# Symmetry of Ground States
# by
# Fourier Rearrangement

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Lars Bugiera
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auf Antrag von

Prof. Dr. Enno Lenzmann
Prof. Dr. Tobias Weth

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<div style="text-align: right">

Prof. Dr. Martin Spiess
Dekan

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*If you really want to be strong... Stop caring about what your surrounding thinks of you!*

Saitama
*A Hero for Fun*

# Acknowledgements

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# Chapter 1

# Motivation and Introduction

The aim of this thesis is to give a comprehensive guideline to symmetry and uniqueness of solutions to various linear and nonlinear PDEs which involve a very general pseudo-differential operator. The main motivation for such results originated in Enno Lenzmann's and Jérémy Sok's article in [24].

The focus of this work lies on the following two recent articles [5] and [6]. Both papers revolve around very similar techniques which will be explained thoroughly in the following pages. This thesis is at it's core a step by step guide to fully grasp the ideas and conclusions of the results in [5] and [6].

In [6] we consider dispersion generalized nonlinear Schrödinger equations of the form

$$i\partial_t u = P(D)u - |u|^{2\sigma}u \tag{gNLS}$$

where $P(D)$ is a very general pseudo-differential operator. In the case of $P(D) = (-\Delta)^s$ this equation naturally occurs as a continuum limit of a discrete model with long-term lattice interactions. A very specific example is given from the point of mathematical biology. We could consider the charge transport in a DNA strand. A possible model for such an object would be the 1-dimensional lattice $h\mathbb{Z}$ with given mesh size $h > 0$. This resembles the distance between the base pairs, whereas those sit on lattice points $x_m := hm$ with $m \in \mathbb{Z}$. As the DNA strand is twisted in a very complicated and somewhat random way it is plausible to think about interactions between each base pairs, hence a long-term interaction. Then we consider a discrete wave function $u_h : \mathbb{R} \times h\mathbb{Z} \to \mathbb{C}$ that satisfies the following discrete nonlocal Schrödinger equation

$$i\frac{d}{dt}u_h(t, x_m) = h \sum_{n \neq m} \frac{u_h(t, x_m) - u_h(t, x_n)}{|x_m - x_n|^{2s+1}} \pm |u_h(t, x_m)|^2 u_h(t, x_m).$$

As long as the interaction term is not too strong the authors showed that solutions of the discrete model converge in a weak sense to solutions of

$$i\partial_t u = (-\Delta)^s u \pm |u|^{2\sigma}u.$$

as the mesh size $h > 0$ of the lattice tends to $0^+$.

In particular, we are interested in traveling solitary waves for gNLS. From a historical point of view, John Scott Russel described such a phenomenon for water waves in the following sense (see [32]):

'I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stoppednot so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called Wave of translation.'

In mathematical terms, a traveling solitary wave is a solutions of the form

$$u(t, x) = e^{i\omega t} Q_{\omega, \mathbf{v}}(x - \mathbf{v}t),$$

with some non-trivial profile $Q : \mathbb{R}^n \to \mathbb{C}$ depending on the given parameters $\omega \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$. Clearly, $\omega$ stands for the frequency of the wave and $\mathbf{v}$ for the velocity. In the case of $P(D) = -\Delta$ there exists a well-known gauge transform which enables us to only consider the case of a standing wave with $\mathbf{v} = 0$. This enables us to study symmetry properties of solutions quite easily. However, it is not known if such a boost transform exists for a more general operator, e.g. for the fractional Laplacian $(-\Delta)^s$.

From a more general point of view consider a functional $E : X \to \mathbb{R}$ defined on some Banach space of complex-valued functions $u : \mathbb{R}^d \to \mathbb{C}$. In many cases of interest the functional $E$ is rotationally and shift invariant, that means

$$E(e^{i\theta}u(\mathsf{R}\cdot)) = E(u)$$

for $\mathsf{R} \in \mathrm{O}(n)$ and $\theta \in \mathbb{R}$. A very natural question is whether optimizers $Q \in X$ of $E$ also share such an invariance property. As a basic model consider the functional

$$E(u) = \|\Delta u\|_{L^2(\mathbb{R}^n)}^2 - \|u\|_{L^p(\mathbb{R}^n)}^p, \tag{1.0.1}$$

where $u \in H^2(\mathbb{R}^n)$ is possibly complex-valued and the normalization $\|u\|_{L^2(\mathbb{R}^n)} = 1$ is assumed. Clearly, $E$ is radially symmetric, hence we ask ourselves whether minimizers of $E$ satisfy this property as well. Those kind of questions can usually be answer by the following three arguments (see [24] for more references)

(I) The Polya-Szegö inequality

$$\|\nabla u^*\|_{L^p(\mathbb{R}^n)} \leqslant \|\nabla u\|_{L^p(\mathbb{R}^n)},$$

where $u^*$ is the symmetric decreasing rearrangement of $u \in W^{1,p}(\mathbb{R}^n)$.

(II) The moving plane method for the corresponding Euler-Lagrange equation.

(III) The inequality given by
$$\|\nabla|u|\|_{L^p(\mathbb{R}^n)} \leqslant \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Neither of these arguments can be applied to (1.0.1). As an example, the corresponding Euler-Lagrange equation is given by a biharmonic nonlinear Schrödinger equation

$$\Delta^2 u + \lambda u - |u|^{p-2}u = 0,$$

where $\lambda > 0$ is some constant. The lack of a maximum principle for the operator $\Delta^2 + \lambda$ readily implies that the second argument does not work. In [24] questions on symmetries were answer with an approach called Fourier rearrangement, i.e. given $f \in L^2(\mathbb{R}^n)$ the Fourier rearrangement is given by

$$f^\sharp = \mathcal{F}^{-1}((\mathcal{F}(f))^*).$$

The main technical part in the proof is to classify the case of equality in the Hardy-Littlewood majorant problem in $\mathbb{R}^n$ for the $L^p$-norms with $p \in 2\mathbb{N} \cup \{\infty\}$. This will be heavily accompanied by the property that the set $\{|\widehat{u}| > 0\}$ is connected in $\mathbb{R}^n$. As a matter of fact, this holds true since $|\widehat{u}| = (\widehat{u})^*$.

