



Boundary Integral Operators for the Heat Equation in Time-Dependent Domains

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Abstract. This article provides a functional analytical framework for boundary integral equations of the heat equation in time-dependent domains. More specifically, we consider a non-cylindrical domain in space-time that is the C^2 -diffeomorphic image of a cylinder, i.e., the tensor product of a time interval and a fixed domain in space. On the non-cylindrical domain, we introduce Sobolev spaces, trace lemmata and provide the mapping properties of the layer operators. Here it is critical that the Neumann trace requires a correction term for the normal velocity of the moving boundary. Therefore, one has to analyze the situation carefully.

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1. Introduction

Boundary integral equations are a well-known technique to solve elliptic partial differential equations, see for example [16, 17]. For parabolic equations on *time-independent*, so-called *cylindrical* domains, Sobolev spaces and the mapping properties of the layer operators for the heat equation are introduced in [3, 15]. To the best of our knowledge, no corresponding theory exists for *time-dependent* or so-called *non-cylindrical* domains, or simply, *tubes*. Some results can be found in e.g. [9, 10]. Therefore, the aim of this article is to close the gaps in the theory also for non-cylindrical domains. To this end, we consider a special class of non-cylindrical domains. We fix a cylindrical domain which serves as a reference domain and define the non-cylindrical domain as the image under a time-dependent C^2 -diffeomorphism. In this setting, different approaches are possible to establish analogue integral equations and properties to integral operators on a cylindrical domain.

A first approach could be to exploit the fact that the fundamental solution does not depend on the boundary data and is thus defined on the

free space $\mathbb{R} \times \mathbb{R}^d$. Therefore, it is the same for a cylindrical and a non-cylindrical domain and allows to state the integral operators in cylindrical and non-cylindrical domains. To establish the mapping properties of the integral operators, one could make use of the equivalence of norms on the tube and on the cylindrical domain by establishing equivalence results of the fundamental solution, evaluated on the tube and on the cylindrical domain. The problem is that the Neumann trace, which will be considered here, contains an additional term involving the normal velocity of the tube. Therefore, one has to come up with a solution how to deal with this.

A second approach could be to map back the partial differential equation from the non-cylindrical domain onto the cylindrical domain. The advantage is that one now considers a cylindrical domain, for which more theory is available. The drawback is that the differential equation in the reference domain is more complicated because of time and space dependent coefficients. Finding a fundamental solution is more difficult and one could for example pursue the parametrix ansatz, taken in [7], and then find the according mapping properties of the respective layer operators.

A third approach considers the partial differential equation on the non-cylindrical domain. Here, the partial differential equation is simple, but the domain is more involved in contrast to the second approach. This approach was used in [18], but without the corresponding Sobolev spaces and mapping properties of the integral operators. Since we already did computations in [1] based on [18], and since [3] provides a self-contained analysis of the mapping properties of the layer operators for the heat equation in a cylindrical domain, we choose this approach for this article.

Although we follow the argumentation line of Costabel [3], we have to modify several proofs for the present non-cylindrical setting. In particular, we have to use the appropriate function spaces and the correct Neumann traces. We would like to emphasize that, once one has the appropriate Neumann trace operator at hand and the mapping properties of the trace operators, the ideas of [3] can basically be followed. We indicate the required adaptations in the article.

The remainder of this article is organized as follows: In Sect. 2, we introduce anisotropic Sobolev spaces on cylindrical and non-cylindrical domain, which are used for the mapping properties. Section 3 is dedicated to the Dirichlet traces and the existence and uniqueness of solutions of the Dirichlet problem. In Sect. 4, we have a look at the appropriate Neumann trace. The main result is presented in Sect. 5, where we establish the mapping properties of the integral operators and the existence and uniqueness of solutions of the Neumann problem. In Sect. 6, we state some concluding remarks.

2. Anisotropic Sobolev Spaces

In order to study the heat equation, we shall introduce appropriate anisotropic Sobolev spaces on cylindrical domains. From these spaces, we will then derive Sobolev spaces on time-dependent domains.

2.1. Anisotropic Sobolev Spaces on Cylindrical Domains

Let $\Omega_0 \subset \mathbb{R}^d$, $d \geq 2$, be a Lipschitz domain in the spatial variable with boundary $\Gamma_0 := \partial\Omega_0$ and let $0 < T < \infty$. Then, the product set $Q_0 := (0, T) \times \Omega_0 \subset \mathbb{R}^{d+1}$ forms a time-space cylinder with the lateral boundary $\Sigma_0 := (0, T) \times \Gamma_0$. The appropriate function spaces for parabolic problems in time-invariant domains, i.e. in cylindrical domains, are the anisotropic Sobolev spaces defined by

$$H^{r,s}(Q_0) := L^2((0, T); H^r(\Omega_0)) \cap H^s((0, T); L^2(\Omega_0))$$

for $r, s \in \mathbb{R}_{\geq 0}$, see, e.g., [2, 3, 12]. The corresponding boundary spaces are

$$H^{r,s}(\Sigma_0) := L^2((0, T); H^r(\Gamma_0)) \cap H^s((0, T); L^2(\Gamma_0))$$

Note that these spaces are well-defined for $r \leq 1$ (while $s \geq 0$ is arbitrary) if Γ_0 is Lipschitz.

Remark 2.1. The space $H^{r,s}(Q_0)$ consists of all functions $u \in L^2(Q_0)$, where the $L^2(Q_0)$ -norm of the partial derivatives $\partial_{\mathbf{x}}^\alpha \partial_t^\beta u(t, \mathbf{x})$ is finite for all $|\alpha| \leq \lambda r$, $\beta \leq (1 - \lambda)s$, and $\lambda \in [0, 1]$.

With these definitions at hand, we can moreover define spaces for functions with zero initial condition by setting

$$H_{;0}^{r,s}(Q_0) := L^2((0, T); H^r(\Omega_0)) \cap H_{;0}^s((0, T); L^2(\Omega_0)),$$

where

$$H_{;0}^s((0, T); L^2(\Omega_0)) := \left\{ u = \tilde{u}|_{(0,T)} : \right. \\ \left. \tilde{u} \in H^s((-\infty, T); L^2(\Omega_0)) : \tilde{u}(t) = 0 \text{ for } t < 0 \right\}.$$

Note that we adopted the notation from [4, 5]. In addition, we can define functions which vanish at $t = T$ by setting

$$H_{;0}^{r,s}(Q_0) := L^2((0, T); H^r(\Omega_0)) \cap H_{;0}^s((0, T); L^2(\Omega_0)),$$

where in complete analogy

$$H_{;0}^s((0, T); L^2(\Omega_0)) := \left\{ u = \tilde{u}|_{(0,T)} : \right. \\ \left. \tilde{u} \in H^s((0, \infty); L^2(\Omega_0)) : \tilde{u}(t) = 0 \text{ for } t > T \right\}.$$

As in the elliptic case, we can also include (spatial) zero boundary conditions into the function spaces by setting

$$H_{;0}^{r,s}(Q_0) := L^2((0, T); H_0^r(\Omega_0)) \cap H_{;0}^s((0, T); L^2(\Omega_0)),$$

$$H_{;0}^{r,s}(Q_0) := L^2((0, T); H_0^r(\Omega_0)) \cap H_{;0}^s((0, T); L^2(\Omega_0)),$$

where the spaces include zero initial and end conditions, respectively. On the boundary, we introduce

$$H_{;0}^{r,s}(\Sigma_0) := L^2((0, T); H^r(\Gamma_0)) \cap H_{;0}^s((0, T); L^2(\Gamma_0)),$$

$$H_{;0}^{r,s}(\Sigma_0) := L^2((0, T); H^r(\Gamma_0)) \cap H_{;0}^s((0, T); L^2(\Gamma_0)).$$

These spaces are the closures of $H^{r,s}(\Sigma_0)$ for zero start and end condition, respectively, compare [4, Section 2.3].

By duality we have

$$H_{;0}^{-r,-s}(Q_0) = [H_{0;0}^{r,s}(Q_0)]' \quad \text{for } r - \frac{1}{2} \notin \mathbb{Z}$$

according to [3]. The anisotropic Sobolev spaces on the boundary with negative smoothness index are defined by

$$\begin{aligned} H_{;0}^{-r,-s}(\Sigma_0) &:= [H_{;0}^{r,s}(\Sigma_0)]', \\ H_{;0}^{-r,-s}(\Sigma_0) &:= [H_{;0}^{r,s}(\Sigma_0)]', \\ \widetilde{H}^{-r,-s}(\Sigma_0) &:= [H^{r,s}(\Sigma_0)]', \end{aligned}$$

see [4, Section 2.3]. Moreover, according to [4, Remark 2.1], for $r \geq 0$ and $0 \leq s < \frac{1}{2}$ it holds $H^{r,s}(\Sigma_0) = H_{;0}^{r,s}(\Sigma_0) = H_{;0}^{r,s}(\Sigma_0)$ and, therefore, the above introduced dual spaces are equal and we simply write $H^{-r,-s}(\Sigma_0)$.

Remark 2.2. We would like to clarify the intuition behind the slightly cumbersome notation. In $H_{0;0}^{r,s}(Q_0)$, a zero before the semicolon indicates a zero boundary condition in space. After the semicolon, a zero initial condition can be indicated by writing a zero between the semicolon and the comma. Whereas, a present zero after the comma stands for a zero end condition. Thus, this notation allows to see the spatial and temporal boundary condition at one glance.

2.2. Anisotropic Sobolev Spaces on Non-Cylindrical Domains

Having at hand the Sobolev spaces defined on cylindrical domains, we can also introduce Sobolev spaces on non-cylindrical domains. Non-cylindrical domains consist of a spatial domain, which we denote by Ω_t . The subscript t indicates that the spatial domain might differ for every point of time. To obtain a non-cylindrical domain Q_T we set

$$Q_T := \bigcup_{0 < t < T} (\{t\} \times \Omega_t).$$

This domain has a lateral boundary Σ_T defined by

$$\Sigma_T := \bigcup_{0 < t < T} (\{t\} \times \Gamma_t),$$

where $\Gamma_t := \partial\Omega_t$.

For every point of time t , we assume to have a smooth diffeomorphism κ , which maps the initial domain Ω_0 onto the time-dependent domain Ω_t . In accordance with [14], we write

$$\kappa: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (t, \mathbf{x}) \mapsto \kappa(t, \mathbf{x}) \tag{2.1}$$

to emphasize the dependence of the mapping κ on the time, where we have $\kappa(t, \Omega_0) = \Omega_t$. Especially, Ω_t is also a Lipschitz domain for all $t \in [0, T]$.

The domains Ω_t each have a spatial normal \mathbf{n}_t , which we will also denote by \mathbf{n} if it is clear from the context. Besides having a spatial normal, we also have a time-space normal ν . We can write the time-space normal as

$$\nu = \frac{1}{\sqrt{1 + v_\nu^2}} \begin{bmatrix} v_\nu \\ \mathbf{n} \end{bmatrix} \tag{2.2}$$

for some appropriate $v_\nu \in \mathbb{R}$. According to [6], it holds $v_\nu = -\langle \mathbf{V}, \mathbf{n} \rangle$ for the vector field \mathbf{V} , which deforms the cylinder Q_0 into the tube Q_T and for which the relation $\mathbf{V} = \partial_t \boldsymbol{\kappa} \circ \boldsymbol{\kappa}^{-1}$ holds.

