Anisotropic vector fields: quantitative estimates and applications to the Vlasov-Poisson equation

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Introduction

The transport equation

$$\partial_t u + b \cdot \nabla u = 0 \tag{0.0.1}$$

is one of the basic building blocks for several (often nonlinear) partial differential equations (PDEs) from mathematical physics, most notably from fluid dynamics, conservation laws, and kinetic theory. In (0.0.1) the vector field $b = b(t,x) : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be given, hence (0.0.1) is a linear equation for the unknown $u = u(t,x) : (0,T) \times \mathbb{R}^N \to \mathbb{R}$, with a prescribed initial datum $u(0,x) = \bar{u}(x)$. Physically, the solution u is advected by the vector field b. In most applications (0.0.1) is coupled to other PDEs, and moreover the vector field is often not prescribed, but rather depends on the other physical quantities present in the problem. Nevertheless, a thorough understanding of the linear equation (0.0.1) is often the basic step for the treatment of such nonlinear cases.

If the vector field is regular enough (Lipschitz in the space variable, uniformly with respect to time) the well-posedness of (0.0.1) is classically well-understood and is based on the theory of characteristics and on the connection with the ordinary differential equation (ODE)

$$\begin{cases} \dfrac{d}{dt} X(s,x) = b(s, X(s,x)) \\[2ex] X(0,x) = x\,. \end{cases} \tag{0.0.2}$$

The map $X = X(t,x) : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is the (classical) flow associated to the vector field b.

When dealing with problems originating from mathematical physics, however, the regularity available on the advecting vector field is often much lower than Lipschitz, and this prevents the application of the classical theory. The low regularity of the vector field usually accounts for "chaotic" and "turbulent" behaviours of the system. This is the reason why in the last few decades a systematic study of (0.0.1) and (0.0.2) out of the Lipschitz regularity setting has been carried out. We mention in particular the seminal papers by DiPerna and Lions [29] and Ambrosio [4], where respectively Sobolev and bounded variation regularity have been assumed on the vector field, together with assumptions of boundedness of the (distributional) spatial divergence and on the growth of the vector field. We will now (briefly and informally) describe the main points of the theory, and we refer for instance to the survey article [7] for more details.

The approach in [29, 4] is based on the notion of renormalized solution of (0.0.1). Formally at least, a strategy to prove uniqueness for (0.0.1) consists in deriving energy estimates: multiplying (0.0.1) by $2u$, integrating in space, and integrating by parts, one obtains

$$\frac{d}{dt} \int_{\mathbb{R}^N} u(t,x)^2 \, dx \leq \|\mathrm{div} b\|_\infty \int_{\mathbb{R}^N} u(t,x)^2 \, dx\,. \tag{0.0.3}$$

If the divergence of the vector field is bounded, Grönwall lemma together with the linearity of (0.0.1) implies uniqueness. However, the formal computations leading to (0.0.3) cannot be made rigorous without any regularity assumptions: when dealing with weak solutions of (0.0.1), which do not enjoy any regularity beyond integrability, it is not justified to apply the chain rule in order to get the identities

$$2u\partial_t u = \partial_t u^2 \qquad \text{and} \qquad 2u\nabla u = \nabla u^2 .$$

Following [29], we say that a bounded weak solution u of (0.0.1) is a renormalized solution if

$$\partial_t \beta(u) + b \cdot \nabla \beta(u) = 0 \tag{0.0.4}$$

holds in the sense of distributions for every smooth function $\beta : \mathbb{R} \to \mathbb{R}$. Roughly speaking, renormalized solutions are the class inside which the energy estimate (0.0.3) can be made rigorous. The problem is then switched to proving that all weak solutions are renormalized. To achieve this, one can regularize (0.0.1) by convolving with a regularization kernel $\rho_\varepsilon(x)$, obtaining

$$\partial_t u_\varepsilon + b \cdot \nabla u_\varepsilon = b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon =: R_\varepsilon ,$$

where we denote $u_\varepsilon = u * \rho_\varepsilon$ and the right hand side R_ε is called commutator. Multiplying this equation by $\beta'(u_\varepsilon)$ we obtain

$$\partial_t \beta(u_\varepsilon) + b \cdot \nabla \beta(u_\varepsilon) = R_\varepsilon \, \beta'(u_\varepsilon) , \tag{0.0.5}$$

which implies (0.0.4) provided the commutator R_ε converges to zero strongly. Such a convergence holds under Sobolev regularity assumptions on the vector field b, as can be proved by rewriting the commutator as an integral involving difference quotients of the vector field. This strategy has been pursued in [29] to show uniqueness and stability of weak solutions of (0.0.1) in the case of Sobolev vector fields, and extended (with several nontrivial modifications) by Ambrosio [4] to the case of vector fields with bounded variation. The convergence to zero of the right hand side of (0.0.5) is more complex in this last setting, and the convolution kernel ρ_ε has to be properly chosen in a way which depends on the vector field itself.

An alternative approach has been developed in [24], working at the level of the ODE (0.0.2) and deriving a priori estimates for the flow which rely only on the Sobolev regularity and growth of b (without assumptions on the divergence). Out of the smooth contest, the notion of classical flow is replaced with that of an almost-everywhere map solving (0.0.2) in a suitable weak sense. This is called *regular Lagrangian flow* and is measure-preserving in the sense that it does not concentrate trajectories. Equivalently there is a constant L such that

$$\mathcal{L}^d(X(t,\cdot)^{-1}(B)) \le L\mathcal{L}^d(B), \qquad \text{for every Borel} \quad B \subset \mathbb{R}^d,$$

a condition which holds for instance for vector fields with bounded divergence. In [24] the authors obtain an upper bound for the difference between two flows, which eventually leads to uniqueness, stability and compactness (and therefore existence) of Lagrangian flows, as well as wellposdness of Lagrangian solutions to the transport equation. This estimate is derived exploiting a functional measuring a "logarithmic distance" between two flows associated to the same vector field, namely

$$\Phi_\delta(s) = \int \log\left(1 + \frac{|X(s,x) - \bar{X}(s,x)|}{\delta}\right) dx , \tag{0.0.6}$$

where $\delta > 0$ is a small parameter which is optimized in the course of the argument. When X and $\bar{X}$ are both flows associated to the same vector field b, differentiating the functional Φ_δ in time one can estimate

$$\Phi'_\delta(s) \lesssim \int \frac{|b(s,X(s,x)) - b(s,\bar{X}(s,x))|}{|X(s,x) - \bar{X}(s,x)|} \, dx \lesssim \int \left[MDb(s,X(s,x)) + MDb(s,\bar{X}(s,x))\right] dx ,$$

where in the second inequality we have estimated the difference quotients of b with the maximal function of Db. Changing variable along the flows X and $\bar{X}$ (which are assumed to have controlled compressibility), and recalling that the maximal function satisfies the so-called strong inequality $\|Mf\|_{L^p} \lesssim \|f\|_{L^p}$ when $1 < p \leq \infty$ (see Lemma 5.2.6), we find that Φ_δ is uniformly bounded in s and in δ if $b \in W^{1,p}$ with $1 < p \leq \infty$. Together with the estimate

$$\mathcal{L}^N\left(\left\{|X(s,x) - \bar{X}(s,x)| > \gamma\right\}\right) \leq \frac{\Phi_\delta(s)}{\log\left(1 + \frac{\gamma}{\delta}\right)} \qquad \forall \gamma > 0\,, \tag{0.0.7}$$

letting $\delta \to 0$ implies that $X = \bar{X}$ almost everywhere.

The main advantage of this approach lies in its quantitative character. Let us mention that the same approach can also be used in some regularity settings not covered by the approach of [29, 4]. In particular, using more sophisticated harmonic analysis tools, the case when the derivative of the vector field is a singular integral of an L^1 function has been considered in [15]. This has been further developed in [11], allowing for singular integrals of a measure, under a suitable condition on splitting of the space in two groups of variables, modeled on the situation for the Vlasov-Poisson characteristics (3.1.5). In order to treat flows associated to such vector fields, the authors of [11] define a new functional

$$\Phi_{\delta_1,\delta_2}(s) = \int \log\left(1 + \frac{|X_1 - \bar{X}_1|}{\delta_1} + \frac{|X_2 - \bar{X}_2|}{\delta_2}\right) dx\,,$$

which will be used also to prove the main results of this thesis, summarized below.

Lagrangian solutions for the Vlasov-Poisson equation with point-charge

In [26] we consider the Cauchy problem for the repulsive Vlasov-Poisson system in the three dimensional space, where the initial datum is the sum of a diffuse density, assumed to be bounded and integrable, and a point charge. Under some decay assumptions for the diffuse density close to the point charge, under bounds on the total energy, and assuming that the initial total diffuse charge is strictly less than one, we prove existence of global Lagrangian solutions. Our result extends the Eulerian theory of [28], proving that solutions are transported by the flow trajectories. The proof is based on the ODE theory developed in [11] in the setting of vector fields with anisotropic regularity, where some components of the gradient of the vector field is a singular integral of a measure.

Flows of partially regular vector field

In [25] we derive quantitative estimates for the Lagrangian flow associated to a partially regular vector field of the form

$$b(t, x_1, x_2) = (b_1(t, x_1), b_2(t, x_1, x_2)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\,, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\,.$$

We assume that the first component b_1 does not depend on the second variable x_2, and has Sobolev $W^{1,p}$ regularity in the variable x_1, for some $p > 1$. On the other hand, the second component b_2 has Sobolev $W^{1,p}$ regularity in the variable x_2, but only fractional Sobolev $W^{\alpha,1}$ regularity in the variable x_1, for some $\alpha > 1/2$. These estimates imply well-posedness, compactness, and quantitative stability for the Lagrangian flow associated to such a vector field.

Plan of the thesis

The plan of the thesis is the following. In Chapter 1 we will recall the Cauchy-Lipschitz theory for ODEs and the theory of characteristics in the classical setting. In addition, we will review the

DiPerna-Lions ([29]) theory of renormalization and wellposedness of bounded weak solutions to the transport equation, and the extension of this theory to partially regular vector fields ([35]). In Chapter 2 we will present the ODE approach initiated in [24] based on quantitative estimates, which leads to wellposedness results for regular Lagrangian flows. First we will focus in the case of Sobolev vector fields, then on vector fields whose derivative is a singular integral of an L^1 function ([15]) and finally on vector fields with different regularity in different directions. In Chapter 3 we describe the initial value problem for the Vlasov-Poisson equation and present some results regarding, in particular, global existence of a solution. In Chapter 4 and Chapter 5 we present, in order, the first and second result of this thesis ([26] and [25]).

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Chapter 1

The transport equation with non-smooth vector field

In Section 1.1 we recall some known results on the ordinary differential equation and its link with the transport equation in the smooth framework. In Section 1.2 we illustrate the theory of renormalized solutions, due by DiPerna and Lions, which allows to prove well-posedness of solutions to the transport equation in the case of Sobolev vector field. In Section 1.3 we show an extension of the previous theory to the case of only partially Sobolev vector field (see [35]).

1.1 Recalls on the smooth setting

1.1.1 The ordinary differential equation

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set and let $b : \Omega \to \mathbb{R}^d$ be a vector field. We want to study the ordinary differential equation (ODE)

$$\dot{\gamma}(t) = b(t, \gamma(t)). \tag{1.1.1}$$

A (classical) solution of (1.1.1) consists of an interval $I \subset \mathbb{R}$ and a function $\gamma \in C^1(I; \mathbb{R}^d)$ which satisfies (1.1.1) for every $t \in I$. In particular $(t, \gamma(t)) \in \Omega$ for every $t \in I$. The solution γ is also called *integral curve* or *characteristic curve* of the vector field b. If we fix $(t_0, x_0) \in \Omega$, we can consider the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = b(t, \gamma(t)) \\ \gamma(t_0) = x_0, \end{cases} \tag{1.1.2}$$

and we notice that γ is a solution to this problem if and only if $\gamma \in C^0(I; \mathbb{R}^d)$ and satisfies

$$\gamma(t) = x_0 + \int_{t_0}^{t} b(s, \gamma(s))\, ds \qquad \text{for every } t \in I.$$

When b enjoys suitable regularity assumptions, mainly in the space variable, the Cauchy-Lipschitz theory ensures well-posedness for the solution to the Cauchy problem. In particular the following theorem provides local existence and uniqueness.

Theorem 1.1.1 (Picard Lindelöf-Cauchy Lipschitz). *Let $b : \Omega \to \mathbb{R}^d$ be continuous and bounded on some region*

$$D = \{(t, x) : |t - t_0| \le \alpha, |x - x_0| \le \beta\}.$$

Assume that b is Lipschitz continuous with respect to x, uniformly in time, on D, i.e.

$$|b(t, x) - b(t, y)| \le L|x - y| \qquad \textit{for every } (t, x),\ (t, y) \in D. \tag{1.1.3}$$

Then there exists $\delta > 0$ *and a function* γ *belonging to* $C^1([t_0 - \delta, t_0 + \delta]; \mathbb{R}^d)$ *which is the unique solution to* (1.1.2).

Proof. Let M be such that $|b(t,x)| \le M$ on D. We choose $\delta < \min\left\{\alpha, \frac{\beta}{M}, \frac{1}{L}\right\}$ and we will show that there exists a unique $\gamma \in C^0(I; \mathbb{R}^d)$ such that

$$\gamma(t) = x_0 + \int_{t_0}^{t} b(s, \gamma(s))\, ds \qquad \text{for every } t \in I_\delta = [t_0 - \delta, t_0 + \delta].$$

We want to use Banach fixed point theorem and construct a solution by iteration. To do this we define a complete metric space X on which the operator

$$T[\gamma](t) = x_0 + \int_{t_0}^{t} b(s, \gamma(s, \gamma(s))\, ds$$

is a contraction. The space X is defined as

$$X = \{\gamma \in C^0(I_\delta; \mathbb{R}^d) : \gamma(t_0) = x_0 \text{ and } |\gamma(t) - x_0| \le \beta \text{ for every } t \in I_\delta\}.$$

It is easy to see that it is complete, since it is a closed subset of the Banach space $C^0(I_\delta; \mathbb{R}^d)$. Moreover, T takes values in X. Indeed, for each $\gamma \in X$, $T[\gamma]$ is a continuous function satisfying $T[\gamma](t_0) = x_0$, and

$$|T[\gamma](t) - x_0| \le \left| \int_{t_0}^{t} |b(s, \gamma(s))| ds \right| \le M|t - t_0| \le M\delta < \beta.$$

Finally, T is a contraction. Take γ_1 and γ_2 in X. Then

$$\begin{aligned} |T[\gamma_1](t) - T[\gamma_2](t)| &\le \left| \int_{t_0}^{t} |b(s, \gamma_1(s)) - b(s, \gamma_2(s))| ds \right| \\ &\le L \left| \int_{t_0}^{t} |\gamma_1(s) - \gamma_2(s)| ds \right| \\ &\le L|t - t_0| \|\gamma_1 - \gamma_2\|_{L^\infty(I_\delta)} \le L\delta \|\gamma_1 - \gamma_2\|_{L^\infty(I_\delta)}. \end{aligned}$$

This implies that

$$\|T[\gamma_1] - T[\gamma_2]\|_{L^\infty(I_\delta)} \le L\delta \|\gamma_1 - \gamma_2\|_{L^\infty(I_\delta)},$$

where $L\delta < 1$. Hence, we apply Banach fixed point theorem ad we get the existence of a unique fixed point for T, which is indeed the unique solution to (1.1.2). □

Still in the classical framework, we have two more general conditions which are sufficient to get uniqueness. These are stated in the following two propositions.

Proposition 1.1.2 (One-sided Lipschitz condition)**.** *Uniqueness forward in time for* (1.1.2) *holds if the Lipschitz continuity condition* (1.1.3) *in Theorem 1.1.1 is replaced by the following one-sided Lipschitz condition:*

$$(b(t,x) - b(t,y)) \cdot (x - y) \le L|x - y|^2 \qquad \textit{for every } (t,x),\ (t,y) \in D.$$

Proposition 1.1.3 (Osgood condition)**.** *Uniqueness for* (1.1.2) *holds if the Lipschitz continuity condition* (1.1.3) *in Theorem 1.1.1 is replaced by the following Osgood condition:*

$$|b(t,x) - b(t,y)| \le \omega(|x - y|) \qquad \textit{for every } (t,x),\ (t,y) \in D,$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function satisfying $\omega(0) = 0$, $\omega(z) > 0$ for every $z > 0$ and

$$\int_0^1 \frac{1}{\omega(z)} dz = \infty. \tag{1.1.4}$$

Remark 1. The integral which appears in (1.1.4) can be interpreted as the amount of time a trajectory takes to enter or exit the origin.

Remark 2. In case b is Lipschitz, then $\omega(z) \backsim z$ and the Osgood condition is trivially verified.

For vector fields with less regularity then those considered above, there are examples that show non-uniqueness of solutions to (1.1.2).

Examples 1.1.1 (The square root example)**.** Let $b(x) := \sqrt{|x|}$ be a continuous vector field defined on $\mathbb{R}$. Notice that b does not satisfy the Lipschitz continuity condition or the Osgood condition ($\omega(z) \backsim \sqrt{z}$, hence the integral in (1.1.4) converges). It is easy to check that the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = \sqrt{|\gamma(t)|} \\ \gamma(0) = 0 \end{cases} \tag{1.1.5}$$

has infinitely many solutions, given by

$$\gamma^c(t) = \begin{cases} 0 & \text{if } t \leq c \\ \frac{1}{4}(t-c)^2 & \text{if } t \geq c, \end{cases}$$

for every $c \in [0, \infty)$. Heuristically, this means that the solution can "stay at rest" in the origin for an arbitrary long time.

The following theorem, however, guarantees local existence of solutions when the vector field is only continuous.

Theorem 1.1.4 (Peano)**.** *Let $b : \Omega \to \mathbb{R}^d$ be continuous and bounded on some region*

$$D = \{(t, y) : |t - t_0| \leq \alpha, |x - x_0| \leq \beta\}.$$

Then there exists a local solution to (1.1.2).

At this point we want to discuss the maximal interval of existence of the solution to (1.1.2). The solution that we constructed in the previous theorems is in fact local in time. We notice that, in order to obtain a global solution (i.e. defined for all $t \in \mathbb{R}$), it is sufficient, for instance, to require that b is bounded on the whole domain Ω. In this way, every local solution $\gamma : (t_1, t_2) \to \mathbb{R}^d$ is Lipschitz continuous, therefore it can be extended to the closed interval $[t_1, t_2]$. Indeed, for every $t_1 < t < t' < t_2$ we have

$$|\gamma(t') - \gamma(t)| \leq \int_t^{t'} |b(s, \gamma(s))| ds \leq M|t' - t|, \tag{1.1.6}$$

where M is an upper bound for $|b|$ on Ω. Hence we can define $\gamma(t_i) = \lim_{t \to t_i} \gamma(t)$ for $i = 1, 2$. If, for instance, $(t_2, \gamma(t_2))$ is not on the boundary of Ω, we can apply again Theorem 1.1.1 to the the ODE coupled with the initial condition $(t_2, \gamma(t_2))$ and iterate until the extended solution touches the boundary.

Combining this argument on the global existence and Theorem 1.1.1, we obtain global existence and uniqueness of solutions to (1.1.2), under the assumption that b is continuous, (globally) bounded in both variables and locally Lipschitz with respect to the spatial variable, uniformly with respect to the time.

1.1.2 The classical flow

Let γ be a solution to the Cauchy problem with initial condition $\gamma(t_0) = x$. If we look at γ as a function of time and initial point, we can define the *classical flow* of a vector field.

Definition 1.1.5. Let $b : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous and bounded vector field, where $I \subset \mathbb{R}$ is an interval. Let $t_0 \in I$. The *(classical) flow of the vector field b* starting at time t_0 is a map

$$X(t,x) : I \times \mathbb{R}^d \to \mathbb{R}^d$$

which satisfies

$$\begin{cases} \frac{\partial X}{\partial t}(t,x) = b(t, X(t,x)) \\ X(t_0, x) = x. \end{cases} \tag{1.1.7}$$

If b is bounded and locally Lipschitz with respect to x, we can immediately deduce existence and uniqueness of the flow from previous arguments. Moreover, the regularity of the vector field in the spatial variable transfers into analogous regularity of the flow (in the spatial variable). The following theorems specify the last statement.

Theorem 1.1.6. *Let $b : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous and bounded vector field, where $I \subset \mathbb{R}$ is an interval. Assume that b is locally Lipschitz continuous with respect to the spatial variable, uniformly with respect to the time. Then for every $t_0 \in I$ there exists a unique classical flow of b starting at time t_0. Moreover, the flow is Lipschitz continuous in t and locally Lipschitz in x.*

Proof. We have already deduced existence and uniqueness of the flow. Recalling (1.1.6), we have the Lipschitz continuity in time of the flow. We are left to show the regularity in space. Take a rectangle subset $D = [t_1, t_2] \times B_r \subset I \times \mathbb{R}^d$. For each $(t,x) \in D$ we have

$$|X(t,x)| \le x + \int_{t_1}^{t_2} |b(s, X(s,x))| ds \le r + |t_1 - t_2| \|b\|_{L^\infty} := R.$$

From the hypotheses we know that b is Lipschitz continuous in the space variable, uniformly in time, on $[t_1, t_2] \times B_R$, with Lipschitz constant L. Hence, for any $(t,x), (t,y) \in D$, we get

$$\begin{aligned} \frac{d}{dt}|X(t,x) - X(t,y)|^2 &= 2 \langle b(t, X(t,x)) - b(t, X(t,y)), X(t,x) - X(t,y) \rangle \\ &\le 2\, L |X(t,x) - X(t,y)|^2, \end{aligned} \tag{1.1.8}$$

Applying Gronwall's Lemma and the square root to (1.1.8) , we obtain

$$|X(t,x) - X(t,y)| \le |x - y| \exp(L \max\{|t_1|, |t_2|\}). \tag{1.1.9}$$

Hence, on every rectangular set $D \subset I \times \mathbb{R}^d$, the flow is Lipschitz continuous in x, uniformly in t, i.e. X is locally Lipschitz in x, uniformly in t. □

Remark 3. Theorem 1.1.6 obviously still holds if we substitute ”locally” with ”globally” Lipschitz.

Theorem 1.1.7. *Let $b : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth and bounded vector field, where $I \subset \mathbb{R}$ is an interval. Then for every $t_0 \in I$ there exists a unique classical flow of b starting at time t_0, which is smooth with respect to t and x.*

Proof. We just give a sketch of the proof. We first assume that b is C^1 to the spatial variable, uniformly in time. Let e be a unit vector in $\mathbb{R}^d$. We observe that differentiating formally (1.1.7) with respect to x in the direction e we obtain the following ordinary differential equation for $D_x X(t,x)e$:

$$\frac{\partial}{\partial t} D_x X(t,x)e = (D_x b)(t, X(t,x)) D_x X(t,x)e. \tag{1.1.10}$$

Motivated by this, we define $w_e(t,x)$ to be the solution of

$$\begin{cases} \frac{\partial w_e}{\partial t}(t,x) = (D_x b)(t, X(t,x)) w_e(t,x) \\ w_e(t_0, x) = e. \end{cases} \tag{1.1.11}$$

It is easy to check that for every $x \in \mathbb{R}^d$ there exists a unique solution w_e and that it depends continuously on the parameter $x \in \mathbb{R}^d$. It can be proved that

$$\frac{X(t, x+he) - X(t,x)}{h} \to w_e(t,x) \qquad as\ h \to 0.$$

This gives $D_x X(t,x)e = w_e(t,x)$ and, since $w_e(t,x)$ is continuous in x, we can conclude that the flow $X(t,x)$ is differentiable with respect to x with continuous differential. By induction we can then deduce that, if b is C^k with respect to the spatial variable, the flow X is C^k with respect to x.

As regards the regularity in the time variable, by induction, it is trivial to show that, if b is C^k, then X is C^{k+1} with respect to t. □

Finally, notice that, as a consequence of the uniqueness of the flow, the map

$$X(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$$

is bijective, for every $t \in I$. Moreover, denoting by $X(t,s,x)$ the flow of b starting at time $s \in I$, the following *semigroup property* holds:

$$X(t_2, t_0, x) = X(t_2, t_1, X(t_1, t_0, x)) \qquad \text{for every } t_0, t_1, t_2 \in I. \tag{1.1.12}$$

1.1.3 The transport equation

The ODE is strictly related to the following linear partial differential equation, known as transport equation. We consider here the Cauchy problem:

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0 \\ u(0,x) = \bar{u}(x) \end{cases} \tag{1.1.13}$$

where $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is the unknown and $\bar{u} : \mathbb{R}^d \to \mathbb{R}$. In the smooth framework, the relation between the *Lagrangian problem* (ODE) and the *Eulerian problem* (PDE) is due to the *theory of characteristics*. Let $X(t,x)$ be a characteristic curve of b, starting at point x at time $t = 0$, and let $u(t,x)$ be a smooth solution of (1.1.13). If we compute the time derivative of $u(t, X(t,x))$, we get

$$\begin{aligned} \frac{d}{dt} u(t, X(t,x)) &= \frac{\partial u}{\partial t}(t, X(t,x)) + \nabla_x u(t, X(t,x)) \cdot \frac{d}{dt} X(t,x) \\ &= \frac{\partial u}{\partial t}(t, X(t,x)) + b(t, X(t,x)) \cdot \nabla_x u(t, X(t,x)) = 0, \end{aligned} \tag{1.1.14}$$

which means that u is constant along the characteristics of b. Hence, we have a formula for the solution to (1.1.13) in terms of the flow of b:

$$u(t,x) = \bar{u}(X(t,\cdot)^{-1}(x)). \tag{1.1.15}$$

This means in particular that a smooth solution to (1.1.13), in case it exists, is unique. In order to check that u, as defined in (1.1.15), is a solution to the transport equation, we observe that the flow $X(t,s,x)$ satisfies the equation

$$\frac{\partial X}{\partial s}(t,s,x) + b(s,x) \cdot \nabla_x X(t,s,x) = 0. \tag{1.1.16}$$

Indeed, exploiting the semigroup property of the flow, we have $\frac{d}{ds}X(t,s,X(s,t,y)) = \frac{d}{ds}x = 0$, which implies (1.1.16), after setting $x = X(s,t,y)$. Therefore, if $\bar{u}$ is C^1, $u(t,x) = \bar{u}(X(0,t,x))$ satisfies the transport equation, as we can compute

$$\frac{\partial}{\partial s}\bar{u}(X(0,s,x))+(b(s,x)\cdot\nabla_x)\bar{u}(X(0,s,x)) = \bar{u}'(X(0,s,x))\cdot\Big(\frac{\partial X}{\partial s}(0,s,x) + b(s,x)\cdot\nabla_x X(0,s,x)\Big) = 0. \tag{1.1.17}$$

1.2 The transport equation in the Sobolev setting

In this Section we describe a strategy, which goes back to DiPerna and Lions (see [29]), that allows to obtain well-posedness for a solution to the transport equation, when the vector field $b(t,x)$ is not Lipschitz continuous in the space variable, but rather has Sobolev regularity.

1.2.1 Weak solutions

We first introduce the weak formulation of the transport equation (1.1.13). Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a locally integrable vector field and denote by div$b$ the divergence of b (with respect to the spatial coordinates) in the sense of distributions.

Definition 1.2.1. Let b, div$b$ and $\bar{u}$ be locally integrable functions. Then a locally bounded function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a weak solution of (1.1.13) if the following identity holds for every function $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$:

$$\int_0^T \int_{\mathbb{R}^d} u\left[\partial_t\varphi + \varphi\mathrm{div}b + b\cdot\nabla\varphi\right] dx\,dt = -\int_{\mathbb{R}^d} \bar{u}(0)\varphi(0,x)dx. \tag{1.2.1}$$

This is the standard notion of weak solution of a PDE and it can be deduced for regular solutions from (1.1.13) multiplying it by φ and integrating by parts. Noticing that functions of the form $\varphi(t,x) = \varphi_1(t)\varphi_2(x)$ are dense in the space of test functions $C_c^\infty((0,T)\times\mathbb{R}^d)$, we are able to give a second equivalent definition of weak solution:

Definition 1.2.2. Let b, div$b$ and $\bar{u}$ be locally integrable functions. We say that a locally bounded function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a weak solution of (1.1.13) if, for every $t\in[0,T]$ and for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\int u(t,x)\varphi(x)dx = \int \bar{u}(x)\varphi(x)dx + \int_0^t\int u(s,x)\varphi(x)\mathrm{div}b(s,x)dx\,ds + \int_0^t\int u(s,x)b(s,x)\cdot\nabla\varphi(x)dx\,ds. \tag{1.2.2}$$

For completeness, we present a third definition, equivalent to the first two. Notice that, if u is merely bounded, the term $\partial_t u$ has a meaning as a distribution, but $b\cdot\nabla u$ is not well defined. Nevertheless, if div$b \in L^1_{\mathrm{loc}}$, we can define the product $b\cdot\nabla u$ as a distribution via the equality

$$\langle b\cdot\nabla u, \phi\rangle := -\langle bu, \nabla\phi\rangle - \langle u\mathrm{div}b, \phi\rangle \qquad \forall\phi\in C_c^\infty((0,T)\times\mathbb{R}^d). \tag{1.2.3}$$

This allows us to give directly a distributional meaning to the transport equation and therefore we have the following

Definition 1.2.3. Suppose b and $\text{div}b$ be locally integrable. Then we say that a locally bounded function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a weak solution of the transport equation if

$$\partial_t u + \text{div}(ub) - u\text{div}b = 0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^d).$$

Concerning the Cauchy problem, it can be proved (see [23]) that, if u is a solution in the sense of Definition 1.2.3, there exists a unique $\tilde{u}$, which is the weak* $-$ L^∞ continuous representative, which means that $t \mapsto \tilde{u}(t,\cdot)$ is weakly* continuous from $[0,T]$ into $L^\infty(\mathbb{R}^d)$. Thus, we can couple the transport equation with $u(0,x) = \bar{u}(x)$ (for a given $\bar{u} : \mathbb{R}^d \to \mathbb{R}$), by simply requiring that $\tilde{u}(0,x) = \bar{u}(x)$. This gives sense to the initial data at $t = 0$.

We remark again that Definition 1.2.1 and Definition 1.2.3 (with initial condition interpreted as in the argument above) are equivalent.

Existence of weak solutions Existence of weak solutions to (1.1.13) is rather easy to prove. A smooth regularization of the vector field and of the initial data enables to construct a sequence of smooth solutions. We then pass to the limit and get a solution thanks to the linearity of the equation.

Theorem 1.2.4. *Let b, $\text{div}b \in L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$ and let $\bar{u} \in L^\infty(\mathbb{R}^d)$. Then there exists a weak solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ to (1.1.13).*

Proof. Let ρ_ε be a standard mollifier on $\mathbb{R}^d$ and let η_ε be a mollifier on $\mathbb{R}^{d+1}$. Denote by $\bar{u}^\varepsilon = \bar{u} * \rho_\varepsilon$ and $b_\varepsilon = b * \eta_\varepsilon$. Since b_ε and $\bar{u}^\varepsilon$ are smooth, there is a unique solution u_ε to the Cauchy problem

$$\begin{cases} \partial_t u + b_\varepsilon \cdot \nabla u = 0 \\ u(0,\cdot) = \bar{u}^\varepsilon. \end{cases} \tag{1.2.4}$$

From the explicit formula for the solution to the transport equation with smooth vector field, we get that $\{u_\varepsilon\}$ is equi-bounded in $L^\infty([0,T]\times\mathbb{R}^d)$. Hence, up to a subsequence, we have that u_ε is weakly* convergent to a limit u in $L^\infty([0,T]\times\mathbb{R}^d)$ which is, clearly, by linearity, a weak solution to (1.1.13). □

1.2.2 A strategy for uniqueness

In the following we want to present a general strategy to show well-posedness of the transport equation. In order to motivate the concept of renormalized solutions, introduced by DiPerna and Lions, we present some formal computations. We start from multiplying both sides of

$$\partial_t u + b\cdot\nabla u = 0$$

by $\beta'(u)$, being $\beta : \mathbb{R}\to\mathbb{R}$ a C^1 function such that $\beta(y) > 0$ for every $y \neq 0$ and $\beta(0) = 0$. We get

$$\beta'(u)\partial_t u + \beta'(u) b\cdot\nabla u = 0. \tag{1.2.5}$$

If b and u were smooth, we could apply the ordinary chain rule and rewrite the last equation as

$$\partial_t\beta(u) + b\cdot\nabla\beta(u) = 0. \tag{1.2.6}$$

The last passage is justified only under regularity assumptions on b and u, and in general is false. Integrating on $\mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d}\partial_t\beta(u(t,x))dx + \int_{\mathbb{R}^d} b(t,x)\cdot\nabla\beta(u(t,x))dx = 0, \tag{1.2.7}$$

and, applying the divergence theorem, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(u(t,x))dx = \int_{\mathbb{R}^d} \beta(u(t,x))\text{div}b(t,x)dx. \tag{1.2.8}$$

Assuming that $\|\text{div}b\|_{L^\infty} \leq C$, for some $C > 0$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(u(t,x))dx \leq C\int_{\mathbb{R}^d} \beta(u(t,x))dx.$$

Using Gronwall's Lemma we obtain

$$\int_{\mathbb{R}^d} \beta(u(t,x))dx \leq e^{Ct}\int_{\mathbb{R}^d} \beta(u(0,x))dx$$

This implies that, if the initial data is $\bar{u} = 0$, then the only solution is $u \equiv 0$. Since the transport equation is linear, this is enough to conclude the uniqueness.

1.2.3 Renormalization

We observe that, in case u and b are not C^1, the computation in the previous Section still holds if we have the following equality (which is "almost a chain rule"):

$$\partial_t\beta(u) + b\cdot\nabla\beta(u) = \beta'(u)[\partial_t u + b\cdot\nabla u],$$

or, alternatively, if

$$\partial_t u + b\cdot\nabla u = 0 \Longrightarrow \partial_t\beta(u) + b\cdot\nabla\beta(u) = 0.$$

This informal argument leads us to introduce a class of weak solutions which satisfy such a rule, in the sense of distributions.

Definition 1.2.5 (Renormalized solutions)**.** Let b and div$b$ be locally integrable functions, and $\bar{u}$ be bounded. We say that a function $u \in L^\infty([0,T]\times\mathbb{R}^d)$ is a renormalized solution to (1.1.13) if it is indeed a weak solution and, for every $\beta \in C^1(\mathbb{R})$, $\beta(u)$ is a weak solution with initial data $\beta(\bar{u})$.

When the renormalization property is satisfied by all bounded weak solutions, it can be transferred to a property of the vector field itself.

Definition 1.2.6 (Renormalization property)**.** Let b, div$b \in L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$. We say that b has the renormalization property if every bounded solution of the transport equation with vector field b is a renormalized solution.

It turns out that this property is intrinsically tied to the well-posedness problem: in particular, renormalization implies well-posedness. Under certain additional assumptions (such as div$b \in L^\infty$) renormalization also implies stability of solutions. The precise statement is the following theorem, which is a minor simplification of Corollary II.1 in [29].

Theorem 1.2.7. *Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a vector field with div$b \in L^1([0,T];L^\infty(\mathbb{R}^d))$ and such that*

$$\frac{b}{1+|x|} = \tilde{b}_1 + \tilde{b}_2 \in L^1([0,T];L^1(\mathbb{R}^d)) + L^1([0,T];L^\infty(\mathbb{R}^d)). \tag{1.2.9}$$

Let $\bar{u} \in L^\infty(\mathbb{R}^d)$. If b has the renormalization property, then there exists a unique weak solution to the transport equation with initial condition $\bar{u}$. Moreover, solutions are stable. By stability we mean that, if b_k and $\bar{u}_k$ are smooth approximating sequences converging strongly in L^1_{loc} to b and $\bar{u}$ respectively, with $\|\bar{u}_k\|_{L^\infty}$ uniformly bounded, then the solutions u_k of the corresponding transport equations converge strongly in L^1_{loc} to the solution u of (1.1.13).

Proof. UNIQUENESS. From the linearity of the equation, it is sufficient to show that $u \equiv 0$ when $\bar{u} = 0$.

We will prove uniqueness for solutions in $L^\infty([0,T]; L^\infty \cap L^1(\mathbb{R}^d))$. The general case, that is when u is only bounded, is done using a duality argument, exploiting the previous case. We take $\varphi \in C_c^\infty$ such that $\operatorname{supp}\varphi \subset B_2$ and $\varphi \equiv 1$ on B_1. We consider the smooth cut-off functions $\varphi_R = \varphi\left(\frac{\cdot}{R}\right)$ for $R \geq 1$. Since b has the renormalized property, we have that, for every $\beta \in C^1(\mathbb{R})$, $\beta(u)$ is a weak solution with initial data $\beta(\bar{u})$. In particular, let us take β such that $\beta > 0$, $\beta(0) = 0$ and test function φ_R. From Definition 1.2.2, we get

$$\int \beta(u(t,x))\varphi_R(x)dx = \int_0^t \int \beta(u(s,x))\varphi_R(x)\operatorname{div} b(s,x)dxds + \int_0^t \int \beta(u(s,x))b(s,x)\cdot\nabla\varphi_R(x)dxds. \tag{1.2.10}$$

For the last integral we can estimate

$$\begin{aligned}
&\left|\int_0^t \int \beta(u(s,x))b(s,x)\cdot\nabla\varphi_R(x)dxds\right| \leq \int_0^t \int \left|\beta(u(s,x))\frac{b(s,x)}{1+|x|}(1+|x|)\cdot\nabla\varphi_R(x)\right| dxds \\
&\leq \|\beta(u)\|_{L^\infty}(1+2R)\|\nabla\varphi_R\|_{L^\infty}\int_0^t \int_{|x|>R} |\tilde{b}_1|dxds + (1+2R)\|\nabla\varphi_R\|_{L^\infty}\int_0^t \|\tilde{b}_2(s,x)\|_{L^\infty_x}\int_{|x|>R}|\beta(u_s)|dxds \\
&\leq \|\beta(u)\|_{L^\infty}\frac{1+2R}{R}\|\nabla\varphi\|_{L^\infty}\|\tilde{b}_1\|_{L^1_s(L^1_{|x|>R})} + \frac{1+2R}{R}\|\nabla\varphi\|_{L^\infty}\int_0^t f(s)\int_{|x|>R}|\beta(u_s)|dxds \\
&\leq C\|\tilde{b}_1\|_{L^1_s(L^1_{|x|>R})} + C\int_0^t f(s)\int_{|x|>R}|\beta(u_s)|dxds = \alpha_R(t),
\end{aligned} \tag{1.2.11}$$

with $f(s) \in L^1([0,T])$. Hence we combine (1.2.10) and (1.2.11), and we get

$$\left|\int \beta(u_t)\varphi_R dx\right| \leq \int_0^t \|\operatorname{div} b(s,x)\|_{L^\infty_x}\left|\int \beta(u_s)\varphi_R dx\right| ds + \alpha_R(t). \tag{1.2.12}$$

Choosing β such that $\beta(u) \leq |u|$ and thereby exploiting the summability of u, we have that $\alpha_R(t) \to 0$ as $R \to \infty$. Therefore, passing to the limit for $R \to \infty$ in (1.2.12) we obtain

$$\left|\int \beta(u_t)dx\right| \leq \int_0^t \|\operatorname{div} b(s,x)\|_{L^\infty_x}\left|\int \beta(u_s)dx\right| ds. \tag{1.2.13}$$

Finally Gronwall's Lemma yields to

$$\int \beta(u_t)dx = 0,$$

which implies $u_t \equiv 0$ for every $t \in [0,T]$.

