reformulation as a local differential equation allows to study the solution u using pseudodifferential techniques developed for boundary problems on singular spaces [4]. The idea to extend to $\mathbb{R}^3_+$ goes back to Caffarelli and Silvestre (2007), but its use for a precise regularity theory seems to be new.

References

[1] G. Acosta, J. P. Borthagaray, *A fractional Laplace equation: regularity of solutions and finite element approximations*, SIAM Journal on Numerical Analysis 55 (2017), 472–495.

[2] I. Babuška, *Finite element method for domains with corners*, Computing 6 (1970), 264–273.

[3] J. P. Borthagaray, W. Li, R. H. Nochetto, *Linear and Nonlinear Fractional Elliptic Problems*, arXiv:1906.04230.

[4] M. Dauge, *Elliptic boundary value problems in corner domains*, Lecture Notes in Mathematics 1341, Springer Verlag, 1988.

[5] G. Estrada-Rodriguez, H. Gimperlein and J. Stocek, *Nonlocal interface problems: Modelling, regularity, finite element approximation*, preprint (2020).

[6] H. Gimperlein, N. Louca, R. Mazzeo, in preparation.

[7] H. Gimperlein, E. P. Stephan and J. Stocek, *Corner Singularities for the Fractional Laplacian and Finite Element Approximation*, preprint (2020).

[8] H. Gimperlein, E. P. Stephan and J. Stocek, in preparation.

[9] J. A. Morrison, J. A. Lewis, *Charge Singularity at the Corner of a Flat Plate*, SIAM Journal on Applied Mathematics 31 (1976), 233-250.

[10] H. Gimperlein, J. Stocek and C. Urzua-Torres, *Optimal operator preconditioning for pseudodifferential boundary problems*, arxiv:1905.03846.

[11] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math. 268 (2015), 478-528.

[12] J. Gwinner, E. P. Stephan, *Advanced Boundary Element Methods – Treatment of Boundary Value, Transmission and Contact Problems*, Springer Series in Computational Mathematics 52, Springer, 2018.

[13] R. Hiptmair, C. Jerez-Hanckes, C. Urzua-Torres, *Closed-form inverses of the weakly singular and hypersingular operators on disks*, Integr. Equ. Oper. Theory 90 (2018), 14 pages.

[14] C. Jerez-Hanckes, J.-C. Nédélec, *Explicit variational forms for the inverses of integral logarithmic operators over an interval*, SIAM Journal on Mathematical Analysis 44 (2012), 2666–2694.

A wavelet-based approach for the optimal control of nonlocal operator equations

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(joint work with Stephan Dahlke and Thomas M. Surowiec)

1. Introduction

We are concerned with a wavelet-based approach for the optimal control of a class of nonlocal operator equations. Namely, we consider a quadratic cost functional where the state equation involves the fractional Laplace operator in integral form. When discretizing this nonlocal operator with standard finite element basis functions, one arrives at a densely populated system matrix. This imposes serious obstructions to the efficient numerical treatment of such problems. Therefore, we use a wavelet basis for discretizing the state equation and its adjoint and apply wavelet matrix compression to arrive at a solver that has linear complexity. In particular, we show how to include box constraints to the optimal control.

2. Optimal control problem

We consider the following optimal control problem which is constrained by a non-local state equation:

$$
\begin{aligned}
&\inf \frac{1}{2}\|Cu-u_d\|_H^2+\frac{\nu}{2}\|z\|_Z^2 \text{ over } (z,u)\in Z_{\mathrm{ad}}\times V\\
&\text{such that}\quad \mathcal{L}u = Bz+f \quad \text{on } \Omega,\\
&\qquad\qquad\quad u=0 \qquad\qquad \text{on } \Omega^c:=\mathbb{R}^n\setminus\overline{\Omega}.
\end{aligned} \tag{1}
$$