We introduce the non-cylindrical analogues of the Sobolev spaces by setting

$$H^{r,s}(Q_T) := \{v \in L^2(Q_T) : v \circ \boldsymbol{\kappa} \in H^{r,s}(Q_0)\}$$

where the composition with $\boldsymbol{\kappa}$ only acts on the spatial component. Due to the chain rule, $v \circ \boldsymbol{\kappa}$ and v have the same Sobolev regularity, provided that the mapping $\boldsymbol{\kappa}$ is smooth enough, see for example [13, Theorem 3.23] for the elliptic case. For what follows, we assume that $\boldsymbol{\kappa} \in C^2([0, T] \times \mathbb{R}^d)$ satisfies

$$\|\boldsymbol{\kappa}(t, \mathbf{x})\|_{C^2([0, T] \times \mathbb{R}^d; \mathbb{R}^d)}, \|\boldsymbol{\kappa}(t, \mathbf{x})^{-1}\|_{C^2([0, T] \times \mathbb{R}^d; \mathbb{R}^d)} \leq C_\boldsymbol{\kappa} \quad (2.3)$$

for some constant $C_\boldsymbol{\kappa} \in (0, \infty)$ as in [8, pg. 826]. We define the norm of $H^{r,s}(Q_T)$ as $\|u\|_{H^{r,s}(Q_T)} := \|u \circ \boldsymbol{\kappa}\|_{H^{r,s}(Q_0)}$ for $r, s \geq 0$. Notice that the Sobolev spaces on the boundary are defined in a similar manner.

Remark 2.3. (i) The space $H^{r,s}(Q_T)$ contains all functions such that $u \circ \boldsymbol{\kappa} \in H^{r,s}(Q_0)$. This means that $\partial_{\mathbf{x}}^\alpha \partial_t^\beta (u \circ \boldsymbol{\kappa}) \in L^2(Q_0)$ for all $|\alpha| \leq \lambda r$, $\beta \leq (1 - \lambda)s$, and $\lambda \in [0, 1]$. According to (2.3), the partial derivatives $\partial_{\mathbf{x}}^\alpha \partial_t^\beta \boldsymbol{\kappa}$ exist and are uniformly bounded for all $|\alpha| + \beta \leq 2$.

(ii) Consider a function $u \in L^2(Q_T)$ with partial derivatives $\partial_{\mathbf{x}}^\alpha \partial_t^\beta u \in L^2(Q_T)$ for all $|\alpha| \leq \lambda r$, $\beta \leq (1 - \lambda)s$, and $\lambda \in [0, 1]$. When computing the time-derivative of $u \circ \boldsymbol{\kappa}$, we obtain also a spatial derivative as the following shows

$$(\partial_t u) \circ \boldsymbol{\kappa} = \partial_t (u \circ \boldsymbol{\kappa}) - \langle (\mathbf{D}\boldsymbol{\kappa})^{-\top} \nabla (u \circ \boldsymbol{\kappa}), \partial_t \boldsymbol{\kappa} \rangle. \quad (2.4)$$

Hence, it holds $u \in H^{r,s}(Q_T)$ only if $r \geq s$ since the temporal derivative $\partial_t^\beta (u \circ \boldsymbol{\kappa})$ involves also spatial partial derivatives $\partial_{\mathbf{x}}^\alpha u$ up to the order $|\alpha| = \beta$ besides the temporal derivative $\partial_t^\beta u$.

(iii) Due to the uniformity condition (2.3), we have as in [8]

$$0 < \underline{\sigma} \leq \min\{\sigma(\mathbf{D}\boldsymbol{\kappa})\} \leq \max\{\sigma(\mathbf{D}\boldsymbol{\kappa})\} \leq \bar{\sigma} < \infty,$$

where $\mathbf{D}\boldsymbol{\kappa}$ denotes the Jacobian of $\boldsymbol{\kappa}$ and $\sigma(\mathbf{D}\boldsymbol{\kappa})$ denotes its singular values. Especially, as in [8, Remark 1, pg. 827], we may assume $\det(\mathbf{D}\boldsymbol{\kappa})$ to be positive. We can thus define the dual of $H_{0,0}^{r,s}(Q_T)$ in two different ways, namely by using either the norm

$$\|u\|_{H_{0,0}^{-r,-s}(Q_T)} = \sup_{\tilde{v} \in H_{0,0}^{r,s}(Q_0)} \int_{Q_0} (u \circ \boldsymbol{\kappa}) \tilde{v} \, d(\mathbf{x}, t)$$

or the norm

$$\| \|u\| \|_{H_{0,0}^{-r,-s}(Q_T)} = \sup_{v \in H_{0,0}^{r,s}(Q_T)} \int_{Q_T} uv \, d(\mathbf{x}, t).$$

However, we show that these norms are equivalent and so the spaces. On one hand, we have

$$\begin{aligned} \|u\|_{H_{0,0}^{-r,-s}(Q_T)} &= \sup_{v \in H_{0,0}^{r,s}(Q_T)} \frac{\int_{Q_0} (u \circ \kappa)(v \circ \kappa) \det(\mathbf{D}\kappa) \, d(\mathbf{x}, t)}{\|v\|_{H_{0,0}^{r,s}(Q_T)}} \\ &\leq \|u\|_{H_{0,0}^{-r,-s}(Q_T)} \sup_{v \in H_{0,0}^{r,s}(Q_T)} \frac{\|(v \circ \kappa) \det(\mathbf{D}\kappa)\|_{H_{0,0}^{r,s}(Q_0)}}{\|v \circ \kappa\|_{H_{0,0}^{r,s}(Q_0)}} \\ &\lesssim \|u\|_{H_{0,0}^{-r,-s}(Q_T)}, \end{aligned}$$

where we used the definition of the norm on $H_{0,0}^{r,s}(Q_T)$ for $s, r \geq 0$ and that the pointwise multiplication with a smooth function is a continuous operation. On the other hand, we likewise find

$$\begin{aligned} \|u\|_{H_{0,0}^{-r,-s}(Q_T)} &= \sup_{\tilde{v} \in H_{0,0}^{r,s}(Q_0)} \frac{\int_{Q_T} u(\tilde{v} \circ \kappa^{-1}) \det(\mathbf{D}\kappa^{-1}) \, d(\mathbf{x}, t)}{\|\tilde{v}\|_{H_{0,0}^{r,s}(Q_0)}} \\ &\leq \|u\|_{H_{0,0}^{-r,-s}(Q_T)} \sup_{\tilde{v} \in H_{0,0}^{r,s}(Q_0)} \frac{\|(\tilde{v} \circ \kappa^{-1}) \det(\mathbf{D}\kappa^{-1})\|_{H_{0,0}^{r,s}(Q_T)}}{\|\tilde{v}\|_{H_{0,0}^{r,s}(Q_0)}} \\ &\lesssim \|u\|_{H_{0,0}^{-r,-s}(Q_T)}. \end{aligned}$$

Hence, both duality pairings result in the same dual spaces and we can say that $H_{0,0}^{r,s}(Q_T)$ and $H_{0,0}^{-r,-s}(Q_T)$ are indeed dual, as likewise for the other pairings.

Finally, let the space $\mathcal{V}(Q_T)$ consist of all the functions v with $v \circ \kappa \in \mathcal{V}(Q_0)$ and

$$\mathcal{V}(Q_0) := \{u \in L^2((0, T); H^1(\Omega_0)) : \partial_t u \in L^2((0, T); H^{-1}(\Omega_0))\}.$$

The norm on this space is given by

$$\|u\|_{\mathcal{V}(Q_0)}^2 := \|u\|_{H^{1,0}(Q_0)}^2 + \|\partial_t u\|_{H^{-1,0}(Q_0)}^2.$$

Note that the space $\mathcal{V}(Q_0)$ is a dense subspace of $H_{0,0}^{1,\frac{1}{2}}(Q_0)$, which follows according to [3, Formula (2.2)] from the interpolation result

$$L^2(I; X) \cap H^1(I; Y) \subset H^{\frac{1}{2}}(I; [X, Y]_{\frac{1}{2}}) \cap C(\bar{I}; [X, Y]_{\frac{1}{2}}) \tag{2.5}$$

for $X \subset Y$ being Hilbert spaces.

3. Dirichlet Problem

3.1. Dirichlet Trace Operator on Cylindrical Domains

We first introduce the notion of traces with respect to cylindrical domains. According to [4, Section 2.3], we can define the (interior) Dirichlet trace for a function $u \in C^1(\bar{Q}_0)$ as

$$\gamma_0 u(t, \mathbf{x}) := \lim_{\Omega_0 \ni \mathbf{y} \rightarrow \mathbf{x} \in \Gamma_0} u(t, \mathbf{y}) \quad \text{for } (t, \mathbf{x}) \in \Sigma_0.$$

We thus have $\gamma_0 u = u|_{\Sigma_0}$. We can introduce a similar operator on anisotropic Sobolev spaces, see the following lemma, being along the lines of [11, Theorem

2.1]. It has been proven for $\Gamma_0 \in C^\infty$, but it is also true for a Lipschitz boundary in accordance with [3, pg. 504ff].

Lemma 3.1. *The map $\gamma_0: H^{1, \frac{1}{2}}(Q_0) \rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$ is linear and continuous.*

We find the following statement in [3, Lemma 2.4], which holds in the case of a Lipschitz domain.

Lemma 3.2. *The Dirichlet trace operator γ_0 is continuous and surjective as an operator from $H_{;0}^{1, \frac{1}{2}}(Q_0)$ to $H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$.*

According to [4, Theorem 2.4], there exists also an extension operator. The extension operator is a right inverse to the surjective Dirichlet trace operator γ_0 and, thus, extends a function defined only on the boundary to the space (see also [5, pg. 12] and [3, Definition 2.17]).

Lemma 3.3. *The Dirichlet trace operator $\gamma_0: H_{;0}^{1, \frac{1}{2}}(Q_0) \rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$ has a continuous right inverse operator $\mathcal{E}_0: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0) \rightarrow H_{;0}^{1, \frac{1}{2}}(Q_0)$, satisfying $\gamma_0 \mathcal{E}_0 v = v$ for all $v \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$.*

3.2. Dirichlet Trace Operator on Non-Cylindrical Domains

In this section, we denote the (interior) Dirichlet trace operator with respect to a non-cylindrical domain by $\gamma_{0,t}$ to distinguish it from the Dirichlet trace operator with respect to a cylindrical domain introduced above. When no confusion can happen, we will drop the subscript t in the trace operator for a non-cylindrical domain.

