From the Hartree equation to the Vlasov-Poisson system: strong convergence for a class of mixed states

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Abstract

We study the semiclassical limit from the time-dependent Hartree equation with Coulomb or gravitational potential to the Vlasov-Poisson equation. We prove convergence in trace norm for mixed states under a technical assumption on the solution of the Vlasov-Poisson equation. We exhibit a special class of mixed quasi-free states which satisfies this assumption.

1 Introduction

In this paper we shall focus on the derivation of the three-dimensional Vlasov-Poisson system

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + E\cdot\nabla_v f(t,x,v) = 0\,, \\ E(t,x) = \left(\nabla\frac{\gamma}{|\cdot|} * \varrho\right)(t,x)\,, & (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3 \\ \varrho(t,x) = \int f(t,x,v)\,dv\,, \end{cases} \tag{1.1}$$

from the quantum N-body dynamics in a joint mean-field and semiclassical regime. More precisely, we will assume the mean-field description given by the Hartree equation to be correct and will address the semiclassical limit from the Hartree dynamics (cf. Eq. (1.4)) towards the Vlasov-Poisson equation (1.1).
The system (1.1) is an effective equation, whose unknown $f:\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ is a time dependent function on the phase space modelling the probability density of particles in a plasma under the effect of a self-induced field E, dependent on the spatial density $\varrho:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$. The parameter γ takes values -1 or 1 if the interaction is respectively gravitational or Coulombian. From a physical viewpoint, the attractive case ($\gamma=-1$) describes the motion of galaxy clusters under the gravitational field with many applications in astrophysics (cf. [28, 29]). The repulsive case ($\gamma=1$) represents the evolution of charged particles in presence of their self-consistent electric field and it is used in plasma physics or in semi-conductor devices (cf. [48]). In this context, the self-induced field $E(t,x)$ is a conservative force, hence there exists a real-valued function of time and space $U(t,x)$ such that $E=\nabla_x U$ which satisfies the Poisson equation $-\Delta_x U=\varrho_t$. More precisely, Eq. (1.1) can be rewritten as a Vlasov equation coupled with a Poisson equation, whence the name Vlasov-Poisson system.

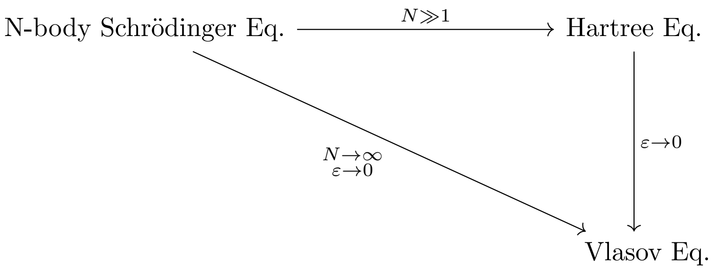
Many authors have been investigating the problem of deriving the Vlasov-Poisson system (1.1) from the many-body quantum dynamics. The aim of this paper is to give a better understanding of the derivation of the Vlasov-Poisson system from the quantum mean-field dynamics given by the Hartree equation (cf. Eq. (1.4)), providing strong convergence and explicit estimates on the semiclassical limit. More precisely, our result focuses on the vertical arrow of the diagram in (1.3) when the interaction among particles is given by the Coulomb or by the gravitational potential.

In particular, our setting applies to bosons, but it is also compatible with the one arising from the mean-field scaling for fermions in which a coupling between the number of particles and the semiclassical parameter is prescribed. More precisely, a parameter ε_N (which plays the role of the Planck constant $\hbar$) is linked to the number of particles N by the identity

$$\varepsilon_N = N^{-\frac{1}{3}} . \tag{1.2}$$

This regime differs from the bosonic one, in which the mean-field approximation and the semiclassical limit are not coupled, and can be considered as two completely different steps. In other words, the mean-field scaling for bosons produces an effective Hartree equation that depends on the Planck constant $\hbar$ (whose role is played by the parameter ε throughout this paper), but not on the number of particles, whereas the Hartree equation for fermions is still N dependent through the relation (1.2). For a rigorous justification of the mean-field limit for fermions interacting through a singular potential and its physical motivations we refer to [44, 45].
For the rest of the paper we will discard the subscript N in ε_N in (1.2) and denote it simply by ε for the sake of readibility. Moreover, we will focus on the fermionic setting and therefore assume the solution of the Hartree equation to be a fermionic operator (cf. next paragraph and Eq. (1.5)). For bosons the exact same proof holds, without the coupling of N and ε given in (1.2).

State of art. The first derivation of the Vlasov equation from many-body quantum dynamics was obtained in the 80s by Narnhofer and Sewell [39] for interaction potentials $V \in C^\omega(\mathbb{R}^3)$ and extended to $V \in C^2(\mathbb{R}^3)$ by Spohn in [47]. These results establish convergence of the dynamics, but no information on the rate of convergence is provided. An analogous result has been obtained by Graffi, Martinez and Pulvirenti in [26] by analysing the dynamics of factored WKB states and combining the mean-field and the semiclassical limit.



(1.3)

A different approach consists in considering the Hartree equation as a bridge between the N-body Schrödinger dynamics and the Vlasov equation, as pictured in (1.3). The Hartree equation reads

$$i\varepsilon\partial_t \, \omega_{N,t} = \left[-\varepsilon^2\Delta + V * \varrho_t \, , \, \omega_{N,t}\right] \tag{1.4}$$

where $\omega_{N,t}$ is a nonnegative trace class fermionic operator over $L^2(\mathbb{R}^3)$, ϱ_t is the diagonal kernel of $\omega_{N,t}$, i.e. $\varrho_t(x) = N^{-1}\omega_{N,t}(x;x)$, and for two operators A and B the standard notation $[A,B]$ stands for the commutator $AB - BA$. More precisely, we say that an operator $\omega_{N,t}$ is fermionic if the bounds

$$0 \leq \omega_{N,t} \leq 1 \tag{1.5}$$

hold. Notice moreover that if we assume (1.5) at time $t = 0$, the equation (1.4) propagates such a bound for positive times.

Looking at the Hartree equation (1.4), one observes that its solution $\omega_{N,t}$ is still N dependent and one expects it to approach a solution to the Vlasov equation as $N \to \infty$. Figure (1.3) above describes two ways of deriving the Vlasov equation from the many-body dynamics: either one performs a direct limit $N = \varepsilon^{-3} \to \infty$, or one first observes that for N large but fixed the many-body Schrödinger equation is approximated by the Hartree equation and then performs the semiclassical limit $\varepsilon \to 0$ recovering the Vlasov equation

$$\begin{cases} \partial_t W_t + 2\, v \cdot \nabla_x W_t - \nabla(V * \varrho_t) \cdot \nabla_v W_t = 0\,, \\ \varrho_t(x) = \int W_t(x,v)\, dv\,. \end{cases}$$

To compare these objects, namely an operator $\omega_{N,t}$ on $L^2(\mathbb{R}^3)$ (solution to the Hartree equation (1.4)) and a function $W_{N,t}$ defined on the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ (solution to the Vlasov system (1.1)), we introduce two standard tools in semiclassical analysis: the Wigner transform of an operator $\omega_{N,t}$ defined as

$$W_{N,t}(x,v) = \left(\frac{\varepsilon}{2\pi}\right)^3 \int \omega_{N,t}\left(x + \varepsilon\frac{y}{2}; x - \varepsilon\frac{y}{2}\right) e^{-iv\cdot y} dy\,, \tag{1.6}$$

and its inverse, known as Weyl quantization, given by

$$\omega_{N,t}(x;y) = N \int W_{N,t}\left(\frac{x+y}{2}, v\right) e^{iv\cdot\frac{(x-y)}{\varepsilon}}\, dv\,. \tag{1.7}$$

We recall that the Wigner transform of a fermionic operator is not always a probability density on the phase space, as in general it is not positive. This issue can be fixed by using the Husimi transform (see [11] for further discussions on this point).

The horizontal line in figure (1.3) in the regime under consideration has been investigated in [15, 12], where it has been shown that for regular interaction potentials the Hartree equation is a good approximation for the many-body Schrödinger evolution when considering zero temperature states enjoying a semiclassical structure. More precisely, in [15] the convergence of the Schrödinger dynamics towards a solution of the Hartree equation in the sense of reduced density matrices has been proved for short times and without a control on the rate of convergence. In [12] the authors prove that such an approximation holds for time intervals of order one and provide explicit estimates on the speed of convergence in terms of the number of particles N. This latter has been extended to positive temperature states in [10] and to fermions with semi-relativistic dispersion relation in [14]. For singular potentials of the form $V(x) = \gamma\,|x|^{-\alpha}$, for $\alpha \in (0,1]$ and $\gamma = \pm 1$, it has been shown in [44, 45] that the Hartree dynamics is still a good approximation of the many-body one, at least for a very special class of initial data, namely translation invariant states.

Different regimes have been considered in [5, 6, 7, 8, 19, 40, 42]. More precisely, states confined in a volume of order $O(N)$ have been studied in [5, 40, 42], while a regime in which the potential energy is sub-leading with respect to the kinetic one has been considered in [6, 7, 8, 19].

Using (1.6), in [36, 20, 38, 17] the vertical line in figure (1.3) has been investigated. The authors prove convergence in weak sense towards the solutions of the Vlasov equation. The analysis in [36, 17] includes the Coulomb potential, but does not provide explicit bounds on the convergence rate, which are fundamental for applications. Indeed, in all relevant situations, the number of particles N is large, but finite. An explicit control on the convergence rate therefore allows to determine how large the system (i.e. the number of particles N) should be in order for the Vlasov equation to be a meaningful approximation.

