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Competitive Information Disclosure to an Auctioneer†

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We analyze how voluntary disclosure of information by bidders affects the outcome of optimally designed auctions. In a single-object auction environment, we assume that before the revenue-maximizing auctioneer designs the auction, bidders noncooperatively choose signal structures that disclose information about their valuations. We show that an equilibrium exists in this two-stage game and that in every equilibrium the object is sold with probability one. Our main result concerns the consequences of information disclosure for the auctioneer's revenue. If in the benchmark without disclosure the object remains unsold with positive probability, then disclosure yields strictly higher revenue in every equilibrium. (*JEL* D44, D82, D83)

Prospective bidders in an auction often try to influence the design of the auction in their favor. Consultations conducted before public assets are auctioned provide an opportunity for this. For example, details of the rules of spectrum auctions, such as reserve prices as well as the configuration and permitted use of the spectrum, are often determined by a consultation process that facilitates lobbying by prospective bidders (Ausubel and Baranov 2017).[1] Importantly, by contributing to consultations, stakeholders may disclose information to the auctioneer and sometimes to the general public.[2] Also in procurement auctions, exertion of influence via market consultations is widespread. In the European Union, public contracting authorities are therefore required to take strict measures to ensure that using advice from a potential bidder does not distort competition (in particular, all relevant information exchanged with that bidder has to be shared with the other bidders; see Directive 2014/24/EU).

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[1] As a case in point, consider the lobbying for favorable rules in the 2000 Dutch UMTS auction reported by Van Damme (2003). Before the 2016 broadcast incentive auction in the United States, the two largest carriers, AT&T and Verizon, lobbied against rules that set aside spectrum for smaller carriers, while smaller carriers, including Sprint and T-Mobile US, lobbied for such rules.

[2] For example, see https://ec.europa.eu/clima/consultations/articles/0002_en for the public contributions to the 2009 consultation by the European Commission on the auctioning rules of the EU Emissions Trading Scheme and https://www.rtr.at/TKP/aktuelles/veroeffentlichungen/veroeffentlichungen/konsultationen/stn_konsult700-1500-2100-mhz.de.html for those on the 2020 5G auction in Austria.

Information revealed in advance by potential participants influences auction rules in many contexts. In indicative bidding, a common practice in takeover auctions, real estate sales, divestiture sales, and utility privatization, prospective buyers first submit nonbinding bids that the auctioneer may use to short-list bidders or set the reserve price in the actual auction (see, e.g., Quint and Hendricks 2018). In common procedures for initial public offerings, such as bookbuilding, the investment banker first solicits indications of interest from institutional investors before fixing pricing and allocation rules.[3] Finally, prospective bidders may be inclined to disclose information even before plans to hold an auction are announced, as, for instance, suppliers in repeated business-to-business procurement who care about their reputation and anticipate that the buyer will demand similar goods or services again in the future.

In this paper, we investigate the incentives of bidders to influence the auction design by disclosing information about their valuations prior to the auction. We find that, quite generally, bidders have an incentive for such persuasion. Importantly, the auctioneer often benefits from the disclosed information.

We consider an auctioneer who seeks to sell an object at the highest possible price without knowing the bidders' valuations for the object, as in the standard optimal auction design framework (Myerson 1981). We augment this framework by an initial stage in which the bidders noncooperatively disclose information about their valuations to the auctioneer in the sense of Bayesian persuasion (Gentzkow and Kamenica 2017; Kamenica and Gentzkow 2011). Specifically, each bidder commits to a signal structure, that is, to a mapping from his valuation to probability distributions over signals. The auctioneer observes the signals, updates his prior beliefs concerning the bidders' valuations, and then designs a revenue-maximizing auction.

The assumption that the bidders can commit to a signal structure breaks with the usual assumption in mechanism design that agents have no commitment power: When reporting to the mechanism, agents are usually assumed to choose reports that are best for them given the content of their private information. For example, an agent cannot commit to withhold information in one state to extract better terms in another state. This is plausible once the mechanism is established. Before, however, agents benefit from commitment power and should therefore try to acquire it. We also assume that the bidders choose the signal structures before they know their valuations. This assumption seems appropriate when the bidders are not anonymous and frequently participate in similar auctions, so that they benefit from a long-run disclosure strategy. Finally, whereas the auctioneer can commit to an auction after information has been disclosed, she cannot make commitments already at the disclosure stage. For example, she cannot commit to hold the auction only if the bidders fully reveal their valuations, which would allow extracting the entire surplus. Our point is that bidders automatically have an incentive to disclose valuable information. Indeed, in reality, information about valuations is often confidential (see, e.g., Rothkopf, Teisberg, and Kahn 1990)—disclosure requirements that go beyond what bidders reveal voluntarily could deter participation in the auction.

[3] See Cornelli and Goldreich (2001), who find evidence that bidders who reveal information are favored when the investment banker allocates shares.

More generally, the disclosure stage represents a point in time at which the bidders anticipate that an auction is going to take place but the details have not been determined yet.[4]

Optimal auctions allocate the object to a bidder with the highest virtual valuation, provided the latter is strictly positive (Myerson 1981). The virtual valuation measures the surplus that the auctioneer can extract from a bidder, and it is lower than the true valuation as the bidder earns an information rent. How much lower it is depends on what information the auctioneer has about that bidder. In our model, the bidders control the auctioneer's information via their disclosure in the first stage. A bidder thus faces the following trade-off when deciding whether to disclose more information: on the one hand, this reduces his information rent, but on the other hand, it increases his virtual valuation and thereby results in a more favorable allocation.[5]

We show that our two-stage game of information disclosure and auction design has an equilibrium, and in every equilibrium, with probability one at least one bidder's virtual valuation is strictly positive. Taken together, these insights imply our main result: if under the auctioneer's prior information the virtual valuation of each bidder can be weakly negative, then our two-stage game results in strictly more expected revenue for the auctioneer than the standard framework without disclosure. In a nutshell, competition in information disclosure erodes the bidders' information rents such that keeping the object is strictly suboptimal for the auctioneer. Remarkably, the anticipation of an optimally designed auction automatically provides bidders with an incentive to reveal valuable information. For real-world auction design, this implies that auctioneers might benefit from announcing plans to hold an auction early and being open for information disclosure, even if they cannot make commitments at the disclosure stage nor design this stage itself. Requiring auctioneers to be immune against lobbying may come at a cost of lower auction revenues or higher procurement prices.

We now sketch why, in every equilibrium, with probability one at least one bidder's virtual valuation is strictly positive. We show later that any signal structure can be modified so as to raise strictly negative virtual valuations to zero without sacrificing any payoff. It can then be modified further by sending, with probability $\epsilon > 0$, an additional signal if the bidder's valuation is his highest possible one. The cost is that, when the signal is sent, the bidder retains no private information and thus gets no information rent. When the signal is not sent, the highest possible valuation appears less likely than before, and as a consequence, the bidder's virtual valuations strictly increase. Now, as $\epsilon \to 0$, the cost of the modification becomes negligible. On the other hand, *any* $\epsilon > 0$ suffices to make all virtual valuations strictly positive—and thus to win with probability one when all other bidders happen to have a weakly negative virtual valuation. Thus, all bidders having a weakly negative virtual valuation cannot happen in equilibrium.

[4] E.g., the 2016 broadcast incentive auction in the United States was mandated by Congress in 2012 but finalized by the Federal Communications Commission only in 2015 (see Milgrom and Segal 2017).

[5] More precisely, what matters are the *ironed* virtual valuations. Since virtual valuations are endogenous in our model, we cannot impose Myerson's regularity condition.

Our equilibrium existence proof has similar arguments at its core. For each profile of posterior beliefs, we fix an optimal auction for the auctioneer. This defines a one-stage game with the bidders as the only players. The existence of a Nash equilibrium in this game implies the existence of subgame-perfect Nash equilibrium in the overall two-stage game. In the one-stage game, the bidders' payoffs can be discontinuous in signal structures, so that standard equilibrium existence proofs do not apply. However, each bidder can modify his virtual valuations at virtually no cost so as to win any tie. We use this insight to show that the one-stage game is better-reply secure as defined by Reny (1999), which guarantees the existence of a Nash equilibrium.

After establishing our main result, we fully characterize equilibrium signal structures for several special cases of the model, which yields insights on bidders' payoffs and efficiency. Whereas the auctioneer typically benefits from information disclosure (and is never worse off), bidders may but need not gain: to improve their chances of winning in the auction, some or all might disclose so much that they would be better off if nobody disclosed anything. Moreover, whereas the auctioneer does not inefficiently retain the object in equilibrium, the auction can nevertheless result in an inefficient allocation in that the winner need not be the bidder with the highest valuation. In a nutshell, some bidders can have a stronger incentive than others to increase virtual valuations through disclosure. Finally, we extend our model and show that it does not matter for the auctioneer's revenue if the bidders disclose information publicly or only to the auctioneer—the revenue is the same in either case.

For analytical convenience, we work with finitely many possible valuations. With continuous valuations, posterior beliefs upon information disclosure can be arbitrary distributions. Optimal auctions for arbitrary distributions are considerably more involved than with discrete distributions or probability density functions, the two cases that most of the literature considers.[6] By assuming a discrete prior for each bidder, we ensure that posteriors are discrete as well and avoid technicalities. The auctioneer benefits from the disclosure because the discontinuities of optimal auctions at ties induce bidders to disclose valuable information. It is worth mentioning that discrete priors are not essential for this argument. More specifically, we do not require symmetry of priors across bidders. Thus, under the prior information, ties can have probability zero.

The most closely related paper is by Bergemann, Brooks, and Morris (2015), who study monopoly pricing under the assumption that the monopolist receives further information about the buyer's valuation, beyond the prior, before setting the price. Allowing for arbitrary further information, they characterize all possible outcomes that can arise, including the buyer-optimal outcome. The buyer-optimal outcome maximizes the social surplus, while the seller just gets the prior-information monopoly profit; all remaining surplus accrues to the buyer. Thus, in particular, the seller does *not* benefit from the additional information she receives. Our model differs from the one by Bergemann, Brooks, and Morris (2015) only in that there is more than one bidder, so that the monopolist sells the good by an auction. Hence, it is solely due to competition among bidders that the auctioneer receives (strictly)

[6] See Skreta (2006) and Monteiro and Svaiter (2010) for optimal auctions with arbitrary distributions.

valuable information. Intuitively, in the one-bidder case, the bidder discloses only to prevent that the auctioneer inefficiently retains the object, and only so much that the auctioneer is just indifferent, whereas in our multibidder case, bidders also disclose to increase their virtual valuations relative to those of their competitors. More specifically, Bergemann, Brooks, and Morris (2015) show that there is a buyer-optimal signal structure in their model under which all possible virtual valuations except for the highest one are zero—which cannot happen in any equilibrium of our model, as explained above.[7] Thus, due to the discontinuities of optimal auctions at ties, competition has a significant effect on information design and the seller's revenue.

Glode, Opp, and Zhang (2018) investigate buyer-optimal information disclosure under monopoly pricing when disclosure can be represented by a deterministic signal conditional on the valuation. They show that buyer-optimal disclosure always results in efficient trade, as in the paper by Bergemann, Brooks, and Morris (2015). But because of the restriction to deterministic signals, the buyer is typically unable to fully extract the additional surplus, so that also the seller benefits. We allow for stochastic signals, as is standard in the literature on Bayesian persuasion. In our model, instead, the auctioneer benefits because the discontinuities in optimal auction design induce bidders to disclose valuable information: these discontinuities arise due to ties and do not exist in monopoly pricing. Stochastic signals enable the bidders to raise virtual valuations at virtually no cost and thereby take full advantage of the discontinuities. In applications, disclosure can often not be represented by a deterministic signal. For instance, a bidder's valuation for winning the auction typically depends on many initially uncertain factors. Naturally, disclosure may then take the form of revealing information about some of those factors, which corresponds to a stochastic signal about the valuation.[8]

Another literature studies the disclosure of (typically verifiable) information by bidders ahead of exogenously given auctions (see, e.g., Benoît and Dubra 2006; Kovenock, Morath, and Münster 2015; Tan 2016). This literature focuses on interdependent valuations, where a bidder's private information can be payoff relevant for their competitors. When the auction is given, bidders may disclose information to their competitors with the aim of influencing their bidding behavior.[9] As in our paper, voluntary disclosure can result in a higher expected revenue for the auctioneer (see in particular Tan 2016).

In the literature on Bayesian persuasion, several papers analyze simultaneous information disclosure by a group of senders to a receiver who then takes an action that is relevant to all players (see, e.g., Boleslavsky and Cotton 2015 and Gentzkow

[7] In Roesler and Szentes (2017), the monopolist receives no information, but, instead, the buyer chooses a signal structure to acquire information about his valuation. Again, there is a buyer-optimal signal structure under which all virtual valuations except for the highest one are zero (see also Condorelli and Szentes 2020). Yang (2019) considers an extension to auctions and shows that as the number of buyers goes to infinity, buyers acquire full information and the auctioneer extracts the first-best surplus. In Terstiege and Wasser (2020), the seller can refine the signal structure chosen by the buyer.

[8] For example, in spectrum auctions, potential bidders may disclose how they assess the installation costs for their new use of spectrum, or in procurement, potential suppliers may provide information about their production efficiency based on experimenting with an early prototype.

[9] In our independent private values model, bidders do not gain by disclosing information to each other. Specifically, an optimal auction can always be found among dominant strategy incentive-compatible auctions, where the beliefs of bidders about their competitors' valuations do not matter.

and Kamenica 2017; see Li and Norman 2021 for sequential disclosure). While our two-stage game has the same structure, with the bidders being the senders and the auctioneer the receiver, none of the existing analyses captures information disclosure before auction design. Most closely related in this literature are the papers by Gentzkow and Kamenica (2017) and Au and Kawai (2020), who show that increased competition in the form of additional senders can increase the information content of the senders' disclosure. The key difference to Gentzkow and Kamenica (2017) is that their senders disclose information about the complete state of the world, which in our setting would correspond to disclosure not only about one's own valuation but also about the valuations of competitors. Au and Kawai (2020) assume a specific choice problem for the receiver (namely, to select a sender) that does not capture our choice problem (selecting an auction mechanism).

The paper is organized as follows. The next section presents the model. In Section II, we discuss an example that illustrates the main result. Section III studies the auction design problem in stage two of our game and Section IV the bidders' choice over signal structures in stage one. In Section V, we show that a subgame-perfect Nash equilibrium exists. Section VI presents our main result on the auctioneer's revenue. Section VII gives full equilibrium characterizations for several special cases of the model, and Section VIII shows that our results are independent of how much the other bidders learn about a bidder's disclosure. Section IX concludes. Most proofs are in the Appendix. The online Appendix contains additional material that complements Section IVB and Section VII.

I. Model

Players and Prior Information.—There is a risk-neutral auctioneer with one object, who seeks to maximize her expected revenue, and there is a set $N = \{1, \ldots, n\}$ of risk-neutral bidders, where $n > 1$. The auctioneer's valuation for the object is normalized to zero. The valuation of each bidder $i \in N$ is initially unknown to all players, including bidder i. The common prior belief is that the valuation is independently drawn from the set $\bar{V}_i$ according to the probability distribution $\bar{p}_i$. We assume that there are finitely many possible valuations, so that $\bar{V}_i = \{\bar{v}_i^1, \ldots, \bar{v}_i^{\bar{m}_i}\}$, where $0 < \bar{v}_i^1 < \cdots < \bar{v}_i^{\bar{m}_i}$ and $\bar{p}_i(v_i) > 0$ for all $v_i \in \bar{V}_i$.

Signal Structures.—Before learning his valuation, and before the auction is designed, each bidder i chooses and commits to a signal structure. A signal structure for bidder i can be understood as a mapping from i's true valuation v_i to distributions over signals. Knowing the signal structure and observing the realized signal allows to update from the prior $\bar{p}_i$ to a posterior $p_i \in \mathcal{P}_i$, where $\mathcal{P}_i$ denotes the set of all probability distributions on $\bar{V}_i$. Following Kamenica and Gentzkow (2011), we abstract from signals and represent any signal structure as a distribution (i.e., a Borel probability measure) b_i on the set of posteriors $\mathcal{P}_i$ such that, to be consistent with Bayesian updating, the expectation equals the prior,

$$\int_{\mathcal{P}_i} p_i(v_i)\, db_i(p_i) = \bar{p}_i(v_i), \quad \forall v_i \in \bar{V}_i. \tag{1}$$

Let B_i be the set of all signal structures of bidder i, that is, the set of all distributions b_i on $\mathcal{P}_i$ that satisfy (1).

Auctions.—Given any distribution $p_i \in \mathcal{P}_i$, let $V_i(p_i)$ be its support, a subset of $\bar{V}_i$. Given any profile $\mathbf{p} = (p_1, \ldots, p_n) \in \mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, let $V(\mathbf{p}) = V_1(p_1) \times \cdots \times V_n(p_n)$.

Fix any profile $\mathbf{p} \in \mathcal{P}$. Suppose each bidder i knows his own valuation and believes that the valuation of any other bidder j is drawn from p_j, and this is common knowledge. A direct auction mechanism at $\mathbf{p}$ is a combination $(\mathbf{q}(\cdot, \mathbf{p}), \mathbf{t}(\cdot, \mathbf{p}))$, where $\mathbf{q}(\cdot, \mathbf{p}) = (q_1(\cdot, \mathbf{p}), \ldots, q_n(\cdot, \mathbf{p}))$ is an *allocation rule* and $\mathbf{t}(\cdot, \mathbf{p}) = (t_1(\cdot, \mathbf{p}), \ldots, t_n(\cdot, \mathbf{p}))$ is a *transfer rule*. The function $q_i(\cdot, \mathbf{p}) : V(\mathbf{p}) \to [0,1]$ determines bidder i's probability of getting the object depending on a profile of reported valuations $\mathbf{v} = (v_1, \ldots, v_n) \in V(\mathbf{p})$, where $\sum_{i \in N} q_i(\mathbf{v}, \mathbf{p}) \leq 1$ for all $\mathbf{v}$. The function $t_i(\cdot, \mathbf{p}) : V(\mathbf{p}) \to \mathbb{R}$ determines a transfer paid by bidder i to the auctioneer. As usual, $(\mathbf{q}(\cdot, \mathbf{p}), \mathbf{t}(\cdot, \mathbf{p}))$ is said to be Bayesian incentive compatible and interim individually rational if truthful reporting and participating in the auction is a Bayes-Nash equilibrium. Invoking the revelation principle, we restrict attention to such auction mechanisms, which we call *auctions*.

The ex ante expected payoff of bidder i in auction $(\mathbf{q}(\cdot, \mathbf{p}), \mathbf{t}(\cdot, \mathbf{p}))$ is

$$\sum_{\mathbf{v} \in V(\mathbf{p})} \left[v_i q_i(\mathbf{v}, \mathbf{p}) - t_i(\mathbf{v}, \mathbf{p})\right] p(\mathbf{v}),$$

where $p(\mathbf{v}) = \prod_{i \in N} p_i(v_i)$, and the ex ante expected revenue of the auctioneer is

$$\sum_{i \in N} \sum_{\mathbf{v} \in V(\mathbf{p})} t_i(\mathbf{v}, \mathbf{p}) p(\mathbf{v}). \tag{2}$$

Timing.—The interaction between the bidders and the auctioneer is a two-stage game. In stage one, each bidder chooses a signal structure. Signal structures are chosen simultaneously. In stage two, each bidder i first observes his valuation v_i and publicly discloses information according to his signal structure.[10] Thus, the players update to a profile of common posteriors, each bidder knowing his own valuation. Afterward, the auctioneer chooses an auction. Following the mechanism-design tradition, we assume that the bidders participate in the auction and report truthfully when indifferent.