For a smooth function $u \in C^1(\overline{Q}_T)$, defined on a non-cylindrical domain, we set

$$\gamma_{0,t}u(t, \mathbf{x}_t) := \lim_{\Omega_t \ni \mathbf{y}_t \rightarrow \mathbf{x}_t \in \Gamma_t} u(t, \mathbf{y}_t).$$

It obviously holds

$$\begin{aligned} \gamma_{0,t}u(t, \mathbf{x}_t) &= \lim_{\substack{\Omega_0 \ni \mathbf{y} \rightarrow \mathbf{x} \in \Gamma_0, \\ \kappa(t, \mathbf{x}) = \mathbf{x}_t}} u(t, \kappa(t, \mathbf{y})) \\ &= \gamma_0(u \circ \kappa)(t, \mathbf{x}) = \gamma_0(u \circ \kappa)(t, \kappa^{-1}(t, \mathbf{x}_t)) \end{aligned}$$

for the diffeomorphism κ from (2.1). By density of the smooth functions in the Sobolev spaces, we can also extend this notion to Sobolev spaces. Moreover, we have the same mapping properties for $\gamma_{0,t}$ as for γ_0 , since

$$\begin{aligned} \|\gamma_{0,t}u\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)} &= \|\gamma_{0,t}u \circ \kappa\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)} \\ &= \|(\gamma_0(u \circ \kappa) \circ \kappa^{-1}) \circ \kappa\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)} \\ &\lesssim \|u \circ \kappa\|_{H^{1, \frac{1}{2}}(Q_0)} \\ &= \|u\|_{H^{1, \frac{1}{2}}(Q_T)}. \end{aligned}$$

Note that the hidden constant changes from line to line and depends on the diffeomorphism κ , because we used the norm equivalence on the cylindrical

and non-cylindrical domain as well as the mapping property of the Dirichlet trace operator on the cylindrical domain.

Due to this consideration, all the properties of Sect. 3.1 remain valid for the Dirichlet trace operator on non-cylindrical domains. The surjectivity follows for example from the following consideration: Let $v \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$. By the definition of the norm, we thus have $v \circ \kappa \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$. By the surjectivity of the Dirichlet trace operator with respect to Q_0 , there exists a $w \in H_{;0}^{1, \frac{1}{2}}(Q_0)$ with $\gamma_0 w = v \circ \kappa$. Due to the bijectivity of κ , we may define

$$\hat{w} := w \circ \kappa^{-1} \in H_{;0}^{1, \frac{1}{2}}(Q_T).$$

We thus have

$$\gamma_{0,t} \hat{w}(t, \mathbf{x}_t) = \gamma_0 w(t, \kappa^{-1}(t, \mathbf{x}_t)) = v(t, \mathbf{x}_t),$$

from where the subjectivity and hence the existence of the right inverse operator \mathcal{E}_0 follows.

3.3. Existence and Uniqueness of Dirichlet Problem

We consider the following Dirichlet problem with homogeneous initial datum

$$\begin{aligned} (\partial_t - \Delta)v &= f && \text{in } Q_T, \\ \gamma_0 v &= q && \text{on } \Sigma_T, \\ v(0, \cdot) &= 0 && \text{in } \Omega_0. \end{aligned} \tag{3.1}$$

We have the following existence and uniqueness theorem for its solution.

Theorem 3.4. *Let $f \in H_{;0}^{-1, -\frac{1}{2}}(Q_T)$ and $q \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$. Then, there exists a unique solution $v \in H_{;0}^{1, \frac{1}{2}}(Q_T)$, satisfying the boundary condition in (3.1) and*

$$\begin{aligned} S(v, \varphi) &:= \int_0^T \int_{\Omega_t} \{ \nabla v \cdot \nabla \varphi + \partial_t v \varphi \} \, d\mathbf{x} dt \\ &= \int_0^T \int_{\Omega_t} f u \, d\mathbf{x} dt \text{ for all } \varphi \in H_{0;0}^{1, \frac{1}{2}}(Q_T). \end{aligned} \tag{3.2}$$

Proof. A proof of this statement based on the proof of [3, Lemma 2.8] can be found in [1]. □

4. Neumann Trace Operator

Similarly as we defined the Dirichlet trace operator, we can also introduce an (interior) Neumann trace operator. In the following, we will first introduce this concept on cylindrical domains. Then, we will introduce the notion of a Neumann trace on a non-cylindrical domain formally and rigorously.

4.1. Neumann Trace Operator on Cylindrical Domains

We first introduce the Neumann trace operator, also called the conormal derivative, on a cylindrical domain along the lines of [3]. Let us define the space

$$H^{1,\frac{1}{2}}(Q_0; \mathcal{L}) := \{u \in H^{1,\frac{1}{2}}(Q_0) : \mathcal{L}u \in L^2(Q_0)\},$$

where $\mathcal{L} := \partial_t - \Delta$ is the partial differential operator under consideration. The norm on this space is given by

$$\|u\|_{H^{1,\frac{1}{2}}(Q_0; \mathcal{L})}^2 := \|u\|_{H^{1,\frac{1}{2}}(Q_0)}^2 + \|(\partial_t - \Delta)u\|_{L^2(Q_0)}^2.$$

According to [3, Lemma 2.16], the bilinear form

$$b(u, v) := \int_{Q_0} \{\langle \nabla u, \nabla v \rangle - (\partial_t - \Delta)uv\} \, d\mathbf{x}dt + d(u, v)$$

is continuous on $H^{1,\frac{1}{2}}(\mathbb{R} \times \Omega_0; \mathcal{L}) \times H^{1,\frac{1}{2}}(\mathbb{R} \times \Omega_0)$, where

$$d(u, v) := \int_{\mathbb{R} \times \Omega_0} \partial_t uv \, d\mathbf{x}dt.$$

The bilinear form $d(u, v)$ has a continuous extension from $C_0^\infty(\mathbb{R}^{d+1}) \times C_0^\infty(\mathbb{R}^{d+1})$ to $H^{\frac{1}{2}}(\mathbb{R}; L^2(\Omega_0)) \times H^{\frac{1}{2}}(\mathbb{R}; L^2(\Omega_0))$ and it holds $d(u, v) = -d(v, u)$ for all $u, v \in H^{\frac{1}{2}}(\mathbb{R}; L^2(\Omega_0))$, compare [3, Lemma 2.6].

The (interior) Neumann trace is defined for $u \in C^1(\overline{Q_0})$ by

$$\gamma_1^{\text{int}}u(t, \mathbf{x}) := \lim_{\Omega_0 \ni \mathbf{y} \rightarrow \mathbf{x} \in \Gamma_0} \langle \nabla_{\mathbf{y}}u(t, \mathbf{y}), \mathbf{n}_{\mathbf{x}} \rangle \quad \text{for } (t, \mathbf{x}) \in \Sigma_0$$

and coincides with the normal derivative on Σ_0 , thus $\gamma_1^{\text{int}}u = \partial u / \partial \mathbf{n}$ on Σ_0 , see [4, Section 3.3] and also [20, Satz 8.7] for the elliptic case. Since it holds

$$b(u, v) = \int_{\Sigma_0} \frac{\partial u}{\partial \mathbf{n}} v \, d\sigma dt$$

for $u, v \in C_0^2(\mathbb{R} \times \overline{\Omega_0})$, we can extend this definition as follows, corresponding to [3, Definition 2.17].

Definition 4.1. Let $u \in H^{1,\frac{1}{2}}(\mathbb{R} \times \Omega_0; \mathcal{L})$. Then, the Neumann trace operator $\gamma_1 u \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)$ is the continuous linear form on $H^{\frac{1}{2}, \frac{1}{2}}(\Sigma_0)$ defined by $\gamma_1^{\text{int}}u : \varphi \mapsto b(u, \mathcal{E}_0\varphi)$, where \mathcal{E}_0 is the extension operator given in Lemma 3.3.

Notice that we can also introduce the conormal derivative $\gamma_1^{\text{int}}u \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)$ as the unique solution of a variational problem, as it is done in [5, Section 3.4]. This variational problem can for example be obtained by applying $\gamma_1^{\text{int}}u$ to φ . According to [3, Proposition 2.18], the Neumann trace has the following properties.

Lemma 4.2. (i) *The map*

$$\gamma_1^{\text{int}} : H^{1,\frac{1}{2}}(\mathbb{R} \times \Omega_0; \mathcal{L}) \rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\mathbb{R} \times \Gamma_0)$$

is continuous and by restriction also the map

$$\gamma_1^{\text{int}} : H^{1,\frac{1}{2}}(Q_0; \mathcal{L}) \rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)$$

is continuous.

(ii) If $u \in C^2(\overline{Q_0})$, then $\gamma_1^{\text{int}} u = (\partial u / \partial \mathbf{n})|_{\Sigma_0}$ due to the Green formula.

4.2. Neumann Trace Operator on Non-Cylindrical Domains

On time-dependent boundaries, one could consider the usual Neumann trace, as it is done for example in [6, Section 6.1]. Instead, we follow here the idea of [18] and employ a velocity corrected Neumann trace. We first formally introduce this Neumann trace and afterwards characterize its properties rigorously.

4.2.1. Formal Derivation. For a time dependent spatial surface we define two Neumann trace operators

$$\gamma_1^\pm \varphi := \frac{\partial \varphi}{\partial \mathbf{n}_t} \mp \frac{1}{2} \langle \mathbf{V}, \mathbf{n}_t \rangle \varphi. \tag{4.1}$$

In order to motivate this definition, consider the boundary value problem

$$\begin{aligned} (\partial_t - \Delta)u &= f && \text{in } Q_T, \\ \gamma_1 u &= g && \text{on } \Sigma_T, \\ u(0, \cdot) &= 0 && \text{in } \Omega_0, \end{aligned} \tag{4.2}$$

where we leave it a priori open what γ_1 means. Let us formally derive the weak formulation of the Neumann problem (4.2) by multiplying with a test function v with $v(T, \cdot) = 0$ in Ω_T and using Reynolds' transport theorem

$$\begin{aligned} \int_0^T \int_{\Omega_t} f v \, d\mathbf{x} dt &= \int_0^T \int_{\Omega_t} (\partial_t - \Delta) u v \, d\mathbf{x} dt \\ &= \int_0^T \int_{\Omega_t} \{ \langle \nabla u, \nabla v \rangle + \partial_t(uv) - u \partial_t v \} \, d\mathbf{x} dt - \int_0^T \int_{\Gamma_t} \frac{\partial u}{\partial \mathbf{n}} v \, d\sigma dt \\ &= \int_0^T \int_{\Omega_t} \{ \langle \nabla u, \nabla v \rangle - u \partial_t v \} \, d\mathbf{x} dt + \int_0^T \frac{d}{dt} \int_{\Omega_t} u v \, d\mathbf{x} dt \\ &\quad - \int_0^T \int_{\Gamma_t} \left\{ \frac{\partial u}{\partial \mathbf{n}} v + u v \langle \mathbf{V}, \mathbf{n} \rangle \right\} \, d\sigma dt. \end{aligned}$$

Due to the fundamental theorem of calculus and the vanishing initial and end condition of u and v , respectively, we obtain the variational equation

$$a(u, v) = \int_0^T \int_{\Omega_t} f v \, d\mathbf{x} dt + \int_0^T \int_{\Gamma_t} \left\{ \frac{\partial u}{\partial \mathbf{n}} v + \frac{1}{2} u v \langle \mathbf{V}, \mathbf{n} \rangle \right\} \, d\sigma dt$$

with bilinear form

$$a(u, v) := \int_0^T \int_{\Omega_t} \{ \langle \nabla u, \nabla v \rangle - u \partial_t v \} \, d\mathbf{x} dt - \frac{1}{2} \int_0^T \int_{\Gamma_t} u v \langle \mathbf{V}, \mathbf{n} \rangle \, d\sigma dt.$$

Thus, if we set the previously unspecified trace in (4.2) as γ_1^- , we arrive at

$$a(u, v) = \int_0^T \int_{\Omega_t} f v \, d\mathbf{x} dt + \int_0^T \int_{\Gamma_t} g v \, d\sigma dt.$$

With the additional boundary integral term it is easy to see that the bilinear form is bounded and coercive in the $H^{1, \frac{1}{2}}$ -norm, so Lion's projection theorem guarantees existence and uniqueness of this Neumann problem. However,

for the cylindrical case, [3, Lemma 2.21] states that this strategy does not yield satisfactory results, since one has to make stronger assumptions on the regularity of the input data. Therefore, as in [3], we will proof the existence and uniqueness of solutions by using a boundary integral formulation (see Corollary 5.14).