The paper [3] was the first of a long list of references in which this aspect has been tackled. Indeed, in [3] Athanassoulis, Paul, Pezzotti and Pulvirenti obtain the convergence of the Wigner transform of a solution to the Hartree equation towards a solution to the Vlasov equation in Hilbert-Schmidt norm with a relative rate $N^{-\frac{2}{21}}$ for $V \in H^1(\mathbb{R}^3)$. In [42] and [1, 2] an expansion of the solution of the Hartree equation in powers of ε has been provided.

More recently, in the same spirit of [3], assumptions on the potentials have been relaxed first to interactions V such that $\nabla V \in \mathrm{Lip}(\mathbb{R}^3)$ [11] and then to inverse power law potentials $V(x) = \gamma|x|^{-\alpha}$, for $\alpha \in (0, 1/2)$ and $\gamma = \pm 1$ (cf. [45]). A key ingredient is to consider the problem from the perspective of the Weyl quantization, instead of the one of the Wigner transform usually adopted in the previous literature. In the same regime, convergence of minimizers of the N-particle energy to the mean-field energy has been proven in [18, 33], respectively for zero and positive temperature. A different approach has been recently proposed by [23] with the introduction of a new pseudo-distance which is reminiscent of the Monge-Kantorovich distance for probability measures in the setting of classical mechanics. Under appropriate conditions on the initial states, such a pseudo-distance metrizes weak convergence. For potentials V such that the force satisfies $\nabla V \in \mathrm{Lip}(\mathbb{R}^3)$, Golse and Paul prove that the Vlasov equation is a good approximation for the Hartree dynamics when considering a special class of bosonic states, defined through Töplitz operators in [23] and they are able to consider projection operators in [24] by introducing the notion of quantum empirical measure. In [25] the convergence result in [23] is proven to be uniform in the Planck constant. On the same line, [23] has been extended by Laflèche in [30, 31] to the case of singular interaction potentials, here included the Coulomb case, providing an explicit convergence rate. For a certain class of initial states, the notion of convergence considered in [30, 31] is equivalent to weak convergence.

In this paper we are interested in giving a strong notion of convergence from the Hartree dynamics towards the Vlasov-Poisson equation, exhibiting explicit control on the convergence rate and thus extending and complementing the previous results [36, 17, 30, 31].

Main result. To state precisely the main result of this paper, we introduce the following notations. For $s \in \mathbb{N}$, let $H^s(\mathbb{R}^3 \times \mathbb{R}^3)$ denote the Hilbert space of real-valued functions f on the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ with finite norm

$$\|f\|_{H^s}^2 := \sum_{|\beta| \le s} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\nabla^\beta f(x,v)|^2 \, dx\, dv \,,$$

where β is a multi-index and ∇^β acts on both x and v. For $s, m \in \mathbb{N}$, let $H^s_m(\mathbb{R}^3 \times \mathbb{R}^3)$ denote the Sobolev space H^s weighted with $(1 + x^2 + v^2)^m$ and define its associated norm

$$\|f\|_{H^s_m}^2 := \sum_{|\beta| \le s} \iint_{\mathbb{R}^3\times\mathbb{R}^3} (1+x^2+v^2)^m |\nabla^\beta f(x,v)|^2 \, dx\, dv \,.$$

We denote the m-th velocity moment associated to a function $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$\mathcal{M}_m(f) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(x,v)\, dx\, dv \,. \tag{1.8}$$

Moreover, we denote by $\varrho_{|[x,\omega_N]|}$ the function associated with the diagonal of the integral kernel of the operator $|[x,\omega_N]|$

$$\varrho_{|[x,\omega_N]|}(x) := |[x,\omega_N]|(x;x)\,, \qquad \text{for } x \in \mathbb{R}^3\,, \tag{1.9}$$

where we recall that given an operator A, $|A|$ is defined as $(A^*A)^{\frac{1}{2}}$.

Moreover, throughout the paper we look at the situation in which a smooth solution to the Vlasov-Poisson equation exists and is unique. This is the case when the following assumptions on the initial datum W_N are satisfied:

i) $W_N \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\mathcal{M}_m(W_N) < \infty$ for all $m < m_0$, with $m_0 > 6$.

ii) For all R, $T > 0$,

$$\begin{aligned} &\operatorname*{ess\,sup}_{y,\,w}\{|\nabla^k W_N|(y+vt,w) \,:\, |y-x| \le R, |w-v| \le R\} \\ &\qquad\qquad \in\; L^\infty((0,T)\times\mathbb{R}^3_x; L^1(\mathbb{R}^3_v) \cap L^2(\mathbb{R}^3_v)) \end{aligned} \tag{1.10}$$

for $k = 0, 1, \ldots, 5$.

iii) There exists $C > 0$ independent of N such that $\|W_N\|_{H^6_4} \leq C$.

In the same spirit of [11, 46], the main result of this paper reads

Theorem 1.1. *Let $\varepsilon = \varepsilon_N$ be defined as in (1.2). Let ω_N be a sequence of fermionic operators on $L^2(\mathbb{R}^3)$, $0 \leq \omega_N \leq 1$, with $\operatorname{tr}\,\omega_N = N$ and with Wigner transform W_N satisfying i), ii), iii). Let $\omega_{N,t}$ denote the solution of the time-dependent Hartree equation (1.4) with initial data $\omega_{N,0} = \omega_N$ and let $\widetilde{W}_{N,t}$ be the solution of the Vlasov-Poisson system (1.1) with initial data $\widetilde{W}_{N,0} = W_N$. Let $\widetilde{\omega}_{N,t}$ denote the Weyl quantization of $\widetilde{W}_{N,t}$ and assume that there exist a time $T > 0$, a number $p > 5$ and a positive constant C such that*

$$\sup_{t\in[0,T]} \sum_{i=1}^{3} \left[\|\varrho_{|[x_i,\widetilde{\omega}_{N,t}]|}\|_{L^1} + \|\varrho_{|[x_i,\widetilde{\omega}_{N,t}}\|_{L^p}\right] \leq CN\varepsilon\,. \tag{1.11}$$

Then

$$\operatorname{tr}\,|\omega_{N,t} - \widetilde{\omega}_{N,t}| \leq C_t\, N\, \varepsilon \left[1 + \sum_{i=1}^{4} \varepsilon^i \sup_N \|W_N\|_{H_4^{i+2}}\right]. \tag{1.12}$$

where C_t is a constant depending only on the time t and on $\|W_N\|_{H^2_4}$.

We comment on the meaning of the Theorem 1.1 above and discuss the assumptions.

- We notice that because of the normalization $\operatorname{tr}\,\omega_{N,t} = N$, Eq. (1.12) proves that $\omega_{N,t}$ and $\widetilde{\omega}_{N,t}$ are close as N is large. Indeed, the trace norm of $\omega_{N,t} - \widetilde{\omega}_{N,t}$ is smaller than the trace of $\omega_{N,t}$ and $\widetilde{\omega}_{N,t}$.

- We observe that assumptions *ii*) and *iii*) restrict our analysis to the case of mixed states, i.e. states at positive temperature. Indeed, at zero temperature a state can be approximated by Slater determinants, whose Wigner transform is in general not smooth. Assumptions of Theorem 1.1 are therefore expected to hold for states describing systems of N particles at positive temperature, but do not include pure states, i.e. states at zero temperature (cf. [11] for details).

- We comment on the assumptions. Hypotheses *i*) and *ii*) (for $k = 0, 1$) where proven in [37] to guarantee existence, uniqueness and propagation of moments of a solution to the Vlasov-Poisson system (1.1). Hypotheses *ii*) (for $k = 2, \ldots, 5$) and *iii*) are crucial to ensure regularity of the solution in H^6_4. The bounds (1.11) are assumed to hold true at positive time. This is a severe restriction of our result. The quantity (1.9) has already played a central role in [12, 44, 45]. We recall that the assumption

  $$\|\varrho_{|[x,\omega_N]|}\|_{L^1} = \operatorname{tr}\,|[x,\omega_N]| \leq CN\varepsilon \tag{1.13}$$

  is equivalent to ask for initial states enjoying a semiclassical structure. More precisely, the Hartree equation is expected to be a good approximation for the many-body Schrödinger dynamics if the kernel of the initial data $\omega_N(x;y)$ is concentrated on the diagonal and decays when $|x - y| \gg \varepsilon$. Thus, as pointed out in [12], the kernel of ω_N should be of the form

  $$\omega_N(x;y) \simeq \frac{1}{\varepsilon^3}\varphi\left(\frac{x-y}{\varepsilon}\right)\varrho\left(\frac{x+y}{2}\right), \tag{1.14}$$

  where ϱ represents the density of particles in space and φ fixes the momentum distribution. In particular, (1.14) is compatible with (1.13). See Chapter 6 in[13] for a detailed explanation and Section 5 in [12] for the propagation in time of (1.13) in the case of smooth interaction potentials.