Later, the authors of [24] asked the natural question if such techniques exist for non-radial Fourier multipliers. Those questions will be the main guidance in this thesis and are thoroughly discussed in Chapter 8. Again, taking a look at (gNLS) and considering traveling solitary waves we can ask the following:

**Question.** *Up to spacial translation and complex phase, i.e. replacing the traveling solitary wave $Q_{\omega,\mathbf{v}}$ by $e^{i\theta}Q_{\omega,\mathbf{v}}(\cdot + x_0)$ with constant phase shift $\theta \in \mathbb{R}$ and translation $x_0 \in \mathbb{R}$, do we have the following symmetries?*

*(S1) $Q_{\omega,\mathbf{v}}$ is **cylindrically symmetric with respect to** $\mathbf{v} \in \mathbb{R}^n$, $n \geqslant 2$, i.e., we have*

$$Q_{\omega,\mathbf{v}}(x) = Q_{\omega,\mathbf{v}}(\mathsf{R}x) \quad \text{for all } \mathsf{R} \in \mathrm{O}(n) \text{ with } \mathsf{R}\mathbf{v} = \mathbf{v}.$$

*(S2) We have the **conjugation symmetry** given by*

$$Q_{\omega,\mathbf{v}}(x) = \overline{Q_{\omega,\mathbf{v}}(-x)}.$$

*That is, $\operatorname{Re} Q_{\omega,\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ is an even function, whereas $\operatorname{Im} Q_{\omega,\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ is an odd function.*

Both symmetry questions will be studied in Chapter 8 and answered for the case of boosted ground states. Those are special solutions to the traveling solitary wave equation

$$P(D)Q_{\omega,\mathbf{v}} + i v \cdot \nabla Q_{\omega,\mathbf{v}} + \omega Q_{\omega,\mathbf{v}} - |Q_{\omega,\mathbf{v}}|^{2\sigma}Q_{\omega,\mathbf{v}} = 0, \qquad (1.0.2)$$

which in addition are obtained as optimizers for a certain variational problem. The arguments are based on rearrangement techniques introduced in Chapter 3 but instead of doing everything in $x$-space, we perform a symmetrization in Fourier space. For $n \geqslant 2$ we can extend the Fourier rearrangement to a Fourier Steiner rearrangement in codimension $n-1$, this is given by

$$u^{\sharp_\mathbf{e}} := \mathcal{F}^{-1}(\mathcal{F}(u)^{*_\mathbf{e}}), \qquad (1.0.3)$$

where $u \in L^2(\mathbb{R}^n)$. In Chapter 8 many properties of this symmetrization are mentioned and proven. The main focus definitely lies on the cylindrical symmetry and closely follows the results from [24]. Note that in one space dimension the question on cylindrical symmetry becomes void but the conjugation symmetry is still valid. A proper symmetrization in that case is given by

$$f^\bullet = \mathcal{F}^{-1}(|\mathcal{F}f|),$$

where $f \in L^2(\mathbb{R})$. Clearly, this symmetrization concept can easily be generalized for the higher dimensional cases. This will be extensively done in Chapter 6.

In both articles, [5] and [6], the arguments are heavily dependent on a topological feature of the set $\{|\widehat{Q_{\omega,\mathbf{v}}}| > 0\}$, i.e. it has to be connected. The core lemma in most of the symmetry results is the following (see [24] for an in-depth discussion):

**Lemma** (Equality in the Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$)**.** *Let $n \geqslant 1$ and $p \in 2\mathbb{N} \cup \{\infty\}$ with $p > 2$. Suppose that $f, g \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$ with $1/p + 1/p' = 1$ satisfy the majorant condition*

$$|\widehat{f}(\xi)| \leqslant \widehat{g}(\xi) \quad \text{for a. e. } \xi \in \mathbb{R}^n.$$

*In addition, we assume that $\widehat{f}$ is continuous and that $\{\xi \in \mathbb{R}^n : |\widehat{f}(\xi)| > 0\}$ is a connected set. Then equality*

$$\|f\|_{L^p} = \|g\|_{L^p}$$

*holds if and only if*

$$\widehat{f}(\xi) = e^{\mathrm{i}(\alpha + \beta \cdot \xi)} \widehat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

*with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$.*

Clearly, the approaches on proving connectedness vary vastly between Chapter 6 and Chapter 8. This can be exemplified via one-dimensional half wave equations of the form

$$i\partial_t u = |\nabla| u - |u|^{2\sigma} u, \tag{HW}$$

where $u : [0, T) \times \mathbb{R} \to \mathbb{C}$ and $\sigma \in \mathbb{N}$. Again, consider boosted ground states (see Chapter 8) $Q_{\omega,\mathbf{v}} \in H^{1/2}(\mathbb{R})$ for (HW). The go-to approach in [5] to show that $\{|\widehat{Q_{\omega,\mathbf{v}}}| > 0\}$ is connected comes from analyticity arguments and Paley-Wiener theory. Clearly, the Fourier symbol of the operator $|\nabla|$ is not analytic and thus we cannot use the arguments from Chapter 6. Instead we can study a Minkowski sum of an open set to conclude a symmetry result. This small lemma is interesting on it's own and shows again how topological aspects play a key role:

**Lemma.** *Let $\Omega \subseteq \mathbb{R}$ be open and not empty. Assume that*

$$\Omega = \bigoplus_{j=1}^{m} \Omega$$

*for some $m \geqslant 2$. Then it holds that*

$$\Omega \in \{\mathbb{R}_{>0}, \mathbb{R}_{<0}, \mathbb{R}\}.$$

Note that openness is absolutely crucial, otherwise $\mathbb{Z}$ would be another solution. In higher dimensions one might still conjecture that $\Omega$ is connected.

Last but not least, in Chapter 6 we study linear Schrödinger equations of the form

$$P(D)\psi + V\psi = E\psi,$$

where $V$ is a given potential and $E$ an eigenvalue. The operator $P(D)$ stands for a self-adjoint, elliptic constant coefficient pseudo-differential operator of order $2s$. Again, we are interested in existence and symmetry questions concerning ground state solutions. In the case of $P(D) = -\Delta$ many results are already known and proofs involve the corresponding heat kernel $e^{t\Delta}$. For higher order operators, e.g. $2s > 1$, uniqueness of ground states might even fail. Under very natural assumption, for example that the Fourier transform of the potential $V$ is negative and $V \in L^{\infty}(\mathbb{R}^n)$, we can still show existence and conjugation symmetry of solutions. This will be done by using the following symmetrization

$$f^{\bullet} = \mathcal{F}^{-1}(|\mathcal{F}f|).$$

In the end, this will be a simple phase retrieval problem, i.e. given the modulus of the Fourier transform of a function one tries to reconstruct its phase. In that case we don't need any advanced argument which are otherwise crucial in the nonlinear case.

# Chapter 2

# Content and Structure

The first parts of this thesis are given by two motivational chapters. These serve the purpose of giving a short introduction to the main ideas behind some symmetry results. After those the main articles given by [5] and [6] are included, each of those chapters is preceded by a simple guideline giving a short breakdown of all ideas.