4.2.2. Rigorous Derivation. We assume κ to be defined on $\mathbb{R} \times \mathbb{R}^d$ and not only on $[0, T] \times \mathbb{R}^d$. Moreover, for sake of simplicity in representation, we always consider functions u and v throughout this section which satisfy

$$\int_{\mathbb{R}} \frac{d}{dt} \int_{\Omega_t} uv \, d\mathbf{x} dt = 0. \quad (4.3)$$

This assumption stems from the fact that we would like to integrate by parts in time. Later on, we will consider a finite time interval $(0, T)$ and equip u and v with the appropriate zero initial and end conditions. Extending u and v by zero for $t < 0$ and $t > T$, respectively, leads then to the fulfillment of (4.3).

Let us define

$$d(u, v) := \int_{\mathbb{R}} \int_{\Omega_t} \partial_t uv \, d\mathbf{x} dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, d\sigma dt. \quad (4.4)$$

Notice that the additional boundary term is a speciality of the time-dependent boundary. We shall first state the analogue of [3, Lemma 2.6].

Lemma 4.3. *The bilinear form $d(u, v)$ has a continuous extension from $C_0^\infty(\mathbb{R}^{d+1}) \times C_0^\infty(\mathbb{R}^{d+1})$ to $H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t)) \times H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t))$, and it holds*

$$d(u, v) = -d(v, u). \quad (4.5)$$

Proof. The use of Reynolds' transport theorem allows us to compute

$$\begin{aligned} d(u, v) &= \int_{\mathbb{R}} \int_{\Omega_t} \partial_t uv \, d\mathbf{x} dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, d\sigma dt \\ &= \int_{\mathbb{R}} \int_{\Omega_t} \{\partial_t(uv) - u\partial_t v\} \, d\mathbf{x} dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, d\sigma dt \\ &= \int_{\mathbb{R}} \frac{d}{dt} \int_{\Omega_t} uv \, d\mathbf{x} dt - \int_{\mathbb{R}} \int_{\Omega_t} u\partial_t v \, d\mathbf{x} dt - \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, d\sigma dt. \end{aligned}$$

The assumption (4.3) hence implies

$$d(u, v) = - \int_{\mathbb{R}} \int_{\Omega_t} u\partial_t v \, d\mathbf{x} dt - \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, d\sigma dt,$$

from where (4.5) follows immediately. The rest is in complete analogy to [3, Lemma 2.6], but we need higher regularity in the spatial variable instead just $L^2(\Omega_0)$ like in [3], because the boundary term in the definition of $d(u, v)$ has to be well-defined. \square

As in Sect. 4.1, we introduce the space

$$H^{1, \frac{1}{2}}(Q_T; \mathcal{L}) := \{u \in H^{1, \frac{1}{2}}(Q_T) : \mathcal{L}u \in L^2(Q_T)\},$$

where $\mathcal{L} := \partial_t - \Delta$ is the differential operator on the non-cylindrical domain. We state the analogue of [3, Lemma 2.16] in the case of a non-cylindrical domain, the proof of which is obvious.

Lemma 4.4. *The bilinear form*

$$b^-(u, v) := \int_0^T \int_{\Omega_t} \{ \langle \nabla u, \nabla v \rangle - (\partial_t - \Delta)uv \} \, dxdt + d(u, v)$$

with $d(u, v)$ being defined in (4.4) is continuous on $H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t); \partial_t - \Delta) \times H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t))$. If $u, v \in C_0^2(\bigcup_{t \in \mathbb{R}}(\{t\} \times \bar{\Omega}_t))$, we have

$$b^-(u, v) = \int_0^T \int_{\Gamma_t} \left\{ \frac{\partial u}{\partial \mathbf{n}} v + \frac{1}{2} \langle \mathbf{V}, \mathbf{n} \rangle uv \right\} \, d\sigma dt$$

by means of Green’s formula.

In complete analogy to the Neumann trace operator in the cylindrical case, we will define $\gamma_1^- u$, which is one of the two required Neumann trace operators.

Definition 4.5. Given $u \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t); \partial_t - \Delta)$, we denote by $\gamma_1^- u \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ the continuous linear form on $H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ defined by $\gamma_1^- u: \varphi \mapsto b^-(u, \mathcal{E}_0 \varphi)$, where \mathcal{E}_0 is the extension operator as mentioned in Sect. 3.2.

The following lemma is the non-cylindrical equivalent to [3, Proposition 2.18].

Lemma 4.6. *The map*

$$\gamma_1^- : H^{1, \frac{1}{2}}\left(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t); \partial_t - \Delta\right) \rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}\left(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Gamma_t)\right)$$

is continuous and by restriction also the map

$$\gamma_1^- : H^{1, \frac{1}{2}}(Q_T; \partial_t - \Delta) \rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$$

is continuous. Moreover, if $u \in C^2(\bar{Q}_T)$, then it holds

$$\gamma_1^- u = \frac{\partial u}{\partial \mathbf{n}} + \frac{1}{2} \langle \mathbf{V}, \mathbf{n} \rangle u.$$

Proof. As in [3], the continuity is a consequence of the continuity of the bilinear form $b(\cdot, \cdot)$ (cf. Lemma 4.4). The second statement follows immediately from Green’s first formula. \square

Remark 4.7. In view of the reformulation of the heat equation in terms of boundary integral equations, we will moreover encounter a second Neumann trace operator, which we denote by γ_1^+ . It can be achieved analogously to above by considering the differential operator $\partial_t + \Delta$ instead of $\partial_t - \Delta$. The former operator for example arises when considering a time reversal of the latter one. With

$$b^+(u, v) := \int_0^T \int_{\Omega_t} \{ \langle \nabla u, \nabla v \rangle + (\partial_t + \Delta)uv \} \, dxdt - d(u, v),$$

we can state the analogue of Lemma 4.4, namely the continuity of $b^+(\cdot, \cdot)$ in the appropriate space and for u and v smooth enough, we have

$$b^+(u, v) = \int_0^T \int_{\Gamma_t} \left\{ \frac{\partial u}{\partial \mathbf{n}} v - \frac{1}{2} \langle \mathbf{V}, \mathbf{n} \rangle uv \right\} d\sigma dt.$$

With this property at hand, we can define the trace operator γ_1^+ in analogy to Definition 4.5. For u smooth enough, it then holds

$$\gamma_1^+ u = \frac{\partial u}{\partial \mathbf{n}} - \frac{1}{2} \langle \mathbf{V}, \mathbf{n} \rangle u.$$

The existence of two Neumann trace operators is a speciality of the time-dependent boundary.

Likewise to [3, Formula (2.35)], given $u \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t); \partial_t - \Delta)$ and $v \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t))$, we obtain Green’s first formula

$$\int_{\mathbb{R}} \int_{\Omega_t} \langle \nabla u, \nabla v \rangle dx dt + d(u, v) = \int_{\mathbb{R}} \int_{\Omega_t} (\partial_t - \Delta) uv dx dt + \langle \gamma_1^- u, \gamma_0 v \rangle. \tag{4.6}$$

By restriction, this formula also holds for $u \in H^{1, \frac{1}{2}}(Q_T; \partial_t - \Delta)$ and $v \in H^{1, \frac{1}{2}}(Q_T)$, but not, as was pointed out in [3], when u, v are both in $H^{1, \frac{1}{2}}(Q_T)$.

In complete analogy to (4.6), Green’s formula for $u \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t); \partial_t + \Delta)$ and $v \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}}(\{t\} \times \Omega_t))$ reads

$$\int_{\mathbb{R}} \int_{\Omega_t} \langle \nabla u, \nabla v \rangle dx dt - d(u, v) = \int_{\mathbb{R}} \int_{\Omega_t} (-\partial_t - \Delta) uv dx dt + \langle \gamma_1^+ u, \gamma_0 v \rangle.$$

Again, by restriction, this formula also holds for $u \in H^{1, \frac{1}{2}}(Q_T; \partial_t + \Delta)$ and $v \in H^{1, \frac{1}{2}}(Q_T)$.

We are now in the position to state Green’s formulas for a finite time interval, the time-independent analogues of which are given in [3, Proposition 2.19]. Notice, however, that [3] introduces a time reversal map. For a time-dependent domain, this approach does not make sense, since the integration over a time forward tube of a time reversed entity is not always well defined. Therefore, we choose a slightly different approach to obtain a further Green formula.