To deal with the singularity of the Coulomb and gravitational potentials, we need also to control L^p norms of $\varrho_{|[x,\widetilde{\omega}_{N,t}]|}$ for $p > 5$, that means to require more structure on the operator $|[x, \widetilde{\omega}_{N,t}]|$. Under the assumptions of Theorem 1.1 we can control the L^1 norm of $\varrho_{|[x,\widetilde{\omega}_{N,t}]|}$, by using that

$$\begin{aligned}\|\varrho_{|[x,\widetilde{\omega}_{N,t}]|}\|_{L^1} &= \mathrm{tr}\ |[x, \widetilde{\omega}_{N,t}]| \\ &\leq \|(1-\varepsilon^2\Delta)^{-1}(1+x^2)^{-1}\|_{\mathrm{HS}}\|(1+x^2)(1-\varepsilon^2\Delta)[x,\widetilde{\omega}_{N,t}]\|_{\mathrm{HS}} \\ &\leq C\sqrt{N}\|(1+x^2)(1-\varepsilon^2\Delta)[x,\widetilde{\omega}_{N,t}]\|_{\mathrm{HS}}\,,\end{aligned}$$

for a positive constant C. In the last line of the above inequality the Hilbert-Schmidt norm (denoted by $\|\cdot\|_{\mathrm{HS}}$) can indeed be written in terms of the L^2 norm of its kernel, that is bounded by $C\sqrt{N}\varepsilon\|\widetilde{W}_{N,t}\|_{H^4_4}$, in turn bounded under the assumptions *i*), *ii*), *iii*) on the initial datum W_N.
This reasoning leaves hope to the possibility of propagating in time the L^p norms of $\varrho_{|[x,\widetilde{\omega}_{N,t}]|}$, with $p > 5$, thus making our conditional assumption (1.11) not unreasonable. However, at the moment we do not know which conditions one has to assume on the initial datum of the Vlasov-Poisson equation in order for $\|\varrho_{|[x,\widetilde{\omega}_{N,t}]|}\|_{L^p}$ to be bounded by $CN\varepsilon$ at positive time when $p > 1$. Theorem 1.1 is therefore a result conditioned to the uniform in time bounds (1.11). Nevertheless, there is one peculiar situation in which it is possible to verify assumption (1.11) holds true for all p and it will be discussed in Section 4.

- Theorem 1.1 is stated and proved for the Hartree dynamics. However, at least heuristically, it is expected to be true also for the Hartree-Fock equation

$$\begin{cases} i\varepsilon\partial_t\ \omega_{N,t} = \left[-\varepsilon^2\Delta + V * \varrho_t - X_t\,,\,\omega_{N,t}\right]\,, \\ X_t(x;y) = \dfrac{1}{N}V(x-y)\omega_N(x;y)\,, \end{cases} \tag{1.15}$$

for in this setting the exchange term should still be sub-leading. This has been proven in [12] in the case of smooth interactions. At the moment, a rigorous proof in the case of singular interactions is missing.

- With respect to the previous literature, Theorem 1.1 provides a strong notion of convergence with explicit rate in the case of Coulomb and gravitational potentials. In particular, a comparison with [30], where the Coulomb and gravitational interactions are considered, is in order. Indeed, in [30] the semiclassical limit is proven in the weak topology induced by the Monge-Kantorovich distance, whereas in the present paper we deal with the trace norm topology. The price to pay for such stronger notion of convergence is that assumption (1.11) seems at the moment difficult to state as an assumption at time zero, whereas in [30] all assumptions are stated on the initial data. However, we will show in Section 4 that, at least for a certain class of initial states, assumption (1.11) is satisfied.

Strategy of the proof. We present here the strategy of the proof in an informal way. We proceed as in [11] by performing a comparison between solutions to the Hartree equation and the Vlasov-Poisson system at the level of operators. This means to consider the Vlasov-Poisson system in its Weyl quantized form (see Eq. (2.4)). More precisely, we consider a sequence of fermionic operators ω_N, i.e. operators such that $0 \leq \omega_N \leq 1$, and look at their evolution first accordingly to the Hartree equation and then according to the Weyl quantized Vlasov-Poisson equation. We denote the solution to the Cauchy problem associated to the Hartree equation with initial data ω_N by $\omega_{N,t}$, whereas the solution of the Weyl quantized Vlasov-Poisson system with initial data ω_N is denoted by $\widetilde{\omega}_{N,t}$. We recall that such a solution exists and it is unique under the assumptions of our theorem, due to the result by Lions and Perthame [37] and its extension to signed measure (see [11] and [45]). We therefore compare $\omega_{N,t}$ and $\widetilde{\omega}_{N,t}$ looking for a Grönwall type inequality on the trace norm of the operator $\omega_{N,t} - \widetilde{\omega}_{N,t}$. The fist difficulty to cope with is a bound on the kinetic energy associated with the difference $\omega_{N,t} - \widetilde{\omega}_{N,t}$. To overcome this issue, in the same spirit of [11] we introduce a reference frame through a unitary dynamics. This suffices to tackle the problem since in this new reference frame the kinetic energy term

$$[\, -\varepsilon^2 \Delta \,,\, \omega_{N,t} - \widetilde{\omega}_{N,t} \,]$$

does not appear anymore (see Lemma 2.1). With respect to [11], where interactions with two bounded derivatives have been treated, the new difficulty here is to deal with the singularity at zero of the Coulomb and gravitational potential. To face this more fundamental point, we make use of an expression for the Coulomb potential introduced by Fefferman and de la Llave (see Lemma 2.2). This turned out to be useful in [44, 45] in the context of the mean-field limit from the many-body Schrödinger equation to the Hartree equation. To be more precise, we employ a smooth version of it (see Eq. (2.9)):

$$\frac{\gamma}{|x-y|} = C\gamma \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{r^5} \chi_{(r,z)}(x)\chi_{(r,z)}(y)\, dz\, dr$$

where $\chi_{(r,z)}(\cdot)$ is a smooth function depending on the distance $|\cdot - z|$ and varying on a scale r, C is a positive numerical constant and $\gamma = \pm 1$. The most important implication of such a rewriting of the Coulomb and gravitational potentials as an integral over all possible spheres of radius $r \geq 0$ consists in isolating the singularity at zero from all the other terms appearing in the Grönwall like type inequality (most of them produce errors which are estimated in Proposition 2.5). Hence, the key idea is to cancel part of the interaction by estimating the trace norm of the commutator $[\chi_{(r,z)}, \widetilde{\omega}_{N,t}]$, where $\chi_{(r,z)}$ acts as a multiplication operator. In order to absorb the singularity at $r = 0$, we need the bound on $\operatorname{tr} |[\chi_{(r,z)}, \widetilde{\omega}_{N,t}]|$ to be sharp in the r variable (see Lemma 2.4). The same strategy applies to the gravitational potential up to a sign that disappears once taking the absolute value to perform the estimates. Such a sharp bound in the r variable forces us to look at quantities (see Eq. (1.9)) that are in general not known to be bounded in terms of the assumptions on the initial data. This is the reason why we need to restrict our analysis to a special class of initial data (see Section 4).

Organisation of the paper. In Section 2 we present some preliminary estimates which will be used in Section 3, where the proof of Theorem 1.1 is presented; we conclude with Section 4, where examples of initial states verifying the assumptions of Theorem 1.1 are analysed, namely steady states for the attractive Vlasov-Poisson system. Hence, Theorem 1.1 shows that when the interaction is the gravitational potential, the Hartree evolution for non stationary states can be approximated by steady states of the Vlasov-Poisson system, thus showing that the Hartree dynamics leaves the state of the system approximately invariant.

2 Auxiliary Lemmas and Propositions

We start by giving a handier expression for the trace norm of the difference of a solution to the Hartree equation (2.1) and the Weyl transform of the solution to the Vlasov-Poisson system (1.1).

Lemma 2.1. *Let ω_N be a sequence of fermionic operators on $L^2(\mathbb{R}^3)$, $0 \leq \omega_N \leq 1$, with $\operatorname{tr} \omega_N = N$ and denote by W_N its Wigner transform. For each $N \in \mathbb{N}$, let $\omega_{N,t}$ be the solution of the time-dependent Hartree equation with Coulomb or gravitational interaction*

$$i\,\varepsilon\, \partial_t \omega_{N,t} = \left[-\varepsilon^2 \Delta + \frac{\gamma}{|\cdot|} * \varrho_t \,,\, \omega_{N,t} \right] \tag{2.1}$$

with initial data $\omega_{N,0} = \omega_N$ and let $\widetilde{W}_{N,t}$ be the solution of the Vlasov-Poisson system (1.1) *with initial data $\widetilde{W}_{N,0} = W_N$. Moreover, let $\widetilde{\omega}_{N,t}$ denote the Weyl quantization of $\widetilde{W}_{N,t}$. Then the following estimate holds true*

$$\operatorname{tr}|\omega_{N,t} - \widetilde{\omega}_{N,t}| \leq \frac{1}{\varepsilon}\int_0^t \operatorname{tr}\left|\left[\frac{\gamma}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s), \widetilde{\omega}_{N,s}\right]\right| ds + \frac{1}{\varepsilon}\int_0^t \operatorname{tr}|B_s|\, ds\,, \tag{2.2}$$

where for every $t \geq 0$, B_t is the operator associated with the kernel

$$B_t(x;y) = \left[\left(\frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t\right)(x) - \left(\frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t\right)(y) - \nabla\left(\frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t\right)\left(\frac{x+y}{2}\right)\cdot(x-y)\right]\widetilde{\omega}_{N,t}(x;y) \tag{2.3}$$

for all $x, y \in \mathbb{R}^3$.