Strategies and Equilibrium.—A strategy of the auctioneer is a (measurable) function

$$f : \mathbf{p} \mapsto \left(\mathbf{q}^f(\cdot, \mathbf{p}), \mathbf{t}^f(\cdot, \mathbf{p})\right)$$

that determines an auction for every profile of posteriors $\mathbf{p} \in \mathcal{P}$. A strategy of bidder i is a signal structure $b_i \in B_i$. We use subgame-perfect Nash equilibrium

[10] In Section VIII, we relax the assumption that the disclosure is public.

(SPNE) as solution concept. An SPNE is a strategy profile $(f^*, \mathbf{b}^*)$ such that $(\mathbf{q}^{f^*}(\cdot, \mathbf{p}), \mathbf{t}^{f^*}(\cdot, \mathbf{p}))$ maximizes the auctioneer's expected revenue (2) at $\mathbf{p}$, subject to Bayesian incentive compatibility and interim individual rationality, and b_i^* maximizes bidder i's payoff

$$\int_{\mathcal{P}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p})} \left[v_i q_i^{f^*}(\mathbf{v}, \mathbf{p}) - t_i^{f^*}(\mathbf{v}, \mathbf{p})\right] p(\mathbf{v})\, db_{-i}^*(\mathbf{p}_{-i})\, db_i(p_i),$$

where $\mathbf{b} = (b_1, \ldots, b_n) \in B = B_1 \times \cdots \times B_n$, b denotes the product distribution constructed from $b_1, \ldots, b_n$ and the symbol $-i$ denotes "all bidders but i."

II. Illustration of the Main Result

We use a simple example to illustrate our main result that bidders disclose valuable information in equilibrium and to highlight the importance of competition among bidders.

Suppose there are two bidders with identical priors over two possible valuations $\bar{V}_1 = \bar{V}_2 = \{v^L, v^H\}$ such that $\bar{p}_1(v^L) = \bar{p}_2(v^L) = \bar{\lambda} \in (0, \lambda^0)$, where $v^L < v^H$ and $\lambda^0 = (v^H - v^L)/v^H$. We simplify the notation in this section and identify any posterior p_i for bidder i with the probability of the low valuation $\lambda_i = p_i(v^L) \in [0,1]$.

One-Bidder Benchmark.—It is instructive to begin with a simple benchmark in which the auctioneer faces only bidder 1. In this case, it is optimal for the auctioneer to just post a price. Price v^L is optimal if $\lambda_1 \geq \lambda^0$, which is equivalent to $v^L \geq (1 - \lambda_1)\, v^H$. Price v^H is optimal if $\lambda_1 < \lambda^0$. Thus, without disclosure and prior $\bar{\lambda} < \lambda^0$, the auctioneer posts price v^H, the object remains unsold with probability $\bar{\lambda}$, and the bidder's payoff is zero. With disclosure, the bidder can ensure that the object is always sold and appropriate the entire additional surplus. Specifically, there is an optimal signal structure b_1^* that draws posterior $\lambda_1' = \lambda^0$ with probability $\bar{\lambda}/\lambda^0$ and posterior $\lambda_1'' = 0$ with probability $1 - \bar{\lambda}/\lambda^0$. As

$$\frac{\bar{\lambda}}{\lambda^0} \lambda_1' + \left(1 - \frac{\bar{\lambda}}{\lambda^0}\right) \lambda_1'' = \bar{\lambda},$$

b_1^* clearly satisfies (1). Intuitively, this corresponds to the bidder revealing his valuation with a certain probability when it is v^H. At posterior $\lambda_1'' = 0$, there is thus no uncertainty, and the auctioneer still charges price v^H. Yet at posterior $\lambda_1' = \lambda^0$, she is now willing to charge only v^L, resulting in a positive payoff for the bidder if his valuation is v^H. As the auctioneer is indifferent between both prices at posterior λ^0, her revenue is the same as when always charging v^H. Hence, the auctioneer does not benefit from the disclosure—a result that generalizes to arbitrary priors $\bar{p}_1$ (see Bergemann, Brooks, and Morris 2015).

Stage Two: Auction Design.—Return to the two-bidder case. As we will see, the prospect of an optimally designed auction gives the bidders an incentive to reveal valuable information, resulting in strictly higher auction revenue. We first need to consider the optimal auction design for any posterior profile (λ_1, λ_2) in stage two. We restrict attention to one specific optimal auction with a particularly simple implementation. If $\max\{\lambda_1, \lambda_2\} < \lambda^0$, the auctioneer uses a posted price equal to v^H as in the one-bidder case. If $\lambda_i \geq \max\{\lambda_j, \lambda^0\}$ for some bidder i, however, it is optimal to first offer the object at price v^H to bidder j and, in case he rejects, to then sell it to bidder i at price v^L.[11] Let the order be random with equal probability if $\lambda_1 = \lambda_2 \geq \lambda^0$. The expected payoff of bidder i given posteriors (λ_1, λ_2) is thus

$$(3) \qquad \begin{cases} \lambda_j(1-\lambda_i)(v^H - v^L), & \text{if } \lambda_i \geq \lambda^0 \text{ and } \lambda_i > \lambda_j; \\ \lambda_j(1-\lambda_i)(v^H - v^L)/2, & \text{if } \lambda_i = \lambda_j \geq \lambda^0; \\ 0, & \text{otherwise.} \end{cases}$$

The auctioneer favors the bidder who is more likely to have a low valuation by offering him the low price, thereby granting him an information rent in case his valuation is high. In disclosing information, bidders may thus compete for being offered the low price.

Stage One: Disclosure.—Considering the bidders' choice of disclosure in the first stage, we next argue that there are profitable deviations if the signal structures of both bidders draw posteriors in $(0, \lambda^0]$. Suppose first that the bidders choose signal structures (b_1, b_2) under which posteriors $\lambda_i \in (0, \lambda^0)$ and $\lambda_j \in (0, \lambda^0]$ have positive probability. At λ_i, i's expected payoff is always zero. Consider a modification b_i' of b_i where posterior $\lambda_i \in (0, \lambda^0)$ is replaced with two posteriors λ_i', λ_i'': instead of drawing posterior λ_i with, say, probability β, b_i' draws $\lambda_i' \in (\lambda^0, 1)$ with probability $\beta\lambda_i/\lambda_i'$ and $\lambda_i'' = 0$ with probability $\beta(1 - \lambda_i/\lambda_i')$. This modification is feasible since

$$\beta\frac{\lambda_i}{\lambda_i'}\lambda_i' + \beta\left(1 - \frac{\lambda_i}{\lambda_i'}\right)\lambda_i'' = \beta\lambda_i,$$

and thus b_i' is consistent with (1) whenever b_i is. Similar to b_1^* in the one-bidder case, conditional on b_i drawing λ_i, b_i' reveals with a certain probability when i's valuation is v^H, resulting in posterior $\lambda_i'' = 0$ if the valuation is revealed and in posterior $\lambda_i' > \lambda^0 > \lambda_i$ if the valuation is not revealed. Bidder i clearly benefits from the modification: whereas his payoff was always zero under λ_i, now his payoff is strictly positive under any posterior profile (λ_i', λ_j) with $\lambda_j \in (0, \lambda^0]$. Hence, (b_1, b_2) cannot be part of an equilibrium.

Now suppose the signal structures (b_1, b_2) are such that each draws posterior λ^0 with positive probability, whereas posteriors in $(0, \lambda^0)$ have probability zero. At $\lambda_i = \lambda_j = \lambda^0$, neither bidder's payoff (3) is zero; however, the seller randomizes to whom to offer the object at the low price v^L. Consider the same modification b_i' of b_i as above: $\lambda_i = \lambda^0$ is replaced with probability λ^0/λ_i' by $\lambda_i' > \lambda^0$ and with

[11] Using the characterization of optimal auctions in Section III, it is easily verified that these sequential offers are optimal. In general, the revenue is at most equal to the expected highest virtual valuation. Depending on his valuation, i's virtual valuation is $J_i(v^L) = v^L - (v^H - v^L)(1-\lambda_i)/\lambda_i$ or $J_i(v^H) = v^H$. Hence, if $\lambda_i \geq \{\lambda_j, \lambda^0\}$, the revenue is at most $(1 - \lambda_i\lambda_j)v^H + \lambda_i\lambda_j J_i(v^L) = (1-\lambda_j)v^H + \lambda_j v^L$, which equals the revenue under the sequential offers.

probability $\left(1 - \lambda^0/\lambda_i'\right)$ by $\lambda_i'' = 0$. With the latter posterior, bidder i's payoff is always zero. However, as $\lambda_i' \to \lambda^0$, this cost of the modification vanishes because $\left(1 - \lambda^0/\lambda_i'\right) \to 0$. On the other hand, *any* $\lambda_i' > \lambda^0$ suffices to get the low price with probability one when $\lambda_j = \lambda^0$. Thus, for small λ_i', the modification benefits bidder i, and (b_1, b_2) again cannot be part of an equilibrium.

Auctioneer's Benefit.—We conclude that equilibrium signal structures (b_1, b_2) are such that in stage two the auctioneer is either certain of a bidder's valuation or assigns a posterior $\lambda_i > \lambda^0$ to at least one bidder i. Accordingly, the auctioneer's revenue must be strictly greater than when she always posts price v^H in stage two. Yet without disclosure and prior $\bar{\lambda} < \lambda^0$, posting price v^H is optimal. Hence, the auctioneer strictly benefits from the disclosure.[12]

Our main result is that the auctioneer strictly benefits from the disclosure in every equilibrium of the general model of Section I if under the prior the virtual valuation of each bidder can be weakly negative. In the example, this condition on the prior corresponds to $\bar{\lambda} \in \left(0, \lambda^0\right]$, which holds by assumption. Beyond the result on the auctioneer's revenue, the example has a number of features that do not generalize, as we will discuss in Section VII. In particular, in the setting considered here, bidders perfectly disclose their valuation and thus obtain zero payoff (as without disclosure) in every equilibrium, whereas in general bidders can both benefit or suffer from the disclosure.

Plan of the Main Analysis.—The following four sections contain our main analysis, culminating in Theorem 1 on the effect of disclosure on the auctioneer's revenue. We proceed in a similar manner as here. We first consider stage two of the game and characterize optimal auctions for arbitrary posteriors in Section III. In Section IV, we then turn to stage one: We first fix an arbitrary optimal strategy for the auctioneer—a strategy that specifies an optimal auction at every posterior profile. This reduces the model to a one-stage game in which the bidders choose their signal structures. To identify profitable deviations, we then generalize the idea of the modifications b_i' discussed above. We thereby show how a bidder can modify his signal structure so as to raise negative (ironed) virtual valuations to zero and win any tie (which is akin to replacing $\lambda_i \leq \lambda^0$). Section V presents an optimal strategy for the auctioneer under which the corresponding one-stage game has a Nash equilibrium. This proves the existence of a subgame-perfect Nash equilibrium in the overall two-stage game. In Section VI, we show that in equilibrium at least one bidder always has a strictly positive virtual valuation under the posterior, which corresponds to $\lambda_i \notin \left(0, \lambda^0\right]$ in the example. By a similar argument as here, we then conclude that the auctioneer's revenue must be strictly greater than under the prior.

[12] As another benchmark besides the one-bidder case, suppose the bidders jointly choose their disclosure. Then both choosing b_1^* as in the one-bidder case is optimal for them: that way, the auctioneer is willing to make sequential offers under posterior profile $\left(\lambda^0, \lambda^0\right)$, ensuring the object is always sold. As the auctioneer is indifferent to always posting price v^H, she does not benefit from the disclosure, and the entire additional surplus goes to the bidders. So, for the auctioneer to benefit, competition is essential.

III. Stage Two: Auction Design

Returning to the general model of Section I, we start our analysis with stage two of the game, in which the auctioneer designs the auction. We characterize the optimal auctions at an arbitrary profile of posteriors $\mathbf{p}$. Since $\mathbf{p}$ remains unchanged in this section, we denote an auction by $(\mathbf{q},\mathbf{t})$ instead of $(\mathbf{q}(\cdot,\mathbf{p}),\mathbf{t}(\cdot,\mathbf{p}))$, and we denote the support of posterior p_i by V_i instead of $V_i(p_i)$.

Let $V_i = \{v_i^1,\ldots,v_i^{m_i}\}$ and $V = V_1 \times \cdots \times V_n$. Let $P_i(v_i^k) = \sum_{l=1}^{k} p_i(v_i^l)$ be the probability that bidder i's valuation is at most v_i^k. Let $Q_i(v_i) = \sum_{\mathbf{v}_{-i}\in V_{-i}} q_i(v_i,\mathbf{v}_{-i})\,p_{-i}(\mathbf{v}_{-i})$ be bidder i's interim allocation probability and $T_i(v_i) = \sum_{\mathbf{v}_{-i}\in V_{-i}} t_i(v_i,\mathbf{v}_{-i})\,p_{-i}(\mathbf{v}_{-i})$ his interim expected transfer.

The auctioneer's goal is to design an auction $(\mathbf{q},\mathbf{t})$ that maximizes her expected revenue subject to Bayesian incentive compatibility and interim individual rationality. Thus, an *optimal auction* solves

$$\max_{(\mathbf{q},\mathbf{t})} \sum_{i\in N}\sum_{\mathbf{v}\in V} t_i(\mathbf{v})\,p(\mathbf{v})$$

subject to

$$v_i Q_i(v_i) - T_i(v_i) \geq v_i Q_i(v_i') - T_i(v_i'), \qquad \forall i \in N, \forall v_i, v_i' \in V_i, \tag{4}$$

$$v_i Q_i(v_i) - T_i(v_i) \geq 0, \qquad \forall i \in N, \forall v_i \in V_i, \tag{5}$$

where (4) ensures Bayesian incentive compatibility and (5) interim individual rationality.

By standard arguments (see, e.g., Vohra 2011, section 6.2), there is a $\mathbf{t}$ such that the auction $(\mathbf{q},\mathbf{t})$ is Bayesian incentive compatible if and only if $Q_i(v_i^{k+1}) \geq Q_i(v_i^k)$ for all $k \in \{1,\ldots,m_i-1\}$ and all $i \in N$. Moreover, for any optimal auction $(\mathbf{q},\mathbf{t})$, all local downward incentive constraints as well as the individual rationality constraint for valuation v_i^1 for each bidder i are binding, yielding interim expected transfers

$$T_i(v_i^k) = v_i^k Q_i(v_i^k) - \sum_{l=1}^{k-1}\left(v_i^{l+1} - v_i^l\right) Q_i(v_i^l). \tag{6}$$

By (6), the ex ante expected transfer from bidder i to the auctioneer can be written as

$$\sum_{v_i\in V_i} T_i(v_i)\,p_i(v_i) = \sum_{v_i\in V_i} J_i(v_i)\,Q_i(v_i)\,p_i(v_i), \tag{7}$$

where the *virtual valuation* $J_i(v_i^k)$ of bidder i with valuation v_i^k is defined as

$$J_i(v_i^k) = \begin{cases} v_i^k - \dfrac{1-P_i(v_i^k)}{p_i(v_i^k)}\left(v_i^{k+1} - v_i^k\right), & \text{if } k < m_i; \\ v_i^k, & \text{if } k = m_i. \end{cases}$$

Consequently, we can state the problem of designing an *optimal allocation rule* as [**P**]:

$$\max_{\mathbf{q}} \sum_{i \in N} \sum_{v_i \in V_i} J_i(v_i)\, Q_i(v_i)\, p_i(v_i)$$

subject to

$$Q_i(v_i^1) \leq \cdots \leq Q_i(v_i^{m_i}) \qquad \forall i \in N. \tag{8}$$

To characterize optimal allocation rules, we follow Elkind (2007), who adapts the ironing procedure of Myerson (1981) to our environment with discrete valuations. Consider any bidder i. Define $G_i(v_i^k) = \sum_{l=1}^{k} J_i(v_i^l)\, p_i(v_i^l)$. The notation $G_i(v_i^0)$ will also be used and means zero. Similarly, $P_i(v_i^0)$ means zero. Define the function $C_i : [0,1] \to \mathbb{R}$ by

$$C_i(z) = \min_{0 \leq k, l \leq m_i, \alpha \in [0,1]} \alpha\, G_i(v_i^k) + (1-\alpha)\, G_i(v_i^l)$$

subject to

$$\alpha\, P_i(v_i^k) + (1-\alpha)\, P_i(v_i^l) = z.$$

Thus, C_i is the highest convex function on $[0,1]$ that is everywhere smaller than or equal to the function that assigns to each $P_i(v_i)$, $v_i \in V_i$, the value $G_i(v_i)$. We also say C_i is the lower convex envelope. See Figure 1 for an illustration. The *ironed virtual valuation* of bidder i with valuation v_i^k is defined as

$$H_i(v_i^k) = \frac{C_i\big(P_i(v_i^k)\big) - C_i\big(P_i(v_i^{k-1})\big)}{p_i(v_i^k)}.$$

Thus, $H_i(v_i^k)$ is equal to the slope of C_i between $P_i(v_i^{k-1})$ and $P_i(v_i^k)$. Since C_i is a convex function, $H_i(v_i^k)$ is nondecreasing in k.

The following proposition fully characterizes optimal allocation rules, which implies a characterization of the set of optimal auctions.[13] Moreover, the proposition gives a convenient representation of bidder i's ex ante expected payoff in an optimal auction. Given a profile of reports $\mathbf{v} \in V$, let

$$W(\mathbf{v}) = \Big\{ i \in N \,|\, H_i(v_i) > 0 \text{ and } i \in \arg\max_j H_j(v_j) \Big\}$$

be the set of bidders whose ironed virtual valuation is strictly positive and the highest among all bidders. Let

$$L(\mathbf{v}) = \Big\{ i \in N \,|\, H_i(v_i) < 0 \text{ or } i \notin \arg\max_j H_j(v_j) \Big\}$$

[13] The proof of part (a) of Proposition 1 is adapted from the proof of Elkind (2007, theorem 2), which identifies one specific allocation rule that is optimal under dominant strategy incentive compatibility.

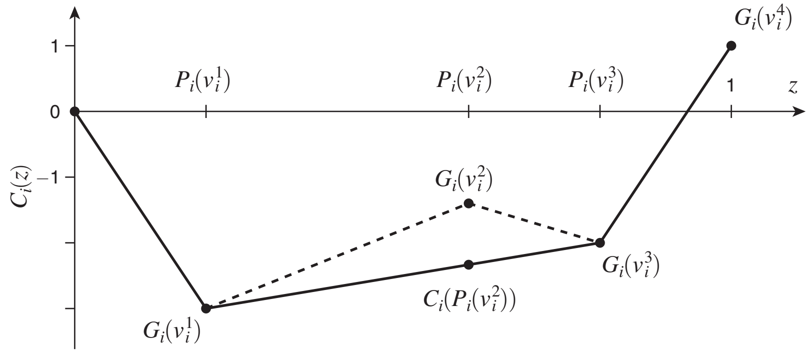

Figure 1. The Function C_i

Notes: The function C_i (solid curve) is displayed for $V_i = \{1,5,6,15\}$ with $p_i(v_i^2) = 0.4$ and $p_i(v_i^1) = p_i(v_i^3) = p_i(v_i^4) = 0.2$. The virtual valuations $J_i(v_i^2) = 4$ and $J_i(v_i^3) = -3$ (slopes of the dashed line segments) are ironed to $H_i(v_i^2) = H_i(v_i^3) = 5/3$.

be the set of bidders whose ironed virtual valuation is strictly negative or not the highest among all bidders. Any optimal auction allocates to a bidder in $W(\mathbf{v})$ if this set is not empty and does not allocate to a bidder in $L(\mathbf{v})$. If $W(\mathbf{v})$ is empty and $N \neq L(\mathbf{v})$, that is, if the highest ironed virtual valuation is equal to zero, it does not matter for the expected revenue if the object is allocated or retained.