Lemma 4.8. (i) Let $u \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}_+}(\{t\} \times \Omega_t); \partial_t - \Delta)$ and $v \in H^{1, \frac{1}{2}}(\bigcup_{-\infty < t < t_0}(\{t\} \times \Omega_t))$. Then, for $t_0 > 0$, there holds Green’s first formula

$$\int_0^{t_0} \int_{\Omega_t} \langle \nabla u, \nabla v \rangle dx dt + d(u, v) = \int_0^{t_0} \int_{\Omega_t} (\partial_t - \Delta) uv dx dt + \langle \gamma_1^- u, \gamma_0 v \rangle.$$

(ii) Let $u \in H^{1, \frac{1}{2}}(\bigcup_{-\infty < t < t_0}(\{t\} \times \Omega_t); \partial_t + \Delta)$ and $v \in H^{1, \frac{1}{2}}(\bigcup_{t \in \mathbb{R}_+}(\{t\} \times \Omega_t))$. Then, for $t_0 > 0$, there holds Green’s alternative first formula

$$\int_0^{t_0} \int_{\Omega_t} \langle \nabla u, \nabla v \rangle dx dt - d(u, v) = \int_0^{t_0} \int_{\Omega_t} (-\partial_t - \Delta) uv dx dt + \langle \gamma_1^+ u, \gamma_0 v \rangle.$$

(iii) Let $u \in H_{;0}^{1,\frac{1}{2}}(\bigcup_{t \in \mathbb{R}_+} (\{t\} \times \Omega_t); \partial_t - \Delta)$ and $v \in H_{;0}^{1,\frac{1}{2}}(\bigcup_{-\infty < t < t_0} (\{t\} \times \Omega_t); \partial_t + \Delta)$. Then, for $t_0 > 0$, there holds Green's second formula

$$\int_0^{t_0} \int_{\Omega_t} \{(\partial_t - \Delta)uv + u(\partial_t + \Delta)v\} \, dxdt = \langle \gamma_0 u, \gamma_1^+ v \rangle - \langle \gamma_1^- u, \gamma_0 v \rangle.$$

Proof. Statements (i) and (ii) are clear. Statement (iii) follows then immediately from these by interchanging v and u in (ii) and using (4.5). \square

We need the tube equivalent of [3, Lemma 2.22]. In there, the space $\tilde{C}^\infty(\bar{Q}_0) := C_0^\infty((0, T] \times \bar{\Omega}_0)$ is defined as the space of the restrictions of functions in $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ to \bar{Q}_0 . This space $\tilde{C}^\infty(\bar{Q}_0)$ is dense in $H_{;0}^{1,\frac{1}{2}}(Q_0; \partial_t - \Delta)$ according to [3, Lemma 2.22]. As we only consider a C^2 -mapping between the reference cylinder and the tube, we will prove the analogue result only for C^2 -functions.

Lemma 4.9. *The space*

$$\tilde{C}^2(\bar{Q}_T) := \left\{ u : u \circ \kappa \in C_0^2((0, T] \times \bar{\Omega}_0) \right\}$$

is dense in $H_{;0}^{1,\frac{1}{2}}(Q_T; \partial_t - \Delta)$.

Proof. We modify the proof of [3, Lemma 2.22] appropriately. According to this proof, $C_0^\infty((0, T] \times \bar{\Omega}_0)$ is dense in $H_{;0}^{1,\frac{1}{2}}(Q_0)$. Therefore, also $C_0^2((0, T] \times \bar{\Omega}_0)$ is dense in $H_{;0}^{1,\frac{1}{2}}(Q_0)$. Due to the definition of the spaces on the tube via the mapping κ and the resulting equivalence of norms, we also obtain that $\tilde{C}^2(\bar{Q}_T)$ is dense in $H_{;0}^{1,\frac{1}{2}}(Q_T)$. Similarly, we obtain that

$$\tilde{C}^2(Q_T) := \left\{ u : u \circ \kappa \in C_0^2((0, T] \times \Omega_0) \right\}$$

is dense in $H_{;0}^{1,\frac{1}{2}}(Q_T)$.

Let \mathcal{R} be an extension operator from $H_{;0}^{1,\frac{1}{2}}(Q_T)$ to $H^{1,\frac{1}{2}}(\mathbb{R}^{d+1})$. It thus holds $(\mathcal{R}u)|_{Q_T} = u$. As in [3], let us choose \mathcal{R} such that $\text{supp } \mathcal{R} \subset [0, \infty) \times \mathbb{R}^d$.¹ In that way, we can identify $H_{;0}^{1,\frac{1}{2}}(Q_T)$ with a closed subspace of $H_{;0}^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$. The map $u \mapsto (\mathcal{R}u, (\partial_t - \Delta)u)$ identifies $H_{;0}^{1,\frac{1}{2}}(Q_T; \partial_t - \Delta)$ with a closed subspace of $H_{;0}^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d) \times L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. Due to this identification, we find for every bounded linear functional $\ell : H_{;0}^{1,\frac{1}{2}}(Q_T; \partial_t - \Delta) \rightarrow \mathbb{R}$ some $f \in (H_{;0}^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d))' = H^{-1,-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $g \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ such that it holds

$$\langle \ell, u \rangle = \langle f, \mathcal{R}u \rangle + \int_{\mathbb{R}_+ \times \mathbb{R}^d} g(\partial_t - \Delta)u \, d(t, \mathbf{x})$$

¹Such an extension operator exists as it can be defined by $\mathcal{R}u = (\tilde{\mathcal{R}}(u \circ \kappa)) \circ \kappa^{-1}$ with $\tilde{\mathcal{R}} : H_{;0}^{1,\frac{1}{2}}(Q_0) \rightarrow H^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ being the extension operator from [3].

for all $u \in H_{0,0}^{1,\frac{1}{2}}(Q_T; \partial_t - \Delta)$. Since ℓ acts only on u , which is supported on \overline{Q}_T , we may assume that $\text{supp } f \subset \overline{Q}_T$ and $\text{supp } g \subset \overline{Q}_T$.

We shall suppose next that it holds $\langle \ell, \varphi \rangle = 0$ for all $\varphi \in \widetilde{C}^2(\overline{Q}_T)$. If we can show $\ell = 0$, we obtain the desired density result in accordance with [19, Korollar III.1.9]. For all $\varphi \in C_0^2(\mathbb{R}_+ \times \mathbb{R}^d)$, we conclude

$$0 = \langle \ell, \varphi \rangle = \langle f, \varphi \rangle + \int_{Q_T} g(\partial_t - \Delta)\varphi \, d(t, \mathbf{x}) = \langle f, \varphi \rangle + \int_{\mathbb{R}_+ \times \mathbb{R}^d} g(\partial_t - \Delta)\varphi \, d(t, \mathbf{x}).$$

This equation states that $f = (\partial_t + \Delta)g$ holds on $\mathbb{R}_+ \times \mathbb{R}^d$ in complete analogy to [3]. Due to $f \in H^{-1,-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and the differential operator, we find $g \in H^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and, thus, $g|_{Q_T} \in H_{0,0}^{1,\frac{1}{2}}(Q_T)$.

On a cylindrical domain, any function $h \in H_{0,0}^{1,\frac{1}{2}}(Q_0)$ can be approximated by a series $h_n \in C_0^\infty((-\infty, T) \times \Omega_0)$ (see [3, Proof of Lemma 2.22]). Hence, by choosing $h := g \circ \kappa$ and setting $g_n := h_n \circ \kappa^{-1}$, we can approximate $g|_{Q_T} \in H_{0,0}^{1,\frac{1}{2}}(Q_T)$ by a series $g_n \in C_0^2(\bigcup_{-\infty < t < T} (\{t\} \times \Omega_t))$ in the norm of $H^{1,\frac{1}{2}}(\bigcup_{0 < t < \infty} (\{t\} \times \Omega_t))$. Thus, denoting by \hat{g}_n the extension of g_n by zero outside of Q_T , we find $(\partial_t + \Delta)\hat{g}_n \rightarrow f$ in $H^{-1,-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$. We then conclude for any $u \in H_{0,0}^{1,\frac{1}{2}}(Q_T; \partial_t - \Delta)$ that

$$\begin{aligned} \langle \ell, u \rangle &= \lim_{n \rightarrow \infty} \left[\langle (\partial_t + \Delta)\hat{g}_n, \mathcal{R}u \rangle + \int_{Q_T} g_n(\partial_t - \Delta)u \, d(t, \mathbf{x}) \right] \\ &= \lim_{n \rightarrow \infty} \left[\int_{Q_T} (\partial_t + \Delta)g_n u \, d(t, \mathbf{x}) + \int_{Q_T} g_n(\partial_t - \Delta)u \, d(t, \mathbf{x}) \right] = 0. \end{aligned}$$

The expression above is equal to zero, because $u = 0$ for $t = 0$, $g_n = 0$ for $t = T$, and g_n has a zero boundary condition. \square

Remark 4.10. If we consider $t \mapsto T - t$ in Lemma 4.9, we obtain that $\widehat{C}^2(\overline{Q}^T)$ is dense in $H_{0,0}^{1,\frac{1}{2}}(Q^T; -\partial_t - \Delta)$, where

$$\widehat{C}^2(\overline{Q}_T) := \left\{ u : u \circ \kappa \in C_0^2([0, T] \times \overline{\Omega}_0) \right\}$$

and Q^T is the time flipped Q_T . Hence, $\widehat{C}^2(\overline{Q}_T)$ is dense in $H_{0,0}^{1,\frac{1}{2}}(Q_T; \partial_t + \Delta)$ as Q_T was arbitrary.

Next, we will introduce a lemma concerning the trace maps, which will be later used in the proof of the jump relations. It is the analogue of [3, Lemma 2.23].

Lemma 4.11. *The combined trace map $(\gamma_0, \gamma_1^+): u \mapsto (\gamma_0 u, \gamma_1^+ u)$ maps $\widehat{C}^2(\overline{Q}_T)$ onto a dense subspace of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$.*

Proof. We mimic the respective proof from [3], but will not use a time reversal map. Let us assume a linear functional $(\chi, \psi) \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ that vanishes on the range of (γ_1^+, γ_0) . We need to show that $(\chi, \psi) = (0, 0)$, since then the density follows by [19, Korollar III.1.9]. To this end, assume

$$\langle \chi, \gamma_1^+ \varphi \rangle = \langle \psi, \gamma_0 \varphi \rangle \quad \text{for all } \varphi \in \widehat{C}^2(\overline{Q}_T). \tag{4.7}$$

Let

$$\mathcal{T} = (g \mapsto \mathcal{T}g) : H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) \rightarrow H^{1, \frac{1}{2}}_{;0}(Q_T)$$

be the solution operator (see Theorem 3.4) of the Dirichlet problem

$$\begin{aligned} (\partial_t - \Delta)(\mathcal{T}g) &= 0 && \text{in } Q_T, \\ \gamma_0(\mathcal{T}g) &= g && \text{on } \Sigma_T. \end{aligned} \tag{4.8}$$

Moreover, let

$$\mathcal{S} = (f \mapsto \mathcal{S}f) : L^2(Q_T) \rightarrow H^{1, \frac{1}{2}}_{0;0}(Q_T)$$

be the solution operator (see Theorem 3.4 used for the substitution $t \mapsto T - t$) of the Dirichlet problem

$$\begin{aligned} (\partial_t + \Delta)(\mathcal{S}f) &= f && \text{in } Q_T, \\ \gamma_0(\mathcal{S}f) &= 0 && \text{on } \Sigma_T. \end{aligned}$$

We can apply Green's second formula from Lemma 4.8 to $u := \mathcal{T}\chi$ and $v := \mathcal{S}f$ for any $f \in L^2(Q_T)$, since $u \in H^{1, \frac{1}{2}}_{;0}(Q_T; \partial_t - \Delta)$ and $(\partial_t + \Delta)v \in L^2(Q_T)$. We obtain

$$\int_{Q_T} \{(\partial_t - \Delta)uv + u(\partial_t + \Delta)v\} d(\mathbf{x}, t) = \langle \gamma_0 u, \gamma_1^+ v \rangle - \langle \gamma_1^- u, \gamma_0 v \rangle.$$

Since $\gamma_0 v = 0$ and $\gamma_0 u = \chi$, as well as $(\partial_t - \Delta)u = 0$ and $(\partial_t + \Delta)v = f$, we obtain

$$\int_{Q_T} uf d(t, \mathbf{x}) = \langle \chi, \gamma_1^+ v \rangle.$$

Due to continuity and Remark 4.10, (4.7) holds also for all $\varphi \in H^{1, \frac{1}{2}}_{;0}(Q_T; \partial_t + \Delta)$ and, thus, also for $\varphi = \mathcal{S}f$. This implies

$$\langle \chi, \gamma_1^+ \mathcal{S}f \rangle = \langle \psi, \gamma_0 \mathcal{S}f \rangle.$$

Since $\gamma_0 \mathcal{S}f = 0$, we thus obtain $\int_{Q_T} uf d(t, \mathbf{x}) = 0$ for all $f \in L^2(Q_T)$. Therefore, $0 = u = \mathcal{T}(\chi)$ and thus $\chi = \gamma_0 u = 0$. Looking again at (4.7) gives

$$\langle \psi, \gamma_0 \varphi \rangle = 0 \quad \text{for all } \varphi \in H^{1, \frac{1}{2}}_{;0}(Q_T).$$

The trace map γ_0 is not only surjective for $\varphi \in H^{1, \frac{1}{2}}_{;0}(Q_T)$ as shown in Sect. 3.2, but also for $\varphi \in H^{1, \frac{1}{2}}_{;0}(Q_T)$ if one considers the backward problem. We may hence conclude that $\psi = 0$. \square

Subsequently, we state the analogue of [3, Proposition 2.24].

