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BOUNDARY INTEGRAL OPERATORS FOR THE HEAT EQUATION IN TIME-DEPENDENT DOMAINS

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Abstract. This article provides a functional analytical framework for boundary integral equations of the heat equation in time-dependent domains. More specifically, we consider a non-cylindrical domain in space-time that is the C^2-diffeomorphic image of a cylinder, i.e., the tensor product of a time interval and a fixed domain in space. On the non-cylindrical domain, we introduce Sobolev spaces, trace lemmata and provide the mapping properties of the layer operators by mimicking the proofs of Costabel [3]. Here it is critical that the Neumann trace requires a correction term for the normal velocity of the moving boundary. Therefore, one has to analyze the situation carefully.

1. Introduction

Boundary integral equations are a well-known technique to solve elliptic partial differential equations, see for example [15, 16]. For parabolic equations on *time-independent*, so-called *cylindrical* domains, Sobolev spaces and the mapping properties of the layer operators for the heat equation are introduced in [3, 14]. To the best of our knowledge, no such theory exists for *time-dependent* or so-called *non-cylindrical* domains, or simply, *tube*. Therefore, the aim of this article is to extend the theory from cylindrical domains to non-cylindrical domains.

To that end, we consider a special class of non-cylindrical domains. We fix a cylindrical domain which serves as a reference domain and define the non-cylindrical domain as the image under a time-dependent C^2-diffeomorphism. In this setting, different approaches are possible to establish analogue integral equations and properties to integral operators on a cylindrical domain.

A first approach could be to exploit the fact that the fundamental solution does not use boundary data and is thus defined on the free space $\mathbb{R} \times \mathbb{R}^d$. Therefore, it is the same for a cylindrical and a non-cylindrical domain and allows to state the integral operators in cylindrical and non-cylindrical domains. To establish the mapping properties of the integral operators, one could make use of the equivalence of norms on the tube and on the cylindrical domain by establishing equivalence results of the fundamental solution, evaluated on the tube and on the cylindrical domain. The problem is that the Neumann trace, which will be considered here, contains an additional term involving the normal velocity of the tube. Therefore, one has to come up with a solution how to deal with this.

A second approach could be to map back the partial differential equation from the non-cylindrical domain onto the cylindrical domain. The advantage is that one now considers a cylindrical domain,

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for which more theory is available. The drawback is that the differential equation in the reference domain is more complicated because of time and space dependent coefficients. Finding a fundamental solution is more difficult and one could for example pursue the parametrix ansatz, taken in [7], and then find the according mapping properties of the respective layer operators.

A third approach considers the partial differential equation on the non-cylindrical domain. Here, the partial differential equation is simple, but the domain is more involved in contrast to the second approach. This approach was used in [17], but without the corresponding Sobolev spaces and mapping properties of the integral operators. Since we already did computations in [1] based on [17], and since [3] provides a self-contained analysis of the mapping properties of the layer operators for the heat equation in a cylindrical domain, we choose this approach for this article.

Although we follow the argumentation line of Costabel [3], we repeat the proofs here in the non-cylindrical setting for the reader's convenience, since we have to use the appropriate function spaces and the correct Neumann traces. We would like to emphasize that, once one has the appropriate Neumann trace operator at hand and the mapping properties of the trace operators, the ideas of [3] can be followed directly. We indicate the needed adaptations in the article.

The remainder of this article is organized as follows: In Section 2, we introduce anisotropic Sobolev spaces on cylindrical and non-cylindrical domain, which are used for the mapping properties. Section 3 is dedicated to the Dirichlet traces and the existence and uniqueness of solutions of the Dirichlet problem. In Section 4, we have a look at the appropriate Neumann trace. The main result is presented in Section 5, where we establish the mapping properties of the integral operators and the existence and uniqueness of solutions of the Neumann problem. In Section 6, we state some concluding remarks.

2. Anisotropic Sobolev spaces

In order to study the heat equation, we shall introduce appropriate anisotropic Sobolev spaces on cylindrical domains. From these spaces, we will then derive Sobolev spaces on time-dependent domains.

2.1. **Anisotropic Sobolev spaces on cylindrical domains.** Let $\Omega_0 \subset \mathbb{R}^d$, $d \geq 2$, be a Lipschitz domain in the spatial variable with boundary $\Gamma_0 := \partial\Omega_0$ and let $0 < T < \infty$. Then, the product set $Q_0 := (0,T) \times \Omega_0 \subset \mathbb{R}^{d+1}$ forms a time-space cylinder with the lateral boundary $\Sigma_0 := (0,T) \times \Gamma_0$. The appropriate function spaces for parabolic problems in time invariant domains, i.e. in cylindrical domains, are the anisotropic Sobolev spaces defined by

$$H^{r,s}(Q_0) := L^2\big((0,T); H^r(\Omega_0)\big) \cap H^s\big((0,T); L^2(\Omega_0)\big)$$

for $r, s \in \mathbb{R}_{\geq 0}$, see, e.g., [2, 3, 11]. The corresponding boundary spaces are

$$H^{r,s}(\Sigma_0) := L^2\big((0,T); H^r(\Gamma_0)\big) \cap H^s\big((0,T); L^2(\Gamma_0)\big)$$

Note that these spaces are well-defined for $r \leq 1$ (while $s \geq 0$ is arbitrary) if Γ_0 is Lipschitz.

Remark 1. *The space $H^{r,s}(Q_0)$ consists of all functions $u \in L^2(Q_0)$, where the $L^2(Q_0)$-norm of the partial derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_t^{\beta} u(t, \mathbf{x})$ is finite for all $|\boldsymbol{\alpha}| \leq \lambda r$, $\beta \leq (1-\lambda)s$, and $\lambda \in [0,1]$.*

With these definitions at hand, we can moreover define spaces for functions with zero initial condition by setting

$$H^{r,s}_{;0,}(Q_0) := L^2\big((0,T); H^r(\Omega_0)\big) \cap H^s_{0,}\big((0,T); L^2(\Omega_0)\big),$$

where

$$H^s_{0,}\big((0,T); L^2(\Omega_0)\big) := \Big\{ u = \tilde{u}|_{(0,T)} : \tilde{u} \in H^s\big((-\infty, T); L^2(\Omega_0)\big) : \tilde{u}(t) = 0 \text{ for } t < 0 \Big\}.$$

Note that we adopted the notation from [4, 5]. In addition, we can define functions which vanish at $t = T$ by setting

$$H^{r,s}_{;,0}(Q_0) := L^2\big((0,T); H^r(\Omega_0)\big) \cap H^s_{,0}\big((0,T); L^2(\Omega_0)\big),$$

where in complete analogy

$$H^s_{,0}\big((0,T); L^2(\Omega_0)\big) := \Big\{ u = \tilde{u}|_{(0,T)} : \tilde{u} \in H^s\big((0,\infty); L^2(\Omega_0)\big) : \tilde{u}(t) = 0 \text{ for } t > T \Big\}.$$

As in the elliptic case, we can also include (spatial) zero boundary conditions into the function spaces by setting

$$H^{r,s}_{0;0,}(Q_0) := L^2\big((0,T); H^r_0(\Omega_0)\big) \cap H^s_{0,}\big((0,T); L^2(\Omega_0)\big),$$
$$H^{r,s}_{0;,0}(Q_0) := L^2\big((0,T); H^r_0(\Omega_0)\big) \cap H^s_{,0}\big((0,T); L^2(\Omega_0)\big),$$

where the spaces include zero initial and end conditions, respectively. On the boundary, we introduce

$$H^{r,s}_{;0,}(\Sigma_0) := L^2\big((0,T); H^r(\Gamma_0)\big) \cap H^s_{0,}\big((0,T); L^2(\Gamma_0)\big),$$
$$H^{r,s}_{;,0}(\Sigma_0) := L^2\big((0,T); H^r(\Gamma_0)\big) \cap H^s_{,0}\big((0,T); L^2(\Gamma_0)\big).$$

These spaces are the closures of $H^{r,s}(\Sigma_0)$ for zero start and end condition, respectively, compare [4, Section 2.3].

By duality we have

$$H^{-r,-s}_{;0,}(Q_0) = \big[H^{r,s}_{0;,0}(Q_0)\big]' \quad \text{for } r - \frac{1}{2} \notin \mathbb{Z}$$

according to [3]. The anisotropic Sobolev spaces on the boundary with negative smoothness index are defined by

$$H^{-r,-s}_{;,0}(\Sigma_0) := \big[H^{r,s}_{;0,}(\Sigma_0)\big]',$$
$$H^{-r,-s}_{;0,}(\Sigma_0) := \big[H^{r,s}_{;,0}(\Sigma_0)\big]',$$
$$\widetilde{H}^{-r,-s}(\Sigma_0) := \big[H^{r,s}(\Sigma_0)\big]',$$

see [4, Section 2.3]. Moreover, according to [4, Remark 2.1], for $r \geq 0$ and $0 \leq s < \frac{1}{2}$ it holds $H^{r,s}(\Sigma_0) = H^{r,s}_{;0,}(\Sigma_0) = H^{r,s}_{;,0}(\Sigma_0)$ and, therefore, the above introduced dual spaces are equal and we simply write $H^{-r,-s}(\Sigma_0)$.

Remark 2. *We would like to clarify the intuition behind the slightly cumbersome notation. In* $H^{r,s}_{0;,}(Q_0)$*, a zero before the semicolon indicates a zero boundary condition in space. After the semicolon, a zero initial condition can be indicated by writing a zero between the semicolon and the comma. Whereas, a present zero after the comma stands for a zero end condition. Thus, this notation allows to see the spatial and temporal boundary condition at one glance.*

2.2. **Anisotropic Sobolev spaces on non-cylindrical domains.** Having at hand the Sobolev spaces defined on cylindrical domains, we can also introduce Sobolev spaces on non-cylindrical domains. Non-cylindrical domains consist of a spatial domain, which we denote by Ω_t. The subscript t indicates that the spatial domain might differ for every point of time. To obtain a non-cylindrical domain Q_T we set

$$Q_T := \bigcup_{0<t<T} \big(\{t\} \times \Omega_t\big).$$

This domain has a lateral boundary Σ_T defined by

$$\Sigma_T := \bigcup_{0<t<T} \big(\{t\} \times \Gamma_t\big),$$

where $\Gamma_t := \partial\Omega_t$.

For every point of time t, we assume to have a smooth diffeomorphism $\boldsymbol{\kappa}$, which maps the initial domain Ω_0 onto the time-dependent domain Ω_t. In accordance with [13], we write

$$\boldsymbol{\kappa}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \quad (t,\mathbf{x}) \mapsto \boldsymbol{\kappa}(t,\mathbf{x}) \tag{2.1}$$

to emphasize the dependence of the mapping $\boldsymbol{\kappa}$ on the time, where we have $\boldsymbol{\kappa}(t,\Omega_0) = \Omega_t$. Especially, Ω_t is also a Lipschitz domain for all $t \in [0,T]$.

The domains Ω_t each have a spatial normal $\mathbf{n}_t$, which we will also denote by $\mathbf{n}$ if it is clear from the context. Besides having a spatial normal, we also have a time-space normal $\boldsymbol{\nu}$. We can write the time-space normal as

$$\boldsymbol{\nu} = \frac{1}{\sqrt{1+v_{\boldsymbol{\nu}}^2}} \begin{bmatrix} v_{\boldsymbol{\nu}} \\ \mathbf{n} \end{bmatrix} \tag{2.2}$$

for some appropriate $v_{\boldsymbol{\nu}} \in \mathbb{R}$. According to [6], it holds

$$v_{\boldsymbol{\nu}} = -\langle \mathbf{V}, \mathbf{n} \rangle$$

for the vector field $\mathbf{V}$, which deforms the cylinder Q_0 into the tube Q_T and for which the relation $\mathbf{V} = \partial_t\boldsymbol{\kappa} \circ \boldsymbol{\kappa}^{-1}$ holds.

We introduce the non-cylindrical analogues of the Sobolev spaces by setting

$$H^{r,s}(Q_T) := \big\{ v \in L^2(Q_T)\colon v \circ \boldsymbol{\kappa} \in H^{r,s}(Q_0) \big\}$$

where the composition with $\boldsymbol{\kappa}$ only acts on the spatial component. Due to the chain rule, $v \circ \boldsymbol{\kappa}$ and v have the same Sobolev regularity, provided that the mapping $\boldsymbol{\kappa}$ is smooth enough, see for example

[12, Theorem 3.23] for the elliptic case. For what follows, we assume that that $\boldsymbol{\kappa} \in C^2\big([0,T] \times \mathbb{R}^d\big)$ satisfies

$$\|\boldsymbol{\kappa}(t,\mathbf{x})\|_{C^2([0,T]\times\mathbb{R}^d;\mathbb{R}^d)}, \|\boldsymbol{\kappa}(t,\mathbf{x})^{-1}\|_{C^2([0,T]\times\mathbb{R}^d;\mathbb{R}^d)} \leq C_{\boldsymbol{\kappa}} \tag{2.3}$$

for some constant $C_{\boldsymbol{\kappa}} \in (0,\infty)$ as in [9, pg. 826]. We define the norm of $H^{r,s}(Q_T)$ as

$$\|u\|_{H^{r,s}(Q_T)} = \|u \circ \kappa\|_{H^{r,s}(Q_0)}$$

for $r, s \geq 0$. Notice that the Sobolev spaces on the boundary are defined in a similar manner.