PROPOSITION 1:

(a) *An allocation rule* $\mathbf{q}$ *is optimal if and only if* (i) *for all* $\mathbf{v} \in V$, $\sum_{i\in W(\mathbf{v})} q_i(\mathbf{v}) = 1$ *if* $W(\mathbf{v}) \neq \emptyset$; (ii) *for all* $\mathbf{v} \in V$, $\sum_{i\in L(\mathbf{v})} q_i(\mathbf{v}) = 0$; *and* (iii) *for all* $i \in N$ *and all* $k < m_i$ *such that* $H_i(v_i^k) = H_i(v_i^{k+1})$, $Q_i(v_i^k) = Q_i(v_i^{k+1})$ *if* $C_i(P_i(v_i^k)) < G_i(v_i^k)$ *and* $Q_i(v_i^k) \leq Q_i(v_i^{k+1})$ *otherwise.*

(b) *An auction* $(\mathbf{q},\mathbf{t})$ *is optimal if and only if* $\mathbf{q}$ *is optimal and* $\mathbf{t}$ *satisfies* (6).

(c) *The ex ante expected payoff of bidder* $i \in N$ *in an optimal auction* $(\mathbf{q},\mathbf{t})$ *is*

$$\sum_{\mathbf{v}\in V}\left[v_i q_i(\mathbf{v}) - t_i(\mathbf{v})\right]p(\mathbf{v}) = \sum_{\mathbf{v}\in V}\left[v_i - H_i(v_i)\right] q_i(v)p(\mathbf{v}).$$

Part (b) of the proposition identifies the auctions that maximize the auctioneer's expected revenue subject to Bayesian incentive compatibility and interim individual rationality. As is well known, some optimal auctions are actually dominant strategy incentive compatible and ex post individually rational; that is, they induce truthful reporting and participation in the auction even if the bidders know the valuations of their competitors (see Gershkov et al. 2013). Formally, an auction $(\mathbf{q},\mathbf{t})$ is dominant strategy incentive compatible if

$$v_i q_i(\mathbf{v}) - t_i(\mathbf{v}) \geq v_i q_i(v_i', \mathbf{v}_{-i}) - t_i(v_i', \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i, v_i' \in V_i, \forall \mathbf{v}_{-i} \in V_{-i},$$

and it is ex post individually rational if

$$v_i q_i(\mathbf{v}) - t_i(\mathbf{v}) \geq 0, \quad \forall i \in N, \forall v_i \in V_i, \forall \mathbf{v}_{-i} \in V_{-i}.$$

For our purposes, it will be important that among such optimal auctions there are ones that do not allocate the object when the highest ironed virtual valuation is weakly negative.

COROLLARY 1: *There is an optimal auction $(\mathbf{q}, \mathbf{t})$ that is dominant strategy incentive compatible and ex post individually rational such that, for all $i \in N$ and all $\mathbf{v} \in V$, $q_i(\mathbf{v}) = 0$ if $i \notin W(\mathbf{v})$.*

IV. Stage One: Information Disclosure

We now turn to stage one of the game, in which the bidders choose the signal structures to disclose information.

A. *Disclosure Games*

Fix an optimal strategy for the auctioneer, that is, a function $f : \mathbf{p} \mapsto (\mathbf{q}^f(\cdot, \mathbf{p}), \mathbf{t}^f(\cdot, \mathbf{p}))$ that determines an optimal auction for every profile of posteriors $\mathbf{p} \in \mathcal{P}$. With the auctions fixed, the bidders' choice of signal structures is a one-stage game, which we call a *disclosure game*. Note that due to the possibility of ties or the highest ironed virtual valuation being zero, optimal auctions need not be unique. Thus, the optimal strategy of the auctioneer is not unique, and one can construct different disclosure games. In any disclosure game, the set of strategies of bidder i is the set of his signal structures B_i. In the disclosure game corresponding to the auctioneer's strategy f, bidder i's ex ante expected payoff in the auction at $\mathbf{p}$ is

$$u_i^f(\mathbf{p}) = \sum_{\mathbf{v} \in V(\mathbf{p})} \left[v_i - H_i(v_i, p_i) \right] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v}), \tag{9}$$

as stated in Proposition 1(c), making the dependence on $\mathbf{p}$ explicit in the notation.[14] His payoff with signal structure b_i when the other bidders play $\mathbf{b}_{-i}$ is

$$U_i^f(b_i, \mathbf{b}_{-i}) = \int_{\mathcal{P}_i} \int_{\mathcal{P}_{-i}} u_i^f(\mathbf{p}) \, db_{-i}(\mathbf{p}_{-i}) \, db_i(p_i).$$

A Nash equilibrium of the disclosure game corresponding to f is a profile $\mathbf{b}^*$ such that

$$U_i^f(\mathbf{b}^*) \geq U_i^f(b_i, \mathbf{b}_{-i}^*) \qquad \forall b_i \in B_i, \forall i \in N.$$

[14] In the Appendix, further notation from Section III is augmented by $\mathbf{p}$ or p_i, respectively.

B. *Improving Ironed Virtual Valuations*

In this subsection, we describe how bidders can improve their ironed virtual valuations by disclosing more information. This will enable us to identify profitable deviations in the following sections. Throughout the paper, we usually denote the support $V_i(p_i)$ of any posterior $p_i \in \mathcal{P}_i$ by $V_i(p_i) = \{v_i^1, \ldots, v_i^{m_i}\}$, where $v_i^1 < \cdots < v_i^{m_i}$ and $m_i \geq 1$. Occasionally, we also write $V_i(p_i) = \{v_i^1(p_i), \ldots, v_i^{m_i}(p_i)\}$ to stress the dependence on p_i. Recall that under posterior p_i, the virtual valuations are $J_i(v_i^{m_i}, p_i) = v_i^{m_i}$ and

$$J_i(v_i^k, p_i) = v_i^k - \frac{1 - P_i(v_i^k)}{p_i(v_i^k)}(v_i^{k+1} - v_i^k) \qquad \text{for } 1 \leq k < m_i.$$

We start with an intuitive description of the main idea. Consider a posterior p_i that is in the support of bidder i's signal structure. Let $\hat{v} \in V_i(p_i)$ and $\xi \in (0,1)$. Suppose bidder i discloses more information by sending an additional signal together with the signal that induces posterior p_i. If bidder i's valuation is $v_i > \hat{v}$, the additional signal reveals with probability ξ that this is the case, and otherwise nothing is revealed. Consequently, instead of p_i, the auctioneer updates with probability $\theta = [1 - P_i(\hat{v})]\xi$ to posterior p_i^I and with probability $1 - \theta$ to posterior p_i^{II}, where

$$p_i^I(v_i) = \begin{cases} 0, & \text{if } v_i \leq \hat{v}; \\ \frac{\xi}{\theta} p_i(v_i), & \text{if } v_i > \hat{v}, \end{cases} \quad \text{and} \quad p_i^{II}(v_i) = \begin{cases} \frac{1}{1-\theta} p_i(v_i), & \text{if } v_i \leq \hat{v}; \\ \frac{1-\xi}{1-\theta} p_i(v_i), & \text{if } v_i > \hat{v}. \end{cases}$$

That is, now either the auctioneer is certain that the valuation is above the cutoff $\hat{v}$ or she remains uncertain but assigns less probability to valuations above $\hat{v}$. As in each case the probabilities $p_i(v_i)$ are multiplied by the same factor for all $v_i > \hat{v}$, the corresponding virtual valuations remain unchanged: $J_i(v_i, p_i^I) = J_i(v_i, p_i^{II}) = J_i(v_i, p_i)$ for $v_i > \hat{v}$. But $J_i(v_i, p_i^{II}) > J_i(v_i, p_i)$ for $v_i \leq \hat{v}$, as valuations $v_i \leq \hat{v}$ get more weight than $v_i > \hat{v}$. So, by replacing p_i with p_i^I and p_i^{II}, bidder i may increase his chance of winning the auction with valuations $v_i \leq \hat{v}$ at the cost of a lower information rent, while leaving valuations $v_i > \hat{v}$ unaffected. Finally, note that the signal structure we obtain when replacing a posterior p_i by p_i^I and p_i^{II} is also consistent with (1) because, by construction, the expected posterior conditional on the original signal structure drawing p_i is p_i (i.e., $\theta p_i^I(v_i) + (1 - \theta_i) p_i^{II}(v_i) = p_i(v_i)$ for all v_i).

We next define two specific modifications of signal structures, called ϵ-extension and δ-extension, that consist of replacing potentially many posteriors p_i with some corresponding p_i^I and p_i^{II}.[15] We show that the effect on J_i translates to the ironed virtual valuations H_i, which are relevant for the allocation, and we characterize the resulting payoff. Before doing so, we establish some basic properties of ironed virtual valuations.

[15] See Section OA2 in the online Appendix for a more formal definition of ϵ- and δ-extensions.

LEMMA 1: *For every* $i \in N$ *and every* $p_i \in \mathcal{P}_i$,

(*a*) $H_i(v_i,p_i) < v_i$ *for* $v_i < v_i^{m_i}$ *and* $H_i(v_i^{m_i},p_i) = v_i^{m_i}$;

(*b*) $H_i(v_i,\cdot)$ *is continuous at* p_i *for every* $v_i \in V_i(p_i)$.

Now, consider any signal structure $b_i \in B_i$ of bidder i. An ϵ-*extension* b_i^ϵ *of* b_i for some $\epsilon \in (0,1)$ replaces every p_i with p_i^I and p_i^{II} for $\hat{v} = v_i^{m_i-1}(p_i)$ and $\xi = \epsilon$. Specifically, whenever b_i draws $p_i \in \mathcal{P}_i$, b_i^ϵ instead draws $p_i' = p_i^I$ with probability $\theta(p_i) = p_i(v_i^{m_i}(p_i))\epsilon$ and $p_i^\epsilon = p_i^{II}$ with probability $1-\theta(p_i)$. Intuitively, for any posterior p_i under b_i, b_i^ϵ perfectly reveals the valuation with probability ϵ if the valuation is the highest possible one under p_i, which is represented by posterior p_i'. Otherwise, posterior p_i^ϵ results where all virtual valuations except for the highest one are strictly greater than under p_i.

The following lemma shows that all ironed virtual valuations $H_i(v_i,p_i^\epsilon)$ (except for $H_i(v_i^{m_i},p_i^\epsilon) = v_i^{m_i}$) are strictly increasing in ϵ. Hence, in consequence of choosing an ϵ-extension b_i^ϵ instead of b_i, bidder i wins any tie that occurs under b_i, and he also wins when his ironed virtual valuation is zero and among the highest ones under b_i. This is true even in the limit when ϵ approaches zero and the cost in terms of a lower information rent vanishes. That is, bidder i's allocation probabilities and thus his payoff are in general not continuous in ϵ at $\epsilon = 0$. The payoff in the limit when ϵ approaches zero can be expressed using the following definition. For $\mathbf{p} \in \mathcal{P}$ and $\mathbf{v} \in V(\mathbf{p})$, let

$$\hat{W}_0(\mathbf{v},\mathbf{p}) = \left\{ i \in N \,|\, H_i(v_i,p_i) \geq 0, i \in \operatorname*{argmax}_j H_j(v_j,p_j), v_i \in V_i(p_i)\backslash\{v_i^{m_i}\} \right\} \tag{10}$$

be the set of bidders i whose ironed virtual valuations are weakly positive, the highest among all bidders, and not their highest possible one at p_i.

LEMMA 2:

(*a*) *For every* $i \in N$, *every* $p_i \in \mathcal{P}_i$, *and every* $v_i \in V_i(p_i)\backslash\{v_i^{m_i}\}$, $H_i(v_i,p_i^\epsilon) > H_i(v_i,p_i)$ *and* $H_i(v_i,p_i^\epsilon)$ *is strictly increasing in* ϵ.

(*b*) *Let* f *be any optimal strategy for the auctioneer. For every* $i \in N$ *and every* $\mathbf{b} \in B$,

$$\lim_{\epsilon \to 0} U_i^f(b_i^\epsilon,\mathbf{b}_{-i}) = \int_{\mathcal{P}_i}\int_{\mathcal{P}_{-i}} \sum_{\mathbf{v}\in V(\mathbf{p})} \left[v_i - H_i(v_i,p_i)\right] \mathbf{1}_{i\in\hat{W}_0(\mathbf{v},\mathbf{p})}\, p(\mathbf{v})\, db_{-i}(\mathbf{p}_{-i})\, db_i(p_i).$$

Observe that $\mathbf{1}_{i\in\hat{W}_0(\mathbf{v},\mathbf{p})} \geq q_i^f(\mathbf{v},\mathbf{p})$ if $v_i < v_i^{m_i}$ and $v_i - H_i(v_i,p_i) = 0$ if $v_i = v_i^{m_i}$. Thus, $\lim_{\epsilon\to 0} U_i^f(b_i^\epsilon,\mathbf{b}_{-i}) \geq U_i^f(\mathbf{b})$: an increase of ϵ from zero results in bidder i winning any tie that can occur under $\mathbf{b}$ as well as winning when his and the highest ironed virtual valuation is zero. If such events have positive probability under $\mathbf{b}$, then by the continuity of ironed virtual valuations, an ϵ-extension b_i^ϵ with small enough ϵ (so that the cost of reducing the information rent vanishes) results in a strictly higher payoff than b_i.

A *δ-extension b_i^δ of b_i* differs from b_i only at posteriors where some ironed virtual valuations are strictly negative: it replaces every posterior p_i such that

$$(11) \quad m_i > 1 \qquad \text{and} \qquad H_i\big(v_i^k, p_i\big) < 0 \leq H_i\big(v_i^{k+1}, p_i\big) \quad \text{for some } k < m_i,$$

with p_i^I and p_i^{II} for $\hat{v} = v_i^k(p_i)$ and $\xi = \delta(p_i) \in (0,1)$. Specifically, whenever b_i draws such a p_i, b_i^δ instead draws $p_i'' = p_i^I$ with probability $\theta(p_i) = \big[1 - P_i\big(v_i^k(p_i)\big)\big]\delta(p_i)$ and $p_i^{\delta(p_i)} = p_i^{II}$ with probability $1 - \theta(p_i)$.

The following lemma shows that each $\delta(p_i)$ can be chosen such that the negative ironed virtual valuation at the cutoff $v_i^k(p_i)$ increases to zero under $p_i^{\delta(p_i)}$. Most importantly, such a δ-extension comes at no cost because all weakly positive ironed virtual valuations remain unchanged, and it thus allows bidder i to weakly increase his payoff.

LEMMA 3:

(*a*) *For every $i \in N$ and every $p_i \in \mathcal{P}_i$ that satisfies* (11), *there is a $\delta(p_i) \in (0,1)$ such that*

$$(12) \qquad H_i\big(v_i, p_i^{\delta(p_i)}\big) = \begin{cases} 0, & \text{if } v_i = v_i^k; \\ H_i(v_i, p_i), & \text{if } v_i \in \big\{v_i^{k+1}, \ldots, v_i^{m_i}\big\}. \end{cases}$$

Moreover, $H_i(v_i, p_i'') = H_i(v_i, p_i)$ for all $v_i \in V_i(p_i'')$.

(*b*) *Let f be any optimal strategy of the auctioneer, $i \in N$, and $\mathbf{b} \in B$. Let b_i^δ be such that for every $p_i \in \mathcal{P}_i$ that satisfies* (11), (12) *holds. Then, $U_i^f\big(b_i^\delta, \mathbf{b}_{-i}\big) \geq U_i^f(\mathbf{b})$.*

V. Existence of Subgame-Perfect Nash Equilibria

This section shows the existence of a subgame-perfect Nash equilibrium in the overall two-stage game of information disclosure and auction design. An SPNE consists of an optimal strategy f for the auctioneer and a profile of signal structures $\mathbf{b}$ that forms a Nash equilibrium in the disclosure game defined by f. Thus, we prove the existence of an SPNE by presenting a disclosure game that has a Nash equilibrium. For $\mathbf{p} \in \mathcal{P}$ and $\mathbf{v} \in V(\mathbf{p})$, let

$$W_0(\mathbf{v}, \mathbf{p}) = \Big\{ i \in N \,|\, H_i(v_i, p_i) \geq 0 \text{ and } i \in \underset{j}{\operatorname{argmax}}\, H_j(v_j, p_j) \Big\}$$

be the set of bidders whose ironed virtual valuation is weakly positive and the highest among all bidders. Consider the allocation rule $\mathbf{q}^h(\cdot, \mathbf{p})$ given by

$$(13) \quad q_i^h(\mathbf{v}, \mathbf{p}) = \begin{cases} 1/|\hat{W}_0(\mathbf{v}, \mathbf{p})|, & \text{if } i \in \hat{W}_0(\mathbf{v}, \mathbf{p}); \\ 1/|W_0(\mathbf{v}, \mathbf{p})|, & \text{if } i \in W_0(\mathbf{v}, \mathbf{p}) \text{ and } \hat{W}_0(\mathbf{v}, \mathbf{p}) = \emptyset; \\ 0, & \text{otherwise}, \end{cases}$$

where $\hat{W}_0(\mathbf{v},\mathbf{p})$ is the subset of bidders in $W_0(\mathbf{v},\mathbf{p})$ whose valuation is not their highest possible one, as defined in (10). Under $\mathbf{q}^h$, the object is allocated also when the highest ironed virtual valuation is zero. Moreover, ties are broken uniformly but not necessarily among all bidders in the tie. Specifically, if there is a tie at $(\mathbf{v},\mathbf{p})$ and the set of bidders $\hat{W}_0(\mathbf{v},\mathbf{p})$ is not empty, then the tie is broken only among the bidders in $\hat{W}_0(\mathbf{v},\mathbf{p})$.

LEMMA 4: *For every* $\mathbf{p} \in \mathcal{P}$, $\mathbf{q}^h(\cdot,\mathbf{p})$ *is an optimal allocation rule.*

Let $\mathbf{t}^h(\cdot,\mathbf{p})$ be an optimal transfer rule corresponding to $\mathbf{q}^h(\cdot,\mathbf{p})$, obtained from Proposition 1(b). We call the disclosure game defined by $h : \mathbf{p} \mapsto (\mathbf{q}^h(\cdot,\mathbf{p}),\mathbf{t}^h(\cdot,\mathbf{p}))$ the *hierarchical* disclosure game.[16] We will show that this game has a Nash equilibrium.

