Birational geometry of sextic double solids with a compound A_n singularity

Erik Paemurru

Departement Mathematik und Informatik
Fachbereich Mathematik
Universität Basel
CH-4051 Basel

Preprint No. 2021-02
January 2021

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Erik Paemurru*†

January 5, 2021

Abstract

Sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, are the lowest degree Gorenstein Fano 3-folds, hence are expected to behave very rigidly in terms of birational geometry. Smooth sextic double solids, and those which are $\mathbb{Q}$-factorial with ordinary double points, are known to be birationally rigid. In this article, we study sextic double solids with an isolated compound A_n singularity. We prove a sharp bound $n \leq 8$, describe models for each n explicitly and prove that sextic double solids with $n > 3$ are birationally non-rigid.

Contents

*The author was supported by the Engineering and Physical Sciences Research Council (grant 1820497) while in Loughborough University, and the London Mathematical Society Early Career Fellowship (grant ECF-1920-24) while in Imperial College London.

†Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland.

1. Introduction

We work with projective varieties over $\mathbb{C}$. Classification of algebraic varieties is one of the fundamental goals in algebraic geometry. The Minimal Model Program says that every variety is birational to either a minimal model or a Mori fibre space. Two Mori fibre spaces are birational if they are connected by a sequence of Sarkisov links (see [Sar89], [Rei91], [Cor95], [HM13]). In the extreme case, the Mori fibre space is *birationally rigid*, meaning that it is essentially the unique Mori fibre space in its birational class.

Examples of Mori fibre spaces include Fano varieties. The first birational rigidity result was in the seminal paper by Iskovskikh and Manin [IM71] for smooth quartic 3-folds in $\mathbb{P}^4$. A wealth of examples of birationally rigid varieties was given in [CPR00] and [CP17], by showing that every quasismooth member of the 95 families of Fano 3-folds that are hypersurfaces in weighted projective spaces is birationally rigid. One major consequence of birational rigidity is nonrationality. Birational rigidity remains an active area of research (see [Pro18], [Kry18], [AO18], [dF17], [CG17], [CS19], [EP18]).

Among smooth Fano 3-folds, the projective space has the highest degree (64), and sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, have the least degree (2). In [Isk80], it is proved that smooth sextic double solids are birationally rigid. It is interesting to see how this changes as we impose singularities on the variety. The paper [Puk97] proved that sextic double solids stays birationally rigid if we impose an ordinary double point, meaning the 3-fold A_1 singularity $x_1^2+x_2^2+x_3^2+x_4^2$. A sextic double solid can have up to 65 singular points (see [Bar96], [Bas06], [JR97], [Wah98]), and for each $n \leq 65$, there exists a sextic double solid with exactly n ordinary double points and smooth otherwise (see [CC82]). A sextic double solid with only ordinary double points is birationally rigid if and only if it is factorial, which is true for example if it has at most 14 ordinary double points (see [CP10]).

The next natural question is to consider more complicated singularities in the Mori category. We study sextic double solids with an isolated *compound* A_n singularity, also called a cA_n singularity, meaning that the general section through the point is the Du Val A_n singularity $x_1x_2+x_3^{n+1}$. A cA_n singularity is locally analytically given by $x_1x_2+h(x_3,x_4)$ where the least degree among monomials in h is $n+1$. The first main result of the paper is describing sextic double solids with an isolated cA_n singularity.

Theorem (see Theorem A)**.** *If a sextic double solid has an isolated cA_n point, then $n \leq 8$.*

Moreover, in Theorem A we explicitly parametrize all sextic double solids with an isolated cA_n singularity for every $n \leq 8$. These form 11 families, as there are 4 families for cA_7. A very general member of every family, except for the family 7.4, is a Mori fibre space.

We say a few words on bounding the number of cA_n singularities. It is clear that an isolated cA_n singularity has Milnor number at least n^2. Since the third Betti number of a smooth sextic double solid is 104 (see [IP99, Table 12.2]), an argument similar to [AK16, Section 3.2] shows that the total Milnor number of a sextic double solid which is a Mori fibre space is at most 104. This gives the bounds that a Mori fibre space sextic double solid can have up to 1 cA_8 singularity, or up to 2 cA_7 singularities, or up to 2 cA_6 singularities, $\dots$, or up to 26 cA_2 singularities. We do not expect these bounds to be sharp, as already for ordinary double points it gives an upper bound of 104, far from the actual 65. Using Theorem A, it is possible to construct sextic double solids with a cA_8 point, a cA_3 point and two ordinary double points with both total Milnor and total Tjurina number at least 66.

The second main result is the following theorem:

Theorem (see Theorem B and Section 5.3)**.** *A general sextic double solid which is a Mori fibre space with an isolated cA_n singularity where $n \geq 4$ is not birationally rigid.*

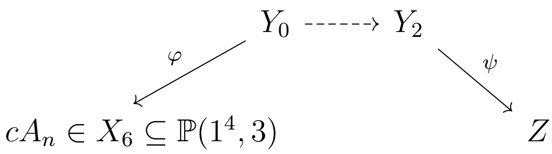
Birational non-rigidity for a sextic double solid X is proved by describing a birational model, meaning a Mori fibre space $T \to S$ such that X and T are birational. We find the birational models by explicitly constructing a Sarkisov link for each family of sextic double solids, under the generality conditions described in Condition 5.1. Table 1 gives an overview of the Sarkisov links $X \leftarrow Y_0 \dashrightarrow Y_2 \to Z$ and the birational models, which are either fibrations $Y_2 \to Z$ or Fano varieties Z. In the latter case, $Y_2 \to Z$ is a divisorial to the given singular point. The morphism $Y_0 \to X$ is a divisorial contraction with centre the cA_n point. The birational maps $Y_0 \dashrightarrow Y_2$ are isomorphisms in codimension 1.

Note that when we say that a birational map $Y_0 \dashrightarrow Y_1$ is k Atiyah flops, then we mean that algebraically it is one flop, contracting k curves to k points and extracting k curves, and locally analytically around each of those points, it is an Atiyah flop. Similarly for flips. Also note that the Sarkisov link to a sextic double solid with a cA_4 singularity was already described in [Oka14, Section 9, No. 9], starting from a general quasismooth complete intersection $X_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4)$.

We briefly describe the proof. The first step in the Sarkisov link starting from a Fano variety X is a divisorial contraction $Y \to X$. Kawakita described divisorial contractions to cA_n points locally analytically, showing that they are certain weighted blowups. To construct Sarkisov links, we need a global description. In Proposition 4.5 and Lemma 4.8, we show how to construct divisorial contractions to cA_n points algebraically on affine hypersurfaces, and use this in Section 5 to construct divisorial contractions $Y \to X$ for (projective) sextic double solids X. Using unprojection techniques (see [PR04] for a general theory of unprojection), we find an embedding of Y inside a toric variety T, such that the 2-ray link of T restricts to a Sarkisov link for X (following [BZ10] and [AZ16]).

If we try the same methods as in the proof of Theorem B on sextic double solids with a cA_n singularity where $n \leq 3$, then we do not find any new birational models. More precisely: a $(3, 1, 1, 1)$-Kawakita blowup of a cA_3 singularity on a general Mori fibre space sextic double solid initiates a Sarkisov link to itself $X \dashrightarrow X$. A $(2, 2, 1, 1)$-Kawakita blowup for a cA_3 singularity, a $(2, 1, 1, 1)$-Kawakita blowup for a $x_1x_2 + x_3^3 + x_4^3$ singularity and the (usual) blowup for an ordinary double point on a general Mori fibre space sextic

Table 1. Birational models for general sextic double solids that are Mori fibre spaces with an isolated cA_n singularity



cA_n	weighted blowup φ	$\dashrightarrow$	weighted blowup or fibration ψ	Z
cA_4	$(3, 2, 1, 1)$	10 Atiyah flops	$(\frac{1}{4}, \frac{1}{4}, \frac{3}{4})$	$\frac{1}{4}(1, 1, 3) \in Z_{5,6} \subseteq \mathbb{P}(1^3, 2, 3, 4)$
cA_5	$(3, 3, 1, 1)$	4 Atiyah flops	$(3, 3, 1, 1)$	$cA_5 \in Z_6 \subseteq \mathbb{P}(1^4, 3)$, $X \not\cong Z$ if general
cA_6	$(4, 3, 1, 1)$	2 Atiyah flops, then $(4, 1, 1, -2, -1; 2)$-flip	$(3, 1, 1, 1)$	$cA_3 \in Z_5 \subseteq \mathbb{P}(1^4, 2)$
cA_7, 1	$(4, 4, 1, 1)$	two $(4, 1, 1, -2, -1; 2)$-flips	$(1, 1, 1, 1)$	ODP $\in Z_{3,4} \subseteq \mathbb{P}(1^4, 2^2)$
cA_7, 2	$(4, 4, 1, 1)$	Atiyah flop, then two $(4, 1, -1, -3)$-flips	$(3, 3, 2, 1)$	$cA_2 \in Z_{2,4} \subseteq \mathbb{P}(1^5, 2)$
cA_7, 3	$(4, 4, 1, 1)$	2 Atiyah flops	dP_2-fibration	$\mathbb{P}^1$
cA_8	$(5, 4, 1, 1)$	$(4, 1, 1, -2, -1; 2)$-flip	$(3, 2, 2, 1, 5)$	$cD_4 \in Z_{3,3} \subseteq \mathbb{P}(1^5, 2)$

double solid initiate 'bad links', which end in either a non-terminal 3-fold or a K3-fibration. These are 2-ray links which are not Sarkisov links, where in the last step of the 2-ray game only K-trivial curves are contracted, leaving the Mori category. We expect that general Mori fibre space sextic double solids with a cA_3 singularity are birationally rigid, and with certain cA_2 or cA_1 singularities are birationally superrigid.

Organization of the paper

In Sections 2.1, 2.2 and 2.3, we give known results that we use respectively in Sections 3, 4 and 5. In Section 3, we construct a parameter space of sextic double solids in Theorem A with an isolated cA_n singularity. In Section 4, we explain the relationship between algebraic and local analytic weighted blowups, and in Proposition 4.5 and the technical Lemma 4.8, show how to construct divisorial contractions to cA_n points algebraically on affine hypersurfaces. In Section 5, we construct birational models for general sextic double solids which are Mori fibre spaces with an isolated cA_n singularity where $n \geq 4$, thereby showing that they are not birationally rigid. We treat the 7 families separately. Appendix A contains the computer code, in particular the splitting lemma from singularity theory (Proposition 3.2) in Listing 1, which we use for constructing the parameter space and the divisorial contractions for sextic double solids.

2. Preliminaries

All varieties we consider are irreducible and over $\mathbb{C}$.

We study sextic double solids, which are double covers of the projective 3-space branched along a sextic surface. We use the following equivalent characterization:

Definition 2.1. A **sextic double solid** is an irreducible hypersurface given by the zero locus of $w^2 + g$ in the weighted projective space $\mathbb{P}(1,1,1,1,3)$ with variables x, y, z, t, w, where $g \in \mathbb{C}[x, y, z, t]$ is a homogeneous polynomial of degree six.

2.1. Singularity theory

We recall some results from the singularity theory of complex analytic spaces and on terminal singularities.

We denote the variables on $\mathbb{C}^n$ by $\boldsymbol{x} = (x_1, \ldots, x_n)$, where n is a positive integer. Let $\mathbb{C}\{\boldsymbol{x}\}$ denote the convergent power series ring. The zero set of an ideal I is denoted by $\mathbb{V}(I)$, where I is either an ideal of regular functions or holomorphic functions, depending on context. Given a regular or holomorphic function f on a variety or space X, denote the non-zero locus of f by X_f. Given positive integer weights $\boldsymbol{w} = (w_1, \ldots, w_n)$ for $\boldsymbol{x}$, we can write a non-zero polynomial or power series f as a sum of its weighted homogeneous parts f_i. Then, the **weight** of f, denoted $\mathrm{wt}(f)$, is the least non-negative integer d such that $f_d \neq 0$. We define $\mathrm{wt}(0) = \infty$. If $\boldsymbol{w} = (1, \ldots, 1)$, then d is called the **multiplicity**, denoted $\mathrm{mult}(f)$. A **hypersurface singularity** is a complex analytic space germ (not necessarily irreducible or reduced) that is isomorphic to $(X, \mathbf{0})$ where $X \subseteq \mathbb{C}^n$ is given by the zero set of some $f \in \mathbb{C}\{\boldsymbol{x}\}$. A singularity is **isolated** if it has a smooth analytic punctured neighbourhood.

Definition 2.2 ([GLS07, Definition 2.9])**.** Let $f, g \in \mathbb{C}\{\boldsymbol{x}\}$.

(a) We say f and g are **right equivalent** if there exists a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $g = f \circ \varphi$.

(b) We say f and g are **contact equivalent** if there exists a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ and a unit $u \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that $g = u(f \circ \varphi)$.

Remark 2.3 ([GLS07, Remark 2.9.1(3)]). Two convergent power series $f, g \in \mathbb{C}\{\boldsymbol{x}\}$ are contact equivalent if and only if the complex analytic space germs $(X, \mathbf{0})$ and $(Y, \mathbf{0})$ are isomorphic, where $X \subseteq \mathbb{C}^n$ is given by the zeros of f and $Y \subseteq \mathbb{C}^n$ is given by the zeros of g.

We use the following proposition in Section 3 to parametrize sextic double solids with a cA_1 singularity:

Proposition 2.4 ([GLS07, Remark 2.50.1])**.** *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be two contact equivalent power series with zero constant term. Then their multiplicity m is the same and furthermore, f_m and g_m are the same up to an invertible linear change of coordinates.*

We use the following proposition in Section 3 to construct sextic double solids with a cA_n singularity where $n \geq 2$, as well as in Section 4 to describe weighted blowups of cA_n points:

Proposition 2.5. *Let $F = x_1^2 + \ldots + x_k^2 + f$ and $G = x_1^2 + \ldots + x_k^2 + g$, where f and g are convergent power series in $\mathbb{C}\{x_{k+1}, \ldots, x_n\}$ with zero constant term. Then F and G are contact (respectively, right) equivalent if and only if f and g are contact (respectively, right) equivalent.*

Proof. By a result of Mather and Yau [MY82] (see also [GLS07, Theorem 2.26]), f and g are contact equivalent if and only the Tjurina algebras T_f and T_g are isomorphic. A simple computation shows that $T_f \cong T_F$ and $T_g \cong T_G$, which proves the proposition for contact equivalence.

The proof for right equivalence is similar. Namely, we use a statement analogous to [MY82]: two elements $h, k \in \mathbb{C}\{\boldsymbol{x}\}$ with zero constant term are right equivalent if and only if the Milnor algebras M_h and M_k are isomorphic as algebras over the ring $\mathbb{C}\{t\}$, where t acts on M_h, respectively M_k, by multiplying by h, respectively k (see [GLS07, Theorem 2.28]). □

Reid defined in [Rei80, Definition 2.1] that a **compound Du Val singularity** is a 3-dimensional singularity where a hypersurface section is a Du Val singularity, also called a surface ADE singularity. The singularity is denoted cA_n, cD_n or cE_n, respectively, if the general hyperplane section is an A_n, D_n or E_n singularity, respectively. Reid showed in [Rei83, Theorem 0.6] that a 3-dimensional hypersurface singularity is terminal if and only if it is an isolated compound Du Val singularity.

In this paper, we focus on the most general class of compound Du Val singularities, namely cA_n singularities. Since a surface A_n singularity is given by $x^2 + y^2 + z^{n+1}$, we have the following almost immediate corollary:

Corollary 2.6. *Let n be a positive integer. A singularity is of type cA_n if and only if it is isomorphic to the complex analytic space germ $(X, \mathbf{0})$ where $X \subseteq \mathbb{C}^4$ is given by the zero set of $x^2 + y^2 + g_{\geq n+1}(z,t)$ with variables x, y, z, t where $g \in \mathbb{C}\{z,t\}$ is a convergent power series of multiplicity $n+1$.*

The simplest example of a cA_1 singularity is the *ordinary double point*, given by $x^2 + y^2 + z^2 + t^2$. Note that terminal sextic double solids have only hypersurface singularities, therefore only cA_n, cD_n and cE_n singularities.

2.2. Divisorial contractions

The first step in a Sarkisov link for a Fano variety is a divisorial contraction.

Definition 2.7. Let $\varphi\colon Y \to X$ be a proper morphism with connected fibres between varieties with terminal singularities. We say φ is a **divisorial contraction** if the exceptional locus of φ is a prime divisor and $-K_Y$ is φ-ample.

Theorem 2.10 says that divisorial contractions to cA_n points are weighted blowups. First, we recall the definition of a weighted blowup in both the algebraic and the analytic categories.

Definition 2.8. We say that two birational morphisms of varieties (or bimeromorphic holomorphisms of complex analytic spaces) $\varphi\colon Y \to X$ and $\varphi'\colon Y' \to X'$ are **equivalent** if there exist isomorphisms $X \cong X'$ and $Y \cong Y'$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \varphi'} \\ X & \longrightarrow & X' \end{array}$$

commutes. We say φ and φ' are **locally equivalent** if there exist isomorphic open subsets $U \subseteq X$ and $U' \subseteq X'$ containing the centres of the morphisms φ and φ' such that the restrictions $\varphi|_{\varphi^{-1}U} : \varphi^{-1}U \to U$ and $\varphi'|_{\varphi'^{-1}U'} : \varphi'^{-1}U' \to U'$ are equivalent.

If we consider the complex analytic space corresponding to a variety or when we wish to emphasize that we are working in the category of complex analytic spaces, we sometimes say **analytically equivalent** or **locally analytically equivalent**.

Definition 2.9. Let $\boldsymbol{w} = (w_1, \dots, w_n)$ be positive integers, called the weights of the blowup. Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by $\lambda \cdot (u, x_1, \dots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n)$ and define T by the geometric quotient $(\mathbb{C}^{n+1} \setminus \mathbb{V}(x_1, \dots, x_n))/\mathbb{C}^*$ (or its analytification). Then, the morphism $\varphi\colon T \to \mathbb{C}^n$, $[u, x_1, \dots, x_n] \mapsto (x_1u^{w_1}, \dots, x_nu^{w_n})$ is called the *$\boldsymbol{w}$-blowup of $\mathbb{C}^n$ at the origin* $\mathbf{0}$. If $Z \subseteq \mathbb{C}^n$ is a subvariety (or a complex analytic subspace) containing $\mathbf{0}$ and $\tilde{Z}$ is its strict transform, then the restriction $\varphi|_{\tilde{Z}} : \tilde{Z} \to Z$ is called the *$\boldsymbol{w}$-blowup of Z at* $\mathbf{0}$. Let $\psi\colon Y \to X$ be a birational morphism of varieties (or bimeromorphic holomorphism of complex analytic spaces). Given an open subset $U \subseteq X$ containing the centre of ψ and an isomorphism $U \cong X' \subseteq \mathbb{C}^n$ taking a point $P \in X$ to $\mathbf{0}$, ψ is called the **$\boldsymbol{w}$-blowup of X at P** if the restriction $\psi|_{\psi^{-1}U} : \psi^{-1}U \to U$ is equivalent, through the given isomorphism $U \cong X'$, to the $\boldsymbol{w}$-blowup of X' at $\mathbf{0}$.

Note that a weighted blowup crucially depends on the choice of coordinates, that is, on the isomorphism $U \cong X'$, even though it is not explicit in the notation.

Kawakita [Kaw03] described divisorial contractions to cA_n points. Notational differences from [Kaw03, Theorem 1.13] are that below we have left out the description for cA_1 singularities and an exceptional case for cA_2. Also, we have written out the converse statement more explicitly (that being a Kawakita blowup implies that it is a divisorial contraction).

Theorem 2.10 ([Kaw03, Theorem 1.13])**.** *Let P be a cA_n point where $n \geq 3$ of a variety X with terminal singularities. Let $\varphi\colon Y \to X$ be a morphism of varieties such that the restriction $\varphi|_{Y \setminus E} : Y \setminus E \to X \setminus \{P\}$ is an isomorphism, where the reduced closed subvariety E is given by $\varphi^{-1}\{P\}$. If φ is a divisorial contraction, then φ is locally analytically equivalent to the $(r_1, r_2, a, 1)$-blowup of $\mathbb{V}(x_1x_2 + g(x_3, x_4)) \subseteq \mathbb{C}^4$ at $\mathbf{0}$ with variables x_1, x_2, x_3, x_4 where*

1. *a divides $r_1 + r_2$ and is coprime to both r_1 and r_2,*
2. *g has weight $r_1 + r_2$, and*
3. *the monomial $x_3^{(r_1+r_2)/a}$ appears in g with non-zero coefficient.*

Moreover, any φ which is locally analytically equivalent to a weighted blowup as above is a divisorial contraction, even for $n = 2$.

Any weighted blowup that is locally analytically equivalent to φ in Theorem 2.10 for $n \geq 2$ is called a $(r_1, r_2, a, 1)$**-Kawakita blowup**, or simply a Kawakita blowup.

2.3. Sarkisov links

One of the possible outcomes of the minimal model program is a Mori fibre space:

Definition 2.11. A **Mori fibre space** is a morphism of normal projective varieties $\varphi\colon X \to S$ with connected fibres such that

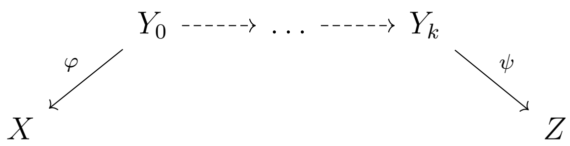
1. X is $\mathbb{Q}$-factorial and has terminal singularities,
2. the anticanonical class $-K_X$ is φ-ample,
3. X/S has relative Picard number 1, and
4. $\dim S < \dim X$.

If $\dim S > 0$, then we say φ is a *strict* Mori fibre space.

The main examples of Mori fibre spaces we see in this paper are Fano 3-folds that are projective, $\mathbb{Q}$-factorial, with terminal singularities and Picard number 1, considered as a morphism over a point.

Any birational map between two Mori fibre spaces is a composition of Sarkisov links (see [Cor95] or [HM13]). Below, we describe the two possible types of Sarkisov links starting from a Fano variety.