## Chapter 3
## -
## Classical Results on Rearrangements

We begin by introducing the main ideas behind most symmetry results in Chapter 8. This readily leads us to understand the symmetric decreasing rearrangement in $\mathbb{R}^n$ and some slight modifications of it, e.g. Schwarz and Steiner symmetrization. Clearly, the aim of this chapter is a step by step proof of the Brascamp-Lieb-Luttinger inequality (see [8]) given by

**Theorem** (Brascamp-Lieb-Luttinger Inequality)**.** *Let $(f_j)_{j \in \mathbb{N}_m}$ be a sequence of nonnegative functions on $\mathbb{R}^n$, vanishing at infinity. Let $k \leqslant m$ and let $A = (a_{ij})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_m}$ be a matrix. Consider*

$$I(f_1, \ldots, f_m) := \int_{(\mathbb{R}^n)^k} \prod_{j=1}^{m} f_j \left( \sum_{i=1}^{k} a_{ij} x_i \right) dx_1 \cdots dx_k.$$

*Then $I(f_1, \ldots, f_m) \leqslant I(f_1^*, \ldots, f_m^*)$.*

## Chapter 4
## -
## Positive Definite Functions

Similar to Chapter 3, we give a basic understanding for properties of positive definite functions. To fully grasp how impactful such a simple generalization of postive definite matrices is we include a proof of Bochner's Theorem.

**Theorem** (Bochner's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous. $f$ is positive semi-definite if and only if there exists a nonnegative Borel measure $\mu$ on $\mathbb{R}^n$ such that*

$$f(x) = \mathcal{F}(\mu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot x} d\mu(\xi).$$

Clearly, many of the smaller results leading to this exact theorem are included and sometimes a simple proof is sketched to get a feeling for many useful properties. Naturally, positive definite functions occur in Chapter 6 and Chapter 8 under very simple assumptions, e.g. if $f \in L^2(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then $f^\bullet : \mathbb{R}^n \to \mathbb{C}$ is continuous, bounded and positive definite.

## Chapter 5
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## A Guideline on Symmetry and Ground States

This chapter is included for the sole purpose of giving a short yet understandable overview to one of the two main articles included in this thesis (see Chapter 6). We introduce the notion of ground states for a wide class of linear and nonlinear PDEs. Outlines of many proofs are included and heavily use the introductory results in Chapter 3 and Chapter 4.

Additionally, we give a gentle introduction to the Hardy-Littlewood majorant property and include a counterexample in the case of the real line $\mathbb{R}$ (see Section 5.3).

**Theorem.** *Suppose $p > 2$ is not an even integer, then there are trigonometric polynomials $P$ and $Q$ with coefficients in $\{-1, 0, 1\}$ such that $|\widehat{P_n}| = \widehat{Q_n}$ and*

$$\|P_\lambda\|_{L^p(\mathbb{R})} > (1 + C)\|Q_\lambda\|_{L^p(\mathbb{R})}$$

*with*

$$|\widehat{P_\lambda}| \leqslant \widehat{Q_\lambda},$$

*where $P_\lambda$ (resp. $Q_\lambda$) is the extension to $\mathbb{R}$ (see Section 5.3) and $C = C(p)$ is a constant only dependent on $p$.*

## Chapter 6
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## On Symmetry and Uniqueness of Ground States for Linear and Nonlinear Elliptic PDEs

This chapter is a direct copy of the article [5] with some minor changes due to formatting.

The aim of this article is to give various uniqueness and symmetry results for ground states that arise from a wide class of linear and nonlinear elliptic PDEs. Instead of using classical methods, we take an approach by Fourier methods, i.e. we consider the following symmetrization

$$f^\bullet = \mathcal{F}^{-1}(|\mathcal{F}f|) \tag{2.0.1}$$

for functions $f \in L^2(\mathbb{R}^n)$. Following some ideas introduced in a recent paper (see [24]) we can conclude an interesting symmetry result.

**Theorem** (Symmetry for Nonlinear Ground States)**.** *Let $n \geqslant 1$, $s > 0$, and $\sigma \in \mathbb{N}$ with $1 \leqslant \sigma < \sigma_*(s, n)$. Suppose $Q \in H^s(\mathbb{R}^n) \backslash \{0\}$ is a ground state solution of (8.1.3) where $\lambda \in \mathbb{R}$ satisfies (6.1.6). Finally, we assume that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$. Then it holds that*

$$Q(x) = e^{i\alpha}Q^\bullet(x + x_0)$$

with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Here $Q^\bullet : \mathbb{R}^d \to \mathbb{C}$ is a smooth, bounded, and positive definite function in the sense of Bochner. As a consequence, it holds that

$$Q^\bullet(-x) = \overline{Q^\bullet(x)} \quad and \quad Q^\bullet(0) \geqslant |Q^\bullet(x)| \quad for \ all \ x \in \mathbb{R}^n.$$

If, in addition, the operator $P(D)$ has an even symbol $p(\xi) = p(-\xi)$, the function $Q^\bullet$ must be real-valued (up to a trivial constant complex phase). Consequently, any ground state $Q$ for (8.1.3) is real and even, i. e., we have $Q(-x) = Q(x)$ for all $x \in \mathbb{R}^n$.


# Chapter 7
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## A Guideline on Symmetry for Traveling Solitary Waves

Similar to Chapter 5, this part of the thesis serves as an overview and introduction as well. Many of the main results in Chapter 8 will be discussed and outlines of proofs will be given.

Additionally, we include a counterexample when imposing non-connectedness on a certain level set. The argument is based on the recent article [24] but adapted to our symmetrization. Let $f$ be a special function (for a detailed construction see Section 7.5). Then we can conclude that $\|f\|_{L^4(\mathbb{R})} = \|f^{\sharp_1}\|_{L^4(\mathbb{R})}$ and

$$\mathcal{F}(f) = e^{i\vartheta}\mathcal{F}(f)^{*_1},$$

where the phase function $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ does not need to be affine in general.

Last but not least, a small section on a numerical scheme will serve the purpose of a visualization. The scheme will be based on a spectral renormalization method which is found in [12].


# Chapter 8
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## On Symmetry for Traveling Solitary Waves for Dispersion Generalized NLS

The last chapter in this thesis contains the article given in [6]. As in Chapter 6, some changes are made due to formatting.