STABILITY. Arguing as in Theorem 1.2.4, we easily deduce that, up to subsequences, u_k converges weakly* in $L^\infty([0,T]\times\mathbb{R}^d)$ to a weak solution. Since the solution is unique, the whole sequence converges to u. Since b_k and u_k are both smooth, u_k is a renormalized solution, therefore u_k^2 solves the transport equation with initial data $\bar{u}_k^2$. Arguing as before, u_k^2 must converge weakly* in $L^\infty([0,T]\times\mathbb{R}^d)$ to the unique solution of (1.1.13) with initial data $\bar{u}^2$. But by the renormalization property, this solution is u^2. Since $u_k \rightharpoonup^\star u$ and $u_k^2 \rightharpoonup^\star u^2$ in $L^\infty([0,T]\times\mathbb{R}^d)$, we deduce by Radon-Riesz theorem that $u_k \to u$ strongly in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$. □

1.2.4 Commutator estimates

From Theorem 1.2.7 we deduce that the renormalization property for a vector field b is enough to prove uniqueness of weak solutions to the relative transport equation. We now come to the seminal result of DiPerna and Lions, in which it is proven that every vector field with Sobolev regularity satisfies the renormalization property.

Proposition 1.2.8. *Let $b \in L^1_{\text{loc}}([0,T]; W^{1,1}_{\text{loc}}(\mathbb{R}^d))$, $\operatorname{div} b \in L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$ and let $u \in L^\infty_{\text{loc}}([0,T]\times\mathbb{R}^d)$ be a weak solution of the transport equation. Then u is a renormalized solution.*

Proof. Let $\{\rho_\varepsilon\}_\varepsilon$ be a family of even convolution kernels in $\mathbb{R}^d$. Denote $u_\varepsilon = u * \rho_\varepsilon$. Convolving the transport equation with ρ_ε, and adding and subtracting the term $b \cdot \nabla u_\varepsilon$, we get that u_ε is a weak solution to the following PDE:

$$\partial_t u_\varepsilon + b \cdot \nabla u_\varepsilon = b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon. \tag{1.2.14}$$

We then define the *commutator* r_ε as the error term in the right hand side of (1.2.14):

$$r_\varepsilon := [b \cdot \nabla, \rho_\varepsilon](u) = b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon, \tag{1.2.15}$$

where $b \cdot \nabla u$ is the distribution defined in (1.2.3). The name *commutator* comes from the fact that this term measures the difference in exchanging the operations of convolution and differentiating in the direction of b. Notice that u_ε is, trivially, smooth in the space variable and $W^{1,1}_{\text{loc}}([0,T])$, as $\partial_t u_\varepsilon = -(b \cdot \nabla u) * \rho_\varepsilon = (ub) * \nabla \rho_\varepsilon + (u \operatorname{div} b) * \rho_\varepsilon$ belongs to L^1_{loc} in time. Thus we can apply Stampacchia's chain rule for Sobolev spaces, to get

$$\partial_t \beta(u_\varepsilon) + b \cdot \nabla \beta(u_\varepsilon) = \beta'(u_\varepsilon) r_\varepsilon. \tag{1.2.16}$$

In order to recover the renormalization property, we would like to pass to the limit, as $\varepsilon \to 0$, showing the convergence to zero of the quantity $\beta'(u_\varepsilon) r_\varepsilon$. The convergence in distribution of the left hand side of the identity above to (1.2.6) is trivial. The convergence of r_ε to 0, in the distributional sense, is also easy to check. However, since $\beta'(u_\varepsilon)$ is locally equibounded, we only need that $r_\varepsilon \to 0$ in L^1_{loc}, in order to ensure distributional convergence of the product $\beta'(u_\varepsilon) r_\varepsilon$. Thanks to the following Proposition, this is indeed the case, if b has Sobolev regularity. □

Lemma 1.2.9 (Strong convergence of the commutator)**.** *Let $b \in L^1_{\text{loc}}([0,T]; W^{1,1}_{\text{loc}}(\mathbb{R}^d))$ and let $u \in L^\infty_{\text{loc}}([0,T]\times\mathbb{R}^d)$. Then $r_\varepsilon \to 0$ strongly in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$, as $\varepsilon \to 0$.*

Proof. From the definition of $b \cdot \nabla u$ we have

$$\begin{aligned} r_\varepsilon &= b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon \\ &= b \cdot \nabla u_\varepsilon + (ub) * \nabla \rho_\varepsilon + (u \operatorname{div} b) * \rho_\varepsilon. \end{aligned} \tag{1.2.17}$$

Recalling some properties of the convolution of a distribution with a C^∞_c function, we get

$$\begin{aligned} r_\varepsilon(t,x) &= -b_t(x) \cdot \int u_t(y) \nabla \rho_\varepsilon(x-y)\,dy + \int u_t(y) b_t(y) \nabla \rho_\varepsilon(x-y)\,dy + (u_t \operatorname{div} b_t) * \rho_\varepsilon \\ &= \int u_t(y)[b_t(y) - b_t(x)] \cdot \nabla \rho_\varepsilon(x-y)\,dy + (u_t \operatorname{div} b_t) * \rho_\varepsilon \\ &= \frac{1}{\varepsilon^d} \int u_t(y)[b_t(y) - b_t(x)] \cdot \nabla \rho\left(\frac{x-y}{\varepsilon}\right) \frac{1}{\varepsilon}\,dy + (u_t \operatorname{div} b_t) * \rho_\varepsilon \\ &= \int u_t(x+\varepsilon z) \left[\frac{b_t(x+\varepsilon z) - b_t(x)}{\varepsilon}\right] \cdot \nabla \rho(z)\,dz + (u_t \operatorname{div} b_t) * \rho_\varepsilon, \end{aligned} \tag{1.2.18}$$

where in the last passage we have used the change of variables $y = x + \varepsilon z$ and the fact that $\nabla\rho$ is odd. Next, it is a standard fact in the theory of Sobolev spaces ([16], Prop. 9.3) that, as $\varepsilon \to 0$,

$$\frac{b_t(x+\varepsilon z) - b_t(x)}{\varepsilon} \to Db_t(x)\cdot z \qquad \text{strongly in } L^1_{\text{loc}}. \tag{1.2.19}$$

Moreover, from the continuity of translations in L^p-spaces, we know that $u_t(x+\varepsilon z) \to u_t(x)$ strongly in L^p_{loc} $\forall p$. Therefore, using an Egorov-like argument, we deduce that the commutator converges strongly in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$ to

$$u_t(x)\int (Db_t(x)\cdot z)\nabla\rho(z)dz + u_t\text{div}b_t.$$

Now we observe that

$$\begin{aligned} u_t(x)\int (Db_t(x)\cdot z)\nabla\rho(z)dz &= u_t(x)\int \sum_{i,j=1}^d \frac{\partial b_i}{\partial x_j}(t,x)z_j\frac{\partial\rho}{\partial z_i}(z)dz \\ &= u_t(x)\sum_{i,j=1}^d \frac{\partial b_i}{\partial x_j}(t,x)\int z_j\frac{\partial\rho}{\partial z_i}(z)dz \\ &= -u_t(x)\text{div}b_t(x) \end{aligned}$$

since, from the divergence theorem, it holds

$$\int z_j\frac{\partial\rho}{\partial z_i}(z)dz = -\delta_{ij}.$$

This concludes the proof. □

We stress the fact that the Sobolev regularity of b only enters in the step (1.2.19) of the commutator proposition. Therefore, the renormalization strategy of Theorem (1.2.7) can be used to prove well-posedness of the transport equation in other contexts. Below we sketch the main idea of paper ([4]), in which Ambrosio extended the DiPerna-Lions theory and showed the renormalization property for vector fields of bounded variation. The main difference is that for BV vector fields we don't have $Db \in L^1_{\text{loc}}$. However, we can decompose the spatial derivative of b as follows:

$$Db = D^a b + D^s b,$$

where $D^a b$ and $D^s b$ denote the absolutely continuous and singular part of Db respectively. The difference quotient (1.2.19) does not converge strongly in L^1_{loc} due to the part of the derivative which is the singular part of the measure. One has instead

$$\frac{b(t+\varepsilon z) - b(t,x)}{\varepsilon} = b^1_{\varepsilon,z}(t,x) + b^2_{\varepsilon,z}(t,x),$$

where

$$\begin{aligned} &b^1_{\varepsilon,z}(t,x) \to D^a b(t,x)\cdot z, \qquad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^d), \\ &\limsup_{\varepsilon\to 0}\int_K |b^2_{\varepsilon,z}(t,x)|dx \le |D^s b(t,\cdot)\cdot z|(K) \qquad \forall K \Subset \mathbb{R}^d. \end{aligned}$$

Hence the commutator r_ε can be divided into two integrals, the first involving $b^1_{\varepsilon,z}(t,x)$, and the other with $b^2_{\varepsilon,z}(t,x)$. Under suitable bounds on the divergence, the first part converges strongly in L^1_{loc} as in the previous proof. The second part however is more complex and relies on an anisotropic regularization procedure on the derivative of b. We state the precise result without proof.

Theorem 1.2.10 (Ambrosio)**.** *Let b be a bounded vector field in $L^1_{\text{loc}}([0,T];BV(\mathbb{R}^d))$, such that $\text{div}b \in L^1([0,T];L^1(\mathbb{R}^d))$. Then b has the renormalization property.*

1.3 Renormalization for partially regular vector fields

In this Section we shortly present another extension of the theory of renormalized solutions to the transport equation introduced by DiPerna and Lions. This extension, due to Le Bris and Lions, considers the case when some coordinates b_i of the vector field b are not $W^{1,1}$ with respect to some space variables x_j. In particular b is in a form such as

$$b(x_1, x_2) = (b_1(x_1), b_2(x_1, x_2)) \quad \text{with } b_1 \in W^{1,1}_{x_1} \text{ and } b_2 \in L^1_{x_1}(W^{1,1}_{x_2}). \tag{1.3.1}$$

Hence the linear transport equation

$$\partial_t u + b \cdot \nabla u = 0$$

can be rewritten as

$$\partial_t u + b_1(x_1) \cdot \nabla_{x_1} u + b_2(x_1, x_2) \cdot \nabla_{x_2} u = 0. \tag{1.3.2}$$

The space variable x is partitioned into $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $N = n_1 + n_2$. Accordingly, the vector field $b = (b_1, b_2)$ is such that $b_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$, and the differential operators gradient and divergence are decomposed as $\nabla = (\nabla_{x_1}, \nabla_{x_2})$, $\mathrm{div}_x = \mathrm{div}_{x_1} + \mathrm{div}_{x_2}$.

The precise setting of [35] includes other technical assumptions which follow the line of those in DiPerna-Lions. The complete assumptions on the vector field are:

(H1) $b_1 = b_1(t, x_1) \in L^1\big([0,T]; W^{1,1}_{x_1,\mathrm{loc}}\big)$ (**it does not depend on** x_2)

(H2) $\frac{b_1}{1+|x_1|} \in L^1\big([0,T]; L^1_{x_1}(\mathbb{R}^{n_1}) + L^\infty_{x_1}(\mathbb{R}^{n_2})\big)$

(H3) $\mathrm{div}_{x_1}\, b_1 \in L^1\big([0,T]; L^\infty_{x_1}(\mathbb{R}^{n_1})\big)$

(H4) $b_2 = b_2(x_1, x_2) \in L^1\Big([0,T]; L^1_{x_1,\mathrm{loc}}\big(\mathbb{R}^{n_1}; W^{1,1}_{x_2,\mathrm{loc}}\big)\Big)$

(H5) $\frac{b_2}{1+|x_2|} \in L^1\Big([0,T]; L^1_{x_1,\mathrm{loc}}\big(\mathbb{R}^{n_1}; L^1_{x_2}(\mathbb{R}^{n_2}) + L^\infty_{x_2}(\mathbb{R}^{n_2})\big)\Big)$

(H6) $\mathrm{div}_{x_2}\, b_2 \in L^1\big([0,T]; L^\infty(\mathbb{R}^N)\big)$

Compared to the theory in [29], the main difference is that regularity of b_2 in the variable x_1 is not required (we require only summability). However, a positive result on the well-posedeness problem is made possible, mainly, by assumption $(H1)$. The authors rely on the renormalization theory, but use two different regularization kernels for x_1 and x_2, namely $\rho_{\varepsilon_1}(x_1)$ and $\rho_{\varepsilon_2}(x_2)$, and eventually they send $\varepsilon_1 \to 0$ first, and then $\varepsilon_2 \to 0$. Roughly speaking, this gives rise to commutators "in x_1 only" for b_1 and "in x_2 only" for b_2.

We can now state the first result, which is the analog of Proposition 1.2.8.

Remark 4. For the sake of simplicity in the following proofs we present the case in which b is not time-dependent and $\mathrm{div}_{x_1} b = \mathrm{div}_{x_2} b = 0$.

Proposition 1.3.1. *We assume (H1) and (H4). Let $u \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^N))$ be a weak solution of* (1.3.2)*. Then u is a renormalized solution.*

Proof. Let ρ_{ε_1} and ρ_{ε_2} be two even regularization kernels in C^∞_c, respectively in the variable x_1 and x_2. We first convolve (1.3.2) with ρ_{ε_2}, obtaining

$$\partial_t(u * \rho_{\varepsilon_2}) + b_1 \cdot \nabla_{x_1}(u * \rho_{\varepsilon_2}) + (b_2 \cdot \nabla_{x_2} u) * \rho_{\varepsilon_2} = 0,$$

where we have used the fact that, in view of (H1), b_1 does not depend on x_2. Denoting by

$$r_{\varepsilon_2} = [b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_2}](u) = b_2 \cdot \nabla_{x_2}(u * \rho_{\varepsilon_2}) - \rho_{\varepsilon_2} * (b_2 \cdot \nabla_{x_2} u), \tag{1.3.3}$$

this can be written

$$\partial_t(u * \rho_{\varepsilon_2}) + b_1 \cdot \nabla_{x_1}(u * \rho_{\varepsilon_2}) + b_2 \cdot \nabla_{x_2}(u * \rho_{\varepsilon_2}) = r_{\varepsilon_2}.$$

Therefore, denoting by $u_{\varepsilon_2} = u * \rho_{\varepsilon_2}$, we have

$$\partial_t u_{\varepsilon_2} + b_1 \cdot \nabla_{x_1} u_{\varepsilon_2} + b_2 \cdot \nabla_{x_2} u_{\varepsilon_2} = r_{\varepsilon_2}. \tag{1.3.4}$$

Next we regularize in the x_1 variable by convolving (1.3.4) with ρ_{ε_1}, obtaining

$$\begin{aligned} \partial_t(u_{\varepsilon_2} * \rho_{\varepsilon_1}) + b_1 \cdot \nabla_{x_1}(u_{\varepsilon_2} * \rho_{\varepsilon_1}) + b_2 \cdot \nabla_{x_2}(u_{\varepsilon_2} * \rho_{\varepsilon_1}) = \\ [b_1 \cdot \nabla_{x_1}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + [b \cdot \nabla_{x_2}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + r_{\varepsilon_2} * \rho_{\varepsilon_1}. \end{aligned} \tag{1.3.5}$$

Then, if we define $u_{\varepsilon_1,\varepsilon_2}$ as the regularization of u with ρ_{ε_1} and ρ_{ε_2}, i.e.

$$u_{\varepsilon_1,\varepsilon_2} = (u * \rho_{\varepsilon_1}) * \rho_{\varepsilon_2},$$

we get that $u_{\varepsilon_1,\varepsilon_2}$ satisfies the PDE

$$\partial_t u_{\varepsilon_1,\varepsilon_2} + b \cdot \nabla u_{\varepsilon_1,\varepsilon_2} = r_{\varepsilon_1,\varepsilon_2}, \tag{1.3.6}$$

where

$$r_{\varepsilon_1,\varepsilon_2} = [b_1 \cdot \nabla_{x_1}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + [b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + r_{\varepsilon_2} * \rho_{\varepsilon_1}. \tag{1.3.7}$$

We can observe that $u_{\varepsilon_1,\varepsilon_2}$ is smooth in the space variable and $W^{1,1}_{\mathrm{loc}}$ in time. In fact $\partial_t u_{\varepsilon_1,\varepsilon_2} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ since

$$\begin{aligned} \partial_t u_{\varepsilon_2} &= -(b_1 \cdot \nabla_{x_1} u) * \rho_{\varepsilon_2} - (b_2 \cdot \nabla_{x_2} u) * \rho_{\varepsilon_2} \\ &= (ub_1) * \nabla_{x_1} \rho_{\varepsilon_2} + (u \operatorname{div}_{x_1} b_1) * \rho_{\varepsilon_2} + (ub_2) * \nabla_{x_2} \rho_{\varepsilon_2} + (u \operatorname{div}_{x_2} b_2) * \rho_{\varepsilon_2} \end{aligned} \tag{1.3.8}$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, it follows from Stampacchia's chain rule for Sobolev spaces that, for every $\beta \in C^1(\mathbb{R}^N)$

$$\partial_t \beta(u_{\varepsilon_1,\varepsilon_2}) + b \cdot \nabla \beta(u_{\varepsilon_1,\varepsilon_2}) = r_{\varepsilon_1,\varepsilon_2} \beta'(u_{\varepsilon_1,\varepsilon_2}). \tag{1.3.9}$$

In order to prove the renormalization property, we want to perform the limit in (1.3.9), as ε_1 and ε_2 go to zero. We notice that the left hand side trivially converges in distribution to the same term not regularized. Hence, it remains to show distributional convergence to zero of the product $r_{\varepsilon_1,\varepsilon_2}\beta'(u_{\varepsilon_1,\varepsilon_2})$. Since $\beta'(u_{\varepsilon_1,\varepsilon_2})$ is bounded, it is sufficient to show that $r_{\varepsilon_1,\varepsilon_2} \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. This is done in the following Lemma. □

Lemma 1.3.2. *We assume (H1) and (H4). Let $u \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^N))$ be a solution of* (1.3.2). *Then*

$$\lim_{\varepsilon_2 \to 0} \lim_{\varepsilon_1 \to 0} r_{\varepsilon_1,\varepsilon_2} = 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

Proof. All the functional spaces here considered are *local*, but we skip the subscript *loc* in order to lighten the notation. It is a standard fact from [29] (see Lemma 1.2.9) that

$$r_{\varepsilon_2} = [b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_2}](u) \to 0 \qquad \text{as } \varepsilon_2 \to 0, \text{ in } L^1. \tag{1.3.10}$$

We recall the definition of $r_{\varepsilon_1,\varepsilon_2}$:

$$r_{\varepsilon_1,\varepsilon_2} = [b_1 \cdot \nabla_{x_1}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + [b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) + r_{\varepsilon_2} * \rho_{\varepsilon_1}. \tag{1.3.11}$$

The first term is the standard error term for the regularization on the variable x_1 of the function $u_{\varepsilon_2} \in L^\infty_{x_1,x_2}$. Therefore, from analog computations to those in Lemma 1.2.9, we have, for fixed ε_2,

$$\lim_{\varepsilon_1 \to 0} [b_1 \cdot \nabla_{x_1}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) = 0, \tag{1.3.12}$$

in $L^1_{x_1,x_2}$, where we used also that b_1 does not depend on x_2. Let us now turn to the second term of (1.3.11). We have

$$\begin{aligned}
[b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) &= b_2 \cdot \nabla_{x_2}(\rho_{\varepsilon_1} * u_{\varepsilon_2}) - \rho_{\varepsilon_1} * (b_2 \cdot \nabla_{x_2} u_{\varepsilon_2}) \\
&= b_2 \cdot ((\nabla_{x_2} u_{\varepsilon_2}) * \rho_{*_1}) - \rho_{\varepsilon_1} * (b_2 \cdot \nabla_{x_2} u_{\varepsilon_2}) = [b_2, \rho_{\varepsilon_1}](\nabla_{x_2} u_{\varepsilon_2}) \\
&= \int_{\mathbb{R}^{n_1}} (b_2(x_1,x_2) - b_2(y_1,x_2)) \cdot \nabla_{x_2} u_{\varepsilon_2}(y_1,x_2) \rho\left(\frac{x_1 - y_1}{\varepsilon_1}\right) dy_1 \\
&= \int_{\mathbb{R}^{n_1}} (b_2(x_1,x_2) - b_2(x_1 + \varepsilon_1 z_1, x_2)) \cdot \nabla_{x_2} u_{\varepsilon_2}(x_1 + \varepsilon_1 z_1, x_2) \rho(z_1) \varepsilon_1^{n_1} dz_1,
\end{aligned} \tag{1.3.13}$$

where in the last passage we perform the change of variables $y_1 = x_1 + \varepsilon_1 z_1$. The continuity of translations in L^p–spaces gives us that $b_2(x_1,x_2) - b_2(x_1 + \varepsilon_1 z_1, x_2) \to 0$ in $L^1_{x_1,x_2}$, as $\varepsilon_1 \to 0$. In the same way, since $\nabla_{x_2} u_{\varepsilon_2}$ belongs to $L^\infty_{x_1,x_2}$, we have that $\nabla_{x_2} u_{\varepsilon_2}(x_1 + \varepsilon_1 z_1, x_2) \to \nabla_{x_2} u_{\varepsilon_2}(x_1,x_2)$ in $L^\infty_{x_1,x_2}$. These arguments allow us to deduce that

$$\lim_{\varepsilon_1 \to 0} [b_2 \cdot \nabla_{x_2}, \rho_{\varepsilon_1}](u_{\varepsilon_2}) = 0 \tag{1.3.14}$$

in $L^1_{x_1,x_2}$, as ε_2 is kept fixed. There remains to treat the third term of (1.3.11) which is the easiest one. Indeed, ε_2 being fixed, it is clear that

$$\lim_{\varepsilon_1 \to 0} r_{\varepsilon_2} * \rho_{\varepsilon_1} = r_{\varepsilon_2} \tag{1.3.15}$$

in L^1. We are now able to complete the proof. We fix ε_2. In view of the convergences (1.3.12) and (1.3.14), the first two terms of (1.3.11) go to zero in $L^1_{x_1.x_2}$, while the third one behaves accordingly to (1.3.15). It follows that

$$\lim_{\varepsilon_1 \to 0} r_{\varepsilon_1,\varepsilon_2} = r_{\varepsilon_2}$$

in L^1. Finally, we let $\varepsilon_2 \to 0$ and use (1.3.10) to get the thesis. □

We are now in the position to state the main result.

Theorem 1.3.3. *We assume (H1) to (H6). Let*

$$\bar{u} \in \left(L^1 \cap L^\infty(\mathbb{R}^N)\right) \cap L^\infty_{x_1}\left(\mathbb{R}^{n_1}; L^1_{x_2}(\mathbb{R}^{n_2})\right). \tag{1.3.16}$$

Then there exists a unique solution

$$u(t,x) \in L^\infty\left([0,T]; L^1 \cap L^\infty(\mathbb{R}^N)\right) \cap L^\infty\Big([0,T]; L^\infty_{x_1}\left(\mathbb{R}^{n_1}; L^1_{x_2}(\mathbb{R}^{n_2})\right)\Big) \tag{1.3.17}$$

to (1.3.2) *with initial condition* $\bar{u}$*.*

Proof. UNIQUENESS: The proof of uniqueness follows exactly the same arguments than the one in 1.2.7: it exploits the fact that, thanks to Proposition 1.3.1, $\beta(u)$ is a solution, if u is such. The only addition is to consider two different cut-off functions, namely $\varphi_{R_1}(x_1)$ and $\psi_{R_2}(x_2)$, when multiplying for $\beta(u)$. As a result we obtain, as before, $\int \beta(u) = 0$, which implies $u = 0$ when $\bar{u} = 0$ and β is positive. This yields to uniqueness, from the linearity of the transport equation.

EXISTENCE: The proof of existence of a solution is analog to the one in Theorem 1.2.4. The only non-standard thing it remains to prove is the fact that the solution in Theorem 1.2.4 necessarily belongs to $L^\infty([0,T]; L^\infty(\mathbb{R}^{n_1}; L^1(\mathbb{R}^{n_2})))$. We will skip this simple check. □

Chapter 2

Flow of non smooth vector fields

Both DiPerna-Lions (in [29]) and Ambrosio (in [4]) derive wellposedness of the ODE from the well-posedness of the transport or continuity equation, in the non smooth setting. Ambrosio, in particular, does that through an extended *theory of characteristics*, in which he establishes a direct connection between the ODE and the PDE. Roughly speaking, he shows that every non-negative solution of the PDE can be written as a superposition of trajectories. He also introduces a new notion of solution to the ODE, that is the one of *regular Lagrangian flow*, which has the property that trajectories do not concentrate in small sets. In this Chapter we will see a new approach to the wellposedness of the ODE, based on quantitative a priori estimates derived directly at the ODE level, with no mention to the PDE theory. This approch has been introduced by Crippa and De Lellis in [24], where they recover, through these estimates, results of existence, uniqueness, stability and compactness of regular Lagrangian flows, when the vector field is $W^{1,p}$ in the space variable, with $p > 1$. An overview of [24] is presented in Section 2.1. In Section 2.2 we recall some useful estimates regarding singular integrals. In Section 2.3 we review a paper by Bouchut and Crippa ([15]), where they recover similar wellposedness results, in the case of vector fields whose gradients are singular integrals of L^1 functions (which includes the case $W^{1,1}$). Finally, in Section 2.4 we consider [11], in which the vector field has different regularity according to different directions.

2.1 Quantitative estimates in the $W^{1,p}$ case, with $p > 1$

We consider the Cauchy problem for the ODE:

$$\begin{cases} \frac{dX(t,x)}{dt} = b(t, X(t,x)) \\ X(0,x) = x. \end{cases} \tag{2.1.1}$$

We underline the fact that in [24] (as well as in [15] and [11]), we do not obtain uniqueness of the flow for "almost every" initial datum x. We consider as admissible solutions only the flows that, in some sense, preserve the measure of sets. We recall the notion of *regular Lagrangian flows* introduced by Ambrosio ([4]). This notion turns out to be the right one in the study of the ordinary differential equation with weakly differentiable vector fields.

Definition 2.1.1. Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. We say that $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow of (2.1.1) if

(i) For a.e. $x \in \mathbb{R}^d$, the function $t \mapsto X(t,x)$ is a solution of the ODE in the integral sense;

(ii) There exists a constant $L > 0$ (indipendent of t) such that

$$\mathcal{L}^d(X(t,\cdot)^{-1}(A)) \leq L\mathcal{L}^d(A)$$

for every Borel set $A \subset \mathbb{R}^d$. The constant L is called *compressibility constant* of X.

Remark 5. Condition (ii) in Definition 2.1.1 is equivalent to

$$\int_{\mathbb{R}^d} \varphi(X(t,x))dx \le L \int_{\mathbb{R}^d} \varphi(x)dx$$

for all measurable non-negative φ.

For simplicity in this Section we only consider globally bounded vector fields. The extension to more general growth conditions is not a problem, but makes the computations much longer. Therefore we assume $b \in W^{1,p} \cap L^\infty$, with $p > 1$. Moreover, we will omit to write sometimes the time dependence; the complete assumption would be $b \in L^1_t W^{1,p}_x \cap L^\infty_{t,x}$. We will consider these extensions in the next sections. Finally, we remark that, differently from [29] and [4], we will require bounds on the spatial divergence only to prove existence.

2.1.1 A strategy for uniqueness: the new integral quantity

When $b(t,x)$ is Lipschitz continuous in x, uniformly in t, a simple way to get uniqueness is to apply Gronwall's Lemma to the estimate

$$\frac{d}{dt}|X_1(t,x) - X_2(t,x)| \le |b(s,X_1(s,x)) - b(s,X_2(s,x))| \le L|X_1(s,x) - X_2(s,x)|, \tag{2.1.2}$$

so that we have $X_1 = X_2$, for any X_1, X_2 solutions of the ODE with vector field b.

If b is only weakly differentiable in space, we need a different strategy. Let X_1 and X_2 be two regular Lagrangian flows, associated to the vector fields b_1 and b_2. Given a small parameter $\delta > 0$ and a truncation radius $r > 0$, we can define the following (time dependent) quantity, which measures the integral distance between the flows:

$$\phi_\delta(t) = \int_{B_r} \log\left(1 + \frac{|X_1(t,x) - X_2(t,x)|}{\delta}\right) dx. \tag{2.1.3}$$

The truncation is necessary in order to make this integral convergent. We observe that, if X_1 and X_2 are distinct regular Lagrangian flows, then there is non negligible set $A \subset \mathbb{R}^d$, such $|X_1(t,x) - X_2(t,x)| \ge \gamma > 0$ for some $t \in [0,T]$ and for all $x \in A$. Therefore

$$\phi_\delta(t) \ge \int_{B_r \cap A} \log\left(1 + \frac{\gamma}{\delta}\right) dx = \mathcal{L}^d(A \cap B_r) \log\left(1 + \frac{\gamma}{\delta}\right),$$

which yields

$$\mathcal{L}^d\left(\{x \in B_r \,:\, |X_1(t,x) - X_2(t,x)| \ge \gamma\}\right) \le \frac{\phi_\delta(t)}{\log\left(1 + \frac{\gamma}{\delta}\right)}. \tag{2.1.4}$$

If the ratio on the right hand side goes to zero as $\delta \to 0$, then we must have that $X_1 = X_2$ almost everywhere. This is achieved if ϕ_δ grows slower than $\log(1/\delta)$ as $\delta \to 0$. This is immediate if, for instance, $\phi_\delta \le C$. More in general from (2.1.4) we understand that a good strategy to prove uniqueness is to provide upper estimates on the functional ϕ_δ.

2.1.2 Upper bound for the integral quantity

The natural computation starts with a time differentiation, aimed at making the difference quotients of the velocity appear. We observe that, if $x \in B_r$, then $X_i(t,x) \in B_{r+T\|b_i\|_{L^\infty}}$, which implies they are $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$ and absolutely continuous in time (AC_t). Hence we can differentiate as follows:

$$
\begin{aligned}
\phi'_\delta(t) &\le \int_{B_r} \frac{|b_1(X_1)-b_2(X_2)|}{\delta+|X_1-X_2|}dx \\
&\le \int_{B_r} \frac{|b_1(X_2)-b_2(X_2)|}{\delta+|X_1-X_2|}dx + \int_{B_r} \frac{|b_1(X_1)-b_1(X_2)|}{\delta+|X_1-X_2|}dx \\
&\le \frac{1}{\delta}\int_{B_r} |b_1(X_2)-b_2(X_2)|dx + \int_{B_r} \min\left\{\frac{2\|b_1\|_{L^\infty}}{\delta};\frac{|b_1(X_1)-b_1(X_2)|}{|X_1-X_2|}\right\}dx.
\end{aligned}
\tag{2.1.5}
$$

To this point, we have not used any regularity assumptions on the vector fields. Notice that, if b_1 is Lipschitz, uniqueness can be recovered bounding the difference quotient with the Lipschitz constant and setting $b_1 = b_2$. However, we have a bound on the different quotient also in the case $b_1 \in W^{1,p}$. This is due by Theorem 5.34 of [8], which states that, for every $b \in BV(\mathbb{R}^d)$ and for a.e. $x, y \in \mathbb{R}^d$, we have

$$
|b(x)-b(y)| \le C_d|x-y|\left(MDb(x)+MDb(y)\right), \tag{2.1.6}
$$

where Mf is the classical *maximal function* of a function f. Given $f \in L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^m)$, we define its maximal function as

$$
Mf(x) = \sup_{r>0}\frac{1}{\mathcal{L}^d(B_r(x))}\int_{B_r(x)}|f(y)|dy, \qquad x\in\mathbb{R}^d. \tag{2.1.7}
$$

Similarly, when μ is a $\mathbb{R}^m$-valued measure in $\mathbb{R}^d$ with locally finite total variation, we define

$$
M\mu(x) = \sup_{r>0}\frac{|\mu|(B_r(x))}{\mathcal{L}^d(B_r(x))}, \qquad x\in\mathbb{R}^d. \tag{2.1.8}
$$

Clearly the two definitions are consistent. Since Sobolev functions have the bounded variation property, from (2.1.5) and (2.1.6) we understand that we need a bound on the L^1 norm of MDb, on the ball B_r. It is immediate to see that

$$
\|Mf\|_{L^\infty} \le \|f\|_{L^\infty}, \tag{2.1.9}
$$

whereas the analogue property involving L^1, unfortunately, does not hold. Only the weak estimate

$$
|||Mf|||_{M^1} \le C_{d,1}\|f\|_{L^1} \tag{2.1.10}
$$

holds, where the *weak Lebesgue space* $M^1(\mathbb{R}^d)$ is defined as the space of all measurable functions g on $\mathbb{R}^d$ such that

$$
|||g|||_{M^1} = \sup_{\lambda>0}\left\{\lambda\mathcal{L}^d\left(\{x : |g(x)|>\lambda\}\right)\right\} < \infty. \tag{2.1.11}
$$

Notice that $|||\cdot|||_{M^1}$ is not a norm, as it lacks the subadditivity, hence the notation with the three bars. By interpolating (2.1.9) and (2.1.10) we can obtain the *strong estimate*

$$
\|Mf\|_{L^p} \le C_{d,p}\|f\|_{L^p}, \tag{2.1.12}
$$

for every $p > 1$ (see Theorem 1 of [48] for a proof). We stress the fact that the strong estimate does not holds for $p = 1$.

We now have the tools to create an upper bound on the the derivative of ϕ_δ. Let L_i be the compressibility constant of the flow X_i. From (2.1.5) we get

$$
\phi'_\delta(t) \le \frac{L_2}{\delta}\|b_1-b_2\|_{L^1(B_{r+T\|\bar{b}\|_{L^\infty}})} + \int_{B_r}\min\left\{\frac{2\|b_1\|_{L^\infty}}{\delta};C\left(MDb_1(X_1)+MDb_1(X_2)\right)\right\}dx, \tag{2.1.13}
$$

and from (2.1.12) we estimate

$$\int_{B_r} MDb(X_1)dx \le L_i \int_{B_{r+T\|\bar{b}_i\|_{L^\infty}}} MDb_1(x)dx \\ \le C\|MDb_1\|_{L^p} \le C\|Db_1\|_{L^p}. \tag{2.1.14}$$

Hens, for $b_1 \in W^{1,p}$ with $p > 1$ we deduce

$$\phi'_\delta(t) \le \frac{C}{\delta}\|b_1 - b_2\|_{L^1_x} + C\|Db_1\|_{L^p_x}, \tag{2.1.15}$$

and so, since $\phi_\delta(0) = 0$, we have

$$\phi_\delta(t) \le \frac{C}{\delta}\|b_1 - b_2\|_{L^1_t(L^1_x)} + C\|Db_1\|_{L^1_t(L^p_x)}, \tag{2.1.16}$$

where the constant C depends on L_i, p, $\|b_i\|_{L^\infty}$, R and T. Putting together this with (2.1.4) we conclude

$$\mathcal{L}^d\big(B_r \cap \{|X_1(t,\cdot) - X_2(t,\cdot)| \ge \gamma\}\big) \\ \le \frac{C}{\delta \log\left(1+\frac{\gamma}{\delta}\right)}\|b_1 - b_2\|_{L^1_t(L^1_x)} + \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)}\|Db_1\|_{L^1_t(L^p_x)}. \tag{2.1.17}$$

This is the fundamental estimate from which many of the wellposedness results descend.

Uniqueness. Setting $b_1 = b_2$ in (2.1.17) we have, for every $\delta > 0$ and $r > 0$,

$$\mathcal{L}^d\big(B_r \cap \{|X_1(t,\cdot) - X_2(t,\cdot)| \ge \gamma\}\big) \le \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)}\|Db_1\|_{L^1_t(L^p_x)}.$$

It suffices to let $\delta \to 0$.

Stability. Consider $b \in L^1_t(W^{1,p}_x)$ and a sequence $\{b_n\}$ convergent to b in $L^1_{\rm loc}$, equibounded in L^∞. Assume that the regular Lagrangian flows X and X_n have equibounded compressibility constants. Then we get

$$\mathcal{L}^d\big(B_r \cap \{|X_n(t,\cdot) - X(t,\cdot)| \ge \gamma\}\big) \\ \le \frac{C}{\delta \log\left(1+\frac{\gamma}{\delta}\right)}\|b_n - b\|_{L^1_t(L^1_{x,\rm loc})} + \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)}\|Db\|_{L^1_t(L^p_x)} = I + II.$$

Given $\gamma, \eta > 0$, we choose $\delta > 0$ so small that $II \le \eta/2$. This fixes the quantity $\frac{C}{\log(1+\gamma/\delta)}$ in I. Therefore we can find $\bar{n}$ so large that $I \le \eta/2$ for all $n \ge \bar{n}$. Hence we get that, given $\gamma > 0$ and $r > 0$, for every $\eta > 0$ we can find $\bar{n}$ such that

$$\mathcal{L}^d\big(B_r \cap \{|X_n(t,\cdot) - X(t,\cdot)| \ge \gamma\}\big) \le \eta \qquad \forall n \ge \bar{n}.$$

Thereby X_n converges to X locally in measure in $\mathbb{R}^d$. Since the flows are locally equibounded, it is easy to check that X_n converges to X in $L^1_{\rm loc}$.