Remark 3. *(i) The space $H^{r,s}(Q_T)$ contains all functions such that $u \circ \boldsymbol{\kappa} \in H^{r,s}(Q_0)$. This means that $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta}(u \circ \boldsymbol{\kappa}) \in L^2(Q_0)$ for all $|\boldsymbol{\alpha}| \leq \lambda r$, $\beta \leq (1-\lambda)s$, and $\lambda \in [0,1]$. According to (2.3), the partial derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta}\boldsymbol{\kappa}$ exist and are uniformly bounded for all $|\boldsymbol{\alpha}| + \beta \leq 2$.*

(ii) Consider a function $u \in L^2(Q_T)$ with partial derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\partial_t^{\beta} u \in L^2(Q_T)$ for all $|\boldsymbol{\alpha}| \leq \lambda r$, $\beta \leq (1-\lambda)s$, and $\lambda \in [0,1]$. When computing the time-derivative of $u \circ \boldsymbol{\kappa}$, we obtain also a spatial derivative as the following shows

$$(\partial_t u) \circ \boldsymbol{\kappa} = \partial_t(u \circ \boldsymbol{\kappa}) - \big\langle (\mathrm{D}\,\boldsymbol{\kappa})^{-\intercal}\nabla(u \circ \boldsymbol{\kappa}), \partial_t \boldsymbol{\kappa}\big\rangle. \tag{2.4}$$

Hence, it holds $u \in H^{r,s}(Q_T)$ only if $r \geq s$ since the temporal derivative $\partial_t^{\beta}(u \circ \kappa)$ involves also spatial partial derivatives $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} u$ up to the order $|\boldsymbol{\alpha}| = \beta$ besides the temporal derivative $\partial_t^{\beta} u$.

(iii) Due to the uniformity condition (2.3), we have as in [9]

$$0 < \underline{\sigma} \leq \min\{\sigma(\mathrm{D}\boldsymbol{\kappa})\} \leq \max\{\sigma(\mathrm{D}\boldsymbol{\kappa})\} \leq \overline{\sigma} < \infty,$$

where $\mathrm{D}\boldsymbol{\kappa}$ denotes the Jacobian of $\boldsymbol{\kappa}$ and $\sigma(\mathrm{D}\boldsymbol{\kappa})$ denotes its singular values. Especially, as in [9, Remark 1, pg. 827], we may assume $\det(\mathrm{D}\boldsymbol{\kappa})$ to be positive.

We can define the dual of $H^{r,s}_{0;,0}(Q_T)$ in two different ways, namely

$$\|u\|_{H^{-r,-s}_{;0,}(Q_T)} = \sup_{\widetilde{v} \in H^{r,s}_{0;,0}(Q_0)} \int_{Q_0} (u \circ \boldsymbol{\kappa})\widetilde{v}\,\mathrm{d}(\mathbf{x},t)$$

and

$$|||u|||_{H^{-r,-s}_{;0,}(Q_T)} = \sup_{v \in H^{r,s}_{0;,0}(Q_T)} \int_{Q_T} uv\,\mathrm{d}(\mathbf{x},t).$$

We show that these norms are equivalent. On one hand, there holds

$$\begin{aligned}
|||u|||_{H^{-r,-s}_{;0,}(Q_T)} &= \sup_{v \in H^{r,s}_{0;,0}(Q_T)} \frac{\int_{Q_0}(u \circ \boldsymbol{\kappa})(v \circ \boldsymbol{\kappa})\det(\mathrm{D}\boldsymbol{\kappa})\,\mathrm{d}(\mathbf{x},t)}{\|v\|_{H^{r,s}_{0;,0}(Q_T)}} \\
&\leq \|u\|_{H^{-r,-s}_{;0,}(Q_T)} \sup_{v \in H^{r,s}_{0;,0}(Q_T)} \frac{\big\|(v \circ \boldsymbol{\kappa})\det(\mathrm{D}\boldsymbol{\kappa})\big\|_{H^{r,s}_{0;,0}(Q_0)}}{\|v \circ \boldsymbol{\kappa}\|_{H^{r,s}_{0;,0}(Q_0)}} \\
&\lesssim \|u\|_{H^{-r,-s}_{;0,}(Q_T)},
\end{aligned}$$

where we used the definition of the norm on $H^{r,s}_{0;,0}(Q_T)$ for $s,r\geq 0$ and that the pointwise multiplication with a smooth function is a continuous operation. On the other hand, we likewise find

$$\begin{aligned}\|u\|_{H^{-r,-s}_{;0,}(Q_T)} &= \sup_{\widetilde{v}\in H^{r,s}_{0;,0}(Q_0)} \frac{\int_{Q_T} u(\widetilde{v}\circ\boldsymbol{\kappa}^{-1})\det(\mathrm{D}\boldsymbol{\kappa}^{-1})\,\mathrm{d}(\mathbf{x},t)}{\|\widetilde{v}\|_{H^{r,s}_{0;,0}(Q_0)}}\\ &\leq |||u|||_{H^{-r,-s}_{;0,}(Q_T)} \sup_{\widetilde{v}\in H^{r,s}_{0;,0}(Q_0)} \frac{\left\|(\widetilde{v}\circ\boldsymbol{\kappa}^{-1})\det(\mathrm{D}\boldsymbol{\kappa}^{-1})\right\|_{H^{r,s}_{0;,0}(Q_T)}}{\|\widetilde{v}\|_{H^{r,s}_{0;,0}(Q_0)}}\\ &\lesssim |||u|||_{H^{-r,-s}_{;0,}(Q_T)},\end{aligned}$$

Hence, both duality pairings result in the same dual spaces and we can say that $H^{r,s}_{0;,0}(Q_T)$ and $H^{-r,-s}_{;0,}(Q_T)$ are indeed dual, as likewise for the other pairings.

Finally, let the space $\mathcal{V}(Q_T)$ consist of all the functions v with $v\circ\boldsymbol{\kappa}\in\mathcal{V}(Q_0)$ and

$$\mathcal{V}(Q_0) := \big\{u\in L^2\big((0,T);H^1(\Omega_0)\big)\colon \partial_t u\in L^2\big((0,T);H^{-1}(\Omega_0)\big)\big\}.$$

The norm on this space is given by

$$\|u\|^2_{\mathcal{V}(Q_0)} := \|u\|^2_{H^{1,0}(Q_0)} + \|\partial_t u\|^2_{H^{-1,0}(Q_0)}.$$

Note that the space $\mathcal{V}(Q_0)$ is a dense subspace of $H^{1,\frac{1}{2}}_{;,}(Q_0)$, which follows according to [3, Formula (2.2)] from the interpolation result

$$L^2(I;X)\cap H^1(I;Y)\subset H^{\frac{1}{2}}\big(I;[X,Y]_{\frac{1}{2}}\big)\cap C\big(\bar{I};[X,Y]_{\frac{1}{2}}\big) \tag{2.5}$$

for $X\subset Y$ being Hilbert spaces.

3. Dirichlet Problem

3.1. **Dirichlet trace operator on cylindrical domains.** We first introduce the notion of traces with respect to cylindrical domains. According to [4, Section 2.3], we can define the (interior) Dirichlet trace for a function $u\in C^1(\overline{Q}_0)$ as

$$\gamma_0 u(t,\mathbf{x}) := \lim_{\Omega_0\ni\mathbf{y}\to\mathbf{x}\in\Gamma_0} u(t,\mathbf{y}) \quad \text{for } (t,\mathbf{x})\in\Sigma_0.$$

We thus have $\gamma_0 u = u|_{\Sigma_0}$. We can introduce a similar operator on anisotropic Sobolev spaces, see the following lemma, being along the lines of [10, Theorem 2.1]. It has been proven for $\Gamma_0\in C^\infty$, but it is also true for a Lipschitz boundary in accordance with [3, pg. 504ff].

Lemma 4. *The map*

$$\gamma_0\colon H^{1,\frac{1}{2}}(Q_0)\to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$$

is linear and continuous.

We find the following statement in [3, Lemma 2.4], which holds in the case of a Lipschitz domain Ω_0.

Lemma 5. *The Dirichlet trace operator* γ_0 *is continuous and surjective as an operator from* $H^{1,\frac{1}{2}}_{;0,}(Q_0)$ *to* $H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$.

According to [4, Theorem 2.4], there exists also an extension operator. The extension operator is a right inverse to the surjective Dirichlet trace operator γ_0 and, thus, extends a function defined only on the boundary to the space (see also [5, pg. 12] and [3, Definition 2.17]).

Lemma 6. *The Dirichlet trace operator*

$$\gamma_0 \colon H^{1,\frac{1}{2}}_{;0,}(Q_0) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$$

has a continuous right inverse operator

$$\mathcal{E}_0 \colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0) \to H^{1,\frac{1}{2}}_{;0,}(Q_0),$$

satisfying $\gamma_0 \mathcal{E}_0 v = v$ *for all* $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$.

3.2. **Dirichlet trace operator on non-cylindrical domains.** In this section, we denote the (interior) Dirichlet trace operator with respect to a non-cylindrical domain by $\gamma_{0,t}$ to distinguish it from the Dirichlet trace operator with respect to a cylindrical domain introduced above. When no confusion can happen, we will drop the subscript t in the trace operator for a non-cylindrical domain.

For a smooth function $u \in C^1(\overline{Q}_T)$, defined on a non-cylindrical domain, we set

$$\gamma_{0,t} u(t, \mathbf{x}_t) := \lim_{\Omega_t \ni \mathbf{y}_t \to \mathbf{x}_t \in \Gamma_t} u(t, \mathbf{y}_t).$$

It obviously holds

$$\gamma_{0,t} u(t, \mathbf{x}_t) = \lim_{\substack{\Omega_0 \ni \mathbf{y} \to \mathbf{x} \in \Gamma_0, \\ \boldsymbol{\kappa}(t,\mathbf{x}) = \mathbf{x}_t}} u\big(t, \boldsymbol{\kappa}(t, \mathbf{y})\big) = \gamma_0 (u \circ \boldsymbol{\kappa})(t, \mathbf{x}) = \gamma_0 (u \circ \boldsymbol{\kappa})\big(t, \boldsymbol{\kappa}^{-1}(t, \mathbf{x}_t)\big)$$

for the diffeomorphism $\boldsymbol{\kappa}$ from (2.1). By density of the smooth functions in the Sobolev spaces, we can also extend this notion to Sobolev spaces. Moreover, we have the same mapping properties for $\gamma_{0,t}$ as for γ_0, since

$$\begin{aligned}
\|\gamma_{0,t} u\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} &= \|\gamma_{0,t} u \circ \boldsymbol{\kappa}\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)} \\
&= \left\| \big(\gamma_0 (u \circ \boldsymbol{\kappa}) \circ \boldsymbol{\kappa}^{-1}\big) \circ \boldsymbol{\kappa} \right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)} \\
&\lesssim \|u \circ \boldsymbol{\kappa}\|_{H^{1,\frac{1}{2}}(Q_0)} \\
&= \|u\|_{H^{1,\frac{1}{2}}(Q_T)}.
\end{aligned}$$

Note that the hidden constant changes from line to line and depends on the diffeomorphism $\boldsymbol{\kappa}$, because we used the norm equivalence on the cylindrical and non-cylindrical domain as well as the mapping property of the Dirichlet trace operator on the cylindrical domain.

Due to this consideration, all the properties of Section 3.1 remain valid for the Dirichlet trace operator on non-cylindrical domains. The surjectivity follows for example from the following consideration: Let $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$. By the definition of the norm, we thus have $v \circ \boldsymbol{\kappa} \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$. By the surjectivity of the Dirichlet trace operator with respect to Q_0, there exists a $w \in H^{1,\frac{1}{2}}_{;0,}(Q_0)$ with $\gamma_0 w = v \circ \boldsymbol{\kappa}$. Due to the bijectivity of $\boldsymbol{\kappa}$, we may define

$$\hat{w} := w \circ \boldsymbol{\kappa}^{-1} \in H^{1,\frac{1}{2}}_{;0,}(Q_T).$$

We thus have

$$\gamma_{0,t}\hat{w}(t,\mathbf{x}_t) = \gamma_0 w\big(t, \boldsymbol{\kappa}^{-1}(t,\mathbf{x}_t)\big) = v(t,\mathbf{x}_t),$$

from where the subjectivity follows and we can also infer the existence of the right inverse operator $\mathcal{E}_0$.

3.3. **Existence and uniqueness of Dirichlet problem.** We consider the following Dirichlet problem with homogeneous initial datum

$$\begin{aligned} (\partial_t - \Delta)v &= f && \text{in } Q_T, \\ \gamma_0 v &= q && \text{on } \Sigma_T, \\ v(0,\cdot) &= 0 && \text{in } \Omega_0. \end{aligned} \tag{3.1}$$

We have the following existence and uniqueness theorem for its solution.

Theorem 7. *Let $f \in H^{-1,-\frac{1}{2}}_{;0,}(Q_T)$ and $q \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$. Then, there exists a unique solution $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$, satisfying the boundary condition in* (3.1) *and*

$$S(v,\varphi) := \int_0^T \int_{\Omega_t} \{\nabla v \cdot \nabla \varphi + \partial_t v \varphi\} \,\mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Omega_t} f u \,\mathrm{d}\mathbf{x}\mathrm{d}t \text{ for all } \varphi \in H^{1,\frac{1}{2}}_{0;,0}(Q_T). \tag{3.2}$$

Proof. A proof of this statement based on the proof of [3, Lemma 2.8] can be found in [1]. □

4. Neumann trace operator

Similarly as we defined the Dirichlet trace operator, we can also introduce an (interior) Neumann trace operator. In the following, we will first introduce this concept on cylindrical domains. Then, we will introduce the notion of a Neumann trace on a non-cylindrical domain formally and rigorously.