The main results consider a class of dispersion generalized nonlinear Schrödinger equations of the form

$$i\partial_t u = P(D)u - |u|^{2\sigma}u,$$

where $P(D)$ denotes a pseudo-differential operator of proper order. Symmetry results for traveling solitary waves with $\sigma \in \mathbb{N}$ are proven with arguments based on a Steiner type rearrangement. One of the main results dealing conjugation symmetry and cylindrical symmetry is the following:

**Theorem** (Symmetry of Boosted Ground States for $n \geqslant 2$)**.** *Let $n \geqslant 2$ and suppose $P(D)$ satisfies Assumptions 4 and 5 with some $s \geqslant \frac{1}{2}$ and $\mathbf{e} \in \mathbb{S}^{n-1}$. Furthermore, let $\mathbf{v} = |\mathbf{v}|\mathbf{e} \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ satisfy the hypotheses in Theorem 8.1.1 and assume $\sigma \in \mathbb{N}$ is an integer with $0 < \sigma < \sigma_*(n,s)$.*

*Then any boosted ground state $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R}^n)$ is of the form*

$$Q_{\omega,\mathbf{v}}(x) = e^{i\alpha}Q^{\sharp\mathbf{e}}(x + x_0)$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. As a consequence, any such $Q_{\omega,\mathbf{v}}$ satisfies (up to a translation and phase) the symmetry properties (P1) and (P2) for almost every $x \in \mathbb{R}^n$.*

# Chapter 3

# Classical Results on Rearrangements

This chapter is devoted to classical results concerning the symmetric decreasing rearrangement which is found in [25]. We will recall some of the basic ideas and definitions, including the Steiner and Schwarz symmetrization. Eventually, a proof of the Brascamp-Lieb-Luttinger inequality will be our goal. The main inspiration for this is found in [8]. We will prove this result in several clear steps and follow the original article quite closely. This serves as a gentle introduction yet giving a very deep and useful result.

In [6] we will extend this idea and use methods that were first investigated in [24]. The following introduction will be very useful to the reader and contains all the techniques needed to successfully understand many ideas in Chapter 8.

## 3.1  Preliminaries and Basic Results

In the following, let $A \subseteq \mathbb{R}^n$ be a Borel measurable set of finite Lebesgue measure. We define $A^*$ as the **symmetric rearrangement** of the set $A$ as the open ball around the origin whose volume is equal to the volume of $A$. To be more specific we have

$$A^* = B_r(0) \quad \text{with} \quad \mathcal{L}^n(A) = \frac{|\mathbb{S}^{n-1}|}{n} r^n,$$

where $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$. One of the main tools in the following results is clearly the layer cake principle. A proof will be included for the sake of completeness.

**Lemma 3.1.1** (Layer Cake Principle). *Let $\nu$ be a measure on the Borel sets of $[0, \infty)$ such that $\varphi(t) := \nu([0, t))$ is finite for all $t > 0$. Let $(\Omega, \Sigma, \mu)$ be any measure space and $f$ a nonnegative measurable function on $\Omega$. Then*

$$\int_\Omega (\varphi \circ f)(x)\mu(dx) = \int_0^\infty \mu(L(f, t))\nu(dt),$$

*where $L(f, t) := \{x \in \mathbb{R}^n | \ f(x) > t\}$ is the superlevel set of $f$ with respect to $t$.*

*Proof.* It's easy to see that

$$\int_0^\infty \mu(L(f, t))\nu(dt) = \int_0^\infty \int_\Omega \chi_{L(f,t)}(x)\mu(dx)\nu(dt) = \int_\Omega \int_0^\infty \chi_{L(f,t)}(x)\nu(dt)\mu(dx)$$

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by Fubini's theorem. The rest follows from rewriting the integral using the definition of the charactic function. $\qquad\square$

**Remark.** *We can generalize Lemma 3.1.1 by the use of a signed measure, e.g. $\nu = \nu_1 - \nu_2$, where $\nu_1$ and $\nu_2$ are positive measures on the positive real half-line $[0, \infty)$. But then one needs to add the following assumption: Either*

$$\int_0^\infty \mu(L(f,t))\nu_1(dt) < +\infty \quad or \quad \int_0^\infty \mu(L(f,t))\nu_2(dt) < +\infty.$$

The layer cake principle from Lemma 3.1.1 allows us to widen the notion of rearrangements from Borel sets in $\mathbb{R}^n$ with finite measure to functions which have similar properties concerning their superlevel sets.

**Definition 3.1.1.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function. We say that $f$ is **vanishing at infinity** if*

$$\mathcal{L}^n(L(|f|,t)) < +\infty \quad for \ all \ t > 0,$$

*where $L(|f|, t) := \{x \in \mathbb{R}^n | \ f(x) > t\}$ is the superlevel set with respect to $t$.*

Before giving the full definition for a rearrangement of a proper function we want to take a look at the simplest one. We define the **symmetric-decreasing rearrangement** of the characteristic function of the set $A$ as

$$\chi_A^* = \chi_{A*}.$$

Now, a completely natural extension using the mentioned layer cake principle in Lemma 3.1.1 is the following.

**Definition 3.1.2.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be Borel measurable and vanishing at infinity. We define the **symmetric-decreasing rearrangement** of $f$ as*

$$f^*(x) := \int_0^\infty \chi_{L(|f|,t)}^*(x)dt. \tag{3.1.1}$$

This symmetrization has a few simple yet important properties. Those are listed in the following lemma, but we will prove only a hand full of them as most techniques are very similar.

**Lemma 3.1.2** (Rearrangement Properties)**.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be Borel measurable and vanishing at infinity, then*

(i) *$f^*$ is nonnegative.*

(ii) *$f^*$ is radially symmetric and nonincreasing.*

(iii) *$f^*$ is lower semi-continuous and $L(f^*, t) = L(|f|, t)^*$ for each $t > 0$.*

(iv) *If $\varphi := \varphi_1 - \varphi_2$ is the difference of two monotone functions $\varphi_1$ and $\varphi_2$ such that $\int_{\mathbb{R}^n} (\varphi_1 \circ f)(x)dx < +\infty$ or $\int_{\mathbb{R}^n} (\varphi_2 \circ f)(x)dx < +\infty$, then*

$$\int_{\mathbb{R}^n} (\varphi \circ |f|)(x)dx = \int_{\mathbb{R}^n} (\varphi \circ f^*)(x)dx.$$

*(v)* *If $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, then $(\Psi \circ |f|)^* = \Psi \circ f^*$.*

*(vi)* *The symmetric-decreasing rearrangement is order preserving, i.e. suppose $g$ has the same regularity as $f$, then*

$$f(x) \leqslant g(x) \quad \forall x \in \mathbb{R}^n \Longrightarrow f^*(x) \leqslant g^*(x) \quad \forall x \in \mathbb{R}^n.$$

*Proof.* We would like to show that $L(f^*, t) = L(|f|, t)^*$ for each $t > 0$. The other statements follow in a similar fashion using the basic definition of a symmetric-decreasing rearrangement in (3.1.1) and the layer cake principle from Lemma 3.1.1.