Lemma 4.12. *Green's first formula given in (4.6) holds for all $u \in H^{1, \frac{1}{2}}_{;0}(Q_T; \partial_t - \Delta)$ and $v \in H^{1, \frac{1}{2}}_{;0}(Q_T)$. If also $v \in H^{1, \frac{1}{2}}_{;0}(Q_T; \partial_t - \Delta)$, we can write the Green's formula as*

$$\begin{aligned} \int_{Q_T} \langle \nabla u, \nabla v \rangle d(t, \mathbf{x}) - d(v, u) + \int_{\Omega_T} u(T, \mathbf{x})v(T, \mathbf{x}) d\mathbf{x} \\ = \langle \gamma_1^- u, \gamma_0 v \rangle + \int_{Q_T} (\partial_t - \Delta)uv d(t, \mathbf{x}). \end{aligned} \tag{4.9}$$

Proof. We adopt the proof of [3, Proposition 2.24]. Given $u \in \tilde{C}^2(\overline{Q}_T)$ and $v \in \tilde{C}^1(\overline{Q}_T)$ ², we find

$$\begin{aligned} \int_{Q_T} \langle \nabla u, \nabla v \rangle d(t, \mathbf{x}) + \int_{Q_T} \partial_t uv d(t, \mathbf{x}) + \frac{1}{2} \int_0^T \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv d\sigma dt \\ = \langle \gamma_1^- u, \gamma_0 v \rangle + \int_{Q_T} (\partial_t - \Delta) uv d(t, \mathbf{x}). \end{aligned} \quad (4.10)$$

All terms are continuous with respect to v in the $H_{;0}^{1, \frac{1}{2}}(Q_T)$ -norm. Thus, by continuity, we can extend (4.10) to all $v \in H_{;0}^{1, \frac{1}{2}}(Q_T)$. Let $v \in H_{;0}^{1, \frac{1}{2}}(Q_T)$ be fixed. Then, all terms in (4.10) except the term containing the $\partial_t u$ are obviously continuous with respect to u in the norm of $H_{;0}^{1, \frac{1}{2}}(Q; \partial_t - \Delta)$. Therefore, also the term containing $\partial_t u$ is continuous. Lemma 4.9 allows to extend (4.10) to all $u \in H_{;0}^{1, \frac{1}{2}}(Q_T; \partial_t - \Delta)$. Thus, Green's first formula holds as given in the claim.

For $u, v \in \tilde{C}^2(\overline{Q}_T)$, (4.9) holds. As in [3], the term $\int_{\Omega_T} u(T, \mathbf{x})v(T, \mathbf{x}) d(t, \mathbf{x})$ is continuous for u and v in the norm of $H_{;0}^{1, \frac{1}{2}}(Q_T; \partial_t - \Delta) \subset \mathcal{V}(Q_T)$ and $\mathcal{V}(Q_T)$ consists of functions φ , which always satisfy $\varphi \circ \kappa \in C([0, T]; L^2(\Omega_0))$. From here, the second claim follows. \square

5. The Calderón Operator

Let us first introduce the fundamental solution for the heat equation, which, in accordance with e.g. [18], reads

$$G(t, \tau, \mathbf{x}, \mathbf{y}) := \begin{cases} \frac{1}{(4\pi(t-\tau))^{d/2}} \exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(t-\tau)}\right), & \text{if } \tau < t, \\ 0, & \text{if } \tau \geq t. \end{cases}$$

Notice that this is equivalent to consider $\overline{G}(t - \tau, \mathbf{x}, \mathbf{y})$, where \overline{G} is given by

$$\overline{G}(\tau, \mathbf{x}, \mathbf{y}) := \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4\tau}\right) \frac{1}{2}(1 + \text{sign } \tau),$$

as introduced in [3, Formula (2.39)]. Moreover, let us denote

$$\tilde{G}(t, \mathbf{x}) = \begin{cases} \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{\|\mathbf{x}\|^2}{4t}\right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \quad (5.1)$$

²The space $\tilde{C}^1(\overline{Q}_T)$ is defined in complete analogy to $\tilde{C}^2(\overline{Q}_T)$ via $\tilde{C}^1(Q_T) := \{u: u \circ \kappa \in C_0^1((0, T] \times \Omega_0)\}$.

For $u \in C^2(\bigcup_{0 < t < \infty} (\{t\} \times \bar{\Omega}_t))$ with $u(0, \mathbf{x}) = 0$ on Ω_0 , we have for $(t_0, \mathbf{x}_0) \in \bigcup_{0 < t < \infty} (\{t\} \times \bar{\Omega}_t)$ that

$$\begin{aligned} u(t_0, \mathbf{x}_0) &= \int_{Q_T} \int_{\Omega_t} (\partial_t - \Delta)u(t, \mathbf{x}) \tilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \, d(t, \mathbf{x}) \\ &\quad + \int_{\Sigma_T} \tilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(t, \mathbf{x}) \, d\sigma_{(t, \mathbf{x})} \\ &\quad - \int_{\Sigma_T} \frac{\partial \tilde{G}}{\partial \mathbf{n}}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) u(t, \mathbf{x}) \, d\sigma_{(t, \mathbf{x})} \\ &\quad + \int_{\Sigma_T} \tilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \langle \mathbf{V}, \mathbf{n} \rangle(t, \mathbf{x}) u(\mathbf{y}, \tau) \, d\sigma_{(t, \mathbf{x})}, \end{aligned}$$

as it can be seen from Lemma 4.8 and the property of the fundamental solution. Moreover, we will only look at the case, for which $(\partial_t - \Delta)u = 0$ holds.

We introduce the single and double layer potentials as

$$\begin{aligned} \tilde{\mathcal{V}}\varphi(t_0, \mathbf{x}_0) &:= \langle \varphi, \gamma_0 G \rangle = \int_{\Sigma_T} G(t_0, t, \mathbf{x}_0, \mathbf{y}) \varphi(t, \mathbf{y}) \, d\sigma_{(t, \mathbf{y})}, \\ \tilde{\mathcal{K}}w(t_0, \mathbf{x}_0) &:= \langle \gamma_1^+ G, w \rangle = \int_{\Sigma_T} \gamma_{1, (t, \mathbf{y})}^+ G(t_0, t, \mathbf{x}_0, \mathbf{y}) w(t, \mathbf{y}) \, d\sigma_{(t, \mathbf{y})}. \end{aligned}$$

Then, similarly to [3, Theorem 2.20], we obtain the representation formula from [18, Equation (6)] given in the following lemma.

Lemma 5.1. *Let $u \in H^{1, \frac{1}{2}}(Q_T)$ with $(\partial_t - \Delta)u = 0$ in Q_T . Then, we have the representation formula*

$$u(t, \tilde{\mathbf{x}}) = \tilde{\mathcal{V}}\gamma_1^- u(t, \tilde{\mathbf{x}}) - \tilde{\mathcal{K}}u(t, \tilde{\mathbf{x}}) \quad \text{for all } (t, \tilde{\mathbf{x}}) \in Q_T. \tag{5.2}$$

As in [3, pg. 514], we can rewrite the definition of the single layer potential by

$$\begin{aligned} \tilde{\mathcal{V}}\varphi(t_0, \mathbf{x}_0) &= \langle \varphi, \gamma_0 G(t, t_0, \mathbf{x}, \mathbf{x}_0) \rangle \\ &= \langle \gamma'_0 \varphi, G(t, t_0, \mathbf{x}, \mathbf{x}_0) \rangle \\ &= \tilde{G} \star (\gamma'_0 \varphi)(t_0, \mathbf{x}_0), \end{aligned} \tag{5.3}$$

where \tilde{G} is given in (5.1) and

$$\langle \gamma'_0 \varphi, \chi \rangle = \langle \varphi, \gamma_0 \chi \rangle = \int_{\Sigma_T} \varphi \chi \, d\sigma_{(t, \mathbf{x})}$$

for all $\chi \in C_0^\infty(\mathbb{R}^{d+1})$. We will use this also for $\chi \in C_0^2(\mathbb{R}^{d+1})$.

We would like to find the mapping properties of the single and double layer potentials, which are the equivalent of the results given in [3, Proposition 3.1, Remark 3.2, and Proposition 3.3].

Lemma 5.2. *The mappings*

$$\begin{aligned} \tilde{\mathcal{V}}: H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) &\rightarrow H_{0, \frac{1}{2}}^1(Q_T; \partial_t - \Delta), \\ \tilde{\mathcal{K}}: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) &\rightarrow H_{0, \frac{1}{2}}^1(Q_T; \partial_t - \Delta) \end{aligned}$$

are continuous.