4.1. **Neumann trace operator on cylindrical domains.** We first introduce the Neumann trace operator, also called the conormal derivative, on a cylindrical domain along the lines of [3]. Let us define the space

$$H^{1,\frac{1}{2}}(Q_0;\mathcal{L}) := \big\{u \in H^{1,\frac{1}{2}}(Q_0) \colon \mathcal{L}u \in L^2(Q_0)\big\},$$

where $\mathcal{L} := \partial_t - \Delta$ is the partial differential operator under consideration. The norm on this space is given by

$$\|u\|^2_{H^{1,\frac{1}{2}}(Q_0;\mathcal{L})} := \|u\|^2_{H^{1,\frac{1}{2}}(Q_0)} + \big\|(\partial_t - \Delta)u\big\|^2_{L^2(Q_0)}.$$

According to [3, Lemma 2.16], the bilinear form

$$b(u,v) := \int_{Q_0} \big\{\langle \nabla u, \nabla v\rangle - (\partial_t - \Delta)uv\big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t + d(u,v)$$

is continuous on $H^{1,\frac{1}{2}}(\mathbb{R}\times\Omega_0;\mathcal{L}) \times H^{1,\frac{1}{2}}(\mathbb{R}\times\Omega_0)$, where

$$d(u,v) := \int_{\mathbb{R}\times\Omega_0} \partial_t uv\,\mathrm{d}\mathbf{x}\mathrm{d}t.$$

The bilinear form $d(u,v)$ has a continuous extension from $C_0^\infty(\mathbb{R}^{d+1})\times C_0^\infty(\mathbb{R}^{d+1})$ to $H^{\frac{1}{2}}\big(\mathbb{R};L^2(\Omega_0)\big)\times H^{\frac{1}{2}}\big(\mathbb{R};L^2(\Omega_0)\big)$ and it holds $d(u,v) = -d(v,u)$ for all $u,v \in H^{\frac{1}{2}}\big(\mathbb{R};L^2(\Omega_0)\big)$, compare [3, Lemma 2.6].

The (interior) Neumann trace is defined for $u \in C^1(\overline{Q}_0)$ by

$$\gamma_1^{\mathrm{int}} u(t,\mathbf{x}) := \lim_{\Omega_0\ni\mathbf{y}\to\mathbf{x}\in\Gamma_0} \big\langle \nabla_{\mathbf{y}} u(t,\mathbf{y}), \mathbf{n}_{\mathbf{x}}\big\rangle \quad \text{for } (t,\mathbf{x})\in\Sigma_0$$

and coincides with the normal derivative on Σ_0, thus $\gamma_1^{\mathrm{int}} u = \partial u/\partial\mathbf{n}$ on Σ_0, see [4, Section 3.3] and also [19, Satz 8.7] for the elliptic case. Since it holds

$$b(u,v) = \int_{\Sigma_0} \frac{\partial u}{\partial \mathbf{n}} v\,\mathrm{d}\sigma\mathrm{d}t$$

for $u,v \in C_0^2(\mathbb{R}\times\overline{\Omega}_0)$, we can extend this definition as follows, which is along the lines of [3, Definition 2.17].

Definition 8. *Let $u \in H^{1,\frac{1}{2}}(\mathbb{R}\times\Omega_0;\mathcal{L})$. Then, the Neumann trace operator $\gamma_1 u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0)$ is the continuous linear form on $H^{\frac{1}{2},\frac{1}{2}}(\Sigma_0)$ defined by*

$$\gamma_1^{\mathrm{int}} u\colon \varphi \mapsto b(u, \mathcal{E}_0\varphi),$$

where $\mathcal{E}_0$ is the extension operator given in Lemma 6.

Notice that we can also introduce the conormal derivative $\gamma_1^{\mathrm{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0)$ as the unique solution of a variational problem, as it is done in [5, Section 3.4]. This variational problem can for example be obtained by applying $\gamma_1^{\mathrm{int}} u$ to φ. According to [3, Proposition 2.18], the Neumann trace has the following properties.

Lemma 9. *(i) The map*

$$\gamma_1^{\mathrm{int}}\colon H^{1,\frac{1}{2}}(\mathbb{R}\times\Omega_0;\mathcal{L}) \to H^{-\frac{1}{2},-\frac{1}{4}}(\mathbb{R}\times\Gamma_0)$$

is continuous and by restriction also the map

$$\gamma_1^{\mathrm{int}}\colon H^{1,\frac{1}{2}}(Q_0;\mathcal{L}) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0)$$

is continuous.

(ii) If $u \in C^2(\overline{Q}_0)$, then $\gamma_1^{\mathrm{int}} u = (\partial u/\partial\mathbf{n})|_{\Sigma_0}$ due to the Green formula.

4.2. **Neumann trace operator on non-cylindrical domains.** On time-dependent boundaries, one could consider the usual Neumann trace, as it is done for example in [6, Section 6.1]. Instead, we follow here the idea of [17] and employ a velocity corrected Neumann trace. We first formally introduce this Neumann trace and afterwards characterize its properties rigorously.

4.2.1. *Formal.* For a time dependent spatial surface we define two Neumann trace operators

$$\gamma_1^{\pm}\varphi := \frac{\partial \varphi}{\partial \mathbf{n}_t} \mp \frac{1}{2}\langle \mathbf{V}, \mathbf{n}_t \rangle \varphi. \tag{4.1}$$

To motivate this definition consider the boundary value problem

$$\begin{aligned} (\partial_t - \Delta)u &= f && \text{in } Q_T, \\ \gamma_1 u &= g && \text{on } \Sigma_T, \\ u(0,\cdot) &= 0 && \text{in } \Omega_0, \end{aligned} \tag{4.2}$$

where we leave it a priori open what γ_1 means. Let us formally derive the weak formulation of the Neumann problem (4.2) by multiplying with a test function v with $v(T,\cdot) = 0$ in Ω_T and using Reynolds' transport theorem

$$\begin{aligned} \int_0^T \int_{\Omega_t} fv \,\mathrm{d}\mathbf{x}\mathrm{d}t &= \int_0^T \int_{\Omega_t} (\partial_t - \Delta)uv \,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \int_0^T \int_{\Omega_t} \big\{\langle \nabla u, \nabla v\rangle + \partial_t(uv) - u\partial_t v\big\} \,\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^T \int_{\Gamma_t} \frac{\partial u}{\partial \mathbf{n}} v \,\mathrm{d}\sigma\mathrm{d}t \\ &= \int_0^T \int_{\Omega_t} \big\{\langle \nabla u, \nabla v\rangle - u\partial_t v\big\} \,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} uv \,\mathrm{d}\mathbf{x}\mathrm{d}t \\ &\qquad - \int_0^T \int_{\Gamma_t} \left\{\frac{\partial u}{\partial \mathbf{n}} v + uv\langle \mathbf{V}, \mathbf{n}\rangle\right\} \mathrm{d}\sigma\mathrm{d}t. \end{aligned}$$

Due to the fundamental theorem of calculus and the vanishing initial and end condition of u and v, respectively, we obtain the variational equation

$$a(u,v) = \int_0^T \int_{\Omega_t} fv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\Gamma_t} \left\{\frac{\partial u}{\partial \mathbf{n}} v + \frac{1}{2} uv\langle \mathbf{V}, \mathbf{n}\rangle\right\} \mathrm{d}\sigma\mathrm{d}t$$

with bilinear form

$$a(u,v) := \int_0^T \int_{\Omega_t} \big\{\langle \nabla u, \nabla v\rangle - u\partial_t v\big\} \,\mathrm{d}\mathbf{x}\mathrm{d}t - \frac{1}{2}\int_0^T \int_{\Gamma_t} uv\langle \mathbf{V}, \mathbf{n}\rangle \,\mathrm{d}\sigma\mathrm{d}t.$$

Thus, if we set the previously unspecified trace in (4.2) as γ_1^-, and we arrive at

$$a(u,v) = \int_0^T \int_{\Omega_t} fv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\Gamma_t} gv \,\mathrm{d}\sigma\mathrm{d}t.$$

With the additonal boundary integral term it is easy to see that the bilinear form is bounded and coercive in the $H^{1,\frac{1}{2}}$-norm, so the Lion's projection theorem guarantees existence and uniqueness of this Neumann problem. However, for the cylindrical case [3, Lemma 2.21] states that this strategy does not yield satisfactory results, since one has to make stronger assumptions on the regularity of the input data. Therefore, as in [3], we will proof the existence and uniqueness of solutions by using a boundary integral formulation (see Corollary 37).

4.2.2. *Rigorous.* We assume $\boldsymbol{\kappa}$ to be defined on $\mathbb{R} \times \mathbb{R}^d$ and not only on $[0,T] \times \mathbb{R}^d$. Moreover, for sake of simplicity in representation, we always consider functions u and v throughout this section which satisfy

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} uv \, \mathrm{d}\mathbf{x}\mathrm{d}t = 0. \tag{4.3}$$

This assumption stems from the fact that we would like to integrate by parts in time. Later on, we will consider a finite time interval $(0,T)$ and equip u and v with the appropriate zero initial and end conditions. Extending u and v by zero for $t < 0$ and $t > T$, respectively, leads then to the fulfillment of (4.3).

Let us define

$$d(u,v) := \int_{\mathbb{R}} \int_{\Omega_t} \partial_t uv \, \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, \mathrm{d}\sigma\mathrm{d}t. \tag{4.4}$$

Notice that the additional boundary term is a speciality of the time-dependent boundary. We shall first state the analogue of [3, Lemma 2.6].

Lemma 10. *The bilinear form $d(u,v)$ has a continuous extension from $C_0^\infty(\mathbb{R}^{d+1}) \times C_0^\infty(\mathbb{R}^{d+1})$ to $H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\} \times \Omega_t)\big) \times H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\} \times \Omega_t)\big)$, and it holds*

$$d(u,v) = -d(v,u). \tag{4.5}$$

Proof. The use of Reynolds' transport theorem allows us to compute

$$\begin{aligned} d(u,v) &= \int_{\mathbb{R}} \int_{\Omega_t} \partial_t uv \, \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, \mathrm{d}\sigma\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_{\Omega_t} \big\{\partial_t(uv) - u\partial_t v\big\} \, \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, \mathrm{d}\sigma\mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} uv \, \mathrm{d}\mathbf{x}\mathrm{d}t - \int_{\mathbb{R}} \int_{\Omega_t} u\partial_t v \, \mathrm{d}\mathbf{x}\mathrm{d}t - \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, \mathrm{d}\sigma\mathrm{d}t. \end{aligned}$$

The assumption (4.3) hence implies

$$d(u,v) = -\int_{\mathbb{R}} \int_{\Omega_t} u\partial_t v \, \mathrm{d}\mathbf{x}\mathrm{d}t - \frac{1}{2} \int_{\mathbb{R}} \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n} \rangle uv \, \mathrm{d}\sigma\mathrm{d}t,$$

from where (4.5) follows immediately. The rest is in complete analogy to [3, Lemma 2.6], but we need higher regularity in the spatial variable instead just $L^2(\Omega_0)$ like in [3], because the boundary term in the definition of $d(u,v)$ has to be well-defined. □

As in Section 4.1, we introduce the space

$$H^{1,\frac{1}{2}}(Q_T;\mathcal{L}) := \big\{u \in H^{1,\frac{1}{2}}(Q_T)\colon \mathcal{L}u \in L^2(Q_T)\big\},$$

where $\mathcal{L} := \partial_t - \Delta$ is the differential operator on the non-cylindrical domain. We state the analogue of [3, Lemma 2.16] in the case of a non-cylindrical domain, the proof of which is obvious.

Lemma 11. *The bilinear form*

$$b^-(u,v) := \int_0^T \int_{\Omega_t} \big\{\langle \nabla u, \nabla v\rangle - (\partial_t - \Delta)uv\big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t + d(u,v)$$

with $d(u,v)$ being defined in (4.4) is continuous on $H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t);\partial_t-\Delta\big)\times H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t)\big)$. If $u,v\in C_0^2\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\overline{\Omega}_t)\big)$, we have

$$b^-(u,v) = \int_0^T \int_{\Gamma_t} \left\{\frac{\partial u}{\partial \mathbf{n}} v + \frac{1}{2}\langle \mathbf{V}, \mathbf{n}\rangle uv\right\}\mathrm{d}\sigma\mathrm{d}t$$

by means of Green's formula.

In complete analogy to the Neumann trace operator in the cylindrical case, we will define $\gamma_1^- u$, which is one of the two required Neumann trace operators.

Definition 12. *Given $u\in H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t);\partial_t-\Delta\big)$, we denote by $\gamma_1^- u\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ the continuous linear form on $H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ defined by*

$$\gamma_1^- u\colon \varphi\mapsto b^-(u,\mathcal{E}_0\varphi),$$

where $\mathcal{E}_0$ is the extension operator as mentioned in Section 3.2.

The following lemma is the non-cylindrical equivalent to [3, Proposition 2.18].