Proof. We perform the Weyl transform of the Vlasov-Poisson system (1.1)

$$\begin{cases} \partial_t \widetilde{W}_{N,t} + v\cdot\nabla_x \widetilde{W}_{N,t} + E\cdot\nabla_v \widetilde{W}_{N,t} = 0\,, \\ E(t,x) = \nabla\left(\frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t\right)(x)\,, \\ \widetilde{\varrho}_t(x) = \int \widetilde{W}_{N,t}(x,v)\, dv\,, \end{cases}$$

and we obtain

$$i\,\varepsilon\,\partial_t \widetilde{\omega}_{N,t} = [-\varepsilon^2\Delta\,,\, \widetilde{\omega}_{N,t}] + A_t \tag{2.4}$$

where $\widetilde{\omega}_{N,t}$ is the Weyl transform of $\widetilde{W}_{N,t}$ and A_t is the operator associated with the kernel

$$A_t(x;y) = \nabla\left(\frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t\right)\left(\frac{x+y}{2}\right)\cdot(x-y)\,\widetilde{\omega}_{N,t}(x;y)\,.$$

Since we are interested in finding an expression for the difference of the operators $\omega_{N,t}$ and $\widetilde{\omega}_{N,t}$, we look for a Grönwall type estimate and compute the quantity $i\varepsilon\partial_t(\omega_{N,t} - \widetilde{\omega}_{N,t})$. To cope with the kinetic terms, we introduce a fictitious unitary dynamics given by the two-parameter group of unitary transformations $\mathcal{U}(t;s)$. Its time evolution is given by

$$i\,\varepsilon\,\partial_t\,\mathcal{U}(t;s) = h_H(t)\,\mathcal{U}(t;s)\,, \tag{2.5}$$

where $h_H = -\varepsilon^2\Delta + \frac{\gamma}{|\cdot|} * \varrho_t$ is the Hartree Hamiltonian.

We then conjugate the operator $(\omega_{N,t} - \widetilde{\omega}_{N,t})$ by $\mathcal{U}(t;s)$. We observe that such a choice makes $\omega_{N,t}$ play the role of a reference frame, thus we get

$$\begin{aligned} &i\,\varepsilon\,\partial_t\,\mathcal{U}^*(t;0)\,(\omega_{N,t} - \widetilde{\omega}_{N,t})\,\mathcal{U}(t;0) \\ &\qquad = -\,\mathcal{U}^*(t;0)\,[h_H(t), \omega_{N,t} - \widetilde{\omega}_{N,t}]\,\mathcal{U}(t;0) \\ &\qquad\quad + \mathcal{U}^*(t;0)\,([h_H(t), \omega_{N,t}] - [-\varepsilon^2\,\Delta, \widetilde{\omega}_{N,t}] - A_t)\,\mathcal{U}(t;0) \\ &\qquad = \mathcal{U}^*(t;0)\,\left(\left[\frac{\gamma}{|\cdot|} * \varrho_t, \widetilde{\omega}_{N,t}\right] - A_t\right)\mathcal{U}(t;0) \\ &\qquad = \mathcal{U}^*(t;0)\,\left(\left[\frac{\gamma}{|\cdot|} * (\varrho_t - \widetilde{\varrho}_t), \widetilde{\omega}_{N,t}\right] + B_t\right)\mathcal{U}(t;0) \end{aligned} \tag{2.6}$$

where B_t denotes the operator with the integral kernel defined in (2.3).

Since at time $t = 0$ $\omega_{N,0} = \widetilde{\omega}_{N,0} = \omega_N$, integration in time gives

$$\begin{aligned}\mathcal{U}^*(t;0)\,(\omega_{N,t} - \widetilde{\omega}_{N,t})\mathcal{U}(t;0) = \frac{1}{i\varepsilon}\int_0^t \mathcal{U}^*(t;s)\,\left[\frac{\gamma}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s), \widetilde{\omega}_{N,s}\right]\mathcal{U}(t;s)\,ds \\ + \frac{1}{i\varepsilon}\int_0^t \mathcal{U}^*(t;s)\,B_s\,\mathcal{U}(t;s)\,ds\,.\end{aligned} \tag{2.7}$$

Taking the trace norm in (2.7), we obtain

$$\operatorname{tr}|\omega_{N,t} - \widetilde{\omega}_{N,t}| \leq \frac{1}{\varepsilon}\int_0^t \operatorname{tr}\left|\left[\frac{\gamma}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s), \widetilde{\omega}_{N,s}\right]\right| ds + \frac{1}{\varepsilon}\int_0^t \operatorname{tr}|B_s|\,ds \tag{2.8}$$

as desired. □

We will estimate the two terms in the right-hand side of (2.8) separately, and conclude by applying Gronwall's lemma. The key idea is to rewrite the interaction as an integral over all possible spheres of radius $r \geq 0$, that is the content of the following Lemma first used in [16] by Fefferman and de la Llave to prove stability of matter in the relativistic case.

Lemma 2.2. *For every $x \in \mathbb{R}^3$ and $r \geq 0$, let $\chi_{(r,x)}(y) := \exp(-|x-y|^2/r^2)$, then*

$$\frac{1}{|x-y|} = \frac{4}{\pi^2}\int_0^\infty \int_{\mathbb{R}^3} \frac{1}{r^5}\,\chi_{(r,z)}(x)\,\chi_{(r,z)}(y)\,dz\,dr\,. \tag{2.9}$$

In the statement by Fefferman and de la Llave the Gaussians $\chi_{(r,z)}(\cdot)$ are replaced by the characteristic functions of the sphere $\{z \in \mathbb{R}^3 : |\cdot - z| \leq r\}$ and the numerical constant $4/\pi^2$ is replaced by $1/\pi$. It is easy to check that the symmetries and scaling of the Coulomb potential allow to replace the characteristic function by any smooth version of it that preserves the same symmetries and scaling, e.g. $\chi_{(r,x)}(y) := \exp(-|x-y|^2/r^2)$, provided the numerical constant in front of the integrals is appropriately modified. A detailed proof that holds also for more general radial potentials decaying at infinity can be found in [27].

Lemma 2.3. *For every $x \in \mathbb{R}^3$ and $r \geq 0$, let $\chi_{(r,x)}(y) := \exp(-|x-y|^2/r^2)$. Then*

$$\operatorname{tr}\left|\left[\frac{\gamma}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s)\,,\,\widetilde{\omega}_{N,s}\right]\right| \leq C\int_0^\infty \frac{1}{r^5}\iint |\varrho_s(y) - \widetilde{\varrho}_s(y)|\,\chi_{(r,z)}(y)\operatorname{tr}|[\chi_{(r,z)}\,,\,\widetilde{\omega}_{N,s}]|\,dz\,dy\,dr\,. \tag{2.10}$$

Proof. The identity (2.9) allows to rewrite the convolution on the l.h.s. of (2.10), for every $x \in \mathbb{R}^3$, as

$$\frac{1}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s)(x) = \frac{4}{\pi^2}\int_0^\infty \iint \frac{1}{r^5}\,\chi_{(r,y)}(x)\,\chi_{(r,z)}(y)\,(\varrho_s(y) - \widetilde{\varrho}_s(y))\,dz\,dy\,dr\,.$$

Therefore, for every $x, x' \in \mathbb{R}^3$, we obtain the following expression for the kernel of the commutator in the l.h.s. of (2.10)

$$\begin{aligned}&\left[\frac{1}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s)\,,\,\widetilde{\omega}_{N,s}\right](x;x') \\ &\quad = \frac{4}{\pi^2}\int_0^\infty \iint \frac{1}{r^5}\,(\varrho_s(y) - \widetilde{\varrho}_s(y))\,\chi_{(r,z)}(y)\,[\chi_{(r,z)}\,,\,\tilde{\omega}_{N,s}](x;x')\,dz\,dy\,dr\,.\end{aligned} \tag{2.11}$$

Taking the trace norm of the operator associated with the kernels in the above expression, the bound (2.10) holds. □

The following Lemma provides a key estimate to deal with the Coulomb singularity at zero. The proof can be found in [44], but we report it here for completeness.

Lemma 2.4 (Lemma 3.1 in [44]). *Let* $\chi_{(r,z)}(x) := \exp(-|x-z|^2/r^2)$ *and, given* $T > 0$, *assume* $[x_i, \widetilde{\omega}_{N,t}]$ *to be a trace class operator for all* $t \in [0,T]$. *Then, for all* $0 < \delta < 1/2$ *there exists* $C > 0$ *such that the following bound holds point-wise*

$$\mathrm{tr}\ \left|[\chi_{(r,z)}, \widetilde{\omega}_{N,t}]\right| \leq C\, r^{\frac{3}{2}-3\delta} \sum_{i=1}^{3} \|\varrho_{|[x_i,\widetilde{\omega}_{N,t}]|}\|_1^{\frac{1}{6}+\delta} \left(\varrho^*_{|[x_i,\widetilde{\omega}_{N,t}]|}(z)\right)^{\frac{5}{6}-\delta}, \tag{2.12}$$

where $\varrho^*_{|[x_i,\widetilde{\omega}_{N,t}]|}$ *denotes the Hardy-Littlewood maximal function of* $\varrho_{|[x_i,\widetilde{\omega}_{N,t}]|}$, *defined by*

$$\varrho^*_{|[x_i,\widetilde{\omega}_{N,t}]|}(z) = \sup_{B: z\in B} \frac{1}{|B|}\int_B \varrho_{|[x_i,\widetilde{\omega}_{N,t}]|}(x)\, dx \tag{2.13}$$

where the supremum is taken over all spheres B *containing the point* $z \in \mathbb{R}^3$, *and* $\varrho_{|[x_i,\widetilde{\omega}_{N,t}]|}$ *is defined in* (1.9).