Proof. The statement for the single layer potential follows as in [3, pg. 514–515] in the case of a cylindrical domain. In there, the claim is proven by considering the problem on \mathbb{R}^{d+1} using Fourier techniques and then restricting it appropriately, which can also be done in the case of a non-cylindrical domain. The statement for the double layer potential is derived in complete analogy to [3, pg. 515]. \square

We can take the traces γ_0 of the single and double layer potential. Let the radius R be large enough such that the boundary Γ_t is contained in the ball $B_R := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| < R\}$ and set $\Omega_t^c := B_R \setminus \overline{\Omega}_t$ and $Q_T^c := \bigcup_{-\infty < t < T} (\{t\} \times \Omega_t^c)$. Lemma 5.2 provides also the continuity of the mappings

$$\begin{aligned}\tilde{\mathcal{V}} &: H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) \rightarrow H_{;0}^{1, \frac{1}{2}}(Q_T^c; \partial_t - \Delta), \\ \tilde{\mathcal{K}} &: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) \rightarrow H_{;0}^{1, \frac{1}{2}}(Q_T^c; \partial_t - \Delta).\end{aligned}$$

In order to state the tube analogue of [3, Theorem 3.4], we define the jumps as in [3, Formula (3.16)] in accordance with

$$[\gamma_0 u] := \gamma_0(u|_{Q_T^c}) - \gamma_0(u|_{Q_T}), \quad [\gamma_1^\pm u] := \gamma_1^\pm(u|_{Q_T^c}) - \gamma_1^\pm(u|_{Q_T}).$$

We then have:

Lemma 5.3. *For all $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ and all $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$, there hold the jump relations*

$$\begin{aligned}[\gamma_0 \tilde{\mathcal{V}}\psi] &= 0, & [\gamma_1^- \tilde{\mathcal{V}}\psi] &= -\psi, \\ [\gamma_0 \tilde{\mathcal{K}}w] &= w, & [\gamma_1^- \tilde{\mathcal{K}}w] &= 0.\end{aligned}$$

Proof. We modify the proof of [3] by not using the time reversal map. Let $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$. We set $u := \tilde{\mathcal{V}}\psi$. Due to the mapping property of the single layer potential, we then have $u \in H_{;0}^{1, \frac{1}{2}}((0, T) \times B_R(0))$ and thus, by the trace lemma, we have $\gamma_0(u|_{Q_T}) = \gamma_0(u|_{Q_T^c})$.

Let us next consider the normal jump of $\tilde{\mathcal{V}}$. From (5.3), we obtain by considering $u = \tilde{\mathcal{V}}\psi$

$$(\partial_t - \Delta)u = \gamma'_0 \psi$$

in $\mathbb{R}_+ \times \mathbb{R}^d$. We consider any test function $\varphi \in C_0^2((0, T) \times B_R)$ and we obtain

$$\langle \psi, \gamma_0 \varphi \rangle = \langle \gamma'_0 \psi, \varphi \rangle = \langle (\partial_t - \Delta)u, \varphi \rangle = -\langle u, (\partial_t + \Delta)\varphi \rangle,$$

where the last equality holds due to the integration by parts on a cylindrical domain. We thus have

$$\langle \psi, \gamma_0 \varphi \rangle = - \int_{(0, T) \times B_R} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}). \quad (5.4)$$

On the other hand, we can use Green's second formula, given in Lemma 4.8 in Q_T and Q_T^c , where we use that $(\partial_t - \Delta)u = 0$ in $Q_T \cup Q_T^c$. This yields

$$\int_{Q_T} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}) = \langle \gamma_0 u, \gamma_1^+ \varphi \rangle - \langle \gamma_1^- u, \gamma_0 \varphi \rangle$$

and

$$\int_{Q_T^c} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}) = -\langle \gamma_0 u, \gamma_1^+ \varphi \rangle + \langle \gamma_1^- u, \gamma_0 \varphi \rangle.$$

Adding these two expressions yields

$$\int_{(0,T) \times B_R} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}) = \langle [\gamma_1^- u], \gamma_0 \varphi \rangle, \tag{5.5}$$

where we used $[\gamma_0 u] = 0 = [\gamma_0 \varphi] = [\gamma_1^+ \varphi]$. Comparing (5.4) with (5.5) results in $[\gamma_1^- u] = -\psi$.

We are left with proving the jump relations for the double layer potential. To that end, we choose $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ and define $u := \tilde{\mathcal{K}}w$. Let $\varphi \in C_0^\infty(\mathbb{R}_+ \times B_R)$ be a test function. As above, we obtain

$$\int_{(0,T) \times B_R} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}) = \langle [\gamma_1^- u], \gamma_0 \varphi \rangle - \langle [\gamma_0 u], \gamma_1^+ \varphi \rangle. \tag{5.6}$$

For $\tilde{\mathcal{K}}$, we obtain $\tilde{\mathcal{K}}w = \tilde{G} \star ((\gamma_1^+)'w)$ similar to (5.3). Therefore, we have $(\partial_t - \Delta)\tilde{\mathcal{K}}w = (\gamma_1^+)'w$ in $\mathbb{R}_+ \times B_R$. From here, it follows that

$$\begin{aligned} -\int_{(0,T) \times B_R} (\partial_t + \Delta)\varphi u \, d(t, \mathbf{x}) &= \langle (\partial_t - \Delta)u, \varphi \rangle \\ &= \langle (\gamma_1^+)'w, \varphi \rangle = \langle w, \gamma_1^+ \varphi \rangle. \end{aligned} \tag{5.7}$$

Comparing (5.6) with (5.7) yields

$$\langle [\gamma_1^- u], \gamma_0 \varphi \rangle = \langle [\gamma_0 u] - w, \gamma_1^+ \varphi \rangle \tag{5.8}$$

for all $\varphi \in C_0^2(\mathbb{R}_+ \times B_R)$. Applying Lemma 4.11 says that both sides of (5.8) have to vanish identically, from where $[\gamma_1^- u] = 0$ and $[\gamma_0 u] = w$ follows. \square

Now, as in [3, Definition 3.5], we are in the position to define the boundary integral operators.

Definition 5.4. Let $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ and $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$. We can then define the single layer operator as

$$\mathcal{V}\psi := \gamma_0 \tilde{\mathcal{V}}\psi,$$

the adjoint double layer operator as

$$\mathcal{K}'\psi := \frac{1}{2} \left(\gamma_1^- (\tilde{\mathcal{V}}\psi)|_{Q_T} + \gamma_1^- (\tilde{\mathcal{V}}\psi)|_{Q_T^c} \right),$$

the double layer operator as

$$\mathcal{K}w := \frac{1}{2} \left(\gamma_0 (\tilde{\mathcal{K}}w)|_{Q_T} + \gamma_0 (\tilde{\mathcal{K}}w)|_{Q_T^c} \right)$$

and the hypersingular operator as

$$\mathcal{D}w := -\gamma_1^- \tilde{\mathcal{K}}w.$$

Likewise to [3, Theorem 3.7], we may conclude the mapping properties of these operators.

Theorem 5.5. *The boundary integral operators from Definition 5.4 are continuous mappings as follows*

$$\begin{aligned}\mathcal{V}: H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) &\rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T), \\ \mathcal{K}': H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) &\rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T), \\ \mathcal{K}: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) &\rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T), \\ \mathcal{D}: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) &\rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T).\end{aligned}$$

Proof. The assertion follows immediately by using the mapping properties of the layer potentials from Lemma 5.2 as well as of the trace operators introduced in Sect. 3.2 and from Lemma 4.6. \square

The analogue of [3, Formulae (3.24)–(3.27)] reads:

Lemma 5.6. *It holds*

$$\begin{aligned}\gamma_0(\tilde{\mathcal{V}}\psi)|_{Q_T} &= \gamma_0(\tilde{\mathcal{V}}\psi)|_{Q_{\tilde{T}}} = \mathcal{V}\psi, \\ \gamma_1^-(\tilde{\mathcal{V}}\psi)|_{Q_T} &= \frac{1}{2}\psi + \mathcal{K}'\psi, \\ \gamma_1^-(\tilde{\mathcal{V}}\psi)|_{Q_{\tilde{T}}} &= -\frac{1}{2}\psi + \mathcal{K}'\psi, \\ \gamma_0(\tilde{\mathcal{K}}w)|_{Q_T} &= -\frac{1}{2}w + \mathcal{K}w, \\ \gamma_0(\tilde{\mathcal{K}}w)|_{Q_{\tilde{T}}} &= \frac{1}{2}w + \mathcal{K}w, \\ \gamma_1^-(\tilde{\mathcal{K}}w)|_{Q_T} &= \gamma_1^-(\mathcal{K}w)|_{Q_{\tilde{T}}} = -\mathcal{D}w.\end{aligned}$$

Proof. We just prove the second statement, as the other statements follow similarly. According to Lemma 5.3, we have

$$[\gamma_1^-\tilde{\mathcal{V}}\psi] = \gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_{\tilde{T}}} - \gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_T} = -\psi.$$

Therefore,

$$\gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_T} = \psi + \gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_{\tilde{T}}}.$$

By Definition 5.4, we have

$$\mathcal{K}'\psi := \frac{1}{2} \left(\gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_T} + \gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_{\tilde{T}}} \right).$$

Substituting this into the expression above yields

$$\gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_T} = \psi + 2\mathcal{K}'\psi - \gamma_1^-\tilde{\mathcal{V}}\psi|_{Q_T},$$

from where the claim follows immediately. \square

Remark 5.7. Following [18, Formulae (7)–(10)], the relations in the interior given in Lemma 5.6 can also be written as

$$\begin{aligned} \gamma_0 \tilde{\mathcal{V}}\psi(t, \mathbf{x}) &= \int_{\Sigma_T} G(t, \tau, \mathbf{x}, \mathbf{y}) \psi(\tau, \mathbf{y}) \, d\sigma_{(\tau, \mathbf{y})}, \\ \gamma_1^- \tilde{\mathcal{V}}\psi(t, \mathbf{x}) &= \frac{1}{2} \psi(t, \mathbf{x}) + \int_{\Sigma_T} \gamma_{1, (t, \mathbf{x})}^- G(t, \tau, \mathbf{x}, \mathbf{y}) \psi(\tau, \mathbf{y}) \, d\sigma_{(\tau, \mathbf{y})}, \\ \gamma_0 \tilde{\mathcal{K}}w(t, \mathbf{x}) &= -\frac{1}{2} w(t, \mathbf{x}) + \int_{\Sigma_T} \gamma_{1, (\tau, \mathbf{y})}^+ G(t, \tau, \mathbf{x}, \mathbf{y}) w(\tau, \mathbf{y}) \, d\sigma_{(\tau, \mathbf{y})}, \\ \gamma_1^- \tilde{\mathcal{K}}w(t, \mathbf{x}) &= -\int_{\Sigma_T} \gamma_{1, (t, \mathbf{x})}^- \gamma_{1, (\tau, \mathbf{y})}^+ G(t, \tau, \mathbf{x}, \mathbf{y}) w(\tau, \mathbf{y}) \, d\sigma_{(\tau, \mathbf{y})}. \end{aligned}$$

We can take the traces in the representation formula (5.2) to obtain the Dirichlet data and the Neumann data of the solution u of the homogeneous heat equation. This yields

$$\gamma_0 u = \frac{1}{2} \gamma_0 u - \mathcal{K} \gamma_0 u + \mathcal{V} \gamma_1^- u, \tag{5.9}$$

$$\gamma_1^- u = \mathcal{D} \gamma_0 u + \frac{1}{2} \gamma_1^- u + \mathcal{K}' \gamma_1^- u, \tag{5.10}$$

compare also [18, Formulae (11) and (12)].