Here, H and Z are a real Hilbert spaces, $Z_{\mathrm{ad}}\subset Z$ is a nonempty, closed, and convex set, $\nu>0$, C is a bounded linear operator whose image represents the observation of the state u, and B is a bounded linear operator that maps the control z into the nonlocal equation. For $\Omega\subset\mathbb{R}^n$, $n\geq 1$, being an open and bounded domain, the fractional Laplacian $\mathcal{L}=(-\Delta)^s$, $0<s<1$, for some function $u:\Omega\to\mathbb{R}$ is given by

$$(\mathcal{L}u)(x) := 2\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\mathrm{d}y,\quad x\in\Omega.$$

The associated state space V is given by

$$V:=\{v\in H^s(\mathbb{R}^n): v=0 \text{ on } \Omega^c\}.$$

In the present situation, it holds $V\cong H^s(\Omega)/\mathbb{R}$ for $0<s<1/2$ and $V\cong H_0^s(\Omega)$ for $1/2<s<1$. In the limit case $s=1/2$, it holds $V\cong H_{00}^{1/2}(\Omega)$, where $H_{00}^{1/2}(\Omega)$ is obtained from interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$, compare [2].

The following theorem is a consequence of the standard theory for optimal control problems, see for example [3].

Theorem 1. *Under the standing assumptions, the optimal control problem* (1) *admits a unique solution* $z^\star\in Z_{\mathrm{ad}}$. *Furthermore, there exists an adjoint state* $\lambda^\star\in V$ *such that*

$$\mathcal{L}u^\star = B\mathfrak{P}\Big(-\frac{1}{\nu}B^\intercal\lambda^\star\Big), \tag{2a}$$

$$\mathcal{L}\lambda^\star = C^\intercal(u_d-Cu^\star). \tag{2b}$$

Here, $\mathfrak{P}: Z\to Z_{\mathrm{ad}}$ *is the usual metric projection onto the closed convex set* Z_{ad}.

Note that in case of $Z=L^2(\Omega)$ and $Z_{\mathrm{ad}}\subset Z$ resulting from the box constraints

$$z_{\min}(x)\leq z(x)\leq z_{\max}(x),\quad x\in\Omega, \tag{3}$$

the projection $\mathfrak{P}z$ is given by

$$\mathfrak{P}z(x)=\begin{cases} z_{\min}(x), & \text{if } z(x)<z_{\min}(x),\\ z(x), & \text{if } z_{\min}(x)\leq z(x)\leq z_{\max}(x),\\ z_{\max}(x), & \text{if } z(x)>z_{\max}(x).\end{cases}$$

3. Wavelet matrix compression

The fractional Laplacian is a *nonlocal operator*. Its discretization will thus amount to a *dense* system matrix, the assembly of which would require large amounts of time and computation capacities. Especially, as the fraction Laplacian is an operator of order $2s$, preconditioning becomes an issue.

We shall hence employ wavelet matrix compression. It employs that the wavelets' vanishing moments lead, in combination with the fact that the integral kernel becomes smoother when getting farther away from the diagonal, to a quasi-sparse system matrix. Moreover, by applying a diagonal scaling, the condition number stays uniformly bounded. Since the number of relevant entries in the system matrix for maintaining the convergence rate of the underlying Galerkin method scales only linearly, wavelet matrix compression leads to a numerical approach that has linear over-all complexity, compare [4] for the details.

4. Primal-dual active set strategy

In case of $H = Z = L^2(\Omega)$ and box constraints (3), we can rewrite the optimal control problem (2) as an equivalent KKT system of the following form:

$$\begin{aligned} \mathcal{L}u^\star &= Bz^\star & \mathcal{L}\lambda^\star &= C^\intercal(u_d - Cu^\star) & &\text{in } \Omega,\\ u^\star &= 0 & \lambda^\star &= 0 & &\text{in } \Omega^c, \end{aligned}$$

$$\lambda^\star + \nu z^\star - \mu^\star_{\min} + \mu^\star_{\max} = 0 \quad \text{in } \Omega,$$

$$\begin{aligned} \mu^\star_{\min} &\geq 0, & z_{\min} - z^\star &\leq 0, & \mu^\star_{\min}(z_{\min} - z^\star) &= 0 & &\text{in } \Omega,\\ \mu^\star_{\max} &\geq 0, & z^\star - z_{\max} &\leq 0, & \mu^\star_{\max}(z^\star - z_{\max}) &= 0 & &\text{in } \Omega. \end{aligned}$$