Proof. We consider the commutator $[\chi_{(r,z)}\,,\, \widetilde{\omega}_{N,t}]$ and we write it as

$$[\chi_{(r,z)}\,,\, \widetilde{\omega}_{N,t}] = \sum_{j=1}^{3} \mathcal{I}_j + \mathcal{J}_j\,,$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are defined as follows

$$\begin{aligned}
\mathcal{I}_j &= -\int_0^1 \chi_{(r/\sqrt{s},z)}(x)\, \frac{(x-z)_j}{r^2}\, [x_j\,,\, \widetilde{\omega}_{N,t}]\, \chi_{(r/\sqrt{1-s},z)}(x)\, ds \\
\mathcal{J}_j &= -\int_0^1 \chi_{(r/\sqrt{s},z)}(x)\, [x_j\,,\, \widetilde{\omega}_{N,t}]\, \frac{(x-z)_j}{r^2}\, \chi_{(r/\sqrt{1-s},z)}(x)\, ds
\end{aligned}$$

for $j = 1,2,3$. As the two terms can be treated in the same way, we focus on $\mathcal{I}_j$. By assumption, $[x_j, \widetilde{\omega}_{N,t}]$ is an antiself-adjoint trace class operator and therefore it has a spectral decomposition. Let $\{f_k\}_k$ be an orthonormal system in $L^2(\mathbb{R}^3)$ and $\{\alpha_k\}_k$ be the associated eigenvalues, where $\alpha_k \in \mathbb{R}$ for all k. Then

$$[x_j\,,\, \widetilde{\omega}_{N,t}] = i \sum_k \alpha_k |\, f_k\rangle\langle f_k\,| \tag{2.14}$$

Eq. (2.14) and the definition of trace norm then leads to

$$\begin{aligned}
\mathrm{tr}\ |\mathcal{I}_j| &\leq \frac{1}{r}\sum_k |\alpha_k| \int_0^1 \frac{1}{\sqrt{s}}\mathrm{tr}\ \left|\left|\chi_{(r/\sqrt{s},z)}(x)\frac{\sqrt{s}|x-z|}{r}\, f_k\right\rangle \left\langle \chi_{(r/\sqrt{1-s},z)}(x)\, f_k\right|\right| ds \\
&\leq \frac{1}{r}\int_0^1 \frac{1}{\sqrt{s}} \left(\sum_k |\alpha_k| \left\|\chi_{(r/\sqrt{s},z)}(x)\frac{\sqrt{s}|x-z|}{r} f_k\right\|_2^2\right)^{\frac{1}{2}} \left(\sum_k |\alpha_k| \left\|\chi_{(r/\sqrt{1-s},z)}(x) f_k\right\|_2^2\right)^{\frac{1}{2}} ds
\end{aligned}$$

where in the last line we used Cauchy-Schwarz inequality.
By using the definition of the kernel of the multiplication operator $\chi_{(r/\sqrt{1-s},z)}(x)$ and again the spectral decomposition (2.14), we get the following bounds: on the one hand, integrating level set by level set, we get

$$\begin{aligned}
\sum_k |\alpha_k| \left\|\chi_{(r/\sqrt{1-s},z)}(x)\, f_k\right\|_2^2 &= \int_0^1 \int_{B(z,\sqrt{r^2\log(1/t)/2(1-s)})} \varrho_{|[x_j\,,\,\widetilde{\omega}_{N,t}]|}(x)\, dx\, dt \\
&\leq \frac{C\, r^3}{(1-s)^{3/2}}\, \varrho^*_{|[x_j,\widetilde{\omega}_{N,t}]|}(z)\,,
\end{aligned} \tag{2.15}$$

where C is a positive constant, $B(z, \sqrt{r^2 \log(1/t)/2(1-s)})$ is the ball centred at z with radius $\sqrt{r^2 \log(1/t)/2(1-s)}$ and $\varrho^*_{|[x_j,\widetilde{\omega}_{N,t}]|}$ is the Hardy-Littlewood maximal function of $\varrho_{|[x_j,\widetilde{\omega}_{N,t}]|}$ defined in (2.13).
On the other hand we have

$$\sum_k |\alpha_k| \left\| \chi_{(r/\sqrt{1-s},z)}(x) f_k \right\|_2^2 \leq C \sum_k |\alpha_k| = \text{tr } |[x_j, \widetilde{\omega}_{N,t}]| = C \|\varrho_{|[x_j,\widetilde{\omega}_{N,t}]|}\|_{L^1} . \tag{2.16}$$

Interpolating between (2.15) and (2.16), we obtain

$$\sum_k |\alpha_k| \left\| \chi_{(r/\sqrt{s},z)}(x) f_k \right\|_2^2 \leq C \frac{r^{3\theta} \|\varrho_{|[x_j,\widetilde{\omega}_{N,t}]|}\|_{L^1}^{1-\theta}}{s^{\frac{3}{2}\theta}} \left(\varrho^*_{|[x_j,\widetilde{\omega}_{N,t}]|}(z) \right)^\theta$$

that, with the choice $\theta = \frac{2}{3} - 2\delta$ yields the desired bound. □

Proposition 2.5. *Let B_t be the operator associated with the kernel* (2.3)*. Then, there exists a constant $C > 0$ depending on $\|\widetilde{W}_{N,t}\|_{H_4^2}$, $\|\widetilde{\rho}_t\|_{L^1}$ and $\|\nabla^2 \widetilde{\rho}_t\|_{L^\infty}$ such that*

$$\text{tr } |B_t| \leq C \, N \, \varepsilon^2 \left(1 + \sum_{k=1}^4 \varepsilon^k \|\widetilde{W}_{N,t}\|_{H_4^{k+2}} \right) . \tag{2.17}$$

Before giving the proof, we remark that the objects on which the constant C depends on are bounded by standard regularity theory for the Vlasov-Poisson system (cf. for instance [21]).

Proof. To bound the trace norm of B_t we introduce the identity operator

$$\mathbf{1} = (1 - \varepsilon^2 \Delta)^{-1} (1 + x^2)^{-1} (1 + x^2)(1 - \varepsilon^2 \Delta).$$

By applying Cauchy-Schwarz inequality we have

$$\text{tr} \, |B_t| \leq \|(1 - \varepsilon^2 \Delta)^{-1} (1 + x^2)^{-1}\|_{\text{HS}} \, \|(1 + x^2)(1 - \varepsilon^2 \Delta) B_t\|_{\text{HS}} . \tag{2.18}$$

We notice that for some $C > 0$ the following bound holds

$$\|(1 - \varepsilon^2 \Delta)^{-1} (1 + x^2)^{-1}\|_{\text{HS}} \leq C \sqrt{N} \,,$$

where we have used the explicit form of the kernel of the operator $(1 - \varepsilon^2 \Delta)^{-1}$ and the fact that $\varepsilon^3 = N^{-1}$.
We denote by U_t the convolution of the interaction with the spatial density at time t

$$U_t := \frac{\gamma}{|\cdot|} * \widetilde{\varrho}_t. \tag{2.19}$$

We introduce the notation

$$\widetilde{B} := (1 - \varepsilon^2 \Delta) B_t$$

and observe that the kernel of $\widetilde{B}$ reads

$$\widetilde{B}(x; x') := \sum_{j=1}^7 \widetilde{B}_j(x; x') \tag{2.20}$$

where

$$\begin{aligned} &\widetilde{B}_1(x; x') \\ &= N \left[U_t(x) - U_t(x') - \nabla U_t \left(\frac{x + x'}{2} \right) \cdot (x - x') \right] \int \widetilde{W}_{N,t} \left(\frac{x + x'}{2}, v \right) e^{i \, v \cdot \frac{(x - x')}{\varepsilon}} dv; \end{aligned}$$

$$\widetilde{B}_2(x;x')$$
$$= - N\varepsilon^2 \left[\Delta U_t(x) - \frac{1}{4}\Delta\nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x') - \frac{1}{2}\Delta U_t\left(\frac{x+x'}{2}\right)\right]\int \widetilde{W}_{N,t}\left(\frac{x+x'}{2},v\right) e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$
$$\widetilde{B}_3(x;x')$$
$$= - \frac{N\varepsilon^2}{4}\left[U_t(x) - U_t(x') - \nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x')\right]\int (\Delta_1\widetilde{W}_{N,t})\left(\frac{x+x'}{2},v\right) e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$
$$\widetilde{B}_4(x;x')$$
$$= N\left[U_t(x) - U_t(x') - \nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x')\right]\int \widetilde{W}_{N,t}\left(\frac{x+x'}{2},v\right) v^2 e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$
$$\widetilde{B}_5(x;x')$$
$$= - \frac{N\varepsilon^2}{2}\left[\nabla U_t(x) - \frac{1}{2}\nabla^2 U_t\left(\frac{x+x'}{2}\right)(x-x') - \nabla U_t\left(\frac{x+x'}{2}\right)\right]\int (\nabla_1\widetilde{W}_{N,t})\left(\frac{x+x'}{2},v\right) e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$
$$\widetilde{B}_6(x;x')$$
$$= - N\varepsilon\left[\nabla U_t(x) - \frac{1}{2}\nabla^2 U_t\left(\frac{x+x'}{2}\right)(x-x') - \nabla U_t\left(\frac{x+x'}{2}\right)\right]\int \widetilde{W}_{N,t}\left(\frac{x+x'}{2},v\right) v e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$
$$\widetilde{B}_7(x;x')$$
$$= - N\varepsilon\left[U_t(x) - U_t(x') - \nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x')\right]\int (v\cdot\nabla_1\widetilde{W}_{N,t})\left(\frac{x+x'}{2},v\right) e^{i\,v\cdot\frac{(x-x')}{\varepsilon}}dv;$$

where we used the notation ∇_1 and Δ_1 to indicate derivatives with respect to the first variable.