Compactness. Consider a sequence $\{b_n\}$ which is equibounded in $L^\infty \cap L^1_t(W^{1,p}_x)$. Assume that there exist associated regular Lagrangian flows X_n with equibounded compressibility constants. Then

$$\begin{aligned}\mathcal{L}^d\big(B_r \cap \{|X_n(t,\cdot) - X_m(t,\cdot)| \geq \gamma\}\big)\\ \leq \frac{C}{\delta \log\left(1+\frac{\gamma}{\delta}\right)} \|b_n - b_m\|_{L^1_t(L^1_x)} + \frac{C}{\log\left(1+\frac{\gamma}{\delta}\right)} = I + II,\end{aligned}$$

where the constant C in II also depends on the equibounds on $\|Db_n\|_{L^1_t(L^p_x)}$. Proceeding as before, for any $\eta > 0$ there is a $\delta > 0$ such that $II \leq \eta/2$, and for that δ we find a $\bar{n}$ such that $I \leq \eta/2$ for every $n, m \geq \bar{n}$. This implies that $\{X_n\}$ is precompact locally in measure. Since X_n are locally equibounded, we obtain also precompactness in L^1_{loc}.

Existence. Consider $b \in L^1_t(W^{1,p}_x) \cap L^\infty$. We need, for the existence, to assume bounds on the divergence. For simplicity we take $\text{div} b \in L^\infty$. We want to use the compactness result, therefore we regularize b by convolution in order to get a sequence b_n of smooth vector fields, which are equibounded in $L^1_t(W^{1,p}_x) \cap L^\infty$ with bounded divergence. In addition the classical flows X_n associated to b_n have equibounded compressibility constants, since we have the estimate

$$\exp\left[-\int_0^t \|[\text{div} b_n]^-\|_{L^\infty} ds\right] \leq JX_n(t,x) \leq \exp\left[\int_0^t \|[\text{div} b_n]^+\|_{L^\infty} ds\right],$$

thanks to the following remark.

Remark 6. If b is regular, we have

$$\frac{d}{dt} JX(t,x) = \text{div} b_t(X(t,x)) JX(t,x),$$

and this can be proved from the ODE

$$\frac{d}{dt} DX = Db_t(X) DX.$$

2.2 Singular integrals and a new maximal function

2.2.1 Singular integrals

In the Section 2.3 we want to deal with vector fields whose gradient is a singular integral of an L^1 function, in the space variable. In order to do this we need to give some definitions and present some properties. Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions on $\mathbb{R}^d$ and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space.

Definition 2.2.1 (Singular kernel)**.** We say that K is a *singular kernel* on $\mathbb{R}^d$ if

(i) $K \in \mathcal{S}'(\mathbb{R}^d)$ and $\widehat{K} \in L^\infty$;

(ii) $K|_{\mathbb{R}^d \setminus \{0\}} \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ and there exists a constant $A \geq 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq A$$

for every $y \in \mathbb{R}^d$.

We next give a sufficient cancellation, growth and regularity condition for kernels $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ so that the associated distribution is a singular kernel.

Proposition 2.2.2. *Consider a function $K \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ satisfying the following conditions:*

(i) There exists a constant $A \geq 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq A \qquad \text{for every } y \in \mathbb{R}^d;$$

(ii) There exists a constant $A_0 \geq 0$ such that

$$\int_{|x|\leq R} |x||K(x)| dx \leq A_0 R \qquad \text{for every } R > 0;$$

(iii) There exists a constant $A_2 \geq 0$ such that

$$\left| \int_{R_1<x<R_2} K(x) dx \right| \leq A_2 \qquad \text{for every } 0 < R_1 < R_2 < \infty.$$

Then K can be extended to a tempered distribution on $\mathbb{R}^d$ which is a singular kernel, unique up to a constant times a Dirac mass at the origin. Conversely, any singular kernel on $\mathbb{R}^d$ has a restriction on $\mathbb{R}^d \setminus \{0\}$ that satisfies conditions (i), (ii), (iii).

For our purpose we introduce a more regular class of kernels.

Definition 2.2.3. A kernel K is a *singular kernel of fundamental type* in $\mathbb{R}^d$ if the following properties hold:

(i) $K|_{\mathbb{R}^d \setminus \{0\}} \in C^1(\mathbb{R}^d \setminus \{0\})$;

(ii) There exists a constant $C_0 \geq 0$ such that

$$|K(x)| \leq \frac{C_0}{|x|^d} \qquad x \in \mathbb{R}^d \setminus \{0\};$$

(iii) There exists a constant $C_1 \geq 0$ such that

$$|\nabla K(x)| \leq \frac{C_1}{|x|^{d+1}} \qquad x \in \mathbb{R}^d \setminus \{0\};$$

(iv) There exists a constant $A_1 \geq 0$ such that

$$\left| \int_{R_1<|x|<R_2} K(x) dx \right| \leq A_1 \qquad \text{for every } 0 < R_1 < R_2 < \infty.$$

These conditions imply in particular those of Proposition 2.2.2. Since $\widehat{K} \in L^\infty(\mathbb{R}^d)$ we may consider the action of a singular kernel on L^2 in Fourier variables. We state the following theorem, the proof can be found in [48].

Theorem 2.2.4 (Calderón Zygmund)**.** *Let K be a singular kernel and define*

$$Su = K * u \quad \text{for } u \in L^2(\mathbb{R}^d),$$

in the sense of multiplication in the Fourier variable. Then for every $1 < p < \infty$ we have the strong estimate

$$\|Su\|_{L^p_{(\mathbb{R}^d)}} \leq C_{d,p}(A + \|\widehat{K}\|_{L^\infty})\|u\|_{L^p(\mathbb{R}^d)}, \quad u \in L^p \cap L^2(\mathbb{R}^d), \tag{2.2.1}$$

and for $p = 1$ the weak estimate

$$|||Su|||_{M^1(\mathbb{R}^d)} \leq C_d(A + \|\widehat{K}\|_{L^\infty})\|u\|_{L^1(\mathbb{R}^d)}, \quad u \in L^1 \cap L^2(\mathbb{R}^d). \tag{2.2.2}$$

Corollary 2.2.5. *The operator S can be extended to the whole $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$, with values in $L^p(\mathbb{R}^d)$, and estimate* (2.2.1) *holds for every $u \in L^p(\mathbb{R}^d)$. Moreover, the operator S can be extended to the whole $L^1(\mathbb{R}^d)$, with values in $M^1(\mathbb{R}^d)$, and estimate* (2.2.4) *holds for every $u \in L^1(\mathbb{R}^d)$.*

Definition 2.2.6. The operator S constructed in Corollary 2.2.5 is called the *singular integral operator* associated to the singular kernel K.

When $p = 1$ the extension S^{M^1} defined on L^1 with values in M^1 can induce some confusion, due to the fact that a function in M^1 is not generally integrable and hence it cannot define a distribution. We can, however, define a tempered distribution $S^D u \in \mathcal{S}'(\mathbb{R}^d)$ via the formula

$$\langle S^D u, \varphi \rangle = \langle u, \tilde{S}\varphi \rangle \tag{2.2.3}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This is well defined, since for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\tilde{S}\varphi \in H^q(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, for $q > d/2$. Since $S^{M^1}u$ is not locally integrable, one cannot identify the values of $S^D u$ as a distribution and $S^{M^1}u$ as an M^1 function. Observe also that $S^D u \in \mathcal{S}'(\mathbb{R}^d)$ can likewise be defined for u finite measure on $\mathbb{R}^d$. Notice also that the definition in (2.2.3) is equivalent to the definition in Fourier variables

$$\widehat{S^D u} = \widehat{K}\widehat{u}$$

where we use that $\widehat{K} \in L^\infty$ and $\widehat{u} \in L^\infty$.

2.2.2 Cancellations in maximal functions and singular integrals

We recall that the case $W^{1,1}$ was not covered by the analysis of the Section 2.1, due to the lack of strong estimates for $p = 1$ for the maximal function. Indeed, (2.1.12) does not hold and only the weak estimate (2.1.10) is available. We observe that a function in $W^{1,1}$ belongs also to the space of functions whose gradient is a singular integral of an L^1 function (simply taking the Dirac delta as kernel), therefore even for this class of functions we do not have the strong estimate. Moreover, for this class we have an additional problem, that is, we cannot even give meaning to the composition between the (classical) maximal function and the gradient of b, since the last one is not locally integrable. As a consequence, a "milder" maximal function is used in [15], called *smooth maximal function*, which will allow such composition.

Given two singular kernels of fundamental type K_1 and K_2 with associated operators S_1 and S_2, we can consider the composition $S_2 S_1 = S$, where S is associated to the kernel defined by $\widehat{K} = \widehat{K_1}\widehat{K_2}$. The cancellations in the convolution $K_2 * K_1$ allow $S_2 S_1$ to be a well defined singular operator. Therefore, $S_2 S_1$ also satisfies the weak estimate (2.1.10). Notice that it is not possible to get this estimate directly by composition, as $S_1 u$ fails to be L^1.

The next theorem states that such cancellations also occur in the composition of a maximal function with a singular integral operator. However, the classical maximal function is too "rough" for such compositions, therefore we consider the smooth maximal function.

Definition 2.2.7. Given a family of functions $\{\rho^\nu\}_\nu \subset L^\infty_c(\mathbb{R}^d)$, for every function $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ we define the $\{\rho^\nu\}$*-maximal function* of u as

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} \rho^\nu_\varepsilon(x-y)u(y)dy \right| = \sup_\nu \sup_{\varepsilon>0} |(\rho^\nu_\varepsilon * u)(x)|. \tag{2.2.4}$$

where

$$\rho^\nu_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho^\nu\left(\frac{x}{\varepsilon}\right).$$

In the case when u is a measure, we take $\{\rho^\nu\}_\nu \subset C^\infty_c(\mathbb{R}^d)$ and define in the distributional sense

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} |\langle u, \rho^\nu_\varepsilon(x - \cdot)\rangle|.$$

The smooth averages and the absence of the absolute value within the integral allows cancellations that take place in the composition of $M_{\{\rho^\nu\}}$ with operator S. This gives rise to a bounded composition operator $M_{\{\rho^\nu\}}S : L^1 \to M^1$.

Theorem 2.2.8. *Let K be a singular kernel of fundamental type, and let $Su = K * u$, for every $u \in L^2(\mathbb{R}^d)$. Let $\{\rho^\nu\}_\nu \subset C_c^\infty(\mathbb{R}^d)$ be a family of kernels such that*

$$\mathrm{spt}\rho^\nu \subset B_1 \qquad \text{and} \qquad \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \leq Q_1 \quad \text{for every } \nu.$$

Then we have the following estimates.

(i) *(a) There exists a constant C_d, depending on the dimension only, such that*

$$|||M_{\rho^\nu}(Su)|||_{M^1(\mathbb{R}^d)} \leq C_d Q_1 (C_0 + C_1 + \|\widehat{K}\|_\infty)\|u\|_{L^1(\mathbb{R}^d)} \tag{2.2.5}$$

for every $u \in L^1 \cap L^2(\mathbb{R}^d)$.

(b) The estimate (2.2.5) *holds also for all finite measures $u \in \mathcal{M}(\mathbb{R}^d)$, where Su is defined as a distribution*

(ii) If $Q_2 = \sup_\nu \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)}$ is finite, then there exists C_d such that

$$\|M_{\rho^\nu}(Su)\|_{L^2(\mathbb{R}^d)} \leq C_d Q_2 \|\widehat{K}\|_\infty \|u\|_{L^2(\mathbb{R}^d)}.$$

2.3 Quantitative estimates in $W^{1,1}$ and for vector fields whose gradient is singular integral of an L^1 function

We present an extension of the wellposedness result of Section 2.1 to the case of vector fields whose derivative is a singular integral of an L^1 function. As previously observed, this class contains trivially $W^{1,1}$. This extension is due to Bouchut and Crippa in [15].

Differently from Section 2.1, where we chose to perform the computations in the simpliest case $b \in L^\infty$, here we consider more general growth conditions. We recall that, one of the main advantages of taking a bounded vector field was that the flow was locally bounded ($x \in B_r$ implies $|X(t,x)| \leq r + T\|b\|_\infty$) and absolutely continuous in time (AC_t) for a.e. x. This in turn implies that the integral quantity ϕ_δ was well defined and we could differentiate in time inside the integral exploiting the weak chain rule. In order to be able to perform similar computations when b is not globally bounded, we need to consider sets where the trajectories of the flows are bounded and integrate only over these sets.

Definition 2.3.1 (Sublevels). Let $X : [t,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable map. For every $\lambda > 0$ we define the *sublevels*

$$G_\lambda = \left\{ x \in \mathbb{R}^d : |X(s,x)| \leq \lambda \text{ for almost all } s \in [t,T] \right\}. \tag{2.3.1}$$

Therefore, if we redefine the functional ϕ_δ as

$$\phi_\delta(s) = \int_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \log\left(1 + \frac{|X(s,x) - \bar{X}(s,x)|}{\delta}\right) dx,$$

we have that $\phi_\delta(s)$ is finite for all s and we can apply the chain rule when we differentiate inside the integral. Clarified this point, we observe that from this truncation comes another problem. Even in the case that we are able to find proper bounds on ϕ_δ, we need to ensure that the complement of the sublevels (called accordingly *superlevels*) become smaller, as $\lambda \to \infty$. Indeed, (2.1.4) becomes

$$\mathcal{L}^d\left(\{x \in B_r \,:\, |X(t,x) - \bar{X}(t,x)| \geq \gamma\}\right) \leq \frac{\phi_\delta(t)}{\log\left(1 + \frac{\gamma}{\delta}\right)} + \mathcal{L}^d(B_r \setminus G_\lambda) + \mathcal{L}^d(B_r \setminus \bar{G}_\lambda). \quad (2.3.2)$$

In Lemma 2.3.3 we will see that $\mathcal{L}^d(B_r \setminus G_\lambda)$ converges to 0, if we consider, for instance, the following growth condition.

(R1) $b(s,x)$ can be decomposed as

$$\frac{b(s,x)}{1+|x|} = \tilde{b}_1(s,x) + \tilde{b}_2(s,x),$$

with

$$\tilde{b}_1 \in L^1((0,T);L^1(\mathbb{R}^d)) \quad \text{and} \quad \tilde{b}_2 \in L^1((0,T);L^\infty(\mathbb{R}^d)).$$

We want to give a new definition of flow of an ODE, which makes sense and allows computations, in the contest of **(R1)**. Indeed, notice that the general Definition 2.1.1 we do not know how to differentiate the flow. When b was bounded however, we could differentiate in a classical way, for almost every x, since X was AC_t. If b is not bounded, we do not even have that X is locally integrable, therefore the standard notion of distributional solution would not make sense. We introduce then the notion of regular Lagrangian flows, in the renormalized sense.

We denote by $L^0(\mathbb{R}^d)$ the space of real-valued measurable functions on $\mathbb{R}^d$, defined a.e. with respect to the Lebesgue measure, endowed with the convergence in measure. We denote by $L^0_{\text{loc}}(\mathbb{R}^d)$ the same space, when we mean that it is endowed with the local convergence in measure. We denote also by $\mathcal{B}(E,F)$ the space of bounded functions from E to F. In addition, we denote by $\log L(\mathbb{R}^d)$ the space of measurable functions such that $\int_{\mathbb{R}^d} \log(1+|u(x)|)dx$ is finite, and the local space $\log L_{\text{loc}}(\mathbb{R}^d)$ is defined accordingly.

Definition 2.3.2 (Regular Lagrangian flow)**.** If b is a vector field satisfying **(R1)**, then for fixed $t \in [0,T)$, a map

$$X \in C([t,T]; L^0_{\text{loc}}(\mathbb{R}^d)) \cap \mathcal{B}([t,T]; \log L_{\text{loc}}(\mathbb{R}^d)) \quad (2.3.3)$$

is a regular Lagrangian flow in the renormalized sense relative to b starting at time t if we have the following:

(i) The equation

$$\partial_s(\beta(X(s,x))) = \beta'(X(s,x))b(s,X(s,x)) \quad (2.3.4)$$

holds in $\mathcal{D}'((t,T)\times\mathbb{R}^d)$, for every function $\beta \in C^1(\mathbb{R}^d;\mathbb{R})$ that satisfies $|\beta(z)| \leq C(1+\log(1+|z|))$ and $|\beta'(z)| \leq \frac{C}{1+|z|}$ for all $z \in \mathbb{R}^d$;

(ii) $X(t,x) = x$ for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$;

(iii) There exists a constant $L \geq 0$ such that $\int_{\mathbb{R}^d} \varphi(X(s,x))dx \leq L \int_{\mathbb{R}^d} \varphi(x)dx$ for all measurable $\varphi : \mathbb{R}^d \to [0,\infty)$.

Remark 7. Note that **(R1)** enables the right hand side of (2.3.4) to be in $L^1((t,T);L^1_{\text{loc}}(\mathbb{R}^d))$. Integrating (2.3.4) in s, this guarantees a bound on $X(s,x)$ in $\log L_{\text{loc}}(\mathbb{R}^d)$ (as stated in (2.3.3)).

We remark that by now this is the usual definition of flows for weakly differentiable vector fields satisfying the general growth condition **(R1)**.

The following lemma gives an estimate for the decay of the superlevels of a regular Lagrangian flow.

Lemma 2.3.3 (Estimate of the superlevels)**.** *Let $b : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a vector field satisfying **(R1)** and let $X : [t,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a regular Lagrangian flow relative to b starting at time t. with compressibility constant L. Then for all $r,\lambda > 0$*

$$\mathcal{L}^d(B_r \setminus G_\lambda) \le g(r,\lambda),$$

where the function g depends only on L, $\|\tilde b_1\|_{L^1((0,T);L^1(\mathbb{R}^d))}$ and $\|\tilde b_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}$ and satisfies $g(r,\lambda)\to 0$ for r fixed and $\lambda\to\infty$.

Proof. The proof can be found [15] or [23], but we write it for completeness.

Step 1. Consider $\beta(z) = \log(1+|z|)$ and the C^1 approximations $\beta_\varepsilon(z) = \log(1+\sqrt{|z|^2+\varepsilon^2})$, which verify the bounds in condition (i) of Definition 2.3.2.

Step 2. Conditions (i) and (ii) of Definition 2.3.2 imply that $\beta_\varepsilon(X)$ is an integral solution of the ODE, for a.e. x, that is

$$\beta_\varepsilon(X(s,x)) = \beta_\varepsilon(X(t,x)) + \int_t^s \beta_\varepsilon'(X(\tau,x))b(\tau,X(\tau,x))d\tau \quad \text{for all } s\in[t,T], \ \text{ for a.e. } x\in\mathbb{R}^d. \tag{2.3.5}$$

Step 3. From (2.3.5) follows

$$\beta_\varepsilon(X(s,x)) \le \beta_\varepsilon(X(t,x)) + \int_t^s \frac{|b(\tau,X(\tau,x))|}{1+|X(\tau,x)|}d\tau.$$

Letting $\varepsilon\to 0$ we have that, for all $s\in[t,T]$ and a.e. $x\in\mathbb{R}^d$,

$$\log(1+|X(s,x)|) \le \log(1+|x|) + \int_t^s \frac{|b(\tau,X(\tau,x))|}{1+|X(\tau,x)|}d\tau. \tag{2.3.6}$$

Step 4. Integrating over $x\in B_r$ we get that, for all $s\in[t,T]$,

$$\int_{B_r} \log\left(\frac{1+|X(s,x)|}{1+r}\right)dx \le L\|\tilde b_1\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \mathcal{L}^d(B_r)\|\tilde b_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}, \tag{2.3.7}$$

from which trivially follows

$$\int_{B_r} \operatorname*{ess\,sup}_{t\le s\le T} \log\left(\frac{1+|X(s,x)|}{1+r}\right)dx \le L\|\tilde b_1\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \mathcal{L}^d(B_r)\|\tilde b_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}.$$

Step 5. Easily we get

$$\int_{B_r} \operatorname*{ess\,sup}_{t\le s\le T} \log\left(\frac{1+|X(s,x)|}{1+r}\right)dx \ge \mathcal{L}^d(B_r\setminus G_\lambda)\log(1+\lambda) - \mathcal{L}^d(B_r)\log(1+r),$$

proving the thesis with

$$g(r,\lambda) = \frac{L\|\tilde b_1\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \mathcal{L}^d(B_r)\|\tilde b_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} + \mathcal{L}^d(B_r)\log(1+r)}{\log(1+\lambda)}.$$

□

As announced before, we are interested in the case when the space derivatives of the vector field can be expressed as singular integrals of L^1 functions, with singular kernel of fundamental type as in Definition 2.2.3. We thus assume that in addition to **(R1)** b satisfies the following assumption.

(R2)

$$\partial_j b = \sum_{k=1}^{m} S_{jk} g_{jk} \qquad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^d)$$

where S_{jk} are singular integral operators of fundamental type in $\mathbb{R}^d$ and the functions $g_j k$ are in $L^1((0,T) \times \mathbb{R}^d)$ for every $j = 1, ..., d$ and $k = 1, ..., m$.

We additionally assume that

(R3)

$$b \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^d) \qquad \text{for some } p > 1.$$

We recall that in Section 2.2, in order to bound $\phi_\delta(t)$, we need an estimate of the difference quotient which is given by the classical maximal function of Db. It tuns out that the same estimate hols true also with the smooth maximal function in the regularity of **(R2)**.

Proposition 2.3.4 (Estimate of difference quotients). *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and assume that for every $j = 1, ..., d$ we have*

$$\partial_j f = \sum_{k=1}^{m} R_{jk} g_{jk}$$

in the sense of distributions, where R_{jk} are singular integrals operators of fundamental type in $\mathbb{R}^d$ and $g_{jk} \in \mathcal{M}(\mathbb{R}^d)$ for $j = 1, ..., d$ and $k = 1, ..., m$, and where $R_{jk} g_{jk}$ is defined in the sense of tempered distributions. Then there exists a nonnegative function $V \in M^1(\mathbb{R}^d)$ such that, for almost every $x, y \in \mathbb{R}^d$, there holds

$$|f(x) - f(y)| \leq |x - y| \big(V(x) + V(y) \big),$$

where V is given by

$$V := \mathcal{V}(R, g) = \sum_{j=1}^{d} \sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}, \xi \in \mathbb{S}^{d-1}\}}(R_{jk} g_{jk}) \tag{2.3.8}$$

and $\Upsilon^{\xi,j}$, for $\xi \in \mathbb{S}^{d-1}$ and $j = 1, ..., d$, is a family of smooth functions explicitly constructed within the proof.

With the idea of creating an upper bound for $\phi_\delta(t)$, we need function $V(x)$ of the previous proposition to be locally integrable. This, however, is not true. Point $(i)(a)$ of Theorem 2.2.8 tells us only that V belongs to the weak Lebesgue space $M^1(\mathbb{R}^d)$. For this issue, the following interpolation Lemma will be useful.

We have defined already, in (2.1.11), the spaceM^1. More in general we can define M^p, for $p > 1$.

Definition 2.3.5. Let u be a measurable function defined on an open set $\Omega \subset \mathbb{R}^d$. For any $1 \leq p < \infty$ we set

$$|||u|||^p_{M^p(\Omega)} = \sup_{\lambda > 0} \left\{ \lambda^p \mathcal{L}^d(\{x \in \Omega : |u(x)| > \lambda\}) \right\},$$

and we define the *weak Lebesgue space* $M^p(\Omega)$ as the space consisting of all u such that $|||u|||^p_{M^p(\Omega)} < \infty$. By convention, for $p = \infty$ we set $M^\infty(\Omega) = L^\infty(\Omega)$.

The Lemma below allows to get an upper bound on the L^1 norm of a function u, provided that it belongs to $M^1 \cap M^p$ for some $p > 1$. It also shows that this bound depends only logarithmically on the M^p pseudonorm. This implies that functions in M^1 are not "too far" from being in L^1. Indeed we have in general only the inclusion $L^1 \subset M^1$, but not the viceversa.

Lemma 2.3.6 (Interpolation Lemma). *Let $u : \Omega \to [0, \infty)$ be a nonnegative measurable function, where $\Omega \subset \mathbb{R}^d$ has finite measure. Then for every $1 < p < \infty$, we have the interpolation estimate*

$$\|u\|_{L^1(\Omega)} \leq \frac{p}{p-1} |||u|||_{M^1(\Omega)} \left[1 + \log\left(\frac{|||u|||_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathcal{L}^d(\Omega)^{1-\frac{1}{p}}\right)\right].$$

Proof. See [15]. □

We also state a crucial lemma on the characterization of a uniformly integrable family of functions. It says that, up to a reminder in L^2, uniformly equiintegrable sequences of functions have arbitrarily small norms in L^1.

Lemma 2.3.7 (Equi-integrability). *Consider a family $\{\varphi\}_{i\in I} \subset L^1(\Omega)$ which is bounded in $L^1(\Omega)$. Then this family is equi-integrable if and only if for every $\varepsilon > 0$, there exists a constant C_ε and a Borel set $A_\varepsilon \subset \Omega$ with finite measure such that for every $i \in I$ one can write*

$$\begin{aligned} &\varphi_i = \varphi_i^1 + \varphi_i^2, \\ &\|\varphi_i^1\|_{L^1(\Omega)} \leq \varepsilon \quad \operatorname{supp}(\varphi_i^2) \subset A_\varepsilon, \quad \|\varphi_i^2\| \leq C_\varepsilon. \end{aligned} \tag{2.3.9}$$

We are now ready to state the main Theorem of [15].

Theorem 2.3.8. *[Fundamenta estimate for flows] Let b and $\bar{b}$ be two vector fields satisfying assumption **(R1)**, and assume that b also satisfies assumptions **(R2)** and **(R3)**. Fix $t \in [0, T)$ and let X and $\bar{X}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compressibility constants L and $\bar{L}$. Then the following holds. For every $\gamma > 0$, $r > 0$ and for every $\eta > 0$ there exists $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^d\left(B_r \cap \{|X(s,\cdot) - \bar{X}(s,\cdot)| > \gamma\}\right) \leq C_{\gamma,r,\eta} \|b - \bar{b}\|_{L^1((0,T)\times B_\lambda} + \eta \tag{2.3.10}$$

for all $s \in [t, T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T); L^1(\mathbb{R}^d))$ of the functions g_{jk} associated to b as in **(R2)**,*
- *The norms of the singular integral operators S^i_{jk}, associated to b as in **(R2)** (i.e. the constants $C_0 + C_1 + \|\widehat{K}\|_\infty$),*
- *The norm in $L^p((0,T) \times B_\lambda)$ of b,*
- *The $L^1((0,T); L^1(\mathbb{R}^d)) + L^1((0,T); L^\infty(\mathbb{R}^d))$ norms of the decomposition of b and $\bar{b}$ as in **(R1)**,*
- *The compressibility constants L and $\bar{L}$.*

Remark 8. As for the corollaries concerning wellposedness of the flow, they follow directly from the fundamental estimate (2.3.10) as in Section 2.1.

In order to improve the readability of the following estimates, we will use the notation "$\lesssim$" to denote an estimate up to a constant only depending on absolute constants and on bounds assumed in Theorem 2.3.8, and the notation "$\lesssim_\lambda$" to mean that the constant could also depend on the truncation parameter λ.

Proof. For any $\delta, \lambda > 0$, $s \in [t, T]$, let

$$\phi_\delta(s) = \int_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \log\left(1 + \frac{|X(s,x) - \bar{X}(s,x)|}{\delta}\right) dx, \tag{2.3.11}$$

where G_λ and $\bar{G}_\lambda$ are the sublevels of X and $\bar{X}$. Following the line of computation (2.1.5) we have

$$\phi'_\delta(s) \leq \frac{\bar{L}}{\delta}\|b(s,\cdot) - \bar{b}(s,\cdot)\|_{L^1(B_\lambda)} + \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))| + |b(s,\bar{X}(s,x))|}{\delta}; \frac{|b(s,X(s,x)) - b(s,\bar{b}(\bar{X}(s,x))|}{|X(s,x) - \bar{X}(s,x)|}\right\} dx.$$

Integrating over $s \in (t,\tau)$ and applying Proposition 2.3.4 for almost every s, we have existence of a function $\mathcal{V}(S,g) := V \in M^1(\mathbb{R}^d)$ (defined as in (2.3.8)) so that

$$\phi_\delta(\tau) \leq \frac{\bar{L}}{\delta}\|b - \bar{b}\|_{L^1((t,\tau)\times B_\lambda)} + \int_t^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))| + |b(s,\bar{X}(s,x))|}{\delta}; V(s,X(s,x)) + V(s,\bar{X}(s,x))\right\} dxds.$$

Fix $\varepsilon > 0$. We apply Lemma 2.3.7 to the finite family $g_{jk} \in L^1((0,T)\times\mathbb{R}^d)$. this gives a constant C_ε and a set of finite measure A_ε such that for each $j = 1,...,d$ and $k = 1,...,m$,

$$g_{jk}(s,x) = g^1_{jk}(s,x) + g^2_{jk}(s,x),$$

with

$$\|g^1_{jk}\|_{L^1((0,T)\times\mathbb{R}^d)} \leq \varepsilon, \quad \mathrm{supp}(g^2_{jk}) \subset A_\varepsilon, \quad \|g^2_{jk}\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C_\varepsilon. \tag{2.3.12}$$

Then we exploit subadditivity of V to get

$$V = \mathcal{V}(S,g) = \mathcal{V}(S,g^1 + g^2) \leq \mathcal{V}(S,g^1) + \mathcal{V}(S,g^2) = V^1 + V^2.$$

Plugging this into the integral gives

$$\begin{aligned}\phi_\delta(\tau) &\leq \frac{\bar{L}}{\delta}\|b - \bar{b}\|_{L^1((t,\tau)\times B_\lambda)} \\ &+ \int_t^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))| + |b(s,\bar{X}(s,x))|}{\delta}; V^1(s,X(s,x)) + V^1(s,\bar{X}(s,x))\right\} dxds \\ &+ \int_t^\tau \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))| + |b(s,\bar{X}(s,x))|}{\delta}; V^2(s,X(s,x)) + V^2(s,\bar{X}(s,x))\right\} dxds \\ &= \frac{\bar{L}}{\delta}\|b - \bar{b}\|_{L^1((t,\tau)\times B_\lambda)} + I_1 + I_2.\end{aligned} \tag{2.3.13}$$

We can disregard the first element of the minimum, change variable and estimate the second integral by

$$\begin{aligned}I_2 &\leq (L+\bar{L})\int_t^\tau ds \int_{B_r\cap G_\lambda\cap\bar{G}_\lambda} V^2(s,x)dx \leq (L+\bar{L})[(\tau - t)\mathcal{L}^d(B_\lambda)]^{1/2}\|V^2\|_{L^2((t,\tau)\times\mathbb{R}^d)} \\ &\leq (L+\bar{L})P_2[(\tau - t)\mathcal{L}^d(B_\lambda)]^{1/2}\|g^2\|_{L^2((t,\tau)\times\mathbb{R}^d)} \lesssim_\lambda C_\varepsilon,\end{aligned} \tag{2.3.14}$$

where in the last line we have applied Theorem 2.2.8 to the operator V^2 and the equiintegrability bound (2.3.12) to g^2. Applying Theorem 2.2.8 to V^1 and (2.3.12) to g^1 we get

$$|||V^1|||_{M^1((t,\tau)\times\mathbb{R}^d} \leq P_1\|g^1\|_{L^1((t,\tau)\times\mathbb{R}^d)} \lesssim \varepsilon. \tag{2.3.15}$$

We apply now the Interpolation Lemma 2.3.6 to the function

$$\varphi(s,x) = \min\left\{\frac{|b(s,X(s,x))| + |b(s,\bar{X}(s,x))|}{\delta}; V^1(s,X(s,x)) + V^1(s,\bar{X}(s,x))\right\}$$

to estimate I_1. After changing variable $X(s,x) \mapsto x$, we get

$$I_1 = \|\varphi\|_{L^1((t,\tau)\times(B_r\cap G_\lambda\cap\bar{G}_\lambda))} \lesssim_{d,p} \|g^1\|_{L^1((t,\tau)\times\mathbb{R}^d)} \left[1 + \log\left(\frac{\|b\|_{L^p((t,\tau)\times B_\lambda)}}{\|g^1\|_{L^1((t,\tau)\times\mathbb{R}^d)}\delta}\right)\right].$$

Plugging this into (2.3.13) and using (2.3.14) and (2.3.15), we deduce that

$$\phi_\delta(\tau) \lesssim_{\lambda,d,p} \frac{\bar{L}}{\delta}\|b-\bar{b}\|_{L^1((t,\tau)\times B_\lambda)} + C_\varepsilon + \varepsilon\left[1+\log\left(\frac{\|b\|_{L^p((t,\tau)\times B_\lambda)}}{\varepsilon\delta}\right)\right]. \tag{2.3.16}$$

Arguing as in (2.1.4) we derive the upper bound

$$\mathcal{L}^d\left(\{x\in B_r\,:\,|X(\tau,x)-\bar{X}(\tau,x)|\geq\gamma\}\right) \leq \frac{\phi_\delta(\tau)}{\log\left(1+\frac{\gamma}{\delta}\right)} + \mathcal{L}^d(B_r\setminus G_\lambda) + \mathcal{L}^d(B_r\setminus\bar{G}_\lambda). \tag{2.3.17}$$

Combining this with (2.3.16) we obtain

$$\begin{aligned}
&\mathcal{L}^d\left(\{x\in B_r\,:\,|X(\tau,x)-\bar{X}(\tau,x)|\geq\gamma\}\right) \lesssim_{\lambda,d,p} \frac{\bar{L}}{\delta\log\left(1+\frac{\gamma}{\delta}\right)}\|b-\bar{b}\|_{L^1((t,\tau)\times B_\lambda)}\\
&+\frac{C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta}\right)} + \frac{\varepsilon\left[1+\log\left(\frac{\|b\|_{L^p((t,\tau)\times B_\lambda)}}{\varepsilon\delta}\right)\right]}{\log\left(1+\frac{\gamma}{\delta}\right)} + \mathcal{L}^d(B_r\setminus G_\lambda)+\mathcal{L}^d(B_r\setminus\bar{G}_\lambda)\\
&= 1)+2)+3)+4)+5).
\end{aligned} \tag{2.3.18}$$

We fix $\eta > 0$. To conclude we chose $\lambda > 0$ large so that by Lemma 2.3.3 4) + 5) $\leq \eta/2$. Then we can find $\varepsilon > 0$ small enough so that 3) $\leq \eta/4$ for every $0 < \delta \leq \gamma$ (notice that 3) is uniformly bounded as $\delta \to 0$). Since at this point λ and ε (and thus C_ε) are fixed, we choose $\delta > 0$ small enough in such a way that 2) $\leq \eta/4$. By setting

$$C_{\gamma,r,\eta} = \frac{\bar{L}}{\delta\log\left(1+\frac{\gamma}{\delta}\right)},$$

where δ has been chosen according to the above discussion, the proof is completed. □

2.4 Quantitative estimates in the anisotropic case

In this Section we will describe the main idea of paper [11], where wellposedness of a regular Lagrangian flow is proved under a specific anisotropic regularity of the vector field, which means that the (weak) regularity has a different character with respect to different directions in space. We split $\mathbb{R}^N$ as $\mathbb{R}^N = \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with variables $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. We denote by $D_1 = D_{x_1}$ the derivative with respect to the first n_1 variables x_1, and by $D_2 = D_{x_2}$ the derivative with respect to the last n_2 variables x_2. Accordingly, we denote $b = (b_1, b_2)(s, x_1, x_2)$. For $X(s, x_1, x_2)$ a regular Lagrangian flow associated to b we denote $X = (X_1, X_2)(s, x_1, x_2)$.

We are going to assume that $D_1 b_2$ is "less regular" than $D_1 b_1$, $D_2 b_1$, $D_2 b_2$: the derivative $D_1 b_2$ is a singular integral of a measure, whereas the other derivatives are singular integrals of L^1 functions. This is made precise as follows:

(R2) Assume that

$$Db = \begin{pmatrix} D_1 b_1 & D_2 b_1 \\ D_1 b_2 & D_2 b_2 \end{pmatrix} = \begin{pmatrix} \gamma^1 S^1\mathtt{p} & \gamma^2 S^2\mathtt{q} \\ \gamma^3 S^3\mathtt{m} & \gamma^4 S^4\mathtt{p} \end{pmatrix},$$

where the submatrices have the representation

$$\begin{array}{ll} i,j \in \{1,...,n_1\}: & i \in \{1,...,n_1\}, \quad j \in \{n_1+1,...,N\}: \\ (D_1 b_1)^i_j = \sum_{k=1}^m \gamma^{1i}_{jk}(s,x_2) S^{1i}_{jk} \mathtt{p}^i_{jk}(s,x_1) & (D_2 b_1)^i_j = \sum_{k=1}^m \gamma^{2i}_{jk}(s,x_2) S^{2i}_{jk} \mathtt{q}^i_{jk}(s,x_1) \\ \\ i \in \{n_1+1,...,N\}, \quad j \in \{1,...,n_1\}: & i \in \{n_1+1,...,N\}: \\ (D_1 b_2)^i_j = \sum_{k=1}^m \gamma^{3i}_{jk}(s,x_2) S^{3i}_{jk} \mathtt{m}^i_{jk}(s,x_1) & (D_2 b_2)^i_j = \sum_{k=1}^m \gamma^{4i}_{jk}(s,x_2) S^{4i}_{jk} \mathtt{r}^i_{jk}(s,x_1). \end{array}$$

In the above assumption we have that:

- $S^{1i}_{jk}, S^{2i}_{jk}, S^{3i}_{jk}, S^{4i}_{jk}$ are singular integrals operators associated to singular kernels of fundamental type in $\mathbb{R}^{n_1}$,
- the functions $\mathtt{p}^i_{jk}, \mathtt{q}^i_{jk}, \mathtt{r}^i_{jk}$ belong to $L^1((0,T); L^1(\mathbb{R}^{n_1}))$,
- $\mathtt{m}^i_{jk} \in L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$,
- the functions $\gamma^{1i}_{jk}, \gamma^{2i}_{jk}, \gamma^{3i}_{jk}, \gamma^{4i}_{jk}$ belong to $L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ for some $q > 1$.

We have denote by $L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ the space of all functions $t \mapsto \mu(t,\cdot)$ taking values in the space $\mathcal{M}(\mathbb{R}^{n_1})$ of finite signed measures on $\mathbb{R}^{n_1}$ such that

$$\int_0^T \|\mu(t,\cdot)\|_{\mathcal{M}(\mathbb{R}^{n_1})} dt < \infty.$$

We can shorten the notation of **(R2)** as follows:

$$Db \in \begin{pmatrix} (S_x * L^1_x) L^q_{\mathrm{loc},y} & (S_x * L^1_x) L^q_{\mathrm{loc},y} \\ (S_x * \mathcal{M}_x) L^q_{\mathrm{loc},y} & (S_x * L^1_x) L^q_{\mathrm{loc},y} \end{pmatrix}. \tag{2.4.1}$$

Remark 9. We anticipate now that the Vlasov-Poisson equation, which we will present in the following Chapter, has a similar structure to the one of assumption **(R2)**. This equation indeed motivates the setting of paper [11].