Denote the vector payoff function of the hierarchical disclosure game by $\mathbf{U}^h : B \to \mathbb{R}^n$, where $\mathbf{U}^h(\mathbf{b}) = (U_1^h(\mathbf{b}),\ldots,U_n^h(\mathbf{b}))$. The graph of $\mathbf{U}^h$ is the set $\{(\mathbf{b},\mathbf{y}) \in B \times \mathbb{R}^n \,|\, \mathbf{y} = \mathbf{U}^h(\mathbf{b})\}$. Endow each set B_i with the weak* topology and Cartesian products with the product topology. According to Reny (1999, theorem 3.1), the hierarchical disclosure game has a Nash equilibrium if it is *better-reply secure*, that is, if whenever $(\mathbf{b}^*,\mathbf{y}^*)$ is in the closure of the graph of the vector payoff function and $\mathbf{b}^*$ is not a Nash equilibrium, then there is a bidder i and a strategy $b_i \in B_i$ such that $U_i^h(b_i,\mathbf{b}_{-i}) > y_i^*$ for all $\mathbf{b}_{-i}$ in some open neighborhood of $\mathbf{b}_{-i}^*$.[17]

LEMMA 5: *The hierarchical disclosure game is better-reply secure. Hence, it has a Nash equilibrium.*

The proof uses the ϵ-extensions of signal structures introduced in Section IVB.[18] For illustration, suppose $(\mathbf{b}^*,\mathbf{y}^*)$ is in the closure of the graph of the vector payoff function, that is, there is a sequence of strategy profiles $(\mathbf{b}^l)$ such that $\lim_{l\to\infty}\mathbf{b}^l = \mathbf{b}^*$ and $\lim_{l\to\infty}\mathbf{U}^h(\mathbf{b}^l) = \mathbf{y}^*$. The key step is to show that if $y_i^* > U_i^h(\mathbf{b}^*)$ for some bidder i, then there is a bidder j and a strategy b_j such that $U_j^h(b_j,\mathbf{b}_{-j}^*) > y_j^*$. If $y_i^* > U_i^h(\mathbf{b}^*)$, we can infer (i) there are ties under $\mathbf{b}^*$ and bidder i wins with strictly higher probability in the limit of $(\mathbf{b}^l)$ than at $\mathbf{b}^*$ and (ii) i's valuation in the ties is not his highest possible one since payoffs do not depend on the allocation probability with the highest valuation (as $H_i(v_i^{m_i},p_i) = v_i^{m_i}$ by Lemma 1(a)). By (i), some bidder j reaches some of the same ties under $\mathbf{b}^*$ and wins with strictly *lower* probability in the limit of $(\mathbf{b}^l)$ than at $\mathbf{b}^*$. By (ii) and the construction of the allocation rules in the hierarchical disclosure game, j's valuation in the ties is also not his highest possible one, so he would benefit from a higher allocation probability. Now, through an ϵ-extension $b_j^{*\epsilon}$, j can raise his ironed virtual valuations and thus his allocation probability. The cost is that he reveals his highest possible valuation with probability ϵ, which reduces his information rent. But by

[16] Measurability of h follows from the continuity of $H_i(v_i,\cdot)$ shown in Lemma 1(b).

[17] Reny (1999, theorem 3.1) requires each B_i to be convex and compact and each function $U_i^h(\cdot,\mathbf{b}_{-i})$ to be quasi-concave for all $\mathbf{b}_{-i} \in B_{-i}$. Clearly, B_i is convex; B_i is compact because it is a closed subset of the (compact) space of all distributions on $\mathcal{P}_i$ endowed with the weak* topology. Quasi-concavity of $U_i^h(\cdot,\mathbf{b}_{-i})$ follows from linearity.

[18] It is related to Reny's (1999) proof that the multiunit first-price auction is better-reply secure.

choosing ϵ very small, this cost becomes negligible, and so $U_j^h(b_j^{*\epsilon}, \mathbf{b}_{-j}^*) > y_j^*$, as required for better-reply security.

We already pointed out that if there is a disclosure game that has a Nash equilibrium, then the overall two-stage game of information disclosure and auction design has an SPNE. Hence, the following proposition directly follows from Lemma 5.

PROPOSITION 2: *A subgame-perfect Nash equilibrium exists.*

VI. Information Disclosure and Expected Revenue

We now present our main result on the auctioneer's expected revenue in any subgame-perfect Nash equilibrium. We first show that in every Nash equilibrium of a disclosure game, with probability one some bidder's ironed virtual valuation is strictly positive.

LEMMA 6: *Suppose* $\mathbf{b}$ *is a Nash equilibrium of a disclosure game. Then, there is a bidder* $i \in N$ *such that*

$$b_i\Big(\big\{p_i \in \mathcal{P}_i \,|\, H_i(v_i, p_i) > 0 \;\forall v_i \in V_i(p_i)\big\}\Big) = 1. \tag{14}$$

The key idea behind this lemma is that competition in information disclosure erodes information rents such that keeping the object is strictly suboptimal for the auctioneer. The proof uses the ϵ- and δ-extensions introduced in Section IVB. For illustration, let f be any optimal strategy for the auctioneer and consider the corresponding disclosure game. Let $\mathbf{b}$ be a profile of signal structures that does not satisfy (14), that is, under which the ironed virtual valuation of every bidder can be weakly negative. Through a δ-extension b_i^δ, any bidder i can raise strictly negative ironed virtual valuations (if any) to zero without sacrificing any payoff. By performing an ϵ-extension of b_i^δ, bidder i can raise these ironed virtual valuations further to a value strictly above zero—with which he wins the auction whenever all other bidders happen to have a weakly negative ironed virtual valuation. The cost of the ϵ-extension is that, with probability ϵ, bidder i reveals his highest possible valuation, which reduces his information rent. But by choosing ϵ very small, this cost becomes negligible. Hence, b_i was not a best response against $\mathbf{b}_{-i}$.

We are now in the position to establish our main result, according to which information disclosure raises the auctioneer's expected revenue. As a benchmark, suppose the bidders cannot disclose information, so that the auctioneer designs the auction based on the prior beliefs. We refer to this as the *model without information disclosure*, as compared to our actual *model with information disclosure*. Suppose that for each bidder i, the prior $\bar{p}_i$ is such that in the model without information disclosure, the ironed virtual valuation for valuation $\bar{v}_i^1$ is weakly negative, $H_i(\bar{v}_i^1, \bar{p}_i) \leq 0$. Then by Corollary 1, in the model without disclosure there is an optimal auction $\big(\hat{\mathbf{q}}(\cdot, \bar{\mathbf{p}}), \hat{\mathbf{t}}(\cdot, \bar{\mathbf{p}})\big)$ that is dominant strategy incentive compatible and ex post individually rational and, with probability one, does not allocate the object when the profile of valuations $\big(\bar{v}_1^1, \ldots, \bar{v}_n^1\big)$ obtains.

Return now to our actual model with information disclosure. Clearly, the auction $\left(\hat{\mathbf{q}}(\cdot,\mathbf{p}),\hat{\mathbf{t}}(\cdot,\mathbf{p})\right)$ with $\hat{\mathbf{q}}(\mathbf{v},\mathbf{p}) = \hat{\mathbf{q}}(\mathbf{v},\bar{\mathbf{p}})$ and $\hat{\mathbf{t}}(\mathbf{v},\mathbf{p}) = \hat{\mathbf{t}}(\mathbf{v},\bar{\mathbf{p}})$ for each $\mathbf{v} \in V(\mathbf{p})$ is dominant strategy incentive compatible and ex post individually rational for every profile of posteriors $\mathbf{p} \in \mathcal{P}$.[19] Hence, it lies in the auctioneer's choice set at every $\mathbf{p}$. Moreover, if she uses it at every $\mathbf{p}$, she obtains the same expected revenue as in the model without information disclosure.[20] However, this auction is not optimal at profiles $\mathbf{p}$ where $\left(\bar{v}_1^1,\ldots,\bar{v}_n^1\right) \in V(\mathbf{p})$ and for at least one bidder all possible ironed virtual valuations are strictly positive: at such profiles, every optimal auction allocates the object with probability one by Proposition 1. Invoking Lemma 6, we conclude that in every subgame-perfect Nash equilibrium $\left(f^*,\mathbf{b}^*\right)$, the auctioneer's expected revenue satisfies

$$\begin{aligned}\int_{\mathbf{p}\in\mathcal{P}}\sum_{i\in N}\sum_{\mathbf{v}\in V(\mathbf{p})} t_i^{f^*}(\mathbf{v},\mathbf{p})p(\mathbf{v})\,db^*(\mathbf{p}) &> \int_{\mathbf{p}\in\mathcal{P}}\sum_{i\in N}\sum_{\mathbf{v}\in V(\mathbf{p})} \hat{t}_i(\mathbf{v},\mathbf{p})p(\mathbf{v})\,db^*(\mathbf{p})\\ &= \sum_{i\in N}\sum_{\mathbf{v}\in\bar{V}}\hat{t}_i(\mathbf{v},\bar{\mathbf{p}})\bar{p}(\mathbf{v}),\end{aligned}$$

where the equality follows from (1). This proves our main result.

THEOREM 1: *Suppose that for each bidder $i \in N$, the prior $\bar{p}_i$ is such that in the model without information disclosure, the ironed virtual valuation for valuation $\bar{v}_i^1$ is weakly negative, $H_i\left(\bar{v}_i^1,\bar{p}_i\right) \leq 0$. Then in every subgame-perfect Nash equilibrium $\left(f^*,\mathbf{b}^*\right)$ of the model with information disclosure, the auctioneer's expected revenue is strictly higher than in the model without information disclosure,*

$$\int_{\mathbf{p}\in\mathcal{P}}\sum_{i\in N}\sum_{\mathbf{v}\in V(\mathbf{p})} t_i^{f^*}(\mathbf{v},\mathbf{p})p(\mathbf{v})\,db^*(\mathbf{p}) > \sum_{i\in N}\sum_{\mathbf{v}\in\bar{V}}\hat{t}_i(\mathbf{v},\bar{\mathbf{p}})\bar{p}(\mathbf{v}).$$

VII. Examples of Equilibrium Information Disclosure

We showed that our two-stage game of information disclosure and auction design has a subgame-perfect Nash equilibrium, and we showed the key property that information disclosure raises the auctioneer's expected revenue in every equilibrium. In this section, we present equilibrium disclosure strategies for several special cases of the model.

Two Possible Valuations.—Consider a generalization of the setting of Section II: there are n bidders with arbitrary priors $\bar{\mathbf{p}}$ with the same support $\bar{V}_i = \left\{v^L,v^H\right\}$ for all $i \in N$.

Suppose bidder i perfectly discloses his valuation. That is, the signal structure b_i draws with probability $\bar{p}_i\left(v^L\right)$ the posterior p_i such that $V_i(p_i) = \left\{v^L\right\}$ and with probability $\bar{p}_i\left(v^H\right)$ the posterior p_i such that $V_i(p_i) = \left\{v^H\right\}$. Clearly, the payoff of bidder i is then zero. But also the payoff of every bidder $j \neq i$ is zero for *every*

[19] If $\left(\hat{\mathbf{q}}(\cdot,\bar{\mathbf{p}}),\hat{\mathbf{t}}(\cdot,\bar{\mathbf{p}})\right)$ is merely *Bayesian* incentive compatible and *interim* individually rational, these properties need not carry over to $\left(\hat{\mathbf{q}}(\cdot,\mathbf{p}),\hat{\mathbf{t}}(\cdot,\mathbf{p})\right)$, as they depend on the bidders' revised beliefs $\mathbf{p}$.

[20] Incidentally, this proves that the auctioneer cannot be worse off through information disclosure.

signal structure b_j: under every posterior p_j such that $p_j(v^L) \in (0,1)$, bidder j never wins the auction with valuation v^L since $H_j(v^L, p_j) < v^L \leq H_i(v_i, p_i)$, and thus his ex ante expected payoff (9) is zero. Intuitively, with binary valuations the auctioneer only leaves an information rent to bidder j if she needs to deter him from reporting v^L when his valuation is v^H. Yet if bidder i's valuation is known, the auctioneer prefers to always let i win when j reports v^L, such that there is no need for such deterrence.

Hence, if bidder i perfectly discloses, every signal structure b_j is a best response of every bidder $j \neq i$. This observation does not depend on which optimal auctions the auctioneer chooses. Accordingly, every profile of signal structures where at least two bidders perfectly disclose is a Nash equilibrium of every disclosure game.

PROPOSITION 3: *Suppose $\bar{V}_i = \{v^L, v^H\}$ for all $i \in N$. Then, $\mathbf{b}^*$ is a Nash equilibrium of every disclosure game if for at least two bidders $i \in N$, b_i^* draws with probability $\bar{p}_i(v^L)$ the posterior p_i' such that $V_i(p_i') = \{v^L\}$ and with probability $\bar{p}_i(v^H)$ the posterior p_i'' such that $V_i(p_i'') = \{v^H\}$. Moreover, the equilibrium payoff is $U_i(\mathbf{b}^*) = 0$ for all $i \in N$.*

In the online Appendix, we further show that there are no other Nash equilibria (see Proposition OA1). When there are only two bidders, both perfectly disclosing is therefore the unique Nash equilibrium of every disclosure game.

With at least two bidders perfectly disclosing, the auctioneer's revenue is equal to the first-best surplus $\sum_{\mathbf{v}\in\bar{V}}(\max_{i\in N} v_i)\bar{p}(\mathbf{v})$. This is always strictly greater than the revenue $\sum_{\mathbf{v}\in\bar{V}}\max\{0, \max_{i\in N} H_i(v_i, \bar{p}_i)\}\bar{p}(\mathbf{v})$ in the model without disclosure, where bidders may earn an information rent. For example, if $\bar{V}_i = \{1,4\}$ and $\bar{p}_i(1) = 3/4$ for $i \in N = \{1,2\}$, then the revenue is $[1-(3/4)^2]\cdot 4 + (3/4)^2 \cdot 1 = 37/16$ with disclosure and $[1-(3/4)^2]\cdot 4 = 28/16$ without disclosure. So the relative gain through disclosure is about 32 percent.

Two Symmetric Bidders and Three Possible Valuations.—Let there be two bidders with identical priors over three possible valuations. The following proposition identifies a symmetric strategy profile that is a Nash equilibrium of every disclosure game. Proposition OA2 in the online Appendix shows that there are no other Nash equilibria.

PROPOSITION 4: *Suppose $N = \{1,2\}$ and $\bar{V}_i = \{v^1, v^2, v^3\}$ with $\bar{p}_i(v^k) = \rho^k > 0$ for $i \in N$ and $k \in \{1,2,3\}$. Let $\underline{y} = (v^3 - v^2)/(v^3 - v^1)$. The following strategy profile (b_1^*, b_2^*) with $b_1^* = b_2^*$ is a Nash equilibrium of every disclosure game: under each b_i^*, posterior p_i such that $V_i(p_i) = \{v^1\}$ is drawn with probability ρ^1, posterior p_i such that $V_i(p_i) = \{v^3\}$ is drawn with probability*

$$\pi^3 = \max\left\{0, \rho^3 - \frac{1-\underline{y}}{\underline{y}}\rho^1 \ln\left[\frac{\rho^1+\rho^2}{\rho^1}\right]\right\} < \rho^3, \tag{15}$$

and with the remaining probability $(1-\rho^1-\pi^3)$ a posterior p_i such that $V_i(p_i) = \{v^2, v^3\}$ and $p_i(v^2) = 1 - p_i(v^3) = y$ is drawn from the continuous *distribution identified with probability density function ϕ of parameter y on $\left[\underline{y},\bar{y}\right]$, where*

$$\phi(y) = \frac{\kappa}{y(1-y)^2} \quad \text{with } \kappa = \frac{\rho^2}{1-\rho^1-\pi^3}\left(\frac{\bar{y}}{1-\bar{y}} - \frac{\underline{y}}{1-\underline{y}}\right)^{-1} \text{ for } y \in \left[\underline{y},\bar{y}\right],$$

and $\bar{y} \in \left(\underline{y},1\right)$ uniquely solves

$$\frac{\rho^3-\pi^3}{\rho^2} = \ln\left[\frac{\bar{y}}{1-\bar{y}}\frac{1-\underline{y}}{\underline{y}}\right]\left(\frac{\bar{y}}{1-\bar{y}} - \frac{\underline{y}}{1-\underline{y}}\right)^{-1}. \tag{16}$$

Moreover, if $\pi^3 > 0$, then the equilibrium payoff is $U_i^f\left(b_1^*,b_2^*\right) = \left(v^2-v^1\right)\rho^1\rho^2$.

In contrast to the case of binary valuations, perfect disclosure is not an equilibrium when there are three possible valuations. Indeed, by retaining some private information, bidder i can earn an information rent if i's competitor perfectly discloses his valuation and this happens to be v^1. Under the equilibrium signal structure b_i^*, bidder i perfectly reveals if he has valuation v^1 but retains some private information regarding whether v^2 or v^3 realized. Specifically, the support of b_i^* contains a continuum of posteriors p_i on $V_i\left(p_i\right) = \left\{v^2,v^3\right\}$ with $p_i\left(v^2\right) \in \left[\underline{y},\bar{y}\right]$, which implies that the virtual valuation $H_i\left(v^2,p_i\right) \geq v^1$. If bidder i plays b_i^*, bidder $j \neq i$ thus faces an opponent whose virtual valuation is always at least v^1. Intuitively, it is thus a best response of bidder j to also perfectly reveal v^1 and to choose a signal structure such that $H_j\left(v^2,p_j\right) \geq v^1$ with positive probability. Moreover, as we show in the proof, the probability density ϕ is constructed in such a way that the distribution of virtual valuations $H_i\left(v^2,p_i\right)$ bidder j faces renders him indifferent between any signal structure that ensures $p_j\left(v^2\right) \in \left[\underline{y},\bar{y}\right]$ when his valuation is v^2 (possibly with the requirement that v^3 is never revealed, $\pi_3 = 0$). Accordingly, b_j^* is indeed a best response.

The payoffs $U_i^f\left(b_1^*,b_2^*\right)$ in the equilibrium of Proposition 4 are the same in every disclosure game: how ties are resolved in the auction if $v_1 = v_2 \in \left\{v^1,v^3\right\}$ does not affect the ex ante expected payoff (9), otherwise ties happen with probability zero, and all virtual valuations are strictly positive. According to Proposition OA2 in the online Appendix, $\left(b_1^*,b_2^*\right)$ is the unique equilibrium of each disclosure game. Thus, the bidders' payoffs (like the revenue) are the same across all subgame-perfect Nash equilibria $\left(f^*,\mathbf{b}^*\right)$. Moreover, the equilibrium disclosure always strictly increases the virtual valuations of v^1 and v^2, resulting in a strictly higher revenue for the auctioneer than in the model without disclosure, independent of whether $H_i\left(\bar{v}_i^1,\bar{p}_i\right) \leq 0$. Further, the resulting allocation is ex post efficient and, unlike in the binary-valuations case, bidders always earn strictly positive payoffs from retaining some relevant private information. As we point out next, these payoffs can be both higher or lower than in the model without disclosure.

To consider a particularly tractable specification, let $\rho^1 = \rho^2 = \rho^3 = 1/3$, $v^1 = 1$, $v^2 = 2$, and $v^3 > 2 + \ln[2]$. Thus, $\pi^3 > 0$ and equilibrium payoffs take the simple form given in Proposition 4, resulting in $U_i^f\left(b_1^*,b_2^*\right) = 1/9$ independent of v^3. As the allocation is ex post efficient, the expected total surplus is $\left(5v^3+7\right)/9$, and the auctioneer's revenue therefore $\left(5v^3+5\right)/9$. In Figure 2, revenue and payoff are depicted by the solid lines in panels A and B, respectively. Next, consider the

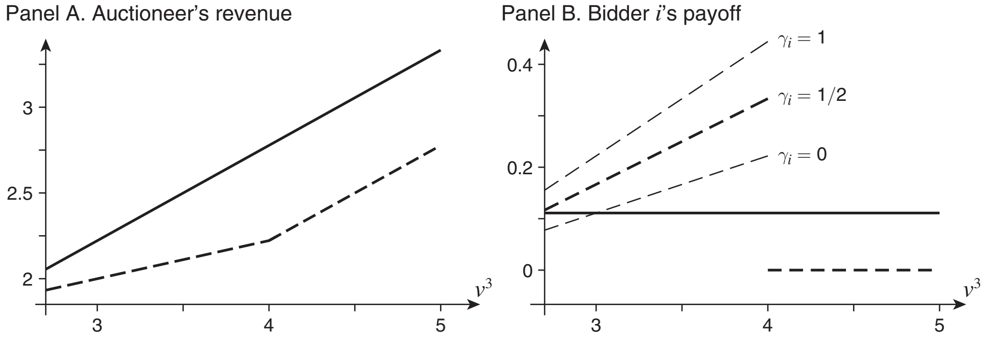

FIGURE 2. COMPARATIVE STATICS WITH RESPECT TO v^3

Notes: The figure shows comparative statics with respect to v^3 given $\rho^1 = \rho^2 = \rho^3 = 1/3$, $v^1 = 1$, and $v^2 = 2$. The solid lines indicate revenue and payoff, respectively, as a function of v^3 in the model with information disclosure. The dashed line segments indicate revenue and payoff (for three values of γ_i), respectively, in the benchmark without disclosure.

benchmark without disclosure, which corresponds to the dashed lines in Figure 2, and note that $J_i\left(v^1,\bar{p}_i\right) = -1$ and $J_i\left(v^2,\bar{p}_i\right) = 4 - v^3$. If $v^3 > 4$, the auctioneer sets a reserve price equal to v^3, resulting in revenue $5v^3/9$ and payoffs $u_i(\bar{\mathbf{p}}) = 0$. Hence, not only the auctioneer but also the bidders strictly benefit from the disclosure. If $v^3 < 4$, the auctioneer's revenue in the benchmark is $5v^3/9 + \left(4 - v^3\right)/3$, and payoffs are $u_i(\bar{\mathbf{p}}) = \left(v^3 - 2\right)\left(1 + \gamma_i\right)/9$, where γ_i is the probability that bidder i wins in case $v_i = v_j = 2$. If tie-breaking is symmetric or if $v^3 \in (3,4)$, both bidders are worse off under disclosure, whereas otherwise always at least one bidder is worse off. The relative gain in revenue through disclosure can be as high as 25 percent, which obtains for $v^3 = 4$.