Lemma 13. *The map*

$$\gamma_1^-: H^{1,\frac{1}{2}}\left(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t);\partial_t-\Delta\right)\to H^{-\frac{1}{2},-\frac{1}{4}}\left(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Gamma_t)\right)$$

is continuous and by restriction also the map

$$\gamma_1^-: H^{1,\frac{1}{2}}(Q_T;\partial_t-\Delta)\to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$$

is continuous. Moreover, if $u\in C^2(\overline{Q}_T)$, then it holds

$$\gamma_1^- u = \frac{\partial u}{\partial\mathbf{n}} + \frac{1}{2}\langle\mathbf{V},\mathbf{n}\rangle u.$$

Proof. As in [3], the continuity is a consequence of the continuity of the bilinear form $b(\cdot,\cdot)$ (cf. Lemma 11). The second statement follows immediately from Green's first formula. □

Remark 14. *In view of the reformulation of the heat equation in terms of boundary integral equations, we will moreover encounter a second Neumann trace operator, which we denote by γ_1^+. It can be achieved analogously to above by considering the differential operator $\partial_t+\Delta$ instead of $\partial_t-\Delta$. The former operator for example arises when considering a time reversal of the latter one. With*

$$b^+(u,v) := \int_0^T \int_{\Omega_t} \big\{\langle \nabla u, \nabla v\rangle + (\partial_t + \Delta)uv\big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t - d(u,v),$$

we can state the analogue of Lemma 11, namely the continuity of $b^+(\cdot,\cdot)$ *in the appropriate space and for* u *and* v *smooth enough, we have*

$$b^+(u,v) = \int_0^T \int_{\Gamma_t} \left\{ \frac{\partial u}{\partial \mathbf{n}} v - \frac{1}{2}\langle \mathbf{V}, \mathbf{n}\rangle uv \right\} \mathrm{d}\sigma \mathrm{d}t.$$

With this property at hand, we can define the trace operator γ_1^+ *in analogy to Definition 12. For* u *smooth enough, it then holds*

$$\gamma_1^+ u = \frac{\partial u}{\partial \mathbf{n}} - \frac{1}{2}\langle \mathbf{V}, \mathbf{n}\rangle u.$$

The existence of two Neumann trace operators is a speciality of the time-dependent boundary.

Likewise to [3, Formula (2.35)], given $u \in H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t);\partial_t - \Delta\big)$ and $v \in H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t)\big)$, we obtain Green's first formula

$$\int_{\mathbb{R}}\int_{\Omega_t} \langle \nabla u, \nabla v\rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t + d(u,v) = \int_{\mathbb{R}}\int_{\Omega_t} (\partial_t - \Delta)uv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \langle \gamma_1^- u, \gamma_0 v\rangle. \tag{4.6}$$

By restriction, this formula also holds for $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$ and $v \in H^{1,\frac{1}{2}}_{;,0}(Q_T)$, but not, as was pointed out in [3], when u, v are both in $H^{1,\frac{1}{2}}_{;0,}(Q_T)$.

In complete analogy, Green's formula for $u \in H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t);\partial_t + \Delta\big)$ and $v \in H^{1,\frac{1}{2}}\big(\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t)\big)$ reads

$$\int_{\mathbb{R}}\int_{\Omega_t} \langle \nabla u, \nabla v\rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t - d(u,v) = \int_{\mathbb{R}}\int_{\Omega_t} (-\partial_t - \Delta)uv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \langle \gamma_1^+ u, \gamma_0 v\rangle.$$

Again, by restriction, this formula also holds for $u \in H^{1,\frac{1}{2}}_{;,0}(Q_T;\partial_t + \Delta)$ and $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$.

We can now formulate Green's formulas for a finite time interval, the time-independent analogues of which are given in [3, Proposition 2.19].

Notice that [3] introduces a time reversal map. For a time-dependent domain, this approach does not make sense, since the integration over a time forward tube of a time reversed entity is not always well defined. Therefore, we choose a slightly different approach to obtain a further Green formula.

Lemma 15.

(i) *Let* $u \in H^{1,\frac{1}{2}}_{;0,}\big(\bigcup_{t\in\mathbb{R}_+}(\{t\}\times\Omega_t);\partial_t - \Delta\big)$ *and* $v \in H^{1,\frac{1}{2}}_{;,0}\big(\bigcup_{-\infty<t<t_0}(\{t\}\times\Omega_t)\big)$. *Then, for* $t_0 > 0$, *there holds Green's first formula*

$$\int_0^{t_0}\int_{\Omega_t} \langle \nabla u, \nabla v\rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t + d(u,v) = \int_0^{t_0}\int_{\Omega_t} (\partial_t - \Delta)uv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \langle \gamma_1^- u, \gamma_0 v\rangle.$$

(ii) *Let* $u \in H^{1,\frac{1}{2}}_{;,0}\big(\bigcup_{-\infty<t<t_0}(\{t\}\times\Omega_t);\partial_t + \Delta\big)$ *and* $v \in H^{1,\frac{1}{2}}_{;0,}\big(\bigcup_{t\in\mathbb{R}_+}(\{t\}\times\Omega_t)\big)$. *Then, for* $t_0 > 0$, *there holds Green's alternative first formula*

$$\int_0^{t_0}\int_{\Omega_t} \langle \nabla u, \nabla v\rangle \,\mathrm{d}\mathbf{x}\mathrm{d}t - d(u,v) = \int_0^{t_0}\int_{\Omega_t} (-\partial_t - \Delta)uv \,\mathrm{d}\mathbf{x}\mathrm{d}t + \langle \gamma_1^+ u, \gamma_0 v\rangle.$$

(iii) Let $u \in H^{1,\frac{1}{2}}_{;0,}\big(\bigcup_{t\in\mathbb{R}_+}(\{t\}\times\Omega_t);\partial_t-\Delta\big)$ *and* $v \in H^{1,\frac{1}{2}}_{;,0}\big(\bigcup_{-\infty<t<t_0}(\{t\}\times\Omega_t);\partial_t+\Delta\big)$. *Then, for* $t_0>0$, *there holds Green's second formula*

$$\int_0^{t_0}\int_{\Omega_t}\big\{(\partial_t-\Delta)uv+u(\partial_t+\Delta)v\big\}\,\mathrm{d}\mathbf{x}\mathrm{d}t = \langle\gamma_0 u,\gamma_1^+ v\rangle - \langle\gamma_1^- u,\gamma_0 v\rangle.$$

Proof. Statements (i) and (ii) are clear. Statement (iii) follows then immediately from these by interchanging v and u in (ii) and using (4.5). □

We need the tube equivalent of [3, Lemma 2.22]. In there, the space

$$\widetilde{C}^\infty(\overline{Q}_0) := C_0^\infty\big((0,T]\times\overline{\Omega}_0\big)$$

is defined as the space of the restrictions of functions in $C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ to $\overline{Q}_0$. This space $\widetilde{C}^\infty(\overline{Q}_0)$ is dense in $H^{1,\frac{1}{2}}_{;0,}(Q_0;\partial_t-\Delta)$ according to [3, Lemma 2.22]. As we only consider a C^2-mapping between the reference cylinder and the tube, we will prove the analogue result only for C^2-functions.

Lemma 16. *Let us define*

$$\widetilde{C}^2(\overline{Q}_T) := \Big\{u\colon u\circ\boldsymbol{\kappa}\in C_0^2\big((0,T]\times\overline{\Omega}_0\big)\Big\}.$$

Then, the space $\widetilde{C}^2(\overline{Q}_T)$ *is dense in* $H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t-\Delta)$.

Proof. We mimic the proof of [3, Lemma 2.22], which is based on a proof of Grisvard in the elliptic case, see [8, Lemma 1.5.3.9]. According to the proof of [3, Lemma 2.22], we have that $C_0^\infty\big((0,T]\times\overline{\Omega}_0\big)$ is dense in $H^{1,\frac{1}{2}}_{;0,}(Q_0)$. Therefore, also $C_0^2\big((0,T]\times\overline{\Omega}_0\big)$ is dense in $H^{1,\frac{1}{2}}_{;0,}(Q_0)$. Due to the definition of the spaces on the tube via the mapping $\boldsymbol{\kappa}$ and the resulting equivalence of norms, we also obtain that $\widetilde{C}^2(\overline{Q}_T)$ is dense in $H^{1,\frac{1}{2}}_{;0,}(Q_T)$. Similarly, we obtain that

$$\widetilde{C}^2(Q_T) := \Big\{u\colon u\circ\boldsymbol{\kappa}\in C_0^2\big((0,T]\times\Omega_0\big)\Big\}$$

is dense in $H^{1,\frac{1}{2}}_{0;0,}(Q_T)$.

Let $\mathcal{R}$ be an extension operator from $H^{1,\frac{1}{2}}_{;0,}(Q_T)$ to $H^{1,\frac{1}{2}}(\mathbb{R}^{d+1})$. It thus holds $(\mathcal{R}u)|_{Q_T}=u$. As in [3], let us choose $\mathcal{R}$ such that $\operatorname{supp}\mathcal{R}\subset[0,\infty)\times\mathbb{R}^d$.[1] In that way, we can identify $H^{1,\frac{1}{2}}_{;0,}(Q_T)$ with a closed subspace of $H^{1,\frac{1}{2}}_{;0,}(\mathbb{R}_+\times\mathbb{R}^d)$. The map $u\mapsto\big(\mathcal{R}u,(\partial_t-\Delta)u\big)$ identifies $H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t-\Delta)$ with a closed subspace of $H^{1,\frac{1}{2}}_{;0,}(\mathbb{R}_+\times\mathbb{R}^d)\times L^2(\mathbb{R}_+\times\mathbb{R}^d)$. Due to this identification, we find for every bounded linear functional $\ell : H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t-\Delta)\to\mathbb{R}$ some $f\in\big(H^{1,\frac{1}{2}}_{;0,}(\mathbb{R}_+\times\mathbb{R}^d)\big)' = H^{-1,-\frac{1}{2}}(\mathbb{R}_+\times\mathbb{R}^d)$ and $g\in L^2(\mathbb{R}_+\times\mathbb{R}^d)$ such that it holds

$$\langle\ell,u\rangle = \langle f,\mathcal{R}u\rangle + \int_{\mathbb{R}_+\times\mathbb{R}^d} g(\partial_t-\Delta)u\,\mathrm{d}(t,\mathbf{x})$$

[1] Such an extension operator exists as it can be defined by $\mathcal{R}u=\big(\widetilde{\mathcal{R}}(u\circ\boldsymbol{\kappa})\big)\circ\boldsymbol{\kappa}^{-1}$ with $\widetilde{\mathcal{R}} : H^{1,\frac{1}{2}}_{;0,}(Q_0)\to H^{1,\frac{1}{2}}(\mathbb{R}_+\times\mathbb{R}^d)$ being the extension operator from [3].

for all $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$. Since ℓ acts only on u, which is supported on $\overline{Q}_T$, we may assume that $\operatorname{supp} f \subset \overline{Q}_T$ and $\operatorname{supp} g \subset \overline{Q}_T$.

We shall suppose next that it holds $\langle \ell, \varphi\rangle = 0$ for all $\varphi \in \widetilde{C}^2(\overline{Q}_T)$. If we can show $\ell = 0$, we obtain the desired density result in accordance with [18, Korollar III.1.9]. For all $\varphi \in C_0^2(\mathbb{R}_+ \times \mathbb{R}^d)$, we conclude

$$\begin{aligned} 0 = \langle \ell, \varphi\rangle &= \langle f, \varphi\rangle + \int_{Q_T} g(\partial_t - \Delta)\varphi \,\mathrm{d}(t,\mathbf{x}) \\ &= \langle f, \varphi\rangle + \int_{\mathbb{R}_+\times\mathbb{R}^d} g(\partial_t - \Delta)\varphi \,\mathrm{d}(t,\mathbf{x}). \end{aligned}$$

This equation states that

$$f = (\partial_t + \Delta)g$$

holds on $\mathbb{R}_+ \times \mathbb{R}^d$ in complete analogy to [3]. Due to $f \in H^{-1,-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and the differential operator, we find $g \in H^{1,\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$ and, thus, $g|_{Q_T} \in H^{1,\frac{1}{2}}_{0;,0}(Q_T)$.

On a cylindrical domain, any function $h \in H^{1,\frac{1}{2}}_{0;,0}(Q_0)$ can be approximated by a series $h_n \in C_0^\infty\big((-\infty, T) \times \Omega_0\big)$ (see [3, Proof of Lemma 2.22]). Hence, by choosing $h := g \circ \boldsymbol{\kappa}$ and setting $g_n := h_n \circ \boldsymbol{\kappa}^{-1}$, we can approximate $g|_{Q_T} \in H^{1,\frac{1}{2}}_{0;,0}(Q_T)$ by a series $g_n \in C_0^2\big(\bigcup_{-\infty<t<T}(\{t\} \times \Omega_t)\big)$ in the norm of $H^{1,\frac{1}{2}}\big(\bigcup_{0<t<\infty}(\{t\} \times \Omega_t)\big)$. Thus, denoting by $\hat{g}_n$ the extension of g_n by zero outside of Q_T, we find $(\partial_t + \Delta)\hat{g}_n \to f$ in $H^{-1,-\frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R}^d)$. We then conclude for any $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$ that

$$\begin{aligned} \langle \ell, u\rangle &= \lim_{n\to\infty}\left[\big\langle(\partial_t + \Delta)\hat{g}_n, \mathcal{R}u\big\rangle + \int_{Q_T} g_n(\partial_t - \Delta)u \,\mathrm{d}(t,\mathbf{x})\right] \\ &= \lim_{n\to\infty}\left[\int_{Q_T}(\partial_t + \Delta)g_n u \,\mathrm{d}(t,\mathbf{x}) + \int_{Q_T} g_n(\partial_t - \Delta)u \,\mathrm{d}(t,\mathbf{x})\right] = 0. \end{aligned}$$

The expression above is equal to zero, because $u = 0$ for $t = 0$, $g_n = 0$ for $t = T$, and g_n has a zero boundary condition. ☐

Remark 17. *If we consider $t \mapsto T-t$ in Lemma 16, we obtain that $\widehat{C}^2(\overline{Q}^T)$ is dense in $H^{1,\frac{1}{2}}_{;,0}(Q^T;-\partial_t-\Delta)$, where*

$$\widehat{C}^2(\overline{Q}_T) := \Big\{u\colon u \circ \boldsymbol{\kappa} \in C_0^2\big([0,T) \times \overline{\Omega}_0\big)\Big\}$$

and Q^T is the time flipped Q_T. Since Q_T was arbitrary, we have $\widehat{C}^2(\overline{Q}_T)$ is dense in $H^{1,\frac{1}{2}}_{;,0}(Q_T;\partial_t+\Delta)$.