We recall shortly the main steps to prove uniqueness in the frameworks of Section 2.1 and 2.3. The key point is to derive upper bounds on the functional $\phi_\delta(s)$, such that they blow up in δ slower than $\log(1/\delta)$ as $\delta \to 0$ (see (2.1.4)). Differentiating in time the functional we get

$$\phi'_\delta(s) \leq \int \frac{|b(X) - b(\bar{X})|}{\delta + |X - \bar{X}|} dx \leq \int \min\left\{ \frac{2\|b\|_\infty}{\delta}; \frac{|b(X) - b(\bar{X})|}{|X - \bar{X}|} \right\} dx.$$

Hence, when b is $W^{1,p}$ with $p > 1$, the estimate of the difference quotient as

$$\frac{|b(X) - b(\bar{X})|}{|X - \bar{X}|} \lesssim MDb(X) + MDb(\bar{X}),$$

toghether with the strong estimate for the maximal function, imply an upper bound on $\phi_\delta(s)$ indipendent on δ. The case $p = 1$ has the issue that only a weak estimate for the maximal function holds. However, the Interpolation Lemma allows us to interpolate this weak estimate and the L^∞ estimate on the first term in the minimum to get an upper bound for $\phi_\delta(s)$, This bound unfortunately is of the order of $\log(1/\delta)$, which is not sufficient to conclude (see (2.1.4)). Therefore we have to play with parameters: up to an L^2-remainder, the L^1-norm of Db can be assumed to be arbitrarily small (we exploit here equiintegrability bounds on $Db \in L^1$). This allows to conclude, noticing that the L^2 part can be treated as in the case $W^{1,p}$ $(p > 1)$. Considering smooth maximal function and more sophisticated tools of harmonic analysis, it is possible also to treat the case $Db \in S * L^1$ with the same strategy.

As already mentioned, we are not able to include the case when Db is a singular integral of a measure, due to the lack of equiintegrability. However, if Db has only a component which is singular integral of a measure, the idea of [11] is to define an anisotropic functional, which can "weight" differently the two (groups of) directions according to the different degree of regularity. This can be done by considering, instead of ϕ_δ, a functional depending on *two* parameters δ_1 and δ_2, namely

$$\phi_{\delta_1,\delta_2}(s) = \int \log\left(1 + \left|\left(\frac{|X_1(s,x) - \bar{X}_1(s,x)|}{\delta_1}, \frac{|X_2(s,x) - \bar{X}_2(s,x)|}{\delta_2}\right)\right|\right) dx. \tag{2.4.2}$$

Following the same strategy as before (estimate of the difference quotients and interpolation in the minimum in $\phi_\delta(s)$), we derive the following bound, which replaces (2.3.16) in this context:

$$\phi_{\delta_1,\delta_2}(s) \lesssim \left[\frac{\delta_1}{\delta_2}\|D_1 b_2\|_{\mathcal{M}} + \frac{\delta_2}{\delta_1}\|D_2 b_1\|_{L^1} + \|D_1 b_1\|_{L^1} + \|D_2 b_2\|_{L^1}\right] \log\left(\frac{1}{\delta_2}\right).$$

Observe that $\|D_2 b_1\|_{L^1}$, $\|D_1 b_1\|_{L^1}$ and $\|D_2 b_2\|_{L^1}$ can be assumed to be small, by the equiintegrability argument as in [15]. This is however not the case for $\|D_1 b_2\|_{\mathcal{M}}$. But we can exploit the presence of the coefficient δ_1/δ_2 multiplying this term: both δ_1 and δ_2 have to be sent to zero, but we can do it with $\delta_1 \ll \delta_2$, so to have ϕ_{δ_1,δ_2} small enough.

We state the main Theorem of [11] without proof. Indeed, the proof involves many technical difficulties, which require, for instance, a definition of an anisotropic smooth maximal function, an anisotropic estimate of the difference quotient and appropriate bounds on this new maximal function. Moreover, the thing that interests the author the most, is the idea of anisotropic functional ϕ_{δ_1,δ_2}, which we have introduced in the argument above. In the papers presented in Chapter 4 and 5 we exploit indeed the same functional. There, we will explain in details how to "play" with δ_1, δ_2 and all the other parameters involved in the fundamental estimate for flows, in order to gain wellposedness.

Theorem 2.4.1. *Let b and $\bar{b}$ be two vector fields satisfying assumption **(R1)**, and assume that b also satisfies assumptions **(R2)** and **(R3)**. Fit $t \in [0,T)$ and let X and $\bar{X}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compression constants L and $\bar{L}$. Then the following holds. For every $\gamma > 0$, $r > 0$ and for every $\eta > 0$ there exists $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^d\left(B_r \cap \{|X(s,\cdot) - \bar{X}(s,\cdot)| > \gamma\}\right) \le C_{\gamma,r,\eta}\|b - \bar{b}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T);L^1(\mathbb{R}^d))$ of the functions p, q, r, as well as the norm in $L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))$ of m (where p, q, r and m are associated to b as in **(R2)**,*

- *The norms of the singular integral operators S^i_{jk}, as well as the norms in $L^\infty((0,T);L^q(\mathbb{R}^{n_2}))$ of γ^i_{jk} (associated to b as in **(R2)**),*

- *The norm in $L^p((0,T)\times B_\lambda)$ of b,*

- *The $L^1((0,T);L^1(\mathbb{R}^d)) + L^1((0,T);L^\infty(\mathbb{R}^d))$ norms of the decomposition of b and $\bar{b}$ as in **(R1)**,*

- *The compressibility constants L and $\bar{L}$.*

Corollaries and transport equation. We underline the fact that, in all three regularity settings analyzed in this Chapter, we have wellposedness of the regular Lagrangian flow and of the Lagrangian solution to the continuity and transport equations follow. The proofs can be found in [15] for the specific setting, and can be adopted easily to the other two cases. In particular we obtain:

- *Uniqueness* of the regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2)** and **(R3**,
- *Stability* (with an esplicit rate) for a sequence X_n of regular Lagrangian flows associated to vector fields b_n, that converge in $L^1([0,T]\times\mathbb{R}^d)$ to a vector field satisfying **(R1)**, **(R2)** and **(R3)**, under the assumption that the decompositions of b_n in **(R1)** and the compressibility constants of X_n satisfy uniform bounds,
- *Compactness* for a sequence X_n of regular Lagrangian flows associated to vector fields b_n satisfying **(R1)**, **(R2)** and **(R3)** with suitable uniform bounds,
- *Existence* of a regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2)** and **(R3)** and such that $[\mathrm{div}]^- \in L^1((0,T);L^\infty(\mathbb{R}^d))$,
- *Lagrangian solutions* to the continuity and transport equations with a vector field b satisfying **(R1)**, **(R2)** and **(R3)** and $\mathrm{div} b \in L^1((0,T);L^\infty(\mathbb{R}^d))$ are well defined and stable. These solutions are in particular solutions in the renormalized sense.

The last item is somehow the analog of the main result in DiPerna-Lion, but for Lagrangian solutions. We notice that wellposedness results for the flow can be used to prove wellposedness of the PDE. Lagrangian solutions of the transport equation are defined as superposition of the initial data with the regular Lagrangian flow:

$$u(t,x) = u^0(X(0,t,x)).$$

This is of course equal to the classical solution in the case of smooth data.

Note that it is not possible in general to exclude non-uniqueness for renormalized or distributional solutions. It may happen that several weak solutions exist, with only one associated to the Lagrangian flow.

Remark 10. In Chapter 4 we will prove existence of Lagrangian solutions to the Vlasov-Poisson equation in a particular anisotropic setting. Uniqueness, unfortunately, does not follow from the corollary above, because of the non-linearity of Vlasov-Poisson.

Chapter 3

Vlasov-Poisson system

3.1 Introduction and physical meaning

The Vlasov-Poisson system belongs to a class of partial differential equations known as kinetic equations. The purpose of kinetic equations is to model a dilute particle gas at an intermediate scale between the microscopic scale and the hydrodynamic scale. With dilute particle gas we mean any system made of a large number of particles (like a gas or a plasma), so that it can be trated as a continuum. Therefore, a description of the position and of the velocity of each particle is irrelevant, but the description cannot be reduced to the computation of an average velocity at any time $t \in \mathbb{R}$ and any position $x \in \mathbb{R}^d$. Kinetic equations take into account more than one possible velocity at each point, and the description is done at a level of the phase space (at a statistical level) by a distribution function $f(t,x,v)$. A minimal assumption that one can make on the distribution function is

$$f(t,\cdot,\cdot) \in L^1_{\mathrm{loc}}\big(\mathbb{R}^d_x; L^1\big(\mathbb{R}^d_v\big)\big),$$

or at least that $f(t,\cdot,\cdot)$ is a finite measure on $K \times \mathbb{R}^d_v$, for any compact set $K \subset \mathbb{R}^d$. This assumption means that a bounded domain in physical space contains only a finite amount of matter. Notice that f can be seen as an approximation of the true density of the gas in phase space (on a scale which is much larger than the typical distance between particles), or it can reflects our lack of knowledge of the true positions of particles, i.e. $f(t,x,v)dxdv$ is the probability of finding particles in an element of volume $dxdv$.

When the collisions between particles are negligible, f has to be constant along the characteristcs $(X(t),V(t))$ in the phase space given by Newton's law:

$$\dot{X} = \frac{dX}{dt} = V, \qquad \dot{V} = \frac{dV}{dt} = F(t, X(t)),$$

where F is the force that moves the particles, so that we have

$$0 = \frac{d}{dt} f(t, X(t), V(t)) = \partial_t f + V(t) \cdot \nabla_x f + F(t, X(t)) \cdot \nabla_v f.$$

Therefore we can say that a collision-less system of particles can be modeled by the transport equation

$$\partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0.$$

In addition, the force F can derive from an internal interaction potential, namely $F = \gamma \nabla_x U$. This is the case of the Vlasov-Poisson system, representing a collision-less plasma, where the force F is self induced, depending on an electric potential which in turn depends on the solution itself:

$$F = \gamma \nabla_x U, \quad -\Delta U = \rho, \quad \text{where } \rho = \int f\, dv \tag{3.1.1}$$

and $\gamma \in \{-1, 1\}$ is a parameter which models the repulsive ($\gamma = 1$) or attractive ($\gamma = -1$) nature of particles. The second equation in (3.1.1) is the Poisson equation (hence the name Vlasov-Poisson), whose solution is $U = \frac{1}{|\cdot|} * \rho$, for $d = 3$, and $U = -\log|x| * \rho$, for $d = 2$. Thus, for $d = 2, 3$, the Vlasov-Poisson system can be written as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0\,, \\ E(t,x) = \gamma \int \frac{x-y}{|x-y|^d} \rho(t,y)\, dy \\ \rho(t,x) = \int f(t,x,v)\, dv\,, \end{cases} \tag{3.1.2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ stands for the non-negative density of particles in a plasma under the effect of a self-induced field E, while $\rho : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is the spatial density. From a physical viewpoint, the repulsive case represents the evolution of charged particles in presence of their self-consistent electric field and it is used in plasma physics or in semi-conductor devices. The attractive case describes the motion of galaxy clusters under the gravitational field with many applications in astrophysics. In this and in the next chapters we consider mainly the repulsive case, by fixing $\gamma = 1$ in (3.1.2).

In the last decades the Vlasov-Poisson system (3.1.2) has been largely investigated. Our focus is in particular on the Cauchy problem. Existence of classical solutions under regularity assumptions on the initial data goes back to Iordanski [34] in dimension one and to Okabe and Ukai [43] in dimension two. The three dimensional case has been addressed first by Bardos and Degond [10] for small initial data, and then extended to a more general class of initial plasma densities by Pfaffelmoser [45] and by Lions and Perthame [36]. Improvements in three dimensions have been obtained in [47, 50, 19, 39, 21]. Global existence of weak solutions has been studied by Arsenev [9] for bounded initial data with finite kinetic energy, while the global existence of renormalized solutions is due to Di Perna and Lions [29], assuming finite total energy and $f_0 \in L \log L(\mathbb{R}^3 \times \mathbb{R}^3)$. The latter assumption has been recently relaxed to $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ in [6] and [12].

One might wonder what happens when $f_0 \notin L^1(\mathbb{R}^d \times \mathbb{R}^d)$. We can assume for instance f_0 to be the sum of an integrable bounded plasma density and a Dirac mass. This is equivalent to studying the Cauchy problem associated with the following system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E+F) \cdot \nabla_v f = 0\,, \\ E(t,x) = \int \frac{x-y}{|x-y|^d} \rho(t,y)\, dy\,, \\ \rho(t,x) = \int f(t,x,v)\, dv\,, \\ F(t,x) = \frac{x-\xi(t)}{|x-\xi(t)|^d}\,, \end{cases} \tag{3.1.3}$$

where the singular electric field $F := F(t,x)$ is induced by a point charge located at a point $\xi(t)$, whose evolution is given by the Newton equations:

$$\begin{cases} \dot{\xi}(t) = \eta(t)\,, \\ \dot{\eta}(t) = E(t, \xi(t))\,. \end{cases} \tag{3.1.4}$$

For every $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote by $f_0(x, v) = f(0, x, v)$ and by $(\xi_0, \eta_0) = (\xi(0), \eta(0))$ respectively the initial density and initial state of the point charge in the phase space $\mathbb{R}^d \times \mathbb{R}^d$. The system (3.1.3)-(3.1.4) can be formally rewritten in the form (3.1.2) for the total density $f(t) + \delta_{\xi(t)} \otimes \delta_{\eta(t)}$.

The model (3.1.3)–(3.1.4) has been recently introduced by Caprino and Marchioro in [17], where they have shown global existence and uniqueness of classical solutions in two dimensions. This result has been extended to the three dimensional case in [40] by Marchioro, Miot and Pulvirenti. Both [17] and [40] require that the initial plasma density does not overlap the point charge. This assumption has been relaxed in [28], where weak solutions of the system (3.1.3)–(3.1.4) have been obtained for initial data which may overlap the point charge, but do have to decay close to it. The price to pay is that the solution is no longer known to be unique and Lagrangian.

Let us make precise what we mean by weak solution and Lagrangian (or classical) solution. For the Vlasov-Poisson system without point charge a *weak solution* is defined, as usual, in the sense of distributions. In the system with point charge we call *weak solution* a couple (f, ξ) such that the first equation in (3.1.3) is satisfied in the sense of distributions and such that (3.1.4) holds in the classical sense. In the following the notions of classical and Lagrangian solution often overlap. We call *Lagrangian solution* a plasma density f and a trajectory (ξ, η) of the Dirac mass, both defined for $t \in \mathbb{R}_+$, such that f is transported by the Lagrangian flow (X, V), solution to the ODE-system

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v) \\ \dot{V}(t,x,v) = E(t, X(t,x,v)) + F(t, X(t,x,v)) \\ \big(X(0,x,v), V(0,x,v)\big) = (x,v), \end{cases} \tag{3.1.5}$$

more precisely

$$f(t,x,v) = f_0\big(X^{-1}(t,\cdot,\cdot)(x,v),\, V^{-1}(t,\cdot,\cdot)(x,v)\big).$$

In the case without point charge, a Lagrangian solution is defined likewise, setting $F = 0$ in (3.1.5). The notion of *classical solution* is slightly stronger: the flow, solution to the characteristic system (3.1.5), is meant in the classical sense.

In Section 3.2 we present some estimates on physical quantities related to the Vlasov-Poisson equation, which will be useful below. In Section 3.3 we explain the idea of controlling large velocities to prove global existence. In Section 3.4 we sketch some results for the problem without point charge. In particular we focus on [45] and [36]. In Section 3.5 we present a selection of results regarding the plasma-charge problem ([17], [40], [28]).

3.2 Conservation laws and a priori bounds

In this Section we want to recall some properties related to the VP system (with or without charge) which will be used in the following to prove existence and uniqueness results.

Notice first of all that the vector field of the transport equation in (3.1.2) or (3.1.3) is divergence free. In fact

$$b(t,x,v) = (v, E(t,x) + F(t,x)) \quad \text{leads to} \quad \mathrm{div} b = 0.$$

This means that, if a flow of b exists, it preserves the measure, hence we can easily perform change of variables.

Proposition 3.2.1 (Mass conservation)**.** *Let f (or (f,ξ)), be a Lagrangian solution to* (3.1.2) *(or* (3.1.3)*). Then*

$$\|f(t)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}$$

is conserved in time. In particular the mass

$$M(t) = \iint f(t,x,v)\, dx dv$$

is conserved in time.

Proof. The conservation of the L^p norms is an immediate consequence of the fact that, if f is Lagrangian, it must be constant along the trajectories of a Lebesgue's measure preserving flow. □

Remark 11. For $p = 1$ the conservation of mass can also be viewed as a consequence of the *local mass conservation* law

$$\partial_t \rho + \mathrm{div}_x j = 0, \tag{3.2.1}$$

where the mass current j is defined as

$$j(t,x) = \int v f(t,x,v)\, dv.$$

Indeed, integrating (3.2.1) w.r.t x we get

$$\frac{d}{dt}\iint f(t,x,v)dxdv = \frac{d}{dt}\int \rho(t,x)dx = 0.$$

Eqn. (3.2.1) follows by formally integrating the VP equation w.r.t v, observing that $E(+F)\cdot\nabla_v f = \mathrm{div}_v(fE(+F))$ vanishes upon integration. Hence we observe that the mass conservation applies regardless of whether f is Lagrangian.

Proposition 3.2.2 (Energy conservation)**.** *Let* (f,ξ) *be a solution of* (3.1.3)-(3.1.4) *(in a weak or classical sense). Then the total energy*

$$H(t) = \iint \frac{|v|^2}{2} f(t,x,v)dxdv + \frac{|\eta(t)|^2}{2} + \frac{1}{2}\iint \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy + \int \frac{\rho(t,x)}{|x-\xi(t)|}dx$$

is conserved in time.

Proof. We compute

$$\begin{aligned}\dot H(t) =& \frac{d}{dt}\left\{\iint \frac{|v|^2}{2} f dxdv + \frac{|\eta|^2}{2} + \frac{1}{2}\iint \frac{\rho(x)\rho(y)}{|x-y|}dxdy + \int \frac{\rho(x)}{|x-\xi|}dx\right\}\\ =& \iint v\cdot(E+F)f dxdv + \eta\cdot E(\xi) - \iint \frac{\nabla_x\cdot(\int vf dv)}{|x-y|}\rho(y)dxdy\\ &- \int \frac{\nabla_x\cdot\int vf dv}{|x-\xi|}dx - \int \rho(x)\eta\cdot\frac{\xi-x}{|\xi-x|^3}dx = 0.\end{aligned}$$

□

Remark 12. Notice that the energy conservation holds trivially also in the simpler case without point charge, with $H(t) = \iint \frac{|v|^2}{2} f(t,x,v)dxdv + \frac{1}{2}\iint \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy$.

In the next sections and in Chapter 4 we will see that, in order to control large velocities, we need some a priori estimates on the spatial density and on the electric field. Notice indeed that the summability of the spatial density ρ follows trivially by the mass conservation, while upper bounds on $\|\rho\|_{L^p}$, for $p > 1$, are not trivial.

Proposition 3.2.3. *Let* $\rho(t,\cdot) \in L^p \cap L^\infty(\mathbb{R}^d)$ *for some* $p \in [1,d)$. *Then*

$$\|E(t,\cdot)\|_\infty \le C\|\rho\|_{L^p}^{p/d}\|\rho\|_\infty^{1-p/d},$$

with C *depending only on* p.

Proof. For any $R > 0$, Hölder's inequality implies

$$|E(t,x)| \leq \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^{d-1}} dy + \int_{|x-y|>R} \frac{\rho(y)}{|x-y|^{d-1}} dy$$
$$\lesssim R\|\rho\|_\infty + (R^{d-(d-1)q})^{1/q}\|\rho\|_{L^p},$$

where $1/p+1/q = 1$. We optimize this estimate by choosing $R = \frac{c\|\rho\|_{L^p}}{\|\rho\|_\infty}$ and we obtain the thesis. □

Proposition 3.2.4. *Let $m \geq 0$, $f(t,\cdot,\cdot) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\rho(t,\cdot) \in L^1(\mathbb{R}^3)$ as in* (3.1.2) *or* (3.1.3). *Then there exists a constant $C > 0$, which only depends on m, such that*

$$\|\rho(t,\cdot)\|_{L^{\frac{m+3}{3}}} \leq C\|f(t,\cdot,\cdot)\|_{L^\infty}^{\frac{m}{m+3}} \left(\iint |v|^m f(t,x,v)\,dx\,dv\right)^{\frac{3}{m+3}}. \tag{3.2.2}$$

Proof. By definition of ρ we have

$$\|\rho(t,\cdot)\|_{L^{\frac{m+3}{3}}} = \left(\int \left|\int f(t,x,v)\,dv\right|^{\frac{m+3}{3}} dx\right)^{\frac{3}{m+3}}. \tag{3.2.3}$$

Fix $R > 0$ and split the integral in the v variable into two pieces:

$$\int f(t,x,v)\,dv = \int_{|v|\leq R} f(t,x,v)\,dv + \int_{|v|>R} f(t,x,v)\,dv$$
$$\leq R^3\|f(t,\cdot,\cdot)\|_{L^\infty} + \frac{1}{R^m}\int |v|^m f(t,x,v)\,dv\,.$$

By optimizing in R in the last line of the above inequality, we get

$$\int f(t,x,v)\,dv \leq \|f(t,\cdot,\cdot)\|_{L^\infty}^{\frac{m}{m+3}} \left(\int |v|^m f(t,x,v)\,dv\right)^{\frac{3}{m+3}}. \tag{3.2.4}$$

We plug (3.2.4) in (3.2.3) and we obtain

$$\|\rho(t,\cdot)\|_{L^{\frac{m+3}{3}}} \leq \|f(t,\cdot,\cdot)\|_{L^\infty}^{\frac{3}{m+3}} \left(\iint |v|^m f(t,x,v)\,dv\,dx\right)^{\frac{3}{m+3}}. \tag{3.2.5}$$

□

Proposition 3.2.5. *Let $f \geq 0$, $f(t,\cdot,\cdot) \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ solution to* (3.1.2) *or* (3.1.3). *Assume the total energy to be initially finite, then $\rho(t,\cdot) \in L^1 \cap L^{5/3}(\mathbb{R}^3)$ and $E(t,\cdot) \in L^q(\mathbb{R}^3)$, for any $\frac{3}{2} < q \leq \frac{15}{4}$.*

Proof. The bound $\rho(t,\cdot) \in L^{5/3}(\mathbb{R}^3)$ follows by Proposition 4.3.4 for $m = 2$. The estimate on the electric field is a consequence of Proposition 4.3.1 for $s = 1$ and $s = \frac{5}{3}$. □

The following proposition regards specifically the case in which we deal with a Dirac mass.

As a consequence of Proposition 3.2.2, we observe that if the energy $H(t)$ is assumed to be initially finite, then it is bounded for all times. This ensures in particular that the velocity of the Dirac mass located at $\xi(t)$ is finite.

Proposition 3.2.6. *Let $T > 0$ such that for all $t \in [0,T]$, $f(t)$ and $\xi(t)$ are solutions of the system* (4.1.1)-(4.1.2) *with finite associated initial energy $H(0)$. Then*

$$|\xi(t)| \leq |\xi_0| + T\sqrt{2H(0)}\,, \tag{3.2.6}$$

$$|\eta(t)| \leq \sqrt{2H(0)}\,. \tag{3.2.7}$$

Proof. We observe that $H(t)$ is a sum of positive terms. Notice that here we are heavily using the electrostatic nature of the particles in the plasma. In the gravitational case, the total energy has a nonpositive term. By Proposition 4.3.2, $H(t) = H(0)$ is finite, hence

$$\frac{|\eta(t)|^2}{2} \leq H(0)\,,$$

from which estimate (4.3.5) easily follows. We can use this bound in the first equation of (4.1.2) to get

$$|\xi(t)| \leq |\xi_0| + \int_0^t |\eta(s)|ds$$

which leads to (4.3.4) when using (4.3.5) and then taking the supremum in $t \in [0,T]$. □

3.3 From local to global existence

From the theory of characteristics in the smooth framework we know that, when $E \in C^1$, there exists a unique local solution $f \in C^1$ for the linear transport equation

$$\partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0, \tag{3.3.1}$$

which is constant along the trajectories, solution to the ODE

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = E(t,x). \tag{3.3.2}$$

The Vlasov-Poisson equation is, however, non linear, as the vector field (v, E) depends in turn on the solution. Nevertheless there is a result which not only provides local existence and uniqueness for a sufficiently large class of initial data, but also says in which way a solution can possibly stop to exist after a finite time.

Theorem 3.3.1. *Every initial datum $f_0 \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$, $f_0 \geq 0$, launches a unique classical solution $f \in C^1$ on some time interval $[0,T)$ with $f(0) = f_0$. For all $t \in [0,T)$ the function $f(t)$ is compactly supported and non-negative. If $T > 0$ is chosen maximal and if*

$$\sup\{|v| \,|\, (x,v) \in \mathrm{supp} f(t),\ 0 \leq t < T\} < \infty$$

or

$$\sup\{\rho(t,x) \,|\, 0 \leq t < T,\ x \in \mathbb{R}^3\} < \infty,$$

then the solution is global, i.e. $T = \infty$.

This means that a classical solution can be extended as long as its velocity support or its spatial density remain bounded. This rules out a breakdown of the solution by shock formation where typically the solution remains bounded but the derivative blows up; if the solution blows up, ρ must blow up due to a concentration effect.

Proof. We give a non-detailed proof.
Existence. To construct the solution, we consider the following iterative scheme. Let the initial datum be such that

$$f_0(x, v) = 0 \quad \text{for } |x| \geq R_0 \text{ or } |v| \geq P_0.$$

The 0th iterate is defined by

$$f_0(t, z) := f_0(z), \quad t \geq 0,\ z \in \mathbb{R}^6.$$

If the nth iterate f_n is already defined, we define

$$\rho_n := \rho_{f_n}, \quad E_n := E_{\rho_n}$$

on $[0, \infty) \times \mathbb{R}^3$, and we denote by

$$Z_n(s, t, z) = (X_n, V_n)(s, t, x, v)$$

the solution to the characteristic system

$$\dot{x} = v, \quad \dot{v} = E_n(s, x)$$

with $Z_n(t, t, z) = z$. Then

$$f_{n+1} := f_0(Z_n(0, t, z)), \quad t \geq 0,\ z \in \mathbb{R}^6$$

defines the next iterate. The idea is to show that the iterates converge on some time interval in a sufficiently strong sense and to identify the limit as the desired solution. Setting, for each $n \in \mathbb{N}$,

$$P_n(t) := \sup \left\{ |V_{n-1}(s, 0, z)| \mid z \in \operatorname{supp} f_0,\ 0 \leq s \leq t \right\},$$

we observe that

$$f_n(t, x, v) = 0 \quad \text{for } |v| \geq P_n(t) \text{ or } |x| \geq R_0 + \int_0^t P_n(s)ds.$$

Moreover, using Lemma 4.2.17, we have

$$\|E_n(t)\|_\infty \leq \|\rho_n(t)\|_{L^1}^{1/3} \|\rho_n(t)\|_\infty^{2/3} = \|f_0\|_{L^1}^{1/3} \|\rho_n(t)\|_\infty^{2/3} \quad \text{and}$$
$$\|\rho_n(t)\|_\infty \leq \frac{4\pi}{3} \|f_0\|_\infty P_n^3(t),$$

that lead to

$$\|E_n(t)\|_\infty \leq C(f_0) P_n^2(t).$$

Denoting by $P : [0, \delta) \to (0, \infty)$ the maximal solution of the integral equation

$$P(t) = P_0 + C(f_0) \int_0^t P^2(s)ds,$$

that is

$$P(t) = \frac{P_0}{1 - P_0 C(f_0) t}, \quad 0 \leq t < \delta := (P_0 C(f_0))^{-1},$$

it can be easily shown by induction that

$$P_n(t) \leq P(t) \quad \text{for } t \in [0, \delta).$$

Therefore

$$\|\rho_n(t)\|_\infty \le \frac{4\pi}{3}\|f_0\|_\infty P^3(t) \quad \text{and} \quad \|E_n(t)\|_\infty \le C(f_0)P^2(t).$$

Now it is possible to show that f_n converges to some f uniformly on $[0,\delta_0]\times\mathbb{R}^6$, for each $\delta_0 < \delta$. Indeed one can see that

$$\begin{aligned}
|f_{n+1}(t,z) - f_n(t,z)| &\le \|\partial_x f_0\|_\infty |Z_n(0,t,z) - Z_{n-1}(0,t,z)| \\
&\le C\int_0^t \|E_n(\tau) - E_{n-1}(\tau)\|_\infty d\tau \\
&\le C\int_0^t \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty^{2/3}\|\rho_n(\tau) - \rho_{n-1}(\tau)\|_{L^1}^{1/3} d\tau \\
&\le C\int_0^t \left(R_0 + \int_0^\tau P(s)ds\right)\|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty d\tau \\
&\le C\delta_0 P(\delta_0)\int_0^t \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty d\tau \\
&\le C\delta_0 P^4(\delta_0)\int_0^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty d\tau.
\end{aligned} \tag{3.3.3}$$

The value of C changes from line to line. Summing up we obtain

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le C_* \int_0^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty d\tau,$$

and by induction

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le C\frac{C_*^n t^n}{n!} \le C\frac{C^n}{n!}, \quad n\in\mathbb{N},\ 0\le t\le\delta_0.$$

This implies that f_n is uniformly Cauchy, hence converges uniformly on $[0,\delta_0]\times\mathbb{R}^6$ to some function $f\in C([0,\delta_0]\times\mathbb{R}^6)$. The fact that $P(t)$ is bounded on $[0,\delta_0]$, i.e. $P(\delta_0)<\infty$, leads also to

$$\rho_n \to \rho := \rho_f, \quad E_n \to E := E_f$$

as $n\to\infty$, uniformly on $[0,\delta_0]\times\mathbb{R}^3$. This in turn implies that

$$Z := \lim_{n\to\infty} Z_n,$$

which is the characteristic flow induced by the limiting field E. Hence

$$f(t,z) = \lim_{n\to\infty} f_0(Z_n(0,t,z)) = f_0(Z(0,t,z)).$$

In order to obtain a smooth flow (i.e. $C^1_{x,v}$), and hence a smooth solution f, additional controls on the derivatives of E are needed. These are obtained through another Gronwall loop as above.

Uniqueness. We notice that if f and g are two classical solutions with $f(0)=g(0)$, we obtain, similarly as above,

$$\|f(t) - g(t)\|_\infty \le C\int_0^t \|f(s) - g(s)\|_\infty ds$$

and uniqueness follows.

Continuation criterion. Let $f\in C^1([0,T)\times\mathbb{R}^6)$ be the maximally extended classical solution obtained above, and assume that

$$P^* := \sup\{|v| \,|\, (t,x,v)\in \operatorname{supp} f\} < \infty,$$

but $T<\infty$. The idea is to use the control of the length δ of the interval on which we constructed the solution to show that, if we use the procedure above for the new initial value problem where we prescribe $f(t_0)$ as initial datum at time $t=t_0$, we extend the solution beyond T if t_0 is chosen sufficiently close to T. This is then the desired contradiction. □

3.4 Vlasov-Poisson without point-charge

The Vlasov-Poisson equation in dimension two has been solved since many years (see for instance [43] and [32]). We sketch the idea.

Let f_0 be the initial distribution, compactly supported in velocity. As in the proof of Theorem 3.3.1, we define P_0 and $P(t)$ as the radius of the minimal sphere containing the support in velocity of f_0 and $f(t)$, respectively. We have

$$P(t) \leq P_0 + \int_0^t \|E(s)\|_\infty ds.$$

On the other hand, from Prop. 3.2.3,

$$\|E(t)\|_\infty \leq C\|\rho(t)\|_\infty^{1-\frac{1}{d}}.$$

Finally, using that

$$\|\rho(t)\|_\infty \leq CP^d(t)$$

we arrive to

$$P(t) \leq P_0 + C\int_0^t P^{d-1}(s)ds. \tag{3.4.1}$$

Eqn. (3.4.1) trivially yields a globally in time control on the velocity (hence also the spatial density) in dimension two only. Thanks to Theorem 3.3.1, this is enough to get global existence and uniqueness of the solution to the Vlasov-Poisson problem in term of characteristics.

In dimension three Eqn. (3.4.1) can be improved by using the energy conservation. Indeed, as shown in Prop. 4.3.5, this leads to additional integrability for ρ, i.e. $\rho \in L^{5/3}$. Moreover Prop. 3.2.3 with $p = \frac{5}{3}$ yields to

$$\|E(t)\|_\infty \leq C\|\rho(t)\|_\infty^{\frac{4}{9}} \leq CP^{\frac{4}{3}}(t). \tag{3.4.2}$$

In conclusion:

$$P(t) \leq P_0 + C\int_0^t P^{\frac{4}{3}}(s)ds \tag{3.4.3}$$

which is better that Eqn. (3.4.1) but still not enough to conclude.

One way to improve this argument is to observe that an a priori bound on a higher order L^p-norm of $\rho(t)$ allows for a smaller power of the L^∞-norm of $\rho(t)$ in the estimate (3.4.2) and thus for a smaller power of $P(s)$ in the Gronwall inequality (3.4.3). In the estimate (3.4.2) we would need an exponent less or equal to $1/3$ on $\|\rho(t)\|_\infty$ to obtain a Gronwall estimate on P leading to a global bound. If we compare this to Prop. 3.2.3 and use Prop. 4.3.7 with $m \geq 3$, we obtain a less demanding continuation criterion which we note for later use:

Proposition 3.4.1. *If for a solution f on its maximal existence interval $[0, T)$ the quantity $\|\rho(t)\|_{L^p}$ or $M_m(t) = \iint |v|^m f(t, x, v)dxdv$ is bounded for some $p \geq 2$ or $m \geq 3$, then the solution is global.*

The strategy described above, of controlling the moments, is adopted by Lions and Perthame in [36] (Eulerian point of view). At about the same time, in the nineties, the three-dimensional Vlasov-Poisson problem was also solved by Pfaffelmoser in [45] by controlling the characteristics (Lagrangian point of view).

3.4.1 Pfaffelmoser

The approach of [45] is to control the growth of the size of the support of a solution, i.e. to control the velocities. The new idea is that the time average of the electric field is better than its maximum. Below we state a slightly different version of the main theorem in [45], where the assumption on the initial distribution is less general but simpler to read. Moreover we follow a simplified version of the proof, due to J. Schaeffer ([47]).

Theorem 3.4.2. *Let $f_0 \in C^1_c(\mathbb{R}^6)$ be the initial distribution. Then the Cauchy problem for the Vlasov-Poisson equation has a global unique classical C^1 solution.*

Proof. We give only an idea of the proof.

Recalling Theorem 3.3.1, it is enough to verify the continuation criterion, that is: if f is a local solution defined on $[0,T)\times\mathbb{R}^6$ for some $T>0$, then $P(t)$ is bounded on $[0,T)$, where

$$P(t) = \sup\{|v| \mid (x,v)\in \mathrm{supp} f(s),\ 0\le s\le t\} = \sup\left\{|V(0,s,x,v)| \mid (x,v)\in \mathrm{supp} f_0,\ 0\le s\le t\right\}.$$

Let us single out one particle in our distribution, the increase in velocity of which we want to control over a certain time interval. Mathematically speaking, we fix a characteristic $(X,V)(t)$ with $(X,V)(0) = (x,v) \in \mathrm{supp} f_0$, and we take $0 \le \Delta \le t < T$. Therefore, from Liouville theorem follows that

$$\begin{aligned} |V(t)-V(t-\Delta)| &\le \int_{t-\Delta}^{t} |E(s,X(s))|ds \\ &\le \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|X(s)-y|^2} dw\,dy\,ds \\ &\le \int_{t-\Delta}^{t} \iint \frac{f(t-\Delta,y,w)}{|X(s)-Y(s)|^2} dw\,dy\,ds, \end{aligned} \tag{3.4.4}$$

where $(Y,W)(s)$ is a characteristic leaving (y,w) at time $t-\Delta$.

Now we remember that, in order to get the bound (3.4.3), we first split x-space to obtain the estimate (3.4.2) for E (see the proof of Prop. 3.2.3) and then split v-space to obtain a bound on $\|\rho\|_{L^{5/3}}$ (see Prop. 4.2.17 with $m=2$). Pfaffelmoser's idea is that, instead of doing one after the other, one should split (x,v)-space in (3.4.4) into suitable chosen sets. Hence, for every parameter $0<p\le P(t)$ and $r>0$, which will be chosen later, we split the domain of integration in (3.4.4) into the following sets, usually called the "good", the "bad" and the "ugly".

The "good" set is the one in which the velocities are bounded, either with respect to our frame of reference or with respect to the one particle which we singled out, namely

$$M_g = \left\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^6 \mid |w|\le p \vee |v-w|\le p\right\},$$

The integration on M_g yields a good bound. Indeed

$$\int_{M_g} \frac{f(t_{i-1},y,w)}{|X(s)-Y(s)|^2} dw\,dy\,ds \le \int_{t_{i-1}}^{t_i}\int \frac{\tilde\rho(s,y)}{|X(s)-y|^2} dy\,ds,$$

where

$$\tilde\rho(s,y) = \int_{|w|\le p \text{ or } |v-w|\le p} f(s,y,w)dw \le Cp^3,$$

and by Proposition 4.3.5,

$$\|\tilde\rho(s)\|_{L^{5/3}} \le \|\rho(s)\|_{L^{5/3}} \le C.$$

Therefore, by the estimate (3.4.2), we get

$$\int_{M_g} \frac{f(t_{i-1}, y, w)}{|X(s) - Y(s)|^2} dw\, dy\, ds \leq C(p^3)^{\frac{4}{9}} \Delta = C p^{4/3} \Delta.$$

Note that this is a good bound in case we set, for instance, $p = P(t)^{3/4}$. Nevertheless, in the proof, the choice of p is postponed after estimating the integrals on the other two sets (and this leads to an even better choice of p). Now, on M_b and M_u, where $|w| > p$ and $|v - w| > p$, we restrict our attention to the time integral

$$\int_{t_{i-1}}^{t_i} \frac{ds}{|X(s) - Y(s)|^2}. \tag{3.4.5}$$

The length of the time interval Δ is chosen in such a way that the relative velocity remains large in that time interval (*stability property*). In this situation the time integral (3.4.5) can be computed almost explicitly, using that $X(s) - Y(s)$ essentially describes a free motion. We notice that the definition of M_b and M_u, which we omit here, involves also a second parameter $r > 0$. The trick, in conclusion, is to optimize on the parameters p and r, which define the sets, in order to have a sublinear Gronwall-type estimate for $P(t)$. More specifically we obtain

$$\frac{1}{\Delta} \int_{t-\Delta}^{t} |E(s, X(s))| ds \leq C P(t)^{16/33} |\log P(t)|^{1/2},$$

which leads to

$$P(t) \leq C(1 + t)^q \quad \text{for} \quad q > \frac{33}{17}.$$

□

Remark 13. We remark that the assumptions on the initial datum can be weakened. For instance f_0 can be an L^∞ function, compactly supported.