Equilibrium Inefficiency and Diverging Benefits.—We now consider an example with two asymmetric bidders. The example will show that disclosure need not result in an efficient allocation of the object and that disclosure may benefit some bidders and at the same time harm others. Suppose

$$\bar{V}_1 = \{1, 7/4\} \quad \text{with } \bar{p}_1(1) = \bar{p}_1(7/4) = 1/2, \tag{17}$$

$$\bar{V}_2 = \{1, 2, 4\} \quad \text{with } \bar{p}_2(1) = \bar{p}_2(2) = \bar{p}_2(4) = 1/3.$$

Then, the virtual valuations are

$$J_1(1,\bar{p}_1) = 1/4, \qquad J_1(7/4,\bar{p}_1) = 7/4,$$

and

$$J_2(1,\bar{p}_2) = -1, \qquad J_2(2,\bar{p}_2) = 0, \qquad J_2(4,\bar{p}_2) = 4.$$

In the model without disclosure, bidder 2 wins the auction if his valuation is 4 and bidder 1 wins otherwise. Thus, only bidder 1 earns an information rent, resulting in payoffs $u_1(\bar{\mathbf{p}}) > 0$ and $u_2(\bar{\mathbf{p}}) = 0$.

Consider the hierarchical disclosure game (i.e., the allocation rule is $\mathbf{q}^h$ given in (13)). The following proposition identifies an equilibrium where bidder 1 perfectly discloses, whereas bidder 2 perfectly reveals valuation 1 and partially reveals valuation 4.[21]

PROPOSITION 5: *Let* $N = \{1,2\}$ *and assume* (17). *The following strategy profile* (b_1^*, b_2^*) *is a Nash equilibrium of the hierarchical disclosure game:* b_1^* *draws posteriors* p_1' *and* p_1'' *each with probability* $1/2$, *where* $V_1(p_1') = \{1\}$ *and* $V_1(p_1'') = \{7/4\}$; b_2^* *draws posterior* p_2', p_2'', *and* p_2''', *respectively, with probability* $1/3$, $1/6$, *and* $1/2$, *where* $V_2(p_2') = \{1\}$, $V_2(p_2'') = \{4\}$, *and* $V_2(p_2''') = \{2,4\}$ *with* $p_2'''(2) = 2/3$.

Clearly, bidder 1's equilibrium payoff is zero. Note that posterior p_2''' is such that $J_2(2, p_2''') = 1$. Under allocation rule $\mathbf{q}^h$, bidder 2 therefore wins the auction if his valuation is 2 and bidder 1's valuation is 1. It follows that bidder 2 obtains an information rent in the auction at posterior profile (p_1', p_2'''). We conclude that the equilibrium payoffs satisfy $U_1(b_1^*, b_2^*) = 0$ and $U_2(b_1^*, b_2^*) > 0$. Comparing these payoffs with those in the model without disclosure, we see that the benefits from disclosure diverge: bidder 1 is worse off and bidder 2 is better off. Moreover, an inefficient allocation may obtain in equilibrium (b_1^*, b_2^*) because in the auction at (p_1'', p_2'''), bidder 1 wins with valuation $7/4$ when bidder 2's valuation is 2.

VIII. Extension: Private Disclosure

So far, we have assumed that bidders publicly disclose information: whatever they reveal to the auctioneer before the auction is also observed by the other bidders. Depending on the context, however, bidders may prefer to privately persuade the auctioneer and minimize the amount of information that is revealed to their competitors. We now relax the assumption of public disclosure and show that our results still hold.

To keep the sequential structure of our model, we continue to assume that, once bidders have simultaneously chosen their signal structures, $\mathbf{b}$ is publicly observed. However, the signal that then realizes from signal structure b_i is no longer public, but is only (perfectly) observed by bidder i and the auctioneer, who, accordingly, update to posterior p_i. This means that the other bidders observe that bidder i communicates with the auctioneer (represented by b_i) but they do not learn the actual information he reveals (represented by p_i). As a consequence, the bidders and the auctioneer need no longer share common posteriors $\mathbf{p}$ about each valuation when the auction takes place.

[21] Lemma OA3 in the online Appendix implies that both bidders perfectly reveal their lowest valuation in every equilibrium of every bilateral disclosure game if the lowest valuations coincide. Using this fact, it is straightforward to extend the Proof of Proposition 5 to show that the equilibrium is unique.

To also cover arbitrary degrees of partially private disclosure, we allow for bidder i to partially learn what the others disclose. That is, while bidder i does not directly observe the realizations from the signal structures $\mathbf{b}_{-i}$ and the resulting posteriors $\mathbf{p}_{-i}$ of the auctioneer, he observes an exogenously fixed imperfect signal of these realizations, which is also observed by the auctioneer. This imperfect signal allows bidder i to update his belief about what the others have disclosed: it induces for bidder i a belief s_i about the posteriors $\mathbf{p}_{-i}$ of the auctioneer. The belief s_i is drawn from a set of possible beliefs S_i according to a distribution σ_i such that, to be consistent with the profile of signal structures $\mathbf{b}_{-i}$ of the other bidders, it holds that

$$\int_{\mathcal{A}_{-i}} db_{-i}(\mathbf{p}_{-i}) = \int_{S_i}\int_{\mathcal{A}_{-i}} ds_i(\mathbf{p}_{-i})\, d\sigma_i(s_i) \tag{18}$$

for any Borel set $\mathcal{A}_{-i} \subseteq \mathcal{P}_{-i}$. Let $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$. Note that fully private disclosure, where each bidder i learns nothing about the realizations $\mathbf{p}_{-i}$ from $\mathbf{b}_{-i}$, corresponds to the distribution σ_i that draws belief $s_i = b_{-i}$ with probability one.

After the chosen signal structures $\mathbf{b}$ have been observed, the auctioneer commits to an auction $(\mathbf{q}(\cdot,\mathbf{p}), \mathbf{t}(\cdot,\mathbf{p}))$ for each $\mathbf{p}$ that may realize from $\mathbf{b}$, that is, for each possible posterior she may hold after the bidders have disclosed information according to their signal structures.[22] She thus solves problem $[\mathbf{b}, \boldsymbol{\sigma}]$:

$$\max_{(\mathbf{q}(\cdot,\mathbf{p}),\mathbf{t}(\cdot,\mathbf{p}))_{\mathbf{p}\in\mathcal{P}}} \int_{\mathcal{P}} \sum_{i\in N} \sum_{\mathbf{v}\in V(\mathbf{p})} t_i(\mathbf{v},\mathbf{p}) p(\mathbf{v})\, db(\mathbf{p}) \tag{19}$$

subject to for each $i \in N$, $p_i \in \mathcal{P}_i$, and $s_i \in S_i$,

$$v_i \check{Q}_i(v_i, p_i, s_i) - \check{T}_i(v_i, p_i, s_i) \geq v_i \check{Q}_i(v_i', p_i, s_i) - \check{T}_i(v_i', p_i, s_i), \quad \forall v_i, v_i' \in V_i(p_i), \tag{20}$$

$$v_i \check{Q}_i(v_i, p_i, s_i) - \check{T}_i(v_i, p_i, s_i) \geq 0, \quad \forall v_i \in V_i(p_i), \tag{21}$$

where $\check{Q}_i(v_i, p_i, s_i) = \int_{\mathcal{P}_{-i}} Q_i(v_i, \mathbf{p})\, ds_i(\mathbf{p}_{-i})$ and $\check{T}_i(v_i, p_i, s_i) = \int_{\mathcal{P}_{-i}} T_i(v_i, \mathbf{p})\, ds_i(\mathbf{p}_{-i})$. Constraint (20) ensures Bayesian incentive compatibility and (21) interim individual rationality. These constraints are weaker than their counterparts with respect to $Q_i(\cdot,\mathbf{p})$ and $T_i(\cdot,\mathbf{p})$ in Section III because (4) and (5) for each $\mathbf{p}$ imply (20) and (21). Hence, the auctioneer cannot do worse than under public disclosure. The following result shows that she can also not do better.

PROPOSITION 6: *Let f be any optimal strategy for the auctioneer under public disclosure; that is, $(\mathbf{q}^f(\cdot,\mathbf{p}), \mathbf{t}^f(\cdot,\mathbf{p}))$ is an optimal auction satisfying Proposition 1(b) for each $\mathbf{p} \in \mathcal{P}$. Then $(\mathbf{q}^f(\cdot,\mathbf{p}), \mathbf{t}^f(\cdot,\mathbf{p}))_{\mathbf{p}\in\mathcal{P}}$ solves problem $[\mathbf{b}, \boldsymbol{\sigma}]$.*

Proposition 6 shows that independent of what the other bidders learn about the information bidder i discloses according to b_i (i.e., independent of $\mathbf{b}$ and $\boldsymbol{\sigma}$), it is optimal for the auctioneer to use the same auctions as under public disclosure. What

[22] The commitment assumption is not crucial, as we argue at the end of this section.

primarily matters for the auctioneer is the information rent she has to grant to each bidder i because he privately knows his valuation. This information rent depends only on p_i, which is why the auctioneer cannot exploit her superior information about the other bidders when dealing with bidder i.

As by Proposition 6 any information other than $\mathbf{p}$ is irrelevant for the auctioneer's design problem, we may again focus on a two-stage game where the auctioneer's strategy is a function f of just $\mathbf{p}$ (cf. Section I). Then, any optimal strategy f gives rise to a disclosure game to which all results in the preceding sections apply. In particular, Theorem 1 extends to any equilibrium $\left(f^*, \mathbf{b}^*\right)$ of the two-stage game with private disclosure.

Note that we have assumed that the auctioneer can commit to auctions before knowing $\mathbf{p}$. Dropping this assumption, we obtain an informed principal problem, as the choice of auction may then convey information about $\mathbf{p}_{-i}$ to bidder i. According to Proposition 6, however, what bidder i learns about $\mathbf{p}_{-i}$ is irrelevant for the auctioneer.[23] Hence, our results also extend to the case without commitment.

IX. Conclusion

In optimal auction design, the auctioneer's problem is to sell the object at the highest possible price without knowing the bidders' valuations. To this end, she designs an auction mechanism that determines transfers as well as the allocation of the object depending on reports by the bidders about their valuations. Often, bidders can anticipate that they will take part in an auction and that this auction is yet to be designed. For example, governments announce long in advance their intention to sell public assets such as electromagnetic spectrum. We augmented a standard model of optimal auction design by a prior stage in which the bidders can disclose information about their valuations to the auctioneer. We showed that, quite generally, the anticipation of optimally designed auctions gives bidders an incentive to disclose valuable information. Importantly, this incentive arises automatically without the auctioneer setting rules for information disclosure or making any commitments at the disclosure stage. Our result suggests that auctioneers may benefit from announcing plans to hold an auction early and being responsive to information disclosure.

Appendix A. Proofs

PROOF OF PROPOSITION 1:

(a) To characterize the optimal allocation rules, that is, the solutions to problem $[\mathbf{P}]$, we first show that for every allocation rule $\mathbf{q}$ that satisfies the monotonicity constraint (8), it holds that

$$\text{(A1)} \qquad \sum_{i \in N} \sum_{v_i \in V_i} J_i(v_i)\, Q_i(v_i)\, p_i(v_i) \;\leq\; \sum_{i \in N} \sum_{v_i \in V_i} H_i(v_i)\, Q_i(v_i)\, p_i(v_i).$$

[23] A closely related information irrelevance result is established in Skreta (2011) for the informed principal problem of an auctioneer who faces bidders with valuations drawn from continuous distributions that satisfy the regularity condition of Myerson (1981).

By (7), the left-hand side of (A1) can be written as $\sum_{i\in N}\hat{T}_i$, where for each i,

$$\hat{T}_i = \sum_{k=1}^{m_i} J_i(v_i^k)\,Q_i(v_i^k)\,p_i(v_i^k)$$

$$= \sum_{k=1}^{m_i}\left[G_i(v_i^k) - G_i(v_i^{k-1})\right]Q_i(v_i^k)$$

$$= -\sum_{k=1}^{m_i-1} G_i(v_i^k)\left[Q_i(v_i^{k+1}) - Q_i(v_i^k)\right] - G_i(v_i^0)\,Q_i(v_i^1) + G_i(v_i^{m_i})\,Q_i(v_i^{m_i}).$$

Analogously, the right-hand side of (A1) can be written as $\sum_{i\in N}\tilde{T}_i$, where for each i,

$$\tilde{T}_i = \sum_{k=1}^{m_i} H_i(v_i^k)\,Q_i(v_i^k)\,p_i(v_i^k)$$

$$= -\sum_{k=1}^{m_i-1} C_i\big(P_i(v_i^k)\big)\left[Q_i(v_i^{k+1}) - Q_i(v_i^k)\right] - C_i\big(P_i(v_i^0)\big)\,Q_i(v_i^1)$$

$$+ C_i\big(P_i(v_i^{m_i})\big)\,Q_i(v_i^{m_i}).$$

Then,

$$\text{(A2)}\quad \tilde{T}_i - \hat{T}_i = -\sum_{k=1}^{m_i-1}\left[C_i\big(P_i(v_i^k)\big) - G_i(v_i^k)\right]\left[Q_i(v_i^{k+1}) - Q_i(v_i^k)\right]$$

$$-\left[C_i\big(P_i(v_i^0)\big) - G_i(v_i^0)\right]Q_i(v_i^1) + \left[C_i\big(P_i(v_i^{m_i})\big) - G_i(v_i^{m_i})\right]Q_i(v_i^{m_i}).$$

Since $C_i\big(P_i(v_i^0)\big) = G_i(v_i^0) = 0$ and $C_i\big(P_i(v_i^{m_i})\big) = G_i(v_i^{m_i})$ and, by definition of C_i, $C_i\big(P_i(v_i^k)\big) \le G_i(v_i^k)$, we have $\tilde{T}_i - \hat{T}_i \ge 0$. This proves (A1).

Secondly, an allocation rule $\mathbf{q}$ maximizes $\sum_{i\in N}\sum_{v_i\in V_i} H_i(v_i)\,Q_i(v_i)\,p_i(v_i)$ if and only if

$$\text{(A3)}\quad \forall \mathbf{v} \in V\colon\quad \sum_{i\in W(\mathbf{v})} q_i(\mathbf{v}) = 1 \text{ if } W(\mathbf{v}) \neq \emptyset \qquad \text{and} \qquad \sum_{i\in L(\mathbf{v})} q_i(\mathbf{v}) = 0.$$

Thirdly, if $\mathbf{q}$ satisfies the monotonicity constraint (8), then by (A2) the equality

$$\sum_{i\in N}\sum_{v_i\in V_i} H_i(v_i)\,Q_i(v_i)\,p_i(v_i) = \sum_{i\in N}\sum_{v_i\in V_i} J_i(v_i)\,Q_i(v_i)\,p_i(v_i)$$

holds if and only if

$$\text{(A4)}\quad \forall i \in N, \forall k < m_i\colon\qquad C_i\big(P_i(v_i^k)\big) < G_i(v_i^k) \Rightarrow Q_i(v_i^k) = Q_i(v_i^{k+1}).$$

We note that if $C_i\left(P_i\left(v_i^k\right)\right) < G_i\left(v_i^k\right)$, then the slope of C_i does not change at $P_i\left(v_i^k\right)$, so that

$$\text{(A5)} \quad \forall i \in N, \forall k < m_i: \qquad C_i\left(P_i\left(v_i^k\right)\right) < G_i\left(v_i^k\right) \Rightarrow H_i\left(v_i^k\right) = H_i\left(v_i^{k+1}\right).$$

Taken together, these three steps imply that if there are allocation rules that satisfy (A3), (A4), and (8), then these are the optimal allocation rules. To see that such allocation rules exist, consider, for example, $\hat{\mathbf{q}}$ given by

$$\text{(A6)} \qquad \hat{q}_i(\mathbf{v}) = \begin{cases} 1/|W(\mathbf{v})|, & \text{if } i \in W(\mathbf{v}); \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\hat{\mathbf{q}}$ satisfies (A3) and (8). Moreover, using (A5), (A4) holds because if $H_i\left(v_i^k\right) = H_i\left(v_i^{k+1}\right)$, then $\hat{Q}_i\left(v_i^k\right) = \hat{Q}_i\left(v_i^{k+1}\right)$. Thus, (A3), (A4), and (8) characterize the optimal allocation rules. The conditions stated in Proposition 1(a) are equivalent because if $H_i\left(v_i^k\right) < H_i\left(v_i^{k+1}\right)$, then (A3) implies $Q_i\left(v_i^k\right) \leq Q_i\left(v_i^{k+1}\right)$.

(b) This is immediate.