Definition 2.12. A **Sarkisov link** of type I (respectively II) between a Fano variety X and a strict Mori fibre space $Y_k \to Z$ (respectively Fano variety Z) is a diagram of the form



where X, Y_0, $\dots$, Y_k, Z are normal, projective and $\mathbb{Q}$-factorial, the varieties X, Y_0, $\dots$, Y_k have terminal singularities, Z has terminal singularities if it 3-dimensional, X has Picard number 1, $\varphi\colon Y_0 \to X$ is a divisorial contraction, $Y_0 \dashrightarrow \dots \dashrightarrow Y_k$ is a sequence of anti-flips, flops and flips, and $\psi\colon Y_k \to Z$ is a strict Mori fibre space (respectively divisorial contraction). If we do not require the varieties $X, Y_0, \dots Y_k$ (respectively $X, Y_0, \dots Y_k, Z$) to be terminal and we do not require $-K_{Y_0}$ to be φ-ample and we do not require $-K_{Y_k}$ to be ψ-ample but all the other properties hold, then the diagram above is called a **2-ray link** ([BZ10, Definition 2.1]).

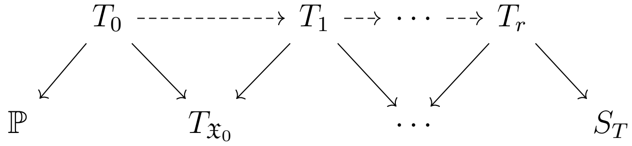
Definition 2.13. A Fano 3-fold X that is a Mori fibre space is **birationally rigid** if for any Mori fibre space $Y \to S$ such that X and Y are birational, we have that S is a point and X and Y are isomorphic.

In Section 5, we show that a general sextic double solid X with a cA_n singularity with $n \geq 4$ which is a Mori fibre space is not birationally rigid. We show this by explicitly constructing a Sarkisov link between X and another Mori fibre space. We find the Sarkisov link by restricting from a toric 2-ray link, as described in Construction 2.14.

See [Cox95] for the definition of Cox rings for toric varieties (where it is called the *homogeneous coordinate ring*), and [HK00, Definition 2.6] for the definition of Cox rings for Mori dream spaces. Note that isomorphic varieties can have different Cox rings. By [Cox95, Theorem 3.7], closed subschemes of a toric variety T with only cyclic quotient singularities are given by homogeneous ideals in the Cox ring $\operatorname{Cox} T$, which is a polynomial ring.

Construction 2.14. Let X be a Fano variety embedded in a weighted projective space $\mathbb{P}$, where X is a Mori fibre space, and let $Y_0 \to X$ be a divisorial contraction from a projective $\mathbb{Q}$-factorial variety Y. By [AK16, Lemma 2.9], the divisorial contraction $Y_0 \to X$ is part of a Sarkisov link only if Y_0 is a Mori dream space.

By [HK00, Proposition 2.11], we can embed a Mori dream space Y_0 into a projective toric variety T_0 with cyclic quotient singularities such that the Mori chambers of Y_0 are unions of finitely many Mori chambers of T_0. Moreover, we can embed Y_0 in such a way that Y_0 is given by a homogeneous ideal I_Y in $\operatorname{Cox} T_0$, and the toric 2-ray link



restricts to a 2-ray link

$$\begin{array}{ccccccccc} & Y_0 & \dashrightarrow & Y_1 & \dashrightarrow & \cdots & \dashrightarrow & Y_r & \\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & & & \searrow \\ X & & \mathfrak{X}_0 & & \cdots & & & & S, \end{array}$$

where each $Y_i \subseteq T_i$ is given by the same ideal $I_Y \subseteq \operatorname{Cox} T_0 = \ldots = \operatorname{Cox} T_r$, and $\mathfrak{X}_i \subseteq T_{\mathfrak{X}_i}$ is given by the ideal $I_Y \cap \mathbb{C}[\nu_0, \ldots, \nu_s]$, where $T_{\mathfrak{X}_i}$ is given by $\operatorname{Proj} \mathbb{C}[\nu_0, \ldots, \nu_s]$ for some polynomials $\nu_j \in \operatorname{Cox} T_0$ that depend on i (see [AZ16, Remark 4]). In this case, $\operatorname{Cox}(T_0)/I_Y$ is a Cox ring for Y_0 and it is said that I_Y **2-ray follows** T_0 ([AZ16, Definition 3.5]).

Note that some of the small birational maps $T_i \dashrightarrow T_{i+1}$ may restrict to isomorphisms $Y_i \to Y_{i+1}$. If all the varieties Y_i are terminal and the anticanonical divisor $-K_{Y_0}$ of Y_0 is inside the interior $\operatorname{int}(\operatorname{Mov} Y_0)$ of the movable cone, then the 2-ray link for Y_0 is a Sarkisov link (see [AK16, Lemma 2.9]), otherwise it is called a **bad link**.

In Section 5, where X is a sextic double solid and the centre of $Y_0 \to X$ is a cA_n point, we use a projective version of Corollary 4.9 to construct the divisorial contraction $Y_0 \to X$, which is the restriction of a toric weighted blowup $\bar{T}_0 \to \mathbb{P}$. This gives us an embedding $Y_0 \to \mathbb{V}(I_{\bar{Y}}) \subseteq \bar{T}_0$ where $I_{\bar{Y}}$ might not 2-ray follow $\bar{T}_0$. We use unprojection to modify $\bar{T}_0$ to find an embedding $Y_0 \to \mathbb{V}(I_Y) \subseteq T_0$ such that I_Y 2-ray follows T_0. See [Rei00, Section 2.1] for a simple example of unprojection, and Sections 5.2, 5.5, 5.6 and 5.8 for applications of unprojection.

To explain the notation we use for 2-ray links, we do an example in detail, namely the 2-ray link for the ambient space of the sextic double solid with a cA_4 singularity in Section 5.2.

Example 2.15 (2-ray link for $\mathbb{P}(1,1,1,1,3,5)$)**.** Denote the variables on $\mathbb{P}(1,1,1,1,3,5)$ by x, y, z, t, α, ξ. We perform the weighted blowup $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ with weights $(1,1,2,3,6)$ for variables y, z, t, α, ξ, where the centre is the point $P_x = [1,0,0,0,0,0]$.

We define T_0 as a geometric quotient. By a slight abuse of notation we denote the variables on $\mathbb{C}^7$ by $u, x, y, z, \alpha, \xi, t$, repeating the symbols for $\mathbb{P}(1,1,1,1,3,5)$. Define a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^7$ for all $(\lambda, \mu) \in (\mathbb{C}^*)^2$ by

$$(\lambda, \mu) \cdot (u, x, y, z, \alpha, \xi, t) = (\mu^{-1} u, \lambda x, \lambda\mu y, \lambda\mu z, \lambda^3\mu^3\alpha, \lambda^5\mu^6\xi, \lambda\mu^2 t).$$

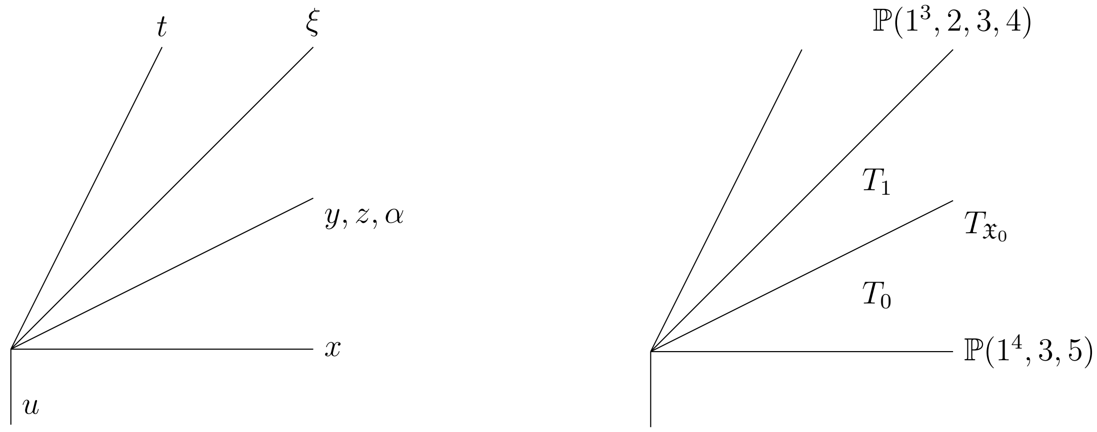
Define the irrelevant ideal $I_0 = (u, x) \cap (y, z, \alpha, \xi, t)$, and define T_0 by the geometric quotient $\mathbb{C}^7 \setminus \mathbb{V}(I_0)/(\mathbb{C}^*)^2$. We use the notation

$$T_0 \colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array}\right).$$

to describe this construction of T_0. Note that we order the variables $u, x, \ldots, t$ such that the corresponding rays $\binom{0}{-1}, \binom{1}{0}, \ldots, \binom{1}{2}$ are ordered anti-clockwise around the origin. The vertical bar indicates that the irrelevant ideal is $(u,x) \cap (y, z, \alpha, \xi, t)$. The Cox ring of T_0 is given by $\operatorname{Cox} T_0 = \mathbb{C}[u, x, y, z, \alpha, \xi, t]$. The weighted blowup $T_0 \to \mathbb{P}(1,1,1,1,3,5)$ is given by

$$[u, x, y, z, \alpha, \xi, t] \mapsto [x, uy, uz, u^2 t, u^3 \alpha, u^6 \xi]. \tag{2.1}$$

We describe the cones of the toric variety T_0. By [HK00], T_0 is a Mori dream space. The Picard group of T_0 is generated by $\mathbb{V}(u)$, the reduced exceptional divisor, and $\mathbb{V}(x)$,



(a) Rays in $N^1(T_0)$ (b) Ample models of T_0

Figure 1. Cones of T_0

the strict transform of a plane not passing through P_x, which have bidegree $\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, respectively. The variety T_0 is $\mathbb{Q}$-factorial, and any two divisors with the same bidegree are linearly equivalent. As in [BZ10, Section 4.1.3], the effective cone $\operatorname{Eff}(T_0)$ is given by $\langle \mathbb{V}(u), \mathbb{V}(x)\rangle$, a cone in the group $N^1(T_0)$ of divisors of T_0 up to numerical equivalence with coefficients in $\mathbb{R}$. As in [AZ16, Section 3.2], the movable cone $\mathrm{Mov}(\mathrm{T}_0)$ is $\langle \mathbb{V}(x), \mathbb{V}(\xi)\rangle$, and it is divided into the nef cone $\operatorname{Nef}(T_0) = \langle \mathbb{V}(x), \mathbb{V}(y)\rangle$ of T_0 and $\langle \mathbb{V}(y), \mathbb{V}(\xi)\rangle$, which is the pull-back of the nef cone of the small $\mathbb{Q}$-factorial modification T_1 of T_0. The cones $\langle \mathbb{V}(x), \mathbb{V}(y)\rangle$ and $\langle \mathbb{V}(y), \mathbb{V}(\xi)\rangle$ are called *Mori chambers*. The variety T_1 is defined by

$$T_1\colon \begin{pmatrix} \overset{u}{0} & \overset{x}{1} & \overset{y}{1} & \overset{z}{1} & \overset{\alpha}{3} & \big| & \overset{\xi}{5} & \overset{t}{1} \\ -1 & 0 & 1 & 1 & 3 & \big| & 6 & 2 \end{pmatrix}.$$

Here, T_1 is the geometric quotient $(\mathbb{C}^7 \setminus I_1)/(\mathbb{C}^*)^2$, where the irrelevant ideal I_1 is given by $(u, x, y, z, \alpha) \cap (\xi, t)$, which is indicated by the position of the vertical bar in the action-matrix. The Cox ring of T_1 is equal to the Cox ring of T_0, namely $\operatorname{Cox} T_1 = \mathbb{C}[u, x, y, z, \alpha, \xi, t]$.

The weighted blowup morphism $T_0 \to \mathbb{P}(1, 1, 1, 1, 3, 5)$ can be read off from the action-matrix of T_0. Consider the ray given by $\mathbb{V}(x)$ in $N^1(T_0)$. The union of the linear systems $|\left(\begin{smallmatrix}n\\0\end{smallmatrix}\right)|$ where $n \geq 0$ has a $\mathbb{C}$-algebra basis $x, uy, uz, u^2t, u^3\alpha, u^6\xi$. So, the ample model (see [BCHM10, Definition 3.6.5]) of the divisor class $\mathbb{V}(x)$ is the morphism

$$T_0 \to \operatorname{Proj} \bigoplus_{n\geq 0} H^0(T_0, \mathcal{O}_{T_0}(n\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))) = \operatorname{Proj} \mathbb{C}[x, uy, uz, u^2t, u^3\alpha, u^6\xi] = \mathbb{P}(1, 1, 1, 1, 3, 5)$$

given by

$$[u, x, y, z, \alpha, \xi, t] \mapsto [x, uy, uz, u^2t, u^3\alpha, u^6\xi],$$

which is precisely the weighted blowup $T_0 \to \mathbb{P}(1, 1, 1, 1, 3, 5)$ given in Equation (2.1).

As in [BZ10, Section 2.1], there are two projective morphisms of relative Picard number 1 from T_0 up to isomorphisms, corresponding to the ample models of divisors in the two edges of the nef cone of T_0. The ample model of any divisor in the interior of the nef cone of T_0 gives an embedding of T_0 into a weighted projective space. The ample model of $\mathbb{V}(y) \in N^1(T_0)$ is given by

$$\begin{aligned} T_0 &\to \operatorname{Proj} \mathbb{C}[y, z, \alpha, u\xi, ut, x\xi, xt] \subseteq \mathbb{P}(1, 1, 3, 5, 1, 6, 2) \\ [u, x, y, z, \alpha, \xi, t] &\mapsto [y, z, \alpha, u\xi, ut, x\xi, xt]. \end{aligned}$$

Denoting $T_{\mathfrak{X}_0} = \operatorname{Proj} \mathbb{C}[y, z, \alpha, u\xi, ut, x\xi, xt]$, we see that the morphism $T_0 \to T_{\mathfrak{X}_0}$ contracts $\mathbb{V}(\xi, t)$ to the surface $\mathbb{P}(1, 1, 3) \subseteq T_{\mathfrak{X}_0}$ and is an isomorphism elsewhere. The ample model of $\mathbb{V}(y) \in N^1(T_1)$ is given similarly by

$$T_1 \to \operatorname{Proj} \mathbb{C}[y, z, \alpha, u\xi, ut, x\xi, xt] = T_{\mathfrak{X}_0},$$

contracting $\mathbb{V}(u, x)$ to $\mathbb{P}(1, 1, 3)$. This induces a birational map $T_0 \dashrightarrow T_1$, a small $\mathbb{Q}$-factorial modification, given by

$$[u, x, y, z, \alpha, \xi, t] \mapsto [u, x, y, z, \alpha, \xi, t].$$

The diagram $T_0 \to T_{\mathfrak{X}_0} \leftarrow T_1$ is a flop.

Note that multiplying the action-matrix of T_0 or T_1 with a matrix in $\mathrm{GL}(2, \mathbb{Q})$ is equivalent to choosing a different basis for the group $(\mathbb{C}^*)^2$, so the geometric quotients T_0 and T_1 stay the same (see [Ahm17, Lemma 2.4]). If we multiply with a matrix with negative determinant, then we change the order of the rays in $N^1(T_0)$ from anti-clockwise to clockwise.

Similarly, there are only two projective morphisms of relative Picard number 1 from T_1: the contraction $T_1 \to T_{\mathfrak{X}_0}$ and the ample model of $\mathbb{V}(\xi)$. We multiply the action-matrix of T_1 by the matrix $\left(\begin{smallmatrix} 6 & -5 \\ 2 & -1 \end{smallmatrix}\right)$ with determinant 4 to find

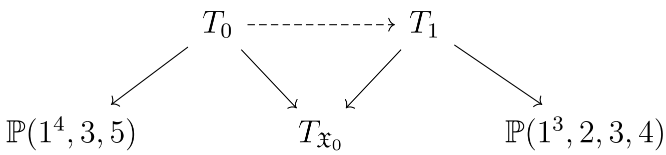
$$T_1\colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 5 & 6 & 1 & 1 & 3 & 0 & -4 \\ 1 & 2 & 1 & 1 & 3 & 4 & 0 \end{array}\right).$$

The ample model of $\mathbb{V}(\xi)$ is given by

$$\begin{gathered} T_1 \to \mathbb{P}(1, 1, 1, 2, 3, 4) \\ [u, x, y, z, \alpha, \xi, t] \mapsto \left[t^{\frac{5}{4}}u, t^{\frac{1}{4}}y, t^{\frac{1}{4}}z, t^{\frac{3}{2}}x, t^{\frac{3}{4}}\alpha, \xi\right]. \end{gathered}$$

Note that this is a morphism of varieties despite having fractional powers (see [BB13]).

The 2-ray link that we have found for $\mathbb{P}(1, 1, 1, 1, 3, 5)$ is summarized by the diagram below.



For more examples on toric 2-ray links, see [BZ10, Section 4].

3. Constructing sextic double solids with a cA_n singularity

In this section, we give a bound $n \leq 8$ for an isolated cA_n singularity on a sextic double solid, and we explicitly describe all sextic double solids that contain an isolated cA_n singularity where $n \leq 8$. The main tool we use for this is the splitting lemma from singularity theory, first introduced in [Tho72], which is used for separating the quadratic terms and the higher order terms of a power series.

3.1. Splitting lemma from singularity theory

Before we go into details, let us recall the statement of the splitting lemma. Here the statement is taken from [GLS07, Theorem 2.47], with a slight modification in notation. Specifically, we write $v(x+p)$ instead of $x+g$, where v is a unit in the power series ring and p does not depend on x, as we use this form in Section 5 for constructing birational models.

Theorem 3.1 (Splitting lemma)**.** *Let m be a positive integer and let $\boldsymbol{y}$ denote variables $(y_1, \ldots, y_m)$. Let $f \in \mathbb{C}\{x, \boldsymbol{y}\}$ be a convergent power series of multiplicity two, with degree two part of the form $x^2 + (\text{terms in } \boldsymbol{y})$. Then, there exist unique $v \in \mathbb{C}[[x, \boldsymbol{y}]]$ and $p, h \in \mathbb{C}[[\boldsymbol{y}]]$, where v is a unit and the multiplicity of p is at least two, such that*

$$f = (v(x+p))^2 + h.$$

Moreover, the power series h, p and v are absolutely convergent around the origin, and the multiplicity of h is at least two. It follows immediately that f is right equivalent to $x^2 + h$.

Proof. It is proved in [GLS07, Theorem 2.47] that there exist unique $g \in \mathbb{C}[[x, \boldsymbol{y}]]$ and $h \in \mathbb{C}[[\boldsymbol{y}]]$, where the multiplicity of g is at least two, such that $f = (x+g)^2 + h$. Moreover, it is proved that the power series g and h are absolutely convergent around the origin, and the multiplicity of h is at least two.

By the Weierstrass preparation theorem (see [GLS07, Theorem 1.6]), there exists a unique unit $v \in \mathbb{C}\{x, \boldsymbol{y}\}$ and a unique $p \in \mathbb{C}\{\boldsymbol{y}\}$ such that $x + g = v(x+p)$. □

Below we give explicit recurrent formulas for g, h, p, v of the splitting lemma in terms of the coefficients of f, which is implemented in Listing 1 in Appendix A.

Proposition 3.2 (Explicit splitting lemma)**.** *Below, we use the same notation as in the splitting lemma Theorem 3.1 and its proof. Denote*

$$f = \sum_{i,d \geq 0} x^i f_{i,d}, \qquad g = \sum_{i,d \geq 0} x^i g_{i,d}, \qquad h = \sum_{d \geq 0} h_d, \qquad p = \sum_{d \geq 0} p_d, \qquad v = \sum_{i,d \geq 0} x^i v_{i,d}$$

where $f_{i,d}, g_{i,d}, h_d, p_d, v_{i,d} \in \mathbb{C}[\boldsymbol{y}]$ are homogeneous of degree d. Then,

$$g_{1,0} = 0,$$

$$g_{i,d} = \frac{1}{2}\left(f_{i+1,d} - \sum_{k=0}^{d} \sum_{j=\max(0,2-k)}^{\min(i+1,i+d-k-1)} g_{j,k} g_{i+1-j,d-k}\right), \quad \text{if } (i,d) \neq (1,0), \tag{3.1}$$

$$h_d = f_{0,d} - \sum_{j=2}^{d-2} g_{0,j} g_{0,d-j}, \tag{3.2}$$

$$p_d = g_{0,d} - \sum_{j=2}^{d-1} v_{0,d-j} p_j, \tag{3.3}$$

$$v_{0,0} = 1,$$

$$v_{i,d} = g_{i+1,d} - \sum_{j=2}^{d} (v_{i+1,d-j} p_j), \quad \text{if } (i,d) \neq (0,0). \tag{3.4}$$

Proof. Taking the degree d part of the coefficient of x^{i+1} in $f = (x+g)^2 + h$ where $i \geq 0$, we find Equation (3.1). Taking all degree d terms of $f = (x+g)^2 + h$ that are not divisible by x, we find Equation (3.2). Taking the degree d part of the coefficient of x^{i+1} in $x + g = v(x+p)$ where $i \geq 0$, we find Equation (3.4), and taking all degree d terms not divisible by x, we find Equation (3.3). □

Example 3.3. Using the notation of Proposition 3.2, the first few homogeneous parts of h are given in terms of coefficients of f by

$$\begin{aligned}
h_2 &= f_{0,2}\\
h_3 &= f_{0,3}\\
h_4 &= f_{0,4} - \frac{f_{1,2}^2}{4}\\
h_5 &= f_{0,5} - \frac{f_{1,2}^2 f_{2,1}}{4} - \frac{f_{1,2}f_{1,3}}{2}\\
h_6 &= f_{0,6} - \frac{f_{1,2}^3 f_{3,0}}{8} + \frac{f_{1,2}^2 f_{2,2}}{4} - \frac{f_{1,2}^2 f_{2,1}^2}{4} + \frac{f_{1,2}f_{1,3}f_{2,1}}{2} - \frac{f_{1,2}f_{1,4}}{2} - \frac{f_{1,3}^2}{4}.
\end{aligned}$$

3.2. Isolated cA_n singularity

Now, we apply the explicit splitting lemma (Proposition 3.2) to the case we are most interested in, that is, sextic double solids. First, we describe the equation of a sextic double solid $X \subseteq \mathbb{P}(1,1,1,1,3)$ that has a singular point P. The following argument shows that without loss of generality, we can move the singular point to $P_x = [1,0,0,0,0]$ using an automorphism of $\mathbb{P}(1,1,1,1,3)$. Denote $P = [P_0, P_1, P_2, P_3, P_4]$. Since X does not contain the point $[0,0,0,0,1]$, there exists $0 \leq i \leq 3$ such that $P_i \neq 0$. After interchanging the variables if necessary, we find $P_0 \neq 0$. Now, the automorphism of $\mathbb{P}(1,1,1,1,3)$ given by

$$(x,y,z,t,w) \mapsto \left(x, y - \frac{P_1}{P_0}x, z - \frac{P_2}{P_0}x, t - \frac{P_3}{P_0}x, w - \frac{P_4}{P_0^3}x^3\right)$$

takes P to P_x.