In order to gain extra powers of ε, we write

$$\begin{aligned}
&U_t(x) - U_t(x') - \nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x') \\
&= \int_0^1 d\lambda\left[\nabla U_t\left(\lambda x + (1-\lambda)x'\right) - \nabla U_t\left(\frac{(x+x')}{2}\right)\right]\cdot(x-x') \\
&= \sum_{i,j=1}^3 \int_0^1 d\lambda\left(\lambda - \frac{1}{2}\right)\int_0^1 d\mu\,\partial_i\partial_j U_t\left(\mu(\lambda x + (1-\lambda)x') + (1-\mu)\frac{(x+x')}{2}\right)(x-x')_i(x-x')_j.
\end{aligned}$$

We notice that U_t defined in (2.19) has a convolution structure. Therefore derivatives of U_t are equivalent to derivatives of the spatial density $\widetilde{\varrho}_t$. Hence, when integrating out the z variable in the Fefferman - de la Llave representation formula (2.9), we are left with

$$\begin{aligned}
&U_t(x) - U_t(x') - \nabla U_t\left(\frac{x+x'}{2}\right)\cdot(x-x') \\
&\quad = \gamma\sum_{i,j=1}^3 \int_0^1 d\lambda\left(\lambda - \frac{1}{2}\right)\int_0^1 d\mu\int_0^\infty \frac{dr}{r^2} \\
&\quad\times\int dy\,\chi_{(r,y)}\left(\mu(\lambda x + (1-\lambda)x') + (1-\mu)\frac{(x+x')}{2}\right)\partial_i\partial_j\widetilde{\varrho}_t(y)\,(x-x')_i(x-x')_j.
\end{aligned} \tag{2.21}$$

Inserting (2.21) into the definition of $\widetilde{B}_1$, using twice the identity

$$(x-x')\int \widetilde{W}_{N,t}\left(\frac{x+x'}{2},v\right) e^{iv\cdot\frac{x-x'}{\varepsilon}}\,dv = -i\varepsilon\int \nabla_v\widetilde{W}_{N,t}\left(\frac{x+x'}{2},v\right) e^{-iv\cdot\frac{x-x'}{\varepsilon}}dv \tag{2.22}$$

we get

$$
\begin{aligned}
&|\widetilde{B}_1(x;x')| \\
&\leq C\,N\,\varepsilon^2 \sum_{i,j=1}^{3} \int_0^1 d\lambda \left|\lambda - \frac{1}{2}\right| \int_0^1 d\mu \left| \int_0^\infty \frac{dr}{r^2} \right. \\
&\qquad \left. \int dy\, \chi_{(r,y)}(\mu(\lambda x + (1-\lambda)x') + (1-\mu)(x+x')/2)\partial^2_{i,j}\widetilde{\varrho}_t(y) \int dv\, \partial^2_{v_i,v_j} \widetilde{W}_{N,t}\left(\frac{x+x'}{2}, v\right) e^{iv\cdot\frac{(x-x')}{\varepsilon}} \right| .
\end{aligned}
$$

Therefore, the Hilbert-Schmidt norm of the operator $(1+x^2)\widetilde{B}_1$, where $(1+x^2)$ is the multiplication operator, can be estimated as follows:

$$
\begin{aligned}
&\|(1+x^2)\widetilde{B}_1\|^2_{\mathrm{HS}} \\
&\qquad \leq C N \varepsilon^4 \int dq \int dp' \left[1 + q^2 + \varepsilon^2 p^2\right]^2 \left| \sum_{i,j=1}^{3} \int_0^1 d\lambda \left(\lambda - \frac{1}{2}\right) \int_0^1 d\mu \int_0^\infty \frac{dr}{r^2} \right. \\
&\qquad\qquad \left. \int dy\, \chi_{(r,y)}(q + \varepsilon\mu(\lambda - 1/2)p)\partial^2_{i,j}\widetilde{\varrho}_t(y) \int dv\, \partial^2_{v_i,v_j}\widetilde{W}_{N,t}\,(q,v)\, e^{iv\cdot p} \right|^2
\end{aligned}
$$

where we performed the change of variables

$$
q = \frac{x+x'}{2}, \qquad p = \frac{x-x'}{\varepsilon} \tag{2.23}
$$

with Jacobian $J = 8\,\varepsilon^3 = 8\,N$.
We fix $k > 0$ and divide the integral into the two sets

$$
A_< := \{r \in \mathbb{R}_+ \mid r \leq k\} \quad \text{and} \quad A_> := \{r \in \mathbb{R}_+ \mid r > k\},
$$

so that

$$
\begin{aligned}
&\|(1+x^2)\widetilde{B}_1\|^2_{\mathrm{HS}} \\
&\leq C N \varepsilon^4 \int dq \int dp [1 + q^2 + \varepsilon^2 p^2]^2 \sum_{i,j=1}^{3} \int_0^1 d\lambda \left|\lambda - \frac{1}{2}\right| \\
&\quad \int_0^1 d\mu \left| \int_{A_<} \frac{dr}{r^2} \int dy\, \chi_{(r,y)}(q + \varepsilon\mu(\lambda - 1/2)p)\partial^2_{ij}\widetilde{\varrho}_t(y) \int dv\, \partial^2_{v_i,v_j}\widetilde{W}_{N,t}\,(q,v)\, e^{iv\cdot p} \right|^2 \\
&+ C N \varepsilon^4 \int dq \int dp [1 + q^2 + \varepsilon^2 p^2]^2 \sum_{i,j=1}^{3} \int_0^1 d\lambda \left|\lambda - \frac{1}{2}\right| \\
&\quad \int_0^1 d\mu \left| \int_{A_>} \frac{dr}{r^2} \int dy\, \chi_{(r,y)}(q + \varepsilon\mu(\lambda - 1/2)p)\partial^2_{ij}\widetilde{\varrho}_t(y) \int dv\, \partial^2_{v_i,v_j}\widetilde{W}_{N,t}\,(q,v)\, e^{iv\cdot p} \right|^2 .
\end{aligned} \tag{2.24}
$$

Denote by $\mathfrak{A}_<$ and $\mathfrak{A}_>$ the first and the second term of the sum on the r.h.s. of (2.24) respectively. For $\mathfrak{A}_<$ we first observe that

$$
\begin{aligned}
&\int_{A_<} \frac{dr}{r^2} \int dy\, \chi_{(r,y)}(q + \varepsilon\mu(\lambda - 1/2)p)\, \partial^2_{ij}\widetilde{\varrho}_t(y) \int dv\, \partial^2_{v_i\,v_j}\widetilde{W}_{N,t}\,(q,v)\; e^{iv\cdot p} \\
&\qquad\qquad\qquad \leq \int_{A_<} r\, \|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty} \int dv\, \partial^2_{v_i\,v_j}\widetilde{W}_{N,t}(q,v)\, e^{iv\cdot p} .
\end{aligned}
$$

Therefore, the singularity in r at zero is solved and

$$\begin{aligned}\mathfrak{A}_< \leq & C\,N\,\varepsilon^4 \int dq \int dp\, [1+q^2+\varepsilon^2 p^2]^2 \sum_{i,j=1}^{3} \int_0^1 d\lambda \left|\lambda - \frac{1}{2}\right| \\ & \int_0^1 d\mu \int dv \int dv' \partial^2_{v_i\,v_j} \widetilde{W}_{N,t}(q,v)\, \partial_{v_i'\,v_j'} \widetilde{W}_{N,t}(q,v')\, e^{i(v-v')\cdot p} \\ \leq & C\,N\,\varepsilon^4 \int dq\, (1+q^2)^2 \int dv\, |\partial^2_{v_i\,v_j} \widetilde{W}_{N,t}(q,v)|^2 + C\,N\,\varepsilon^8 \int dq \int dv\, |\nabla_v^4 \widetilde{W}_{N,t}(q,v)|^2\,,\end{aligned}$$

where in the last inequality we used that $[1+q^2+\varepsilon^2p^2]^2 \leq C[(1+q^2)^2+\varepsilon^4p^4]$ and we have integrated by parts twice in v in the last term on the r.h.s. in the same spirit of identity (2.22). We therefore get the bound

$$\mathfrak{A}_< \leq CN\varepsilon^4\, \|\widetilde{W}_{N,t}\|^2_{H^2_2} + CN\varepsilon^8\, \|\widetilde{W}_{N,t}\|^2_{H^4} \tag{2.25}$$

where C depends on $\|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty}$.

For $\mathfrak{A}_>$, we integrate by parts twice in the y variable and recall that $e^{-|z-y|^2/r^2}(1+|z-y|^2/r^2)$ is bounded uniformly in $z \in \mathbb{R}^3$. Since $\widetilde{\varrho}_t \in L^1(\mathbb{R}^3)$ we get the bound

$$\mathfrak{A}_> \leq CN\varepsilon^4 \int dq \int dv (1+q^2)^2 |\nabla_v^2 \widetilde{W}_{N,t}(q,v)|^2 + CN\varepsilon^8 \int dq \int dv\, |\nabla_v^4 \widetilde{W}_{N,t}(q,v)|^2\,, \tag{2.26}$$

where C depends on $\|\widetilde{\varrho}_t\|_{L^1}$.

Whence, considering the two estimates (2.25), (2.26) together, we get

$$\|(1+x^2)\widetilde{B}_1\|_{\mathrm{HS}} \leq C\sqrt{N}\varepsilon^2 \|\widetilde{W}_{N,t}\|_{H^2_2} + C\sqrt{N}\varepsilon^4 \|\widetilde{W}_{N,t}\|_{H^4} \tag{2.27}$$

where $C = C(\|\widetilde{\varrho}_t\|_{L^1}, \|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty})$.