(c) By (7),

$$\sum_{\mathbf{v} \in V}\left[v_i q_i(\mathbf{v}) - t_i(\mathbf{v})\right] p(\mathbf{v}) = \sum_{\mathbf{v} \in V}\left[v_i - J_i(v_i)\right] q_i(v) p(\mathbf{v}).$$

By part (a), an optimal allocation rule satisfies (A4), which implies that the difference $\tilde{T}_i - \hat{T}_i$ defined in (A2) is zero and thus

$$\sum_{\mathbf{v} \in V} J_i(v_i)\, q_i(v) p(\mathbf{v}) = \sum_{\mathbf{v} \in V} H_i(v_i)\, q_i(v) p(\mathbf{v}). \blacksquare$$

PROOF OF COROLLARY 1:

Consider the optimal allocation rule $\hat{\mathbf{q}}$ given by (A6) in the Proof of Proposition 1. Note that this $\hat{\mathbf{q}}$ satisfies

$$\text{(A7)} \qquad \hat{q}_i\left(v_i^k, \mathbf{v}_{-i}\right) \leq \hat{q}_i\left(v_i^{k+1}, \mathbf{v}_{-i}\right) \qquad \forall i \in N, \forall k < m_i, \forall \mathbf{v}_{-i} \in V_{-i}.$$

Consider the transfer rule $\hat{\mathbf{t}}$ given by

$$\hat{t}_i\left(v_i^k, \mathbf{v}_{-i}\right) = v_i^k \hat{q}_i\left(v_i^k, \mathbf{v}_{-i}\right) - \sum_{l=1}^{k-1}\left(v_i^{l+1} - v_i^l\right) \hat{q}_i\left(v_i^l, \mathbf{v}_{-i}\right).$$

Taking expectations over $\mathbf{v}_{-i}$, $\hat{\mathbf{t}}$ clearly satisfies (6). Hence, $\left(\hat{\mathbf{q}}, \hat{\mathbf{t}}\right)$ is an optimal auction. Moreover, $\left(\hat{\mathbf{q}}, \hat{\mathbf{t}}\right)$ is ex post individually rational since

$$v_i^k \hat{q}_i\left(v_i^k, \mathbf{v}_{-i}\right) - \hat{t}_i\left(v_i^k, \mathbf{v}_{-i}\right) = \sum_{l=1}^{k-1}\left(v_i^{l+1} - v_i^l\right) \hat{q}_i\left(v_i^l, \mathbf{v}_{-i}\right) \geq 0$$

for each i, k, and $\mathbf{v}_{-i}$. To see that $(\hat{\mathbf{q}}, \hat{\mathbf{t}})$ is dominant strategy incentive compatible, note that the gain in ex post payoff of bidder i with valuation v_i^k from reporting any v_i^h instead is

$$v_i^k \hat{q}_i(v_i^h, \mathbf{v}_{-i}) - \hat{t}_i(v_i^h, \mathbf{v}_{-i}) - \sum_{l=1}^{k-1} (v_i^{l+1} - v_i^l)\, \hat{q}_i(v_i^l, \mathbf{v}_{-i})$$

$$= \begin{cases} \sum_{l=k}^{h-1} (v_i^{l+1} - v_i^l)\left[\hat{q}_i(v_i^l, \mathbf{v}_{-i}) - \hat{q}_i(v_i^h, \mathbf{v}_{-i})\right] \le 0, & \text{if } h > k; \\ \sum_{l=h}^{k-1} (v_i^{l+1} - v_i^l)\left[\hat{q}_i(v_i^h, \mathbf{v}_{-i}) - \hat{q}_i(v_i^l, \mathbf{v}_{-i})\right] \le 0, & \text{if } h < k, \end{cases}$$

where the inequalities follow from (A7). ∎

PROOF OF LEMMA 1:

(a) As the posterior p_i remains fixed in this part of the proof, we omit the dependence on p_i in the notation as in Section III.

Let $k < m_i$. Recall that

$$H_i(v_i^k) = \frac{C_i\big(P_i(v_i^k)\big) - C_i\big(P_i(v_i^{k-1})\big)}{p_i(v_i^k)}$$

is the slope between $P_i(v_i^{k-1})$ and $P_i(v_i^k)$ of the lower convex envelope C_i of the function that assigns to each $P_i(v_i)$, $v_i \in V_i$, the value $G_i(v_i)$. Hence, there are l, l' with $l < k \le l'$ such that $C_i\big(P_i(v_i^l)\big) = G_i(v_i^l)$, $C_i\big(P_i(v_i^{l'})\big) = G_i(v_i^{l'})$, and

$$H_i(v_i^k) = \frac{G_i(v_i^{l'}) - G_i(v_i^l)}{P_i(v_i^{l'}) - P_i(v_i^l)}.$$

Since $C_i\big(P_i(v_i^{l+1})\big) \le G_i(v_i^{l+1})$, we have

$$H_i(v_i^{l+1}) = \frac{C_i\big(P_i(v_i^{l+1})\big) - C_i\big(P_i(v_i^l)\big)}{p_i(v_i^{l+1})} \le \frac{G_i(v_i^{l+1}) - G_i(v_i^l)}{p_i(v_i^{l+1})} = J_i(v_i^{l+1}).$$

Since $H_i(v_i^k) = H_i(v_i^{l+1})$ and $J_i(v_i^{l+1}) < v_i^k$, it follows that $H_i(v_i^k) < v_i^k$.

Clearly, $C_i\big(P_i(v_i^{m_i})\big) = G_i(v_i^{m_i})$. By contradiction, suppose $C_i\big(P_i(v_i^{m_i-1})\big) < G_i(v_i^{m_i-1})$. That is, there is a $k < m_i - 1$ such that

$$C_i\big(P_i(v_i^{m_i-1})\big) = \alpha\, G_i(v_i^k) + (1-\alpha)\, G_i(v_i^{m_i}) < G_i(v_i^{m_i-1}),$$

where

$$\alpha P_i(v_i^k) + (1-\alpha)\, P_i(v_i^{m_i}) = P_i(v_i^{m_i-1}) \Leftrightarrow \alpha = \frac{p_i(v_i^{m_i})}{1 - P_i(v_i^k)}.$$

This leads to a contradiction because

$$\begin{aligned} C_i\Big(P_i\big(v_i^{m_i-1}\big)\Big) &= G_i\big(v_i^{m_i}\big) - \frac{G_i\big(v_i^{m_i}\big) - G_i\big(v_i^k\big)}{1 - P_i\big(v_i^k\big)} p_i\big(v_i^{m_i}\big) \\ &= G_i\big(v_i^{m_i}\big) - \frac{\sum_{l=1}^{m_i} J_i\big(v_i^l\big)\, p_i\big(v_i^l\big) - \sum_{l=1}^{k} J_i\big(v_i^l\big)\, p_i\big(v_i^l\big)}{1 - P_i\big(v_i^k\big)} p_i\big(v_i^{m_i}\big) \\ &= G_i\big(v_i^{m_i}\big) - \frac{\sum_{l=k+1}^{m_i} J_i\big(v_i^l\big)\, p_i\big(v_i^l\big)}{1 - P_i\big(v_i^k\big)} p_i\big(v_i^{m_i}\big) \\ &> G_i\big(v_i^{m_i}\big) - v_i^{m_i} p_i\big(v_i^{m_i}\big) = G_i\big(v_i^{m_i-1}\big), \end{aligned}$$

where the inequality follows from $v_i^{m_i} > J_i\big(v_i^l\big)$ for $l = k+1, \ldots, m_i - 1$. Hence, $C_i\Big(P_i\big(v_i^{m_i-1}\big)\Big) = G_i\big(v_i^{m_i-1}\big)$, and consequently, $H_i\big(v_i^{m_i}\big) = J_j\big(v_i^{m_i}\big) = v_i^{m_i}$.

(b) For this part of the proof, we need to accommodate the fact that the support of the posterior p_i can change as p_i changes. We therefore redefine the functions G_i and C_i with respect to the support $\bar{V}_i = \big\{\bar{v}_i^1, \ldots, \bar{v}_i^{\bar{m}_i}\big\}$ of the prior $\bar{p}_i$.

Suppose $\bar{m}_i > 1$, for otherwise there is nothing to prove. The notation $\tilde{G}_i\big(\bar{v}_i^0\big)$ will mean zero. For $k \in \big\{1, \ldots, \bar{m}_i\big\}$ and $p_i \in \mathcal{P}_i$, define

$$\tilde{G}_i\big(\bar{v}_i^k, p_i\big) = \tilde{G}_i\big(\bar{v}_i^{k-1}, p_i\big) + \begin{cases} p_i\big(\bar{v}_i^k\big)\, \bar{v}_i^k - \Big[1 - P_i\big(\bar{v}_i^k\big)\Big]\big(\bar{v}_i^{k+1} - \bar{v}_i^k\big), & \text{if } k < \bar{m}_i; \\ p_i\big(\bar{v}_i^k\big)\, \bar{v}_i^k, & \text{if } k = \bar{m}_i. \end{cases}$$

Define $\tilde{C}_i(\cdot, p_i) : [0,1] \to \mathbb{R}$ by

$$\tilde{C}_i(z, p_i) = \min_{0 \le k, l \le \bar{m}_i, \alpha \in [0,1]} \alpha\, \tilde{G}_i\big(\bar{v}_i^k, p_i\big) + (1-\alpha)\, \tilde{G}_i\big(\bar{v}_i^l, p_i\big)$$

subject to

$$\alpha P_i\big(\bar{v}_i^k\big) + (1-\alpha) P_i\big(\bar{v}_i^l\big) = z.$$

Note that $\tilde{C}_i(z, p_i) = C_i(z, p_i)$ for all $z \in [0,1]$. Consequently, for any $\bar{v}_i^k \in \bar{V}_i$ and any $p_i \in \mathcal{P}_i$ such that $p_i\big(\bar{v}_i^k\big) > 0$, we have

$$\frac{\tilde{C}_i\Big(P_i\big(\bar{v}_i^k\big), p_i\Big) - \tilde{C}_i\Big(P_i\big(\bar{v}_i^{k-1}\big), p_i\Big)}{p_i\big(\bar{v}_i^k\big)} = H_i\big(\bar{v}_i^k, p_i\big).$$

Since $\tilde{G}_i(v_i, \cdot)$ is continuous at p_i for each $v_i \in \bar{V}_i$, Berge's Maximum Theorem implies that $\tilde{C}_i(z, \cdot)$ is continuous at p_i for each $z \in [0,1]$, which implies the continuity of $H_i\big(\bar{v}_i^k, \cdot\big)$ at p_i. ∎

PROOF OF LEMMA 2:

(a) As in the Proof of Lemma 1, recall that $H_i(v_i^k, p_i)$ is the slope between $P_i(v_i^{k-1})$ and $P_i(v_i^k)$ of the lower convex envelope of the function that assigns to each $P_i(v_i)$, $v_i \in V_i(p_i)$, the value $G_i(v_i, p_i)$. Hence, for each $k = 1, \ldots, m_i$, there are l, l' such that $0 \leq l < k \leq l' \leq m_i$ and

$$H_i(v_i^k, p_i) = \frac{G_i(v_i^{l'}, p_i) - G_i(v_i^l, p_i)}{P_i(v_i^{l'}) - P_i(v_i^l)},$$

where

$$l \in \operatorname*{argmax}_{0 \leq l'' < k} \frac{G_i(v_i^{l'}, p_i) - G_i(v_i^{l''}, p_i)}{P_i(v_i^{l'}) - P_i(v_i^{l''})}, \tag{A8}$$

$$l' \in \operatorname*{argmin}_{k \leq l'' \leq m_i} \frac{G_i(v_i^{l''}, p_i) - G_i(v_i^l, p_i)}{P_i(v_i^{l''}) - P_i(v_i^l)}. \tag{A9}$$

By definition,

$$p_i^\epsilon(v_i) = \begin{cases} \frac{1}{1 - p_i(v_i^{m_i})\epsilon} p_i(v_i), & \text{if } v_i \in V_i(p_i) \backslash \{v_i^{m_i}\}; \\ \frac{1 - \epsilon}{1 - p_i(v_i^{m_i})\epsilon} p_i(v_i^{m_i}), & \text{if } v_i = v_i^{m_i}. \end{cases}$$

For any $k < m_i$, we have $P_i^\epsilon(v_i^k) = P_i(v_i^k) / [1 - p_i(v_i^{m_i})\epsilon]$ and

$$J_i(v_i^k, p_i^\epsilon) = v_i^k - \frac{1 - P_i(v_i^k) - \epsilon p_i(v_i^{m_i})}{p_i(v_i^k)} (v_i^{k+1} - v_i^k).$$

For any $\epsilon'' > \epsilon' \geq 0$ and any $v_i' < v_i'' < v_i^{m_i}$, it follows that

$$\frac{G_i(v_i'', p_i^{\epsilon''}) - G_i(v_i', p_i^{\epsilon''})}{P_i^{\epsilon''}(v_i'') - P_i^{\epsilon''}(v_i')} > \frac{G_i(v_i'', p_i^{\epsilon'}) - G_i(v_i', p_i^{\epsilon'})}{P_i^{\epsilon'}(v_i'') - P_i^{\epsilon'}(v_i')}. \tag{A10}$$

Consider any $v_i \in V_i(p_i) \backslash \{v_i^{m_i}\}$. For $\epsilon'' > \epsilon' \geq 0$, let $\hat{v}'' \geq v_i > \hat{v}'$ and $v'' \geq v_i > v'$ with

$$H_i(v_i, p_i^{\epsilon''}) = \frac{G_i(\hat{v}_i'', p_i^{\epsilon''}) - G_i(\hat{v}_i', p_i^{\epsilon''})}{P_i^{\epsilon''}(\hat{v}_i'') - P_i^{\epsilon''}(\hat{v}_i')}$$

and

$$H_i(v_i, p_i^{\epsilon'}) = \frac{G_i(v_i'', p_i^{\epsilon'}) - G_i(v_i', p_i^{\epsilon'})}{P_i^{\epsilon'}(v_i'') - P_i^{\epsilon'}(v_i')}.$$

Then

$$H_i\big(v_i,p_i^{\epsilon''}\big) \;\geq\; \frac{G_i\big(\hat{v}_i'',p_i^{\epsilon''}\big) - G_i\big(v_i',p_i^{\epsilon''}\big)}{P_i^{\epsilon''}\big(\hat{v}_i''\big) - P_i^{\epsilon''}\big(v_i'\big)} \;>\; \frac{G_i\big(\hat{v}_i'',p_i^{\epsilon'}\big) - G_i\big(v_i',p_i^{\epsilon'}\big)}{P_i^{\epsilon'}\big(\hat{v}_i''\big) - P_i^{\epsilon'}\big(v_i'\big)} \;\geq\; H_i\big(v_i,p_i^{\epsilon'}\big),$$

where the first inequality follows from (A8), the second one from (A10), and the third one from (A9). Hence, $H_i\big(v_i,p_i^{\epsilon}\big) > H_i\big(v_i,p_i\big)$ and $H_i\big(v_i,p_i^{\epsilon}\big)$ is strictly increasing in ϵ.

(b) Consider the ex ante expected payoff of bidder i in the optimal auction at $\big(p_i^{\epsilon},\mathbf{p}_{-i}\big)$,

$$u_i^f\big(p_i^{\epsilon},\mathbf{p}_{-i}\big) \;=\; \sum_{\mathbf{v}\in V(\mathbf{p})}\Big[v_i - H_i\big(v_i,p_i^{\epsilon}\big)\Big]\,q_i\Big(\mathbf{v},\big(p_i^{\epsilon},\mathbf{p}_{-i}\big)\Big)\,p_{-i}\big(\mathbf{v}_{-i}\big)\,p_i^{\epsilon}\big(v_i\big).$$

By the continuity of ironed virtual valuations shown in Lemma 1(b), $\lim_{\epsilon\to 0}H_i\big(v_i,p_i^{\epsilon}\big) = H_i\big(v_i,p_i\big)$. For all $v_i \in V_i\big(p_i\big)\backslash\big\{v_i^{m_i}\big\}$,

$$\lim_{\epsilon\to 0} q_i^f\Big(\mathbf{v},\big(p_i^{\epsilon},\mathbf{p}_{-i}\big)\Big) \;=\; \begin{cases} 1, & \text{if } i \in \hat{W}_0\big(\mathbf{v},\mathbf{p}\big); \\ 0, & \text{otherwise,} \end{cases}$$

by Proposition 1(a) because $H_i\big(v_i,p_i^{\epsilon}\big) > H_i\big(v_i,p_i\big)$ for all ϵ. Finally, $\lim_{\epsilon\to 0}p_i^{\epsilon} = p_i$. Hence, using that $v_i^{m_i} - H_i\big(v_i^{m_i},p_i\big) = 0$ by Lemma 1(a),

$$\lim_{\epsilon\to 0} u_i^f\big(p_i^{\epsilon},\mathbf{p}_{-i}\big) \;=\; \sum_{\mathbf{v}\in V(\mathbf{p})}\Big[v_i - H_i\big(v_i,p_i\big)\Big]\,\mathbf{1}_{i\in\hat{W}_0(\mathbf{v},\mathbf{p})}\,p\big(\mathbf{v}\big).$$

Since

$$U_i^f\big(b_i^{\epsilon},\mathbf{b}_{-i}\big) \;=\; \int_{\mathcal{P}_i}\Big[1 - p_i\big(v_i^{m_i}\big)\epsilon\Big]\int_{\mathcal{P}_{-i}} u_i^f\big(p_i^{\epsilon},\mathbf{p}_{-i}\big)\,db_{-i}\big(\mathbf{p}_{-i}\big)\,db_i\big(p_i\big),$$

the Dominated Convergence Theorem gives the desired equation

$$\lim_{\epsilon\to 0} U_i^f\big(b_i^{\epsilon},\mathbf{b}_{-i}\big) \;=\; \int_{\mathcal{P}_i}\int_{\mathcal{P}_{-i}}\sum_{\mathbf{v}\in V(\mathbf{p})}\Big[v_i - H_i\big(v_i,p_i\big)\Big]\,\mathbf{1}_{i\in\hat{W}_0(\mathbf{v},\mathbf{p})}\,p\big(\mathbf{v}\big)\,db_{-i}\big(\mathbf{p}_{-i}\big)\,db_i\big(p_i\big). \;\blacksquare$$

PROOF OF LEMMA 3:

(a) Suppose $p_i \in \mathcal{P}_i$ satisfies (11). Consider any $v_i \in V_i\big(p_i''\big)$. Since $H_i\big(v_i^k,p_i\big) < H_i\big(v_i^{k+1},p_i\big)$, we have $C_i\Big(P_i\big(v_i^k\big),p_i\Big) = G_i\big(v_i^k,p_i\big)$, which implies

$$\text{(A11)}\qquad C_i\Big(P_i\big(v_i\big),p_i\Big) \;=\; \min_{k\leq l,l'\leq m_i,\alpha\in[0,1]} \alpha\,G_i\big(v_i^l,p_i\big) + \big(1-\alpha\big)\,G_i\big(v_i^{l'},p_i\big)$$

subject to

$$\alpha\,P_i\big(v_i^l\big) + \big(1-\alpha\big)\,P_i\big(v_i^{l'}\big) \;=\; P_i\big(v_i\big).$$

Since $V_i\big(p_i''\big) = \big\{v_i^{k+1},\ldots,v_i^{m_i}\big\}$, we have

$$C_i\Big(P_i''\big(v_i\big),p_i''\Big) \;=\; \min_{k\leq l,l'\leq m_i,\alpha\in[0,1]} \alpha\,G_i\big(v_i^l,p_i''\big) + \big(1-\alpha\big)\,G_i\big(v_i^{l'},p_i''\big)$$

subject to

$$\alpha P_i''(v_i^l) + (1-\alpha) P_i''(v_i^{l'}) = P_i''(v_i).$$

By definition, for all $v_i \in V_i(p_i'')$

$$p_i''(v_i) = \frac{p_i(v_i)}{1 - P_i(v_i^k)},$$

which implies

$$G_i(v_i, p_i'') = \frac{1}{1 - P_i(v_i^k)}\left[G_i(v_i, p_i) - G_i(v_i^k, p_i)\right].$$

Hence,

$$C_i(P_i''(v_i), p_i'') = \frac{1}{1 - P_i(v_i^k)}\left[C_i(P_i(v_i), p_i) - G_i(v_i^k, p_i)\right],$$

and so $H_i(v_i, p_i'') = H_i(v_i, p_i)$.