We define the Calderón projector and the associated involution \mathcal{A} as

$$\mathcal{C}_{Q_T} := \frac{1}{2} \text{id} + \mathcal{A} := \frac{1}{2} \text{id} + \begin{bmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{D} & \mathcal{K}' \end{bmatrix}$$

and state the analogue of [3, Theorem 3.9] and [3, Corollary 3.10], the proof of which stays the same.

Theorem 5.8. *The operator \mathcal{C}_{Q_T} is a projection operator in the space*

$$\mathcal{H} := H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$$

while the operator $\mathcal{A}: \mathcal{H} \rightarrow \mathcal{H}$ is an isomorphism, satisfying

$$\mathcal{A}^2 = \frac{1}{4} \text{id}. \tag{5.11}$$

Moreover, the following statements are equivalent for $(w, \psi) \in \mathcal{H}$:

- (i) There is a $u \in H_{0,0}^{1, \frac{1}{2}}(Q_T)$ with $(\partial_t - \Delta)u = 0$ in Q_T and $w = \gamma_0 u$, $\psi = \gamma_1^- u$ on Σ_T .
- (ii) It holds

$$\begin{bmatrix} w \\ \psi \end{bmatrix} = \mathcal{C}_{Q_T} \begin{bmatrix} w \\ \psi \end{bmatrix}.$$

As in [3], we can interchange the columns of the operator \mathcal{A} to define the operator

$$A := \begin{bmatrix} \mathcal{V} & -\mathcal{K} \\ \mathcal{K}' & \mathcal{D} \end{bmatrix},$$

which is an isomorphism of the space

$$\mathcal{H}' := H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) \times H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$$

onto its dual space \mathcal{H} . Furthermore, we define the duality product between \mathcal{H}' and \mathcal{H} according to

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, \begin{bmatrix} v \\ \varphi \end{bmatrix} \right\rangle := \langle \psi, v \rangle + \langle \varphi, w \rangle$$

for all $v, w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ and $\varphi, \psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$. We are now in the position to state the analogue of [3, Theorem 3.11], which is the positive definiteness of the operator A .

Theorem 5.9. *There exists a constant $\alpha > 0$ such that*

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, A \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle \geq \alpha \left(\|\psi\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)}^2 + \|w\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)}^2 \right)$$

for all $[\psi, w]^\top \in \mathcal{H}'$.

Proof. The proof of this theorem follows in complete analogy to the proof of [3, Theorem 3.11], when one uses Lemma 5.10 below which constitutes the tube analogue of [3, Lemma 2.15].³ □

Lemma 5.10. *Let $u \in \mathcal{V}(Q_T)$ such that $(\partial_t - \Delta)u = 0$ in Q_T . Then, there exist constants m_1, m_2 , and m_3 such that*

$$\|u\|_{H^{1,0}(Q_T)} \leq m_1 \|u\|_{H^{1, \frac{1}{2}}(Q_T)} \leq m_2 \|u\|_{\mathcal{V}(Q_T)} \leq m_3 \|u\|_{H^{1,0}(Q_T)}.$$

In other words, for functions $u \in \mathcal{V}(Q_T)$ satisfying the homogeneous heat equation, we have the equivalence of the norms in $\mathcal{V}(Q_T)$, $H^{1,0}(Q_T)$, and $H^{1, \frac{1}{2}}(Q_T)$.

Proof. The first and second inequality follow directly from the equivalence of norms on Q_T and Q_0 , since the proof of [3] is based on the definition of the norm for the first inequality and the interpolation result (2.5) for the second inequality. Nonetheless, we cannot apply the equivalence of norms on Q_T and norms on Q_0 directly for the third inequality, because we have $(\partial_t - \Delta)u = 0$ as an assumption, which is needed to show the third inequality. Mapping this differential operator from the tube onto the cylinder or vice versa will alter it. Therefore, we use the ideas of the proof of [3, Lemma 2.15], but adapt them to our context.

Transforming the partial differential equation $(\partial_t - \Delta)u = 0$ from Q_T back to Q_0 via the weak formulation (see [1]) yields

$$\partial_t(u \circ \kappa) - \mathcal{M}(u \circ \kappa) = 0 \quad \text{in } Q_0,$$

where \mathcal{M} is defined as

$$\begin{aligned} \mathcal{M}(u \circ \kappa) := & \operatorname{div} \left((D\kappa)^{-1} (D\kappa)^{-\top} \nabla(u \circ \kappa) \right) + (D\kappa)^{-\top} \nabla(u \circ \kappa) \cdot \partial_t \kappa \\ & + \frac{1}{\det(D\kappa)} \nabla(\det(D\kappa)) \cdot (D\kappa)^{-1} (D\kappa)^{-\top} \nabla(u \circ \kappa). \end{aligned}$$

³At this point, it is crucial that we split the term $\langle \mathbf{V}, \mathbf{n} \rangle$ in (4.1) with the factor $\frac{1}{2}$. If we choose the factor differently, say λ and $1 - \lambda$, we would obtain a boundary term in [3, Formula (3.48)] involving $\int_0^T \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle (\gamma_0^{\text{int}} u)^2 \, d\sigma dt$, which would require an appropriate, non-obvious treatment.

By the standard theory, for fixed $t \in (0, T)$, we have that $\mathcal{M}: H^1(\Omega_0) \rightarrow H^{-1}(\Omega_0)$ is bounded. Thus, for $u \in H^{1,0}(Q_0)$, we obtain $\mathcal{M}u \in H^{-1,0}(Q_0) := [H^{1,0}(Q_0)]'$ and we conclude

$$\begin{aligned} \|u\|_{\mathcal{V}(Q_T)}^2 &= \|u \circ \kappa\|_{\mathcal{V}(Q_0)}^2 = \|u \circ \kappa\|_{H^{1,0}(Q_0)}^2 + \|\partial_t(u \circ \kappa)\|_{H^{-1,0}(Q_0)}^2 \\ &= \|u \circ \kappa\|_{H^{1,0}(Q_0)}^2 + \|\mathcal{M}(u \circ \kappa)\|_{H^{-1,0}(Q_0)}^2 \\ &\lesssim \|u \circ \kappa\|_{H^{1,0}(Q_0)}^2 \\ &= \|u\|_{H^{1,0}(Q_T)}^2. \end{aligned} \quad \square$$

Having the main result Theorem 5.9 at hand, we can state a few corollaries along the lines of [3, Corollary 3.13, Corollary 3.14, Remark 3.15, Corollary 3.16, Corollary 3.17].

Corollary 5.11. *The single layer operator*

$$\mathcal{V}: H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) \rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$$

is an isomorphism and there exists $\alpha > 0$ such that

$$\langle \mathcal{V}\psi, \psi \rangle \geq \alpha \|\psi\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)}^2 \quad \text{for all } \psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T). \quad (5.12)$$

The hypersingular operator

$$\mathcal{D}: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) \rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$$

is an isomorphism and there exists $\alpha > 0$ such that

$$\langle \mathcal{D}w, w \rangle \geq \alpha \|w\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)}^2 \quad \text{for all } w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T). \quad (5.13)$$

Proof. As in the proof of [3, Corollary 3.13], the coercivity estimates (5.12) and (5.13) result from Theorem 5.9 by using the special cases $w = 0$ and $\psi = 0$, respectively. In view of the continuity of \mathcal{V} and \mathcal{D} , this leads to the invertibility of the operators. \square

Corollary 5.12. *The operators*

$$\begin{aligned} \frac{1}{2} \text{id} + \mathcal{K}, \quad \frac{1}{2} \text{id} - \mathcal{K}: H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T) &\rightarrow H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T), \\ \frac{1}{2} \text{id} + \mathcal{K}', \quad \frac{1}{2} \text{id} - \mathcal{K}': H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) &\rightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T) \end{aligned}$$

are isomorphisms. In addition, there holds the identity $\mathcal{V}^{-1}\mathcal{K}\mathcal{V} = \mathcal{K}' = \mathcal{D}\mathcal{K}\mathcal{D}^{-1}$.

Proof. The claim is implied by (5.11), compare the proof of [3, Corollary 3.14]. \square

Corollary 5.13. *The unique solution $u \in H_{0,0}^{1,\frac{1}{2}}(Q_T)$ of the Dirichlet problem*

$$\begin{aligned} (\partial_t - \Delta)u &= 0 \quad \text{in } Q_T, \\ \gamma_0 u &= g \quad \text{on } \Sigma_T, \end{aligned}$$

with $g \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ can be represented

(i) as $u = \tilde{\mathcal{V}}\psi - \tilde{\mathcal{K}}g$, where $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{V}\psi = \left(\frac{1}{2}\text{id} + \mathcal{K}\right)g.$$

(ii) as $u = \tilde{\mathcal{V}}\psi - \tilde{\mathcal{K}}g$, where $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\text{id} - \mathcal{K}'\right)\psi = \mathcal{D}g.$$

(iii) as $u = \tilde{\mathcal{V}}\psi$, where $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{V}\psi = g.$$

(iv) as $u = \tilde{\mathcal{K}}w$, where $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\text{id} - \mathcal{K}\right)w = -g.$$

In (i) and (ii), it particularly holds $\psi = \gamma_1^- u$ on Σ_T .

Proof. We can again use directly the idea of the proof of [3, Corollary 3.16], which means the uniqueness results from above and the jump relations given in Lemma 5.6. \square

Corollary 5.14. *The unique solution $u \in H_{;0}^{1, \frac{1}{2}}(Q_T)$ of the Neumann problem*

$$\begin{aligned} (\partial_t - \Delta)u &= 0 && \text{in } Q_T, \\ \gamma_1^- u &= h && \text{on } \Sigma_T, \end{aligned}$$

with $h \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ can be represented

(i) as $u = \tilde{\mathcal{V}}h - \tilde{\mathcal{K}}w$, where $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\text{id} + \mathcal{K}\right)w = \mathcal{V}h.$$

(ii) as $u = \tilde{\mathcal{V}}h - \tilde{\mathcal{K}}w$, where $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{D}w = \left(\frac{1}{2}\text{id} - \mathcal{K}'\right)h.$$

(iii) as $u = \tilde{\mathcal{V}}\psi$, where $\psi \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\text{id} + \mathcal{K}'\right)\psi = h.$$

(iv) as $u = \tilde{\mathcal{K}}w$, where $w \in H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{D}w = -h.$$

In (i) and (ii), we have that $w = \gamma_0 u$ on Σ_T .

Proof. The proof follows by the same arguments as in the proof of [3, Corollary 3.17], which is similar to the respective proof for the Dirichlet problem. \square

6. Conclusion

In this article, we considered the heat equation on a time-varying (so-called non-cylindrical) domain. In contrast to the problem on a cylindrical domain, we used a modified Neumann trace operator containing a term which is dependent on the velocity of the moving surface. We were able to show the mapping properties of the layer operators by following the proofs of Costabel [3]. To this end, we heavily used the fact that the non-cylindrical domain is a mapped cylindrical domain. Then, using mapped anisotropic Sobolev spaces, we obtain analogous mapping properties and are also able to prove existence and uniqueness of solutions of the Dirichlet and of the Neumann problem.

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