Below, the subindices denote degree and $\mathbb{V}(f)$ denotes the zero locus of a polynomial f.

Notation 3.4. Define the variety X by

$$X \colon \mathbb{V}(f) \subseteq \mathbb{P}(1,1,1,1,3),$$

with variables x, y, z, t, w where

$$\begin{aligned}
f = &-w^2 + x^4t^2 + x^4\xi_2\\
&+ x^3(4t^3a_0 + 4t^2a_1 + 2ta_2 + a_3)\\
&+ x^2(2t^4b_0 + 2t^3b_1 + 2t^2b_2 + 2tb_3 + b_4)\\
&+ x(2t^5c_0 + 2t^4c_1 + 2t^3c_2 + 2t^2c_3 + 2tc_4 + c_5)\\
&+ t^6d_0 + 2t^5d_1 + t^4d_2 + 2t^3d_3 + t^2d_4 + 2td_5 + d_6,
\end{aligned}$$

where the polynomials $\xi_j, a_j, b_j, c_j, d_j \in \mathbb{C}[y,z]$ are homogeneous of degree j.

Now, define the following technical conditions:

2. $\xi_2 = 0$.
3. $a_3 = 0$.
4. $b_4 = a_2^2$.
5. $c_5 = 2a_2b_3 - 4a_1a_2^2$.
6. $d_6 = 2a_2c_4 + b_3^2 - 8a_1a_2b_3 - 2a_2^2b_2 + 4a_0a_2^3 + 16a_1^2a_2^2$.

7. There exist unique $q, r, s, e \in \mathbb{C}[y, z]$ where r and s do not have a common prime divisor, and q and e do not have a common prime divisor, such that

$$\begin{aligned} a_2 &= qr \\ b_3 &= qs + 4a_1qr \\ c_4 &= 2a_1qs - 6a_0q^2r^2 + 8a_1^2qr + er \\ d_5 &= 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr, \end{aligned}$$

and q, r, s, e are respectively homogeneous of degrees

(7.1) $0, 2, 3, 2$ and $q = 1$,

(7.2) $1, 1, 2, 3$ and the leading coefficient of q under the ordering $y < z$ is one,

(7.3) $2, 0, 1, 4$ and $r = 1$, or

(7.4) $3, *, 0, 5$ and $r = 0$ and $s = 1$ (since $r = 0$, '$*$' denotes that r is homogeneous of any non-negative degree).

8. Condition (7.1) holds and there exists a unique $A_0 \in \mathbb{C}$ and a unique polynomial $B_1 \in \mathbb{C}[y, z]$ homogeneous of degree 1 such that

$$\begin{aligned} e_2 &= 4A_0r_2 + b_2 - 6a_1^2 \\ c_3 &= 6a_0s_3 - 4A_0s_3 + 4a_0a_1r_2 - 8A_0a_1r_2 + B_1r_2 + 2a_1b_2 - 4a_1^3 \\ d_4 &= -2s_3B_1 + 16r_2^2A_0^2 - 8b_2r_2A_0 + 16a_1^2r_2A_0 + 4b_1s_3 \\ &\quad - 8a_0a_1s_3 - 2b_0r_2^2 + 2c_2r_2 + b_2^2 - 4a_1^2b_2 + 4a_1^4. \end{aligned}$$

Notation 3.4 describes 11 families of sextic double solids, namely when conditions 1 to n are satisfied for some $n \leq 8$. For $n = 7$, there are 4 families. Below, *general* means 'in a Zariski open dense subset of the family', and *very general* means 'outside a countable union of proper Zariski closed subsets of the family'. Using the above notation, we state the main theorem of this section, describing sextic double solids with an isolated cA_n singularity.

Theorem A.

(a) If a sextic double solid has an isolated cA_n singularity, then $n \leq 8$.

Furthermore, for every positive integer $n \leq 8$:

(b) Every sextic double solid with an isolated cA_n singularity P is isomorphic to some X satisfying conditions 2 to n in Notation 3.4, with the isomorphism sending $P \mapsto P_x = [1, 0, 0, 0, 0]$.

(c) Every X that satisfies conditions 2 to n in Notation 3.4 and has otherwise general coefficients is smooth outside a cA_n singularity at P_x.

(d) A very general sextic double solid with an isolated cA_n singularity is factorial and has Picard number 1, except for the family (7.4) in Notation 3.4. No member of the family (7.4) is $\mathbb{Q}$-factorial.

Remark 3.5. Note that if $n = 1$, then the set of conditions 2 to n is empty, so every variety with a cA_1 singularity is isomorphic to some X in Notation 3.4, and every X in Notation 3.4 that has general coefficients has a cA_1 singularity at P_x and is smooth elsewhere. Note that zero is homogeneous of every non-negative degree, so for example in condition (7.1) of Notation 3.4, the term e can be zero. Also, note that in conditions (7.1) and (7.2), the terms r and s must both be non-zero, otherwise either r or s is a common divisor of both r and s.

Before we prove Theorem A, we state a few lemmas needed for the proof.

Lemma 3.6. *If X in Notation 3.4 satisfies conditions 2 to 6 and P_x is an isolated singularity of X, then either $a_2 \neq 0$ or $b_3 \neq 0$.*

Proof. If conditions 2 to 6 hold and $a_2 = b_3 = 0$, then $a_3 = b_4 = c_5 = d_6 = 0$. Let C be the curve defined by the ideal $(t, w, 2xc_4 + 2d_5)$. Taking partial derivatives, we see that every point of C is a singular point of X. Since C passes through P_x, X does not have an isolated singularity at P_x, a contradiction. □

Lemma 3.7. *Let $r, s \in \mathbb{C}[y, z]$ have no common prime divisors, and let $q \in \mathbb{C}[y, z]$ be non-zero. Let $h_n \in \mathbb{C}[y, z]$ be of the form $h_n = q^\alpha(r^\beta C_r - s^\gamma C_s)$ where $C_r, C_s \in \mathbb{C}[y, z]$ and α, β, γ are non-negative integers. Then*

$$h_n = 0 \iff \textit{there exists } C \in \mathbb{C}[y, z] \textit{ such that } C_r = s^\gamma C \textit{ and } C_s = r^\beta C.$$

Lemma 3.8. *If X in Notation 3.4 satisfies conditions 2 to 7 and P_x is an isolated singularity of X, then q and e do not have a common prime divisor in $\mathbb{C}[y, z]$.*

Proof. If q and e have a common prime divisor D, then D divides a_2, b_3, c_4, d_5, and D^2 divides a_3, b_4, c_5, d_6. Let C be the curve defined by the ideal (D, t, w). Taking partial derivatives, we see that X is singular at every point of C, so P_x is not isolated, a contradiction. □

Lemma 3.9. *Denote the parameter space of all possible f in Notation 3.4 satisfying conditions 2 to n by $\mathcal{P}_n$. Denote the parameter space of all $f \in \mathcal{P}_n$ where $\mathbb{V}(f)$ has a singular point with x, t and w-coordinate zero by $\mathcal{A}_n$. Then $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.*

Proof. Let $P = [0, \beta, \gamma, 0, 0]$ be a singular point of $\mathbb{V}(f)$ where $f \in \mathcal{A}_n$ and $\beta, \gamma \in \mathbb{C}$. We find

$$f(P) = d_6(P),\ \frac{\partial f}{\partial x}(P) = c_5(P),\ \frac{\partial f}{\partial y}(P) = \frac{\partial d_6}{\partial y}(P),\ \frac{\partial f}{\partial z}(P) = \frac{\partial d_6}{\partial z}(P),\ \frac{\partial f}{\partial t}(P) = 2d_5(P).$$

Define the polynomial $l = \gamma y - \beta z$. Since P is a singular point, we have

$$l \text{ divides } c_5,\ d_5 \text{ and } d_6, \text{ and } l^2 \text{ divides } d_6. \tag{3.5}$$

We use the divisibility constraint (3.5) repeatedly below.

1. If $n \leq 5$, then there are no restrictions on d_6 or d_5 in $\mathcal{P}_n$. For $\mathcal{A}_n$, we have the restrictions that $l^2 \mid d_6$ and $l \mid d_5$. In particular, d_6 has a square factor which is also a factor of d_5. We find that $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.
2. If $n = 6$, then there are no restrictions on d_5, a_2, b_3 or c_4 in $\mathcal{P}_n$. We have $c_5 = a_2(2b_3 - a_1a_2)$ and $d_6 = a_2 \cdot (\dots) + b_3^2$. Below we consider $f \in \mathcal{A}_n$.

   If l divides a_2, then using the divisibility constraint (3.5), we find that l divides b_3. So, there are at least two less degrees of freedom in choosing a_2, b_3 and d_5.

   If l does not divide a_2, then l divides $2b_3 - a_1a_2$, and from $l \mid d_6$ we find that l also divides $8c_4 - 4a_2b_2 + 8a_0a_2^2 + a_1^2a_2$. So, after fixing a_0, a_1, a_2 and b_2, there are at least two less degrees of freedom in choosing b_3, c_4 and d_5.

   In either case, we see that $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.

3. If $n = 7$, then

$$c_5 = 4q^2r(2s + a_1r)$$
$$d_5 = -es + q(2b_2s - a_1^2s - 4b_1qr^2 + c_3r)$$
$$d_6 = 4q(er^2 + q(s^2 + a_1rs - 8a_0qr^3 - b_2r^2 + a_1^2r^2)).$$

Let us consider $f \in \mathcal{A}_n$. If $l \mid q$, then since q and e are coprime, we have $l \mid r$ and $l \mid s$, a contradiction. If $l \mid r$, then since $l \mid d_6$, we find $l \mid s$, a contradiction. Therefore, l divides neither q nor r.

So, l divides $2s + a_1r$. Using $l^2 \mid d_6$, we see that l^2 divides $-32a_0q^2r - 4b_2q + 3a_1^2q + 4e$. After fixing a_0, a_1, b_2, q and r, we see that there are at least two less degrees of freedom in choosing s and e. So, we have $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.

4. If $n = 8$, then

$$c_5 = 2r_2(s_3 + 2a_1r_2)$$
$$d_5 = r_2(r_2B_1 - 8s_3A_0 - 8a_1r_2A_0 + 6a_0s_3 - b_1r_2 + 4a_0a_1r_2 + 2a_1b_2 - 4a_1^3) + s_3(b_2 - 2a_1^2)$$
$$d_6 = r_2(8r_2^2A_0 + 4a_1s_3 - 8a_0r_2^2 + 4a_1^2r_2) + s_3^2$$

We consider $f \in \mathcal{A}_n$. If $l \mid r_2$, then $l \mid s_3$, a contradiction. So, l divides $s_3 + 2a_1r_2$. Since l divides d_6, we have $l \mid r_2^3(A_0 - a_0)$. So, $A_0 = a_0$. Since l divides d_5, we see that $l \mid r_2^2(B_1 - b_1)$. We find that the coefficients of f have at least three less degrees of freedom, namely $A_0 = a_0$, and the polynomials B_1, b_1 and $s_3 + 2a_1r_2$ have a common prime divisor. So, we have $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$. □

Lemma 3.10. *Denote the parameter space of all possible f in Notation 3.4 satisfying conditions 2 to n by $\mathcal{P}_n$. Denote the parameter space of all $f \in \mathcal{P}_n$ such that $\mathbb{V}(f)$ has a singular point at $P_t = [0,0,0,1,0]$ by $\mathcal{B}_n$. Then, $\dim \mathcal{B}_n = \dim \mathcal{P}_n - 4$.*

Proof. We find

$$f(P_t) = d_0, \quad \frac{\partial f}{\partial x}(P_t) = 2c_0, \quad \frac{\partial f}{\partial y}(P_t) = 2\frac{\partial d_1}{\partial y}, \quad \frac{\partial f}{\partial z}(P_t) = 2\frac{\partial d_1}{\partial z}, \quad \frac{\partial f}{\partial t}(P_t) = 6d_0.$$

For $\mathcal{B}_n$, we have $d_0 = c_0 = d_1 = 0$. So, there are 4 less degrees of freedom in choosing coefficients for $f \in \mathcal{B}_n$, therefore $\dim \mathcal{B}_n = \dim \mathcal{P}_n - 4$. □

To show $\mathbb{Q}$-factoriality and that the Picard number is 1, we use the following proposition by Namikawa:

Proposition 3.11 ([Nam97, Proposition 2])**.** *Let X be a Fano 3-fold with Gorenstein terminal singularities and D its general anti-canonical divisor. Then, the natural homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is an injection.*

Corollary 3.12. *In the notation of Proposition 3.11, let X be smooth along D. Then $\mathrm{Cl}(X) \to \mathrm{Pic}(D)$ is an injection.*

Proof. Let U be the smooth locus of X. Since $X \setminus U$ is of codimension at least 2 in X, we have $\mathrm{Cl}(X) \cong \mathrm{Cl}(U)$. It follows from the proof of Proposition 3.11 that $\mathrm{Pic}(U)$ injects into $\mathrm{Pic}(D)$. □

Proof of Theorem A. Note that every X above with a cA_n singularity at P_x is irreducible.

(b) We prove that every sextic double solid $Y \subseteq \mathbb{P}(1,1,1,1,3)$ with a cA_n singularity is isomorphic to some X above, with the isomorphism sending the cA_n point to $P_x = [1,0,0,0,0]$. After applying a suitable automorphism of $\mathbb{P}(1,1,1,1,3)$, the cA_n point is at P_x. From Corollary 2.6, we know that a cA_n singularity is isomorphic to a complex analytic space germ $(\mathbb{V}(\alpha^2+\beta^2+H), \mathbf{0})$ with variables $\alpha, \beta, \gamma, \delta$ where $H \in \mathbb{C}\{\gamma, \delta\}$ has multiplicity $n+1$. Consider the affine patch Y_x, given by inverting x. Using Proposition 2.4, we find that after a suitable invertible linear coordinate change, Y_x is given by $\mathbb{V}(-w^2 + t^2 + g_{\geq 2}(y,z,t))$ in $\mathbb{C}^4$ with variables y,z,t,w, where $g \in \mathbb{C}[y,z,t]$ has multiplicity at least two and the degree 2 part g_2 is contained in $\mathbb{C}[y,z]$. So, Y has the required form $\mathbb{V}(f)$, proving the case $n=1$.

Applying the splitting lemma on the affine patch where x is non-zero, we find that X is locally analytically of the form $\mathbb{V}(-w^2+t^2+h(y,z))$ in $\mathbb{C}^4$ with variables y,z,t,w for some $h \in \mathbb{C}\{y,z\}$ of multiplicity at least two. Any cA_n singularity is isomorphic to a complex analytic space germ $(\mathbb{V}(\alpha^2+\beta^2+H), \mathbf{0})$ with variables $\alpha,\beta,\gamma,\delta$ where $H \in \mathbb{C}\{\gamma,\delta\}$ has multiplicity $n+1$. By Proposition 2.5, X has a cA_n singularity at P_x if and only if $h_2 = h_3 = \ldots = h_n = 0$ and $h_{n+1} \neq 0$ as polynomials in $\mathbb{C}[y,z]$. We have $h_2 = \xi_2$, so $h_2 = 0$ is equivalent to condition 2, namely $\xi_2 = 0$. We can show that $h_3 = a_3$, so $h_3 = 0$ is equivalent to condition 3, namely $a_3 = 0$. Similarly, using the explicit splitting lemma (Proposition 3.2), it is straightforward to compute that $h_2 = \ldots = h_n = 0$ is equivalent to satisfying conditions 2 to n when $n \leq 6$, even if P_x is not isolated. This proves part (b) for $n \leq 6$.

In the rest of the proof of part (b), using that fact that the singularity P_x of X is isolated, we show that $h_2 = \ldots = h_n = 0$ is equivalent to satisfying conditions 2 to n for any $n \leq 8$.

By Lemma 3.6, either $a_2 \neq 0$ or $b_3 \neq 0$. We write $a_2 = qr$ and $b_3 = qs + 4a_1qr$, where $q \in \mathbb{C}[y,z]$ is a (homogeneous) greatest common divisor of a_2 and b_3, and r and $s \in \mathbb{C}[y,z]$ have no common prime divisor. In the rest of the proof of parts (a) and (b), we repeatedly use Lemma 3.7.

If conditions 2 to 6 hold, then using the explicit splitting lemma (Proposition 3.2), we compute in Listing 3 of Appendix A that

$$h_7 = q(r(-12a_0q^2rs + 4b_2qs - 2b_1q^2r^2 + 2c_3qr - 2d_5) - s(2c_4 - 4a_1qs)).$$

Using Lemma 3.7, we find that $h_7 = 0$ is equivalent to the existence of a homogeneous $e \in \mathbb{C}[y,z]$ such that

$$\begin{aligned} c_4 &= 2a_1qs - 6a_0q^2r^2 + 8a_1^2qr + er \\ d_5 &= 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr. \end{aligned}$$

We defined q as a homogeneous greatest common divisor of a_2 and b_3. Every non-zero complex multiple of q is another greatest common divisor. Therefore, there is redundancy in choosing q,r,s,e. We eliminate this redundancy by choosing $q=1$ in condition (7.1), leading coefficient of q equal to one in condition (7.2), $r=1$ in condition (7.3), and $s=1$ in condition (7.4). By Lemma 3.8, q and e have no common prime divisor in $\mathbb{C}[y,z]$.

This proves part (b) for $n=7$, namely that $h_2 = \ldots = h_7 = 0$ is equivalent to conditions 2 to 7 if P_x is an isolated singularity of X.

Now, we show that if $h_2 = \ldots = h_8 = 0$ and one of the conditions (7.2) to (7.4) holds, then the singularity P_x is not isolated. In condition (7.2), we calculate that $h_8 + e^2r^2$ is divisible by q, giving $r = Cq$ for some $C \in \mathbb{C}$. Substituting into h_8, we compute that $h_8 - 2qes^2$ is divisible by q^2. Therefore q and s have a common prime divisor, giving that r and s have a common prime divisor, a contradiction. So, P_x is not an isolated singularity of X. Conditions (7.3) and (7.4) are similar.

Hence, if $h_2 = \ldots = h_8 = 0$ and P_x is an isolated singularity, then condition (7.1) holds. Using the explicit splitting lemma, we calculate h_8 in Listing 3 in Appendix A, and using Lemma 3.7, we can show that $h_2 = \ldots = h_8 = 0$ is equivalent to conditions 2 to 8.

(a) If conditions 2 to 8 are satisfied, then similarly to proof of part (b), P_x being a cA_n singularity where $n \geq 9$ implies that $h_9 = 0$. Using the explicit splitting lemma, we compute h_9 in Listing 3 in Appendix A, and using Lemma 3.7, we find that this implies that there exists $B_0 \in \mathbb{C}$ such that

$$\begin{aligned}
A_0 &= a_0 \\
B_1 &= b_1 \\
d_3 &= -s_3 B_0 + 2b_0 s_3 - 2a_0^2 s_3 + c_1 r_2 - 4a_0 b_1 r_2 \\
&\quad + 16a_0^2 a_1 r_2 + b_1 b_2 - 4a_0 a_1 b_2 - 2a_1^2 b_1 + 8a_0 a_1^3 \\
c_2 &= r_2 B_0 - 6a_0^2 r_2 + 2a_0 b_2 + 2a_1 b_1 - 12a_0 a_1^2.
\end{aligned}$$

Substituting into f gives

$$\begin{aligned}
x^3 a_3 + x^2 b_4 + x c_5 + d_6 &= (s_3 + 2a_1 r_2 + x r_2)^2 \\
x^3 a_2 + x^2 b_3 + x c_4 + d_5 &= (s_3 + 2a_1 r_2 + x r_2)(-2a_0 r_2 + b_2 - 2a_1^2 + 2x a_1 + x^2).
\end{aligned}$$

Define the curve C by the ideal $(w, t, s_3 + 2a_1 r_2 + x r_2)$. Taking partial derivatives, we find that X is singular at every point of C. Therefore, P_x is not an isolated singularity of X.

(c) We consider varieties X satisfying conditions 1 to n. We show that a general X has no other singular points apart from $P_x = [1, 0, 0, 0, 0]$. Denote the parameter space of all possible f in Notation 3.4 satisfying conditions 2 to n by $\mathcal{P}_n$.

If $P \neq P_x$ is a singular point of $\mathbb{V}(f)$ with t-coordinate zero, then one of y or z-coordinate is non-zero. A suitable change of coordinates of the form $x \mapsto x + \alpha y$ or $x \mapsto x + \alpha z$, where $\alpha \in \mathbb{C}$, takes the point P to P' with x, t and w-coordinate zero. Note that this coordinate change fixes the point P_x, keeps the form of f given in Notation 3.4, and f will continue to satisfy conditions 2 to n after this coordinate change. Using Lemma 3.9, we find that the parameter space of all f such that $\mathbb{V}(f)$ has a singular point $P \neq P_x$ with t-coordinate zero is at most $(\dim \mathcal{P}_n - 1)$-dimensional.