The Hilbert-Schmidt norms $\|(1+x^2)\widetilde{B}_3\|_{\mathrm{HS}}$, $\|(1+x^2)\widetilde{B}_4\|_{\mathrm{HS}}$ and $\|(1+x^2)\widetilde{B}_7\|_{\mathrm{HS}}$ can be handled analogously, thus obtaining

$$\|(1+x^2)\widetilde{B}_3\|_{\mathrm{HS}} \leq C\sqrt{N}\varepsilon^4 \|\widetilde{W}_{N,t}\|_{H^4_4} + C\sqrt{N}\varepsilon^6 \|\widetilde{W}_{N,t}\|_{H^6_4}\,, \tag{2.28}$$

$$\|(1+x^2)\widetilde{B}_4\|_{\mathrm{HS}} \leq C\sqrt{N}\varepsilon^2 \|\widetilde{W}_{N,t}\|_{H^2_4} + C\sqrt{N}\varepsilon^4 \|\widetilde{W}_{N,t}\|_{H^4_4}\,, \tag{2.29}$$

$$\|(1+x^2)\widetilde{B}_7\|_{\mathrm{HS}} \leq C\sqrt{N}\varepsilon^3 \|\widetilde{W}_{N,t}\|_{H^2_3} + C\sqrt{N}\varepsilon^5 \|\widetilde{W}_{N,t}\|_{H^5_2}\,. \tag{2.30}$$

To bound the $\widetilde{B}_6$ term in which a higher order derivative of U_t appears, we proceed as for $\widetilde{B}_1$: we first use (2.21) and then divide the integral in the r variable into two parts, according to the definition of the sets $A_<$ and $A_>$:

$$\begin{aligned}&|\widetilde{B}_6(x;x')| \\ &\leq C\,N\,\varepsilon^3 \int_0^1 d\lambda \left|\lambda-\frac{1}{2}\right| \int_0^1 d\mu \left| \int_{A_<} \frac{dr}{r^2} \int dy\, \nabla\chi_{(r,y)} \left((2\lambda-1)\frac{\mu}{2}(x-x') + \frac{(x+x')}{2}\right) \nabla^2\widetilde{\varrho}_t(y) \right. \\ &\qquad\qquad \left. \int dv\, \nabla_v^2 \widetilde{W}_{N,t}\left(\frac{x+x'}{2}, v\right) e^{iv\cdot\frac{(x-x')}{\varepsilon}} \right| \\ &+ C\,N\,\varepsilon^3 \int_0^1 d\lambda \left|\lambda-\frac{1}{2}\right| \int_0^1 d\mu \left| \int_{A_>} \frac{dr}{r^2} \int dy\, \nabla^3\chi_{(r,y)} \left((2\lambda-1)\frac{\mu}{2}(x-x') + \frac{(x+x')}{2}\right) \widetilde{\varrho}_t(y) \right. \\ &\qquad\qquad \left. \int dv\, \nabla_v^2 \widetilde{W}_{N,t}\left(\frac{x+x'}{2}, v\right) e^{iv\cdot\frac{(x-x')}{\varepsilon}} \right|\end{aligned} \tag{2.31}$$

where in the second term we have integrated by parts twice in the y variable.

We denote by $\widetilde{B}_6^<$ and $\widetilde{B}_6^>$ the operators with kernels defined respectively by the first and second term in the r.h.s. of (2.31). We consider $\|(1+x^2)\widetilde{B}_6^<\|_{\rm HS}$, perform the change of variables (2.23) and choose $k=1$. Then we can apply Jensen's inequality with measure dr and we get the bound

$$\begin{aligned}\|(1+x^2)\widetilde{B}_6^<\|_{\rm HS}^2 \le C\,N\,\varepsilon^6 \int dq \int dp [1+q^2+\varepsilon^2p^2]^2 \int_0^1 d\lambda \left|\lambda-\frac{1}{2}\right|^2 \int_0^1 d\mu \int_0^1 \frac{dr}{r^4} \\ \int dy\, \chi_{(r,y)}(q+\varepsilon\mu(\lambda-1/2)p)\frac{|q+\varepsilon\mu(\lambda-1/2)p-y|}{r} \\ \int dy'\, \chi_{(r,y')}(q+\varepsilon\mu(\lambda-1/2)p)\frac{|q+\varepsilon\mu(\lambda-1/2)p-y'|}{r} \\ \iint dv\, dv'\, \nabla_v^2 \widetilde{W}_{N,t}(q,v)\, \nabla_{v'}^2 \widetilde{W}_{N,t}(q,v')\, e^{i(v-v')\cdot p} \\ \le C\,N\,\varepsilon^6 \|\widetilde{W}_{N,t}\|_{H_2^2}^2 + C\,N\,\varepsilon^{10} \|\widetilde{W}_{N,t}\|_{H^4}^2\end{aligned} \tag{2.32}$$

where C depends on $\|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty}$.
For $r \in A_>$, we consider $\|(1+x^2)\widetilde{B}_6^>\|_{\rm HS}$. We perform the change of variables (2.23) and recall that $e^{-|z-y|^2/r^2}(|z-y|/r)^n \le C$ for every $z \in \mathbb{R}^3$ and $n \in \mathbb{N}$. Since $\widetilde{\varrho}_s \in L^1(\mathbb{R}^3)$ we get the bound

$$\|(1+x^2)\widetilde{B}_6^>\|_{\rm HS}^2 \le C\,N\,\varepsilon^6 \|\widetilde{W}_{N,t}\|_{H_2^2}^2 + C\,N\,\varepsilon^{10} \|\widetilde{W}_{N,t}\|_{H^4}^2 \tag{2.33}$$

where C depends on $\|\widetilde{\varrho}_t\|_{L^1}$.

Thus considering the two terms together we get the desired bound

$$\|(1+x^2)\widetilde{B}_6\|_{\rm HS} \le C\sqrt{N}\varepsilon^3 \|\widetilde{W}_{N,t}\|_{H_2^2} + C\sqrt{N}\varepsilon^5 \|\widetilde{W}_{N,t}\|_{H^4} \tag{2.34}$$

where $C = C(\|\widetilde{\varrho}_t\|_{L^1}, \|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty})$.

The norms $\|(1+x^2)\widetilde{B}_j\|_{\rm HS}$, $j=2,5$, can be dealt analogously, thus obtaining

$$\|(1+x^2)\widetilde{B}_2\|_{\rm HS} \le C\sqrt{N}\varepsilon^4 \|\widetilde{W}_{N,t}\|_{H_2^2} + C\sqrt{N}\varepsilon^6 \|\widetilde{W}_{N,t}\|_{H^4} \tag{2.35}$$

and

$$\|(1+x^2)\widetilde{B}_5\|_{\rm HS} \le C\sqrt{N}\varepsilon^4 \|\widetilde{W}_{N,t}\|_{H_2^4} + C\sqrt{N}\varepsilon^6 \|\widetilde{W}_{N,t}\|_{H^6} \tag{2.36}$$

where $C = C(\|\widetilde{\varrho}_t\|_{L^1}, \|\nabla^2 \widetilde{\varrho}_t\|_{L^\infty})$.

Gathering together all the terms, we get

$$\begin{aligned}&\|(1+x^2)\widetilde{B}\|_{\rm HS} \\ &\quad \le C\sqrt{N}\left[\varepsilon^2\|\widetilde{W}_{N,t}\|_{H_4^2} + \varepsilon^3\|\widetilde{W}_{N,t}\|_{H_4^3} + \varepsilon^4\|\widetilde{W}_{N,t}\|_{H_4^4} + \varepsilon^5\|\widetilde{W}_{N,t}\|_{H_4^5} + \varepsilon^6\|\widetilde{W}_{N,t}\|_{H_4^6}\right].\end{aligned} \tag{2.37}$$

□

3 Proof of Theorem 1.1

From Lemma 2.1 Eq. (2.2) and Proposition 2.5, we know that

$$\mathrm{tr}\,|\omega_{N,t} - \widetilde{\omega}_{N,t}| \le \frac{1}{\varepsilon}\int_0^t \mathrm{tr}\left|\left[\frac{\gamma}{|\cdot|} * (\varrho_s - \widetilde{\varrho}_s)\,,\, \widetilde{\omega}_{N,s}\right]\right| ds + C\,N\,\varepsilon \int_0^t \left(1 + \sum_{k=1}^4 \varepsilon^k \|\widetilde{W}_{N,s}\|_{H_4^{k+2}}\right) ds. \tag{3.1}$$

We focus on the first term on the r.h.s. of (3.1). Recalling Lemma 2.3 Eq. (2.10), we fix a positive real number k and we write

$$\begin{aligned}\operatorname{tr}\left|\left[\frac{\gamma}{|\cdot|}*(\varrho_s-\widetilde{\varrho}_s)\,,\,\widetilde{\omega}_{N,s}\right]\right| &\le C\int_0^\infty \frac{1}{r^5}\iint |\varrho_s(y)-\widetilde{\varrho}_s(y)|\,\chi_{(r,z)}(y)\operatorname{tr}|[\chi_{(r,z)}\,,\,\widetilde{\omega}_{N,s}]|\,dz\,dy\,dr\\ &\le C\int_0^k \frac{1}{r^{\frac{7}{2}+3\delta}}\int |\varrho_s(y)-\widetilde{\varrho}_s(y)|\,g_r(y)\sum_{i=1}^3\|\varrho_{|[x_i,\widetilde{\omega}_{N,s}]|}\|_{L^1}^{\frac16+\delta}\,dy\,dr\\ &\quad + C\sum_{i=1}^3\|\varrho_{|[x_i,\widetilde{\omega}_{N,s}]|}\|_{L^1}\int_k^\infty\frac{1}{r^6}\int|\varrho_s(y)-\widetilde{\varrho}_s(y)|\,dy\,dr\end{aligned} \tag{3.2}$$

where

$$g_r(y)=\int \chi_{(r,z)}(y)\,\left(\varrho^*_{|[x,\widetilde{\omega}_{N,s}]|}(z)\right)^{\frac56-\delta}\,dz\,,$$

and we used Lemma 2.4 in the second line of equation (3.2) and the bound (2.16) in the last line of equation (3.2).