Fix $t > 0$ and let $q \in L(|f|, t)^*$, then for all $s \in (0, t]$ we have

$$L(|f|, t) \subseteq L(|f|, s)$$

and hence it follows that

$$L(|f|, t)^* \subseteq L(|f|, s)^*.$$

Clearly, from this we deduce that $\chi_{L(|f|,s)*}(q) = 1$ for all $s \in (0, t]$ and therefore $f^*(q) > t$ which implies

$$L(|f|, t)^* \subseteq L(f^*, t).$$

For the other inclusion we assume that $q \notin L(|f|, t)^*$. Next for all $s > 0$ such that $q \in L(|f|, s)^*$ we conclude that $0 < s \leqslant t$ holds true. Hence $f^*(q) \leqslant t$, which readily gives $q \notin L(f^*, t)$. Upon taking the complement in $\mathbb{R}^n$ we find

$$L(f^*, t) \subseteq L(|f|, t)^*.$$

$\square$

## 3.2 Basic Inequalities for Rearrangements

In this section we recall some of the most basic inequalities dealing with the symmetric-decreasing rearrangement (e.g. see [25]). With the given theorems it will be easy to understand how the Brascamp-Lieb-Luttinger inequality in Section 3.2.1 emerges, which is ultimately proven within the last part of Chapter 3. The following introduction will give a clear idea on how those inequalities work but many techniques in proving those will be provided later in the proof of Theorem 3.2.4.

The first theorem is the most basic version of an inequality containing symmetric-decreasing rearrangements. This is the foundation and even serves as a guideline for Chapter 8.

**Theorem 3.2.1.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative and vanishing at infinity. Then the following inequality holds*

$$\int_{\mathbb{R}^n} f(x)g(x)dx \leqslant \int_{\mathbb{R}^n} f^*(x)g^*(x)dx.$$

For the next theorem we notice the following generalization: Let $f$ and $g$ be nonnegative functions in $L^2(\mathbb{R}^n)$ then we have

$$\int_{\mathbb{R}^n} |f^*(x) - g^*(x)|^2 dx \leqslant \int_{\mathbb{R}^n} |f(x) - g(x)|^2 dx.$$

This readily follows from applying Theorem 3.2.1. Clearly, an obvious generalization would be

$$\|f^* - g^*\|_{L^p(\mathbb{R}^n)} \leqslant \|f - g\|_{L^p(\mathbb{R}^n)},$$

which actually turns out to be true as well. We henceforth say that the symmetric-decreasing rearrangement from Definition 3.1.1 is non-expansive on $L^p(\mathbb{R}^n)$. This fact follows basically from the convexity of $|\cdot|^p$. The following theorem proves this fact and even gives a slightly more general result.

**Theorem 3.2.2.** *Let $f$ and $g$ be nonnegative functions on $\mathbb{R}^n$ which are vanishing at infinity. Additionally assume that $J : \mathbb{R} \to \mathbb{R}$ is a nonnegative convex function with $J(0) = 0$. Then*

$$\int_{\mathbb{R}^n} J \circ (f^* - g^*)(x) dx \leqslant \int_{\mathbb{R}^n} J \circ (f - g)(x) dx.$$

The next result is basically the prototype for the Brascamp-Lieb-Luttinger inequality in Section 3.2.1 and uses convolutions instead of simple products. It's called the Riesz inequality (see [25]).

**Theorem 3.2.3** (Riesz inequality)**.** *Let $f$, $g$ and $h$ be nonnegative functions on $\mathbb{R}^n$ which are vanishing at infinity. Then we have*

$$\int_{\mathbb{R}^n} f(x)(g * h)(x) dx \leqslant \int_{\mathbb{R}^n} f^*(x)(g^* * h^*)(x) dx.$$

### 3.2.1 The Brascamp-Lieb-Luttinger Inequality

In this part of the introduction we are dealing with a step by step proof of the Brascamp-Lieb-Luttinger inequality given in Theorem 3.2.4. We will be using this theorem on many occasions and show it's usefulness when introducing the main result in Chapter 8. This inequality was first shown in [8] and proved by induction over the dimension. We will follow the steps therein and fill out some details which were not included in the original article.

**Theorem 3.2.4** (Brascamp-Lieb-Luttinger Inequality)**.** *Let $(f_j)_{j \in \mathbb{N}_m}$ be a sequence of nonnegative functions on $\mathbb{R}^n$, vanishing at infinity. Let $k \leqslant m$ and let $A = (a_{ij})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_m}$ be a matrix. Consider*

$$I(f_1, \ldots, f_m) := \int_{(\mathbb{R}^n)^k} \prod_{j=1}^{m} f_j \left( \sum_{i=1}^{k} a_{ij} x_i \right) dx_1 \cdots dx_k.$$

*Then $I(f_1, \ldots, f_m) \leqslant I(f_1^*, \ldots, f_m^*)$.*

Before going to prove this result we need some advanced knowledge concerning convex sets. The next lemma covers Brunn's part on a general problem dealing with convexity (see [8] for references).

**Lemma 3.2.1.** *Let $C \subseteq \mathbb{R}^{n+1}$ be a convex set, let $v \in \mathbb{R}^{n+1}$ and $V(t) := \{x \in \mathbb{R}^{n+1} \mid v \cdot x = t\}$ for $t \in \mathbb{R}$. Additionally, let $S(t) := \mathcal{L}^n(V(t) \cap C)$. Then $S(t)^{1/n}$ is a concave function of $t$ in a interval where $S(t) > 0$.*

**Corollary 3.2.1.** *Let $C$, $v$ and $S(t)$ be as in Lemma 3.2.1. Assume that $C$ is also a balanced set. Then we have $S(t) = S(-t)$ and whenever $0 \leqslant t_1 \leqslant t_2$ one has $S(t_2) \leqslant S(t_1)$.*

The proof of the Brascamp-Lieb-Luttinger inequality in Theorem 3.2.4 is made in several steps and is at its core a proof by induction over the dimension $n$. The hard part will be going from dimension $n = 1$ to $n = 2$, the rest follows in a more or less straightforward manner. Nevertheless this will be included as well. Without further ado, assume $n = 1$.