Here, $\mu_{\min}$ and $\mu_{\max}$ are Lagrange multipliers. In order to compute the solution to this KKT system, we apply the primal-dual active set strategy as introduced in [1]. The essential idea of this iterative solution strategy is to replace successively the inequality constraints by the related equality constraints for all the indices where the constraint becomes active. Since it can be reinterpreted as a semi-smooth Newton method, the primal-dual active set strategy converges superlinearly, see [5].

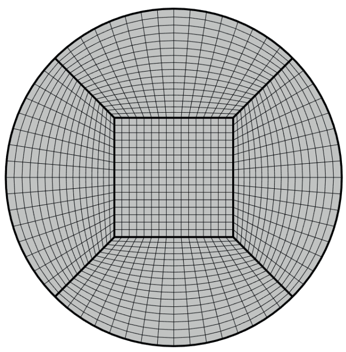

Figure 1. The domain Ω under consideration with mesh on level 4. The operator C is the projection onto the interior square.

Numerical results in case of Ω being the unit circle and $s = 1/4$ are given in Figure 2. Here, we computed the solution for about 80 000 piecewise constant ansatz functions each for the state and for the control (indeed, we use Haar wavelets for the discretization), where $z_{\min} = -0.1$, $z_{\max} = 0.1$, $\nu = 10^{-3}$, B is the identity, and C is the projection onto the square $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^2$, which is the interior patch seen in Figure 2.

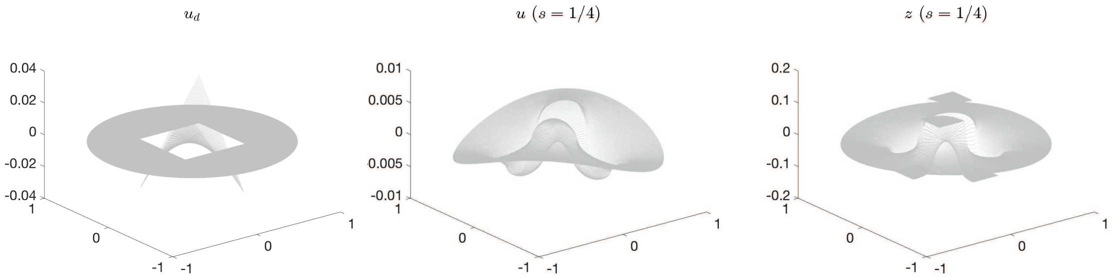

FIGURE 2. The desired state (left), the optimal state u (middle), and the optimal control z (right) in case of $s = 1/4$.

References

[1] M. Bergounioux, K. Ito, and K. Kunisch. Primal-dual strategy for constrained optimal control problems. *SIAM J. Contr. Optim.*, 37:1176–1194, 1999.

[2] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7–9):1245–1260, 2007.

[3] J.L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*, volume 170 of *Grundlehren der mathematischen Wissenschaften.* Springer, Berlin–Heidelberg, 1971.

[4] H. Harbrecht and R. Schneider. Rapid solution of boundary integral equations by wavelet Galerkin schemes. In R. DeVore and A. Kunoth, editors, *Multiscale, Nonlinear and Adaptive Approximation*, pages 249–294. Springer, Berlin–Heidelberg, 2009.

[5] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.*, 13:865–888, 2003.

Electromagnetic Force Computation in the Boundary Element Method

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(joint work with Piyush Panchal)

Electrostatic boundary value problem. As a simple model problem we consider a conducting body in the interior of a metallic box. A fixed voltage drop U_0 between both is imposed so that the electrostatic potential u in the space Ω between both objects can be recovered as the solution of the Dirichlet boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega\,, \quad u = g \quad \text{on } \partial\Omega\,, \quad g := \begin{cases} U_0 & \text{on conductor}\,, \\ 0 & \text{on box.} \end{cases} \tag{1}$$