Next, we will introduce a lemma concerning the trace maps, which will be later used in the proof of the jump relations. It is the analogue of [3, Lemma 2.23].

Lemma 18. *The combined trace map $(\gamma_0, \gamma_1^+)\colon u \mapsto (\gamma_0 u, \gamma_1^+ u)$ maps $\widehat{C}^2(\overline{Q}_T)$ onto a dense subspace of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$.*

Proof. We again mimic the respective proof from [3], but will not use a time reversal map. Let us assume a linear functional $(\chi, \psi) \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ that vanishes on the range of (γ_1^+, γ_0). We need to show that $(\chi, \psi) = (0, 0)$, since then the density follows by [18, Korollar III.1.9]. To this end, assume

$$\langle \chi, \gamma_1^+ \varphi \rangle = \langle \psi, \gamma_0 \varphi \rangle \quad \text{for all } \varphi \in \widehat{C}^2(\overline{Q}_T). \tag{4.7}$$

Let

$$\mathcal{T} = (g \mapsto \mathcal{T}g)\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T)$$

be the solution operator (see Theorem 7) of the Dirichlet problem

$$\begin{aligned} (\partial_t - \Delta)(\mathcal{T}g) &= 0 \qquad \text{in } Q_T, \\ \gamma_0(\mathcal{T}g) &= g \qquad \text{on } \Sigma_T. \end{aligned} \tag{4.8}$$

Moreover, let

$$\mathcal{S} = (f \mapsto \mathcal{S}f)\colon L^2(Q_T) \to H^{1,\frac{1}{2}}_{0;,0}(Q_T)$$

be the solution operator (see Theorem 7 used for the substitution $t \mapsto T - t$) of the Dirichlet problem

$$\begin{aligned} (\partial_t + \Delta)(\mathcal{S}f) &= f \qquad \text{in } Q_T, \\ \gamma_0(\mathcal{S}f) &= 0 \qquad \text{on } \Sigma_T. \end{aligned}$$

We can apply Green's second formula from Lemma 15 to $u := \mathcal{T}\chi$ and $v := \mathcal{S}f$ for any $f \in L^2(Q_T)$, since $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T; \partial_t - \Delta)$ and $(\partial_t + \Delta)v \in L^2(Q_T)$. We obtain

$$\int_{Q_T} \big\{(\partial_t - \Delta)uv + u(\partial_t + \Delta)v\big\}\, \mathrm{d}(\mathbf{x}, t) = \langle \gamma_0 u, \gamma_1^+ v\rangle - \langle \gamma_1^- u, \gamma_0 v\rangle.$$

Since $\gamma_0 v = 0$ and $\gamma_0 u = \chi$, as well as $(\partial_t - \Delta)u = 0$ and $(\partial_t + \Delta)v = f$, we obtain

$$\int_{Q_T} uf\, \mathrm{d}(t, \mathbf{x}) = \langle \chi, \gamma_1^+ v\rangle.$$

Due to continuity and Remark 17, (4.7) holds also for all $\varphi \in H^{1,\frac{1}{2}}_{;,0}(Q_T; \partial_t + \Delta)$ and, thus, also for $\varphi = \mathcal{S}f$. This implies

$$\langle \chi, \gamma_1^+ \mathcal{S}f\rangle = \langle \psi, \gamma_0 \mathcal{S}f\rangle.$$

Since $\gamma_0 \mathcal{S}f = 0$, we thus obtain $\int_{Q_T} uf\, \mathrm{d}(t, \mathbf{x}) = 0$ for all $f \in L^2(Q_T)$. Therefore, $0 = u = \mathcal{T}(\chi)$ and thus $\chi = \gamma_0 u = 0$. Looking again at (4.7) gives

$$\langle \psi, \gamma_0 \varphi\rangle = 0 \quad \text{for all } \varphi \in H^{1,\frac{1}{2}}_{;,0}(Q_T).$$

The trace map γ_0 is not only surjective for $\varphi \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ as shown in Section 3.2, but also for $\varphi \in H^{1,\frac{1}{2}}_{;,0}(Q_T)$ if one considers the backward problem. We may hence conclude that $\psi = 0$. □

In the following, we state the analogue of [3, Proposition 2.24].

Lemma 19. *Green's first formula given in* (4.6) *holds for all* $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$ *and* $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$. *If also* $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$, *we can write the Green's formula as*

$$\begin{aligned}\int_{Q_T} \langle \nabla u, \nabla v\rangle \,\mathrm{d}(t,\mathbf{x}) - d(v,u) + \int_{\Omega_T} u(T,\mathbf{x})v(T,\mathbf{x})\,\mathrm{d}\mathbf{x} \\ = \langle \gamma_1^- u, \gamma_0 v\rangle + \int_{Q_T} (\partial_t - \Delta)uv\,\mathrm{d}(t,\mathbf{x}).\end{aligned} \tag{4.9}$$

Proof. We again mimic the proof of [3, Proposition 2.24]. Given $u \in \widetilde{C}^2(\overline{Q}_T)$ and $v \in \widetilde{C}^1(\overline{Q}_T)^2$, we find

$$\begin{aligned}\int_{Q_T} \langle \nabla u, \nabla v\rangle \,\mathrm{d}(t,\mathbf{x}) + \int_{Q_T} \partial_t uv\,\mathrm{d}(t,\mathbf{x}) + \frac{1}{2}\int_0^T \int_{\Gamma_t} \langle \mathbf{V}, \mathbf{n}\rangle uv \mathrm{d}\sigma \mathrm{d}t \\ = \langle \gamma_1^- u, \gamma_0 v\rangle + \int_{Q_T} (\partial_t - \Delta)uv\,\mathrm{d}(t,\mathbf{x}).\end{aligned} \tag{4.10}$$

All terms are continuous with respect to v in the $H^{1,\frac{1}{2}}_{;0,}(Q_T)$-norm. Thus, by continuity, we can extend (4.10) to all $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$. Let $v \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ be fixed. Then, all terms in (4.10) except the term containing the $\partial_t u$ are obviously continuous with respect to u in the norm of $H^{1,\frac{1}{2}}_{;0,}(Q;\partial_t - \Delta)$. Therefore, also the term containing $\partial_t u$ is continuous. Lemma 16 allows to extend (4.10) to all $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$. Thus, Green's first formula holds as given in the claim.

For $u, v \in \widetilde{C}^2(\overline{Q}_T)$, (4.9) holds. As in [3], the term $\int_{\Omega_T} u(T,\mathbf{x})v(T,\mathbf{x})\,\mathrm{d}(t,\mathbf{x})$ is continuous for u and v in the norm of $H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta) \subset \mathcal{V}(Q_T)$ and $\mathcal{V}(Q_T)$ consists of functions φ, which always satisfy $\varphi \circ \boldsymbol{\kappa} \in C\big([0,T]; L^2(\Omega_0)\big)$. From here, the second claim follows. ☐

5. The Calderón operator

Let us first introduce the fundamental solution for the heat equation, which, in accordance with e.g. [17], reads

$$G(t,\tau,\mathbf{x},\mathbf{y}) := \begin{cases} \frac{1}{(4\pi(t-\tau))^{\frac{d}{2}}} \exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(t-\tau)}\right), & \text{if } \tau < t, \\ 0, & \text{if } \tau \geq t, \end{cases}$$

Notice that this is equivalent to consider $\overline{G}(t-\tau,\mathbf{x},\mathbf{y})$, where $\overline{G}$ is given by

$$\overline{G}(\tau,\mathbf{x},\mathbf{y}) := \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4\tau}\right) \frac{1}{2}(1 + \operatorname{sign}\tau),$$

[2] The space $\widetilde{C}^1(\overline{Q}_T)$ ist defined in complete analogy to $\widetilde{C}^2(\overline{Q}_T)$ via

$$\widetilde{C}^1(Q_T) := \big\{u \colon u \circ \boldsymbol{\kappa} \in C_0^1\big((0,T] \times \Omega_0\big)\big\}$$

as introduced in [3, Formula (2.39)]. Moreover, let us denote

$$\widetilde{G}(t,\mathbf{x}) = \begin{cases} \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{\|\mathbf{x}\|^2}{4t}\right), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \tag{5.1}$$

For $u \in C^2\big(\bigcup_{0<t<\infty}(\{t\} \times \overline{\Omega}_t)\big)$ with $u(0,\mathbf{x}) = 0$ on Ω_0, we have for $(t_0,\mathbf{x}_0) \in \bigcup_{0<t<\infty}(\{t\} \times \Omega_t)$ that

$$\begin{aligned} u(t_0,\mathbf{x}_0) = &\int_{Q_T}\int_{\Omega_t} (\partial_t - \Delta)u(t,\mathbf{x})\widetilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \,\mathrm{d}(t,\mathbf{x}) \\ &+ \int_{\Sigma_T} \widetilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(t,\mathbf{x}) \,\mathrm{d}\sigma_{(t,\mathbf{x})} \\ &- \int_{\Sigma_T} \frac{\partial \widetilde{G}}{\partial \mathbf{n}}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) u(t,\mathbf{x}) \,\mathrm{d}\sigma_{(t,\mathbf{x})} \\ &+ \int_{\Sigma_T} \widetilde{G}(t_0 - t, \mathbf{x}_0 - \mathbf{x}) \langle \mathbf{V}, \mathbf{n}\rangle(t,\mathbf{x}) u(\mathbf{y},\tau) \mathrm{d}\sigma_{(t,\mathbf{x})}, \end{aligned}$$

as it can be seen from Lemma 15 and the property of the fundamental solution. Moreover, we will only look at the case, for which $(\partial_t - \Delta)u = 0$ holds.

We introduce the single and double layer potentials as

$$\begin{aligned} \widetilde{\mathcal{V}}\varphi(t_0,\mathbf{x}_0) &:= \langle \varphi, \gamma_0 G\rangle = \int_{\Sigma_T} G(t_0, t, \mathbf{x}_0, \mathbf{y})\varphi(t,\mathbf{y}) \,\mathrm{d}\sigma_{(t,\mathbf{y})}, \\ \widetilde{\mathcal{K}}w(t_0,\mathbf{x}_0) &:= \langle \gamma_1^+ G, w\rangle = \int_{\Sigma_T} \gamma_{1,(t,\mathbf{y})}^+ G(t_0, t, \mathbf{x}_0, \mathbf{y}) w(t,\mathbf{y}) \,\mathrm{d}\sigma_{(t,\mathbf{y})}. \end{aligned}$$

Then, similarly to [3, Theorem 2.20], we obtain the representation formula from [17, Equation (6)] given in the following lemma.

Lemma 20. *Let $u \in H^{1,\frac{1}{2}}(Q_T)$ with $(\partial_t - \Delta)u = 0$ in Q_T. Then, we have the representation formula*

$$u(t,\widetilde{\mathbf{x}}) = \widetilde{\mathcal{V}}\gamma_1^- u(t,\widetilde{\mathbf{x}}) - \widetilde{\mathcal{K}}u(t,\widetilde{\mathbf{x}}) \quad \text{for all } (t,\widetilde{\mathbf{x}}) \in Q_T. \tag{5.2}$$

As in [3, pg. 514], we can rewrite the definition of the single layer potential by

$$\begin{aligned} \widetilde{\mathcal{V}}\varphi(t_0,\mathbf{x}_0) &= \langle \varphi, \gamma_0 G(t, t_0, \mathbf{x}, \mathbf{x}_0)\rangle \\ &= \langle \gamma_0' \varphi, G(t, t_0, \mathbf{x}, \mathbf{x}_0)\rangle \\ &= \widetilde{G} \star (\gamma_0'\varphi)(t_0,\mathbf{x}_0), \end{aligned} \tag{5.3}$$

where $\widetilde{G}$ is given in (5.1) and

$$\langle \gamma_0'\varphi, \chi\rangle = \langle \varphi, \gamma_0\chi\rangle = \int_{\Sigma_T} \varphi\chi \,\mathrm{d}\sigma_{(t,\mathbf{x})}$$

for all $\chi \in C_0^\infty(\mathbb{R}^{d+1})$. We will use this also for $\chi \in C_0^2(\mathbb{R}^{d+1})$.

We would like to find the mapping properties of the single and double layer potentials, which are the equivalent of the results given in [3, Proposition 3.1, Remark 3.2, and Proposition 3.3].

Lemma 21. *The mapping*

$$\widetilde{\mathcal{V}}\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$$

is continuous.

Proof. The proof follows as in [3, pg. 514–515] in the case of a cylindrical domain. In there, the claim is proven by considering the problem on $\mathbb{R}^{d+1}$ using Fourier techniques and then restricting it appropriately, which can also be done in the case of a non-cylindrical domain. □

Lemma 22. *The mapping*

$$\widetilde{\mathcal{K}}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta)$$

is continuous.