If P is a singular point of $\mathbb{V}(f)$ with t-coordinate non-zero and $n \geq 2$, then a suitable change of coordinates given by $x \mapsto x + \alpha_x t$, $y \mapsto y + \alpha_y t$ and $z \mapsto z + \alpha_z t$, where $\alpha_x, \alpha_y, \alpha_z \in \mathbb{C}$, takes the point P to $P_t = [0, 0, 0, 1, 0]$. Note that this coordinate change fixes the point P_x, keeps the form of f given in Notation 3.4, and f will continue to satisfy conditions 2 to n after this coordinate change. Using Lemma 3.10, we find that the parameter space of all f such that $\mathbb{V}(f)$ has a singular point P with t-coordinate non-zero is at most $(\dim \mathcal{P}_n - 1)$-dimensional, under the condition $n \geq 2$. If $n = 1$, then instead we perform a suitable coordinate change given by $x \mapsto x + \alpha_x t$, $y \mapsto y + \alpha_y t$, $z \mapsto z + \alpha_z t$ and $t \mapsto t$ or $t \mapsto 2t$, composed with a coordinate change of the form $t \mapsto \beta_y y + \beta_z z + \beta_t t$ where β_y, β_z and $\beta_t \in \mathbb{C}$ depend only on α_x, α_y, α_z and the coefficients of $x^4 \xi_2$, such that this composition takes the point P to P_t, fixes the point P_x and keeps the form of f given in Notation 3.4. This extra coordinate change $t \mapsto \beta_y y + \beta_z z + \beta_t t$ is needed to keep the form of f in Notation 3.4, namely to diagonalize the quadratic part with respect to t, that is, to remove the quadratic monomials yt and zt, and set the coefficient of t^2 to one. Similarly, using Lemma 3.10, we find that the parameter space of all f such that $\mathbb{V}(f)$ has a singular point P with t-coordinate non-zero is at most $(\dim \mathcal{P}_n - 1)$-dimensional when $n = 1$.

This shows that a general X satisfying 2 to n is smooth outside a cA_n singularity at P_x.

(d) Since X has local complete intersection singularities, it is Gorenstein ([Eis95, Corollary 21.19]). A terminal Gorenstein Fano 3-fold is $\mathbb{Q}$-factorial if and only if it is factorial [Kaw88, Lemma 6.3]. To see that in family (7.4) we do not have any $\mathbb{Q}$-factorial members, suffices to note that $\mathbb{V}(t, q - w)$ and $\mathbb{V}(t, q + w)$ are not Cartier on the sextic double solid.

In all other families apart from (7.4), a very general sextic double solid X satisfies that the hyperplane section $\mathbb{V}(x)$ is a very general sextic double plane. More precisely, fix a positive integer $n \leq 8$ and a connected component of the parameter space of sextic double solids with an isolated cA_n singularity described in Remark 3.13, other than cA_7 family 4. Consider the 28-dimensional parameter space of sextic double planes

$$\mathbb{V}(-w^2 + g) \subseteq \mathbb{P}(1, 1, 1, 3)$$

with variables y, z, t, w where $g \in \mathbb{C}[y, z, t]$ is homogeneous of degree 6, where the sextic double plane corresponds to a point in $\mathbb{C}^{28}$ given by the coefficients of y^6, $y^5 z$, …, t^6. For cA_4, cA_5, cA_6 and cA_7 families 1–3, the image of the parameter space of sextic double solids, under taking the hyperplane section $\mathbb{V}(x)$, contains a Zariski open dense set of the parameter space of sextic double planes. We show this by computing the rank of the Jacobian matrix corresponding to this projection morphism, and showing it is 28 for some particular point. For cA_7 family 4, respectively cA_8, it gives set which is open dense in a subvariety of codimension 3, respectively 1. For cA_8, using additionally the coordinate transformation $t \mapsto \alpha y + \beta z + t$ where $\alpha, \beta \in \mathbb{C}$ on the image of the parameter space of sextic double solids, we get a Zariski open dense set of the parameter space of sextic double planes.

Since a very general sextic double plane has Picard number 1, by Corollary 3.12 a very general sextic double solid X has Class number 1, except for cA_7 family 4. Therefore, X is factorial and has Picard number 1. □

Remark 3.13. Consider the tuples η of coefficients of $\xi_2, a_i, b_i, c_i, d_i$ in Notation 3.4. These form the parameter space $\mathbb{C}^{77}$. As in the proof of Theorem A, we can locally analytically write X with a cA_n singularity by $\mathbb{V}(wt + h) \subseteq \mathbb{C}^4$ where $h \in \mathbb{C}\{y, z\}$ has multiplicity $n + 1$. Requiring that all the coefficients of h_{n+1} are zero gives us $n + 2$ conditions. If these conditions are algebraically independent and if h_{n+2} is not zero after imposing these, then we would expect the parameter space for cA_{n+1} to have dimension $n + 2$ less. Counting parameters in Notation 3.4, we see that this is precisely the case. Table 2 shows the dimensions of the parameter spaces.

The parameter space for $n+1$ lies in the Zariski closure of the parameter space for n. Note that the parameter space for cA_7 has four connected components, each of dimension 44.

The automorphisms of $\mathbb{P}(1, 1, 1, 1, 3)$ that keep the form of a general f when $n \geq 2$ are of the form

$$\begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \\ & \alpha_5 & \alpha_6 & \alpha_7 & \\ & \alpha_8 & \alpha_9 & \alpha_{10} & \\ & & & \alpha_1^{-2} & \\ & & & & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix}.$$

These automorphisms form a 10-dimensional algebraic group. When $n = 1$, instead of polynomials f, we can consider polynomials F of the form $-w^2 + x^4 E_2 + x^3 A_3 + x^2 B_4 +$

$xC_5 + D_6$ where $E_i, A_i, B_i, C_i, D_i \in \mathbb{C}[y, z, t]$ are homogeneous of degree i. The parameter space of polynomials F is 80-dimensional, and the automorphisms of $\mathbb{P}(1, 1, 1, 1, 3)$ that keep the form of a general F form a 13-dimensional algebraic group. If the course moduli space of sextic double solids with an isolated cA_n singularity exists, then we expect it to have dimension 10 less than the parameter space in Notation 3.4.

Table 2. Dimension of the space of sextic double solids with an isolated cA_n

n	1	2	3	4	5	6	7	8
parameter space dim	77	74	70	65	59	52	44	35
expected moduli space dim	67	64	60	55	49	42	34	25

Remark 3.14. In some cases it suffices if X is *general* in Theorem A part (d), as opposed to very general. For example, if $n = 1$, then a general X has only one singularity which is an ordinary double point, and every such X is factorial and has Picard number 1, that is, X is a Mori fibre space. If $n = 2$, then a general X has the singularity given by $x_1x_2 + x_3^3 + x_4^3$, and is a Mori fibre space by [CM04, Remark 1.2]. If $n = 4$, a general X is a Mori fibre space, since in Section 5.2 we construct a Sarkisov link to a complete intersection $Z_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4)$ which is a Mori fibre space if it is general.

3.3. Other cA_n singularities

Although the primary interest is in isolated cA_n singularities since these are terminal, it is also possible to study non-isolated singularities with the same methods.

Remark 3.15. It follows from the proof of Theorem A that every f that satisfies conditions 2 to n defines a sextic double solid with a singularity at P_x which is either cA_m (possibly non-isolated) where $m \geq n$, or it is the germ $(Z, \mathbf{0})$ where $Z = \mathbb{V}(x_1^2 + x_2^2) \subseteq \mathbb{C}^4$ with variables x_1, x_2, x_3, x_4.

We describe a family of examples of sextic double solids with a non-isolated cA_n singularity for all $9 \leq n \leq 11$.

Proposition 3.16. *Let* $9 \leq n \leq 11$. *If* X *in Notation 3.4 satisfies conditions* 2 *to* 8*, and in addition satisfies conditions* 9 *to* n *and does not satisfy condition* $n+1$ *from the following:*

9. *there exists* $B_0 \in \mathbb{C}$ *such that*

$$\begin{aligned}
A_0 &= a_0 \\
B_1 &= b_1 \\
d_3 &= -s_3B_0 + 2b_0s_3 - 2a_0^2s_3 + c_1r_2 - 4a_0b_1r_2 \\
&\quad + 16a_0^2a_1r_2 + b_1b_2 - 4a_0a_1b_2 - 2a_1^2b_1 + 8a_0a_1^3 \\
c_2 &= r_2B_0 - 6a_0^2r_2 + 2a_0b_2 + 2a_1b_1 - 12a_0a_1^2.
\end{aligned}$$

10. $B_0 = b_0$

$$\begin{aligned}
d_2 &= 2c_0r_2 - 8a_0b_0r_2 + 16a_0^3r_2 + 2b_0b_2 - 4a_0^2b_2 + b_1^2 - 8a_0a_1b_1 - 4a_1^2b_0 + 24a_0^2a_1^2 \\
c_1 &= 2a_0b_1 + 2a_1b_0 - 12a_0^2a_1,
\end{aligned}$$

11. $c_0 = 2a_0b_0 - 4a_0^3$
$d_1 = b_0b_1 - 2a_0^2b_1 - 4a_0a_1b_0 + 8a_0^3a_1$,

12. $d_0 = b_0^2 - 4a_0^2b_0 + 4a_0^4$,

then P_x is a non-isolated cA_n singularity of a non-terminal sextic double solid X.

Proof. Repeatedly applying the divisibility condition (3.5) similarly to the proof of part (b) of Theorem A. □

Remark 3.17. If $9 \leq n \leq 11$, then the variety X in Proposition 3.16 is singular along the curve $C: \mathbb{V}(t, w, s_3 + 2a_1r_2 + xr_2)$ passing through P_x (see the proof of part (a) of Theorem A). We can compute that at a general point of C, the singularity is locally analytically $\mathbb{C}^1 \times$ODP, that is, it is isomorphic to the germ $(Z, \mathbf{0})$ where Z is $\mathbb{V}(x_1^2 + x_2^2 + x_3^2) \subseteq \mathbb{C}^4$ with variables x_1, x_2, x_3, x_4.

Remark 3.18. Translating the point $P_t = [0, 0, 0, 1, 0]$ to $[1, 0, 0, 0, 0]$, we find similar conditions to Theorem A for having a cA_n singularity at $P_t \in X$, which can be used to construct general sextic double solids with two cA_n singularities. It is also easy to construct simple examples with only two cA_5 singularities, such as the variety below with cA_5 singularities at P_x and at P_t,

$$\mathbb{V}(-w^2 + x^4t^2 + x^2t^4 + y^6 + z^6) \subseteq \mathbb{P}(1, 1, 1, 1, 3).$$

4. Weighted blowups

In this section, we discuss weighted blowups from both algebraic and local analytic points of view. In Proposition 4.5 we show that to check whether a weighted blowup is a Kawakita blowup (see Theorem 2.10), it suffices to compute the weight of the defining power series. Using this, in the technical Lemma 4.8 we show how to algebraically construct Kawakita blowups of cA_n points on affine hypersurfaces.

4.1. Weight-respecting maps

Let n and m be positive integers. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ denote the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Choose positive integer weights for $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 4.1. Let $X \subseteq \mathbb{C}^n$, $X' \subseteq \mathbb{C}^m$ be complex analytic spaces containing the origins. We say a biholomorphic map $\psi : X \to X'$ is **weight-respecting** if denoting its inverse by θ, we can locally analytically around the origins write $\psi = (\psi_1, \ldots, \psi_m)$ and $\theta = (\theta_1, \ldots, \theta_n)$ where for all i and j, the power series $\psi_j \in \mathbb{C}\{\boldsymbol{x}\}$ and $\theta_i \in \mathbb{C}\{\boldsymbol{y}\}$ satisfy $\mathrm{wt}(\psi_j) \geq \mathrm{wt}(y_j)$ and $\mathrm{wt}(\theta_i) \geq \mathrm{wt}(x_i)$.

It is known that a biholomorphic map taking the origin to the origin lifts to a unique biholomorphic map of the blown-up spaces under the usual weights $(1, \ldots, 1)$ (see for example [GLS07, Remark 3.17.1(4)]). It is easy to come up with examples where a biholomorphic map does not lift under weighted blowups. We give one example below.

Example 4.2. Let $X \subseteq \mathbb{C}^3$ be the complex analytic space given by $\mathbb{V}(f)$ where

$$f = x_2^2x_3 + x_1^3 + ax_1x_3^2 + bx_3^3$$

for some $a, b \in \mathbb{C}^*$. Define $X' \subseteq \mathbb{C}^3$ by $\mathbb{V}(f')$ where $f' = f(x_1, x_2, -x_2 + x_3)$. Choose weights $(1, 1, 2)$ for (x_1, x_2, x_3). Then, X and X' are biholomorphic and $\mathrm{wt}\, f = \mathrm{wt}\, f'$, but the weighted blowups of X and X' are not locally analytically equivalent.

Proof. Let $\psi\colon X \to X'$ be any local biholomorphism taking the origin to the origin. Composing with a suitable weight-respecting biholomorphic map and using Lemma 4.3, it suffices to consider the case where ψ is a linear biholomorphism. Since the elliptic curve defined by f in $\mathbb{P}^2$ with variables x_1, x_2, x_3 has only two automorphisms, there are only four possibilities for a linear biholomorphism $X \to X'$, namely $(x_1, x_2, x_3) \mapsto (x_1, \pm x_2, \pm x_2 + x_3)$.

Let $Y \to X$ and $Y' \to X'$ be the $(1, 1, 2)$-blowups of X and X' respectively. Then Y is given by $\mathbb{V}(g)$ where

$$g(u, x_1, x_2, x_3) = ux_2^2x_3 + x_1^3 + au^2x_1x_3^2 + bu^3x_3^3.$$

Denoting the points of Y and Y' by $[u, x_1, x_2, x_3]$, the lifted map $\psi_Y\colon Y \to Y'$ is given by $[u, x_1, x_2, x_3] \mapsto [u, x_1, \pm x_2, \pm x_2/u + x_3]$, which is not holomorphic on the exceptional locus $\mathbb{V}(u)$. □

On the other hand, a weight-respecting coordinate change does lift to weighted blowups:

Lemma 4.3. *The weighted blowups of $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ at the origin are analytically equivalent if there exists a weight-respecting biholomorphic map $X \to X'$ taking $\mathbf{0}$ to $\mathbf{0}$.*

Proof. Let $\varphi\colon Y \to X$ and $\varphi'\colon Y' \to X'$ be the weighted blowups at the origin and let $\psi\colon X \to X'$ be a weight-respecting biholomorphic map. We define the holomorphism $\psi_Y = (\psi_{Y,0}, \psi_{Y,1}, \ldots, \psi_{Y,m})\colon Y \to Y'$ by choosing $\psi_{Y,0} = u$ and for all $j \geq 1$, $\psi_{Y,j} = (\psi_j \circ \varphi)/u^{\mathrm{wt}(y_j)}$. Similarly, we define $\theta\colon Y' \to Y$ by $\theta_0 = u$ and $\theta_i = (\psi_i^{-1} \circ \varphi')/u^{\mathrm{wt}(x_i)}$. Since ψ and ψ^{-1} are weight-respecting, the maps ψ_Y and θ are indeed holomorphic.

The map $\theta \circ \psi_Y$ coincides with the identity map on a dense open subset of Y: namely, for all $[1, \boldsymbol{x}] \in Y$, we have $(\theta \circ \psi_Y)[1, \boldsymbol{x}] = \theta[1, \psi(\boldsymbol{x})] = [1, \boldsymbol{x}]$. Since coincidence sets are closed, the map $\theta \circ \psi_Y$ is the identity. Similarly, $\psi_Y \circ \theta$ is the identity, giving $\theta = \psi_Y^{-1}$.

Also, we have $(\varphi' \circ \psi_Y)[1, \boldsymbol{x}] = \psi(\boldsymbol{x}) = (\psi \circ \varphi)[1, \boldsymbol{x}]$, showing that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi_Y} & Y' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ X & \xrightarrow{\psi} & X' \end{array}$$

commutes. Therefore, φ and φ' are analytically equivalent. □

4.2. Kawakita blowup in analytic neighbourhoods

In the following, we focus on Kawakita blowups (see Theorem 2.10). Unlike Example 4.2, for cA_n singularities, having the correct weight for the defining power series is enough for the local analytic equivalence of weighted blowups.

Notation 4.4. We choose positive integer weights $\boldsymbol{w} = (r_1, r_2, a, 1)$ for variables $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ on $\mathbb{C}^4$ and define $n = (r_1 + r_2)/a - 1$ such that

- a divides $r_1 + r_2$ and is coprime to both r_1 and r_2,
- $r_1 \geq r_2$, and
- $n \geq 2$.

Proposition 4.5. *Using Notation 4.4, let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be such that $\mathbb{V}(f)$ has an isolated cA_n singularity at the origin and f has weight $r_1 + r_2$. Then, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f) \subseteq \mathbb{C}^4$ is a $\boldsymbol{w}$-Kawakita blowup.*

Proof. First, we remind that the terms *homogeneous*, *degree* and *multiplicity* are with respect to the standard weights $(1, \dots, 1)$. Let the *quadratic part* of f denote the homogeneous part of f of degree 2. After a suitable invertible linear weight-respecting coordinate change, the quadratic part of f is $x_1 x_2$.

We find that $f = x_1 x_2 + x_1 G + H$, where $G \in \mathbb{C}\{x_1, \dots, x_4\}$ has weight at least r_2 and multiplicity $m \geq 2$, and $H \in \mathbb{C}\{x_2, x_3, x_4\}$. The coordinate change $x_2 \mapsto x_2 - G_m$, where G_m is the homogeneous degree m part of G, takes f to $x_1 x_2 + x_1 G' + H'$, where G' has multiplicity at least $m+1$. By induction, this defines the unique formal power series $K \in \mathbb{C}[[x_1, \dots, x_4]]$ of multiplicity at least 2 and weight at least r_2 such that the transformation $x_2 \mapsto x_2 + K$ takes f to the form $x_1 x_2 + H''$ where $H'' \in \mathbb{C}[[x_2, x_3, x_4]]$. Similarly, we transform f into $x_1 x_2 + h$ where $h \in \mathbb{C}[[x_3, x_4]]$, using $x_1 \mapsto x_1 + L$ where $L \in \mathbb{C}[[x_2, x_3, x_4]]$. At the end of the proof, we show how to find a convergent weight-respecting coordinate change.

Since the singularity is cA_n where $n = (r_1 + r_2)/a - 1$, h must contain a monomial of degree $(r_1 + r_2)/a$. Since $x_1 x_2 + h$ has weight $r_1 + r_2$, if $a > 1$, then the coefficient of $x_3^{(r_1+r_2)/a}$ in h is non-zero. If $a = 1$, then after a suitable invertible linear coordinate change on $\mathbb{C}\{x_3, x_4\}$, the coefficient of $x_3^{(r_1+r_2)/a}$ in h is non-zero.

We found that we can transform f into the form $x_1 x_2 + h$ where the coefficient of $x_3^{(r_1+r_2)/a}$ in h is non-zero, by using only weight-respecting coordinate changes. By Lemma 4.3, the weighted blowup of f is locally analytically equivalent to the weighted blowup of $x_1 x_2 + h$, which is precisely a Kawakita blowup.

Lastly, we discuss convergence. Instead of the coordinate changes $x_2 \mapsto x_2 + K$, $x_1 \mapsto x_1 + L$, which might not be convergent, we do a coordinate change with truncated power series $K_{\leq N}$ and $L_{\leq N}$ of homogeneous parts of K and L of degree at most N. The coordinate change $\Psi\colon x_1 \mapsto x_1 + i x_2$, $x_2 \mapsto x_1 - i x_2$ takes $x_1 x_2$ into $x_1^2 + x_2^2$. Now we use the splitting lemma, which gives a convergent coordinate change Φ which respects the weighting when N is large enough, to give f the form $x_1^2 + x_2^2 + h(x_3, x_4)$ where h converges. Applying Ψ^{-1}, we get $x_1 x_2 + h$. Note that the coordinate changes Ψ and Ψ^{-1} might not respect the weighting $\boldsymbol{w}$, but the total coordinate change $\Psi^{-1} \circ \Phi \circ \Psi$ is weight-respecting if N is large enough. □

Given a variety X with an isolated cA_n point P, we show that any two $\boldsymbol{w}$-Kawakita blowups $Y \to X$ and $Y' \to X$ of the point P are locally analytically equivalent. Note that they need not be globally algebraically equivalent. For example, [CM04, Remark 2.4] describes two different $(2,1,1,1)$-Kawakita blowups of a cA_2 singularity on a quartic 3-fold.

Proposition 4.6. *Any two $\boldsymbol{w}$-Kawakita blowups of locally biholomorphic singularities are locally analytically equivalent.*

Proof. Let $f = x_1 x_2 + g(x_3, x_4)$ and $f' = x_1 x_2 + g'(x_3, x_4)$ be contact equivalent, where $g, g' \in \mathbb{C}\{x_3, x_4\}$ have weight $r_1 + r_2$ and $x_3^{(r_1+r_2)/a}$ appears in both g and in g' with non-zero coefficient. Suffices to show that the $\boldsymbol{w}$-blowups of $\mathbb{V}(f) \subseteq \mathbb{C}^4$ and $\mathbb{V}(f') \subseteq \mathbb{C}^4$ are locally analytically equivalent.

Since f and f' are contact equivalent, there exists a unit $u \in \mathbb{C}\{\boldsymbol{x}\}$ and a local biholomorphism $\psi\colon (\mathbb{C}^4, \mathbf{0}) \to (\mathbb{C}^4, \mathbf{0})$ such that $f' = u(f \circ \psi)$. Note that f' and $f \circ \psi$ have the same weight $r_1 + r_2$, and $x_3^{(r_1+r_2)/a}$ appears in $f \circ \psi$ with non-zero coefficient. Since the germs $(\mathbb{V}(f'), \mathbf{0})$ and $(\mathbb{V}(f \circ \psi), \mathbf{0})$ are equal, it suffices to show that the $\boldsymbol{w}$-blowups of $\mathbb{V}(f)$ and $\mathbb{V}(f \circ \psi)$ are locally analytically equivalent.

Using arguments similar to the proof of Proposition 4.5, we can find a weight-respecting biholomorphic map germ $\theta\colon (\mathbb{C}^4, \mathbf{0}) \to (\mathbb{C}^4, \mathbf{0})$ such that $f \circ \psi \circ \theta$ is of the form $x_1x_2 + g''$ where $g'' \in \mathbb{C}\{x_3, x_4\}$ contains $x^{(r_1+r_2)/a}$ and has weight $r_1 + r_2$. It suffices to show that the $\boldsymbol{w}$-blowups of $\mathbb{V}(f \circ \psi \circ \theta)$ and $\mathbb{V}(f)$ are locally analytically equivalent.