3.4.2 Lions and Perthame

We present the idea developed in [36], which leads to the verification of the continuation criterion in Proposition 3.4.1. Here there is no need for the initial data to be compactly supported.

Theorem 3.4.3. *Let $f_0 \geq 0$ be a function in $L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and assume that*

$$\iint f_0(x, v)|v|^m dx dv < +\infty$$

for some $m > 3$. Then there exists a weak solution to the Vlasov-Poisson system in $C(\mathbb{R}^+; L^p(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $p \in [1, +\infty)$ satisfying

$$\sup_{t \in (0,T)} \iint f(t, x, v)|v|^m dx dv \leq C(T) \quad \textit{for any} \quad T > 0,$$

$$\rho(t, x) = \int f(t, x, v) dv \in C(\mathbb{R}^+; L^q(\mathbb{R}^3)), \quad 1 \leq q < \frac{3 + m}{3},$$

$$E(t, x) \in C(\mathbb{R}^+; L^q(\mathbb{R}^3)), \quad \frac{3}{2} < q < 3\frac{3 + m}{6 - m}.$$

Proof. The main estimate is the propagation of moments.

$$\begin{aligned} f(t, x, v) &= \int_0^t \mathrm{div}_v(E)(t - s, x - vs) f(t - s, x - vs, v) ds + f_0(x - vt, v) \\ &= \int_0^t \mathrm{div}_v \left[E f(t - s, x - vs, v) \right] ds + \int_0^t s\, \mathrm{div}_x \left[E f(t - s, x - vs, v) \right] ds + f_0(x - vt, v). \end{aligned}$$

If $\rho_0(t,x) = \int f_0(x - vt, v)dv$, then

$$\rho(t,x) = \rho_0(t,x) + \int_0^t s \operatorname{div}_x [Ef(t-s, x-vs, v)]\, ds$$

and according to Hardy-Littlewood-Sobolev inequalities with $\frac{1}{p} = \frac{1}{r} - \frac{1}{3}$, $\frac{3}{2} < p < +\infty$,

$$\|E(t)\|_{L^p} \leq \|\rho_0(t)\|_{L^r} + C \left\| \int_0^t s \int Ef(t-s, x-vs, v) dv dx \right\|_{L^p}.$$

For $p = m+3$, $r = \frac{3(m+3)}{m+6}$, $m \geq 3$,

$$\|\rho_0(t)\|_{L^r} \leq C \left(\iint f_0(x,v)|v|^m dx dv \right)^{\frac{3}{m+3}} = \text{const.}$$

$$\text{and} \quad \|E(t)\|_{L^{m+3}} \leq C \left(1 + \left\| \int_0^t s \int Ef(t-s, x-vs, v) dv dx \right\|_{L^{m+3}} \right). \quad \text{But}$$

$$\frac{d}{dt} \left(\iint |v|^k f(t,x,v) dx dv \right) \leq C \, \|E(t)\|_{L^{k+3}} \left(\iint |v|^k f(t,x,v) dx dv \right)^{\frac{k+2}{k+3}}$$

which (roughly speaking) closes the system of Gronwall estimates. □

Remark 14. Moreover if $m > 6$, adding other regularity assumptions on the initial density, it is possible to prove uniqueness of a weak solution (see [36]). In addition, the theory of DiPerna and Lions ([29]) ensures that such solutions are transported by characteristics which are defined in a weak sense.

3.5 Vlasov-Poisson with point-charge

The study of the modified Vlasov-Poisson system with macroscopic point charges ((3.1.3)-(3.1.4)) was initiated by Caprino and Marchioro ([17]) who solved the problem in dimension two, in case the charges are initially apart from the plasma. The difficulty here consists in the addiction to the electric field produced by the plasma of a singular field, due to the presence of the charges. This could push plasma and charges to get close to each other, causing an infinite velocity field. The key idea in [17] is the introduction of the energy of a single plasma trajectory (see definition (3.5.1) below) which controls the motion of a plasma particle and prevents its approach to the point charge. Combining this with static estimates on the electric field, the authors of [17] prove global existence and uniqueness of solutions à la Pfaffelmoser.

The problem in dimension three was studied, under particular assumptions on the inital data, by Marchioro, Miot, Pulvirenti in [40] and by Desvillettes, Miot, Saffirio in [28].

3.5.1 Marchioro-Miot-Pulvirenti

The plasma-charge problem in dimension three is more involved. On one hand, when we raise the dimension, the a priori bound on the electric field given by Prop. 3.2.3 is not anymore sufficient to allow a linear (or sublinear) Gronwall-type estimate on the velocity support. Indeed we recall that it leads only to

$$P(t) \leq P(0) + \int_0^t P(s)^2 ds.$$

Therefore, following the line of Pfeffelmoser, the approach in [40] will be to estimate not the maximum, but averages of the electric field on a proper partition of the time interval.

On the other hand, when a point charge is present, the stability property for the trajectories of the plasma fails: from (3.1.4) we have

$$\dot{V} = E(X) + \frac{X-\xi}{|X-\xi|^3}.$$

Indeed, the velocity of the plasma particles can change extremely quickly if one of them collides with (or approaches very close to) the point charge. As a consequence, following [17], in [40] the authors find convenient to exploit the notion of *microscopic energy*, defined by

$$h(t,x,v) = \frac{|v-\eta(t)|^2}{2} + \frac{1}{|x-\xi(t)|}. \tag{3.5.1}$$

Differentiating along the trajectories we get

$$\frac{d}{dt}h(t,X(t),V(t)) = (V(t)-\eta(t))\cdot(E(X(t)) - E(\xi(t)),$$

from which

$$\left|\frac{d}{dt}\sqrt{h}(t,X,V)\right| \le |E(X)| + |E(\xi)|.$$

Note that the variation of h is controlled by the smooth part of the electric field, as the singular part does not appear. Therefore h is stable (for small times we expect a little change), while V is not. In conclusion, considering the partition of the time interval

$$(0,T] = \cup_{i=1}^n (t_{i-1},t_i],$$

we want to have a control on the quantities

$$\int_{t_{i-1}}^{t_i} |E(X(t))|dt \quad \text{and} \quad \int_{t_{i-1}}^{t_i} |E(\xi(t))|dt.$$

This clearly allows to have an upper bound on $\sqrt{h}$, once we assume that the plasma is initially apart from ξ_0 (recall also Prop. 4.3.3). This in turn implies that, if the initial density does not overlap with the charge, this property remains true at later times so that the field induced by the charge is bounded on the support of the density and the velocities of the plasma particles do not blow up. In other words, it is possible to adapt Pfaffelmoser's arguments, replacing $P(t)$ with $Q(t)$, showing that the quantity

$$Q(t) = \sup\left\{ |V(0,t,x,v)| + \frac{1}{|X(0,t,x,v)-\xi(t)|} \mid t\in[0,T],\ (x,v)\in \text{supp}(f_0)\right\}$$

is bounded on $[0,T]$. Once this fact is proved, existence and uniqueness of a classical solution follows by rather standard arguments. Let us state the theorem of [40].

Theorem 3.5.1. *Let $f_0 \in L^\infty$ be compactly supported. Let $(\xi_0,\eta_0)\in\mathbb{R}^3\times\mathbb{R}^3$. Assume that there exists a $\delta_0>0$ such that*

$$\min\{|x-\xi_0| \mid (x,v)\in\text{supp}(f_0)\} \ge \delta_0.$$

For all time $T>0$ there exists a unique classical solution (ξ,f) to (3.1.3)-(3.1.4) *on* $[0,T]$ *with this initial datum.*

3.5.2 Desvillettes-Miot-Saffirio

In [28] the authors want to extend the study of the Vlasov-Poisson system with point-charge to the case of initial densities which are not compactly supported and can overlap with the charge. Since the electric field is now a priori unbounded with a singular component, the authors find more convenient to adapt the PDE point of view from [36]. They show existence of a weak solution propagating the energy moments. Notice that their result holds for initial densities that do not necessarily vanish in a neighborhood of the charge, but that have to decay close to it in some sense.

The idea of the proof is the following: we consider a sequence of mollifier $f_{0,\varepsilon}$ of f_0 which vanishes in a small ε- neighborhood of ξ_0. Then the existence of a classical solution to (3.1.3)-(3.1.4) is provided by [40]. Through iteration of some arguments used in [36], the authors are able to recover some a priori estimates for (f, ξ) which will eventually lead to the existence of a solution by compactness.

We state the theorem.

Proposition 3.5.2 (Theorem 1.1. in [28]). *Let* $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *non-negative,* $(\xi_0, \eta_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ *and* $H(0)$ *finite. Assume further that*

(i) $M(0) < 1$,

(ii) *There exists* $m_0 > 6$ *such that for all* $m < m_0$

$$\iint \left(|v|^2 + \frac{1}{|x - \xi_0|} \right)^{m/2} f_0(x, v) dx dv < +\infty .$$

Then there exists a global weak solution (f, ξ) *to the system* (4.1.1)-(4.1.2)*, with* $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ *for any* $1 \leq p < +\infty$*,* $\xi \in C^2(\mathbb{R}_+)$*, and* $E \in L^\infty([0, T], C^{0,\alpha}(\mathbb{R}^3))$ *for all* $T > 0$.

Moreover, for all $t \in \mathbb{R}_+$ *and for all* $m < \min(m_0, 7)$*,*

$$\iint \left(|v|^2 + \frac{1}{|x - \xi(t)|} \right)^{m/2} f(t, x, v)\, dx\, dv \leq C(1 + t)^c , \tag{3.5.2}$$

where C *and* c *only depend on the initial data.*

Chapter 4

Lagrangian solution to the Vlasov-Poisson system with a point charge

In [26] we consider the Cauchy problem for the repulsive Vlasov-Poisson system in the three dimensional space, where the initial datum is the sum of a diffuse density, assumed to be bounded and integrable, and a point charge. Under some decay assumptions for the diffuse density close to the point charge, under bounds on the total energy, and assuming that the initial total diffuse charge is strictly less than one, we prove existence of global Lagrangian solutions. Our result extends the Eulerian theory of [28], proving that solutions are transported by the flow trajectories. The proof is based on the ODE theory developed in [11] in the setting of vector fields with anisotropic regularity, where some components of the gradient of the vector field is a singular integral of a measure.

4.1 Introduction and main result

The three-dimensional Vlasov-Poisson system for initial data containing one Dirac mass writes

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + (E+F)\cdot\nabla_v f = 0\,, \\ E(t,x) = \int \frac{x-y}{|x-y|^3}\rho(t,y)\,dy\,, \\ \rho(t,x) = \int f(t,x,v)\,dv\,, \\ F(t,x) = \frac{x-\xi(t)}{|x-\xi(t)|^3}\,. \end{cases} \tag{4.1.1}$$

Here $f = f(t,x,v) \in L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)$ corresponds to a nonnegative density of charged particles in a plasma, subjected to a self-induced electric force field $E = E(t,x)$. The plasma interacts with a point charge, located at $\xi(t)$ with velocity $\eta(t)$, which induces the singular electric field $F = F(t,x)$. The evolution of the charge is itself given by

$$\begin{cases} \dot\xi(t) = \eta(t)\,, \\ \dot\eta(t) = E(t,\xi(t))\,. \end{cases} \tag{4.1.2}$$

The initial conditions associated to (4.1.1)-(4.1.2) are

$$(\xi(0),\eta(0)) = (\xi(0),\eta(0)), \qquad f(0,x,v) = f_0(x,v). \tag{4.1.3}$$

As observed in Chapter 3, the system (4.1.1)-(4.1.2) can also be thought of as the standard Vlasov-Poisson system (3.1.2) for the total density $f(t) + \delta_{\xi(t)} \otimes \delta_{\eta(t)}$.

In [26] we prove, under particular assumptions on the initial density, existence of *Lagrangian solutions*, *i.e.* a plasma density f and a trajectory (ξ, η) of the Dirac mass, both defined for $t \in \mathbb{R}_+$, such that f is transported by the Lagrangian flow (X, V), solution to the ODE-system

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v) \\ \dot{V}(t,x,v) = E(t, X(t,x,v)) + F(t, X(t,x,v)) \\ \big(X(0,x,v), V(0,x,v)\big) = (x,v)\,, \end{cases} \tag{4.1.4}$$

more precisely

$$f(t,x,v) = f_0\big(X^{-1}(t,\cdot,\cdot)(x,v)\,,\, V^{-1}(t,\cdot,\cdot)(x,v)\big)\,.$$

This is a finer physical structural information on the solution than the mere fact that f and (ξ, η) are weak solutions of (4.1.1)–(4.1.2).

In the framework of classical solutions, the Eulerian description and the Lagrangian evolution of particles given by the system of characteristics are completely equivalent. When dealing with weak or renormalized solutions, the correspondence between the Eulerian and Lagrangian formulations is non trivial and requires a careful analysis of the Lagrangian structure of transport equations with non-smooth vector fields. Indeed, without any regularity assumptions, it is not even clear whether the flow associated with the vector field generated by a weak solution exists.

In recent years the theory of transport and continuity equations with non-smooth vector fields has witnessed a massive amount of progress, also due to the large number of applications to nonlinear PDEs. In the seminal paper by DiPerna and Lions [29] the theory has been first developed in the context of Sobolev vector fields, with suitable bounds on space divergence and under suitable growth assumptions. This has been extended by Ambrosio [4] to the setting of vector field with bounded variation (BV), roughly speaking allowing for discontinuities along codimension-one hypersurfaces.

In the context of the Vlasov-Poisson system with a Dirac mass considered in this paper ((4.1.1)-(4.1.2)) the system of characteristics is given by (4.1.4). The singular electric field F generated by the Dirac mass is not regular, and it does not even belong to any Sobolev space of order one or to the BV space. Therefore the theory of [29, 4] cannot be directly applied to this case. However, a related theory of Lagrangian flows for non-smooth vector fields has been initiated in [24]. In a nutshell, the approach in [24] provides a suitable extension of Grönwall-like estimates to the context of Sobolev vector fields, by introducing a suitable functional measuring a logarithmic distance between Lagrangian flows. In addition, the theory in [24] has a quantitative character, providing explicit rates in the stability and compactness estimates, and it has been pushed even to situations out of the Sobolev or BV contexts of [29, 4]. In particular, using more sophisticate harmonic analysis tools, the case when the derivative of the vector field is a singular integral of an L^1 function has been considered in [15]. This has been further developed in [11], allowing for singular integrals of a measure, under a suitable condition on splitting of the space in two groups of variables, modelled on the situation for the Vlasov-Poisson characteristics (4.1.4). This theory has been applied to the study of the Euler equation with L^1 vorticity [13] and of the Vlasov-Poisson equation with L^1 density [12]. The latter has also been studied in [6], using the theory of maximal Lagrangian flows developed in [5].

The purpose of [26] is to recover the relation between the Eulerian and the Lagrangian picture for solutions provided in [28] by exploiting the transport structure of the equation. In other words we aim to prove existence of Lagrangian solutions to the Vlasov-Poisson system (3.1.2) with $\gamma = 1$ and initial data $f_0 + \delta_{\xi_0} \otimes \delta_{\eta_0}$, where f_0 satisfies the assumptions of [28].

Our main result is the following

Theorem 4.1.1. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, such that the initial total charge*

$$M(0) = \iint f_0(x, v)\, dx dv < 1 \tag{4.1.5}$$

and the total energy

$$H(0) = \iint \frac{|v|^2}{2} f_0(x, v) dx dv + \frac{|\eta_0|^2}{2} + \frac{1}{2} \iint \frac{\rho(0, x)\rho(0, y)}{|x - y|} dx dy + \iint \frac{\rho(0, x)}{|x - \xi_0|} dx \tag{4.1.6}$$

is finite. Assume that there exists $m_0 > 6$ such that for all $m < m_0$ the energy moments

$$\mathcal{H}_m(0) = \iint \left(|v|^2 + \frac{1}{|x - \xi_0|} \right)^{m/2} f_0(x, v) dx dv \tag{4.1.7}$$

are finite. Then there exists a global Lagrangian solution to the system (4.1.1)–(4.1.2).

Some remarks are in order:

1. The moments (4.1.7) are propagated in time (see Proposition 4.3.7 for the precise statement and [28] for details). This implies $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ for $1 \le p < \infty$, $E \in L^\infty([0, T], C^{0,\alpha}(\mathbb{R}^3))$ and $\xi \in C^2(\mathbb{R}_+)$.

2. We observe that the hypothesis (4.1.5) is needed only to get a control on the electric field generated by the point charge (see Proposition 4.3.6). This means that the charge of the plasma has to be smaller than the charge associated with the Dirac mass. From the viewpoint of physics, this is a purely technical and too restrictive condition. In a forthcoming paper, we plan to remove this constraint.

3. When considering the Cauchy problem associated with (3.1.2) with $\gamma = -1$ (attractive case) and initial data $f_0 + \delta_{\xi_0} \otimes \delta_{\eta_0}$, the whole strategy fails. This is due to a crucial change of sign in the total energy H and in $\mathcal{H}_m$. More precisely, the last two terms in (4.1.6) and the last term in (4.1.7), representing respectively the potential energy of the system and the potential energy per particle, come with a negative sign. This prevents to establish a control on the trajectory of the point charge as in Proposition 4.3.3 and to prove Proposition 4.3.7.
   The simpler case of a system in which the particles in the plasma are interacting through a repulsive potential while the point charge generates an attractive force field has been treated in [18] in dimension two. Notice that, even in this case, the existence of solutions in three dimensions remains an interesting open problem.

4. Theorem 4.1.1 does not imply uniqueness of the Lagrangian solution. In analogy to [45], where uniqueness of compactly supported classical solutions of (3.1.2) has been proved, uniqueness of solutions to (4.1.1)-(4.1.2) which do not overlap with the point charge and have compact support in phase space has been established in [40]. In the context of weak solutions to (3.1.2), sufficient conditions for uniqueness have been proved in [36] and later extended to weak measure-valued solutions with bounded spatial density by Loeper [39]. Recently Miot [41] generalised the latter condition to a class of solutions whose L^p norms of spatial density grow at most linearly w.r.t. p, then extended to spatial densities belonging to some Orlicz space in [33]. Unfortunately, it seems that none of these conditions apply to our setting and new ideas are needed.

Let us informally describe the main steps of our proof. We rely on the result in [40], which guarantees existence of a (unique) Lagrangian solution to the Cauchy problem for the Vlasov-Poisson system (4.1.1)-(4.1.2), provided that at initial time the plasma density has a positive distance from the

Dirac mass and bounded support in the phase space. We therefore approximate the plasma density f_0 at initial time by a sequence f_0^n obtained by cutting off f_0 close to the Dirac mass in the space variable and out of a compact set in phase space. We use [40] to construct a Lagrangian flow (X_n, V_n) and a trajectory for the Dirac mass (ξ_n, η_n) corresponding to the initial data f_0^n and (ξ_0, η_0). The assumptions of Theorem 4.1.1 together with the propagation of the moments $\mathcal{H}$ from [28] entail some additional integrability of the densities ρ_n, which in turn implies uniform Hölder estimates on the electric fields E_n. Moreover, assumption (4.1.5) allows to prove some uniform decay of the superlevels of the Lagrangian flows (X_n, V_n), which combined with an extension of the Lagrangian theory developed in [11] gives compactness of the Lagrangian flows (X_n, V_n). Finally, standard energy estimates guarantee the uniform continuity of the trajectories ξ_n uniformly in n. All this enables us to pass to the limit in the Lagrangian formulation of the problem, eventually giving a Lagrangian solution corresponding to the initial plasma density f_0.

One of the main technical difficulties of our analysis is the control on large velocities. In this work, this reflects in the necessity of some control on the superlevels of the Lagrangian flows. This was already an issue in [12] and here the situation is made even more complicated by the presence of the singular field generated by the point charge. We tackle this problem by weighting superlevels with the measure given by the initial distribution of charges $f_0(x, v)\,dx\,dv$ (see Lemma 5.2.2). In this way the control on the superlevels can be proven exploiting virial type estimates on the time integral of the electric field generated by the diffuse charge and evaluated in the point charge (see Proposition 4.3.6). This carries the physical meaning that it is only relevant to control the flow starting from points in the support of the initial density of charge.

In connection to the theory of [11], this weighted estimates manifests in the presence of the density $h = f_0$ in the functional (4.2.12) measuring the compactness of the flows. Moreover, in contrast to [12], which was based on the isotropic analysis of [15], here we strongly rely on the anisotropic theory of [11] in which some components of the gradient of the velocity field are allowed to be singular integrals of measures, accounting for the presence of the point charge.

The plan of the chapter is the following: in Section 4.2 we present and prove the key theorem on Lagrangian flows; in Section 4.3 we recall some useful properties related to solutions of the Vlasov-Poisson system; in Section 4.4 we give the proof of Theorem 4.1.1, which follows from compactness arguments by using the results established in Section 4.2 and 4.3.

4.2 Lagrangian flows

Consider a smooth solution u to a transport equation in $\mathbb{R}^+ \times \mathbb{R}^d$

$$\partial_t u + b \cdot \nabla_z u = 0\,,$$

where $b = b(t, z)$ is a smooth vector field. Then u is constant along the characteristics $s \mapsto Z(s, t, z)$, exiting from z at time t, i.e. solutions to the equation

$$\frac{dZ}{ds}(s, t, z) = b(s, Z(s, t, z)), \tag{4.2.1}$$

with initial data $Z(t, t, z) = z$. Thus the solution can be expressed as $u(t, z) = u_0(Z(0, t, z))$.

For simplicity from now on we will consider the initial time t in (4.2.1) fixed and denote the flow $Z(s, t, z)$ by $Z(s, z)$.

In [26] we deal with flows of non-smooth vector fields. In order to extend the usual notion of characteristics to our case, we extend the definition of *regular Lagrangian flows* in a renormalized sense by introducing a reference measure with bounded density. This turns out to be convenient in the estimates involving the superlevels of the flow (see Lemma 5.2.2).

Definition 4.2.1 (μ-regular Lagrangian flow)**.** Given an absolutely continuous measure μ with bounded density, a vector field $b(s,z) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, and $t\in[0,T)$, a map

$$Z \in C([t,T]_s; L^0_{\mathrm{loc}}(\mathbb{R}^d_z, d\mu)) \cap \mathcal{B}([t,T]_s; \log\log L_{\mathrm{loc}}(\mathbb{R}^d_z, d\mu))$$

is a μ-regular Lagrangian flow in the renormalized sense starting at time t relative to b if we have the following:

(1) The equation

$$\partial_s(\beta(Z(s,z))) = \beta'(Z(s,z))b(s,Z(s,z)) \tag{4.2.2}$$

holds in $\mathcal{D}'((t,T))$ for μ-a.e. z, for every function $\beta\in C^1(\mathbb{R}^d;\mathbb{R})$ that satisfies

$$|\beta(z)| \le C(1+\log(1+\log(1+|z|^2))) \quad \text{and} \quad |\beta'(z)| \le \frac{C\,|z|}{(1+|z|^2)(1+\log(1+|z|^2))}$$

for all $z\in\mathbb{R}^d$;

(2) $Z(t,z)=z$ for μ-a.e. $z\in\mathbb{R}^d$;

(3) There exists a $L\ge 0$, called compressibility constant, such that, for every $s\in[t,T]$,

$$Z(s,\cdot)_\#\mu \le L\mu, \tag{4.2.3}$$

i.e.

$$\mu(\{z\in\mathbb{R}^d : Z(s,z)\in B\}) \le L\mu(B) \qquad \text{for every Borel set } B\subset\mathbb{R}^d.$$

We have denoted with L^0_{loc} the space of measurable functions endowed with the local convergence in measure, by $\log\log L_{\mathrm{loc}}$ the space of measurable functions u such that $\log(1+\log(1+|u|^2))$ is locally integrable, and by $\mathcal{B}$ the space of bounded functions. When the reference measure μ is not explicitly specified, the spaces under consideration are endowed with the Lebesgue measure.

Remark 15. Our definition of μ-regular Lagrangian flow slightly differs from the one in [11]. On the one hand we change the reference measure from the Lebesgue measure to μ. On the other hand we consider a different class of β's, which grow slower at infinity.

Definition 4.2.2. Let $Z : [t,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a measurable map. For every $\lambda>0$, we define the sublevel of Z as

$$G_\lambda = \left\{ z\in\mathbb{R}^d : |Z(s,z)|\le\lambda \text{ for almost all } s\in[t,T] \right\}. \tag{4.2.4}$$

4.2.1 Setting and result of [11]

We summarize here the regularity setting and the stability estimate of [11]. We say that a vector field b satisfies **(R1)** if b can be decomposed as

$$\frac{b(t,z)}{1+|z|} = \tilde b_1(t,z) + \tilde b_2(t,z) \tag{4.2.5}$$

where $\tilde b_1 \in L^1((0,T);L^1(\mathbb{R}^d))$, $\tilde b_2 \in L^1((0,T);L^\infty(\mathbb{R}^d))$. Notice that this hypothesis leads to an estimate for the decay of the superlevels of a regular Lagrangian flow. In fact Lemma 3.2 of [11] tells us that, if b satisfies (R1) and Z is a regular Lagrangian flow associated with b starting at time t, with compressibility constant L, then $\mathcal{L}^d(B_r\setminus G_\lambda)\le g(r,\lambda)$ for any $r,\lambda>0$, where g depends only on L, $\|\tilde b_1\|_{L^1((0,T);L^1(\mathbb{R}^d))}$ and $\|\tilde b_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}$ and satisfies $g(r,\lambda)\downarrow 0$ for r fixed and $\lambda\uparrow\infty$.

(R2) We want to consider a vector field $b(t,z)$ such that its regularity changes with respect to different directions of the variable $z\in\mathbb{R}^d$, that is we consider $\mathbb{R}^d=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and $z=(z_1,z_2)$ with

$z_1 \in \mathbb{R}^{n_1}$ and $z_2 \in \mathbb{R}^{n_2}$. We denote with D_1 the derivative with respect to z_1 and D_2 the derivative with respect to z_2. Accordingly we denote $b = (b_1, b_2)(s, z) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $Z = (Z_1, Z_2)(s, z) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Therefore we assume that the elements of the matrix Db, denoted as $(Db)^i_j$, are in the form

$$(Db)^i_j = \sum_{k=1}^{m} \gamma^i_{jk}(s, z_2) S^i_{jk} \mathsf{m}^i_{jk}(s, z_1) \tag{4.2.6}$$

where

- S^i_{jk} are singular integral operators associated with singular kernels of fundamental type in $\mathbb{R}^{n_1}$ (see [48]),
- the functions γ^i_{jk} belong to $L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ for some $q > 1$,
- $\mathsf{m}^i_{jk} \in L^1((0,T); L^1(\mathbb{R}^{n_1}))$ for all the elements of the submatrices $D_1 b_1$, $D_2 b_1$ and $D_2 b_2$, while $\mathsf{m}^i_{jk} \in L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ if $(Db)^i_j$ is an element of $D_1 b_2$.

We have denoted by $L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ the space of all functions $t \mapsto \mu(t, \cdot)$ taking values in the space $\mathcal{M}(\mathbb{R}^{n_1})$ of finite signed measures on $\mathbb{R}^{n_1}$ such that

$$\int_0^T \|\mu(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^{n_1})} dt < \infty.$$

Moreover, we assume condition **(R3)**, that is

$$b \in L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^d) \qquad \text{for some } p > 1. \tag{4.2.7}$$

We recall the main theorem from [11].

Theorem 4.2.3. *Let b and $\bar{b}$ be two vector fields satisfying assumption (R1), where b satisfies also (R2), (R3). Fix $t \in [0,T]$ and let Z and $\bar{Z}$ be regular Lagrangian flows starting at time t associated with b and $\bar{b}$ respectively, with compressibility constants L and $\bar{L}$. Then the following holds. For every γ, r, $\eta > 0$ there exist λ, $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^d \left(B_r \cap \{ |Z(s,\cdot) - \bar{Z}(s,\cdot)| > \gamma \} \right) \le C_{\gamma,r,\eta} \|b - \bar{b}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T); L^1(\mathbb{R}^{n_1}))$ of all the m^i_{jk} which belong to this set, as well as the norm in $L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ of the remaining m^i_{jk} (where these functions are associated with b as in (R2)),*
- *The norms of the singular integrals operators S^i_{jk}, as well as the norms of γ^i_{jk} in $L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ (associated with b as in (R2)),*
- *The norm in $L^p((0,T) \times B_\lambda)$ of b,*
- *The $L^1((0,T); L^1(\mathbb{R}^d)) + L^1((0,T); L^\infty(\mathbb{R}^d))$ norms of the decomposition of b and $\bar{b}$ as in (R1),*
- *The compressibility constants L and $\bar{L}$.*

4.2.2 Flow estimate in the new setting

We are going now to state a variant of this theorem, where (R1) and (R2) are replaced by (R1a) and (R2a) below. The dimension d will be here equal to $2N$, instead of $n_1 + n_2$, and the variable z will be in the form $z = (x, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

We consider the following assumptions, that are adapted to our setting of the Vlasov-Poisson system with a point charge:

(R1a) For all μ-regular Lagrangian flow $Z : [t, T] \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ relative to b starting at time t with compression constant L, and for all r, $\lambda > 0$,

$$\mu(B_r \setminus G_\lambda) \le g(r, \lambda), \qquad \text{with } g(r, \lambda) \to 0 \text{ as } \lambda \to \infty \text{ at fixed } r, \tag{4.2.8}$$

where G_λ denotes the sublevel of the flow Z defined in (4.2.2).

(R2a) Motivated by the particular structure of the Vlasov-Poisson system (4.1.1)-(4.1.2), we assume b to have the following structure:

$$b(t, x, v) = (b_1, b_2)(t, x, v) = (b_1(v), b_2(t, x)), \tag{4.2.9}$$

with

$$b_1 \in \mathrm{Lip}(\mathbb{R}^N_v), \tag{4.2.10}$$

and where b_2 is such that for every $j = 1, \dots, N$,

$$\partial_{x_j} b_2 = \sum_{k=1}^{m} S_{jk} \mathsf{m}_{jk}, \tag{4.2.11}$$

where S_{jk} are singular integrals of fundamental type on $\mathbb{R}^N$ and $\mathsf{m}_{jk} \in L^1((0, T); \mathcal{M}(\mathbb{R}^N))$.

Theorem 4.2.4. *Let $\mu = h\mathcal{L}^{2N}$ with $h \in L^1 \cap L^\infty$ and non-negative. Let b and $\bar{b}$ be two vector fields satisfying (R1a), b satisfying also (R2a), (R3). Given $t \in [0, T]$, let Z and $\bar{Z}$ be μ-regular Lagrangian flows starting at time t associated with b and $\bar{b}$ respectively, with sublevels G_λ and $\bar{G}_\lambda$, and compressibility constants L and $\bar{L}$. Then the following holds.*
For every γ, r, $\eta > 0$, there exist λ, $C_{\gamma,r,\eta} > 0$ such that

$$\mu(B_r \cap \{|Z(s, \cdot) - \bar{Z}(s, \cdot)| > \gamma\}) \le C_{\gamma,r,\eta} \|b - \bar{b}\|_{L^1((0,T)\times B_\lambda)} + \eta$$

uniformly in s, $t \in [0, T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The norms of the singular integral operators S_{jk} from (R2a),*
- *The norms in $L^1((0, T); \mathcal{M}(\mathbb{R}^N))$ of m_{jk} from (R2a),*
- *The Lipschitz constant of b_1 from (R2a),*
- *The norm in $L^p((0, T) \times B_\lambda)$ of b corresponding to (R3),*
- *The rate of decay of $\mu(B_r \setminus G_\lambda)$ and $\mu(B_r \setminus \bar{G}_\lambda)$ from (R1a),*
- *The norm in $L^\infty(\mathbb{R}^{2N})$ of the function h defined in (R1a),*
- *The compressibility constants L and $\bar{L}$.*

Proof. The proof follows the same line as in Theorem 4.2.3 (see [11]), with some modifications due to the different hypotheses. Given $\delta_1, \delta_2 > 0$, let A be the constant $2N \times 2N$ matrix

$$A = \mathrm{Diag}(\underbrace{\delta_1, \dots, \delta_1}_{N \text{ times}}, \underbrace{\delta_2, \dots, \delta_2}_{N \text{ times}}),$$

that means $A(x, v) = (\delta_1 x, \delta_2 v)$. We consider the following functional depending on the two parameters δ_1 and δ_2, with $\delta_1 \le \delta_2$:

$$\Phi_{\delta_1,\delta_2}(s) = \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \log(1 + |A^{-1}[Z(s, x, v) - \bar{Z}(s, x, v)]|)\, h(x, v)\, dx\, dv. \tag{4.2.12}$$

In order to improve the readability of the following estimates, we will use the notation "$\lesssim$" to denote an estimate up to a constant only depending on absolute constants and on the bounds assumed in Theorem 4.2.4, and the notation "$\lesssim_\lambda$" to mean that the constant could also depend on the truncation parameter for the superlevels of the flow λ. The norm of the measure m however will be written explicitly.

Step 1: Differentiating Φ_{δ_1,δ_2}. Differentiating with respect to time and taking out of the integral the L^∞ norm of h, we get

$$\begin{aligned}\Phi'_{\delta_1,\delta_2}(s) &\le \|h\|_{L^\infty(\mathbb{R}^{2N})} \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \frac{|A^{-1}[b(s, Z(s, x, v)) - \bar{b}(s, \bar{Z}(s, x, v))]|}{1 + |A^{-1}[Z(s, x, v) - \bar{Z}(s, x, v)]|}\, dx\, dv \\ &\lesssim \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \frac{|A^{-1}[b(s, Z(s, x, v)) - \bar{b}(s, \bar{Z}(s, x, v))]|}{1 + |A^{-1}[Z(s, x, v) - \bar{Z}(s, x, v)]|}\, dx\, dv\,.\end{aligned}$$

Then we set $Z(s, x, v) = Z$ and $\bar{Z}(s, x, v) = \bar{Z}$ and we estimate

$$\Phi'_{\delta_1,\delta_2}(s) \lesssim \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} |A^{-1}[b(s, \bar{Z}) - \bar{b}(s, \bar{Z})]|\, dx\, dv + \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \frac{|A^{-1}[b(s, Z) - b(s, \bar{Z})]|}{1 + |A^{-1}[Z - \bar{Z}]|}\, dx\, dv.$$

After a change of variable along the flow $\bar{Z}$ in the first integral, and noting that $\delta_1 \le \delta_2$, we further obtain

$$\begin{aligned}\Phi'_{\delta_1,\delta_2}(s) &\lesssim \frac{\bar{L}}{\delta_1} \|b(s, \cdot) - \bar{b}(s, \cdot)\|_{L^1(B_\lambda)} \\ &\quad + \iint\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \min\left\{ |A^{-1}[b(s, Z) - b(s, \bar{Z})]|, \frac{|A^{-1}[b(s, Z) - b(s, \bar{Z})]|}{|A^{-1}[Z - \bar{Z}]|} \right\} dx\, dv\,.\end{aligned}$$

Step 2: Splitting the quotient. Using the special form of b from (R2a) and the action of the matrix A^{-1}, we have

$$A^{-1}[Z - \bar{Z}] = \left(\frac{X - \bar{X}}{\delta_1}, \frac{V - \bar{V}}{\delta_2} \right)$$

and

$$A^{-1}[b(s, Z) - b(s, \bar{Z})] = \left(\frac{b_1(V) - b_1(\bar{V})}{\delta_1}, \frac{b_2(s, X) - b_2(s, \bar{X})}{\delta_2} \right).$$

Therefore

$$
\begin{aligned}
\Phi'_{\delta_1,\delta_2}(s) &\lesssim \frac{\bar L}{\delta_1}\|b(s,\cdot)-\bar b(s,\cdot)\|_{L^1(B_\lambda)}+\\
&\iint\limits_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\left\{|A^{-1}[b(s,Z)-b(s,\bar Z)]|, \frac{1}{\delta_1}\frac{|b_1(V)-b_1(\bar V)|}{|A^{-1}[Z-\bar Z]|}+\frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar X)|}{|A^{-1}[Z-\bar Z]|}\right\} dx\,dv\\
&\le \frac{\bar L}{\delta_1}\|b(s,\cdot)-\bar b(s,\cdot)\|_{L^1(B_\lambda)}+\\
&\iint\limits_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\left\{|A^{-1}[b(s,Z)-b(s,\bar Z)]|, \frac{\delta_2}{\delta_1}\frac{|b_1(V)-b_1(\bar V)|}{|V-\bar V|}+\frac{\delta_1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar X)|}{|X-\bar X|}\right\} dx\,dv\\
&\le \frac{\bar L}{\delta_1}\|b(s,\cdot)-\bar b(s,\cdot)\|_{L^1(B_\lambda)}+\frac{\delta_2}{\delta_1}\mathrm{Lip}(b_1)\mathcal{L}^{2N}(B_r)+\\
&\iint\limits_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\left\{|A^{-1}[b(s,Z)-b(s,\bar Z)]|, \frac{\delta_1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar X)|}{|X-\bar X|}\right\} dx\,dv\\
&\le \frac{\bar L}{\delta_1}\|b(s,\cdot)-\bar b(s,\cdot)\|_{L^1(B_\lambda)}+\frac{\delta_2}{\delta_1}\mathrm{Lip}(b_1)\mathcal{L}^{2N}(B_r)+\iint\limits_{B_r\cap G_\lambda\cap\bar G_\lambda}\Psi(s,z)\,dx\,dv,
\end{aligned}
$$

where we denoted

$$
\Psi(s,z)=\min\left\{|A^{-1}[b(s,Z(s,z))-b(s,\bar Z(s,z))]|, \frac{\delta_1}{\delta_2}\frac{|b_2(s,X(s,z))-b_2(s,\bar X(s,z))|}{|X(s,z)-\bar X(s,z)|}\right\}.
$$

Step 3: Definition of the function $\mathfrak{U}$. Using assumption (R2a), we can now use the estimate of [15] on the difference quotient of b_2,

$$
\frac{|b_2(s,X(s,z))-b_2(s,\bar X(s,z))|}{|X(s,z)-\bar X(s,z)|}\le \mathfrak{U}(s,X(s,z))+\mathfrak{U}(s,\bar X(s,z)), \tag{4.2.13}
$$

where $\mathfrak{U}$ for fixed s is given by

$$
\mathfrak{U}(s,x)=\sum_{j=1}^{N}\sum_{k=1}^{m} M_j(S_{jk}\mathsf{m}_{jk}(s,x)),
$$

with M_j a certain smooth maximal operator on $\mathbb{R}^N_x$.