Proof. The proof is in complete analogy to [3, pg. 515], but we repeat it for the convenience of the reader. We consider the solution operator $\mathcal{T}$, which maps the Dirichlet data g to the solution $u := \mathcal{T}g$ of the partial differential equation (4.8). According to Theorem 7, the solution operator $\mathcal{T}$ is a continuous mapping

$$\mathcal{T}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T;\partial_t - \Delta). \tag{5.4}$$

The representation formula (5.2) yields $u(t,\widetilde{\mathbf{x}}) = \widetilde{\mathcal{V}}\gamma_1^- u(t,\widetilde{\mathbf{x}}) - \widetilde{\mathcal{K}}u(t,\widetilde{\mathbf{x}})$ and thus $\mathcal{T}g = \widetilde{\mathcal{V}}\gamma_1^-\mathcal{T}g - \widetilde{\mathcal{K}}g$ for all $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$. Rearranging gives hence

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{V}}\gamma_1^-\mathcal{T} - \mathcal{T}.$$

The claim follows now by using the mapping property (5.4) of $\mathcal{T}$, $\widetilde{\mathcal{V}}$ (Lemma 21), and γ_1^- (Lemma 13). □

We can take the traces γ_0 of the single and double layer potential. Let the radius R be large enough such that the boundary Γ_t is contained in the ball $B_R := \{\mathbf{x} \in \mathbb{R}^d\colon \|\mathbf{x}\| < R\}$ and set $\Omega_t^c := B_R \backslash \overline{\Omega}_t$ and $Q_T^c := \bigcup_{-\infty<t<T}(\{t\} \times \Omega_t^c)$. Lemmata 21 and 22 provide also the continuity of the mappings

$$\widetilde{\mathcal{V}}\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T^c;\partial_t - \Delta)$$

and

$$\widetilde{\mathcal{K}}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{1,\frac{1}{2}}_{;0,}(Q_T^c;\partial_t - \Delta).$$

In order to state the tube analogue of [3, Theorem 3.4], we define the jumps as in [3, Formula (3.16)] in accordance with

$$[\gamma_0 u] := \gamma_0(u|_{Q_T^c}) - \gamma_0(u|_{Q_T}), \quad [\gamma_1^\pm u] := \gamma_1^\pm(u|_{Q_T^c}) - \gamma_1^\pm(u|_{Q_T}).$$

We then have:

Lemma 23. *For all $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ and all $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$, there hold the jump relations*

$$[\gamma_0\widetilde{\mathcal{V}}\psi] = 0, \qquad [\gamma_1^-\widetilde{\mathcal{V}}\psi] = -\psi,$$
$$[\gamma_0\widetilde{\mathcal{K}}w] = w, \qquad [\gamma_1^-\widetilde{\mathcal{K}}w] = 0.$$

Proof. We mimic the proof of [3] without using the time reversal map. Let $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$. We set $u := \widetilde{\mathcal{V}}\psi$. Due to the mapping property of the single layer potential, we then have $u \in H^{1,\frac{1}{2}}_{;0,}\big((0,T)\times B_R(0)\big)$ and thus, by the trace lemma, we have $\gamma_0(u|_{Q_T}) = \gamma_0(u|_{Q_T^c})$.

Let us next consider the normal jump of $\widetilde{\mathcal{V}}$. From (5.3), we obtain by considering $u = \widetilde{\mathcal{V}}\psi$

$$(\partial_t - \Delta)u = \gamma_0'\psi$$

in $\mathbb{R}_+ \times \mathbb{R}^d$. We consider any test function $\varphi \in C_0^2\big((0,T)\times B_R\big)$ and we obtain

$$\langle \psi, \gamma_0\varphi\rangle = \langle \gamma_0'\psi, \varphi\rangle = \big\langle (\partial_t - \Delta)u, \varphi\big\rangle = -\big\langle u, (\partial_t + \Delta)\varphi\big\rangle,$$

where the last equality holds due to the integration by parts on a cylindrical domain. We thus have

$$\langle \psi, \gamma_0\varphi\rangle = -\int_{(0,T)\times B_R} (\partial_t + \Delta)\varphi u \,\mathrm{d}(t,\mathbf{x}). \tag{5.5}$$

On the other hand, we can use Green's second formula, given in Lemma 15 in Q_T and Q_T^c, where we use that $(\partial_t - \Delta)u = 0$ in $Q_T \cup Q_T^c$. This yields

$$\int_{Q_T} (\partial_t + \Delta)\varphi u \,\mathrm{d}(t,\mathbf{x}) = \langle \gamma_0 u, \gamma_1^+\varphi\rangle - \langle \gamma_1^- u, \gamma_0\varphi\rangle$$

and

$$\int_{Q_T^c} (\partial_t + \Delta)\varphi \; u \,\mathrm{d}(t,\mathbf{x}) = -\langle \gamma_0 u, \gamma_1^+\varphi\rangle + \langle \gamma_1^- u, \gamma_0\varphi\rangle.$$

Adding these two expressions yields

$$\int_{(0,T)\times B_R} (\partial_t + \Delta)\varphi u \,\mathrm{d}(t,\mathbf{x}) = \big\langle [\gamma_1^- u], \gamma_0\varphi\big\rangle, \tag{5.6}$$

where we used $[\gamma_0 u] = 0 = [\gamma_0\varphi] = [\gamma_1^+\varphi]$. Comparing (5.5) with (5.6) results in $[\gamma_1^- u] = -\psi$.

We are left with proving the jump relations for the double layer potential. To that end, we choose $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ and define $u := \widetilde{\mathcal{K}}w$. Let $\varphi \in C_0^\infty(\mathbb{R}_+ \times B_R)$ be a test function. As above, we obtain

$$\int_{(0,T)\times B_R} (\partial_t + \Delta)\varphi u \,\mathrm{d}(t,\mathbf{x}) = \big\langle [\gamma_1^- u], \gamma_0\varphi\big\rangle - \big\langle [\gamma_0 u], \gamma_1^+\varphi\big\rangle. \tag{5.7}$$

For $\widetilde{\mathcal{K}}$, we obtain $\widetilde{\mathcal{K}}w = \widetilde{G} \star \big((\gamma_1^+)'w\big)$ similar to (5.3). Therefore, we have $(\partial_t - \Delta)\widetilde{\mathcal{K}}w = (\gamma_1^+)'w$ in $\mathbb{R}_+ \times B_R$. From here, it follows that

$$-\int_{(0,T)\times B_R} (\partial_t + \Delta)\varphi u \,\mathrm{d}(t,\mathbf{x}) = \big\langle (\partial_t - \Delta)u, \varphi\big\rangle = \big\langle (\gamma_1^+)'w, \varphi\big\rangle = \langle w, \gamma_1^+\varphi\rangle. \tag{5.8}$$

Comparing (5.7) with (5.8) yields

$$\big\langle [\gamma_1^- u], \gamma_0\varphi\big\rangle = \big\langle [\gamma_0 u] - w, \gamma_1^+\varphi\big\rangle \tag{5.9}$$

for all $\varphi \in C_0^2(\mathbb{R}_+ \times B_R)$. Applying Lemma 18 says that both sides of (5.9) have to vanish identically, from where $[\gamma_1^- u] = 0$ and $[\gamma_0 u] = w$ follows. $\square$

Now, as in [3, Definition 3.5], we are in the position to define the boundary integral operators.

Definition 24. *Let $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ and $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$. We can then define the single layer operator as*

$$\mathcal{V}\psi := \gamma_0 \widetilde{\mathcal{V}}\psi,$$

the adjoint double layer operator as

$$\mathcal{K}'\psi := \frac{1}{2}\left(\gamma_1^-(\widetilde{\mathcal{V}}\psi)|_{Q_T} + \gamma_1^-(\widetilde{\mathcal{V}}\psi)|_{Q_T^c}\right),$$

the double layer operator as

$$\mathcal{K}w := \frac{1}{2}\left(\gamma_0(\widetilde{\mathcal{K}}w)|_{Q_T} + \gamma_0(\widetilde{\mathcal{K}}w)|_{Q_T^c}\right)$$

and the hypersingular operator as

$$\mathcal{D}w := -\gamma_1^- \widetilde{\mathcal{K}}w.$$

As in [3, Theorem 3.7], we have the following mapping properties of these operators.

Theorem 25. *The boundary integral operators from Definition 24 are continuous mappings as follows*

$$\begin{aligned}
\mathcal{V}\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) &\to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T),\\
\mathcal{K}'\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) &\to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T),\\
\mathcal{K}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) &\to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T),\\
\mathcal{D}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) &\to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T).
\end{aligned}$$

Proof. The assertion follows immediately by using the mapping properties of the layer potentials from Lemma 21 and Lemma 22 as well as of the trace operators introduced in Section 3.2 and from Lemma 13. □

We can state the analogue of [3, Formulae (3.24)–(3.27)].

Lemma 26. *It holds*

$$\begin{aligned}
\gamma_0(\widetilde{\mathcal{V}}\psi)|_{Q_T} = \gamma_0(\widetilde{\mathcal{V}}\psi)|_{Q_T^c} &= \mathcal{V}\psi,\\
\gamma_1^-(\widetilde{\mathcal{V}}\psi)|_{Q_T} &= \frac{1}{2}\psi + \mathcal{K}'\psi,\\
\gamma_1^-(\widetilde{\mathcal{V}}\psi)|_{Q_T^c} &= -\frac{1}{2}\psi + \mathcal{K}'\psi,\\
\gamma_0(\widetilde{\mathcal{K}}w)|_{Q_T} &= -\frac{1}{2}w + \mathcal{K}w,\\
\gamma_0(\widetilde{\mathcal{K}}w)|_{Q_T^c} &= \frac{1}{2}w + \mathcal{K}w,\\
\gamma_1^-(\widetilde{\mathcal{K}}w)|_{Q_T} = \gamma_1^-(\mathcal{K}w)|_{Q_T^c} &= -\mathcal{D}w.
\end{aligned}$$

Proof. We just prove the second statement, as the other statements follow similarly. According to Lemma 23, we have

$$[\gamma_1^- \widetilde{\mathcal{V}}\psi] = \gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T^c} - \gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T} = -\psi.$$

Therefore,

$$\gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T} = \psi + \gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T^c}.$$

By Definition 24, we have

$$\mathcal{K}'\psi := \frac{1}{2}\left(\gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T} + \gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T^c}\right).$$

Substituting this into the expression above yields

$$\gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T} = \psi + 2\mathcal{K}'\psi - \gamma_1^- \widetilde{\mathcal{V}}\psi|_{Q_T},$$

from where the claim follows immediately. □

Remark 27. *Following* [17, Formulae (7)–(10)], *the relations in the interior given in Lemma 26 can also be written as*

$$\begin{aligned}
\gamma_0 \widetilde{\mathcal{V}}\psi(t,\mathbf{x}) &= \int_{\Sigma_T} G(t,\tau,\mathbf{x},\mathbf{y})\psi(\tau,\mathbf{y})\,\mathrm{d}\sigma_{(\tau,\mathbf{y})},\\
\gamma_1^- \widetilde{\mathcal{V}}\psi(t,\mathbf{x}) &= \frac{1}{2}\psi(t,\mathbf{x}) + \int_{\Sigma_T} \gamma_{1,(t,\mathbf{x})}^- G(t,\tau,\mathbf{x},\mathbf{y})\psi(\tau,\mathbf{y})\,\mathrm{d}\sigma_{(\tau,\mathbf{y})},\\
\gamma_0 \widetilde{\mathcal{K}}w(t,\mathbf{x}) &= -\frac{1}{2}w(t,\mathbf{x}) + \int_{\Sigma_T} \gamma_{1,(\tau,\mathbf{y})}^+ G(t,\tau,\mathbf{x},\mathbf{y})w(\tau,\mathbf{y})\,\mathrm{d}\sigma_{(\tau,\mathbf{y})},\\
\gamma_1^- \widetilde{\mathcal{K}}w(t,\mathbf{x}) &= -\int_{\Sigma_T} \gamma_{1,(t,\mathbf{x})}^- \gamma_{1,(\tau,\mathbf{y})}^+ G(t,\tau,\mathbf{x},\mathbf{y})w(\tau,\mathbf{y})\,\mathrm{d}\sigma_{(\tau,\mathbf{y})}.
\end{aligned}$$

We can take the traces in the representation formula (5.2) to obtain the Dirichlet data and the Neumann data of the solution u of the homogeneous heat equation. This yields

$$\gamma_0 u = \frac{1}{2}\gamma_0 u - \mathcal{K}\gamma_0 u + \mathcal{V}\gamma_1^- u, \tag{5.10}$$

$$\gamma_1^- u = \mathcal{D}\gamma_0 u + \frac{1}{2}\gamma_1^- u + \mathcal{K}'\gamma_1^- u, \tag{5.11}$$

compare also [17, Formulae (11) and (12)].

As in [3, pg. 518], we can define the Calderón projector and the associated involution $\mathcal{A}$ as

$$\mathcal{C}_{Q_T} := \frac{1}{2}\,\mathrm{id} + \mathcal{A} := \frac{1}{2}\,\mathrm{id} + \begin{bmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{D} & \mathcal{K}' \end{bmatrix}.$$

We state next the analogue of [3, Theorem 3.9].

Theorem 28. *The operator $\mathcal{C}_{Q_T}$ is a projection operator in the space*

$$\mathcal{H} := H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T).$$

The following statements are equivalent for $(w,\psi) \in \mathcal{H}$:

- *(i) There is a $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ with $(\partial_t - \Delta)u = 0$ in Q_T and $w = \gamma_0 u$, $\psi = \gamma_1^- u$ on Σ_T.*

(ii) It holds

$$\begin{bmatrix} w \\ \psi \end{bmatrix} = \mathcal{C}_{Q_T} \begin{bmatrix} w \\ \psi \end{bmatrix}.$$

Proof. We again follow the proof of [3, Thereom 3.9].