We now compute the L^∞ norm of $g_r(y)$:

$$\|g_r\|_{L^\infty}\le Cr^{\frac3q}\|\varrho^*_{|[x,\widetilde{\omega}_{N,s}]|}\|^{\frac56-\delta}_{L^{(\frac56-\delta)q'}}\le Cr^{\frac3q}\|\varrho_{|[x,\widetilde{\omega}_{N,s}]|}\|^{\frac56-\delta}_{L^{(\frac56-\delta)q'}}$$

where q and q' are conjugated Hölder exponents and we have used the L^s boundedness of the Hardy-Littlewood maximal operator in the last inequality, for $s=(\frac56-\delta)q'>1$.

To deal with the singularity at zero in the r variable in (3.2), we choose $q'>6$ and $q<6/5$. Hence, there exist a constant $C_{t,1}$, depending on time but independent on N, such that

$$\begin{aligned}\operatorname{tr}\left|\left[\frac{\gamma}{|\cdot|}*(\varrho_s-\widetilde{\varrho}_s)\,,\,\widetilde{\omega}_{N,s}\right]\right| &\le C\|\varrho_s-\widetilde{\varrho}_s\|_{L^1}\sum_{i=1}^3\left(\|\varrho_{|[x_i,\widetilde{\omega}_{N,s}]|}\|_{L^1}^{\frac16+\delta}\|\varrho_{|[x_i,\widetilde{\omega}_{N,s}]|}\|_{L^p}^{\frac56-\delta}+\|\varrho_{|[x_i,\widetilde{\omega}_{N,s}]|}\|_{L^1}\right)\\ &\le C_{t,1}\,\varepsilon\operatorname{tr}|\omega_{N,s}-\widetilde{\omega}_{N,s}|\,.\end{aligned} \tag{3.3}$$

In the last inequality we used assumption (1.11) with $p>5$ and the fact that

$$\begin{aligned}\|\varrho_s-\widetilde{\varrho}_s\|_{L^1} &= \sup_{\substack{O\in L^\infty(\mathbb{R}^3)\\ \|O\|_{L^\infty}\le 1}}\left|\int O(z)\,(\varrho_s(z)-\widetilde{\varrho}_s(z))\,dz\right|\\ &\le \frac1N\sup_{O\,:\,\|O\|\le1}|\operatorname{tr}O(\omega_{N,s}-\widetilde{\omega}_{N,s})|\\ &\le \frac1N\operatorname{tr}|\omega_{N,s}-\widetilde{\omega}_{N,s}|\,,\end{aligned}$$

where in the second line the supremum is taken over all bounded operators O with operator norm $\|O\|\le 1$.

We now analyse the second term on the r.h.s. of (3.1). Using assumptions $i)$, $ii)$ and $iii)$ in Theorem 1.1 and a trivial adaptation of Appendix A in [46], we can bound the weighted Sobolev norms $\|\widetilde{W}_{N,t}\|_{H_4^{k+2}}$, for $k=1,\dots,4$, in terms of the initial data W_N:

$$\|\widetilde{W}_{N,t}\|_{H_4^{k+2}}\le C_t\|W_N\|_{H_4^{k+2}}\,, \tag{3.4}$$

where C_t is a time dependent constant, for $t\in[0,T]$.

Therefore, Eq. (3.3) and Eq. (3.4) leads to the Grönwall type estimate

$$\text{tr } |\omega_{N,t} - \widetilde{\omega}_{N,t}| \leq C_{t,1} \int_0^t \text{tr } |\omega_{N,s} - \tilde{\omega}_{N,s}| \, ds + C_{t,2} \, N \, \varepsilon \int_0^t \left(1 + \sum_{k=1}^{4} \varepsilon^k \|W_N\|_{H_4^{k+2}} \right) ds \,,$$

where, for every fixed $T > 0$ and for all $t \in [0, T]$, $C_{t,1}$ is proportional to the constant appearing in assumption (1.11) and $C_{t,2}$ depends on $t \in [0, T]$ and on $\|W_N\|_{H_4^2}$. Both $C_{t,1}$ and $C_{t,2}$ are independent of N. Hence, by Grönwall Lemma, we conclude the proof of (1.12).

4 Steady states of the Vlasov-Poisson system

In general, for $T > 0$ fixed, we do not know which hypotheses ω_N should satisfy at time $t = 0$ in order for the bounds (1.11) to hold for all $t \in [0, T]$. In this section, we are interested in identifying a special class of states which satisfy the assumptions of Theorem 1.1.

We start by considering a sequence of fermionic reduced densities ω_N which are superpositions of coherent states

$$f_{q,p}(x) = \varepsilon^{-\frac{3}{2}} e^{-ip\cdot x/\varepsilon} G(x - q)$$

where, for every $\delta > 0$, G is the Gaussian defined as follows

$$G(x - q) = \frac{e^{-|x-q|^2/2\delta^2}}{(2\pi\delta^2)^{\frac{3}{4}}} \,.$$

Namely, for $M : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ probability density such that $0 \leq M(q, p) \leq 1$ for all $(q, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\iint M(q, p) \, dq \, dp = 1$, we define the sequence of fermionic operators

$$\omega_N = \iint M(q, p) \, |f_{(q,p)}\rangle\langle f_{(q,p)}| \, dq \, dp \,, \tag{4.1}$$

with kernel

$$\omega_N(x; y) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} M(q, p) \, f_{(q,p)}(x) \, \overline{f_{(q,p)}(y)} \, dq \, dp \,, \tag{4.2}$$

where in formula (4.1) we used the bra-ket notation.

We notice that, if $M \in \mathcal{W}^{1,1}(\mathbb{R}^3 \times \mathbb{R}^3)$, the Sobolev space of functions such that the norm $\|\nabla M\|_{L^1}$ is finite, then the sequence ω_N defined as in (4.1) satisfies the bound

$$\|\varrho_{|[x,\omega_N]|}\|_{L^1(\mathbb{R}^3)} = \text{tr } |[x, \omega_N]| \leq CN\varepsilon \,,$$

where $C = \|\nabla_v M\|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)}$. Indeed, consider the kernel of the commutator $[x, \omega_N]$:

$$\begin{aligned} [x, \omega_N](x; y) &= \iint (x - y) \, M(q, p) \, f_{(q,p)}(x) \, \overline{f_{(q,p)}(y)} \, dq \, dp \\ &= \frac{N\varepsilon}{i} \iint \nabla_p M(q, p) \, f_{(q,p)}(x) \, \overline{f_{(q,p)}(y)} \, dq \, dp \end{aligned}$$

thus

$$[x, \omega_N] = \frac{N\varepsilon}{i} \iint \nabla_p M(q, p) \, |f_{(q,p)}\rangle\langle f_{(q,p)}| \, dq \, dp$$

hence the trace norm is easily bounded as follows

$$\text{tr } |[x, \omega_N]| \leq N\varepsilon \iint |\nabla_p M(q, p)| \, \|f_{(q,p)}\|^2_{L^2(\mathbb{R}^3)} \, dq \, dp = N\varepsilon \|\nabla_v M\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \,.$$

We do expect that, under suitable regularity and integrability assumptions on M, there exists a finite positive constant C such that $\|\varrho_{|[x,\omega_N]|}\|_{L^p(\mathbb{R}^3)} \leq CN\varepsilon$ for some $p > 5$.

Now consider ω_N defined in (4.1) and satisfying the bound

$$\|\varrho_{|[x,\omega_N]|}\|_{L^1} + \|\varrho_{|[x,\omega_N]|}\|_{L^p} \leq C\,N\varepsilon\,, \qquad p > 5 \tag{4.3}$$

to be the initial datum of the Cauchy problem associated with Eq. (2.4). Its evolution is denoted by $\widetilde{\omega}_{N,t}$. If we restrict to attractive interactions, then the existence of steady states for the Vlasov-Poisson system (cf. [9]) allows to sigle out a specific class of initial data for which assumption (4.3) can be trivially propagated at time t to satisfy (1.11). Indeed, if $\widetilde{\omega}_{N,t}$ is a steady state for the Weyl transformed Vlasov-Poisson system then, for every fixed $T > 0$, $\widetilde{\omega}_{N,t}$ automatically satisfies the bound

$$\|\varrho_{|[x,\widetilde{\omega}_{N,t}]|}\|_{L^1} + \|\varrho_{|[x,\widetilde{\omega}_{N,t}]|}\|_{L^p} \leq C\,N\varepsilon\,, \qquad p > 5 \tag{4.4}$$

for all $t \in (0, T]$, if it does at time $t = 0$.

More precisely, if $\widetilde{\omega}_N = \widetilde{\omega}_{N,t}$ is a steady state for Eq. (2.4) with gravitational interaction, then its Wigner transform $\widetilde{W}_N$ solves the equation

$$\begin{cases} v \cdot \nabla_x \widetilde{W}_N - \nabla_x U \cdot \nabla_v \widetilde{W}_N = 0\,, \\ -\Delta_x U(x) = \widetilde{\varrho}(x)\,, \\ \widetilde{\varrho}(x) = \int \widetilde{W}_N(x,v)\,dv\,. \end{cases} \tag{4.5}$$

One example of states which satisfy (4.5) are functions of the form

$$\widetilde{W}_N(x,v) = \Phi \circ H(x,v)$$

where Φ is a smooth function of the local energy

$$H(x,v) = \frac{|v|^2}{2} - U(x)\,. \tag{4.6}$$

The existence of this class of steady states and their stability properties have been addressed in [9, 32]. For such states, the associated fermionic operator ω_N is independent of time. Hence, if we assume (4.3), the bounds (4.4) are satisfied.

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