Using the layer-cake principle from Lemma 3.1.1 and Fubini's theorem we can restrict ourselves to a finite sequence of characteristic functions. We will call them $F_1, ..., F_m$ and for easier notation shall use the same letters for their corresponding sets. The distinction of those will be clear form the context.
Recalling the outer regularity of the Lebesgue measure (see [25]) we find for all $F_j$ a sequence $(F_{j,l})_{l \in \mathbb{N}}$ of open sets such that

$$F_j \subseteq F_{j,l} \subseteq F_{j,l-1}$$

and additionally $\lim_{l \to \infty} F_{j,l} = F_j$. So upon using the dominated convergence theorem we have

$$\lim_{l \to \infty} I(F_{1,l}, ..., F_{m,l}) = I(F_1, ..., F_m).$$

Recall that every open set of the reals $\mathbb{R}$ is given as a disjoint union of countably many open intervals. A simple proof of this fact goes as follows:

*Proof.* Let $O \subseteq \mathbb{R}$ be open and nonempty. For $x, y \in O$ we define the following equivalence relation

$$x \sim y :\Longleftrightarrow [\min(x, y), \max(x, y)] \subseteq O.$$

Those equivalence classes are pairwise disjoint open intervals in $\mathbb{R}$, possibly being unbounded. Let $\mathcal{E}$ be the set of equivalence classes, then $O = \bigcup_{I \in \mathcal{E}} I$. Clearly for each class $I$ we can choose $r_I \in I \cap \mathbb{Q}$ and find that the map $c : \mathcal{E} \to \mathbb{Q}$ with $c(I) = r_I$ is injective. Hence by definition $\mathcal{E}$ needs to be countable. $\square$

Using the monotone convergence theorem we conclude that it is enough to show a proof for characteristic functions of finite disjoint unions of open intervals which is a standard procedure laid out in [25]. This observation leads to the first part of the proof.

**Lemma 3.2.2.** *Let* $(f_j)_{j \in \mathbb{N}_m}$ *be a finite sequence of characteristic functions of intervals with the following form*

$$I_j := (b_j - c_j, b_j + c_j).$$

*Furthermore, assume that* $k \leqslant m$, $A = (a_{ij})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_m}$ *is a matrix. Let* $t \in [0, 1]$ *and define*

$$f_j(x|t) := f_j(x + b_j t)$$

*Then*

$$I(t) := \int_{\mathbb{R}^k} \prod_{j=1}^{m} f_j \left( \sum_{i=1}^{k} a_{ij} x_i | t \right) dx_1 \cdots dx_k$$

*is a nondecreasing function of* $t \in [0, 1]$.

*Proof.* As far as the proof goes, note that $I(t)$ is the volume of

$$S := \bigcap_{j=1}^{m} S_j,$$

where

$$S_j := \{x \in \mathbb{R}^k | \sum_{i=1}^{k} a_{ij}x_i \in (b_j(1-t) - c_j, b_j(1-t) + c_j)\}.$$

Next, consider the following set

$$C := \bigcap_{j=1}^{m} \{x \in \mathbb{R}^{k+1} | -c_j < \sum_{i=1}^{k} a_{ij}x_i - b_j x_{k+1} < c_j\}.$$

It is clear that $C$ is a convex and balanced set. We remark that $I(t)$ is also the volume of the intersection of $C$ with the plane where $x_{k+1} = 1 - t$, for $t \in \mathbb{R}$. Recalling Corollary 3.2.1 we see that $I(t)$ has all the desired properties needed. $\qquad\square$

**Remark 3.2.5.** *The technique above is called a sliding argument and was introduced in [8]. One easily sees that for each $j \in \mathbb{N}_m$ we have $f_j(x|0) = f_j(x)$ and $f_j(x|1) = f_j^*(x)$ by recalling the definition of a symmetric-decreasing rearrangement in (3.1.1).*

Next we let the $f_j$'s be characteristic functions of a finite union of open sets. This will be the second step in the proof of Theorem 3.2.4. One can already guess how further advancements will work out.

**Lemma 3.2.3.** *Lemma 3.2.2 from above holds if $(f_j)_{j \in \mathbb{N}_m}$ is a sequence of characteristic functions of a finite union of disjoint open intervals.*

*Proof.* Let $f_j$ be a characteristic function of $n_j$ open and disjoint intervals where $j \in \mathbb{N}_m$. To prove the claim we make an induction over $\mathcal{N} := \{n_1, ..., n_m\}$. We define that $\mathcal{M} \prec \mathcal{N}$ if $m_j \leq n_j$ for $j \in \mathbb{N}_k$ and $m_i < n_i$ for some $i$. We have already seen that the claim holds true for $\mathcal{N} = \{1, ..., 1\}$, now assume the claim holds for all $\mathcal{M} \prec \mathcal{N}$. Hence let $f_j$ be the characteristic function of the following set

$$\bigcup_{q=1}^{n_j} (b_{jq} - c_{jq}, b_{jq} + c_{jq}), \tag{3.2.1}$$

where $b_{jq} + c_{jq} < b_{j,q+1} - c_{j,q+1}$ for $j \in \mathbb{N}_m$ and $p \in \mathbb{N}_{n_j}$. Furthermore define $f_j(.|t)$ as the characteristic function of a shifted version of (3.2.1), i.e.

$$\bigcup_{q=1}^{n_j} (b_{jq}(1-t) - c_{jq}, b_{jq}(1-t) + c_{jq})$$

for $t \in [0, \tau]$ with

$$\tau = \min_{j,q} \left( 1 - (b_{j,q+1} - b_{jq})^{-1}(c_{j,q+1} + c_{jq}) \right) > 0.$$

For $0 \leq t < \tau$ we can apply Lemma 3.2.2 interval by interval because those a disjoint and hence it follows that

$$\int_{\mathbb{R}^k} \prod_{j=1}^{m} f_j(\sum_{i=1}^{k} a_{ij}x_i)dx_1 \cdots dx_k \leq I(\tau).$$

At $t = \tau$ some intervals might intersect, but then we apply Lemma 3.2.2 again to $(f_j(.|\tau))_{j \in \mathbb{N}_m}$ but for some $\mathcal{N}$ that has been reduces to some $\mathcal{M} \prec \mathcal{N}$. Henceforth the claim follows from the fact that $f_j(.|\tau)^* = f_j^*$, i.e.

$$\int_{\mathbb{R}^k} \prod_{j=1}^m f_j(\sum_{i=1}^k a_{ij}x_i|\tau)dx_1\cdots dx_k \leqslant \int_{\mathbb{R}^k} \prod_{j=1}^m f_j^*(\sum_{i=1}^k a_{ij}x_i)dx_1\cdots dx_k.$$

$\square$

The third part uses the Steiner symmetrization for functions to apply Lemma 3.2.3 in each dimension once. We will give the proper definition and some remark about rotations. This will be absolutely crucial in the proof of the Brascamp-Lieb-Luttinger inequality in Theorem 3.2.4 and is later used on several other occations.