For $\delta(p_i) = 0$, $H_i(v_i^k, p_i^{\delta(p_i)}) = H_i(v_i^k, p_i) < 0$, whereas for $\delta(p_i) = 1$, $H_i(v_i^k, p_i^{\delta(p_i)}) = v_i^k > 0$ by Lemma 1(a). Since ironed virtual valuations are continuous in posteriors by Lemma 1(b), it follows that there exists a $\delta(p_i) \in (0,1)$ such that $H_i(v_i^k, p_i^{\delta(p_i)}) = 0$. By definition,

$$p_i^{\delta(p_i)}(v_i) = \begin{cases} \dfrac{p_i(v_i)}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)}, & \text{if } v_i \le v_i^k; \\ \dfrac{\left[1 - \delta(p_i)\right] p_i(v_i)}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)}, & \text{if } v_i > v_i^k, \end{cases}$$

which implies, for any $v_i \in \{v_i^{k+1}, \dots, v_i^{m_i}\}$,

$$G_i(v_i, p_i^{\delta(p_i)}) = \frac{1}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)} G_i(v_i^k, p_i) + \frac{1 - \delta(p_i)}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)}\left[G_i(v_i, p_i) - G_i(v_i^k, p_i)\right].$$

Since $C_i(P_i(v_i^k), p_i) = G_i(v_i^k, p_i)$ and $C_i(\cdot, p_i^{\delta(p_i)})$ is a lower convex envelope, it holds that $G_i(v_i, p_i) \ge G_i(v_i^k, p_i) + H_i(v_i^{k+1}, p_i)\left[P_i(v_i) - P_i(v_i^k)\right]$. Thus,

$$\begin{aligned} G_i(v_i, p_i^{\delta(p_i)}) \ge\ & \frac{1}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)} G_i(v_i^k, p_i) \\ & + \frac{1 - \delta(p_i)}{1 - \left[1 - P_i(v_i^k)\right]\delta(p_i)} H_i(v_i^{k+1}, p_i)\left[P_i(v_i) - P_i(v_i^k)\right] \\ =\ & G_i(v_i^k, p_i^{\delta(p_i)}) + H_i(v_i^{k+1}, p_i)\left[P_i^{\delta(p_i)}(v_i) - P_i^{\delta(p_i)}(v_i^k)\right] \\ & \forall v_i \in \{v_i^{k+1}, \dots, v_i^{m_i}\}. \end{aligned}$$

Since $H_i\left(v_i^k, p_i^{\delta(p_i)}\right) = 0 \leq H_i\left(v_i^{k+1}, p_i\right)$ and $C_i\left(\cdot, p_i^{\delta(p_i)}\right)$ is a lower convex envelope, it follows that $C_i\left(P_i^{\delta(p_i)}\left(v_i^k\right), p_i^{\delta(p_i)}\right) = G_i\left(v_i^k, p_i^{\delta(p_i)}\right)$. Hence, analogously to (A11), we have for any $v_i \in \left\{v_i^{k+1}, \ldots, v_i^{m_i}\right\}$,

$$C_i\left(P_i^{\delta(p_i)}(v_i), p_i^{\delta(p_i)}\right)$$
$$= \min_{k \leq l, l' \leq m_i, \alpha \in [0,1]} \alpha\, G_i\left(v_i^l, p_i^{\delta(p_i)}\right) + (1-\alpha)\, G_i\left(v_i^{l'}, p_i^{\delta(p_i)}\right)$$

subject to

$$\alpha\, P_i^{\delta(p_i)}\left(v_i^l\right) + (1-\alpha)\, P_i^{\delta(p_i)}\left(v_i^{l'}\right) = P_i^{\delta(p_i)}(v_i).$$

Using the definitions of $p_i^{\delta(p_i)}(v_i)$ and $G_i\left(v_i, p_i^{\delta(p_i)}\right)$,

$$C_i\left(P_i^{\delta(p_i)}(v_i), p_i^{\delta(p_i)}(v_i)\right)$$
$$= \frac{1-\delta(p_i)}{1-\left[1-P_i\left(v_i^k\right)\right]\delta(p_i)} C_i\left(P_i(v_i), p_i\right) + \frac{\delta(p_i)}{1-\left[1-P_i\left(v_i^k\right)\right]\delta(p_i)} G_i\left(v_i^k, p_i\right).$$

Consequently, $H_i\left(v_i, p_i^{\delta(p_i)}\right) = H_i(v_i, p_i)$.

(b) Let $\hat{\mathcal{P}}_i = \left\{p_i \in \mathcal{P}_i \,|\, (11) \text{ holds}\right\}$. We have

$$\begin{aligned} U_i^f\left(b_i^\delta, \mathbf{b}_{-i}\right) = & \int_{\mathcal{P}_i \setminus \hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p})} \left[v_i - H_i(v_i, p_i)\right] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v})\, db_{-i}(\mathbf{p}_{-i})\, db_i(p_i) \\ & + \int_{\hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i > v_i^k} \left[v_i - H_i(v_i, p_i)\right] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v})\, db_{-i}(\mathbf{p}_{-i})\, db_i(p_i) \\ & + \int_{\hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i = v_i^k} v_i\, q_i^f\left(\mathbf{v}, \left(p_i^{\delta(p_i)}, \mathbf{p}_{-i}\right)\right) p(\mathbf{v})\, db_{-i}(\mathbf{p}_{-i})\, db_i(p_i) \\ \geq & \; U_i^f(\mathbf{b}). \blacksquare \end{aligned}$$

PROOF OF LEMMA 4:

Let $\mathbf{p} \in \mathcal{P}$. Suppose $H_i(v_i, p_i) = H_i(v_i', p_i)$ for $v_i, v_i' \in V_i(p_i)$. We show that $Q_i^h(v_i, \mathbf{p}) = Q_i^h(v_i', \mathbf{p})$. By Proposition 1(a), this is all we need to show.

Since $H_i(v_i, p_i) = H_i(v_i', p_i)$, $W_0\left((v_i, \mathbf{v}_{-i}), \mathbf{p}\right) = W_0\left((v_i', \mathbf{v}_{-i}), \mathbf{p}\right)$ for all $\mathbf{v}_{-i}$. By Lemma 1(a), either $H_i(v_i, p_i) < v_i < v_i^{m_i}$ or $H_i(v_i, p_i) = v_i = v_i^{m_i}$. Hence, $v_i < v_i^{m_i}$ if and only if $v_i' < v_i^{m_i}$, which implies $\hat{W}_0\left((v_i, \mathbf{v}_{-i}), \mathbf{p}\right) = \hat{W}_0\left((v_i', \mathbf{v}_{-i}), \mathbf{p}\right)$ for all $\mathbf{v}_{-i}$. Consequently, $q_i^h\left((v_i, \mathbf{v}_{-i}), \mathbf{p}\right) = q_i^h\left((v_i', \mathbf{v}_{-i}), \mathbf{p}\right)$ for all $\mathbf{v}_{-i}$, and thus $Q_i^h(v_i, \mathbf{p}) = Q_i^h(v_i', \mathbf{p})$. $\blacksquare$

PROOF OF LEMMA 5:

We start with an auxiliary result.[24]

[24] Claim A1 holds for any disclosure game, not just the hierarchical one.

CLAIM A1: *For every* $\mathbf{b} \in B$, *every* $i \in N$, *and every* $\eta > 0$, *there exists a* $b_i' \in B_i$ *and an open neighborhood of* $\mathbf{b}_{-i}$ *such that* $U_i^h(b_i', \mathbf{b}'_{-i}) > U_i^h(\mathbf{b}) - \eta$ *for all* $\mathbf{b}'_{-i}$ *in the neighborhood.*

PROOF:

Let $\mathbf{b} \in B, i \in N, \eta > 0$. For $\epsilon \in (0,1)$, let $\hat{b}_i^\epsilon \in B_i$ be the distribution on $\mathcal{P}$ generated by first drawing an ϵ'-extension $b_i^{\epsilon'}$ of b_i from the uniform distribution over $\epsilon' \in [\epsilon/2, \epsilon]$ and then drawing a posterior $\hat{p}_i = p_i^{\epsilon'}$ or $\hat{p}_i = p_i'$ from $b_i^{\epsilon'}$. Since the ironed virtual valuations $H_i(v_i, p_i^{\epsilon'})$, $v_i \in V_i(p_i) \backslash \{v_i^{m_i}\}$, are strictly increasing in ϵ' by Lemma 2(a), for any $x \in \mathbb{R}$ posteriors $p_i^{\epsilon'}$ such that $H_i(v_i, p_i^{\epsilon'}) = x$ for $v_i \in V_i(p_i) \backslash \{v_i^{m_i}\}$ have probability zero under $\hat{b}_i^\epsilon$. (That is, ties occur with probability zero at valuations that matter for i's ex ante expected payoff in the auction.) Hence, the ex ante expected payoff, $u_i^h(\hat{p}_i, \cdot)$, is continuous in $\mathbf{p}_{-i}$ for $\hat{b}_i^\epsilon$-almost all $\hat{p}_i$, and so $\int_{\mathcal{P}_i} u_i^h(\hat{p}_i, \cdot)\, d\hat{b}_i^\epsilon(\hat{p}_i)$ is continuous in $\mathbf{p}_{-i}$ by the Dominated Convergence Theorem. Since $\int_{\mathcal{P}_i} u_i^h(\hat{p}_i, \cdot)\, d\hat{b}_i^\epsilon(\hat{p}_i)$ is also bounded, the definition of the weak* topology implies that, for every $\eta' > 0$, there is an open neighborhood of $\mathbf{b}_{-i}$ such that $U_i^h(\hat{b}_i^\epsilon, \mathbf{b}'_{-i}) > U_i(\hat{b}_i^\epsilon, \mathbf{b}_{-i}) - \eta'$ for all $\mathbf{b}'_{-i}$ in the neighborhood. Since $\lim_{\epsilon \to 0} U_i^h(\hat{b}_i^\epsilon, \mathbf{b}_{-i}) \geq U_i^h(\mathbf{b})$ by Lemma 2(b), we can choose ϵ small enough such that $U_i^h(\hat{b}_i^\epsilon, \mathbf{b}'_{-i}) > U_i^h(\mathbf{b}) - \eta$ for all $\mathbf{b}'_{-i}$ in an open neighborhood of $\mathbf{b}_{-i}$.

Let $(\mathbf{b}^*, \mathbf{y}^*)$ be in the closure of the graph of U^h. Since the weak* topology on B_i is metrizable (see Dudley 2002, theorem 11.3.3), there is a sequence $(\mathbf{b}^l)$ in B such that $\lim_{l \to \infty} \mathbf{b}^l = \mathbf{b}^*$ and $\lim_{l \to \infty} \mathbf{U}^h(\mathbf{b}^l) = \mathbf{y}^*$(see Dudley 2002, theorem 2.1.3). Suppose $\mathbf{b}^*$ is not a Nash equilibrium. To verify better-reply security, we must show that

(A12) there is a bidder i and a strategy $b_i \in B_i$ such that
$$U_i^h(b_i, \mathbf{b}_{-i}) > y_i^* \text{ for all } \mathbf{b}_{-i} \text{ in an open neighborhood of } \mathbf{b}^*_{-i}.$$

Suppose first that $U_i^h(\mathbf{b}^*) \geq y_i^*$ for all bidders $i \in N$. Then, there is a bidder i and a $b_i \in B_i$ such that $U_i^h(b_i, \mathbf{b}^*_{-i}) > U_i^h(\mathbf{b}^*) \geq y_i^*$ because $\mathbf{b}^*$ is not a Nash equilibrium. Hence, (A12) holds by Claim A1.

For the rest of the proof, define $\hat{u}_i(\mathbf{p}) = \sum_{\mathbf{v} \in V(\mathbf{p})} [v_i - H_i(v_i, p_i)]\, \mathbf{1}_{i \in \hat{W}_0(\mathbf{v}, \mathbf{p})} p(\mathbf{v})$. Let $\hat{U}_i(\mathbf{b}) = \int_{\mathcal{P}} \hat{u}_i(\mathbf{p})\, db(\mathbf{p})$. Note that

$$\hat{U}_i(\mathbf{b}) \geq U_i^h(\mathbf{b}), \quad \forall \mathbf{b} \in B. \tag{A13}$$

Moreover, since $\hat{u}_i$ is upper semicontinuous, the Portmanteau Theorem (see Bogachev 2007, corollary 8.2.5) implies

$$\hat{U}_i(\mathbf{b}^*) \geq \limsup_{l \to \infty} \hat{U}_i(\mathbf{b}^l). \tag{A14}$$

Now suppose that $y_i^* > U_i^h(\mathbf{b}^*)$ for some bidder i. Then, the set of discontinuity points of u_i^h has positive probability under $\mathbf{b}^*$ by the Portmanteau Theorem (see Klenke 2020, theorem 13.16). Thus, there is an $x \in \{1/n, \ldots, 1/2, 1\}$ and a $\mathbf{v} \in \bar{V}$

such that for $M_i = \{\mathbf{p} \in \mathcal{P} | q_i^h(\mathbf{v},\mathbf{p}) = x, p(\mathbf{v}) > 0, v_i \in V_i(p_i) \setminus \{v_i^{m_i}\}\}$, we have $b^*(M_i) > 0$. Indeed, we can assume this for $x \neq 1$, for otherwise $U_i^h(\mathbf{b}^*) = \hat{U}_i(\mathbf{b}_i^*)$, which contradicts $y_i^* > U_i^h(\mathbf{b}^*)$ since $\hat{U}_i(\mathbf{b}^*) \geq \limsup_{l\to\infty} \hat{U}_i(\mathbf{b}^l) \geq \limsup_{l\to\infty} U_i^h(\mathbf{b}^l) = y_i^*$ by (A14) and (A13). Since $x \neq 1$, there are bidders $i' \neq i$ such that $b^*(M_{i'}) > 0$. For $\eta > 0$, define $M_{i'}^\eta = \{\mathbf{p} \in \mathcal{P} | \exists \mathbf{p}' \in M_{i'}$ such that $||\mathbf{p} - \mathbf{p}'|| < \eta\}$. Since $M_{i'}^\eta$ is open, $\liminf_{l\to\infty} b^l(M_{i'}^\eta) \geq b^*(M_{i'}^\eta) \geq b^*(M_{i'}) > 0$ by the Portmanteau Theorem. Since $q_j^h(\mathbf{v},\mathbf{p}) = 1$ for at most one bidder $j \in N$, it follows that there is a bidder j such that

$$\text{(A15)} \quad \liminf_{l\to\infty} b^l\left(\{\mathbf{p} \in M_j^\eta | q_j^h(\mathbf{v},\mathbf{p}) < 1, p(\mathbf{v}) > 0, v_j \in V_j(p_j) \setminus \{v_j^{m_i}\}\}\right) > 0 \quad \forall \eta > 0.$$

Then $y_j^* < \limsup_{l\to\infty} \hat{U}_j(\mathbf{b}^l) \leq \hat{U}_j(\mathbf{b}^*) = \lim_{\epsilon\to 0} U_j^h(b_j^{*\epsilon}, \mathbf{b}_{-j}^*)$, where the first inequality follows from (A15), the second one from (A14), and the equality from Lemma 2(b). Thus, $U_j^h(b_j^{*\epsilon}, \mathbf{b}_{-j}^*) > y_j^*$ for small ϵ, and so (A12) holds by Claim A1.

Hence, the hierarchical disclosure game is better-reply secure. Thus, it has a Nash equilibrium by Reny (1999, theorem 3.1). ∎

PROOF OF PROPOSITION 2:

In the main text. ∎

PROOF OF LEMMA 6:

Let f be any optimal strategy for the auctioneer. By contradiction, suppose $\mathbf{b}$ is a Nash equilibrium of the disclosure game defined by f and (14) does not hold. That is,

$$\text{(A16)} \quad b_i\left(\{p_i \in \mathcal{P}_i | \exists v_i \in V_i(p_i) \text{ such that } H_i(v_i,p_i) \leq 0\}\right) > 0, \quad \forall i \in N.$$

Suppose first that for some bidder i, we have in addition $b_i\left(\{p_i \in \mathcal{P}_i | \exists v_i \in V_i(p_i)\right.$ such that $\left.H_i(v_i,p_i) < 0\}\right) > 0$. For every $p_i \in \mathcal{P}_i$ such that $H_i(v_i,p_i) < 0$ for some $v_i \in V_i(p_i)$, (11) holds. Consider a δ-extension b_i^δ of b_i. By Lemma 3, we can choose δ such that for all p_i that satisfy (11),

$$H_i\left(v_i, p_i^{\delta(p_i)}\right) = \begin{cases} 0, & \text{if } v_i = v_i^k; \\ H_i(v_i,p_i), & \text{if } v_i \in \{v_i^{k+1}, \ldots, v_i^m\}, \end{cases}$$

$H_i(v_i,p_i'') = H_i(v_i,p_i)$ for all $v_i \in V_i(p_i'')$, and $U_i^f(b_i^\delta, \mathbf{b}_{-i}) \geq U_i^f(\mathbf{b})$. Applying δ-extensions iteratively if necessary, we obtain a signal structure b_i' such that $b_i'\left(\{p_i \in \mathcal{P}_i | H_i(v_i,p_i) \geq 0 \ \forall v_i \in V_i(p_i)\}\right) = 1$ and $U_i^f(b_i', \mathbf{b}_{-i}) \geq U_i^f(\mathbf{b})$.

Thus, when examining whether b_i is a best response against $\mathbf{b}_{-i}$ for any bidder i, we can assume without loss of generality that $b_i\left(\{p_i \in \mathcal{P}_i | H_i(v_i,p_i) \geq 0 \ \forall v_i \in V_i(p_i)\}\right) = 1$. Consider an ϵ-extension b_i^ϵ of b_i. By Lemma 2(b),

$$\lim_{\epsilon\to 0} U_i^f(b_i^\epsilon, \mathbf{b}_{-i}) = \int_{\mathcal{P}_i}\int_{\mathcal{P}_{-i}} \sum_{\mathbf{v}\in V(\mathbf{p})} \left[v_i - H_i(v_i,p_i)\right] \mathbf{1}_{i\in \hat{W}_0(\mathbf{v},\mathbf{p})} p(\mathbf{v}) \, db_{-i}(\mathbf{p}_{-i}) \, db_i(p_i).$$

By (A16) and since $q_i(\mathbf{v},\mathbf{p}) = 1$ for at most one bidder $i \in N$, it follows that there are bidders i such that $\lim_{\epsilon\to 0} U_i^f(b_i^\epsilon, \mathbf{b}_{-i}) > U_i^f(\mathbf{b})$. Hence, for small ϵ, $U_i^f(b_i^\epsilon, \mathbf{b}_{-i}) > U_i^f(\mathbf{b})$. Thus, b_i is not a best response against $\mathbf{b}_{-i}$, and consequently, $\mathbf{b}$ is not a Nash equilibrium, a contradiction. ∎

PROOF OF THEOREM 1:

In the main text. ∎

PROOF OF PROPOSITION 3:

In the main text. ∎

PROOF OF PROPOSITION 4:

We first show that the signal structure b_i^* as given in the proposition is well defined. The function ϕ is a probability density on $[\underline{y},\bar{y}]$ as

$$\int_{\underline{y}}^{\bar{y}} \phi(y)\,dy = \kappa\left(\frac{\bar{y}}{1-\bar{y}} - \frac{\underline{y}}{1-\underline{y}} + \ln\left[\frac{\bar{y}}{1-\bar{y}}\frac{1-\underline{y}}{\underline{y}}\right]\right) = 1,$$

where the second equality is implied by the definition of κ and (16). Next, we prove that there is a unique $\bar{y} \in (\underline{y},1)$ that solves (16). Let $\bar{z} = \bar{y}/(1-\bar{y})$ and $\underline{z} = \underline{y}/(1-\underline{y})$ so that the right-hand side of (16) can be written as $R(\bar{z},\underline{z}) = (\ln[\bar{z}] - \ln[\underline{z}])/(\bar{z}-\underline{z})$. Note that R is continuous and strictly decreasing in $\bar{z}$, $\lim_{\bar{z}\to\underline{z}} R(\bar{z},\underline{z}) = 1/\underline{z}$, and $\lim_{\bar{z}\to\infty} R(\bar{z},\underline{z}) = 0$. By (15), the left side of (16) satisfies

$$\frac{\rho^3-\pi^3}{\rho^2} = \min\left\{\frac{\rho^3}{\rho^2}, \frac{1}{\underline{z}}\frac{\rho^1}{\rho^2}\ln\left[\frac{\rho^1+\rho^2}{\rho^1}\right]\right\} < \frac{1}{\underline{z}}$$

since $\ln[(\rho^1+\rho^2)/\rho^1] < \rho^2/\rho^1$. Hence, there is a unique $\bar{z} \in (\underline{z},\infty)$ and therefore a unique $\bar{y} \in (\underline{y},1)$ that solves (16). Finally, we verify that $b_i^* \in B_i$, that is, b_i^* satisfies (1). For valuation v^1, which is perfectly disclosed, (1) clearly holds. For valuation v^2,

$$\begin{aligned}\int_{\mathcal{P}_i} p_i(v^2)\,db_i^*(p_i) &= (1-\rho^1-\pi^3)\int_{\underline{y}}^{\bar{y}} y\phi(y)\,dy \\ &= (1-\rho^1-\pi^3)\left(\frac{\bar{y}}{1-\bar{y}} - \frac{\underline{y}}{1-\underline{y}}\right)\kappa = \rho^2,\end{aligned}$$

and for valuation v^3,

$$\int_{\mathcal{P}_i} p_i(v^3)\,db_i^*(p_i) = \pi^3 + (1-\rho^1-\pi^3)\int_{\underline{y}}^{\bar{y}}(1-y)\phi(y)\,dy = \rho^3$$

because $\int_{\underline{y}}^{\bar{y}}\phi(y)\,dy = 1$ and $(1-\rho^1-\pi^3)\int_{\underline{y}}^{\bar{y}} y\phi(y)\,dy = \rho^2$.