By Proposition 2.5, g and g'' are right equivalent, meaning there exists an automorphism Φ of $\mathbb{C}\{x_3, x_4\}$ such that $\Phi(g) = g''$. Since $x_3^{(r_1+r_2)/a}$ has non-zero coefficient in both g and g'', and both g and g'' have weight $r_1 + r_2$, the image of x_3 has weight a under both Φ and Φ^{-1}. Define the biholomorphic map germ $\varphi\colon (\mathbb{V}(f \circ \psi \circ \theta), \mathbf{0}) \to (\mathbb{V}(f), \mathbf{0})$ by $\boldsymbol{x} \mapsto (x_1, x_2, \Phi(x_3), \Phi(x_4))$. By Lemma 4.3, the $\boldsymbol{w}$-blowups of $\mathbb{V}(f \circ \psi \circ \theta) \subseteq \mathbb{C}^4$ and $\mathbb{V}(f) \subseteq \mathbb{C}^4$ are locally analytically equivalent. □

4.3. Kawakita blowups on affine hypersurfaces

In this section, we see how to construct weighted blowups for affine hypersurfaces with a cA_n singularity where $n \geq 2$ such that locally analytically they are Kawakita blowups.

Most cA_n singularities do not admit $(r_1, r_2, a, 1)$-Kawakita blowups where $a \geq 2$. Below we define the *type* of an isolated cA_n singularity, which for $n \geq 2$ is equal to the highest integer a such that it admits some $(r_1, r_2, a, 1)$-Kawakita blowup locally analytically. General sextic double solids with an isolated cA_n singularity have a type 1 cA_n singularity.

Definition 4.7. Let (X, P) be the complex analytic space germ of an isolated cA_n singularity. Let a be the largest integer such that (X, P) is isomorphic to some germ $(\mathbb{V}(x_1x_2 + g), \mathbf{0})$ where $g \in \mathbb{C}\{x_3, x_4\}$ has weight $a(n + 1)$ under the weighting $(a, 1)$ for (x_3, x_4). Then, we say that the cA_n singularity is of **type** a.

It is not obvious how to globally algebraically construct a Kawakita blowup for variety with a cA_n singularity. We show this for affine hypersurfaces in the technical Lemma 4.8. We use a projectivization of Corollary 4.9 in Section 5 for constructing Kawakita blowups of sextic double solids.

We describe the notation for Lemma 4.8. Choose $n \geq 2$ and weights

$$\boldsymbol{w} = \mathrm{wt}(\alpha, \beta, x_3, x_4) = (r_1, r_2, a, 1)$$

as in Notation 4.4. Let $F \in \mathbb{C}[x_1, x_2, x_3, x_4]$ have multiplicity at least 3, and let

$$f = -x_1^2 + x_2^2 + F$$

be such that $\mathbb{V}(f) \subseteq \mathbb{C}^4$ has terminal singularities and has a cA_n singularity of type at least a at the origin. Let q, w be the power series when splitting with respect to x_1 (Theorem 3.1), and p, v be the power series when splitting with respect to x_2, that is,

$$f = -((x_1 + q)w)^2 + ((x_2 + p)v)^2 + h \tag{4.1}$$

where $q \in \mathbb{C}\{x_2, x_3, x_4\}$ and $p \in \mathbb{C}\{x_3, x_4\}$ both have multiplicity at least 2, and $w \in \mathbb{C}\{x_1, x_2, x_3, x_4\}$ and $v \in \mathbb{C}\{x_2, x_3, x_4\}$ are units, and $h \in \mathbb{C}\{x_3, x_4\}$ has multiplicity at least 3. Choose weights

$$\boldsymbol{w}' = \mathrm{wt}(\alpha, \beta, x_1, x_2, x_3, x_4) = (r_1,\ r_2,\ m,\ \min(r_2, \operatorname{mult} p),\ a,\ 1)$$

for the variables on $\mathbb{C}^6$, where $m = \min(r_2, \operatorname{mult} q)$. If $a > 1$, then perform a coordinate change on x_3, x_4 for f such that h has weight $r_1 + r_2$. Writing a power series $s \in$

$\mathbb{C}\{x_1, x_2, x_3, x_4\}$ as a sum of its $\boldsymbol{w}'$-weighted homogeneous parts $s = \sum_{i=0}^{\infty} s_i$, let $s_{<k}$ denote $\sum_{i<k} s_i$ and $s_{\geq k}$ denote $\sum_{i \geq k} s_i$. Define the ideal

$$I = (f,\ -\alpha + (x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2},\ -\beta + x_2 + p_{<r_2}),$$

where $v_{<r_1-r_2}$ is defined to be 1 when $r_1 = r_2$, and $w_{<r_1-m}$ is defined to be 1 when $r_1 = m$. Note that the affine varieties $\mathbb{V}(f) \subseteq \mathbb{C}^4$ and $\mathbb{V}(I) \subseteq \mathbb{C}^6$ are isomorphic.

Lemma 4.8. *Using the notation above, the $\boldsymbol{w}'$-blowup of $\mathbb{V}(I)$ is a $\boldsymbol{w}$-Kawakita blowup.*

Proof. The morphism

$$\begin{aligned} &\varphi\colon \mathbb{C}^4 \to \mathbb{C}^4 \\ &(x_1, x_2, x_3, x_4) \mapsto ((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2},\ x_2 + p_{<r_2},\ x_3,\ x_4) \end{aligned}$$

has a local analytic inverse φ^{-1}, given by

$$\begin{aligned} &\varphi^{-1}\colon (\mathbb{C}^4, \mathbf{0}) \to (\mathbb{C}^4, \mathbf{0}) \\ &(\alpha, \beta, x_3, x_4) \mapsto ((\alpha - (\beta - p_{<r_2} + p_{<r_1})v')u - q',\ \beta - p_{<r_2},\ x_3,\ x_4) \end{aligned}$$

where $u \in \mathbb{C}\{\alpha, \beta, x_3, x_4\}$ is a unit, $v' = v_{<r_1-r_2}(\beta - p_{<r_2}, x_3, x_4)$ and $q' = q_{<r_1}(\beta - p_{<r_2}, x_3, x_4)$. Define the map germ

$$\begin{aligned} \psi\colon (\mathbb{C}^4, \mathbf{0}) &\to (\mathbb{C}^6, \mathbf{0}) \\ (\alpha, \beta, x_3, x_4) &\mapsto (\alpha, \beta, \varphi^{-1}(\alpha, \beta, x_3, x_4)). \end{aligned}$$

The restriction of ψ to $\mathbb{V}(I) \to \mathbb{V}(f \circ \psi)$ is a weight-respecting local biholomorphism, whose inverse is a projection. Therefore, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f \circ \psi)$ is equivalent to the $\boldsymbol{w}'$-blowup of $\mathbb{V}(I)$. If the $\boldsymbol{w}$-weight of $f \circ \psi$ is $r_1 + r_2$, then by Proposition 4.5, the $\boldsymbol{w}$-blowup of $\mathbb{V}(f \circ \psi)$ is the $\boldsymbol{w}$-Kawakita blowup map germ. Using Equation (4.1), it suffices to show that

$$\operatorname{wt}[((x_1 + q)w + (x_2 + p)v) \circ \psi] = r_1 \tag{4.2}$$

$$\operatorname{wt}[(-(x_1 + q)w + (x_2 + p)v) \circ \psi] = r_2. \tag{4.3}$$

Since ψ is weight-respecting, we have

$$\begin{aligned} \operatorname{wt}[(x_1 + q)w_{\geq r_1-m} \circ \psi)] &\geq r_1 \\ \operatorname{wt}[q_{\geq r_1} w_{<r_1-m} \circ \psi)] &\geq r_1 \\ \operatorname{wt}[(x_2 + p)v_{\geq r_1-r_2} \circ \psi] &\geq r_1 \\ \operatorname{wt}[p_{\geq r_1} v_{<r_1-r_2} \circ \psi] &\geq r_1. \end{aligned}$$

Since $((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2}) \circ \psi = \alpha$, this proves Equation (4.2). Using in addition that $\operatorname{wt}[(x_2 + p_{<r_1})v_{<r_1-r_2} \circ \psi] = r_2$, Equation (4.3) follows. □

Corollary 4.9. *Using the notation above, if $F \in \mathbb{C}[x_2, x_3, x_4]$, or equivalently, if $q = 0$ and $w = 1$, then define the ideal $J \subseteq \mathbb{C}[\alpha, \beta, x_2, x_3, x_4]$ by*

$$J = (-(\alpha - (x_2 + p_{<r_1})v_{<r_1-r_2})^2 + x_2^2 + F,\ -\beta + x_2 + p_{<r_2}), \tag{4.4}$$

where $v_{<r_1-r_2}$ is defined to be 1 if $r_1 = r_2$. Then, $\mathbb{V}(J)$ and $\mathbb{V}(f)$ are isomorphic affine varieties, and the $(r_1, r_2, \min(r_2, \operatorname{mult} p), a, 1)$-blowup of $\mathbb{V}(J)$ is a $\boldsymbol{w}$-Kawakita blowup. If in addition $r_1 = r_2$, then define the ideal $J' \subseteq \mathbb{C}[x_1, \beta, x_2, x_3, x_4]$ by

$$J' = (f, -\beta + x_2 + p_{<r_2}). \tag{4.5}$$

Then, $\mathbb{V}(J')$ and $\mathbb{V}(f)$ are isomorphic affine varieties, and the $(r_1, r_2, \min(r_2, \operatorname{mult} p), a, 1)$-blowup of $\mathbb{V}(J')$ is a $\boldsymbol{w}$-Kawakita blowup.

Proof. The isomorphism between $\mathbb{V}(I)$ and $\mathbb{V}(J)$ is a projection, with inverse given by $x_1 \mapsto \alpha - (\beta - p_{<r_2} + p_{<r_1})v_{<r_1-r_2}$, which is weight-respecting. If $r_1 = r_2$, the isomorphism between $\mathbb{V}(J)$ and $\mathbb{V}(J')$ is given by $x_1 \mapsto \alpha - \beta$, which is weight-respecting. □

The power series p, v, q, w can be expressed in terms of the coefficients of F using the explicit splitting lemma, Proposition 3.2.

5. Birational models of sextic double solids

In this section, we prove Theorem B on birational non-rigidity of certain sextic double solids. First, we give generality conditions we use.

Condition 5.1. Let the sextic double solid X be given as in Notation 3.4, and let $\mathbb{P}(1,1,3)$ have variables y, z, w and $\mathbb{P}^1$ have variables y, z. Then we have the following conditions, depending on the family that X lies in:

(cA_4) $\mathbb{V}(2wa_2 + c_5, w^2 - d_6) \subseteq \mathbb{P}(1,1,3)$ is 10 distinct points,

(cA_5) $\mathbb{V}(a_2, -w^2 + d_6) \subseteq \mathbb{P}(1,1,3)$ is 4 distinct points,

(cA_6) $c_4 - 2a_1b_3 - a_2b_2 + 2a_0a_2^2 + 6a_1^2a_2 \in \mathbb{C}[y,z]$ is non-zero, and $\mathbb{V}(a_2) \subseteq \mathbb{P}^1$ is two distinct points, and for both of these points P, either $b_3(P)$, $c_4(P)$ or $d_5(P)$ is non-zero,

(cA_7, 1) $\mathbb{V}(-e_2 + 4a_0r_2 + b_2 - 6a_1^2) \subseteq \mathbb{P}^1$ is two distinct points,

(cA_7, 2) r_1 and q_1 are coprime in $\mathbb{C}[y,z]$,

(cA_7, 3) $q_2 \in \mathbb{C}[y,z]$ is not a square,

(cA_8) $a_0 \neq A_0$.

Theorem B. *A sextic double solid, which is a Mori fibre space containing an isolated cA_n singularity with $n \geq 4$ and satisfying Condition 5.1, has a Sarkisov link starting with a weighted blowup of the cA_n point.*

We treat each of the 7 families separately. We use the notation in Construction 2.14 and Example 2.15 for the 2-ray links. We write the cA_4 case in more detail. Below, when we say that a birational map is k Atiyah flops, then we mean that the base of the flop is k points, above each we are contracting a curve and extracting a curve, and locally analytically above each of the points it is an Atiyah flop (see [Rei92, Section 1.3] for Atiyah flop). Similarly for flips. Below, for a morphism $\Phi\colon T_0 \to \mathbb{P}$, $\Phi^*\colon \operatorname{Cox}\mathbb{P} \to \operatorname{Cox} T_0$ denotes a corresponding $\mathbb{C}$-algebra homomorphism of Cox rings (described explicitly in the proof of Proposition 5.4).

5.1. Singularities after divisorial contraction

Before proving Theorem B, we show that for any Kawakita blowup $Y_0 \to X$ (Theorem 2.10) of a sextic double solid X with an isolated cA_n singularity, the variety Y_0 has only up to two singular points if X is general, which are quotient singularities. We do not give the generality conditions of Proposition 5.3 explicitly. We do not use this proposition in the proof of Theorem B. First, we give an elementary lemma:

Lemma 5.2. *Let $a, b \in \mathbb{C}[y,z]$ be non-zero homogeneous polynomials with $\deg a \geq \deg b$ such that for every homogeneous polynomial $c \in \mathbb{C}[y,z]$ of degree $\deg a - \deg b$, the polynomial $a + bc$ is divisible by the square of a linear form. Then a and b are both divisible by the square of the same linear form.*

Proof. Suffices to prove that for non-zero polynomials $f, g \in \mathbb{C}[x]$, if $f + \lambda g$ has a repeated root for all $\lambda \in \mathbb{C}$, then f and g have a common repeated root. This holds if there exists $x_0 \in \mathbb{C}$ which is as a repeated root of $f + \lambda g$ for infinitely many λ. Since g and $f/g + \lambda$ have only finitely many repeated roots, the claim follows. □

Proposition 5.3. *Let X be a general sextic double solid with an isolated cA_n singularity P and $Y_0 \to X$ a divisorial contraction with centre P, which is a $(r_1, r_2, 1, 1)$-Kawakita blowup. Then, Y_0 has a quotient singularity $1/r_1(1, 1, r_1 - 1)$ if $r_1 > 1$ and a quotient singularity $1/r_2(1, 1, r_2 - 1)$ if $r_2 > 1$, and is smooth elsewhere.*

Proof. By Theorem A, a general X is smooth outside P. So, it suffices to show that Y_0 has only up to two quotient singularities on the exceptional divisor and is smooth elsewhere. Since $Y_0 \to X$ is a $(r_1, r_2, 1, 1)$-Kawakita blowup, we can consider the local analytic coordinate system around P where X is given by $wt + h(y, z)$ where $h \in \mathbb{C}\{y, z\}$ has multiplicity $n + 1$. The variety Y_0 is locally analytically around the exceptional divisor given by $wt + \frac{1}{u^{n+1}} h(uy, uz)$ inside the geometric quotient $(\mathbb{C}^5 \setminus \mathbb{V}(w, t, y, z))/\mathbb{C}^*$ where the $\mathbb{C}^*$-action is given by $\lambda \cdot (u, w, t, y, z) = (\lambda^{-1}u, \lambda^{r_1}w, \lambda^{r_2}t, \lambda y, \lambda z)$. Taking partial derivatives, the singular locus of Y_0 is given by

$$\operatorname{Sing} Y_0 = \mathbb{V}\left(u, w, t, h_{n+1}, \frac{\partial h_{n+1}}{\partial y}, \frac{\partial h_{n+1}}{\partial z}, h_{n+2}\right) \cup \{P_w\}_{\text{if } r_1 > 1} \cup \{P_t\}_{\text{if } r_2 > 1},$$

where h_i denotes the homogeneous degree i part of h, and P_w and P_t are the points $[0, 1, 0, 0, 0]$ and $[0, 0, 1, 0, 0]$, respectively. To prove the claim, it suffices to show that if X is general, then no square of a linear form divides h_{n+1}, that is, h_{n+1} is squarefree.

Considering the 11 families of Theorem A separately, it is easy to compute using the explicit splitting lemma (Proposition 3.2) and Lemma 5.2 that h_{n+1} is squarefree when X is general. For example, for a cA_8 singularity, we compute that

$$h_9 = Q - 2d_3 r_2^3 = 8(a_0 - A_0)s_3^3 + r_2 R,$$

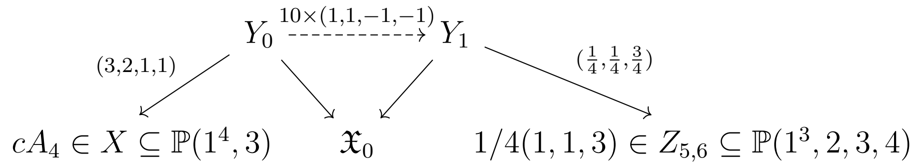
where $Q, R \in \mathbb{C}[y, z]$ are homogeneous of degrees 9 and 7 respectively, and Q does not contain the polynomial d_3. If the affine cone of Y_0 is not smooth for a general X, then the affine cone of Y_0 is singular for all X. In that case, Lemma 5.2 shows that a prime factor of r_2 divides h_9, which implies that it divides s_3, a contradiction. Similarly for the other 10 families. □

5.2. cA_4 model

Note that Okada described a Sarkisov link starting from a general complete intersection $Z_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4)$ to a sextic double solid (see entry No. 9 of the table in [Oka14, Section 9]). We show the converse:

Proposition 5.4. *A sextic double solid with a cA_4 singularity satisfying Condition 5.1 has a Sarkisov link to a complete intersection $Z_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4)$, starting with a $(3, 2, 1, 1)$-blowup of the cA_4 point, then 10 Atiyah flops, and finally a Kawamata divisorial contraction (see [Kaw96]) to a terminal quotient $1/4(1, 1, 3)$ point. Under further generality conditions (Proposition 5.3), Z is quasismooth.*

Proof. We exhibit the diagram below.



The corresponding diagram for the ambient toric spaces is given in detail in Example 2.15.

First, we describe the sextic double solid X. By Theorem A, any sextic double solid $\hat{X}$ with an isolated cA_4 singularity can be given by

$$\hat{X}\colon \mathbb{V}(\hat{f}) \subseteq \mathbb{P}(1,1,1,1,3)$$

with variables x, y, z, t, w where

$$\hat{f} = -w^2 + x^4t^2 + 2x^3ta_2 + x^3t^2A_1 + x^2a_2^2 + x^2tB_3 + xC_5 + D_6,$$

where $a_2 \in \mathbb{C}[y,z]$ is homogeneous of degree 2, and $A_i, B_i, C_i, D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define the bidegree $(5,6)$ complete intersection X, isomorphic to $\hat{X}$, by

$$X\colon \mathbb{V}(f, -x\xi + \alpha^2 - D_6) \subseteq \mathbb{P}(1,1,1,1,3,5)$$

with variables x, y, z, t, α, ξ, where

$$f = -\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2A_1 + xtB_3 + C_5.$$

The isomorphism is given by

$$\begin{aligned} \hat{X} &\to X \\ [x,y,z,t,w] &\mapsto \left[x,y,z,t,\alpha', 2\alpha' a_2 + 2\alpha' xt + x^2t^2A_1 + xtB_3 + C_5\right] \end{aligned}$$

where $\alpha' = w + x^2t + xa_2$, with inverse

$$[x,y,z,t,\alpha,\xi] \mapsto [x,y,z,t,\alpha - x^2t - xa_2].$$

We describe the divisorial contraction $\varphi\colon Y_0 \to X$. Define the toric variety

$$T_0\colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array}\right),$$

as in Example 2.15. Let Φ be the ample model of $\mathbb{V}(x)$, that is,

$$\begin{aligned} \Phi\colon T_0 &\to \mathbb{P}(1,1,1,1,3,5) \\ [u,x,y,z,\alpha,\xi,t] &\mapsto [x,uy,uz,u^2t,u^3\alpha,u^6\xi]. \end{aligned}$$

Let Y_0 be the strict transform of X. Let Φ^* denote the corresponding $\mathbb{C}$-algebra homomorphism, namely

$$\begin{aligned} \Phi^*\colon \mathbb{C}[x,y,z,t,\alpha,\xi] &\to \mathbb{C}[u,x,y,z,\alpha,\xi,t] \\ \Phi^*\colon x \mapsto x,\ y \mapsto uy,\ z \mapsto uz,\ t &\mapsto u^2t,\ \alpha \mapsto u^3\alpha,\ \xi \mapsto u^6\xi. \end{aligned}$$

Define

$$A_Y = A_1(y, z, ut), \qquad B_Y = B_3(y, z, ut), \qquad C_Y = C_5(y, z, ut), \qquad D_Y = D_6(y, z, ut)$$

and define the polynomial $g = \Phi^* f / u^5$, that is,

$$g = -u\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2 A_Y + xtB_Y + C_Y.$$

Then, Y_0 is given by

$$Y_0 \colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (g,\, -x\xi + \alpha^2 - D_Y).$$

We will see later that I_Y 2-ray follows T_0. Note that there exist other ideals that define the same variety $Y_0 \subseteq T_0$ (see [Cox95, Corollary 3.9]), but where the ideal might not 2-ray follow T_0. Also note that we have not (and do not need to) prove that the ideal I_Y is saturated with respect to u, although in general, saturating might help in finding the ideal that 2-ray follows T_0. The morphism $Y_0 \to X$ is the restriction of $T_0 \to \mathbb{P}(1,1,1,1,3,5)$. Locally, $(Y_0)_x \to X_x$ is the $(3,2,1,1)$-blowup of $\mathbb{V}(f') \subseteq \mathbb{C}^4$ with variables α, t, y, z, where

$$f' = -\alpha^2 + 2\alpha a_2 + 2\alpha t + t^2 A_1 + tB_3 + C_5 + D_6.$$

Since $\operatorname{wt} f' = 5$, by Proposition 4.5, $(Y_0)_x \to X_x$ is a $(3,2,1,1)$-Kawakita blowup.

The first diagram in the 2-ray game for Y_0 is 10 Atiyah flops, under Condition 5.1. We describe the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ globally. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, define

$$T_1 \colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

Define Y_1 by $\mathbb{V}(I_Y) \subseteq T_1$. Define the morphisms $Y_0 \to \mathfrak{X}_0$ and $Y_1 \to \mathfrak{X}_0$ as the ample models of $\mathbb{V}(y)$. The exceptional locus of $Y_0 \to \mathfrak{X}_0$ is $E_0^- = \mathbb{V}(\xi, t) \subseteq Y_0$, the exceptional locus of $Y_1 \to \mathfrak{X}_0$ is $E_1^+ = \mathbb{V}(u, x) \subseteq Y_1$, and the base of the flop is

$$\{P_i\} = \mathbb{V}(2\alpha a_2 + C_5(y,z,0),\, \alpha^2 - D_6(y,z,0)) \subseteq \mathbb{P}(1,1,3) \subseteq \mathfrak{X}_0,$$

where $\mathbb{P}(1,1,3)$ has variables y, z, α. If a_2, $C_5(y,z,0)$ and $D_5(y,z,0)$ are general enough, that is, if Condition 5.1 is satisfied, then the base of the flop is 10 points $\{P_i\}_{1 \le i \le 10}$, and both E_0^- and E_1^+ are 10 disjoint curves mapping to $\{P_i\}_{1 \le i \le 10}$.