**Definition 3.2.1.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be Borel measurable and vanishing at infinity, then we define the following symmetrization for the index set $I := \{2, \dots, n\}$*

$$f^{*_I}(x) := f(\cdot, x_2, \dots, x_n)^*(x_1),$$

*where $f(\cdot, x_2, \dots, x_n)^*$ is the symmetric-decreasing rearrangement from (3.1.1). We call $f^{*_I}$ the **Steiner symmetrization** of $f$ with respect to $I$.*

From a simple point of view, the Steiner symmetrization basically does a symmetric-decreasing rearrangement in $x_1$ but fix all the other coordinates. So we could say that it's a symmetric-decreasing rearrangement in codimension 1. This terminology is will be used in Chapter 8 as well.

**Remark 3.2.6.** *Clearly, the rearrangement operator $*_I$ can easily be generalized to any arbitrary coordinate direction. For this let $V$ be a $(n-1)$-dimensional plane through the origin, then choose an orthogonal coordinate system in $\mathbb{R}^n$ such that the $x_1$-axis is perpendicular to $V$. This can be achieved by a proper rotation $\mathsf{R} \in \mathrm{O}(n)$. To be more precise, let $\mathbf{e} \in \mathbb{S}^{n-1}$ such that $\mathsf{R}\mathbf{e} = \mathbf{e}_1$. Let $(\mathsf{R}f)(x) := f(\mathsf{R}^{-1}x)$ be the action of $\mathsf{R}$ on functions $f : \mathbb{R}^n \to \mathbb{C}$, then we define*

$$f^{*_V} := \mathsf{R}^{-1}((\mathsf{R}f)^{*_I}),$$

*where $\mathsf{R}$ is dependent on the plane $V$.*

**Lemma 3.2.4.** *Let $(f_j)_{j \in \mathbb{N}_m}$ be a finite sequence of nonnegative Borel measurable functions on $\mathbb{R}^n$, vanishing at infinity. Furthermore, assume that $k \leqslant m$ and let $A = (a_{ij})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_m}$ be a matrix. Additionally let $V$ be any plane through the origin of $\mathbb{R}^n$. Then*

$$I(f_1, \dots, f_m) \leqslant I(f_1^{*_V}, \dots, f_m^{*_V}).$$

*Proof.* First choose proper orthogonal coordinates in $x \in \mathbb{R}^n$ such that the $x_1$-axis is orthogonal to $V$. Then one is in the same position as in Lemma 3.2.3 and hence a proof is a simple adaption. $\square$

To complete the proof for Theorem 3.2.4 we need some kind of induction step to go from dimension $n = 1$ to $n = 2$. Instead of symmetrizing around one variable, like we did for the Steiner symmetrization, we take all the other variables and use a similar technique. This is called a Schwarz symmetrization and is provided in the following definition.

**Definition 3.2.2.** *Let $n \geqslant 2$. Consider $f : \mathbb{R}^n \to \mathbb{C}$ to be Borel measurable and vanishing at infinity, then for $x = (x_1, x') \in \mathbb{R}^n$ we define*

$$f^{*_1}(x) := f(x_1, \cdot)^*(x').$$

*We call $f^{*_1}$ the **Schwarz symmetrization** or **Steiner symmetrization in codimension $\mathbf{n-1}$**.*

**Remark 3.2.7.** *As previously done for the Steiner symmetrization in Definition 3.2.1 we can generalize the rearrangement operator $*_1$ to any arbitrary direction with the same trick. Let $\mathsf{R} \in \mathrm{O}(n)$ and $\mathbf{e} \in \mathbb{S}^{n-1}$ with $\mathsf{R}\mathbf{e} = \mathbf{e}_1$. Then we define*

$$f^{*\mathbf{e}} := \mathsf{R}^{-1}((\mathsf{R}f)^{*_1}).$$

*For a shorter notation we will write $*_j$ instead of $*_{\mathbf{e}_j}$, where $\mathbf{e}_j$ is the unit vector in direction $j$.*

Now we are finally ready to give a proof of the Brascamp-Lieb-Luttinger inequality in 3.2.4. We will closely follow the original paper [8] and let us influence from simpler results on rearrangements in [25].

*Proof.* First of all, let $n = 2$. Again, we restrict ourselves to characteristic functions as previously done in Lemma 3.2.3.
Fix a rotation $\mathsf{R}_\alpha \in \mathrm{O}(2)$ with angle $\alpha$, where $\alpha = 2\pi r$ with $r \in \mathbb{R} \backslash \mathbb{Q}$. Now choose any set $F \subseteq \mathbb{R}^2$ with finite Lebesgue measure and define the following operation

$$F^1 := TS\mathsf{R}_\alpha F,$$

where $S$ is the Steiner symmetrization around the $x$-axis and $T$ the one around the $y$-axis. Clearly $SF$ is given by it's characteristic function, i.e.

$$\chi_{SF} = \chi_{F*_1} = \chi_F^{*_1}.$$

Using the definition we see that $\mathcal{L}^2(F_1) = \mathcal{L}^2(F)$ for the Lebesgue measure in $\mathbb{R}^2$. Inductively we define the set $F^q$ by applying $TS\mathsf{R}_\alpha$ $q$ times to $F$.

To prove the theorem we need a finite sequence of sets $(F_j)_{j \in \mathbb{N}_m}$ with finite Lebesgue measure. Using the procedure above, we define $(F_j^q)_{j \in \mathbb{N}_m}$.
By Theorem 3.2.2 we note that

$$\|T\chi_{F_i} - T\chi_{F_j}\|_{L^2(\mathbb{R}^2)} \leqslant \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^2)}$$

and

$$\|S\chi_{F_i} - S\chi_{F_j}\|_{L^2(\mathbb{R}^2)} \leqslant \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^2)}$$

hold true for $i, j \in \mathbb{N}_m$. Additionaly, recalling that rotations are measure preserving we find

$$\|\mathsf{R}_\alpha \chi_{F_i} - \mathsf{R}_\alpha \chi_{F_j}\|_{L^2(\mathbb{R}^2)} = \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^2)} \quad \text{for } i, j \in \mathbb{N}_m.$$

In the end, we want to show that all $F_j^q$ converge strongly in $L^2(\mathbb{R}^2)$ to some ball of the same volume. Note that the short remark from before implies that we can restrict ourselves to bounded sets. Indeed, for a given $\epsilon > 0$ we find some $\tilde{F}_j$ contained in some centered ball such that $\|\chi_{F_j} - \chi_{\tilde{F}_j}\|_{L^2(\mathbb{R}^2)} < \epsilon$, hence

$$\|\chi_{F_j^q} - \chi_{\tilde{F}_j^q}\|_{L^2(\mathbb{R}^2)} < \epsilon$$

for all $q \in \mathbb{N}$. Once we have shown that $\tilde{F}_j^q$ converges, it follows immediately for $F_j^q$ as well. So from now on, assume that $(F_j)_{j \in \mathbb{N}_m}$ is a sequence of bounded sets with finite Lebesgue measure.