(i) ⇒ (ii) follows by the considerations above, especially in (5.10) and (5.11).

For the proof of (ii) ⇒ (i), ψ and w are given and we define

$$u := \widetilde{\mathcal{V}}\psi - \widetilde{\mathcal{K}}w. \tag{5.12}$$

Using the mapping properties of the potentials implies that $u \in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ and we obtain

$$\begin{bmatrix} \gamma_0 u \\ \gamma_1^- u \end{bmatrix} = \mathcal{C}_{Q_T} \begin{bmatrix} w \\ \psi \end{bmatrix}. \tag{5.13}$$

Since the right hand side equals to $[w, \psi]^\intercal$ according to (ii), the claim follows immediately.

It remains to show the projection property of $\mathcal{C}_{Q_T}$. Since on the one hand $[\gamma_0 u, \gamma_1^- u]^\intercal = \mathcal{C}_{Q_T}[\gamma_0 u, \gamma_1^- u]^\intercal$ holds according to (5.10), (5.11) and on the other hand (5.13) holds for any $[w, \psi]^\intercal$ and u given by (5.12), we obtain $\mathcal{C}_{Q_T}[w, \psi]^\intercal = \mathcal{C}_{Q_T}^2[w, \psi]^\intercal$ for any $[w, \psi]^\intercal$ and thus

$$\mathcal{C}_{Q_T}^2 = \mathcal{C}_{Q_T}. \tag{5.14}$$

□

We can state the following corollary in analogy to [3, Corollary 3.10].

Corollary 29. *The operator $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is an isomorphism.*

Proof. We use the same argument as in the proof of [3, Corollary 3.10]. Notice that we can reformulate (5.14) as follows:

$$\left(\frac{1}{2}\,\mathrm{id} + \mathcal{A}\right)^2 = \frac{1}{4}\,\mathrm{id} + \mathcal{A} + \mathcal{A}^2 = \frac{1}{2}\,\mathrm{id} + \mathcal{A}.$$

We hence conclude

$$\mathcal{A}^2 = \frac{1}{4}\,\mathrm{id}, \tag{5.15}$$

which is equivalent to

$$\mathcal{A}^{-1} = 4\mathcal{A}.$$

□

As in [3], we can interchange the columns of the operator $\mathcal{A}$ to define the operator

$$A := \begin{bmatrix} \mathcal{V} & -\mathcal{K} \\ \mathcal{K}' & \mathcal{D} \end{bmatrix},$$

which is an isomorphism of the space

$$\mathcal{H}' := H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$$

onto its dual space $\mathcal{H}$. Following [3], we define the duality product between $\mathcal{H}'$ and $\mathcal{H}$ in accordance with

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, \begin{bmatrix} v \\ \varphi \end{bmatrix} \right\rangle := \langle \psi, v \rangle + \langle \varphi, w \rangle$$

for all v, $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ and φ, $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$. We are now in the position to state the analogue of [3, Theorem 3.11], which is the positive definiteness of the operator A.

Theorem 30. *There exists a constant $\alpha > 0$ such that*

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, A \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle \geq \alpha \left(\|\psi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)} + \|w\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} \right)$$

for all $[\psi, w]^\intercal \in \mathcal{H}'$.

For the proof, we again mimic the proof of [3, Theorem 3.11], which is based on the following lemma (see [3, Lemma 3.12]). Its proof can be found in [3].

Lemma 31. *Let $A\colon X \to X'$ be a bounded linear operator, where X' is the dual space of the Hilbert space X. With a compact operator $T\colon X \to X'$ and a constant α, let A satisfy*

$$\big\langle (A+T)x, x \big\rangle \geq \alpha \|x\|_X^2 \quad \textit{for all } x \in X$$

and

$$\langle Ax, x \rangle > 0 \quad \textit{for all } x \in X \backslash \{0\}. \tag{5.16}$$

Then, there exists a constant $\alpha_1 > 0$ such that

$$\langle Ax, x \rangle \geq \alpha_1 \|x\|_X^2 \quad \textit{for all } x \in X.$$

Moreover, we need the following analogue of [3, Lemma 2.15].

Lemma 32. *Let $u \in \mathcal{V}(Q_T)$ such that $(\partial_t - \Delta)u = 0$ in Q_T. Then, there exist constants m_1, m_2, and m_3 such that*

$$\|u\|_{H^{1,0}(Q_T)} \leq m_1 \|u\|_{H^{1,\frac{1}{2}}(Q_T)} \leq m_2 \|u\|_{\mathcal{V}(Q_T)} \leq m_3 \|u\|_{H^{1,0}(Q_T)}.$$

In other words, for functions $u \in \mathcal{V}(Q_T)$ satisfying the homogeneous heat equation, we have the equivalence of the norms in $\mathcal{V}(Q_T)$, $H^{1,0}(Q_T)$, and $H^{1,\frac{1}{2}}(Q_T)$.

Proof. The first and second inequality follow directly from the equivalence of norms on Q_T and Q_0, since the proof of [3] is based on the definition of the norm for the first inequality and the interpolation result (2.5) for the second inequality. Nonetheless, we cannot apply the equivalence of norms on Q_T and norms on Q_0 directly for the third inequality, because we have $(\partial_t - \Delta)u = 0$ as an assumption, which is needed to show the third inequality. Mapping this differential operator from the tube onto the cylinder oder vice versa will alter it. Therefore, we use the ideas of the proof of [3, Lemma 2.15], but adapt them to our context.

Transforming the partial differential equation $(\partial_t - \Delta)u = 0$ from Q_T back to Q_0 via the weak formulation (see [1]) yields

$$\partial_t(u \circ \boldsymbol{\kappa}) - \mathcal{M}(u \circ \boldsymbol{\kappa}) = 0 \quad \text{in } Q_0,$$

where $\mathcal{M}$ is defined as

$$\begin{aligned}\mathcal{M}(u \circ \boldsymbol{\kappa}) := \operatorname{div}\Big((\mathrm{D}\boldsymbol{\kappa})^{-1}(\mathrm{D}\boldsymbol{\kappa})^{-\intercal}\nabla(u \circ \boldsymbol{\kappa})\Big) + (\mathrm{D}\boldsymbol{\kappa})^{-\intercal}\nabla(u \circ \boldsymbol{\kappa}) \cdot \partial_t \boldsymbol{\kappa} \\ + \frac{1}{\det(\mathrm{D}\boldsymbol{\kappa})}\nabla\big(\det(\mathrm{D}\boldsymbol{\kappa})\big) \cdot (\mathrm{D}\boldsymbol{\kappa})^{-1}(\mathrm{D}\boldsymbol{\kappa})^{-\intercal}\nabla(u \circ \boldsymbol{\kappa}).\end{aligned}$$

By the standard theory, for fixed $t \in (0, T)$, we have that $\mathcal{M}\colon H^1(\Omega_0) \to H^{-1}(\Omega_0)$ is bounded. Thus, for $u \in H^{1,0}(Q_0)$, we obtain $\mathcal{M}u \in H^{-1,0}(Q_0) := [H^{1,0}(Q_0)]'$ and we conclude

$$\begin{aligned}\|u\|^2_{\mathcal{V}(Q_T)} = \|u \circ \boldsymbol{\kappa}\|^2_{\mathcal{V}(Q_0)} &= \|u \circ \boldsymbol{\kappa}\|^2_{H^{1,0}(Q_0)} + \big\|\partial_t(u \circ \boldsymbol{\kappa})\big\|^2_{H^{-1,0}(Q_0)} \\ &= \|u \circ \boldsymbol{\kappa}\|^2_{H^{1,0}(Q_0)} + \big\|\mathcal{M}(u \circ \boldsymbol{\kappa})\big\|^2_{H^{-1,0}(Q_0)} \\ &\lesssim \|u \circ \boldsymbol{\kappa}\|^2_{H^{1,0}(Q_0)} \\ &= \|u\|^2_{H^{1,0}(Q_T)}.\end{aligned}$$

□

Proof of Theorem 30. We follow the proof of [3, Theorem 3.11]. As above, we let the radius $R > 0$ be big enough such that the ball B_R contains the boundary Γ_t for all t. We then write $\Omega^c_{t,R} = B_R \backslash \overline{\Omega}_t$ and $Q^c_{T,R} = \bigcup_{0<t<T}(\{t\} \times \Omega^c_{t,R})$.

Let $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ and $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$. Apart from the boundary Σ_T, we define

$$u := \widetilde{\mathcal{V}}\psi - \widetilde{\mathcal{K}}w. \tag{5.17}$$

From the jump relations (see Lemma 23), we obtain

$$[\gamma_0 u] = -w, \quad [\gamma_1^- u] = -\psi. \tag{5.18}$$

Using Definition 24 immediately yields

$$\frac{1}{2}\left(\begin{bmatrix}\gamma_0 u|_{Q_T} \\ \gamma_1^- u|_{Q_T}\end{bmatrix} + \begin{bmatrix}\gamma_0 u|_{Q^c_{T,R}} \\ \gamma_1^- u|_{Q^c_{T,R}}\end{bmatrix}\right) = A\begin{bmatrix}\psi \\ w\end{bmatrix}. \tag{5.19}$$

In view of (5.18) and (5.19), we can rewrite the bilinear form as

$$\begin{aligned}\left\langle \begin{bmatrix}\psi \\ w\end{bmatrix}, A\begin{bmatrix}\psi \\ w\end{bmatrix}\right\rangle &= \frac{1}{2}\left\langle \begin{bmatrix}\gamma_1^- u|_{Q_T} \\ \gamma_0 u|_{Q_T}\end{bmatrix} - \begin{bmatrix}\gamma_1^- u|_{Q^c_{T,R}} \\ \gamma_0 u|_{Q^c_{T,R}}\end{bmatrix}, \begin{bmatrix}\gamma_0 u|_{Q_T} \\ \gamma_1^- u|_{Q_T}\end{bmatrix} + \begin{bmatrix}\gamma_0 u|_{Q^c_{T,R}} \\ \gamma_1^- u_{Q^c_{T,R}}\end{bmatrix}\right\rangle \\ &= \langle \gamma_1^- u|_{Q_T}, \gamma_0 u|_{Q_T}\rangle - \langle \gamma_1^- u|_{Q^c_{T,R}}, \gamma_0 u|_{Q^c_{T,R}}\rangle,\end{aligned} \tag{5.20}$$

where we used (5.16).

Since u satisfies $(\partial_t - \Delta)u = 0$ in Q_T and also in $Q^c_{T,R}$, we apply Green's first formula (4.6) (see Lemma 15) to obtain

$$\int_{Q_T} \|\nabla u\|^2 \,\mathrm{d}(\mathbf{x}, t) + d(u, u) = \langle \gamma_1^- u|_{Q_T}, \gamma_0 u|_{Q_T}\rangle,$$

while applying (4.9) (see Lemma 19) yields

$$\int_{Q_T} \|\nabla u\|^2 \,\mathrm{d}(t,\mathbf{x}) - d(u,u) + \int_{\Omega_T} |u(T,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \langle \gamma_1^- u|_{Q_T}, \gamma_0 u|_{Q_T}\rangle.$$

Adding the two expressions together gives[3]

$$\begin{aligned} \langle \gamma_1^- u|_{Q_T}, \gamma_0 u|_{Q_T}\rangle &= \int_{Q_T} \|\nabla u\|^2 \,\mathrm{d}(t,\mathbf{x}) + \frac{1}{2}\int_{\Omega_T} |u|^2 \,\mathrm{d}\mathbf{x} \\ &\geq \int_{Q_T} \|\nabla u\|^2 \,\mathrm{d}(t,\mathbf{x}). \end{aligned} \tag{5.21}$$

On $Q_{T,R}^c$, we obtain analogously

$$\begin{aligned} -\langle \gamma_1^- u|_{Q_{T,R}^c}, \gamma_0 u|_{Q_{T,R}^c}\rangle &= \int_{Q_{T,R}^c} \|\nabla u\|^2 \,\mathrm{d}(\mathbf{x},t) \\ &\quad + \frac{1}{2}\int_{\Omega_{T,R}^c} |u(T,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} - \int_0^T \int_{\partial B_R} u\partial_r u \,\mathrm{d}\sigma \mathrm{d}t, \end{aligned} \tag{5.22}$$

where $\partial_r u$ denotes the normal derivative of u at the boundary $(0,T)\times\partial B_R$. Inserting (5.21) and (5.22) into (5.20) yields

$$\left\langle \begin{bmatrix}\psi\\ w\end{bmatrix}, A\begin{bmatrix}\psi\\ w\end{bmatrix}\right\rangle \geq \int_{Q_T\cup Q_{T,R}^c} \|\nabla u\|^2 \,\mathrm{d}(\mathbf{x},t) - \int_0^T \int_{\partial B_R} u\partial_r u \,\mathrm{d}\sigma \mathrm{d}t. \tag{5.23}$$

According to (5.17), $u|_{(0,T)\times\partial B_R}$ and $\partial_r u|_{(0,T)\times\partial B_R}$ are defined from $[w,\psi]^\intercal$ by the action of integral operators with smooth kernels. These integral operators as well as their adjoints are compact and therefore, using Young's inequality, there exists a compact operator $T_1 : \mathcal{H}\to\mathcal{H}$ such that

$$\left|\int_0^T \int_{\partial B_R} u\partial_r u \,\mathrm{d}\sigma \mathrm{d}t\right| \leq \left\langle \begin{bmatrix}\psi\\ w\end{bmatrix}, T_1\begin{bmatrix}\psi\\ w\end{bmatrix}\right\rangle.$$