Step 4: Estimates on Ψ. Let $\Omega=(t,\tau)\times B_r\cap G_\lambda\cap\bar G_\lambda\subset\mathbb{R}^{2N+1}$ and $\Omega'=(t,\tau)\times B_\lambda\subset\mathbb{R}^{2N+1}$. We can estimate the $L^p(\Omega)$ norm of Ψ by considering the first element of the minimum and changing variables along the flows:

$$
\|\Psi\|_{L^p(\Omega)}\le\frac{L+\bar L}{\delta_1}\|b\|_{L^p(\Omega')}\lesssim_\lambda \frac{1}{\delta_1}. \tag{4.2.14}
$$

Considering now the second element of the minimum and eq.n (4.2.13), we can also bound the $M^1(\Omega)$ pseudo-norm of Ψ (where M^1 is the Lorentz space):

$$
\begin{aligned}
|||\Psi|||_{M^1(\Omega)} &\le \frac{\delta_1}{\delta_2}|||\mathfrak{U}(s,X)+\mathfrak{U}(s,\bar X)|||_{M^1(\Omega)}\le\frac{\delta_1}{\delta_2}(L+\bar L)|||\mathfrak{U}|||_{M^1(\Omega')}\\
&\lesssim \frac{\delta_1}{\delta_2}\|\,|||\mathfrak{U}(s,x)|||_{M^1_{x,v}(B_\lambda)}\|_{L^1((t,\tau))}\le\frac{\delta_1}{\delta_2}\|\,|||\mathfrak{U}(s,x)|||_{M^1(B^x_\lambda\times B^v_\lambda)}\|_{L^1((t,\tau))}\\
&\le \frac{\delta_1}{\delta_2}\|\|\,|||\mathfrak{U}(s,x)|||_{M^1_x(B_\lambda)}\|_{L^1_v(B_\lambda)}\|_{L^1((t,\tau))}\le\frac{\delta_1}{\delta_2}(2\lambda)^N\|\,|||\mathfrak{U}(s,x)|||_{M^1(\mathbb{R}^N)}\|_{L^1((t,\tau))},
\end{aligned}
$$

that is

$$|||\Psi|||_{M^1(\Omega)} \lesssim_\lambda \frac{\delta_1}{\delta_2} \| \, |||\mathfrak{U}(s,x)|||_{M^1(\mathbb{R}^N)} \|_{L^1((t,\tau))}.$$

From Theorem 2.10 in [11], we know

$$|||\mathfrak{U}(s,\cdot)|||_{M^1(\mathbb{R}^N)} \lesssim \|\mathsf{m}(s,\cdot)\|_{\mathcal{M}(\mathbb{R}^N)},$$

and thus

$$|||\Psi|||_{M^1(\Omega)} \lesssim_\lambda \frac{\delta_1}{\delta_2} \|\mathsf{m}\|_{L^1((t,\tau);\mathcal{M}(\mathbb{R}^N))}. \tag{4.2.15}$$

Step 5: Interpolation. We have now the ingredients to apply the Interpolation Lemma 2.2 in [15], which allows to bound the norm in $L^1(\Omega)$ of Ψ using $\|\Psi\|_{L^p(\Omega)}$ and $|||\Psi|||_{M^1(\Omega)}$ as follows:

$$\|\Psi\|_{L^1(\Omega)} \lesssim |||\Psi|||_{M^1(\Omega)} \left[1 + \log \left(\frac{\|\Psi\|_{L^p(\Omega)}}{|||\Psi|||_{M^1(\Omega)}} \right) \right]. \tag{4.2.16}$$

Therefore, using the monotonicity of the functions $\log(y)$ and $y\left[1+\log(\frac{1}{y})\right]$ and the bounds (4.2.14) and (4.2.15), we get

$$\|\Psi\|_{L^1(\Omega)} \lesssim_\lambda \frac{\delta_1}{\delta_2} \|\mathsf{m}\| \left[1 + \log \left(\frac{\delta_2}{\delta_1^2 \|\mathsf{m}\|} \right) \right]. \tag{4.2.17}$$

Step 5: Upper bound for Φ_{δ_1,δ_2}. Integrating in time, from t to τ, the last inequality of Step 1, we obtain

$$\Phi_{\delta_1,\delta_2}(\tau) \lesssim \frac{\bar{L}}{\delta_1} \|b - \bar{b}\|_{L^1(\Omega)} + T \frac{\delta_2}{\delta_1} \mathrm{Lip}(b_1) \mathcal{L}^{2N}(B_r) + \int_\Omega \Psi(s,z)\, dz\, ds \tag{4.2.18}$$

$$\lesssim \frac{1}{\delta_1} \|b - \bar{b}\|_{L^1(\Omega)} + \frac{\delta_2}{\delta_1} + \|\Psi\|_{L^1(\Omega)}. \tag{4.2.19}$$

Therefore, applying (4.2.17) and setting $\frac{\delta_1}{\delta_2} = \alpha$, we get

$$\Phi_{\delta_1,\delta_2}(\tau) \lesssim_\lambda \frac{1}{\delta_1} \|b - \bar{b}\|_{L^1(\Omega)} + \frac{1}{\alpha} + \alpha \|\mathsf{m}\| \left[1 + \log \left(\frac{1}{\delta_1 \alpha \|\mathsf{m}\|} \right) \right]. \tag{4.2.20}$$

Step 6: Final estimate. Fix $\gamma > 0$. By definition of Φ_{δ_1,δ_2} and μ, since h is non negative, we have

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) &\geq \int_{B_r \cap \{|Z(\tau,z) - \bar{Z}(\tau,z)| > \gamma\} \cap G_\lambda \cap \bar{G}_\lambda} \log \left(1 + \frac{\gamma}{\delta_2} \right) h(s,z)\, dz \\ &= \log \left(1 + \frac{\gamma}{\delta_2} \right) \mu \left(B_r \cap \{|Z(\tau,z) - \bar{Z}(\tau,z)| > \gamma\} \cap G_\lambda \cap \bar{G}_\lambda \right).\end{aligned}$$

This implies that

$$\mu \left(B_r \cap \{|Z(\tau,z) - \bar{Z}(\tau,z)| > \gamma\} \right) \leq \frac{\Phi_{\delta_1,\delta_2}(\tau)}{\log\left(1 + \frac{\gamma}{\delta_2}\right)} + \mu(B_r \setminus G_\lambda) + \mu(B_r \setminus \bar{G}_\lambda). \tag{4.2.21}$$

Combining (4.2.20) and (4.2.21) we obtain

$$
\begin{aligned}
&\mu\left(B_r \cap \{|Z(\tau,z)-\bar{Z}(\tau,z)|>\gamma\}\right)\\
&\le C_\lambda \left\{ \frac{\frac{\|b-\bar{b}\|}{\delta_1}+\frac{1}{\alpha}+\alpha\|\mathsf{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathsf{m}\|}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)} \right\} + \mu(B_r\setminus G_\lambda)+\mu(B_r\setminus \bar{G}_\lambda)\\
&= C_\lambda \left\{ \frac{\|b-\bar{b}\|}{\delta_1\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{1}{\alpha\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{\alpha\|\mathsf{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathsf{m}\|}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)} \right\} + \mu(B_r\setminus G_\lambda)+\mu(B_r\setminus \bar{G}_\lambda)\\
&= 1)+2)+3)+4)+5).
\end{aligned}
$$

Fix $\eta > 0$. Since b and $\bar{b}$ satisfy assumption (R1a), we can choose $\lambda > 0$ large enough so that $4)+5) \le \frac{2\eta}{4}$. Then, replacing δ_1 with $\alpha\cdot\delta_2$, we notice that 3) is uniformly bounded for $\delta_2 \to 0$, so we can choose α small enough in order to get $3) \le \frac{\eta}{4}$. Now λ and α are fixed, but δ_1 and δ_2 are free to be chosen as long as the ratio equals α. Hence we choose δ_2 small enough so that $2) \le \frac{\eta}{4}$. This fixes all parameters.

Setting

$$
C_{\gamma,r,\eta} = \frac{C_\lambda}{\delta_1\log\left(1+\frac{\gamma}{\delta_2}\right)}
$$

we have proven our statement. □

4.2.3 Uniqueness, stability and compactness

In this subsection we use the result obtained in Theorem 4.2.4 to show uniqueness, stability, and compactness of the regular Lagrangian flow.

Corollary 4.2.5 (Uniqueness)**.** *Let b be a vector field satisfying assumptions (R1a), (R2a) and (R3), and fix $t \in [0,T]$. Then, the μ-regular Lagrangian flow associated with b starting at time t, if it exists, is unique μ-a.e..*

Proof. Let Z and $\bar{Z}$ be two μ-regular Lagrangian flows associated with the same vector field b. Then from Theorem 4.2.4, setting $b=\bar{b}$, we have

$$
\mu(B_r \cap \{|Z(s,\cdot)-\bar{Z}(s,\cdot)|>\gamma\}) \le \eta, \tag{4.2.22}
$$

for all γ, r, $\eta > 0$ and for all $s \in [0,T]$. This implies $Z = \bar{Z}$ μ-a.e.. □

Corollary 4.2.6 (Stability)**.** *Let $\{b_n\}$ be a sequence of vector fields satisfying assumption (R1a), converging in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^{2N})$ to a vector field b which satisfies assumptions (R1a), (R2a) and (R3). Assume that there exist Z_n and Z μ-regular Lagrangian flows starting at time t associated with b_n and b respectively, and denote by L_n and L the compressibility constants of the flows. Suppose that:*

- *The measure of the superlevels associated with Z_n in hypothesis (R1a) is bounded by some functions $g_n(r,\lambda)$ which go to zero uniformly in n as $\lambda\to\infty$ at fixed r,*
- *The sequence $\{L_n\}$ is equi-bounded.*

Then the sequence $\{Z_n\}$ converges to Z locally in measure with respect to μ in $\mathbb{R}^{2N}$, uniformly in s and t.

Proof. We set $\bar{b} = b_n$ and $\bar{Z} = Z_n$ in Theorem 4.2.4, then there exist two positive constants λ and $C_{\gamma,r,\eta}$, which are independent of n, such that for all $s \in [0, T]$ it holds

$$\mu(B_r \cap \{|Z(s,\cdot) - Z_n(s,\cdot)| > \gamma\}) \leq C_{\gamma,r,\eta}\|b - b_n\|_{L^1((0,T)\times B_\lambda)} + \eta.$$

In particular, for any r, $\gamma > 0$ and any $\eta > 0$, we can choose $\bar{n}$ large enough so that

$$\mu(B_r \cap \{|Z(s,\cdot) - Z_n(s,\cdot)| > \gamma\}) \leq 2\eta \qquad \text{for all } n \geq \bar{n} \text{ and } s \in [t, T],$$

which is the thesis. □

Corollary 4.2.7 (Compactness)**.** *Let $\{b_n\}$ be a sequence of vector fields satisfying assumptions (R1a), (R2a) and (R3), converging in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^{2N})$ to a vector field b which satisfies assumptions (R1a), (R2a) and (R3). Assume that there exist Z_n μ-regular Lagrangian flows starting at time t associated with b_n. Suppose that:*

- *The measure of the superlevels associated with Z_n in hypothesis (R1a) is bounded by some functions $g_n(r,\lambda)$ which go to zero uniformly in n as $\lambda \to \infty$ at fixed r,*
- *For any compact subset K of $\mathbb{R}^{2N}$,*
$$\int_K \log(1 + \log(1 + |Z_n(s,z)|))\, d\mu(z) \tag{4.2.23}$$
is equi-bounded in n and s, t,
- *For some $p > 1$ the norms $\|b_n\|_{L^p((0,T)\times B_r)}$ are equi-bounded for any fixed $r > 0$,*
- *The norms of the singular integral operators associated with the vector fields b_n (as well as their number m) are equi-bounded,*
- *The norms of m^n_{jk} in $L^1((0,T);\mathcal{M}(\mathbb{R}^N))$ are equi-bounded in n.*

Then as $n \to \infty$ the sequence $\{Z_n\}$ converges to some Z locally in measure with respect to μ, uniformly with respect to s and t, and Z is a regular Lagrangian flow starting at time t associated with b.

Proof. We apply Theorem 4.2.4 with $b = b_n$ and $\bar{b} = b_m$. Observe that the compressibility constants L and $\bar{L}$ of the same theorem are equal to 1. Indeed b and $\bar{b}$ are divergence free as they both satisfy assumption (R2a). Hence we have for any $r, \gamma > 0$

$$\mu(B_r \cap \{|Z_n(s,\cdot) - Z_m(s,\cdot)| > \gamma\}) \to 0 \quad \text{as } m, n \to \infty, \text{ uniformly in } s, t.$$

Thus it follows that Z_n converges to some $Z \in C([t,T]; L^0_{\mathrm{loc}}(\mathbb{R}^{2N}, d\mu))$ locally in measure with respect to μ, uniformly in s, t. The uniformity in n and s, t of the bound (4.2.23) implies $Z \in \mathcal{B}([t,T]; \log\log L_{\mathrm{loc}}(\mathbb{R}^{2N}, d\mu))$. We notice that conditions (2) and (3) in Definition 4.2.1 are satisfied, since thanks to (R2a) the vector fields b_n are divergence free. We are left with the proof of condition (1). Observe that a $\beta \in C^1(\mathbb{R}^{2N})$ can be approximated by a sequence of $\beta_\epsilon \in C^1_c(\mathbb{R}^{2N})$, therefore it suffices to show condition (1) for this latter class of functions. To this end we want to perform the limit in n of equation (4.2.2) written for Z_n and b_n. From the convergence in measure of Z_n to Z and the fact that Z_n and Z lie in a fixed ball B_r (the support of β_ϵ) it follows the convergence in distributional sense of $\beta_\epsilon(Z_n)$ to $\beta_\epsilon(Z)$ and of $\beta'_\epsilon(Z_n)$ to $\beta'_\epsilon(Z)$. While using the uniform bound of $\|b_n\|_{L^p((0,T)\times B_r)}$ and Lusin's Theorem, we get convergence in L^1_{loc} of $b_n(Z_n)$ to $b(Z)$. Thus we have convergence in the sense of distribution to equation (4.2.2). □

The above compactness statement does not directly translate into an existence result for Lagrangian flows, since in general it is not trivial to find a sequence b_n approximating b as in the hypotheses of Corollary 4.2.7. This is due to the fact that the function $g(r, \lambda)$ in Lemma 5.2.2 does not depend only on bounds on the vector field, but also on bounds on the density of charge. We are able to do this in the specific case of the flow associated with the Vlasov-Poisson equation (solution to (4.1.4)) and therefore we postpone this to Section 4.4.

4.3 Useful estimates

In this Section we recall some a priori estimates related to the Vlasov-Poisson equation and we adapt them to the context of the system (4.1.1)-(4.1.2). The estimates which are stated without proof have already been proved in Chapter 3 (Section 3.2).

Proposition 4.3.1. *Let $\rho(t,\cdot) \in L^s(\mathbb{R}^3)$, for some s such that $1 \le s \le \infty$. Then*

$$\|E(t,\cdot)\|_{L^{3s/(3-s)}} \le C\|\rho(t,\cdot)\|_{L^s}\,, \qquad \text{if } s \in (1,3)\,, \tag{4.3.1}$$

$$\|E(t,\cdot)\|_{C^{0,\alpha}} \le C\|\rho(t,\cdot)\|_{L^s}\,, \qquad \text{if } s > 3\,,\ \text{with } \alpha = 1 - \frac{3}{s}\,, \tag{4.3.2}$$

$$|||E(t,\cdot)|||_{M^{3/2}} \le C\|\rho(t,\cdot)\|_{L^1}\,, \tag{4.3.3}$$

where C is a constant depending only on s.

Proof. We observe that the electric field can be written as $E(t,x) = 4\pi\nabla_x\Delta_x^{-1}\rho(t,x)$. Eq.ns (4.3.1) and (4.3.2) easily follow respectively from Gagliardo–Nirenberg–Sobolev and Morrey inequalities in dimension three (see for instance [30]). Inequality (4.3.3) is a direct consequence of Hardy-Littlewood-Sobolev inequality. □

Proposition 4.3.2 (Mass and energy conservation)**.** *Let*

$$M(t) = \iint f(t,x,v)\,dxdv\,,$$
$$H(t) = \iint \frac{|v|^2}{2} f(t,x,v)dxdv + \frac{|\eta(t)|^2}{2} + \frac{1}{2}\iint \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy + \int \frac{\rho(t,x)}{|x-\xi(t)|}dx\,,$$

be respectively the total mass and the total energy associated with the system (4.1.1)-(4.1.2)*. If $f(t)$ and $\xi(t)$ are solutions to* (4.1.1)-(4.1.2) *on* $[0,T]$*, then $M(t)$ and $H(t)$ are conserved quantities w.r.t. time.*

From Proposition 4.3.2 follows that, if the energy $H(t)$ is initially finite, then the velocity of the Dirac mass is finite.

Proposition 4.3.3. *Let $T > 0$ such that for all $t \in [0,T]$, $f(t)$ and $\xi(t)$ are solutions of the system* (4.1.1)-(4.1.2) *with finite associated initial energy $H(0)$. Then*

$$|\xi(t)| \le |\xi_0| + T\sqrt{2H(0)}\,, \tag{4.3.4}$$

$$|\eta(t)| \le \sqrt{2H(0)}\,. \tag{4.3.5}$$

Proposition 4.3.4. *Let $m \ge 0$, $f(t,\cdot,\cdot) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\rho(t,\cdot) \in L^1(\mathbb{R}^3)$ as in* (4.1.1)*. Then there exists a constant $C > 0$, which only depends on m, such that*

$$\|\rho(t,\cdot)\|_{L^{\frac{m+3}{3}}} \le C\|f(t,\cdot,\cdot)\|_{L^\infty}^{\frac{m}{m+3}} \left(\iint |v|^m f(t,x,v)\,dx\,dv\right)^{\frac{3}{m+3}}\,. \tag{4.3.6}$$

Proposition 4.3.5. *Let $f \geq 0$, $f(t,\cdot,\cdot) \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ solution to* (4.1.1)*. Assume the total energy to be initially finite, then $\rho(t,\cdot) \in L^1 \cap L^{5/3}(\mathbb{R}^3)$ and $E(t,\cdot) \in L^q(\mathbb{R}^3)$, for any $\frac{3}{2} < q \leq \frac{15}{4}$.*

Proof. The bound $\rho(t,\cdot) \in L^{5/3}(\mathbb{R}^3)$ follows by Proposition 4.3.4 for $m = 2$. The estimate on the electric field is a consequence of Proposition 4.3.1 for $s = 1$ and $s = \frac{5}{3}$. □

The following two propositions regard specifically the case in which we deal with a Dirac mass and their proof relies on the condition that the total charge $M(0)$ has to be strictly less than one. This is the only reason why we need to assume (4.1.5) in Theorem 4.1.1.

Proposition 4.3.6 (Proposition 2.9 in [28])**.** *Let $M(0) < 1$, $H(0) < +\infty$ and (f,ξ) a classical solution to* (4.1.1)-(4.1.2) *on* $[0,T]$*. Then for all $t \in [0,T]$ there is a constant depending only on $M(0)$ and $H(0)$ such that*

$$\int_0^t \iint \frac{f(s,x,v)}{|x-\xi(s)|^2}\,dx\,dv\,ds \leq C(1+t)\,. \tag{4.3.7}$$

Proof. For $s \in [0,T]$, consider $(X(s,x,v), V(s,x,v))$ solution to the characteristic system (4.1.4) with initial data (x,v). We now use the shorter notation $(X(s),V(s))$ and compute

$$\frac{d^2}{ds^2}|X(s)-\xi(s)| = \frac{1}{|X(s)-\xi(s)|^2} + \frac{(X(s)-\xi(s))\cdot(E(s,X(s))-E(s,\xi(s)))}{|X(s)-\xi(s)|}\,.$$

Then we obtain

$$\frac{1}{|X(s)-\xi(s)|^2} \leq \frac{d^2}{ds^2}|X(s)-\xi(s)| + |E(s,X(s))| + |E(s,\xi(s))|\,. \tag{4.3.8}$$

By integrating the above expression w.r.t. time and the measure $f_0(x,v)\,dx\,dv$, we get

$$\begin{aligned}\int_0^t \iint \frac{f_0(x,v)}{|X(s)-\xi(s)|^2}\,dx\,dv\,ds \leq{}& \int_0^t \iint \frac{d^2}{ds^2}|X(s)-\xi(s)|\,f_0(x,v)\,dx\,dv\,ds\\ &+ \int_0^t \iint |E(s,X(s))|\,f_0(x,v)\,dx\,dv\,ds\\ &+ M(0)\int_0^t |E(s,\xi(s))|\,ds\,.\end{aligned} \tag{4.3.9}$$

The first term in the r.h.s. of (4.3.9) can be bounded as follows

$$\begin{aligned}\iint f_0(x,v)\int_0^t \frac{d^2}{ds^2}|X(s)-\xi(s)|\,ds\,dx\,dv &= \iint f_0(x,v)\left[\frac{d}{ds}|X-\xi|\right]_{s=0}^{s=t} dx\,dv\\ &\leq 2\sup_{t\in[0,T]}\iint f_0(x,v)\,|V(t,x,v)-\eta(t)|\,dx\,dv\\ &= 2\sup_{t\in[0,T]}\iint f(t,x,v)\,|v-\eta(t)|\,dx\,dv\\ &\leq C\,M(0)^{1/2}(H(0)+H(0)M(0))^{1/2}\,,\end{aligned} \tag{4.3.10}$$

where we used Hölder inequality and the conservation of mass and energy in the latter estimate.

The second term in (4.3.9) is bounded by means of Hölder inequality and Proposition 4.3.5:

$$\begin{aligned}\int_0^t \iint |E(s,X(s))|\,f_0(x,v)\,dx\,dv\,ds &= \int_0^t \iint |E(s,x)|\,f(s,x,v)\,dx\,dv\,ds\\ &\leq C\int_0^t \|\rho(s)\|_{L^{\frac{5}{3}}}\|E(s)\|_{L^{\frac{5}{2}}}\,ds \leq Ct\,.\end{aligned} \tag{4.3.11}$$

We use (4.3.10) and (4.3.11) in the r.h.s. of (4.3.9) and we obtain

$$\int_0^t \iint \frac{f(s,x,v)}{|x-\xi(s)|^2}\,dx\,dv\,ds \le C(1+t) + M(0)\int_0^t \frac{f(s,x,v)}{|x-\xi(s)|^2}\,dx\,dv\,ds\,,$$

that concludes the proof since $M(0) < 1$. □

Proposition 4.3.7 (Theorem 1.1. in [28])**.** *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ non-negative, $(\xi_0,\eta_0) \in \mathbb{R}^3\times\mathbb{R}^3$ and $H(0)$ finite. Assume further that*

(i) $M(0) < 1$,

(ii) *There exists $m_0 > 6$ such that for all $m < m_0$*

$$\iint \left(|v|^2 + \frac{1}{|x-\xi_0|}\right)^{m/2} f_0(x,v)\,dx\,dv < +\infty\,.$$

Then there exists a global weak solution (f,ξ) to the system (4.1.1)-(4.1.2), with $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3\times\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$ for any $1 \le p < +\infty$, $\xi \in C^2(\mathbb{R}_+)$, and $E \in L^\infty([0,T], C^{0,\alpha}(\mathbb{R}^3))$ for all $T > 0$.

Moreover, for all $t \in \mathbb{R}_+$ and for all $m < \min(m_0, 7)$,

$$\iint \left(|v|^2 + \frac{1}{|x-\xi(t)|}\right)^{m/2} f(t,x,v)\,dx\,dv \le C(1+t)^c\,, \tag{4.3.12}$$

where C and c only depend on the initial data.

Remark 16. Observe that thanks to Proposition 4.3.4, condition (4.3.12) implies $\rho(t) \in L^s(\mathbb{R}^3)$, for $s > 3$. Hence the Hölder continuity of the electric field follows directly by Proposition 4.3.1.

4.4 Proof of the Theorem 4.1.1

4.4.1 Existence of the Lagrangian flow

In this subsection we shall use the results obtained in Section 4.2 for a general flow solution to equation (4.2.1) and apply them to the context of the Vlasov-Poisson system (4.1.1)-(4.1.2), namely to the ODE (4.1.4). In particular we will prove existence of a flow associated with the vector field $b(s,x,v) = (v, E(s,x)+F(s,x))$, using the compactness result provided by Corollary 4.2.7. To this end it suffices to construct a sequence b_n which approximates b and satisfies the hypotheses of Corollary 4.2.7.

Let f_0 and (ξ_0,η_0) be the initial data of system (4.1.1), satisfying the hypotheses of Theorem 4.1.1. We consider the approximating initial densities given by

$$f_0^n(x,v) = f_0(x,v)\mathbb{1}_{\{(x,v):\,\frac{1}{n}<|x-\xi_0|<n,\,|v-\eta_0|<n\}}(x,v). \tag{4.4.1}$$

Thanks to [40], this choice ensures existence and uniqueness of f_n and (ξ_n,η_n), solutions to the Vlasov-Poisson system (4.1.1)-(4.1.2). Moreover f_n is a Lagrangian solution, i.e.

$$f_n(s, X_n(s,x,v), V_n(s,x,v)) = f_0^n(x,v), \tag{4.4.2}$$

where (X_n, V_n) satisfy

$$\begin{cases} \dot{X}_n(s,x,v) = V_n(s,x,v) \\ \dot{V}_n(s,x,v) = E_n(s,X_n(s,x,v)) + F_n(s,X_n(s,x,v)), \end{cases} \tag{4.4.3}$$

with

$$E_n(s,x) = \left(\nabla \frac{1}{|\cdot|} * \rho_n\right)(s,x), \quad \rho_n(s,x) = \int f_n(s,x,v)\,dv, \quad F_n(s,x) = \frac{x-\xi_n(s)}{|x-\xi_n(s)|^3}. \tag{4.4.4}$$

From now on the abstract measure μ of Section 2 will be set as $\mu = f_0\,\mathcal{L}^{2N}$, where f_0 is the initial density of our problem. In order to apply Corollary 4.2.7, we need then the approximating vector fields $b_n(s,x,v) = (v, E_n(s,x) + F_n(s,x))$ to satisfy hypotheses (R1a), (R2a), and (R3) "uniformly" in n (with equi-bounds on the quantities involved) and the bound (4.2.23). Furthermore we set the dimension N equal to 3.

Proof of (R1a) + equibound: control of superlevels

In [11] a control on the superlevels was obtained using hypothesis (R1) which provided an upper bound on the integral of $\log(1+|Z|)$. Without assumption (R1), we need estimates on $|V|^2$ in order to control the superlevels. This requires integrating a function which grows slower than $\log(1+|V|)$ at infinity. Furthermore, differently from [12], we will bound the superlevels of Z with respect to the measure $\mu = f_0\,\mathcal{L}^6$. For the sake of clarity we will use the notation $f_0(B)$ to indicate the measure μ of a set $B \subseteq \mathbb{R}^6$. The result is the following lemma, whose proof is postponed to Subsection 4.2.

Lemma 4.4.1. *Let $b(t,x,v) = (v, E(t,x) + F(t,x))$ and let $Z : [t,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ be the μ-regular Lagrangian flow relative to b starting at time t, with sublevel G_λ. Assume $M(0) < 1$. Then, for all $r, \lambda > 0$, we have*

$$f_0(B_r \setminus G_\lambda) \le g(r,\lambda),$$

where the function g depends only on $\|E\|_{L^\infty_t(L^2_x)}$, $\|E\|_{L^\infty_t(L^{5/2}_x)}$, $\|U\|_{L^1_t(L^\infty_x)}$, $\|F\|_{L^\infty_t(M^{3/2}_x)}$, $\|f\|_{L^\infty_t(L^\infty_{x,v})}$, $\|f\|_{L^\infty_t(L^1_{x,v})}$, $H(0)$, and $g(r,\lambda) \downarrow 0$ for r fixed and $\lambda \uparrow \infty$.

Notice that this lemma holds also for the regularized problem (system (4.1.1)-(4.1.2) with initial density f_0^n). Therefore we have, for all $r, \lambda > 0$,

$$f_0^n(B_r \backslash G_\lambda^n) \le g_n(r,\lambda), \tag{4.4.5}$$

where g_n converges to zero for r fixed and $\lambda \uparrow \infty$. Moreover, this convergence is uniform in n. Indeed the proof of Lemma 5.2.2 entails the functions g_n to be increasing with respect to the norms of E_n, U_n, F_n, f_n, and with respect to $H_n(0)$. These quantities are in turn all bounded by the same quantities without the index n. Therefore, due to the choice of the initial densities of the regularized problem, we have

$$\begin{aligned} f_0(B_r \backslash G_\lambda^n) &\le f_0^n(B_r \backslash G_\lambda^n) + f_0\Big(\mathbb{R}^6 \backslash \big\{(x,v) : \frac{1}{n} < |x-\xi_0| < n, |v-\eta_0| < n\big\}\Big) \\ &\le g_n(r,\lambda) + f_0\Big(\big\{(x,v) : |x-\xi_0| \le \frac{1}{n} \text{ or } |x-\xi_0| \ge n\big\}\Big) \\ &\quad + f_0\big(\big\{(x,v) : |v-\eta_0| \ge n\big\}\big), \end{aligned} \tag{4.4.6}$$

where $g_n(r,\lambda)$ depends on the norms of E, U, F, f and on $H(0)$, and tends to zero as $\lambda \to \infty$ uniformly in n. Moreover the last two terms tend to zero as $n \to \infty$ by Lebesgue's Dominate Convergence Theorem. Hence we have, for any fixed ϵ, $r > 0$, that there exist $\lambda > 0$ and $N \in \mathbb{N}$ such that

$$f_0(B_r \backslash G_\lambda^n) \le \epsilon \tag{4.4.7}$$

for each $n \ge N$.

Proof of (R2a): spatial regularity

Since $b_n(t,x,v) = (b_1^n(v), b_2^n(t,x))$ with $b_1^n(v) = v$ and $b_2^n(t,x) = E_n(t,x) + F_n(t,x)$, we observe that the Lipschitz constants of b_1^n and b_1 are trivially equi-bounded. We are left to show that the derivatives of b_2^n and b_2 are singular integrals of fundamental type on $\mathbb{R}^3$ of finite measures, and that the norms of the kernels associated with the singular integral operators and those of the measures in $L^1((0,T);\mathcal{M}(\mathbb{R}^3))$ are equi-bounded. We compute, outside of the origin,

$$\begin{aligned}\partial_{x_j}(b_2)_i(x) = \partial_{x_j}(E+F)_i(x) &= \partial_{x_j}\left(\frac{\cdot}{|\cdot|^3} * \rho(t,\cdot)\right)_i(x) + \partial_{x_j}\left(\frac{\cdot}{|\cdot|^3} * \delta_{\xi(t)}\right)_i(x) \\ &= \left(\partial_{x_j}\frac{(\cdot)_i}{|\cdot|^3} * \left(\rho(t,\cdot)+\delta_{\xi(t)}\right)\right)(x) \\ &= \left(\frac{\delta_{ij}|\cdot|^2 - 3\cdot_i\cdot_j}{|\cdot|^5} * \left(\rho(t,\cdot)+\delta_{\xi(t)}\right)\right)(x).\end{aligned}$$

Therefore $\partial_{x_j}(b_2)_i$ is a singular integral of the finite measure $\rho + \delta_{\xi(t)}$, with kernel

$$K_{ij}(y) = \frac{\delta_{ij}|y|^2 - 3y_iy_j}{|y|^5}.$$

The kernel satisfies conditions of Def.2.13 in [15], therefore it is a singular kernel of fundamental type. Similarly we have $\partial_{x_j}(b_2^n)_i = K_{ij}(\cdot) * (\rho(t,\cdot) + \delta_{\xi_n(t)})$, hence also $\partial_{x_j}(b_2^n)_i$ are singular integrals of finite measures, with equi-bounded kernels and equi-bounds on the measures' norms.

Proof of (R3)

We shall prove now that the L^p-norms of b and b_n in $(0,T)\times B_r$ are equi-bounded, for some $p > 1$ and for any fixed $r > 0$. Through an easy computation we notice that the $M^{3/2}$-pseudo-norms of F and F_n are equi-bounded and uniform in t:

$$|||F_n(t,\cdot)|||_{M^{3/2}}^{3/2} = \sup_{\lambda>0}\left\{\lambda^{3/2}\mathcal{L}^3\left(\{x : \frac{1}{|x-\xi_n(t)|^2} > \lambda\}\right)\right\} = \sup_{\lambda>0}\left\{\lambda^{3/2}\int_{|x-\xi_n(t)|<\frac{1}{\sqrt{\lambda}}} 1\,dx\right\} \le C\,.$$

Similarly we have that the L^1-norms of F and F_n are equi-bounded in $(0,T)\times B_r$ for any $r > 0$:

$$\sup_{t\in[0,T]}\|F_n\|_{L^1(B_r)} = \sup_{t\in[0,T]}\int_{B_r}\frac{1}{|x-\xi_n(t)|^2}dx = \sup_{t\in[0,T]}\int_{B_r(\xi_n(t))}\frac{1}{|y|^2}dy \le C\,.$$

Furthermore Propositions 4.3.1 tells us that E and E_n belong to $L^\infty((0,T);M^{3/2}(\mathbb{R}^3))$, with the respective pseudo-norms which are equi-bounded in n. Therefore the second component of the vector fields b and b_n (i.e. $E+F$, E_n+F_n) are equi-bounded in the space $L^\infty((0,T);M^{3/2}(\mathbb{R}^3)) \subset L^p_{\text{loc}}((0,T)\times\mathbb{R}^3)$ for any $1 \le p < \frac{3}{2}$. Since $v \in L^p_{\text{loc}}((0,T)\times\mathbb{R}^3)$ for any p, we conclude that b, b_n belong to $L^p_{\text{loc}}((0,T)\times\mathbb{R}^3)$ for any $1 \le p < \frac{3}{2}$, with uniform bound on the norms.

Proof of the equi-boundedness of (4.2.23)

We observe that

$$|Z_n| \le |X_n| + |V_n| \le |x| + (1+T)|V_n|\,. \tag{4.4.8}$$

Thus it suffices to prove the equi-boundedness of (4.4.16) for the regularised flow V_n. This is a byproduct of the proof of Lemma 5.2.2, where we show that the constant A depends on quantities which are uniformly bounded in n.

4.4.2 Conclusion of the proof of Theorem 4.1.1: existence of Lagrangian solutions to the Vlasov-Poisson system

Let f_0 be as in Theorem 4.1.1. In order to prove existence of a Lagrangian solution to system (4.1.1)-(4.1.2), we use a compactness argument. For each n, we consider the initial datum f_0^n defined in (4.4.1), which converges to f_0. The result in [40] ensures existence and uniqueness of the classical Lagrangian solution f_n, (ξ_n, η_n) to the Vlasov-Poisson system with point charge

$$\begin{cases} \partial_t f_n + v \cdot \nabla_x f_n + (E_n + F_n) \cdot \nabla_v f_n = 0\,, \\ f_n(0,x,v) = f_0^n(x,v)\,, \\ E_n(t,x) = \int \frac{x-y}{|x-y|^3} \rho_n(t,y)\,dy\,, \\ \rho_n(t,x) = \int f_n(t,x,v)\,dv\,, \\ F_n(t,x) = \frac{x-\xi_n(t)}{|x-\xi_n(t)|^3}\,, \end{cases} \tag{4.4.9}$$

where $(\xi_n(t), \eta_n(t))$ evolves according to

$$\begin{cases} \dot{\xi}_n(t) = \eta_n(t)\,, \\ \dot{\eta}_n(t) = E_n(t, \xi_n(t))\,, \\ (\xi_n(0), \eta_n(0)) = (\xi_0, \eta_0)\,. \end{cases} \tag{4.4.10}$$

Therefore, there exists a unique flow $Z_n = (X_n, V_n) : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ associated with the vector field $(V_n, E_n(X_n) + F_n(X_n))$, such that $f^n = Z_{n\#} f_0^n$ is the push-forward of f_0^n through Z_n, i.e.

$$f_n(t, X_n(t,x,v), V_n(t,x,v)) = f_0^n(x,v)\,.$$

From Subsection 4.4.1, there exists Z such that $Z_n \to Z$ in measure, with respect to $\mu = f_0 \mathcal{L}^6$. Therefore we define a density f which is the push forward of the initial data f_0 through the limiting flow Z, i.e.

$$f := Z_\# f_0\,.$$

The aim of this subsection is to verify that the above defined f is indeed a solution to (4.1.1)-(4.1.2). In other words, we want to perform the limit $n \to \infty$ in (4.4.9)-(4.4.10) and get (4.1.1)-(4.1.2). This will conclude the proof of Theorem 4.1.1. To this end we observe that, up to subsequences:

- $f_n \rightharpoonup f$ weakly in $L^1_{x,v}$ and weakly* in $L^\infty_{x,v}$, uniformly in t.
  Indeed, $f_0^n \to f_0$ in $L^1_{x,v}$ and $Z_n \to Z$ in measure μ. Since the latter limit is uniform in s and t, we define the inverse of the flow $Z_n^{-1}(t,s,x,v) := Z_n(s,t,x,v)$ and observe that $Z_n^{-1} \to Z^{-1}$ in measure and therefore μ-a.e., uniformly in t. Given $\varphi \in C_c(\mathbb{R}^3 \times \mathbb{R}^3)$, we can estimate

$$\begin{aligned} &\iint \varphi(x,v)\,(f_n(t,x,v) - f(t,x,v))\,dx\,dv \\ &= \iint \varphi(x,v)\,\left(f_0^n(Z_n^{-1}(t,x,v)) - f_0(Z^{-1}(t,x,v))\right)\,dx\,dv \\ &= \iint \varphi(Z_n(t,x,v))\,f_0^n(x,v)\,dx\,dv - \iint \varphi(Z(t,x,v))\,f_0(x,v)\,dx\,dv \\ &= \iint \left(\varphi(Z_n(t,x,v)) - \varphi(Z(t,x,v))\right)\,f_0(x,v)\,dx\,dv \\ &+ \iint \varphi(Z_n(t,x,v))\,(f_0^n(x,v) - f_0(x,v))\,dx\,dv\,. \end{aligned}$$

The first term in the r.h.s. converges to zero, since $Z_n \to Z$ μ-a.e. The second term also converges to zero because φ is bounded and $f_0^n \to f_0$ in $L^1_{x,v}$. Moreover, since f_n is equi-bounded in $L^1_{x,v} \cap L^\infty_{x,v}$, uniformly in t, we obtain weak convergence in $L^1_{x,v}$ and weak* convergence in $L^\infty_{x,v}$ of f_n to f, uniformly in t.