We show that locally analytically, the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is 10 Atiyah flops. Let $P \in \mathfrak{X}_0$ be any point in the base of the flop. Then, P has either y or z coordinate non-zero. We consider the case where the y-coordinate is non-zero, the other case is similar. Since the base of the flop is 10 points, the point P is smooth in $\mathbb{P}(1,1,3)$. By the implicit function theorem, we can locally analytically equivariantly express α and z in terms of the variables u, x, ξ, t on the patches $(Y_0)_y$, $(\mathfrak{X}_0)_y$ and $(Y_1)_y$. So, the flop $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is locally analytically a $(1,1,-1,-1)$-flop, the so-called Atiyah flop, around P.

The last morphism $Y_1 \to Z$ in the link for X is a divisorial contraction. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 6 & -5 \\ 2 & -1 \end{smallmatrix}\right)$ with determinant 4, we see that

$$T_1 \cong \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \xi & t \\ 5 & 6 & 1 & 1 & 3 & 0 & -4 \\ 1 & 2 & 1 & 1 & 3 & 4 & 0 \end{array}\right).$$

Let $Y_1 \to Z$ be the ample model of $\frac{1}{4}\mathbb{V}(\xi)$, that is,

$$\begin{gathered} Y_1 \to Z \\ [u,x,y,z,\alpha,\xi,t] \mapsto \left[t^{\frac{5}{4}}u, t^{\frac{1}{4}}y, t^{\frac{1}{4}}z, t^{\frac{3}{2}}x, t^{\frac{3}{4}}\alpha, \xi\right]. \end{gathered}$$

Then Z is the bidegree $(5,6)$ complete intersection

$$Z\colon \mathbb{V}(h, -x\xi + \alpha^2 - D_6(y,z,u)) \subseteq \mathbb{P}(1,1,1,2,3,4)$$

with variables u, y, z, x, α, ξ, where the h is given by applying the $\mathbb{C}$-algebra homomorphism $t \mapsto 1$ to g. The morphism $Y_1 \to Z$ contracts the exceptional divisor $\mathbb{V}(t) \subseteq Y_1$ to the point $P_\xi = [0,0,0,0,0,1]$. On the quasiprojective patch $(Y_1)_\xi$, we can express u and x locally analytically equivariantly in terms of y, z, α, t. So, the morphism $Y_1 \to Z$ is locally analytically the Kawamata weighted blowdown (see [Kaw96]) to the terminal quotient singular point P_ξ of type $1/4(1,1,3)$. □

Remark 5.5. We explain below how we found the variety X. We start with the variety $\hat{X}$, given by Theorem A. Next, we perform the coordinate change $\hat{X} \to \bar{X}$ given in Equation (4.4) of Corollary 4.9, with $(r_1, r_2, a, 1) = (3,2,1,1)$, $p_2 = a_2$ and $v_0 = 1$. We see that $\hat{X}$ is isomorphic to

$$\bar{X}\colon \mathbb{V}(\bar{f}) \subseteq \mathbb{P}(1,1,1,1,3)$$

with variables x, y, z, t, α, where

$$\bar{f} = \alpha(-\alpha + 2x^2t + 2xa_2) + x^3t^2A_1 + x^2tB_3 + xC_5 + D_6.$$

We construct a $(3,2,1,1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define the toric variety $\bar{T}_0$ by

$$\bar{T}_0\colon \left(\begin{array}{cc|cccc} u & x & y & z & \alpha & t \\ 0 & 1 & 1 & 1 & 3 & 1 \\ -1 & 0 & 1 & 1 & 3 & 2 \end{array}\right).$$

In other words, $\bar{T}_0$ is given by the geometric quotient

$$\bar{T}_0 = \frac{\mathbb{C}^6 \setminus \mathbb{V}((u,x) \cap (y,z,\alpha,t))}{(\mathbb{C}^*)^2}.$$

Let $\bar{\Phi}$ be the ample model of $\mathbb{V}(x)$, and let $\bar{Y}_0 \subseteq \bar{T}_0$ be the strict transform of $\bar{X}$. By Corollary 4.9, $\bar{Y}_0 \to \bar{X}$ is a $(3,2,1,1)$-Kawakita blowup. Alternatively, we use Proposition 4.5 on the patch $(\bar{Y}_0)_x \to \bar{X}_x$ to show it is a $(3,2,1,1)$-Kawakita blowup, like in Proposition 5.4.

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$. We describe the next (and the final) map in the 2-ray game for $\bar{T}_0$. Acting by the matrix $\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$, we can write $\bar{T}_0$ by

$$\bar{T}_0 \cong \left(\begin{array}{cc|cccc} u & x & y & z & \alpha & t \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 0 \end{array}\right).$$

The ample model of the divisor $\mathbb{V}(y)$ is the weighted blowup

$$\begin{gathered} \bar{T}_0 \to \mathbb{P}(1,1,1,2,3) \\ [u,x,y,z,\alpha,t] \mapsto [y,z,ut,xt,\alpha], \end{gathered}$$

where the centre is the surface $\mathbb{P}(1,1,3)$ given by $\mathbb{V}(u,x) \subseteq \mathbb{P}(1,1,1,2,3)$ with variables y,z,u,x,α. Above every point in $\mathbb{P}(1,1,3)$, the fibre is $\mathbb{P}^1$. Define

$$\bar{Z}\colon \mathbb{V}(\bar{h}) \subseteq \mathbb{P}(1,1,1,2,3)$$

where

$$\bar{h} = \alpha(-u\alpha + 2x^2 + 2xa_2) + x^3 A_Z + x^2 B_Z + xC_Z + uD_Z,$$

where

$$A_Z = A_1(y,z,u), \qquad B_Z = B_3(y,z,u), \qquad C_Z = C_5(y,z,u), \qquad D_Z = D_6(y,z,u).$$

We show that when restricting the weighted blowup to $\bar{Y}_0 \to \bar{Z}$, the exceptional locus is 1-dimensional. After restricting to $\bar{Y}_0$, the exceptional divisor $\mathbb{V}(t)$ becomes $\mathbb{V}(t, x(2\alpha a_2 + C_5(y,z,0)) + u(-\alpha^2 + D_6(y,z,0)))$. By Condition 5.1, there are exactly 10 points $P_1,\dots,P_{10} \in \mathbb{P}(1,1,3) \subseteq \bar{Z}$ such that $2\alpha a_2 + C_5(y,z,0)$ and $-\alpha^2 + D_6(y,z,0)$ have a common solution. Above each of those points, the fibre is $\mathbb{P}^1$. Above any other point, the fibre is just one point. Therefore, the morphism $\bar{Y}_0 \to \bar{Z}$ contracts 10 curves onto 10 points, and is an isomorphism elsewhere. This shows that $\bar{Y}_0$ does not 2-ray follow $\bar{T}_0$, since a 2-ray link ends with either a fibration or a divisorial contraction.

The problem with the previous embedding was that $\bar{g}$ belonged to the irrelevant ideal (u,x). We "unproject" the divisor $\mathbb{V}(u,x)$, to embed $\bar{Y}_0$ into a toric variety T_0 such that Y_0 2-ray follows T_0. The varieties $Y_0 \subseteq T_0$ are defined as in the proof of Proposition 5.4. We see that $\bar{Y}_0$ is isomorphic to Y_0 through the map

$$[u,x,y,z,\alpha,t] \mapsto \left[u,x,y,z,\alpha,\frac{\alpha^2 - D_Y}{x},t\right].$$

The map is a morphism, since we have the equality

$$\frac{\alpha^2 - D_Y}{x} = \frac{2\alpha a_2 + 2\alpha x t + x^2t^2 A_Y + xtB_Y + C_Y}{u}$$

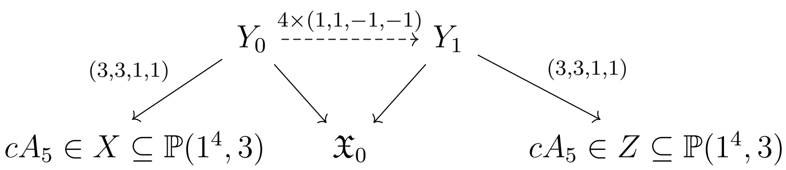
in the field of fractions of $\mathbb{C}[u,x,y,z,\alpha,t]$, and either x or u is non-zero at every point of T_0. For more details on this kind of "unprojection", see [Rei00, Section 2] or [PR04, Section 2.3].

Now, the coordinate change $\bar{Y}_0 \to Y_0$ induces a coordinate change $\bar{X} \to X$, where X is defined as in the proof of Proposition 5.4.

5.3. cA_5 model

Proposition 5.6. *A sextic double solid X which is a Mori fibre space with a cA_5 singularity satisfying Condition 5.1 has a Sarkisov link to a sextic double solid Z with a cA_5 singularity, starting with a $(3,3,1,1)$-blowup of the cA_5 point in X, then four Atiyah flops, and finally a $(3,3,1,1)$-blowdown to a cA_5 point. If in addition c_4 is general after fixing a_i, b_i and d_6 in Notation 3.4, then X and Z are not isomorphic. Under further generality conditions, both X and Z are smooth outside the cA_5 point.*

Proof. We exhibit the diagram below.



We construct X and a $(3,3,1,1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A, and performing the coordinate change in Equation (4.5) of Corollary 4.9 (with $p_2 = a_2$), we can write a sextic double solid X with a cA_5 singularity by

$$X \colon \mathbb{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1,1,1,1,2,3),$$

with variables x, y, z, t, β, w where

$$f = -w^2 + x\beta(2b_3 - 4\beta a_1 + 8xta_1 + x\beta) + 4x^3t^3a_0 + x^2t^2B_2 + xtC_4 + D_6,$$

where B_i, C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degrees i. Define T_0 by

$$T_0 \colon \left(\begin{array}{cc|ccccc} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 3 & 3 & 2 \end{array}\right).$$

Let $\Phi \colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ be the ample model of $\mathbb{V}(x)$, $Y_0 \subseteq T_0$ the strict transform of X, and $Y_0 \to X$ the restriction of Φ. Then, Y_0 is given by

$$Y_0 \colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f / u^6,\ -u\beta + xt + a_2),$$

and $Y_0 \to X$ is a $(3,3,1,1)$-Kawakita blowup.

We show that the first map in the 2-ray game for Y_0 is a flop, locally analytically 4 Atiyah flops, under Condition 5.1. Acting by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|ccccc} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

The base of the flop in $\mathbb{P}(1,1,3) \subseteq \mathfrak{X}_0$ is given by $\mathbb{V}(a_2, -w^2 + D_6(y,z,0)) \subseteq \mathbb{P}(1,1,3)$. If a_2 and $D_6(y,z,0)$ are general, that is, Condition 5.1 is satisfied, then this is exactly 4 points. In this case, any such point P is a smooth point in $\mathbb{P}(1,1,3)$. Consider the case where the y-coordinate of P is non-zero, the case where z is non-zero is similar. Locally analytically equivariantly, we can express z and w in terms of u, x, β, t in Y_0, $\mathfrak{X}_0$ and Y_1. So, the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is locally analytically four Atiyah flops.

The last map in the 2-ray game of Y_0 is a weighted blowdown $Y_1 \to Z$. After acting by $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial matrix of T_0, we find that T_1 is given by

$$T_1 \colon \left(\begin{array}{ccccc|cc} u & x & y & z & w & \beta & t \\ 2 & 3 & 1 & 1 & 3 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 1 & 0 \end{array}\right).$$

We see that $Z \subseteq \mathbb{P}(1,1,1,1,2,3)$ with variables β, u, y, z, x, w is given by the ideal

$$I_Z = (h, -u\beta + x + a_2),$$

where h is given by sending t to 1 in $\Phi^* f / u^6$, namely

$$h = -w^2 + x\beta(2b_3 - 4u\beta a_1 + 8xa_1 + x\beta) + 4x^3a_0 + x^2B_Z + xC_Z + D_Z$$

and

$$B_Z = B_2(y,z,u), \qquad C_Z = C_4(y,z,u), \qquad D_Z = D_6(y,z,u).$$

Substituting $x = u\beta - a_2$ into h, we find that Z is a sextic double solid. Applying the explicit splitting lemma (Proposition 3.2), we find that the complex analytic space germ (Z, P_β) is isomorphic to $(\mathbb{V}(h_{\text{ana}}), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ with variables w, u, y, z, where

$$h_{\text{ana}} = -w^2 + u^2 + d_6 - (b_3 - 2a_1a_2)^2 + (\text{h.o.t in } y, z),$$

where (h.o.t in y, z) stands for higher order terms in the variables y, z. So, $P_\beta \in Z$ is a cA_5 singularity. On the patch where β is non-zero, we can substitute $u = xt + a_2$, so the morphism $(Y_1)_\beta \to Z_\beta$ is a weighted blowup of a hypersurface given by a weight 6 polynomial. By Proposition 4.5, $Y_1 \to Z$ is a $(3,3,1,1)$-Kawakita blowup.

We show that X and Z are not isomorphic when $a_2 \neq 0$ and c_4 is general, using a dimension counting argument similar to [GLS07, Theorem 2.55]. Using the explicit splitting lemma, we find that the complex analytic space germ (X, P_x) is isomorphic to $(\mathbb{V}(f_{\text{ana}}), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ with variables w, t, y, z where

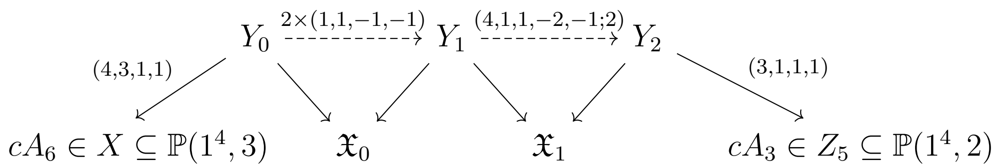
$$f_{\text{ana}} = -w^2 + t^2 + d_6 - 2a_2c_4 + 2a_2^2b_2 - 4a_0a_2^3 - (b_3 - 4a_1a_2)^2 + (\text{h.o.t in } y, z).$$

If X and Z are isomorphic, then this implies that the complex analytic space germs (X, P_x) and (Z, P_β) are isomorphic, implying by Propositions 2.5 and 2.4 that the degree 6 parts of $f_{\text{ana}}(0,0,y,z)$ and $h_{\text{ana}}(0,0,y,z)$ are the same up to an invertible linear coordinate change on y, z. Fixing a_0, a_1, a_2, b_2, b_3 and d_6, we see that $h_{\text{ana}}(0,0,y,z)$ is fixed, but $f_{\text{ana}}(0,0,y,z)$ has 5 degrees of freedom. Since there are only 4 degrees of freedom in picking an element of $\mathrm{GL}(\mathbb{C}, 2)$, the polynomials $f_{\text{ana}}(0,0,y,z)$ and $h_{\text{ana}}(0,0,y,z)$ are not related by an invertible linear coordinate change when c_4 is general. This shows that if X is general, then the varieties X and Z are not isomorphic. □

5.4. cA_6 model

Proposition 5.7. *A sextic double solid that is a Mori fibre space with a cA_6 singularity satisfying Condition 5.1 has a Sarkisov link to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with a cA_3 singularity, starting with a $(4,3,1,1)$-blowup of the cA_6 point, then two $(1,1,-1,-1)$-flops, then a $(4,1,1,-2,-1;2)$-flip, and finally a $(2,2,1,1)$-blowdown to a cA_3 point. Under further generality conditions, the singular locus of Z consists of 3 points, namely the cA_3 point, the $1/2(1,1,1)$ quotient singularity and an ordinary double point.*

Proof. We exhibit the diagram below.



We construct X and a $(4,3,1,1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A and Corollary 4.9 with $p_2 = a_2$ and $p_3 = b_3 - 4a_1a_2$, we can write a sextic double solid X with a cA_6 singularity by

$$X\colon \mathbb{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1,1,1,1,2,3),$$

with variables x, y, z, t, β, w where

$$\begin{aligned} f &= \alpha(-\alpha + 2(b_3 - 4\beta a_1 + 4xta_1 + x\beta)) \\ &\quad + 2\beta(c_4 - \beta b_2 + 2xtb_2 + 2x\beta a_1 + 2\beta^2 a_0 - 6xt\beta a_0 + 6x^2t^2a_0) \\ &\quad + x^2t^3B_1 + xt^2C_3 + tD_5 \end{aligned}$$

where B_i, C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define T_0 by

$$T_0 \colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 3 & 2 \end{array}\right).$$

Let $\Phi \colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ be the ample model of $\mathbb{V}(x)$, $Y_0 \subseteq T_0$ the strict transform of X, and $Y_0 \to X$ the restriction of Φ. Then, Y_0 is given by

$$Y_0 \colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^7,\ -u\beta + xt + a_2),$$

and $Y_0 \to X$ is a $(4,3,1,1)$-Kawakita blowup.

We show that the first diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ in the 2-ray game for Y_0 is locally analytically two Atiyah flops under Condition 5.1, namely that $\mathbb{V}(a_2) \subseteq \mathbb{P}^1$ with variables y,z consists of exactly two points, and for both of the points P, either $b_3(P)$, $c_4(P)$ or $d_5(P)$ is non-zero, where $D_5 = t^5 d_0 + 2t^4 d_1 + t^3 d_2 + 2t^2 d_3 + t d_4 + 2d_5$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \colon \left(\begin{array}{cc|ccccc} u & x & y & z & \alpha & \beta & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

Under the above condition, after a suitable linear change of coordinates on y,z, we find that $a_2 = yz$. Let $P = \mathbb{V}(z) \in \mathbb{P}^1 \subseteq \mathfrak{X}_0$, the case where $P = \mathbb{V}(y)$ is similar. On the patch where y is non-zero, we can substitute $z = u\beta - xt$. The contracted locus is $\mathbb{P}^1 \cong \mathbb{V}(\alpha,\beta,t) \subseteq (Y_0)_y$, and the extracted locus is $\mathbb{V}(u,x) = \mathbb{V}(u,x,\alpha b_3(1,0) + \beta c_4(1,0) + t d_5(1,0)) \subseteq (Y_1)_y$. By Condition 5.1, we can express one of α,β,t equivariantly locally analytically in the other variables. So, the flop diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is locally analytically a $(1,1,-1,-1)$-flop above both of the points.

We show that the next diagram in the 2-ray game of Y_0 is a $(4,1,1,-2,-1;2)$-flip (this is case (1) in [Bro99, Theorem 8]). The toric variety T_1 is given by

$$T_1 \colon \left(\begin{array}{cccc|ccc} u & x & y & z & \alpha & \beta & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

The base of the flip is $P_\alpha = [0,0,0,0,1,0,0]$. On the patch where α is non-zero, we can express u locally analytically and equivariantly in terms of x,y,z,β,t. After substitution, the ideal is principal, with generator $f' = -\beta \cdot (2x + \dots) + xt + a_2$. Under Condition 5.1, a_2 has a non-zero coefficient in f', so the flip diagram corresponds to case (1) in [Bro99, Theorem 8]. The flips contracts a curve containing a $1/4(1,1,3)$ singularity and extracts a curve containing a $1/2(1,1,1)$ singularity and an ordinary double point. The ordinary double point on Y_2 is at $[u_0,0,0,0,2,1,1]$ for some $u_0 \in \mathbb{C}$.

We show that the last map in the 2-ray game of Y_0 is a weighted blowup $Y_2 \to Z$, where Z is isomorphic to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with variables u,y,z,β,α. Acting by the matrix $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial action-matrix of T_0, we find that T_2 is given by

$$T_2 \colon \left(\begin{array}{ccccc|cc} u & x & y & z & \alpha & \beta & t \\ 2 & 3 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 1 & 0 \end{array}\right).$$

Define the bidegree $(5,2)$ complete intersection $Z\colon \mathbb{V}(h, a_2 - u\beta + x) \subseteq \mathbb{P}(1,1,1,1,2,2)$ with variables $u, y, z, \beta, x, \alpha$, where

$$\begin{aligned} h = {} & \alpha(-u\alpha + 2(b_3 - 4u\beta a_1 + 4xa_1 + x\beta)) \\ & + 2\beta(c_4 - u\beta b_2 + 2xb_2 + 2x\beta a_1 + 2u^2\beta^2 a_0 - 6ux\beta a_0 + 6x^2 a_0) \\ & + x^2 B_Z + xC_Z + D_Z, \end{aligned}$$

where

$$B_Z = B_1(y,z,u), \qquad C_Z = C_3(y,z,u), \qquad D_Z = D_5(y,z,u).$$

The morphism $Y_2 \to Z$ given by the ample model of $\mathbb{V}(\beta)$ is a weighted blowdown with centre P_β and exceptional locus $\mathbb{V}(t)$. Substituting

$$x = u\beta - a_2 \tag{5.1}$$

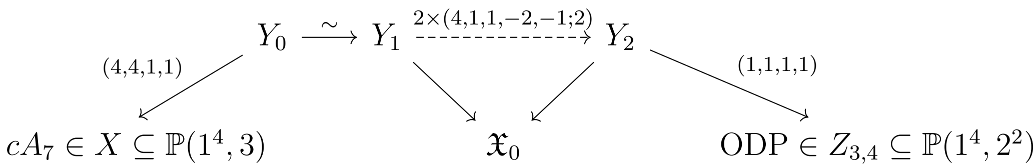
into h, we find that Z is isomorphic to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,1,2)$ with variables u, y, z, β, α. The substitution (5.1) does not lift onto Y_2. Instead, on the patch Z_β, we can substitute $u = (a_2 + x)/\beta$. This substitution lifts to $(Y_2)_\beta$. By Condition 5.1, $P_\beta \in Z$ is a cA_3 singularity and the hypersurface Z_β is given by a weight 4 polynomial. By Proposition 4.5, $(Y_2)_\beta \to Z_\beta$ is a $(3,1,1,1)$-Kawakita blowup.