According to [3], $H_{;0,}^{1,\frac{1}{2}}(Q_0)$ embeds compactly into $L^2(Q_0)$ and, therefore, also $H_{;0,}^{1,\frac{1}{2}}(Q_T)$ embeds compactly into $L^2(Q_T)$ due to the smooth mapping $\boldsymbol{\kappa}$. Then, in view of the mapping properties of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{K}}$ (see Lemma 21 and Lemma 22), we obtain the existence of another compact operator $T_2\colon \mathcal{H}\to\mathcal{H}$, such that, due to the definition of the norm, we have

$$\begin{aligned} \int_{Q_T\cup Q_{T,R}^c} \|\nabla u\|^2 \,\mathrm{d}(t,\mathbf{x}) &= \|u|_{Q_T}\|_{H^{1,0}(Q_T)}^2 + \big\|u|_{Q_{T,R}^c}\big\|_{H^{1,0}(Q_{T,R}^c)}^2 - \int_{Q_T\cup Q_{T,R}^c} |u|^2 \,\mathrm{d}(t,\mathbf{x}) \\ &= \|u|_{Q_T}\|_{H^{1,0}(Q_T)}^2 + \big\|u|_{Q_{T,R}^c}\big\|_{H^{1,0}(Q_{T,R}^c)}^2 - \left\langle \begin{bmatrix}\psi\\ w\end{bmatrix}, T_2\begin{bmatrix}\psi\\ w\end{bmatrix}\right\rangle. \end{aligned}$$

[3] At this point, it is crucial that we split the term $\langle \mathbf{V},\mathbf{n}\rangle$ in (4.1) with the factor $\frac{1}{2}$. If we choose the factor differently, say λ and $1-\lambda$, we would obtain a boundary term involving $\int_0^T\int_{\Gamma_t}\langle \mathbf{V},\mathbf{n}\rangle(\gamma_0^{\mathrm{int}}u)^2 \,\mathrm{d}\sigma\mathrm{d}t$, which would require an appropriate, not straight-forward treatment.

Due to Lemma 32, the norms of $u|_{Q_T}$ in $H^{1,0}(Q_T)$ and $H^{1,\frac{1}{2}}(Q_T)$ are equivalent, and likewise those of $u|_{Q^c_{T,R}}$. Hence, (5.23) induces

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, A \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle \geq \alpha \left(\|u|_{Q_T}\|^2_{H^{1,\frac{1}{2}}(Q_T)} + \left\|u|_{Q^c_{T,R}}\right\|^2_{H^{1,\frac{1}{2}}(Q^c_{T,R})} \right) - \left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, (T_1 + T_2) \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle. \tag{5.24}$$

Using the jump relations (5.18) and then the trace lemmata (see Subsection 3.2 and Lemma 13) yields

$$\begin{aligned} \|w\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} &= \left\|\gamma_0 u|_{Q_T} - \gamma_0 u|_{Q^c_{T,R}}\right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} \\ &\lesssim \left(\left\|u|_{Q_T}\right\|_{H^{1,\frac{1}{2}}(Q_T)} + \left\|u|_{Q^c_{T,R}}\right\|_{H^{1,\frac{1}{2}}(Q^c_{T,R})} \right) \end{aligned}$$

and similarly

$$\begin{aligned} \|\psi\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)} &= \left\|\gamma_1^- u|_{Q_T} - \gamma_1^- u|_{Q^c_{T,R}}\right\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)} \\ &\lesssim \left(\left\|u|_{Q_T}\right\|_{H^{1,\frac{1}{2}}(Q_T)} + \left\|u|_{Q^c_{T,R}}\right\|_{H^{1,\frac{1}{2}}(Q^c_{T,R})} \right). \end{aligned}$$

Looking at (5.24) we thus have the existence of a constant $\alpha > 0$ with

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, (A + T_1 + T_2) \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle \geq \alpha \left(\|\psi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)} + \|w\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} \right),$$

which is the first assumption in Lemma 31.

It remains to prove the positivity assumption in Lemma 31. To that end, we show that the term $\int_0^T \int_{\partial B_R} u \partial_r u \, \mathrm{d}\sigma \mathrm{d}t$ in (5.23) goes to zero as $R \to \infty$. Let $0 < R_0 < R$ such that $\overline{\Omega}_t \subset B_{R_0}$ for all t and set $Q^c_{T,R_0} = (0,T) \times (B_{R_0} \backslash \overline{\Omega}_0)$. We can use Green's second formula from Lemma 15 for $v = \widetilde{G}(T-t, \mathbf{x})$ with $(t_0, \mathbf{x}_0) \notin Q^c_{T,R_0}$. Thus, outside of Q^c_{T,R_0} for $\|\mathbf{x}\| > R_0$, the function u coincides with

$$u_0 := \widetilde{\mathcal{V}}\psi_0 - \widetilde{\mathcal{K}} w_0,$$

where the potentials take the densities on $\Sigma_{R_0} := (0,T) \times \partial B_{R_0}$ given by

$$w_0 := u|_{\Sigma_{R_0}}, \quad \psi_0 := \partial_r u|_{\Sigma_{R_0}}.$$

Since the singularity of u lies on the boundary Σ_T, the densities w_0 and ψ_0 are smooth and also the boundary Σ_{R_0} is smooth. Therefore, we can estimate $u|_{\Sigma_R} = u_0|_{\Sigma_R}$ and $\partial_r u|_{\Sigma_R} = \partial_r u_0|_{\Sigma_R}$ for $R > R_0$ by looking at the behaviour of the fundamental solution G. Because the fundamental solution is the same for the cylindrical and the non-cylindrical case, we estimate

$$|G(t, \mathbf{x})| \leq C_\mu t^{-\mu} \|\mathbf{x}\|^{2\mu - d} \quad \text{for all } \mu \in \mathbb{R}$$

and we obtain a similar estimate for ∇G. Then, for finite T, we have

$$u = \mathcal{O}(R^{-d}), \quad \partial_r u = \mathcal{O}(R^{-d-1}) \quad \text{as } \|\mathbf{x}\| = R \to \infty.$$

Therefore, since the integrand is of order $\mathcal{O}(R^{-d-d-1})$ and the measure of the boundary ∂B_R is of order $\mathcal{O}(R^{d-1})$, we obtain

$$\int_0^T \int_{\partial B_R} u\partial_r u \, \mathrm{d}\sigma \mathrm{d}t = \mathcal{O}(R^{-d-2}) \to 0 \quad \text{as } \|\mathbf{x}\| = R \to \infty.$$

Since the left hand side in (5.23) is independent of R, we can conclude that

$$\lim_{R\to\infty} \int_{Q^c_{T,R}} \|\nabla u\|^2 \, \mathrm{d}(t, \mathbf{x})$$

is finite and

$$\left\langle \begin{bmatrix} \psi \\ w \end{bmatrix}, A \begin{bmatrix} \psi \\ w \end{bmatrix} \right\rangle \geq \int_{(0,T)\times(\mathbb{R}^d\setminus\Gamma_t)} \|\nabla u\|^2 \, \mathrm{d}(t, \mathbf{x}).$$

Assume that the right hand side vanishes. Then, since u is smooth enough, we obtain that $u(t, \cdot)$ is constant on Ω_t and $\mathbb{R}^d\setminus\overline{\Omega}_t$ for every $t \in (0, T)$. Since $u = 0$ on Ω_0, we thus obtain that $u \equiv 0$ on $(0, T) \times \mathbb{R}^d$. From the jump relations (5.18), we obtain $w = 0$ and $\psi = 0$. This implies the positivity assumption (5.16) of Lemma 31 and the claim in the theorem follows immediately. □

Having the main result Theorem 30 at hand, we can state a few corollaries along the lines of [3, Corollary 3.13, Corollary 3.14, Remark 3.15, Corollary 3.16, Corollary 3.17].

Corollary 33. *The single layer operator*

$$\mathcal{V}\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$$

is an isomorphism and there exists $\alpha > 0$ such that

$$\langle \mathcal{V}\psi, \psi\rangle \geq \alpha\|\psi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)} \quad \text{for all } \psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T). \tag{5.25}$$

The hypersingular operator

$$\mathcal{D}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$$

is an isomorphism and there exists $\alpha > 0$ such that

$$\langle \mathcal{D}w, w\rangle \geq \alpha\|w\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)} \quad \text{for all } w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T). \tag{5.26}$$

Proof. As in the proof of [3, Corollary 3.13], the coercivity estimates (5.25) and (5.26) result from Theorem 30 by using the special cases $w = 0$ and $\psi = 0$, respectively. In view of the continuity of $\mathcal{V}$ and $\mathcal{D}$, this leads to the invertibility of the operators. □

Corollary 34. *The operators*

$$\frac{1}{2}\,\mathrm{id} + \mathcal{K},\quad \frac{1}{2}\,\mathrm{id} - \mathcal{K}\colon H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T),$$
$$\frac{1}{2}\,\mathrm{id} + \mathcal{K}',\quad \frac{1}{2}\,\mathrm{id} - \mathcal{K}'\colon H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$$

are isomorphisms.

Proof. We again follow the proof of [3, Corollary 3.14] directly. From the projection property (5.14), more specifically from (5.15), we obtain

$$\left(\frac{1}{2}\operatorname{id}+\mathcal{K}\right)\left(\frac{1}{2}\operatorname{id}-\mathcal{K}\right)=\mathcal{V}\mathcal{D},$$
$$\left(\frac{1}{2}\operatorname{id}+\mathcal{K}'\right)\left(\frac{1}{2}\operatorname{id}-\mathcal{K}'\right)=\mathcal{W}\mathcal{V}.$$

Since the right hand sides are isomorphisms, we immediately arrive at the claim. □

Remark 35. *The other two relations gained from* (5.15) *lead to*

$$\mathcal{V}^{-1}\mathcal{K}\mathcal{V}=\mathcal{K}'=\mathcal{D}\mathcal{K}\mathcal{D}^{-1}.$$

Corollary 36. *The unique solution $u\in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ of the Dirichlet problem*

$$(\partial_t-\Delta)u=0 \qquad \text{in } Q_T,$$
$$\gamma_0 u=g \qquad \text{on } \Sigma_T,$$

with $g\in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ can be represented

(i) as $u=\widetilde{\mathcal{V}}\psi-\widetilde{\mathcal{K}}g$, where $\psi\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{V}\psi=\left(\frac{1}{2}\operatorname{id}+\mathcal{K}\right)g.$$

(ii) as $u=\widetilde{\mathcal{V}}\psi-\widetilde{\mathcal{K}}g$, where $\psi\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\operatorname{id}-\mathcal{K}'\right)\psi=\mathcal{D}g.$$

(iii) as $u=\widetilde{\mathcal{V}}\psi$, where $\psi\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ is the unique solution of the first kind integral equation

$$\mathcal{V}\psi=g.$$

(iv) as $u=\widetilde{\mathcal{K}}w$, where $w\in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ is the unique solution of the second kind integral equation

$$\left(\frac{1}{2}\operatorname{id}-\mathcal{K}\right)w=-g.$$

In (i) and (ii), it particularly holds $\psi=\gamma_1^- u$ on Σ_T.

Proof. We can again use directly the idea of the proof of [3, Corollary 3.16], which are the uniqueness results from above and the jump relations given in Lemma 26. □

Corollary 37. *The unique solution $u\in H^{1,\frac{1}{2}}_{;0,}(Q_T)$ of the Neumann problem*

$$(\partial_t-\Delta)u=0 \qquad \text{in } Q_T,$$
$$\gamma_1^- u=h \qquad \text{on } \Sigma_T,$$

with $h\in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ can be represented *(i)* *as* $u = \widetilde{\mathcal{V}}h - \widetilde{\mathcal{K}}w$*, where* $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ *is the unique solution of the second kind integral equation*

$$\left(\frac{1}{2}\,\mathrm{id} + \mathcal{K}\right) w = \mathcal{V}h.$$

(ii) *as* $u = \widetilde{\mathcal{V}}h - \widetilde{\mathcal{K}}w$*, where* $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ *is the unique solution of the first kind integral equation*

$$\mathcal{D}w = \left(\frac{1}{2}\,\mathrm{id} - \mathcal{K}'\right) h.$$

(iii) *as* $u = \widetilde{\mathcal{V}}\psi$*, where* $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_T)$ *is the unique solution of the second kind integral equation*

$$\left(\frac{1}{2}\,\mathrm{id} + \mathcal{K}'\right) \psi = h.$$

(iv) *as* $u = \widetilde{\mathcal{K}}w$*, where* $w \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma_T)$ *is the unique solution of the first kind integral equation*

$$\mathcal{D}w = -h.$$

In (i) and (ii), we have that $w = \gamma_0 u$ *on* Σ_T*.*

Proof. The proof follows by the same arguments as in the proof of [3, Corollary 3.17], which is similar to the respective proof for the Dirichlet problem. □

6. Conclusion

In this article, we considered the heat equation on a time-varying (so-called non-cylindrical) domain. In contrast to the problem on a cylindrical domain, we used a modified Neumann trace operator containing a term which is dependent on the velocity of the moving surface. We were able to show the mapping properties of the layer operators by following the proofs of Costabel [3]. To this end, we heavily used the fact that the non-cylindrical domain is a mapped cylindrical domain. Then, using mapped anisotropic Sobolev spaces, we obtain analogous mapping properties and are also able to prove existence and uniqueness of solutions of the Dirichlet and of the Neumann problem.

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