}(b)=b$ for $\left.j \neq i\right)$.

[^5]:    ${ }^{9}$ One difficulty with $W^{I I}(\mathbf{q})$ is that a local change of $Q_{i}\left(\theta_{i}\right)$ for one type $\theta_{i}$ affects the term $\int_{0}^{\theta_{i}^{\prime}} Q_{i}(z) \mathrm{d} z$ for all types $\theta_{i}^{\prime} \geq \theta_{i}$.
    10 For $n=2$, Gershkov et al. (2021, Theorem 2) implies that $W^{I}(\mathbf{q})$ is uniquely maximized by the ex post efficient symmetric allocation if $F(\theta)>\theta f(\theta)$ for all $\theta$. Consider $f$ given in example (10) and modify it with some small $\epsilon>0$ to obtain $\tilde{f}(\theta)=f(\theta)+\epsilon(175 / 512-\theta)$ for $\theta \leq 7 / 16$ and $\tilde{f}(\theta)=$ $f(\theta)-\epsilon(49 / 512)$ for $\theta>7 / 16$. Then $\tilde{F}$ satisfies both $\tilde{F}(\theta)>\theta \tilde{f}(\theta)$ and condition (ii) in Theorem 1. Hence, even though the symmetric allocation maximizes $W^{I}$, differential treatment strictly increases $W$ because of $W^{I I}$.

[^6]:    11 See also Chawla et al. (2019, Theorem 3.6) for a sketch of an optimal symmetric reserve contest that maximizes the expected highest effort.
    12 Part (ii) of Proposition 2 always applies in this case because $\psi(0)<0$, which implies $\bar{\psi}(0)<0$.
    ${ }^{13}$ Chawla et al. (2019) present a numerical example where the expected highest effort is higher in an asymmetric reserve contest than in an optimal symmetric one. The example meets condition (ii) of Theorem 1R.