Note that Z has an ordinary double point at $[u_0, 0, 0, 1, 2]$ for some $u_0 \in \mathbb{C}$. ☐

5.5. cA_7 family 1 model

Proposition 5.8. *A Mori fibre space sextic double solid with a cA_7 singularity in family 1 satisfying Condition 5.1 has a Sarkisov link to $Z_{3,4} \subseteq \mathbb{P}(1,1,1,1,2,2)$ with an ordinary double point, starting with a $(4,4,1,1)$-blowup of the cA_7 point, then two $(4,1,1,-2,-1;2)$-flips, and finally a blowdown (with standard weights $(1,1,1,1)$) to an ordinary double point. Under further generality conditions, Z has exactly five singular points, namely two $1/2(1,1,1)$ singularities and three ordinary double points.*

Proof. We exhibit the diagram below.



We construct X and a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$. We can write a sextic double solid X with an isolated cA_7 singularity in family 1 by

$$X\colon \mathbb{V}(f, \beta - xt - r_2, \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,2,3,3)$$

with variables $x, y, z, t, \beta, \gamma, w$, where

$$\begin{aligned} f = {} & -w^2 + \gamma^2 - 2t\gamma e_2 + 2\beta^2 e_2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4xt^2\beta b_1 \\ & + 2x^2t^4 b_0 - 16t\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 - 8\beta^3 a_0 + 12xt\beta^2 a_0 + xt^3 C_2 + t^2 D_4, \end{aligned}$$

where C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 2 \end{array}\right).$$

Define Y_0 by

$$Y_0\colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^8,\ u\beta - r_2 - xt,\ u\gamma - s_3 - x\beta).$$

The ample model of $\mathbb{V}(x) \subseteq Y_0$ is a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$.

We show that the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ induces an isomorphism $Y_0 \to Y_1$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

Define T_1, respectively T_2, with the same action as T_0 but with irrelevant ideal $(u,x,y,z) \cap (w,\gamma,\beta,t)$, respectively $(u,x,y,z,w,\gamma) \cap (\beta,t)$. Define $Y_1 \subseteq T_1$ and $Y_2 \subseteq T_2$ by the same ideal I_Y. The base of the flop $T_0 \to T_{\mathfrak{X}_0} \leftarrow T_1$ restricts to $\mathbb{V}(r_2, s_3) \subseteq \mathbb{P}^1 \subseteq \mathfrak{X}_0$, which is empty. Therefore, $Y_0 \to \mathfrak{X}_0$ and $\mathfrak{X}_0 \leftarrow Y_1$ are isomorphisms.

We show that the next diagram $Y_1 \to \mathfrak{X}_1 \leftarrow Y_2$ in the 2-ray game of Y_0 is locally analytically two $(4,1,1,-2,-1;2)$-flips. The only monomials in $\Phi^* f/u^8$ that are not in (u,x,y,z,β,t) are $-w^2$ and γ^2. Therefore, the base of the flip is two points, $[1,1]$ and $[-1,1] \in \mathbb{P}^1$ with variables w and γ inside $\mathfrak{X}_1$. We make a change of coordinates $w' = w - \gamma$, respectively $w' = w + \gamma$, for the flip above $[1,1]$, respectively $[-1,1]$. On the patch where γ is non-zero, we can substitute $u = s_3 + x\beta$ in $\Phi^* f/u^8$, and express w' locally analytically and equivariantly above $[1,1]$, respectively $[-1,1]$, in terms of x,y,z,β,t. After projecting away the variables u and w', we are left with the principal ideal $(\beta s_3 - r_2 + x\beta^2 - xt)$. Since it contains both r_2 and xt, by case (1) in [Bro99, Theorem 8], it is a terminal $(4,1,1,-2,-1;2)$-flip above both $[1,1]$ and $[-1,1]$. The flip contracts two curves, both containing a $1/4(1,1,3)$ singularity, and extracts two curves, both containing a $1/2(1,1,1)$ singularity and a cA_1 singularity. The cA_1 points are both ordinary double points if r_2 is not a square of a linear form, and are both 3-fold A_2 singularities (given by $x_1^2 + x_2^2 + x_3^2 + x_4^3$) otherwise. On Y_2, the cA_1 singularities are at $[0,0,0,0,1,1,1,1]$ and $[0,0,0,0,-1,1,1,1]$.

We show that the last map in the link for X is a divisorial contraction $Y_2 \to Z'$. Acting by the matrix $\left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ on the initial action-matrix of T_0, we see that

$$T_2 \cong \left(\begin{array}{cccccc|cc} u & x & y & z & w & \gamma & \beta & t \\ 2 & 3 & 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \end{array}\right).$$

Define $Z' \subseteq \mathbb{P}(1,1,1,1,2,2,2)$ with variables u,y,z,β,w,γ,x by the ideal $I_{Z'}$, where $I_{Z'}$ is the image of the ideal I_Y under the homomorphism $t \mapsto 1$. Let $Y_2 \to Z'$ be the ample model of $\mathbb{V}(\beta)$. On the affine patch Z'_β, we can express u and x locally analytically and equivariantly in terms of y,z,w,γ,β,t. This coordinate change lifts to Y_2. By Condition 5.1, we can compute that $P_\beta \in Z'$ is an ordinary double point, and $Y_2 \to Z'$ is locally analytically the (usual) blowup with centre P_β.

The variety Z' is isomorphic to a complete intersection $Z_{3,4} \subseteq \mathbb{P}(1^4, 2^2)$, by projecting away from x. The variety Z is given by

$$Z_{3,4}\colon \mathbb{V}(-s_3 + \beta r_2 + u\gamma - u\beta^2,\ h) \subseteq \mathbb{P}(1,1,1,1,2,2)$$

with variables u,y,z,β,w,γ, where

$$\begin{aligned} h = &-w^2 + \gamma^2 + 2b_0 r_2^2 - 4\beta b_1 r_2 - 4u\beta b_0 r_2 - 12\beta^2 a_0 r_2 - 2\gamma e_2 + 2\beta^2 e_2 + 2\beta c_3 + 4\gamma b_2 \\ &- 2\beta^2 b_2 + 2u\beta^2 b_1 + 2u^2\beta^2 b_0 - 16\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 + 4u\beta^3 a_0 + (u\beta - r_2)C_Z + D_Z, \end{aligned}$$

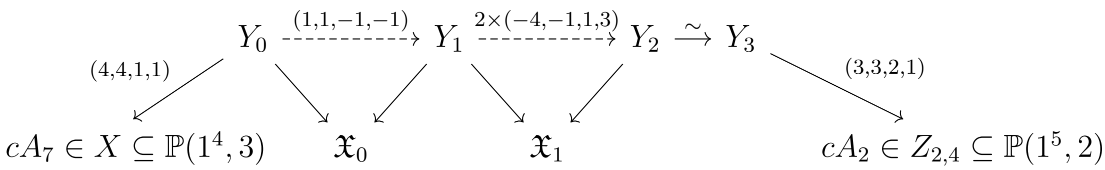
where $C_Z = C_2(y,z,u)$ and $D_Z = D_4(y,z,u)$. The variety Z has two cA_1 singularities at $[0,0,0,1,1,1]$ and $[0,0,0,1,-1,1]$. □

Remark 5.9. We explain how we found the variety X. Using $p_2 = r_2$ and $p_3 = s_3$, we can write a sextic double solid with an isolated cA_7 in family 1 by $\bar{X}\colon \mathbb{V}(\bar{f}, x^2t + xr_2 + s_3 - \bar{\gamma})$ inside $\mathbb{P}(1,1,1,1,3,3)$ with variables $x, y, z, t, w, \bar{\gamma}$, where $\bar{f}$ is given as in Theorem A. The $(1,1,4,4,2)$-blowup $\bar{Y}_0 \to \bar{X}$ for variables $y, z, w, \bar{\gamma}, t$ is a $(4,4,1,1)$-Kawakita blowup, but the 2-ray game of $\bar{Y}_0$ does not follow the ambient toric variety $\bar{T}_0$. Namely, the toric anti-flip $\bar{T}_0 \to \bar{T}_{\bar{\mathfrak{X}}_0} \leftarrow \bar{T}_1$ restricts to $\bar{Y}_0 \to \bar{\mathfrak{X}}_0 \leftarrow \bar{Y}_1$, where $\bar{Y}_0 \to \bar{\mathfrak{X}}_0$ is an isomorphism and $\bar{\mathfrak{X}}_0 \leftarrow \bar{Y}_1$ extracts $\mathbb{P}^2$, a divisor on $\bar{Y}_1$. The reason why $\bar{Y}_0$ was not the correct variety is that one of the generators of the ideal of $\bar{Y}_0$ is $\bar{g}_1 = x^2t + xr_2 + us_3 - u\bar{\gamma}$, which is inside the irrelevant ideal (u,x). We find the correct variety Y_0 by "unprojecting" $\bar{g}_1 = 0$ with respect to u, x. By "unprojection", we mean the coordinate change $\bar{Y}_0 \to Y_0$, an isomorphism. See [Rei00, Section 2] or [PR04, Section 2.3] for more details on this type of unprojection. This coordinate change induces the coordinate change $\bar{X} \to X$, where X is given as in the proof of Proposition 5.8.

5.6. cA_7 family 2 model

Proposition 5.10. *A Mori fibre space sextic double solid with a cA_7 singularity in family 2 satisfying Condition 5.1 has a Sarkisov link to a complete intersection $Z_{2,4} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with a cA_2 singularity, starting with a $(4,4,1,1)$-blowup of the cA_7 point, followed by one Atiyah flop, then two $(4,1,-1,-3)$-flips, and finally a $(3,3,2,1)$-blowdown to a cA_2 point. Under further generality conditions, the variety Z is smooth outside the cA_2 point.*

Proof. We exhibit the diagram below.



We describe the sextic double X. Define $X \subseteq \mathbb{P}(1,1,1,1,2,3,3,3)$ with variables x, y, z, t, β, w, γ, ξ by the ideal

$$I_X = (f - 2e_3\xi,\ \beta - q_1r_1 - xt,\ \gamma - q_1s_2 - x\beta,\ -\xi + ts_2 - \beta r_1), \tag{5.2}$$

where

$$f = -w^2 + \gamma^2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4xt^2\beta b_1 + 2x^2t^4 b_0$$
$$- 16t\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta\gamma a_1 - 8\beta^3 a_0 + 12xt\beta^2 a_0 + xt^3 C_2 + t^2 D_4$$

where C_i, $D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i.

We describe the weighted blowup $Y_0 \to X$, restriction of $\Phi\colon T_0 \to \mathbb{P}(1,1,1,1,2,3,3,3)$. Define T_0 by

$$T_0\colon \left(\begin{array}{cc|ccccccc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 5 & 2 \end{array}\right).$$

Define $Y_0 \subseteq T_0$ by the ideal I_Y with the 6 generators

$$g - 2e_3\xi, \quad u\beta - q_1r_1 - xt, \quad u\gamma - q_1s_2 - x\beta,$$
$$-u\xi + ts_2 - \beta r_1, \quad -x\xi + \beta s_2 - \gamma r_1, \quad -q_1\xi + t\gamma - \beta^2,$$

where $g = \Phi^* f/u^8$. On the affine patch X_x, we can express β, t and ξ in terms of w, γ, y, z, to get a hypersurface in $\mathbb{C}^4$ given by $f_{\text{hyp}} \in \mathbb{C}[w, \gamma, y, z]$. Note that these coordinate changes lift to $(Y_0)_x$. Since f_{hyp} has weight 8, by Proposition 4.5, $Y_0 \to X$ is a $(4, 4, 1, 1)$-Kawakita blowup.

We show that the first diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ in the 2-ray game of Y_0 is an Atiyah flop, provided that r_1 and q_1 are coprime in $\mathbb{C}[y, z]$. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$ on the action-matrix of T_0, define T_1 by

$$T_1 \colon \left(\begin{array}{cccc|ccccc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -3 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{array}\right).$$

Define $Y_1 \subseteq T_1$ by the ideal I_Y. The base of the flop is $\mathbb{V}(q_1) \subseteq \mathbb{P}^1$ with variables y, z, which is one point. Perform a suitable invertible linear coordinate change on y, z such that $q_1 = z$ and $r_1 = y$. Since $u\beta - q_1 r_1 - xt$ is in I_Y, we can substitute $z = u\beta - xt$ on the patch where y is non-zero. The coefficients of β in $-u\xi + ts_2 - \beta y \in I_Y$ and γ in $-x\xi + \beta s_2 - \gamma y \in I_Y$ are non-zero on the patch where y is non-zero. Therefore, we can locally analytically equivariantly express β and γ in terms of u, x, w, t. After substituting z, β, γ, we find that the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is locally analytically the Atiyah flop.

The next diagram in the 2-ray game of Y_0 is the flip $Y_1 \to \mathfrak{X}_1 \leftarrow Y_2$. The base of the flip is $\mathbb{V}(\gamma^2 - w^2) \subseteq \mathbb{P}^1$ with variables w, γ, which is two points $[1, 1]$ and $[-1, 1]$. We consider the point $P = [1, 1]$, the flip for the other point is similar. Perform a coordinate change $w' = w - \gamma$. On the patch where γ is non-zero, we find $u = q_1 s_2 + x\beta$ and $t = q_1\xi + \beta^2$. Writing $q_1 = z$ and $r_1 = y$ as before, we find $y = -x\xi + \beta s_2$. We are left with the principal ideal in $\mathbb{C}[x, z, w', \beta, \xi]$ generated by $-w'(2 + w') +$ terms not involving w'. So, we can locally analytically equivariantly express w' in terms of x, z, β, ξ. So, the diagram $Y_1 \to \mathfrak{X}_1 \leftarrow Y_2$ is locally analytically two $(-4, -1, 1, 3)$-flips.

The next diagram in the toric 2-ray game $T_2 \to T_{\mathfrak{X}_2} \leftarrow T_3$ restricts to isomorphisms $Y_2 \to \mathfrak{X}_2 \leftarrow Y_3$. The reason is that the base of the toric flip P_β restricts to an empty set in $\mathfrak{X}_2$, since I_Y contains the polynomial $t\gamma - \beta^2 - q\xi$.

We show that the last diagram in the 2-ray game of Y_0 is a divisorial contraction $Y_3 \to Z$. Multiplying the action-matrix of T_1 by $\left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)$, we see that T_3 is given by

$$T_3 \colon \left(\begin{array}{ccccccc|cc} u & x & y & z & w & \gamma & \beta & \xi & t \\ 3 & 5 & 2 & 2 & 3 & 3 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 \end{array}\right).$$

Consider the variety $Z \subseteq \mathbb{P}(1, 1, 1, 1, 1, 2, 2, 2)$ with variables $\xi, u, y, z, \beta, x, w, \gamma$ where $Y_3 \to Z$ is the ample model of $\mathbb{V}(\xi)$. On the patch Z_ξ, we can substitute $u = s_2 - \beta r_1$, $x = \beta s_2 - \gamma r_1$ and $z = \gamma - \beta^2$, and compute that Z_ξ is a hypersurface given by a weight 6 polynomial, with a cA_2 singularity at $P_\xi \in Z_\xi$, of type at least 2 (see Definition 4.7). These substitutions lift to $(Y_3)_\xi$, showing that $Y_3 \to Z$ is a $(3, 3, 2, 1)$-Kawakita blowup with centre P_ξ. If the coefficients are general, namely when

$$-2e_\beta + 8\beta^4 a_0 r_\beta - 2\beta^2 b_\beta + 12\beta^2 a_\beta^2 \in \mathbb{C}[y, \beta]$$

is not a full square, where $r_\beta = r_1(y, -\beta^2)$, $e_\beta = e_3(y, -\beta^2)$, $a_\beta = a_1(y, -\beta^2)$ and $b_\beta = b_2(y, -\beta^2)$, then the point P_ξ is exactly of type 2.

The variety Z is isomorphic to a complete intersection $Z_{2,4} \subseteq \mathbb{P}(1, 1, 1, 1, 1, 2)$ with variables u, y, z, β, ξ, w. We see this by substituting $x = u\beta - q_1 r_1$ and $\gamma = q_1\xi + \beta^2$. We find that Z is isomorphic to $Z_{2,4}\colon \mathbb{V}(-u\xi + s_2 - \beta r_1,\, h)$, where

$$\begin{aligned} h = &-w^2 + \xi^2 q_1^2 - 2e_3\xi + \beta^4 + 2b_0 q_1^2 r_1^2 - 4\beta b_1 q_1 r_1 - 4u\beta b_0 q_1 r_1 - 12\beta^2 a_0 q_1 r_1 + 4\xi b_2 q_1 \\ &- 16\xi a_1^2 q_1 + 4\beta\xi a_1 q_1 + 2\beta^2\xi q_1 + 2\beta c_3 + 2\beta^2 b_2 + 2u\beta^2 b_1 + 2u^2\beta^2 b_0 + 4\beta^3 a_1 + 4u\beta^3 a_0 \\ &+ (u\beta - q_1 r_1)C_Z + D_Z, \end{aligned}$$

where $C_Z = C_2(y,z,u)$ and $D_Z = D_4(y,z,u)$. □

Remark 5.11. We explain below how we found the embedding of X. Using Theorem A and the coordinate change in cA_7 family 1, we can write a sextic double solid $\bar{X}$ with an isolated cA_7 in family 2 by

$$\bar{X}\colon \mathbb{V}(f - 2e_3(ts_2 - \beta r_1),\, \beta - xt - q_1 r_1,\, \gamma - x\beta - q_1 s_2) \subseteq \mathbb{P}(1,1,1,1,2,3,3)$$

with variables $x, y, z, t, \beta, \gamma, w$.

We construct a $(4,4,1,1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define $\bar{T}_0$ by

$$\bar{T}_0\colon \left(\begin{array}{cc|cccccc} u & x & y & z & w & \gamma & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 2 \end{array}\right).$$

Let $T_0 \to \mathbb{P}(1,1,1,1,2,3,3)$ be the ample model of $\mathbb{V}(x)$ and $Y_0 \subseteq T_0$ the strict transform of X. Then $\bar{Y}_0$ is given by the ideal $I_{\bar{Y}} = (\bar{g}_1, \dots, \bar{g}_5)$, where

$$\begin{aligned} &\bar{g}_1 = ug + 2e_3(\beta r_1 - ts_2), \quad \bar{g}_2 = u\beta - q_1 r_1 - xt, \qquad \bar{g}_3 = u\gamma - q_1 s_2 - x\beta, \\ &\bar{g}_4 = xg + 2e_3(\gamma r_1 - \beta s_2), \quad \bar{g}_5 = q_1 g + 2e_3(\beta^2 - t\gamma)). \end{aligned}$$

The morphism $\bar{Y}_0 \to \bar{X}$ is a $(4,4,1,1)$-Kawakita blowup, as can be checked on the patch $(\bar{Y}_0)_x \to \bar{X}_x$.

Note that we do not prove that $I_{\bar{Y}}$ is saturated with respect to u. In fact, the saturation will not be I_Y if we do not use assume some generality conditions, similarly to cA_6 and cA_7 family 1. As a heuristic argument to see why $I_{\bar{Y}}$ might be saturated in the general case ("general" meaning a Zariski open dense set of the parameter space), we can use computer algebra software, pretend that a_i, b_i, c_i, d_i, q_1, r_1, s_2, e_3 are algebraically independent variables of a polynomial ring over $\mathbb{Q}$ or $\mathbb{Z}_p$ for a large prime p, and calculate that the saturation in that case indeed equals the ideal $I_{\bar{Y}}$.

Similarly to the diagram $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ in the proof of Proposition 5.10, the diagram $\bar{Y}_0 \to \bar{\mathfrak{X}}_0 \leftarrow \bar{Y}_1$ is an Atiyah flop, provided r_1 and q_1 are coprime.

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$, namely that the diagram $\bar{Y}_1 \to \bar{\mathfrak{X}}_1 \leftarrow Y_2$ contracts a curve and extracts a divisor. Acting by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ -1 & 1 \end{smallmatrix}\right)$ on the action-matrix of $\bar{T}_0$, define $\bar{T}_1$ by

$$\bar{T}_1\colon \left(\begin{array}{cccc|cccc} u & x & y & z & w & \gamma & \beta & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right),$$

and define $\bar{Y}_1 \subseteq \bar{T}_1$ by the zeros of $I_{\bar{Y}}$. We consider the toric flip $\bar{T}_1 \to \bar{T}_{\bar{\mathfrak{X}}_1} \leftarrow \bar{T}_2$ and restrict it to $\bar{Y}_1 \to \bar{\mathfrak{X}}_1 \leftarrow \bar{Y}_2$. Since $I_{\bar{Y}}$ is the zero ideal when restricting to $\mathbb{V}(u,x,y,z,\beta,t)$, the base $\mathbb{P}^1 \subseteq \bar{T}_{\bar{\mathfrak{X}}_1}$ of the toric flip restricts to $\mathbb{P}^1 \subseteq \bar{\mathfrak{X}}_1$ with variables w, γ. The morphism $\bar{Y}_1 \to \bar{\mathfrak{X}}_1$ contracts a curve $\mathbb{P}^1$ to both of the points $[1,1]$ and $[1,-1]$ in the base $\mathbb{P}^1 \subseteq \bar{\mathfrak{X}}_1$ and is an isomorphism elsewhere. The morphism $\bar{\mathfrak{X}}_1 \leftarrow \bar{Y}_2$ extracts a curve $\mathbb{P}^1$ for every point in the base $\mathbb{P}^1 \subseteq \bar{\mathfrak{X}}_1$, so extracts a divisor on $\bar{Y}_2$. The diagram $\bar{Y}_1 \to \bar{\mathfrak{X}}_1 \leftarrow \bar{Y}_2$ is not a step in the 2-ray game of $\bar{Y}_0$, so $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$. The reason for this was that the ideal $I_{\bar{Y}}$ is contained in (u, x, y, z), so the surface $\mathbb{V}(u, x, y, z) \subseteq \bar{T}_2$ exists on $\bar{Y}_2$, but does not exist on $\bar{T}_1$.