- $\rho_n \rightharpoonup \rho$ weakly in L^1_x. It follows from the weak $L^1_{x,v}$ convergence of f_n to f. Moreover, thanks to Remark 16, $\rho_n \rightharpoonup \rho$ weakly in L^s_x, for some $s > 3$.
- $\partial_t f_n$ converges to $\partial_t f$ in $\mathcal{D}'$ and $v \cdot \nabla_x f_n$ converges to $v \cdot \nabla_x f$ in $\mathcal{D}'$.
- $E_n \to E$ uniformly. This is a consequence of Proposition 4.3.7. Indeed, the r.h.s. of equation (4.3.12) is uniformly bounded in n. Therefore, by Proposition 4.3.4, $\|\rho_n\|_{L^{\frac{m+3}{3}}}$ is uniformly bounded and Proposition 4.3.1 yields $\{E_n\}_n$ equi-Hölder. Ascoli-Arzelà Theorem guarantees the existence of a uniformly convergent subsequence. The limit couple (E, ρ) satisfies $E(t,x) = \int \frac{x-y}{|x-y|^3}\rho(t,y)\,dy$, since $E \in M^{3/2}$ and decays at infinity, while $\rho \in L^s$, for some $s > 3$.
- $E_n \cdot \nabla_v f_n \to E \cdot \nabla_v f$ in $\mathcal{D}'$. This follows by rewriting $E_n \cdot \nabla_v f_n = \mathrm{div}_v(E_n f_n)$ and $E \cdot \nabla_v f = \mathrm{div}_v(E f)$, and by the facts that $E_n \to E$ uniformly and $f_n \rightharpoonup f$ weakly in $L^1_{x,v}$.

We are left with the part of the system (4.4.9)-(4.4.10) which involves the point charge. In particular, we define

$$\gamma_n(t) = (\xi_n(t), \eta_n(t)) \tag{4.4.11}$$

and set

$$(\xi(t), \eta(t)) := \lim_{n\to\infty} \gamma_n(t)\,. \tag{4.4.12}$$

Observe that the limit in (4.4.12) exists. Indeed, $\gamma_n(t)$ is equi-Lipschitz because of the following estimate:

$$\mathrm{Lip}(\gamma_n) \le \|\dot\gamma_n\|_{L^\infty} \le \sup_t |\eta_n(t)| + \sup_t |E_n(t, \xi_n(t))|\,, \tag{4.4.13}$$

where $\mathrm{Lip}(\gamma_n)$ is the Lipschitz constant of γ_n. Proposition 4.3.3 yields a uniform bound on the first term in the r.h.s. of (4.4.13), that combined with the uniform bounds on E_n proved in this subsection, implies γ_n equi-Lipschitz. By Ascoli-Arzelà Theorem, there exists a subsequence $\{(\xi_{n_k}(t), \eta_{n_k}(t))\}_k$ which converges uniformly to $(\xi(t), \eta(t))$. To perform the limit in (4.4.9)-(4.4.10), we observe that

- $(\dot\xi_n(t), \dot\eta_n(t)) \to (\dot\xi(t), \dot\eta(t))$. Indeed, $(\xi_n(t), \eta_n(t))$ converges to $(\xi(t), \eta(t))$ uniformly and
$$\sup_t |\dot\gamma_n(t) - (\eta(t), E(t, \xi(t)))| \le \sup_t |\eta_n(t) - \eta(t)| + \sup_t |E_n(t, \xi_n(t)) - E(t, \xi(t))|\,. \tag{4.4.14}$$
The first term in the r.h.s. of (4.4.14) converges to zero uniformly. As for the second term, we use that
$$\begin{aligned}\sup_t |E_n(t,\xi_n(t)) &- E(t,\xi(t))| \\ &\le \sup_t |E_n(t,\xi_n(t)) - E(t,\xi_n(t))| + \sup_t |E(t,\xi_n(t)) - E(t,\xi(t))| \\ &\le \sup_{t,x} |E_n(t,x) - E(t,x)| + \sup_t |E(t,\xi_n(t)) - E(t,\xi(t))|.\end{aligned} \tag{4.4.15}$$
Combining the facts that $E_n \to E$, $\xi_n \to \xi$ and E is uniformly continuous, the last line in (4.4.15) vanishes as $n \to \infty$.
- $F_n \to F$ in $L^1_{x,\mathrm{loc}}$. Indeed, $F_n \to F$ pointwise, by the uniform convergence of $\xi_n(t)$ to $\xi(t)$ up to subsequences, and F_n, $F \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore, we conclude by Dominated Convergence's Theorem.
- $F_n \cdot \nabla_v f_n \to F \cdot \nabla_v f$ in $\mathcal{D}'$. This follows by rewriting $F_n \cdot \nabla_v f_n = \mathrm{div}_v(F_n f_n)$ and $F \cdot \nabla_v f = \mathrm{div}_v(F f)$, and by the facts that $F_n \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $f_n \overset{*}{\rightharpoonup} f$ weakly* in $L^\infty_{x,v}$.

4.4.3 Proof of Lemma 5.2.2

We call $\tilde{G}_\lambda$ the sublevel of V and we remark that by the first equation in (4.2.1), whenever $(x, v) \in \tilde{G}_\lambda$ one has $|X(s, x, v)| \leq |x| + |s - t|\lambda$ and $|Z(s, x, v)| \leq |x| + (1 + T)\lambda$. Thus for $\lambda > r$ one has $B_r \backslash G_\lambda \subset B_r \backslash \tilde{G}_{(\lambda - r)/(1+T)}$, while for $\lambda \leq r$ we can just use that $B_r \backslash G_\lambda \subset B_r$, so to conclude the proof it suffices to bound the superlevels of V. In order to do this we will first prove that

$$\iint_{B_r} \sup_{s \in [t,T]} \log\left(1 + \log\left(1 + \frac{|V(s, x, v)|^2}{2}\right)\right) f_0(x, v)\, dx\, dv \leq A \tag{4.4.16}$$

where A is a constant depending on $\|E\|_{L^\infty_t(L^2_x)}$, $\|E\|_{L^\infty_t(L^{5/2}_x)}$, $\|U\|_{L^1_t(L^\infty_x)}$, $|||F|||_{L^\infty_t(M^{3/2}_x)}$, $\|f\|_{L^\infty_t(L^\infty_{x,v})}$, $\|f\|_{L^\infty_t(L^1_{x,v})}$ and $H(0)$. Once one has shown that (4.4.16) holds, the statement of the lemma follows simply by the following inequality:

$$\iint_{B_r} \sup_{s \in [t,T]} \log\left(1 + \log\left(1 + \frac{|V(s, x, v)|^2}{2}\right)\right) f_0(x, v)\, dx\, dv$$
$$\geq f_0(B_r \setminus \tilde{G}_\lambda) \log\left(1 + \log\left(1 + \frac{\lambda^2}{2}\right)\right). \tag{4.4.17}$$

Consider the ODE system (4.1.4) and recall the Definition 4.2.1. Let $\beta(z) = \log\left(1 + \log\left(1 + \frac{|z|^2}{2}\right)\right)$, then

$$\beta'(z) = \frac{z}{\left(1 + \log\left(1 + \frac{|z|^2}{2}\right)\right)(1 + \frac{|z|^2}{2})}. \tag{4.4.18}$$

Using (4.2.2) and (4.4.18), we compute

$$\partial_s[\beta(V(s, x, v))] = \frac{(E(s, X(s, x, v)) + F(s, X(s, x, v))) \cdot V(s, x, v)}{\left(1 + \log\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\right)\left(1 + \frac{|V(s,x,v)|^2}{2}\right)}.$$

Observe that, by definition of the electric potential U,

$$E(s, X(s, x, v)) \cdot V(s, x, v) = -\partial_s[U(s, X(s, x, v))] + (\partial_s U)(s, X(s, x, v)),$$

thus

$$\begin{aligned}
\partial_s[\beta(V(s, x, v))] =& -\partial_s\left[\frac{U(s, X(s, x, v))}{\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\left(1 + \log\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\right)}\right] \\
&- \frac{U(s, X(s, x, v))}{\left(1 + \frac{|V(s,x,v)|^2}{2}\right)^2\left(1 + \log\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\right)^2} \\
&\times \left[V(s, x, v) \cdot (E(s, X(s, x, v)) + F(s, X(s, x, v))\left(1 + \log\left(1 + \frac{|V(s, x, v)|^2}{2}\right)\right)\right. \\
&\left. + \left(1 + \frac{|V(s, x, v)|^2}{2}\right)\left(\frac{V(s, x, v) \cdot (E(s, X(s, x, v)) + F(s, X(s, x, v))}{1 + \frac{|V(s,x,v)|^2}{2}}\right)\right] \\
&+ \frac{(\partial_s U)(s, X(s, x, v))}{\left(1 + \log\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\right)\left(1 + \frac{|V(s,x,v)|^2}{2}\right)} \\
&+ \frac{F(s, X(s, x, v)) \cdot V(s, x, v)}{\left(1 + \log\left(1 + \frac{|V(s,x,v)|^2}{2}\right)\right)\left(1 + \frac{|V(s,x,v)|^2}{2}\right)},
\end{aligned}$$

whence

$$
\begin{aligned}
&\partial_s[\beta(V(s,x,v))] \\
&\quad = -\,\partial_s\left[\frac{U(s,X(s,x,v))}{\left(1+\frac{|V(s,x,v)|^2}{2}\right)\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)}\right] \\
&\qquad - \frac{U(s,X(s,x,v))V(s,x,v)\cdot(E(s,X(s,x,v))+F(s,X(s,x,v))\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)}{\left(1+\frac{|V(s,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^2} \\
&\qquad - \frac{U(s,X(s,x,v))V(s,x,v)\cdot(E(s,X(s,x,v))+F(s,X(s,x,v))\left(1+\frac{|V(s,x,v)|^2}{2}\right)}{\left(1+\frac{|V(s,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^2\left(1+\frac{|V(s,x,v)|^2}{2}\right)} \\
&\qquad + \frac{(\partial_s U)(s,X(s,x,v))+F(s,X(s,x,v))\cdot V(s,x,v)}{\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(s,x,v)|^2}{2}\right)}\,.
\end{aligned}
\tag{4.4.19}
$$

By integrating (4.4.19) w.r.t. time we get

$$
\begin{aligned}
&\log\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right) \\
&\quad = -\frac{U(s,X(s,x,v))}{\left(1+\frac{|V(t,x,v)|^2}{2}\right)\left(1+\log\left(1+\frac{|V(t,x,v)|^2}{2}\right)\right)} \\
&\qquad + \frac{U(t,x)}{\left(1+\frac{|v|^2}{2}\right)\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)} + \log\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right) \\
&\qquad + \int_t^s \left\{ -\frac{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot(E(\tau,X(\tau,x,v))+F(\tau,X(\tau,x,v)))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)} \right. \\
&\qquad\qquad - \frac{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot(E(\tau,X(\tau,x,v))+F(\tau,X(\tau,x,v)))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)^2} \\
&\qquad\qquad + \frac{(\partial_s U)(\tau,X(\tau,x,v))}{\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)} \\
&\qquad\qquad \left. + \frac{F(t,X(t,x,v))\cdot V(t,x,v)}{\left(1+\log\left(1+\frac{|V(t,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(t,x,v)|^2}{2}\right)} \right\} d\tau\,.
\end{aligned}
\tag{4.4.20}
$$

We have by the Sobolev embedding in $\mathbb{R}^3$ that

$$U(t,\cdot) = \nabla^{-1}E(t,\cdot) \in L^6(\mathbb{R}^3)$$

since $E(t,\cdot) \in L^2(\mathbb{R}^3)$ thanks to Propositions 4.3.5. Thus clearly

$$\frac{U(t,x)}{1+\frac{|v|^2}{2}} \in L^6(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)\,.$$

It follows that the first two terms in the r.h.s. of (4.4.20) are bounded in $L^6(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$. The inequality

$$\log\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right) \le \frac{|v|^2}{2}$$

allows to estimate the third term multiplied by f_0 with the kinetic energy, that is in turn bonded by the initial total energy $H(0)$. Thus the third term belongs to $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v, d\mu)$ with $\mu = f_0\,\mathcal{L}^6$, so it remains to compute the terms in the integral. Let

$$\begin{aligned}
\Phi := \int_t^s \Bigg(& -\frac{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot(E(\tau,X(\tau,x,v))+F(\tau,X(\tau,x,v)))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)} \\
& -\frac{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot(E(\tau,X(\tau,x,v))+F(\tau,X(\tau,x,v)))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)^2\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)^2} \\
& +\frac{(\partial_\tau U)(\tau,X(\tau,x,v))}{\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)} \\
& +\frac{F(\tau,X(\tau,x,v))\cdot V(\tau,x,v)}{\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)}\,d\tau\Bigg) \\
= \int_t^s & (\Phi_1+\Phi_2+\Phi_3+\Phi_4)\,d\tau\,.
\end{aligned} \tag{4.4.21}$$

Observe that we only have to bound the integrals of Φ_1, Φ_3, Φ_4, since Φ_2 is easily estimated by Φ_1. We start with Φ_1 and we split it into $\Phi_{1,1}$, which contains E, and $\Phi_{1,2}$, which contains F. Taking absolute value and changing variables $(X(t,x,v), V(t,x,v)) \mapsto (x,v)$ we have

$$\begin{aligned}
\left\|\int_t^s \Phi_{1,1}(\tau)d\tau\right\|_{L^{3/2}_{x,v}} &\le \int_t^s \|\Phi_{1,1}(\tau)\|_{L^{3/2}_{x,v}} d\tau \\
&= \int_t^s \left(\iint \left|\frac{U(\tau,x)\,v\cdot E(\tau,x)}{\left(1+\frac{|v|^2}{2}\right)^2\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)}\right|^{3/2} dxdv\right)^{2/3} d\tau \\
&\le \int_t^s \left(\int |U(\tau,x)E(\tau,x)|^{3/2}dx\right)^{2/3} d\tau \left(\int \frac{v^{3/2}}{\left(1+\frac{|v|^2}{2}\right)^3\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3/2}}dv\right)^{2/3} \\
&\le C\|U\|_{L^1_t(L^6_x)}\|E\|_{L^\infty_t(L^2_x)}\int_0^\infty \frac{r^{7/2}dr}{\left(1+r^2\right)^3\left(1+\log\left(1+r^2\right)\right)^{3/2}} \\
&\le C\|U\|_{L^1_t(L^6_x)}\|E\|_{L^\infty_t(L^2_x)}
\end{aligned}$$

Since the integral in dr is convergent, we have that $\int_t^s \Phi_{1,1}(\tau)d\tau \in L^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$. We now estimate the $L^1_t(M^{3/2}_{x,v})$ pseudo-norm of $\Phi_{1,2}$. We obtain that $\Phi_{1,2}(\tau) \in M^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, since

$$\int_t^s |||\Phi_{1,2}(\tau)|||_{M^{3/2}_{x,v}} d\tau$$
$$\leq \int_t^s |||U(\tau)F(\tau)|||_{M^{3/2}_x} \left\| \frac{v}{\left(1+\frac{|v|^2}{2}\right)^2 \left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)} \right\|_{L^{3/2}_v} d\tau$$
$$\leq C \int_t^s \|U(\tau)\|_{L^\infty_x} |||F(\tau)|||_{M^{3/2}_x} d\tau \leq C\|U\|_{L^1_t(L^\infty_x)} \|F\|_{L^\infty_t(M^{3/2}_x)},$$

where $U \in L^\infty_x$ by Sobolev embedding. However, notice that in this case the previous estimate does not imply directly that $\int_t^s \Phi_{1,2}(\tau)d\tau \in M^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, since $|||\cdot|||_{M^{3/2}}$ is not subadditive. In order to obtain $\int_t^s \Phi_{1,2}(\tau)d\tau \in M^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, which we need for the bound (4.4.16), it suffices to recall that, if $p > 1$, the pseudo-norm $|||\cdot|||_{M^p}$ is well known to be equivalent to a norm $\|\cdot\|_*$ defined as

$$\|u\|_* := \sup_{0<|E|<\infty} \frac{\int_E |u|}{|E|^{1-\frac{1}{p}}}, \qquad \text{for any measurable function } u.$$

For Φ_3 we observe that

$$\partial_s U = \Delta^{-1/2} j(s,x), \qquad \text{where } j(s,x) = \int v f(s,x,v)dv \tag{4.4.22}$$

hence by Hölder inequality in the v variable

$$|j(s,x)| \leq \left(\int f(s,x,v)dv\right)^{1/2} \left(\int |v|^2 f(s,x,v)dv\right)^{1/2}. \tag{4.4.23}$$

Integrating over x, using Hölder inequality and the bounds $f(s) \in L^1_{x,v}$ and $\int |v|^2 f(s)dxdv \leq H(0)$, we get that $j(s) \in L^1_x$. Thus by Hardy-Littlewood-Sobolev inequality,

$$|||\partial_s U(s)|||_{M^{3/2}_x} = |||\,|x|^{-2} * j(s)|||_{M^{3/2}_x} \leq c\|j(s)\|_{L^1_x}.$$

Therefore, changing variables, we obtain that $\Phi_3(\tau) \in M^{3/2}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) + L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, in fact

$$\begin{aligned}
|||\Phi_3(\tau)|||_{M^{3/2}_{x,v}} &= \left|\left|\left| \frac{(\partial_\tau U)(\tau, X(\tau,x,v))}{\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)} \right|\right|\right|_{M^{3/2}_{x,v}} \\
&= \left|\left|\left| \frac{(\partial_\tau U)(\tau,x)}{\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)\left(1+\frac{|v|^2}{2}\right)} \right|\right|\right|_{M^{3/2}_{x,v}} \\
&\leq \left|\left|\left| \frac{(\partial_\tau U)(\tau,x)}{\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)\left(1+\frac{|v|^2}{2}\right)} \right|\right|\right|_{M^{3/2}_x(L^{3/2}_v)} \\
&\leq C|||\partial_\tau U(\tau)|||_{M^{3/2}_x} \int_{\mathbb{R}^3} \frac{1}{\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3/2}\left(1+\frac{|v|^2}{2}\right)^{3/2}} dv \\
&\leq C\|j(\tau)\|_{L^1_x} \int_0^\infty \frac{r^2\, dr}{\left(1+\log\left(1+r^2\right)\right)^{3/2}(1+r)^3} \\
&\leq C\|j(\tau)\|_{L^1_x},
\end{aligned} \tag{4.4.24}$$

where in the inequality we used that the integral in the r variable is convergent. Given that f_0 is bounded, in order to have (4.4.16), we are left with the term Φ_4, which does not belong to $L^1(\mathbb{R}^3 \times \mathbb{R}^3) + L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

For Φ_4 we compute

$$\begin{aligned}
&\int_0^T \iint \Phi_4(\tau, x, v)\, f_0(x, v)\, dx\, dv\, d\tau \\
&\qquad = \int_0^T \iint \frac{F(\tau, X(\tau, x, v)) \cdot V(\tau, x, v)}{\left(1 + \log\left(1 + \frac{|V(\tau,x,v)|^2}{2}\right)\right)\left(1 + \frac{|V(\tau,x,v)|^2}{2}\right)}\, f_0(x, v)\, dx\, dv\, d\tau\,.
\end{aligned} \tag{4.4.25}$$

Since the denominator of the integrand is bounded, we can estimate the above quantity as follows:

$$\begin{aligned}
(4.4.25) &\le C \int_0^T \iint F(\tau, X(\tau, x, v))\, f_0(x, v)\, dx\, dv\, d\tau \\
&\le C \int_0^T \iint \frac{f_0(x, v)}{|X(\tau, x, v) - \xi(\tau)|^2} dx\, dv\, d\tau \\
&= C \int_0^T \iint \frac{f(\tau, X(\tau, x, v), V(\tau, x, v))}{|X(\tau, x, v) - \xi(\tau)|^2} dx\, dv\, d\tau \\
&= C \int_0^T \iint \frac{f(\tau, x, v)}{|x - \xi(\tau)|^2} dx\, dv\, d\tau \le C(1 + T),
\end{aligned}$$

where in the last inequality we used Proposition 4.3.6.

Thus, condition (4.4.16) is satisfied and the proof is completed thanks to (4.4.17).

Chapter 5

Flows of partially regular vector fields

In [25] we derive quantitative estimates for the Lagrangian flow associated to a partially regular vector field of the form

$$b(t, x_1, x_2) = (b_1(t, x_1), b_2(t, x_1, x_2)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\,, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\,.$$

We assume that the first component b_1 does not depend on the second variable x_2, and has Sobolev $W^{1,p}$ regularity in the variable x_1, for some $p > 1$. On the other hand, the second component b_2 has Sobolev $W^{1,p}$ regularity in the variable x_2, but only fractional Sobolev $W^{\alpha,1}$ regularity in the variable x_1, for some $\alpha > 1/2$. These estimates imply well-posedness, compactness, and quantitative stability for the Lagrangian flow associated to such a vector field.

5.1 Introduction

Regarding the problem of wellposdness for transport equation and ODE, we would like to remark that both approaches of renormalization (due to DiPerna-Lions) and of a priori estimates of the flow, require information on a full derivative of the vector field, even though in a suitable weak sense (Sobolev or BV regularity, derivative which is a singular integral of an integrable function…), with an integrable control with respect to time. This kind of assumption is in general sharp, as shown by various counterexamples ([27, 22, 1, 29, 3, 2]). However, under more special "structural" conditions on the vector field, wellposedness can be proved even for vector fields with "less than one derivative", see for instance [3, 2] in the two-dimensional setting and [20] for the Hamiltonian case in general dimension.

A further case enjoying a "special structure" is that of partially regular vector fields as in [35, 37, 38] (see also Section 3.2.2 in [14]). Let us describe this case in some more detail. We assume to have a splitting of the space as $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and we denote the variable by $x = (x_1, x_2)$. We consider a vector field of the form

$$b = (b_1, b_2)\,, \qquad \text{with} \qquad b_1 = b_1(t, x_1)\,, \qquad b_2 = b_2(t, x_1, x_2)\,, \tag{5.1.1}$$

where b_1 is assumed to be Sobolev (respectively, BV) in x_1, and b_2 is assumed to be Sobolev (respectively, BV) in x_2, but merely integrable in x_1, see [35, 37] (respectively, [38]). Compared to the theory in [29, 4], no regularity is required for b_2 in the variable x_1; this is due to the strong requirement that b_1 does not depend on x_2. The authors in [35, 37, 38] address the PDE problem relying on the renormalization theory, with the additional idea to use two regularization kernels, namely $\rho_{\varepsilon_1} = \rho_{\varepsilon_1}(x_1)$ and $\rho_{\varepsilon_2} = \rho_{\varepsilon_2}(x_2)$, and to eventually send $\varepsilon_1 \to 0$ first, and then $\varepsilon_2 \to 0$. Roughly speaking, this gives rise to commutators "in x_1 only" for b_1 and "in x_2 only" for b_2.

In [25] we exploit the Lagrangian approach from [29] in order to derive well-posedness and quantitative estimate for the flow associated to a vector field of the form (5.1.1). As in [35, 37, 38] we exploit the anisotropy of the problem and we employ different scales in x_1 and x_2. However, this is not done by convolving the PDE with the two kernels $\rho_{\varepsilon_1}(x_1)$ and $\rho_{\varepsilon_2}(x_2)$, but rather relying on an anisotropic variant (introduced in [11]) of the Lagrangian functional (0.0.6), namely

$$\Phi_{\delta_1,\delta_2}(s) = \int \log\left(1 + \frac{|X_1 - \bar{X}_1|}{\delta_1} + \frac{|X_2 - \bar{X}_2|}{\delta_2}\right) dx\,, \tag{5.1.2}$$

where $\delta_1 \leq \delta_2$ (see (5.3.5) below for the exact expression of the functional we will use).

In fact, due to the structure of the proof, we cannot send the two parameters δ_1 and δ_2 to zero one after the other; they are however related, and δ_1 will be taken to be much smaller than δ_2. This will reflect in the need for some regularity on b_2 in the variable x_1; however, we will need only a derivative of fractional order (more specifically, higher that $1/2$, see assumption **(R2)** in Section 5.3.1 for the precise statement).

Let us explain the key steps in our argument. Directly differentiating Φ_{δ_1,δ_2} in time and arguing as in [11] we get

$$\Phi_{\delta_1,\delta_2}(s) \lesssim \|D_{x_1} b_1\| + \frac{\delta_1}{\delta_2}\|D_{x_1} b_2\| + \|D_{x_2} b_2\|\,,$$

with suitable norms on the right hand side, which depend on which exact regularity we assume on the vector field. The ratio δ_1/δ_2 can indeed be taken very small, but since b_2 does not possess a full derivative with respect to x_1, the term $\|D_{x_1} b_2\|$ is not bounded.

We can fix this issue by regularizing b_2 in the variable x_1 at scale $\varepsilon > 0$. In this way we get:

$$\Phi_{\delta_1,\delta_2}(s) \lesssim \frac{\|b_2^\varepsilon - b_2\|}{\delta_1} + \|D_{x_1} b_1\| + \frac{\delta_1}{\delta_2}\|D_{x_1} b_2^\varepsilon\| + \|D_{x_2} b_2\| \lesssim C + \frac{\varepsilon^\alpha}{\delta_1} + \frac{\delta_1}{\delta_2}\varepsilon^{\alpha-1}\,, \tag{5.1.3}$$

where in the second inequality we used that

$$\|b_2^\varepsilon - b_2\| \sim \varepsilon^\alpha \qquad \text{and} \qquad \|D_{x_1} b_2^\varepsilon\| \sim \varepsilon^{\alpha-1}\,,$$

assuming that b_2 possesses a derivative of order α in x_1 (see Lemma 5.2.4). Taking $\delta_1 = \delta_2\varepsilon^{1-\alpha}$ the right hand side of (5.1.3) takes the form $C + \varepsilon^{2\alpha-1}/\delta_2$, which can be made bounded as $\delta_2 \to 0$ by a suitable choice of ε provided $\alpha > 1/2$. This is the reason why, with this approach, we need some fractional regularity of b_2 in x_1. From this bound on Φ_{δ_1,δ_2} all results on the well-posedness and further properties of the flow follow as in [15], see Section 5.3.3 for the precise statements.

5.2 Preliminaries

5.2.1 Regular Lagrangian flows

In the context of non-Lipschitz vector fields, the right concept of solution of the ordinary differential equation (0.0.2) is that of regular Lagrangian flow (see [29, 4, 7]). In the following, we are going to assume that the vector field $b : (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following growth condition:

$$\begin{aligned}\textbf{(R1)} : \quad & \frac{b(s,x)}{1+|x|} = c_1(s,x) + c_2(s,x)\,, \\ & \text{with} \quad c_1 \in L^1((0,T);L^1(\mathbb{R}^N)) \quad \text{and} \quad c_2 \in L^1((0,T);L^\infty(\mathbb{R}^N))\,.\end{aligned} \tag{5.2.1}$$

Definition 5.2.1 (Regular Lagrangian flow)**.** Let b be a vector field satisfying **(R1)**. A map

$$X \in C([0,T];L^0_{\mathrm{loc}}(\mathbb{R}^N)) \cap \mathcal{B}([0,T];\log L_{\mathrm{loc}}(\mathbb{R}^N))$$

is a regular Lagrangian flow in the renormalized sense relative to b if:

1. The equation
$$\partial_s\big(\beta(X(s,x))\big) = \beta'(X(s,x))b(s,X(s,x))$$
holds in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$, for every function $\beta \in C^1(\mathbb{R}^N;\mathbb{R})$ that satisfies
$$|\beta(z)| \le C(1+\log(1+|z|)) \quad \text{and} \quad |\beta'(z)| \le \frac{C}{1+|z|} \quad \text{for all } z\in\mathbb{R}^N,$$

2. $X(0,x) = x$ for a.e $x \in \mathbb{R}^N$,

3. There exists a constant $L \ge 0$ such that $\int_{\mathbb{R}^N} \varphi(X(s,x))dx \le L\int_{\mathbb{R}^N}\varphi(x)dx$ for all continuous functions $\varphi : \mathbb{R}^N \to [0,\infty)$. The constant L is called *compressibility constant* of the flow.

In the above definition, L^0_{loc} denotes the space of measurable functions endowed with the local convergence in measure, $\mathcal{B}$ denotes the space of bounded functions, and $\log L_{\text{loc}}$ denotes the space of locally logarithmically integrable functions.

Given a vector field satisfying **(R1)**, we can estimate the measure of the superlevels of the associated regular Lagrangian flow thanks to the following lemma:

Lemma 5.2.2. *Let $b : (0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ be a vector field satisfying* **(R1)** *and let $X : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ be a regular Lagrangian flow relative to b with compressibility constant L. Define the sublevels of the flow as*
$$G_\lambda = \{x \in \mathbb{R}^N : |X(s,x)| \le \lambda \text{ for almost all } s \in [0,T]\}\,. \tag{5.2.2}$$
Then for all $r, \lambda > 0$ it holds
$$\mathcal{L}^N(B_r \setminus G_\lambda) \le g(r,\lambda)\,,$$
where the function g depends only on L, $\|c_1\|_{L^1((0,T);L^1(\mathbb{R}^N))}$ and $\|c_2\|_{L^1((0,T);L^\infty(\mathbb{R}^N))}$ and satisfies $g(r,\lambda) \downarrow 0$ for r fixed and $\lambda \uparrow \infty$.

5.2.2 Fractional Sobolev spaces

We will make use of fractional Sobolev spaces according to the Sobolev–Slobodeckij definition:

Definition 5.2.3 (Fractional Sobolev–Slobodeckij space)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, $f \in L^1(\mathbb{R}^n)$. Given $0 < s < 1$ and $1 \le p < \infty$, we say that $f \in W^{s,p}(\mathbb{R}^n)$ if
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+n}}\,dy\,dx < \infty\,.$$

The following lemma gives a rate of convergence of the convolution to the original function, and a rate of blow-up of the derivative of the function, under the assumption of fractional Sobolev regularity.

Lemma 5.2.4. *Let $f \in W^{s,p}(\mathbb{R}^n)$ and let f^ε be the convolution of f with the standard mollifier φ^ε. Then we have*
$$\|f - f^\varepsilon\|_{L^p(\mathbb{R}^n)} \le C\varepsilon^s\|f\|_{W^{s,p}(\mathbb{R}^n)} \qquad \textit{and} \qquad \|Df^\varepsilon\|_{L^p(\mathbb{R}^n)} \le C\varepsilon^{s-1}\|f\|_{W^{s,p}(\mathbb{R}^n)}. \tag{5.2.3}$$

Proof. For the first estimate we compute

$$
\begin{aligned}
\|f - f^\varepsilon\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |f(x) - f^\varepsilon(x)|^p dx = \int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)\,dy \right|^p dx \\
&= \int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} f(x-y)\varphi\left(\frac{y}{\varepsilon}\right)\frac{1}{\varepsilon^n}\,dy \right|^p dx = \int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} f(x-\varepsilon z)\varphi(z)\frac{1}{\varepsilon^n}\,\varepsilon^n dz \right|^p dx \\
&= \int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} f(x-\varepsilon z)\varphi(z)\,dz \right|^p dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} [f(x)\varphi(z) - f(x-\varepsilon z)\varphi(z)] dz \right|^p dx \\
&= \int_{\mathbb{R}^n} \left| \int [f(x) - f(x-\varepsilon z)]\varphi(z) dz \right|^p dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(x-\varepsilon z)|^p \varphi(z)\,dz\,dx \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(x-\varepsilon z)|^p}{|\varepsilon z|^{sp+n}} |\varepsilon z|^{sp+n}\varphi(z)\,dz\,dx \\
&\le \varepsilon^{sp+n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(x-\varepsilon z)|^p}{|\varepsilon z|^{sp+n}} \sup_z\{|z|^{sp+n}\varphi(z)\}dz\,dx \\
&\le C\varepsilon^{sp+n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(x-\varepsilon z)|^p}{|\varepsilon z|^{sp+n}} dz\,dx = C\varepsilon^{sp+n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{sp+n}} \frac{1}{\varepsilon^n} dy\,dx \\
&\le C\varepsilon^{sp} \|f\|^p_{W^{s,p}},
\end{aligned}
$$

where in the forth line we used Jensen's inequality applied with the measure $\varphi \cdot \mathcal{L}^n$. This proves the first inequality in the statement.

For the second estimate we compute

$$
\begin{aligned}
\|Df^\varepsilon\|^p_{L^p(\mathbb{R}^n)} &= \|f * D\varphi^\varepsilon\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) D\varphi^\varepsilon(y)\,dy \right|^p dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y) D_y\left(\frac{1}{\varepsilon^n}\varphi\left(\frac{y}{\varepsilon}\right)\right) dy \right|^p dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-\varepsilon z) D_z\varphi(z)\frac{1}{\varepsilon^{n+1}}\varepsilon^n\,dz \right|^p dx \\
&= \frac{1}{\varepsilon^p}\int_{\mathbb{R}^n} \left| \int_{B_1} f(x-\varepsilon z) D_z\varphi(z)\,dz \right|^p dx = \frac{1}{\varepsilon^p}\int_{\mathbb{R}^n} \left| \int_{B_1} f(x-\varepsilon z) D_z\varphi(z)\,dz - \int_{\mathbb{R}^n} f(x) D_z\varphi(z)\,dz \right|^p dx \\
&= \frac{1}{\varepsilon^p}\mathcal{L}^n(B_1)^p \int_{\mathbb{R}^n} \left| \int_{B_1} [f(x-\varepsilon z) - f(x)] D_z\varphi(z)\,\frac{dz}{\mathcal{L}^n(B_1)} \right|^p dx \\
&\le \frac{1}{\varepsilon^p}\mathcal{L}^n(B_1)^p \int_{\mathbb{R}^n}\int_{B_1} \left|[f(x-\varepsilon z) - f(x)] D_z\varphi(z)\right|^p \frac{dz}{\mathcal{L}^n(B_1)} dx \\
&= \frac{1}{\varepsilon^p}\mathcal{L}^n(B_1)^{p-1} \int_{\mathbb{R}^n}\int_{B_1} \left|[f(x-\varepsilon z) - f(x)] D_z\varphi(z)\right|^p dz\,dx \\
&= \frac{1}{\varepsilon^p}\mathcal{L}^n(B_1)^{p-1} \int_{\mathbb{R}^n}\int_{B_1} \frac{|f(x-\varepsilon z) - f(x)|^p}{|\varepsilon z|^{sp+n}} |\varepsilon z|^{sp+n} |D_z\varphi(z)|^p dz\,dx \\
&\le \frac{1}{\varepsilon^p}\varepsilon^{sp+n} C_n \int_{\mathbb{R}^n}\int_{B_1} \frac{|f(x-\varepsilon z) - f(x)|^p}{|\varepsilon z|^{sp+n}} \sup_z\{|z|^{sp+n}|D_z\varphi(z)|^p\} dz\,dx \\
&\le C\varepsilon^{p(s-1)}\varepsilon^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{sp+n}} \frac{1}{\varepsilon^n} dy\,dx \\
&\le C\varepsilon^{p(s-1)} \|f\|^p_{W^{s,p}},
\end{aligned}
$$

where in the third line we used that $D_z\varphi$ has zero average, and in the fifth line we used Jensen's inequality for the measure $\frac{1}{\mathcal{L}^n(B_1)} \cdot \mathcal{L}^n$. □

5.2.3 Maximal estimates

In the course of the proof of our main theorem we will several times need to estimate difference quotients of the vector field. We will follow the strategy in [24] and rely on suitable maximal estimates. We now briefly recall the main definitions, the most classical version of these estimates, and some anisotropic variants proved in [11].

Definition 5.2.5. For any integrable function $u : \mathbb{R}^n \to \mathbb{R}$ the maximal function of u is defined as

$$Mu(x) = \sup_{r>0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |u(z)|\, dz\,, \qquad x \in \mathbb{R}^n\,.$$

It can be shown that, for $u \in L^1(\mathbb{R}^n)$, the maximal function Mu is a.e. finite. Moreover, the following norm estimates hold (see [48, 49] for a proof):

Lemma 5.2.6. *For any* $1 < p \le \infty$ *the strong estimate*

$$\|Mu\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{L^p(\mathbb{R}^n)}$$

holds, where C depends on p and n only. For $p = 1$ *only the weak etimate*

$$|||Mu|||_{M^1(\mathbb{R}^n)} \le C\|u\|_{L^1(\mathbb{R}^n)}$$

holds, with C depending on n only. In the above we denoted by

$$|||f|||_{M^1(\mathbb{R}^n)} = \sup_{\lambda>0} \left\{ \lambda\, \mathcal{L}^n(\{x : |f| > \lambda\}) \right\} \tag{5.2.4}$$

the weak-L^1 *norm.*

The basic maximal estimate for the difference quotients of a Sobolev function is the following one. We recall its classical proof for the reader's convenience.