Now, we will show that b_i^* is a best response against itself. So, suppose bidder j plays the equilibrium strategy b_j^*. Accordingly, he perfectly discloses his valuation when it is v^1, which implies $H_j\left(v^1,p_j\right) = v^1$. Moreover, for all posteriors p_j with $V(p_j) = \left\{v^2,v^3\right\}$, $p_j\left(v^2\right) = y \geq \underline{y}$, which implies

$$H_j\left(v^2,p_j\right) = v^2 - \frac{1-y}{y}\left(v^3-v^2\right) \geq v^2 - \frac{1-\underline{y}}{\underline{y}}\left(v^3-v^2\right) = v^1,$$

whereas $H_j\left(v^2,p_j\right) < H_j\left(v^3,p_j\right) = v^3$ for $y < 1$. Now, consider bidder $i \neq j$ who uses an arbitrary strategy b_i. Recall that his ex ante expected payoff in the auction at $\left(p_i,p_j\right)$ is

$$u_i^f\left(p_i,p_j\right) = \sum_{\mathbf{v}\in V(\mathbf{p})}\left[v_i - H_i\left(v_i,p_i\right)\right] q_i^f(\mathbf{v},\mathbf{p})p(\mathbf{v}).$$

As the virtual valuation of i's opponent is always at least v^1, $\left[v_i - H_i\left(v_i,p_i\right)\right] q_i^f(\mathbf{v},\mathbf{p}) = 0$ whenever $v_i = v^1$. Hence, it is a best response for i to also perfectly disclose v^1: when doing so, the contribution of $v_i = v^1$ to the auction payoff is still zero, whereas the contribution of $v_i > v^1$ remains unaffected. To see that the latter is true, note that for $v_i > v^1$, either $H_i\left(v_i,p_i\right) = J_i\left(v_i,p_i\right)$ and thus $H_i\left(v_i,p_i\right)$ is independent of $p_i\left(v^1\right)$ or $v^1 > H_i\left(v^1,p_i\right) = H_i\left(v_i,p_i\right) > J_i\left(v_i,p_i\right)$, $v_i = v^2$, and thus $q_i^f(\mathbf{v},\mathbf{p}) = 0$ independent of whether v^1 is perfectly disclosed.

Moreover, we also have $\left[v_i - H_i\left(v_i,p_i\right)\right] q_i^f(\mathbf{v},\mathbf{p}) = 0$ if $v_i = v^3$ and if $v_i = v^2$ and $p_i\left(v^2\right) < \underline{y}$ or $p_i\left(v^2\right) = 1$. Hence, bidder i expects a nonzero payoff only from posteriors p_i with $p_i\left(v^2\right) \in \left[\underline{y},1\right)$. Conditional on drawing the posterior p_i with $p_i\left(v^2\right) = y_i \geq \underline{y}$, bidder i's expected payoff is

$$\begin{aligned}
&\int_{\mathcal{P}_j} u_i^f\left(p_i,p_j\right) db_j^*\left(p_j\right) \\
&\quad = \left[v^2 - H_i\left(v^2,p_i\right)\right] p_i\left(v^2\right) \int_{\mathcal{P}_j}\left[p_j\left(v^1\right) + p_j\left(v^2\right)\mathbf{1}_{p_j(v^2)<p_i(v^2)}\right] db_j^*\left(p_j\right) \\
&\quad = \left(1-y_i\right)\left(v^3-v^2\right)\left(\rho^1 + \left(1-\rho^1-\pi^3\right)\int_{\underline{y}}^{y_i} y\phi(y)\,dy\right),
\end{aligned}$$

where

$$\left(1-\rho^1-\pi^3\right)\int_{\underline{y}}^{y_i} y\phi(y)\,dy = \begin{cases} \left(1-\rho^1-\pi^3\right)\left[\dfrac{y_i}{1-y_i} - \dfrac{\underline{y}}{1-\underline{y}}\right]\kappa, & \text{if } y_i \in \left[\underline{y},\bar{y}\right]; \\ \rho^2, & \text{if } y_i \geq \bar{y}. \end{cases}$$

Let $\hat{\mathcal{P}}_i$ denote a set of posteriors such that $p_i\left(v^2\right) = 1 - p_i\left(v^3\right) = y_i \in \left(\bar{y},1\right)$. Consider a strategy b_i such that $\int_{p_i\in\hat{\mathcal{P}}_i} db_i\left(p_i\right) > 0$. As $1 - y_i = p_i\left(v^3\right)$, the posteriors in $\hat{\mathcal{P}}_i$ contribute

$$\int_{p_i\in\hat{\mathcal{P}}_i}\int_{\mathcal{P}_j} u_i^f\left(p_i,p_j\right) db_j^*\left(p_j\right) db_i\left(p_i\right) = \left(v^3-v^2\right)\left(\rho^1+\rho^2\right)\int_{p_i\in\hat{\mathcal{P}}_i} p_i\left(v^3\right) db_i\left(p_i\right)$$

to i's expected payoff. Consider another strategy b_i' that coincides with b_i except that it does not draw any $p_i \in \hat{\mathcal{P}}_i$: instead, b_i' draws the posterior such that $p_i\left(v^2\right) = 1 - p_i\left(v^3\right) = \bar{y}$ with probability $\int_{p_i\in\hat{\mathcal{P}}_i} p_i\left(v^3\right) db_i\left(p_i\right)/\left(1-\bar{y}\right)$ and the posterior such that $p_i\left(v^2\right) = 1$ with probability $\int_{p_i\in\hat{\mathcal{P}}_i} db_i\left(p_i\right) - \int_{p_i\in\hat{\mathcal{P}}_i} p_i\left(v^3\right) db_i\left(p_i\right)/\left(1-\bar{y}\right)$. Note that b_i' yields the same expected payoff as b_i. Hence, it is a best response for i to use a strategy under which $p_i(v^2) \in (\bar{y}, 1)$ has probability zero.

So, suppose bidder i plays a strategy b_i such that $p_i(v^2) \in [0, \bar{y}] \cup \{1\}$ with probability one and v^1 is perfectly disclosed. Only posteriors p_i such that $p_i(v_2) = 1 - p_i(v^3) = y_i \in [\underline{y}, \bar{y}]$ contribute nonzero payoff. For $y_i \in [\underline{y}, \bar{y}]$, we can write $\int_{\mathcal{P}_j} u_i^f(p_i, p_j)\, db_j^*(p_j) = (v^3 - v^2)[\psi(1 - y_i) + \omega y_i]$, where

$$\psi = \rho^1 - (1 - \rho^1 - \pi^3)\frac{\underline{y}}{1 - \underline{y}}\kappa \quad \text{and} \quad \omega = (1 - \rho^1 - \pi^3)\kappa.$$

Then, i's payoff in the disclosure game is

$$\text{(A17)} \qquad U_i^f(b_i, b_j^*) = \int_{\mathcal{P}_i}\int_{\mathcal{P}_j} u_i^f(p_i, p_j)\, db_j^*(p_j)\, db_i(p_i)$$
$$= (v^3 - v^2)\int_{\{p_i | p_i(v^2) \in [\underline{y}, \bar{y}]\}} [\psi p_i(v^3) + \omega p_i(v^2)]\, db_i(p_i).$$

Clearly, $\omega > 0$. We next show that depending on π^3, either $\psi = 0$ or $\psi > 0$. Consider definition (15). If

$$\pi^3 =(>)\ \rho^3 - \frac{1 - \underline{y}}{\underline{y}}\rho^1 \ln\left[\frac{\rho^1 + \rho^2}{\rho^1}\right],$$

which means $\pi^3 \geq (=)\ 0$, then (16) yields

$$\frac{1 - \underline{y}}{\underline{y}}\frac{\rho^1}{\rho^2}\ln\left[\frac{\rho^1 + \rho^2}{\rho^1}\right] =(>)\ \ln\left[\frac{\bar{y}}{1 - \bar{y}}\frac{1 - \underline{y}}{\underline{y}}\right]\left(\frac{\bar{y}}{1 - \bar{y}} - \frac{\underline{y}}{1 - \underline{y}}\right)^{-1}$$

$$\Leftrightarrow \quad \frac{\bar{y}}{1 - \bar{y}} =(>)\ \frac{\rho^1 + \rho^2}{\rho^1}\frac{\underline{y}}{1 - \underline{y}}$$

$$\text{(A18)} \qquad \Rightarrow \quad \kappa =(<)\ \frac{\rho^1}{1 - \rho^1 - \pi^3}\frac{1 - \underline{y}}{\underline{y}}$$

and therefore $\psi =(>)\ 0$. So, suppose $\pi^3 \geq 0$ and $\psi = 0$. It then follows from (A17) that $U_i^f(b_i, b_j^*)$ is maximized by any b_i such that $\int\{p_i | p_i(v^2) \in [\underline{y}, \bar{y}]\} p_i(v^2)\, db_i(p_i) = \rho^2$. Hence, $b_i = b_j^*$ is indeed a best response. Now, suppose $\pi^3 = 0$ and $\psi > 0$. By (A17), $U_i^f(b_i, b_j^*)$ is maximized by any b_i such that $\int\{p_i | p_i(v^2) \in [\underline{y}, \bar{y}]\} p_i(v^2)\, db_i(p_i) = \rho^2$ and $\int\{p_i | p_i(v^2) \in [\underline{y}, \bar{y}]\} p_i(v^3)\, db_i(p_i) = \rho^3$. Hence, again, $b_i = b_j^*$ is a best response.

We are left to prove the claim regarding the equilibrium payoff. If $\pi^3 > 0$, then $\psi = 0$ and (A18) holds with equality. Hence, by (A17),

$$U_i^f(b_1^*, b_2^*) = (v^3 - v^2)\omega\rho^2 = (v^3 - v^2)\frac{1 - \underline{y}}{\underline{y}}\rho^1\rho^2 = (v^2 - v^1)\rho^1\rho^2. \blacksquare$$

PROOF OF PROPOSITION 5:

We will show that b_1^*, b_2^* are best responses against each other. If bidder 2 plays b_2^*, his virtual valuation is always at least 1 because $J_2(2, p_2''') = 1$. Thus, bidder 1's payoff is zero under any signal structure, and b_1^* is a best response.

Similarly, if bidder 1 plays b_1^*, bidder 2 can win the auction only if his virtual valuation is at least 1. Thus, it is a best response to perfectly reveal valuation 1, that is, to draw p_2' with probability $1/3$.[25] Moreover, bidder 2 then only obtains an information rent in the auction at posterior profiles (p_1, p_2) such that $V_2(p_2) = \{2,4\}$ and he wins the auction with valuation 2. This is the case at (p_1', p_2) such that $p_2(2) \geq p_2'''(2) = 2/3$, which ensures $J_2(2, p_2) \geq 1$, and at (p_1'', p_2) such that $p_2(2) \geq 8/9$, which ensures $J_2(2, p_2) \geq 7/4$. Bidder 2's payoff at such profiles is $u_2^h(p_1, p_2) = 2 - 2p_2(2)$.

Any posterior p_2 such that $p_2(2) \in (0, 2/3)$ yields zero payoff, and bidder 2 can do better by performing an ϵ-extension to replace p_2 with p_2''' and p_2''. So, a best response b_2 of bidder 2 draws among posteriors such that $V_2(p_2) = \{2,4\}$ only those that satisfy $p_2(2) \geq 2/3 > \bar{p}_2$, which implies that b_2 must also draw p_2'' with positive probability (to ensure consistency (1)). Suppose under b_2 posterior p_2 such that $p_2(2) \in (2/3, 8/9)$ is drawn with some probability β. Then bidder 2 can improve by performing the opposite of an ϵ-extension: drawing p_2''' with a probability greater than β and reducing the probability of p_2''. This is profitable, as (p_1', p_2''') both realizes with higher probability and yields a higher payoff than (p_1', p_2). Now, suppose p_2 such that $p_2(2) \geq 8/9$ is drawn with some probability β. Accordingly, the payoff in the auction given p_2 is always positive, so that p_2 contributes $\beta \cdot [2 - 2p_2(2)]$ to the expected payoff. Again, bidder 2 can replace p_2 and instead reduce the probability of p_2'' and draw p_2''' with probability $\beta p_2(2)/p_2'''(2) > \beta$. Note that the auction payoff given p_2''' is positive only when bidder 1's valuation is 1. Thus p_2''' contributes $\beta p_2(2)/p_2'''(2) \cdot [2 - 2p_2'''(2)]/2$ to the expected payoff. Since $p_2(2) \geq 8/9$ and $p_2'''(2) = 2/3$, the contribution of p_2''' is strictly greater than the contribution of p_2; that is, replacing p_2 is profitable. We conclude that by choosing b_2^*, bidder 2 can improve upon any b_2 that with positive probability draws posteriors p_2 such that $p_2(2) \in (0, 2/3) \cup (2/3, 1]$. Consequently, b_2^* is a best response against b_1^*. ∎

PROOF OF PROPOSITION 6:

As in Section III, by standard arguments, there are transfers such that incentive compatibility (20) holds if and only if $\check{Q}_i(v_i^{k+1}, p_i, s_i) \geq \check{Q}_i(v_i^k, p_i, s_i)$ for all $k \in \{1, \ldots, m_i - 1\}$ and $i \in N$. Moreover, noting that by (18) the objective (19) can be written as

$$\sum_{i \in N} \int_{\mathcal{P}_i} \int_{S_i} \sum_{v_i \in V_i(p_i)} \check{T}_i(v_i, p_i, s_i)\, p_i(v_i)\, d\sigma_i(s_i)\, db_i(p_i),$$

for any solution to problem $[\mathbf{b}, \boldsymbol{\sigma}]$, again all the local downward incentive constraints as well as individual rationality (21) for valuation v_i^1 for (σ_i, b_i)-almost all (s_i, p_i) and all $i \in N$ are binding, yielding interim expected transfers

$$\text{(A19)} \qquad \check{T}_i(v_i^k, p_i, s_i) = v_i^k \check{Q}_i(v_i^k, p_i, s_i) - \sum_{l=1}^{k-1} (v_i^{l+1} - v_i^l)\, \check{Q}_i(v_i^l, p_i, s_i).$$

[25] For details, see the Proof of Proposition 4, which involves a similar argument.

Hence, bidder i's contribution to the auctioneer's objective can be written as

$$\int_{\mathcal{P}_i}\int_{S_i}\sum_{v_i\in V_i(p_i)}\breve{T}_i(v_i,p_i,s_i)p_i(v_i)\,d\sigma_i(s_i)\,db_i(p_i)$$
$$= \int_{\mathcal{P}_i}\int_{S_i}\sum_{v_i\in V_i(p_i)}J_i(v_i,p_i)\,\breve{Q}_i(v_i,p_i,s_i)\,p_i(v_i)\,d\sigma_i(s_i)\,db_i(p_i),$$

where $J_i(v_i^k,p_i)$ is the virtual valuation defined in Section III. Consequently, the solutions to problem $[\mathbf{b},\boldsymbol{\sigma}]$ satisfy (A19) and solve problem $[\mathbf{P}']$:

$$\max_{(\mathbf{q}(\cdot,\mathbf{p}))_{\mathbf{p}\in\mathcal{P}}}\sum_{i\in N}\int_{\mathcal{P}_i}\int_{S_i}\sum_{v_i\in V_i(p_i)}J_i(v_i,p_i)\,\breve{Q}_i(v_i,p_i,s_i)\,p_i(v_i)\,d\sigma_i(s_i)\,db_i(p_i),$$

subject to

$$\text{(A20)}\quad \breve{Q}_i(v_i^1,p_i,s_i)\ \le\ \cdots\ \le\ \breve{Q}_i(v_i^{m_i},p_i,s_i)\qquad \forall i\in N, p_i\in\mathcal{P}_i, s_i\in S_i.$$

The remainder of the proof consists in adapting arguments from the Proof of Proposition 1(a) to show that the solutions to problem $[\mathbf{P}]$ also solve $[\mathbf{P}']$. We will first show that for any allocation rule that satisfies the monotonicity constraint (A20),

$$\text{(A21)}\quad \sum_{i\in N}\int_{\mathcal{P}_i}\int_{S_i}\sum_{v_i\in V_i(p_i)}J_i(v_i,p_i)\,\breve{Q}_i(v_i,p_i,s_i)\,p_i(v_i)\,d\sigma_i(s_i)\,db_i(p_i)$$
$$\le \sum_{i\in N}\int_{\mathcal{P}_i}\int_{S_i}\sum_{v_i\in V_i(p_i)}H_i(v_i,p_i)\,\breve{Q}_i(v_i,p_i,s_i)\,p_i(v_i)\,d\sigma_i(s_i)\,db_i(p_i).$$

As in the Proof of Proposition 1(a), define

$$\hat{T}_i(\mathbf{p})\ :=\ \sum_{k=1}^{m_i}J_i(v_i^k,p_i)\,Q_i(v_i^k,\mathbf{p})\,p_i(v_i^k)$$

and

$$\tilde{T}_i(\mathbf{p})\ :=\ \sum_{k=1}^{m_i}H_i(v_i^k,p_i)\,Q_i(v_i^k,\mathbf{p})\,p_i(v_i^k).$$

From (A2), it follows that (suppressing the dependence of C_i and G_i on p_i)

$$\int_{\mathcal{P}_{-i}}\left[\tilde{T}_i(\mathbf{p})-\hat{T}_i(\mathbf{p})\right]ds_i(\mathbf{p}_{-i})$$
$$= -\sum_{k=1}^{m_i-1}\left[C_i\big(P_i(v_i^k)\big)-G_i(v_i^k)\right]\left[\breve{Q}_i(v_i^{k+1},p_i,s_i)-\breve{Q}_i(v_i^k,p_i,s_i)\right].$$

This is positive whenever monotonicity (A20) holds, which implies (A21).

Now, it follows from the Proof of Proposition 1(a) that the optimal allocation rules identified there maximize the right-hand side of (A21) and that (A21) holds with equality for these allocation rules. Hence these allocation rules solve problem $[\mathbf{P}']$.

Finally, note that (6) for all $\mathbf{p} \in \mathcal{P}$ implies (A19). Thus, $\left(\mathbf{q}(\cdot,\mathbf{p}),\mathbf{t}(\cdot,\mathbf{p})\right)_{\mathbf{p}\in\mathcal{P}}$ satisfies (A19) and solves problem $[\mathbf{P}']$ if $\left(\mathbf{q}(\cdot,\mathbf{p}),\mathbf{t}(\cdot,\mathbf{p})\right)$ satisfies Proposition 1(b) for all $\mathbf{p}$. ■

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