We "unproject" $\bar{g}_1 = \bar{g}_4 = \bar{g}_5 = 0$ with respect to u, x, y, z in $\bar{Y}_1 \subseteq \bar{T}_1$, to find a variety $Y_1 \subseteq T_1$. We explain below what we mean by this. We can write the system of equations $\bar{g}_1 = \bar{g}_4 = \bar{g}_5 = 0$ in the matrix form

$$\begin{pmatrix} g & 0 & 0 & \beta r_1 - t s_2 \\ 0 & g & 0 & \gamma r_1 - \beta s_2 \\ 0 & 0 & g & \beta^2 - t\gamma \end{pmatrix} \begin{pmatrix} u \\ x \\ q_1 \\ 2e_3 \end{pmatrix} = \mathbf{0}.$$

If the above equations hold, then we have

$$\frac{\left|\begin{pmatrix} 0 & 0 & \beta r_1 - t s_2 \\ g & 0 & \gamma r_1 - \beta s_2 \\ 0 & g & \beta^2 - t\gamma \end{pmatrix}\right|}{u} = \frac{\left|\begin{pmatrix} g & 0 & \beta r_1 - t s_2 \\ 0 & 0 & \gamma r_1 - \beta s_2 \\ 0 & g & \beta^2 - t\gamma \end{pmatrix}\right|}{-x} = \frac{\left|\begin{pmatrix} g & 0 & \beta r_1 - t s_2 \\ 0 & g & \gamma r_1 - \beta s_2 \\ 0 & 0 & \beta^2 - t\gamma \end{pmatrix}\right|}{q_1} = \frac{\left|\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix}\right|}{-2e_3}.$$

Calculating the determinants and dividing by $-g^2$, we find the equalities

$$\frac{t s_2 - \beta r_1}{u} = \frac{\beta s_2 - \gamma r_1}{x} = \frac{t\gamma - \beta^2}{q_1} = \frac{g}{2e_3}, \tag{5.3}$$

between elements of the field of fractions of $\mathbb{C}[u, x, y, z, w, \gamma, \beta, t]/I_{\bar{Y}}$. Using the Equations 5.3, we see that the morphism $\bar{Y}_1 \to Y_1$ given by

$$[u, x, y, z, w, \gamma, \beta, t] \mapsto [u, x, y, z, w, \gamma, \beta, \frac{t s_2 - \beta r_1}{u}, t]$$

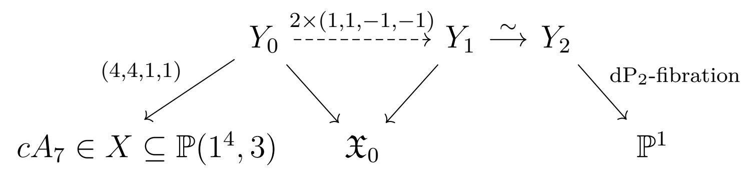
is an isomorphism, where Y_1 is described in the proof of Proposition 5.10.

The coordinate change $\bar{Y}_1 \to Y_1$ induces an isomorphism $\bar{X} \to X$, giving the variety X.

5.7. cA_7 family 3 model

Proposition 5.12. *A Mori fibre space sextic double solid with a cA_7 singularity in family 3 satisfying Condition 5.1 has a Sarkisov link to a degree 2 del Pezzo fibration, starting with a $(4, 4, 1, 1)$-blowup of the cA_7 point and followed by two Atiyah flops.*

Proof. We exhibit the diagram below.



First, we define X and a $(4, 4, 1, 1)$-Kawakita blowup $Y_0 \to X$. Any sextic double solid with an isolated cA_7 family 3 can be given by a bidegree $(6, 2)$ complete intersection

$$X\colon \mathbb{V}(f, -\xi + t s_1 - q_2 - xt) \subseteq \mathbb{P}(1, 1, 1, 1, 2, 3)$$

with variables x, y, z, t, ξ, w, where

$$\begin{aligned} f = &-w^2 + x^2\xi^2 - 2\xi e_4 + \xi^2(s_1^2 + 4a_1s_1 + 2xs_1 - 2b_2 + 16a_1^2 + 4xa_1 + 8\xi a_0) \\ &+ t(ts_1^4 + 4ta_1s_1^3 - 8t^2a_0s_1^3 - 2\xi s_1^3 + 2tb_2s_1^2 - 2t^2b_1s_1^2 - 8\xi a_1s_1^2 + 24t\xi a_0s_1^2 \\ &+ 12xt^2a_0s_1^2 - 2x\xi s_1^2 + 2tc_3s_1 + 4t\xi b_1s_1 + 4xt^2b_1s_1 - 16\xi a_1^2s_1 - 4x\xi a_1s_1 \\ &- 24\xi^2a_0s_1 - 24xt\xi a_0s_1 - 2\xi c_3 - 4x\xi b_2 - 2\xi^2b_1 - 4xt\xi b_1 + 2x^2t^3b_0 + 16x\xi a_1^2 \\ &+ 12x\xi^2a_0 + xt^2C_2 + tD_4), \end{aligned}$$

where C_i, $D_i \in \mathbb{C}[y, z, t]$ are homogeneous of degree i. Define

$$T_0 \colon \left(\begin{array}{cc|ccccc} u & x & y & z & w & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 2 \end{array}\right).$$

Define $\Phi \colon T_0 \to \mathbb{P}(1,1,1,1,2,3)$ by the ample model of $\mathbb{V}(x)$, and define Y_0 as the strict transform of X. Then, Y_0 is given by

$$Y_0 \colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^8,\ -u^2\xi + uts_1 - q_2 - xt),$$

Using Proposition 4.5, we see that $Y_0 \to X$ is a $(4,4,1,1)$-Kawakita blowup.

We describe the flop $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$. Multiplying the action-matrix of T_0 by $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, we find

$$T_0 \cong \left(\begin{array}{cc|ccccc} u & x & y & z & w & \xi & t \\ 1 & 1 & 0 & 0 & -1 & -2 & -1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 2 \end{array}\right).$$

The base of the flop is given by $\mathbb{V}(q_2) \subseteq \mathbb{P}^1 \subseteq \mathfrak{X}_0$. After a suitable coordinate change on y, z, we find $q_2 = yz$. Consider the flop over $\mathbb{V}(y)$, the flop over the other point is similar. Since q_2 and e_4 have no common divisor, on the patch where z is non-zero, we can express y and ξ locally analytically equivariantly in terms of u, x, t, w. So, $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$ is locally analytically two Atiyah flops.

The morphisms $Y_1 \to \mathfrak{X}_1 \leftarrow Y_2$ are isomorphisms, since w^2 has a non-zero coefficient in $\Phi^* f/u^8$.

We show that Y_2 is a degree 2 del Pezzo fibration. Multiplying the original action-matrix of T_0 by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 2 & -1 \end{smallmatrix}\right)$ with determinant -1, we find

$$T_2 \colon \left(\begin{array}{ccccc|cc} u & x & y & z & w & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 & 0 & 0 \end{array}\right).$$

The ample model of $\mathbb{V}(t)$ is

$$\begin{aligned} Y_2 &\to \mathbb{P}(2,1) \\ [u, x, y, z, w, \xi, t] &\mapsto [\xi, t]. \end{aligned}$$

Since $\mathbb{P}(2,1)$ is isomorphic to $\mathbb{P}^1$, we see that Y_2 is a fibration onto $\mathbb{P}^1$. On the patch $(Y_2)_t$, we can substitute $x = us_1 - q_2 - u^2\xi$, to find that the general fibre is a weighted degree 4 hypersurface in $\mathbb{P}(1,1,1,2)$, so a degree 2 del Pezzo surface. ☐

5.8. cA_8 model

Proposition 5.13. *A Mori fibre space sextic double solid with a cA_8 singularity satisfying Condition 5.1 has a Sarkisov link to a complete intersection $Z_{3,3} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with a cD_4 singularity, starting with a $(5,4,1,1)$-blowup of the cA_8 point, followed by a $(4,1,1,-1,-2;2)$-flip, and finally a $(3,2,2,1,5)$-blowdown to the cD_4 singularity. Under further generality conditions, the singular locus of Z consists of 3 points, namely the cD_4 point, the $1/2(1,1,1)$ singularity and an ordinary double point.*

Proof. We exhibit the diagram below.

$$\begin{array}{ccccccccc} & & Y_0 & \xrightarrow{\sim} & Y_1 & \overset{(4,1,1,-1,-2;2)}{\dashrightarrow} & Y_2 & \xrightarrow{\sim} & Y_3 & & \\ & \swarrow^{(1,1,4,5)} & & & & \searrow \quad \swarrow & & & & \searrow^{(3,2,2,1,5)} & \\ cA_8 \in X_6 \subseteq \mathbb{P}(1^4,3) & & & & & \mathfrak{X}_1 & & & & & cD_4 \in Z_{3,3} \subseteq \mathbb{P}(1^5,2) \end{array}$$

First, we describe X and the weighted blowup $Y_0 \to X$. A sextic double solid with a cA_8 singularity can be given by a multidegree $(6,2,3)$ complete intersection

$$X \colon \mathbb{V}(f,\ \beta - xt - r_2,\ \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,2,3,3),$$

with variables $x, y, z, t, \beta, \gamma, \xi$ where

$$\begin{aligned} f = {} & 8\beta^3(A_0 - a_0) + \xi(-\xi + 2\gamma - 8tA_0r_2 + 2tb_2 - 4ta_1^2 + 4\beta a_1) \\ & + t(-16t\beta A_0^2 r_2 + 2t\beta c_2 + 4t\gamma b_1 - 2\beta^2 b_1 - 2t\beta^2 b_0 + 4xt^2\beta b_0 - 8t\gamma a_0 a_1 + 8\beta^2 a_0 a_1 \\ & + 12\beta\gamma a_0 - 2t\gamma B_1 + 2\beta^2 B_1 + 16t\beta^2 A_0^2 - 16xt^2\beta A_0^2 - 8\beta\gamma A_0 + xt^3 C_1 + t^2 D_3) \end{aligned}$$

where $C_i,\ D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree i. Note that $B_1 \in \mathbb{C}[y,z]$. Define

$$T_0 \colon \left(\begin{array}{cc|cccccc} u & x & y & z & \gamma & \beta & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 & 4 & 3 & 5 & 2 \end{array}\right).$$

Let $\Phi \colon T_0 \to \mathbb{P}(1,1,1,1,2,3,3)$ be the ample model of $\mathbb{V}(x)$ and let $Y_0 \subseteq T_0$ be the strict transform of X. Then Y_0 is given by

$$Y_0 \colon \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = \left(\Phi^* f/u^9,\ u\beta - xt - r_2,\ u\gamma - x\beta - s_3\right),$$

and $Y_0 \to X$ is a $(5,4,1,1)$-Kawakita blowup.

The first diagram in the 2-ray game of T_0 restricts to a isomorphisms $Y_0 \to \mathfrak{X}_0 \leftarrow Y_1$, since r_2 and s_3 are coprime.

The second diagram in the 2-ray game of T_0 restricts to a $(4,1,1,-1,-2;2)$-flip $Y_1 \to \mathfrak{X}_1 \leftarrow Y_2$. Define the toric variety T_1 by multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 4 & -3 \\ 3 & -2 \end{smallmatrix}\right)$,

$$T_1 \colon \left(\begin{array}{cccc|cccc} u & x & y & z & \gamma & \beta & \xi & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -3 & -2 \\ 2 & 3 & 1 & 1 & 1 & 0 & -1 & -1 \end{array}\right).$$

On the patch where γ is non-zero, we have $u = x\beta + s_3$ and we can write ξ locally analytically equivariantly in terms of x, y, z, β, t. We are left with the hypersurface given by $x\beta^2 + \beta s_3 - xt - r_2$ in $\mathbb{C}^5$ with variables x, y, z, β, t with weights $(4,1,1,-1,-2)$. The polynomial contains xt and r_2, so this corresponds to case (1) in [Bro99, Theorem 8], a $(4,1,1,-1,-2;2)$-flip. Similarly to Proposition 5.8, the flip contracts a curve containing a $1/4(1,1,3)$ singularity, and extracts a curve containing a $1/2(1,1,1)$ singularity and a cA_1 singularity, which is an ordinary double point if r_2 is not a square and is a 3-fold A_2 singularity otherwise. The cA_1 singularity on Y_2 is at $[0,0,0,0,1,1,-2a_0,1]$.

The third diagram in the 2-ray game of T_0 restricts to isomorphisms $Y_2 \to \mathfrak{X}_2 \leftarrow Y_3$, under Condition 5.1, namely that $a_0 \neq A_0$. On the patch where β is non-zero, the base of the toric flip restricts to $\mathbb{V}(A_0 - a_0, u, x, y, z, \gamma, \xi, t) \subseteq \mathfrak{X}_2$.

We describe the weighted blowdown $Y_3 \to Z$. Multiplying the action matrix of T_0 by the matrix $\left(\begin{smallmatrix} 5 & -3 \\ 2 & -1 \end{smallmatrix}\right)$, the toric variety T_3 is given by

$$T_3\colon \begin{array}{c} \\ \left(\begin{array}{cccccc|cc} u & x & y & z & \gamma & \beta & \xi & t \\ 3 & 5 & 2 & 2 & 3 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 0 \end{array}\right). \end{array}$$

The ample model of $\mathbb{V}(\xi)$ is $Y_3 \to Z$ where Z is the tridegree $(3,2,3)$ complete intersection

$$Z\colon \mathbb{V}(h,\ u\beta - x - r_2,\ u\gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,1,2,2)$$

with variables $u, y, z, \beta, \xi, x, \gamma$, where

$$\begin{aligned} h &= 8\beta^3(A_0 - a_0) + \xi(-u\xi + 2\gamma - 8A_0 r_2 + 2b_2 - 4a_1^2 + 4\beta a_1) \\ &\quad - 16\beta A_0^2 r_2 + 2\beta c_2 + 4\gamma b_1 - 2\beta^2 b_1 - 2u\beta^2 b_0 + 4x\beta b_0 - 8\gamma a_0 a_1 + 8\beta^2 a_0 a_1 \\ &\quad + 12\beta\gamma a_0 - 2\gamma B_1 + 2\beta^2 B_1 + 16u\beta^2 A_0^2 - 16x\beta A_0^2 - 8\beta\gamma A_0 + xC_Z + D_Z \end{aligned}$$

where $C_Z = C_1(y,z,u)$ and $D_Z = D_3(y,z,u)$. Substituting $x = u\beta - r_2$, we see that Z is isomorphic to a complete intersection of bidegree $(3,3)$ in $\mathbb{P}(1^5, 2)$ with variables $u, y, z, \beta, \xi, \gamma$. The variety Z has a cA_1 singularity at $[0,0,0,1,-2a_0,1]$. We can compute that the point $P_\xi \in Z$ is a cD_4 point, by showing the complex analytic space germ (Z, P_ξ) is isomorphic to $(\mathbb{V}(u^2 + 2\beta r_2 - s_3 + \text{h.o.t}), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$ with variables u, β, y, z, where h.o.t are higher order terms in y, z, β. We can compute that $Y_3 \to Z$ is the divisorial contraction to a cD_4 point described in [Yam18, Theorem 2.3]. □

A. Computer code

The code below is for the computer algebra system Maxima [Max18]. To use the splitting lemma library, copy the code below to a file named "Splitting lemma.mac", start Maxima in the same folder as that file, and load the library using `load("Splitting lemma.mac");`. Alternatively, just copy-paste the code below to Maxima.

Listing 1. Splitting lemma library

```
/* Language: Maxima 5.42.1 */
splitting(str, poly, inDeg, splitVar, dummyVar, outDeg) := block(
  /* Assume f[0, 0] = f[1, 0] = f[0, 1] = f[1, 1] = 0 and f[2, 0] = 1 */
  [simpPoly, outFun, outPoly],
  local(f, h, g, p, v),
  simpPoly : ratexpand(poly),
  /* Memoizing functions f[i, d] instead of f(i, d) for performance */
  f[i, d] := coeff(coeff(simpPoly, splitVar, i), dummyVar, inDeg-i-d),
```

```
  /* Use apply + makelist instead of sum to avoid dynamic scoping issues */
  h[d] := ratexpand(
    f[0, d] - apply("+", makelist(g[0, j]*g[0, d-j], j, 2, d-2))
  ),
  g[i, d] := if i = 1 and d = 0 then
      0
    else
      ratexpand(1/2*(f[i+1, d] - apply("+", makelist(apply("+", makelist(
        g[j, k]*g[i+1-j, d-k], j, max(0, 2-k), min(i+1, i+d-k-1)
      )), k, 0, d)))),
  p[d] := ratexpand(g[0, d] - apply("+", makelist(v[0, d-j]*p[j], j, 2, d-1))),
  v[i, d] := if i = 0 and d = 0 then
      1
    else
      ratexpand(g[i+1, d] - apply("+", makelist(v[i+1, d-j]*p[j], j, 2, d))),
  for i : 1 thru 4 do
    if str = ["h", "g", "p", "v"][i] then outFun : [h, g, p, v][i],
  outPoly : if member(str, ["h", "p"]) then
      apply("+", makelist(dummyVar^(outDeg-d)*outFun[d], d, 0, outDeg))
    else if member(str, ["g", "v"]) then
      apply("+", makelist(apply("+", makelist(
        dummyVar^(outDeg-k) * splitVar^i * outFun[i, k-i], i, 0, k
      )), k, 0, outDeg))
    else
      "Splitting error: first argument must be 'h', 'g', 'p' or 'v'.",
  return(outPoly)
);
```

We give an example below how to use the splitting lemma library.

Listing 2. Splitting lemma example — quartic surface

```
/*
 * Language: Maxima 5.42.1
 *
 * Example of a quartic surface in projective space with an
 * A_{19} singularity. We use the splitting lemma twice to verify that
 * the singularity type is A_{19}, so it is locally analytically given
 * by x^2 + y^2 + z^20.
 *
 * The quartic polynomial is taken from M.~Kato, I.~Naruki, \emph{Depth
 * of rational double points on quartic surfaces}, Proc.~Japan
 * Acad.~Ser.~A Math.~Sci.~\textbf{58} (1982), no 2, p 72--75.
 * doi:10.3792/pjaa.58.72,
 * \url{https://projecteuclid.org/euclid.pja/1195516147}.
 *
 * Here t is the dummy homogenizing variable, x and y are the splitting
 * variables. We check the singularity type of the point [0, 0, 0, 1].
 */

load("Splitting lemma.mac");
f : 1/16*(16*(x^2 + y^2)*t^2 + 32*x*z^2*t - 16*y^3*t + 16*z^4 - 32*y*z^3
  + 8*(2*x^2 - 2*x*y + 5*y^2)*z^2 + 8*(2*x^3 - 5*x^2*y - 6*x*y^2 - 7*y^3)*z
```

```
  + 20*x^4 + 44*x^3*y + 65*x^2*y^2 + 40*x*y^3 + 41*y^4);
splitQuartic(poly, outDeg) := block(
  [splitPoly],
  splitPoly : splitting("h", poly, 4, x, t, outDeg),
  return(subst(1, t, splitting("h", splitPoly, outDeg, y, t, outDeg)))
);
/* Output: 0 */
splitQuartic(f, 19);
/* Output: z^20 */
splitQuartic(f, 20);
```

We use the code below in Section 3 to find the equations of sextic double solids with a cA_n singularity.

Listing 3. Construct sextic double solids with a cAn singularity

```
/* Language: Maxima 5.42.1 */
load("Splitting lemma.mac");
splitSDS(poly, n) := subst(1, x, splitting("h", poly + w^2, 6, t, x, n));
fGen : -w^2 + x^4*t^2
  + x^3*(4*t^3*a_0 + 4*t^2*a_1 + 2*t*a_2 + a_3)
  + x^2*(2*t^4*b_0 + 2*t^3*b_1 + 2*t^2*b_2 + 2*t*b_3 + b_4)
  + x*(2*t^5*c_0 + 2*t^4*c_1 + 2*t^3*c_2 + 2*t^2*c_3 + 2*t*c_4 + c_5)
  + t^6*d_0 + 2*t^5*d_1 + t^4*d_2 + 2*t^3*d_3 + t^2*d_4 + 2*t*d_5 + d_6;
h_3 = splitSDS(fGen, 3);
sub3(poly) := ratexpand(subst(0, a_3, poly));
h_4 = splitSDS(sub3(fGen), 4);
sub4(poly) := ratexpand(subst(a_2^2, b_4, sub3(poly)));
h_5 = splitSDS(sub4(fGen), 5);
sub5(poly) := ratexpand(subst(2*a_2*b_3 - 4*a_1*a_2^2, c_5, sub4(poly)));
h_6 = splitSDS(sub5(fGen), 6);
sub6(poly) := ratexpand(subst(2*a_2*c_4 + b_3^2 - 8*a_1*a_2*b_3 - 2*a_2^2*b_2
  + 4*a_0*a_2^3 + 16*a_1^2*a_2^2, d_6, sub5(poly)));
h_7 = splitSDS(sub6(fGen), 7);
sub7(poly) := ratexpand(
  subst(q*r, a_2,
  subst(q*s + 4*a_1*q*r, b_3,
  subst(2*a_1*q*s - 6*a_0*q^2*r^2 + 8*a_1^2*q*r + e*r, c_4,
  subst(2*b_2*q*s - 8*a_1^2*q*s - e*s - b_1*q^2*r^2 + c_3*q*r, d_5,
  sub6(poly)))))
);
sub71(poly) := ratexpand(subst(1, q, subst(r_2, r, subst(s_3, s, subst(e_2, e,
  sub7(poly))))));
h_8Family1 = splitSDS(sub71(fGen), 8);
sub72(poly) := ratexpand(subst(q_1, q, subst(r_1, r, subst(s_2, s,
  subst(e_3, e, sub7(poly))))));
h_8Family2 = splitSDS(sub72(fGen), 8);
sub73(poly) := ratexpand(subst(1, r, subst(q_2, q, subst(s_1, s, subst(e_4, e,
  sub7(poly))))));
h_8Family3 = splitSDS(sub73(fGen), 8);
sub74(poly) := ratexpand(subst(1, s, subst(0, r, subst(q_3, q, subst(e_5, e,
  sub7(poly))))));
h_8Family4 = splitSDS(sub74(fGen), 8);
```

```
sub8(poly) := ratexpand(
  subst(4*A_0*r_2 + b_2 - 6*a_1^2, e_2,
  subst(r_2*B_1 - 4*s_3*A_0 + 6*a_0*s_3 + 4*a_0*a_1*r_2 - 2*a_1*e_2 + 4*a_1*b_2
    - 16*a_1^3, c_3,
  subst(-2*s_3*B_1 + 16*r_2^2*A_0^2 - 8*b_2*r_2*A_0 + 16*a_1^2*r_2*A_0
    + 4*b_1*s_3 - 8*a_0*a_1*s_3- 2*b_0*r_2^2 + 2*c_2*r_2 + b_2^2 - 4*a_1^2*b_2
    + 4*a_1^4, d_4,
  sub71(poly))))
);
h_9 = splitSDS(sub8(fGen), 9);
```

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