Lemma 5.2.7. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function in* $W^{1,1}(\mathbb{R}^n)$*. Then for a.e.* $x, y \in \mathbb{R}^n$*,*

$$|f(x) - f(y)| \le C_n |x-y| \big(MDf(x) + MDf(y) \big)\,.$$

Proof. First we prove the estimate for $f \in C^1$. We denote

$$\begin{aligned} A = B\left(\frac{x+y}{2}, \frac{|x-y|}{2}\right) \qquad & A_{t,x} = tx + (1-t)A \qquad & B_{t,x} = B(x, (1-t)|x-y|) \\ & A_{t,y} = ty + (1-t)A \qquad & B_{t,y} = B(y, (1-t)|x-y|)\,. \end{aligned} \tag{5.2.5}$$

We note that $A_{t,x} \subset B_{t,x}$ and $A_{t,y} \subset B_{t,y}$. We estimate

$$\begin{aligned}
|f(x) - f(y)| &= \fint_A |f(x) - f(y)|\, dz \le \fint_A |f(x) - f(z)|\, dz + \fint_A |f(y) - f(z)|\, dz \\
&= \fint_A \left| \int_0^1 \frac{d}{dt}[f(tx + (1-t)z] dt \right| dz + \fint_A \left| \int_0^1 \frac{d}{dt}[f(ty + (1-t)z] dt \right| dz \\
&\le \frac{1}{\mathcal{L}^n(A)} \int_A \int_0^1 \left| \frac{d}{dt}[f(tx + (1-t)z] \right| dt\, dz + \frac{1}{\mathcal{L}^n(A)} \int_A \int_0^1 \left| \frac{d}{dt}[f(ty + (1-t)z] \right| dt\, dz \\
&\le \frac{1}{\mathcal{L}^n(A)} \left[\int_0^1 \int_A |Df(tx + (1-t)z)|\, |x - z| dz dt + \int_0^1 \int_A |Df(ty + (1-t)z)|\, |y - z| dz dt \right] \\
&\le \frac{1}{\mathcal{L}^n(A)} |x-y| \left[\int_0^1 \int_A |Df(tx + (1-t)z)| dz dt + \int_0^1 \int_A |Df(ty + (1-t)z)| dz dt \right].
\end{aligned}$$

We apply a change of variable and we obtain that the last line equals

$$
\begin{aligned}
&\frac{1}{\mathcal{L}^n(A)}|x-y|\left[\int_0^1\int_{A_{t,x}}|Df(w)|\frac{dw}{1-t}dt+\int_0^1\int_{A_{t,y}}|Df(w)|\frac{dw}{1-t}dt\right]\\
&\le \frac{1}{\mathcal{L}^n(A)}|x-y|\left[\int_0^1\frac{\mathcal{L}^n(B_{t,x})}{1-t}\frac{1}{\mathcal{L}^n(B_{t,x})}\int_{B_{t,x}}|Df(w)|dw\,dt\ +\ \text{symmetric}\right]\\
&\le \frac{n}{\frac{|x-y|^n}{2^n}(2\pi)^{\frac{n}{2}}}|x-y|\left[\int_0^1\frac{\frac{(1-t)^n}{n}|x-y|^n(2\pi)^{\frac{n}{2}}}{1-t}\sup_{r>0}\fint_{B(x,r)}|Df(w)|dw\,dt\ +\ \text{symmetric}\right]\\
&=2^n|x-y|\int_0^1(1-t)^{n-1}dt\left[MDf(x)+MDf(y)\right]\\
&=C_n|x-y|\left[MDf(x)+MDf(y)\right],
\end{aligned}
$$

where we used $\mathcal{L}^n(B(x,r))=\frac{r^n(2\pi)^{\frac{n}{2}}}{n}$.

To conclude the proof for $f\in W^{1,1}(\mathbb{R}^n)$ it suffices to approximate f with a sequence $(f_\varepsilon)\subset C^1(\mathbb{R}^n)$ which converges to f in $W^{1,1}(\mathbb{R}^n)$ as $\varepsilon\to 0$. □

In our main result we will deal with a vector field with partial regularity. This assumption entails a splitting of the space as $\mathbb{R}^N=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ (with $N=n_1+n_2$). We will denote the variable $x\in\mathbb{R}^N$ by $x=(x_1,x_2)$, where $x_1\in\mathbb{R}^{n_1}$ and $x_2\in\mathbb{R}^{n_2}$. Following [11], for $\delta_1,\delta_2>0$ we consider the $N\times N$ diagonal matrix

$$
A=\begin{bmatrix}\delta_1&&&&\\&\delta_1&&&\\&&\ddots&&\\&&&\delta_2&\\&&&&\delta_2\end{bmatrix}, \tag{5.2.6}
$$

where δ_1 appears at the first n_1 entries on the diagonal, and δ_2 at the remaining n_2. In other words, we have

$$
A(x_1,x_2)=(\delta_1x_1,\delta_2x_2),\qquad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}.
$$

The next two lemmas have been proved in larger generality in [11]. We state them in our setting and give a simpler proof for the reader's convenience.

Lemma 5.2.8. *Let $f:\mathbb{R}^N\to\mathbb{R}$ be a function in $W^{1,1}(\mathbb{R}^N)$. Let A be the matrix defined in* (5.2.6). *Then there exists a nonnegative function U such that for a.e. $x,y\in\mathbb{R}^N$,*

$$
|f(x)-f(y)|\lesssim|A^{-1}[x-y]|\,(U(x)+U(y)),
$$

with

$$
U(x)=M\left(\sum_{j=1}^N|\partial_jf(A\cdot)|A_{jj}\right)(A^{-1}x).
$$

Proof. The result follows from Lemma 5.2.7 above. We denote $\tilde f(z)=f(Az)$. Then we know that, for a.e. z, w,

$$
|\tilde f(z)-\tilde f(w)|\le C_N|z-w|\big(MD\tilde f(z)+MD\tilde f(w)\big), \tag{5.2.7}
$$

where in addition we notice

$$
MD\tilde f(z)\le M\left(\sum_{j=1}^N|\partial_j\tilde f|\right)(z)=M\left(\sum_{j=1}^N(|\partial_jf(A\cdot)|A_{jj})\right)(z). \tag{5.2.8}
$$

Combining (5.2.7) and (5.2.8) we have, for a.e. z, w,

$$|f(Az) - f(Aw)| \leq C_N |z-w| \left(M \sum_{j=1}^{N} (|\partial_j f(A\cdot)| A_{jj})(z) + M \sum_{j=1}^{N} (|\partial_j f(A\cdot)| A_{jj})(w) \right). \quad (5.2.9)$$

Now from the last inequality, taking x and y such that $z = A^{-1}x$ and $w = A^{-1}y$, we obtain the thesis. □

Lemma 5.2.9 (Operator bounds). *Let U be defined as in Lemma 5.2.8. Then we have the estimates*

$$|||U|||_{M^1(\mathbb{R}^N)} \leq C \left(\delta_1 \sum_{j=1}^{n_1} ||\partial_j f||_{L^1(\mathbb{R}^N)} + \delta_2 \sum_{j=n_1+1}^{N} ||\partial_j f||_{L^1(\mathbb{R}^N)} \right) \quad (5.2.10)$$

for $\partial_j f \in L^1$, and

$$||U||_{L^p(\mathbb{R}^N)} \leq C \left(\delta_1 \sum_{j=1}^{n_1} ||\partial_j f||_{L^p(\mathbb{R}^N)} + \delta_2 \sum_{j=n_1+1}^{N} ||\partial_j f||_{L^p(\mathbb{R}^N)} \right) \quad (5.2.11)$$

for $\partial_j f \in L^p$.

Proof. As in Lemma 5.2.8 we consider $\tilde{f}(z) = f(Az)$. We exploit the estimates in Lemma 5.2.6 to the effect that

$$\begin{aligned} |||U|||_{M^1(\mathbb{R}^N)} &= \left|\left|\left| M \sum_{j=1}^{N} (|\partial_j f(A\cdot)| A_{jj})(A^{-1}\cdot) \right|\right|\right|_{M^1(\mathbb{R}^N)} = \left|\left|\left| M \sum_{j=1}^{N} |\partial_j \tilde{f}|(A^{-1}\cdot) \right|\right|\right|_{M^1(\mathbb{R}^N)} \\ &\leq C \left\| \sum_{j=1}^{N} |\partial_j \tilde{f}|(A^{-1}\cdot) \right\|_{L^1(\mathbb{R}^N)} \leq C \sum_{j=1}^{N} \|(\partial_j \tilde{f})(A^{-1}\cdot)\|_{L^1(\mathbb{R}^N)} \\ &= C \sum_{j=1}^{N} \|(\partial_j f(A\cdot) A_{jj})(A^{-1}\cdot)\|_{L^1(\mathbb{R}^N)} = C \sum_{j=1}^{N} A_{jj} \|\partial_j f\|_{L^1(\mathbb{R}^N)}, \end{aligned}$$

which is equation (5.2.10). With similar computations we can obtain (5.2.11). □

We close this section with the following interpolation lemma, which allows to estimate the L^1 norm in terms of the weak-L^1 norm defined in (5.2.4), with a logarithmic dependence on higher integrability norms.

Lemma 5.2.10 (Interpolation). *Let $u : \Omega \to [0, +\infty)$ be a nonnegative measurable function, where $\Omega \subset \mathbb{R}^n$ has finite measure. Then for every $1 < p < \infty$, we have the interpolation estimate*

$$\|u\|_{L^1(\Omega)} \leq \frac{p}{p-1} |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{\|u\|_{L^p(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathcal{L}^n(\Omega)^{1-\frac{1}{p}} \right) \right],$$

and analogously for $p = \infty$

$$\|u\|_{L^1(\Omega)} \leq |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{\|u\|_{L^\infty(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathcal{L}^n(\Omega) \right) \right].$$

5.3 Main result and corollaries

5.3.1 Assumptions on the vector field

We recall that we consider a splitting of the space as $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and that we denote the variable by $x = (x_1, x_2)$, with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. We are dealing with a vector field $b : (0,T) \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ for which we assume the following regularity:

$$\begin{aligned}
(\mathbf{R2}):\quad & b(s,x_1,x_2) = \big(b_1(s,x_1), b_2(s,x_1,x_2)\big) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^N \\
& b_1(s,x_1) \in L^1\big((0,T); W^{1,p}_{x_1}(\mathbb{R}^{n_1})\big) \\
& b_2(s,x_1,x_2) \in L^1\big((0,T) \times \mathbb{R}^{n_2}_{x_2}; W^{\alpha,1}_{x_1}(\mathbb{R}^{n_1})\big) \cap L^1\big((0,T) \times \mathbb{R}^{n_1}_{x_1}; W^{1,p}_{x_2}(\mathbb{R}^{n_2})\big)\,,
\end{aligned} \tag{5.3.1}$$

for some given $p > 1$ and $1/2 < \alpha < 1$.

Moreover, we will assume that

$$(\mathbf{R3}):\quad b(t,x_1,x_2) \in L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^N)\,. \tag{5.3.2}$$

Also recall that suitable growth conditions on b have been assumed in **(R1)**.

Let us introduce some further notation that will be used in the following.

We denote by $D_i b_j = D_{x_i} b_j$ the partial derivatives in distributional sense. We set $D_1 b_1 = p(t,x_1)$, $D_1 b_2 = q(t,x_1,x_2)$, and $D_2 b_2 = r(t,x_1,x_2)$. Then we have

$$Db = \begin{pmatrix} D_1 b_1 & D_2 b_1 \\ D_1 b_2 & D_2 b_2 \end{pmatrix} = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \in \begin{pmatrix} L^1_{x_2,\mathrm{loc}} L^p_{x_1} & 0 \\ \text{distribution} & L^1_{x_1} L^p_{x_2} \end{pmatrix}. \tag{5.3.3}$$

5.3.2 Main estimate for the Lagrangian flow

Theorem 5.3.1. *Let b and $\bar b$ be two vector fields satisfying assumptions* **(R1)**. *Assume the following:*

- *The second component of $\bar b$ satisfies $\bar b_2 \in L^1\big((0,T) \times \mathbb{R}^{n_2}_{x_2}; W^{\alpha,1}_{x_1}(\mathbb{R}^{n_1})\big)$,*
- *The vector field b satisfies* **(R2)** *and* **(R3)**.

Let X and $\bar X$ be regular Lagrangian flows associated to b and $\bar b$ respectively, with compressibility constants L and $\bar L$. Then the following holds. For every positive γ, r and η there exists $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that

$$\mathcal{L}^N\left(B_r \cap \left\{|X(s,\cdot) - \bar X(s,\cdot)| > \gamma\right\}\right) \le C_{\gamma,r,\eta} \|b - \bar b\|_{L^1((0,T)\times B_\lambda)} + \eta \tag{5.3.4}$$

for all $s \in [0,T]$, where $C_{\gamma,r,\eta}$ depends on L, $\bar L$, the bound for $\bar b_2$ in $L^1\big((0,T)\times\mathbb{R}^{n_2}_{x_2}; W^{\alpha,1}_{x_1}(\mathbb{R}^{n_1})\big)$, the bound for the decomposition of $\bar b$ as in **(R1)**, *and the various bounds for b involved in the assumptions* **(R1)**, **(R2)**, *and* **(R3)**.

Proof. We exploit the anisotropic functional

$$\Phi_{\delta_1,\delta_2}(s) = \int_{B_r \cap G_\lambda \cap \bar G_\lambda} \log\left(1 + |A^{-1}[X(s,x) - \bar X(s,x)]|\right)\,dx\,, \tag{5.3.5}$$

where the matrix A has been defined in (5.2.6) and G_λ (respectively, $\bar G_\lambda$) are the sublevels of the regular Lagrangian flow X (respectively, $\bar X$) defined as in (5.2.2).

Step 1: *Regularization of the vector field.* We regularize b_2 by convolution in x_1. Let φ^ε be a standard mollifier in $\mathbb{R}^{n_1}$. We denote the regularization of b_2 by

$$b_2^\varepsilon(t,x_1,x_2) = b_2(t,x_1,x_2) *_{x_1} \varphi^\varepsilon(x_1)\,, \qquad \text{for } t \text{ and } x_2 \text{ fixed,}$$

and we further denote $b^\varepsilon = (b_1, b_2^\varepsilon)$. Moreover, q^ε and r^ε are associated to b^ε as in (5.3.3).

Due to standard properties of the convolution we have that $b^\varepsilon \to b$ and $r^\varepsilon \to r$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Also recall the rates of convergence and blow-up proved in Lemma 5.2.4.

Step 2: *Time differentiation.* By differentiating the functional $\Phi_{\delta_1,\delta_2}(s)$ with respect to time we get

$$\begin{aligned}
\Phi'_{\delta_1,\delta_2}(s) &\le \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|A^{-1}[b(s,X)-\bar b(s,\bar X)]|}{1+|A^{-1}[X-\bar X]|}dx \\
&\le \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|A^{-1}[b(s,X)-b^\varepsilon(s,X)]|}{1+|A^{-1}[X-\bar X]|} + \frac{|A^{-1}[\bar b^\varepsilon(s,\bar X)-\bar b(s,\bar X)]|}{1+|A^{-1}[X-\bar X]|} + \frac{|A^{-1}[b^\varepsilon(s,X)-\bar b^\varepsilon(s,\bar X)]|}{1+|A^{-1}[X-\bar X]|}dx \\
&\le \frac{L}{\delta_1}\|b-b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} + \frac{\bar L}{\delta_1}\|\bar b-\bar b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} \\
&\quad + \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar X)+b^\varepsilon(s,\bar X)-\bar b^\varepsilon(s,\bar X)]|}{1+|A^{-1}[X-\bar X]|}\,dx \\
&\le \frac{L}{\delta_1}\|b-b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} + \frac{\bar L}{\delta_1}\|\bar b-\bar b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} + \frac{\bar L}{\delta_1}\|b^\varepsilon-\bar b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} \\
&\quad + \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \frac{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar X)]|}{1+|A^{-1}[X-\bar X]|}\,dx \\
&\le \frac{L}{\delta_1}\|b-b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} + \frac{\bar L}{\delta_1}\|\bar b-\bar b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} + \frac{\bar L}{\delta_1}\|b^\varepsilon-\bar b^\varepsilon(s,\cdot)\|_{L^1(B_\lambda)} \\
&\quad + \int_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\left\{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar X)]|, \frac{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar X)]|}{|A^{-1}[X-\bar X]|}\right\}dx.
\end{aligned}$$

Step 3: *Bounds with maximal operators.* Integrating in time and recalling the definition of the matrix A in 5.2.6 we get

$$\begin{aligned}
\Phi_{\delta_1,\delta_2}(\tau) &\le \frac{L}{\delta_1}\|b-b^\varepsilon\|_{L^1((0,\tau)\times B_\lambda)} + \frac{\bar L}{\delta_1}\|\bar b-\bar b^\varepsilon\|_{L^1((0,\tau)\times B_\lambda)} + \frac{\bar L}{\delta_1}\|b^\varepsilon-\bar b^\varepsilon\|_{L^1((0,\tau)\times B_\lambda)} \\
&\quad + \int_0^\tau\int_{B_r\cap G_\lambda\cap\bar G_\lambda} \min\left\{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar X)]|, \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\bar X)|}{|A^{-1}[X-\bar X]|}\right. \\
&\qquad\qquad\left. + \frac{1}{\delta_2}\frac{|b_2^\varepsilon(s,X)-b_2^\varepsilon(s,\bar X)|}{|A^{-1}[X-\bar X]|}\right\}dxds.
\end{aligned} \tag{5.3.6}$$

Lemmas 5.2.8 and 5.2.9 can be easily extended to vector valued functions. We would like to apply these lemmas to b^ε, which is only locally $W^{1,1}$ in $\mathbb{R}^N$, as the first component b_1 does not depend on x_2. This can be done by defining a new vector field $\tilde b^\varepsilon$ as the smooth cut-off of b^ε on the ball of radius 2λ, i.e. $\tilde b^\varepsilon = b^\varepsilon \cdot \chi_{B_\lambda} = (b_1\cdot\chi_{B_\lambda}, b_2^\varepsilon\cdot\chi_{B_\lambda}) = (\tilde b_1, \tilde b_2^\varepsilon)$, where χ_{B_λ} is a smooth function with value 1 on $B_{2\lambda}$ and 0 on $\mathbb{R}^N\setminus B_{2\lambda+1}$, and by using suitable truncated maximal functions in the maximal estimates. We define $\tilde p$, $\tilde q$, $\tilde r$, $\tilde q^\varepsilon$ and $\tilde r^\varepsilon$ as the partial derivatives of $\tilde b$ $(= b\cdot\chi_{B_\lambda})$ and $\tilde b^\varepsilon$.

Lemma 5.2.8 applied to $\tilde b_1$ and $\tilde b_2^\varepsilon$ yields

$$\frac{|\tilde b_1(s,x)-\tilde b_1(s,\bar x)|}{|A^{-1}[x-\bar x]|} \lesssim U_{\tilde p}(x) + U_{\tilde p}(\bar x), \tag{5.3.7}$$

and

$$\frac{|\tilde{b}_2^\varepsilon(s,x)-\tilde{b}_2^\varepsilon(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \lesssim U_{\tilde{q}^\varepsilon,\tilde{r}^\varepsilon}(x)+U_{\tilde{q}^\varepsilon,\tilde{r}^\varepsilon}(\bar{x}) \tag{5.3.8}$$

for $s\in[0,T]$, and for a.e. $x,\bar{x}\in\mathbb{R}^N$.

By subadditivity of U we can estimate

$$U_{\tilde{q}^\varepsilon,\tilde{r}^\varepsilon} \le U_{\tilde{q}^\varepsilon}+U_{\tilde{r}^\varepsilon},$$

implying that

$$\begin{aligned}\frac{|\tilde{b}_1(s,x)-\tilde{b}_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} &\lesssim U_{\tilde{p}}(x)+U_{\tilde{p}}(\bar{x}),\\ \frac{|\tilde{b}_2^\varepsilon(s,x)-\tilde{b}_2^\varepsilon(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} &\lesssim U_{\tilde{q}^\varepsilon}(x)+U_{\tilde{r}^\varepsilon}(x)+U_{\tilde{q}^\varepsilon}(\bar{x})+U_{\tilde{r}^\varepsilon}(\bar{x}).\end{aligned}$$

Step 4: *Estimates for the maximal operators.* Let $\Omega=(0,\tau)\times\left(B_r\cap G_\lambda\cap\bar{G}_\lambda\right)\subset\mathbb{R}^{N+1}$. We can estimate the last term of the sum (5.3.6) with

$$\begin{aligned}\int_\Omega \min\Big\{ |A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar{X})]|, \frac{1}{\delta_1}\left(U_{\tilde{p}}(s,X)+U_{\tilde{p}}(s,\bar{X})\right)\\ +\frac{1}{\delta_2}\left((U_{\tilde{q}^\varepsilon}+U_{\tilde{r}^\varepsilon})(s,X)+(U_{\tilde{q}^\varepsilon}+U_{\tilde{r}^\varepsilon})(s,\bar{X})\right)\Big\} dx\,ds =: \tilde{\Phi}_{\delta_1,\delta_2}(\tau).\end{aligned}$$

Lemma 5.2.9 implies

$$|||U_{\tilde{q}^\varepsilon}|||_{M^1((0,T)\times B_\lambda)} \lesssim \delta_1\|\tilde{q}^\varepsilon\|_{L^1((0,T)\times\mathbb{R}^N)}=\delta_1\|\tilde{q}^\varepsilon\|_{L^1((0,T)\times B_{2\lambda+1})}\le\delta_1\|q^\varepsilon\|_{L^1((0,T)\times B_{2\lambda+1})}=:\delta_1\psi(\varepsilon).$$

Notice that the quantity $\psi(\varepsilon)$ at the right hand side could a priori blow up as $\varepsilon\to0$, as we are not assuming that $q=D_1b_2$ is integrable.

Splitting the minima once again, we obtain

$$\begin{aligned}\tilde{\Phi}_{\delta_1,\delta_2}(\tau) \le& \int_\Omega \min\left\{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar{X})]|, \frac{1}{\delta_2}\left(U_{\tilde{q}^\varepsilon}(s,X)+U_{\tilde{q}^\varepsilon}(s,\bar{X})\right)\right\} dx\,ds\\ &+\int_\Omega \min\left\{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar{X})]|, \frac{1}{\delta_2}\left(U_{\tilde{r}^\varepsilon}(s,X)+U_{\tilde{r}^\varepsilon}(s,\bar{X})\right)\right\} dx\,ds\\ &+\int_\Omega \min\left\{|A^{-1}[b^\varepsilon(s,X)-b^\varepsilon(s,\bar{X})]|, \frac{1}{\delta_1}\left(U_{\tilde{p}}(s,X)+U_{\tilde{p}}(s,\bar{X})\right)\right\} dx\,ds\\ =& \int_\Omega \varphi_1(s,X,\bar{X})\,dxds+\int_\Omega \varphi_2(s,X,\bar{X})\,dxds+\int_\Omega \varphi_3(s,X,\bar{X})\,dxds.\end{aligned}$$

Let $\Omega'=(0,\tau)\times B_\lambda\subset\mathbb{R}^{N+1}$. Using the first element of the minimum and relying on assumption **(R3)** we can estimate

$$\|\varphi_1\|_{L^p(\Omega)} \le \frac{L^{1/p}+\bar{L}^{1/p}}{\delta_1}\|b^\varepsilon\|_{L^p(\Omega')} \lesssim \frac{1}{\delta_1}\|b^\varepsilon\|_{L^p(\Omega')} \lesssim \frac{1}{\delta_1}\|b\|_{L^p(\Omega')} \simeq \frac{1}{\delta_1}.$$

Exploiting the second term of the minimum, we get

$$|||\varphi_1|||_{M^1(\Omega)} \le \frac{1}{\delta_2}|||U_{\tilde{q}^\varepsilon}(X)+U_{\tilde{q}^\varepsilon}(\bar{X})|||_{M^1(\Omega)} \lesssim \frac{1}{\delta_2}|||U_{\tilde{q}^\varepsilon}|||_{M^1(\Omega')} \lesssim \frac{\delta_1}{\delta_2}\|q_\varepsilon\|_{L^1((0,T)\times B_{2\lambda+1})}=\frac{\delta_1}{\delta_2}\psi(\varepsilon).$$

For φ_2 and φ_3 using assumption **(R2)** we have

$$\|\varphi_2\|_{L^1(\Omega)} \lesssim \frac{1}{\delta_2}\|U_{\tilde{r}^\varepsilon}\|_{L^1(\Omega')} \lesssim_\lambda \frac{1}{\delta_2}\|U_{\tilde{r}^\varepsilon}\|_{L^1((0,T);L^p(B_\lambda))} \lesssim \frac{\delta_2}{\delta_2}\|\tilde{r}_2^\varepsilon\|_{L^1((0,T);L^p(\mathbb{R}^N))} \lesssim C \tag{5.3.9}$$

and

$$\|\varphi_3\|_{L^1(\Omega)} \lesssim \frac{1}{\delta_1}\|U_{\tilde{p}}\|_{L^1(\Omega')} \lesssim_\lambda \frac{1}{\delta_1}\|U_{\tilde{p}}\|_{L^1((0,T);L^p(B_\lambda))} \lesssim \frac{\delta_1}{\delta_1}\|\tilde{p}_2\|_{L^1((0,T);L^p(\mathbb{R}^N))} \lesssim C\,. \tag{5.3.10}$$

Step 5: *Interpolation Lemma.* We can apply now Lemma 5.2.10 to φ_1, to the effect that

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) &\lesssim_\lambda \frac{1}{\delta_1}\|b^\varepsilon - \bar{b}^\varepsilon\|_{L^1(\Omega')} + \frac{1}{\delta_1}\sigma(\varepsilon) + \frac{1}{\delta_1}\bar{\sigma}(\varepsilon) + \frac{\delta_1}{\delta_2}\psi(\varepsilon)\log\left(\frac{1}{\frac{\delta_1}{\delta_2}\psi(\varepsilon)\delta_1}\right) + C\\ &\lesssim \frac{1}{\delta_1}\|b - \bar{b}\|_{L^1(\Omega')} + \frac{1}{\delta_1}\left[\sigma(\varepsilon) + \bar{\sigma}(\varepsilon)\right] + \frac{\delta_1}{\delta_2}\psi(\varepsilon)\log\left(\frac{1}{\frac{\delta_1}{\delta_2}\psi(\varepsilon)\delta_1}\right) + C\,,\end{aligned}$$

where $\sigma(\varepsilon) = \|b - b^\varepsilon\|_{L^1(\Omega')}$ and $\bar{\sigma}(\varepsilon) = \|\bar{b} - \bar{b}^\varepsilon\|_{L^1(\Omega')}$ tend to 0 as $\varepsilon \to 0$. Lemma 5.2.4 implies that

$$\sigma(\varepsilon) + \bar{\sigma}(\varepsilon) \lesssim \left(\|b_2\|_{L^1_{t,x_2}W^{\alpha,1}_{x_1}} + \|\bar{b}_2\|_{L^1_{t,x_2}W^{\alpha,1}_{x_1}}\right)\varepsilon^\alpha \quad \text{and} \quad \psi(\varepsilon) \lesssim \left(\|b_2\|_{L^1_{t,x_2}W^{\alpha,1}_{x_1}}\right)\varepsilon^{\alpha-1}. \tag{5.3.11}$$

Therefore

$$\begin{aligned}&\mathcal{L}^N\left(B_r \cap \left\{|X(s,\cdot) - \bar{X}(s,\cdot)| > \gamma\right\}\right)\\ &\lesssim_\lambda \frac{\|b - \bar{b}\|_{L^1(\Omega')}}{\delta_1\log(1+\frac{\gamma}{\delta_2})} + \frac{\sigma(\varepsilon)+\bar{\sigma}(\varepsilon)}{\delta_1\log(1+\frac{\gamma}{\delta_2})} + \frac{\frac{\delta_1}{\delta_2}\psi(\varepsilon)\log\left(\frac{1}{\frac{\delta_1}{\delta_2}\psi(\varepsilon)\delta_1}\right)}{\log(1+\frac{\gamma}{\delta_2})} + \frac{C}{\log(1+\frac{\gamma}{\delta_2})}\\ &\qquad + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus \bar{G}_\lambda)\\ &\lesssim \frac{\|b - \bar{b}\|_{L^1((0,T)\times B_\lambda)}}{\delta_1\log(1+\frac{\gamma}{\delta_2})} + \frac{\varepsilon^\alpha}{\delta_1\log(1+\frac{\gamma}{\delta_2})} + \frac{\frac{\delta_1}{\delta_2}\varepsilon^{\alpha-1}\log\left(\frac{1}{\frac{\delta_1}{\delta_2}\varepsilon^{\alpha-1}\delta_1}\right)}{\log(1+\frac{\gamma}{\delta_2})} + \frac{C}{\log(1+\frac{\gamma}{\delta_2})}\\ &\qquad + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus \bar{G}_\lambda)\\ &= \frac{\|b - \bar{b}\|_{L^1((0,T)\times B_\lambda)}}{\delta_1\log(1+\frac{\gamma}{\delta_2})} + 1) + 2) + 3) + 4) + 5)\,.\end{aligned}$$

Step 6: *Choice of the parameters and conclusion.* Fix $\eta > 0$. By choosing λ sufficiently large we can make $4) + 5) \leq 2\eta/5$.

Define

$$\beta = \frac{\delta_1}{\delta_2} \ll 1\,, \qquad \text{so that } \delta_1 = \beta\delta_2.$$

We need to choose $\varepsilon > 0$, $\beta > 0$, and $\delta_2 > 0$ in such a way that

$$1) + 2) + 3) = \frac{\varepsilon^\alpha}{\beta\delta_2\log(1+\frac{\gamma}{\delta_2})} + \frac{\beta\varepsilon^{\alpha-1}\log\left(\frac{1}{\beta^2\varepsilon^{\alpha-1}\delta_2}\right)}{\log(1+\frac{\gamma}{\delta_2})} + \frac{C}{\log(1+\frac{\gamma}{\delta_2})} \leq \frac{3\eta}{5}\,.$$

The term 3) can be made smaller than $\eta/5$ by choosing $\delta_2 > 0$ sufficiently small. We fix $0 < \mu < 1$ to be determined later (depending on the exponent $\alpha > 1/2$ in assumption **(R2)** only) and choose $\varepsilon > 0$ such that

$$\varepsilon^{\alpha-1} = \beta^{\mu-1}, \qquad \text{that is, } \varepsilon = \beta^{\frac{1-\mu}{1-\alpha}}.$$

In this way we get

$$2) = \frac{\beta^{\mu} \log\left(\frac{1}{\beta^{\mu+1}\delta_2}\right)}{\log(1+\frac{\gamma}{\delta_2})} = \frac{\beta^{\mu} \log\left(\frac{1}{\beta^{\mu+1}}\right)}{\log(1+\frac{\gamma}{\delta_2})} + \frac{\beta^{\mu} \log\left(\frac{1}{\delta_2}\right)}{\log(1+\frac{\gamma}{\delta_2})},$$

which can be made smaller that $\eta/5$ if $\beta > 0$ is chosen to be small enough.

With the above choices the term 1) becomes

$$1) = \frac{\beta^{\frac{1-\mu}{1-\alpha}\alpha}}{\beta\delta_2 \log(1+\frac{\gamma}{\delta_2})} = \frac{\beta^{\frac{2\alpha-\alpha\mu-1}{1-\alpha}}}{\delta_2 \log(1+\frac{\gamma}{\delta_2})},$$

which can be made smaller than $\eta/5$ by a suitable choice of $\beta > 0$, provided the exponent of β at the numerator is positive, that is,

$$\frac{2\alpha - \alpha\mu - 1}{1-\alpha} > 0 \qquad \Longleftrightarrow \qquad \alpha > \frac{1}{2-\mu}. \tag{5.3.12}$$

Since $\alpha > 1/2$, we see that we can choose $\mu > 0$ small enough in such a way that (5.3.12) holds. This gives $1) + 2) + 3) + 4) + 5) \leq \eta$ and therefore concludes the proof. □

5.3.3 Well-posedness and further properties of the Lagrangian flow

Estimate (5.3.4) in Theorem 5.3.1 is the key information which guarantees existence, uniqueness, and stability of the regular Lagrangian flow. The proof of these results as a consequence of estimate (5.3.4) is by now quite standard, see the theory developed in [24, 15, 11]. We begin with the uniqueness.

Corollary 5.3.2 (Uniqueness)**.** *Let b be a vector field satisfying assumptions* **(R1)***,* **(R2)***, and* **(R3)***. Then, the regular Lagrangian flow associated to b, if it exists, is unique.*

It is indeed very easy to see that uniqueness follows from estimate (5.3.4). We consider $b = \bar{b}$, then the right hand side of (5.3.4) can be made arbitrarily small, for any $\gamma > 0$ fixed. This readily implies uniqueness.

Remark 17. We observe that, in contrast to the PDE theory in [35, 37, 38], no assumptions on the divergence of the vector field are required for the uniqueness of the regular Lagrangian flow. The divergence will play a role for the existence only.

The main advantage of the quantitative theory of ODEs, in contrast to the PDE theory, is that it provides an explicit rate for the compactness and the stability, depending on the uniform bounds that are assumed on the sequence of vector fields. The following two results can be proven arguing as in [15], as a consequence of the main estimate (5.3.4).

Corollary 5.3.3 (Stability)**.** *Let $\{b_n\}$ be a sequence of vector fields satisfying assumption* **(R1)***, converging in $L^1_{\rm loc}([0,T] \times \mathbb{R}^N)$ to a vector field b which satisfies assumptions* **(R1)***,* **(R2)***, and* **(R3)***. Assume that there exist X_n and X regular Lagrangian flows associated to b_n and b respectively, and denote by L_n and L the compressibility constants of the flows. Suppose that:*

- *For some decomposition $b_n/(1+|x|) = c_{n,1} + c_{n,2}$ as in assumption* **(R1)***, we have that*

$$\|c_{n,1}\|_{L^1_t(L^1_x)} + \|c_{n,2}\|_{L^1_t(L^\infty_x)} \quad \textit{is equi-bounded;}$$

- *The sequence $\{L_n\}$ is equi-bounded;*
- *The norm of $b_{n,2}(s, x_1, x_2)$ in $L^1\big((0,T)\times\mathbb{R}^{n_2}_{x_2}; W^{\alpha,1}_{x_1}(\mathbb{R}^{n_1})\big)$ is equi-bounded.*

Then the sequence $\{X_n\}$ converges to X locally in measure in $\mathbb{R}^N$, uniformly with respect to time.

In the above corollary, the assumption in the third bullet is necessary in order to have a uniform estimate on the quantity $\sigma_n(\varepsilon)$ associated to b_n (as in the proof of Theorem 5.3.1).

Corollary 5.3.4 (Compactness)**.** *Let $\{b_n\}$ be a sequence of vector fields satisfying assumption* **(R1)***,* **(R2)***, and* **(R3)***, converging in $L^1_{\rm loc}([0,T]\times\mathbb{R}^N)$ to a vector field b which satisfies assumptions* **(R1)***,* **(R2)***, and* **(R3)***. Assume that there exist X_n regular Lagrangian flows associated to b_n, and denote by L_n the compressibility constants of the flows. Suppose that:*

- *For some decomposition $b_n/(1+|x|) = c_{n,1} + c_{n,2}$ as in assumption* **(R1)***, we have that*

$$\|c_{n,1}\|_{L^1_t(L^1_x)} + \|c_{n,2}\|_{L^1_t(L^\infty_x)} \quad \textit{is equi-bounded;}$$

- *The sequence $\{L_n\}$ is equi-bounded;*
- *The norms of the vector fields $\{b_n\}$ involved in the assumptions* **(R2)** *and* **(R3)** *are equi-bounded.*

Then the sequence $\{X_n\}$ is pre-compact locally in measure in $\mathbb{R}^N$, uniformly with respect to time, and converges to a regular Lagrangian flow X associated to b.

By a simple regularization procedure Corollary 5.3.4 implies existence of the regular Lagrangian flow, under the assumption of boundedness of the divergence of the vector field. Such an assumption is needed in order to have equi-boundedness of the compressibility constants for the sequence of approximated regular Lagrangian flows X_n in Corollary 5.3.4.

Corollary 5.3.5 (Existence)**.** *Let b be a vector field satisfying assumptions* **(R1)***,* **(R2)***, and* **(R3)***. Assume that the (distributional) spatial divergence of b is bounded. Then, there exists a regular Lagrangian flow associated to b.*

Remark 18. Arguing as in [15], it is also possible to develop a theory of Lagrangian solutions of the continuity equations, that is, solutions that are transported by the regular Lagrangian flow.

5.3.4 Remarks and possible extensions

We conclude by listing a few remarks and questions concerning the results and the approach in this work:

(1) The same proof for Theorem 5.3.1 works if we assume only local regularity bounds in assumption **(R2)**. We omitted this just for simplicity of notation.

(2) Compared to the PDE theory in [35, 37, 38], we need to assume some fractional Sobolev regularity of b_2 with respect to the variable x_1. This seems unavoidable for our strategy of proof, since we cannot send to zero the two parameters δ_1 and δ_2 one after the other, but we rather need to send them together to zero, under a condition on their ratio $\beta = \delta_1/\delta_2$. Is it possible to modify our proof and remove this assumption, that is, is it possible to derive an estimate like (5.3.4) under the only assumption of integrable depencence of b_2 with respect to x_1?

(3) Is it possible to treat the case $p = 1$ in assumption **(R2)**? We briefly explain here what is the obstruction with the present approach. In the case $p = 1$, in Step 4 of the main proof the operators $U_{\tilde{r}^\varepsilon}$ and $U_{\tilde{p}}$ cannot be directly estimated in L^1 as in (5.3.9) and (5.3.10) (recall Lemma 5.2.9). One needs to argue as done in the same step for $U_{\tilde{q}^\varepsilon}$ exploiting the equi-integrability and the interpolation from Lemma 5.2.10. After some computations we would obtain that, for every $\theta > 0$, there is a constant $C_\theta > 0$ so that the term

$$\frac{C}{\log(1 + \frac{\gamma}{\delta_2})}$$

in the last estimate at the end of Step 5 is replaced by the sum

$$\frac{\theta \log\left(\frac{1}{\theta\beta\delta_2}\right)}{\log(1 + \frac{\gamma}{\delta_2})} + \frac{C_\theta}{\log(1 + \frac{\gamma}{\delta_2})} .$$

We need to make also this sum small, exploiting the arbitrariness of θ. We see that, in order to make the first term small, we need to take θ coupled to β. Choosing $\varepsilon^{\alpha-1} = \beta^{\mu-1}$ as in the proof of Theorem 5.3.1, we see that we still have β and δ_2 as free parameters, and eventually we need to make small the sum

$$\frac{\beta^{\frac{2\alpha-\alpha\mu-1}{1-\alpha}}}{\delta_2 \log(1 + \frac{\gamma}{\delta_2})} + \frac{C_\beta}{\log(1 + \frac{\gamma}{\delta_2})}$$

(as now θ is coupled to β). However, since C_β blows up for $\beta \to 0$ (depending on the equi-integrability rate), with this strategy there is in general no choice of such parameters which makes the last sum small.

(4) Can one relax the strong requirement that b_1 does not depend on the variable x_2, and require instead (for instance) that b_1 has a smooth dependence on x_2?

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