Next consider the upper half-space part of $F_j^q$. This set is bounded by a graph of a symmetric, nonincreasing function $h_{j,q}$ which can be chosen to be lower semicontinuous and

uniformly bounded (see [25] and the proof of Lemma 3.2.2). Then there exists a subsequence given by $h_{j,q(l)}$ that converges everywhere to a lower semicontinuous function $h_j$ which bounds the upper half-space part of a set $D_j$. We want to show that each $D_j$ is a disk.

Henceforth, consider any strictly symmetric-decreasing function $g_j$ for every $j \in \mathbb{N}_m$ and define
$$\Delta_j^q := \|g_j - \chi_{F_j^q}\|_{L^2(\mathbb{R}^2)}.$$

Using Definitions 3.2.1 and 3.2.2 we see that $g^{*_1} = g^{*_2} = \mathsf{R}_\alpha g = g$. For later use, we shall write $g^{*_2} = Tg$ and $g^{*_1} = Sg$. Again, by Theorem 3.2.2 we find that $\Delta_j^q$ is nonincreasing for each $j$ and $q$, hence has a limit denoted by $\Delta_j$. Using the previous thoughts on $h_{j,q}$, we find that $\chi_{F_j^q}$ converges pointwise a.e. to $\chi_{D_j}$. Since $\chi_{F_j^{q(l)}}$ is bounded by $\chi_{D_j}$ for each $l$ we can use the dominated convergence theorem to show that
$$\Delta_j = \|g_j - \chi_{D_j}\|_{L^2(\mathbb{R}^2)}.$$

Following the simple inequalities already given in the proof, we conclude that
$$\|\chi_{F_j^{q(l)+1}} - TS\mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)} = \|TS\mathsf{R}_\alpha \chi_{F_j^{q(l)}} - TS\mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } l \to +\infty.$$

Hence by monotonicity of $\Delta_j^q$ we have
$$\Delta_j = \|g_j - TS\mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)}.$$

Upon recalling the definition of $g_j$ we find it actually is rotationally invariant. Hence
$$\|g_j - \mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)} = \|g_j - \chi_{D_j}\|_{L^2(\mathbb{R}^2)} = \Delta_j,$$

and so it's easy to see that
$$\|g_j - \mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)} = \|g_j - TS\mathsf{R}_\alpha \chi_{D_j}\|_{L^2(\mathbb{R}^2)}.$$

Again, by Theorem 3.2.2 and recalling that $g$ is strictly decreasing, we see that $TS\mathsf{R}_\alpha \chi_{D_j} = \mathsf{R}_\alpha \chi_{D_j}$ almost everywhere, but then $\mathsf{R}_\alpha \chi_{D_j}$ has a symmetry with respect to a reflection $P_j$ around the $x$-axis. This implies that $\mathsf{R}_\alpha \chi_{D_j} = P_j \mathsf{R}_\alpha \chi_{D_j} = \mathsf{R}_{-\alpha} P_j \chi_{D_j} = \mathsf{R}_{-\alpha} \chi_{D_j}$. Whence it readily gives the invariance of $D_j$ under the rotation $\mathsf{R}_{2\alpha}$, which is a rotation with an irrational angle. Defining the function $\mu_j(\theta) := \|\chi_{D_j} - \mathsf{R}_\theta \chi_{D_j}\|_{L^2(\mathbb{R}^2)}$ and using a density argument, we find $\{\mu_j = 0\}$ is dense in $[0, 2\pi)$. Upon showing that $\mu$ is continuous we find $\chi_{D_j} = \mathsf{R}_\theta \chi_{D_j}$ a.e. for every $\theta$ and so $D_j = F_j^*$.

Clearly, it is sufficient to show continuity of
$$r_j(\theta) = \int_{\mathbb{R}^2} \chi_{D_j} \mathsf{R}_\theta \chi_{D_j} dx.$$

Thanks to the approximation of $L^2(\mathbb{R}^2)$ functions using $C^\infty(\mathbb{R}^2)$ functions by mollification (see [31]) there exists a sequence $(u_{j,q})_{j \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^2)$ such that
$$\|\chi_{D_j} - u_{j,q}\|_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } q \to +\infty.$$

Using Schwarz's inequality we have
$$\left| \int_{\mathbb{R}^2} (\chi_{D_j} - u_{j,q}) \mathsf{R}_\theta \chi_{D_j} dx \right| \leq \|\chi_{D_j} - u_{j,q}\|_{L^2(\mathbb{R}^2)} \|\chi_{D_j}\|_{L^2(\mathbb{R}^2)}$$

and so $r_{j,q}(\theta) = \int_{\mathbb{R}^2} u_{j,q} \mathsf{R}_\theta \chi_{D_j} dx$ converges uniformly to $r_j(\theta)$. It is easily seen that

$$r_{j,q}(\theta) = \int_{\mathbb{R}^2} (\mathsf{R}_{-\theta} u_{j,q}) \chi_{D_j} dx$$

and so $r_j$ is continuous. Hence we have that $D_j$ is a ball for each $j \in \mathbb{N}_m$.

Recall that there exists a subsequence of $\chi_{F_j^q}$ converging pointwise a.e. to $\chi_{D_j}$, where each $F_j^q$ is contained in some fixed ball dependent on $j$. But then using the dominated convergence theorem we easily see that this subsequence converges to $\chi_{D_j}$ in $L^2(\mathbb{R}^2)$. Therefore we find that $\|\chi_{F_j^q} - \chi_{D_j}\|_{L^2(\mathbb{R}^2)}$ is a decreasing sequence by Theorem 3.2.2. So not only subsequences converge to $\chi_{D_j}$ but also the whole sequence.

Upon inspecting $D_j$ again, we find that $\chi_{F_j}$ converges strongly to $\chi_{F_j}^*$ in $L^2(\mathbb{R}^2)$ for each $j \in \mathbb{N}_m$. But then one deduces that

$$\lim_{q \to \infty} I(F_1^q, \ldots, F_m^q) = I(F_1^*, \ldots, F_m^*).$$

Upon using Lemma 3.2.4, this proves that $I(F_1^q, \ldots, F_m^q)$ is nondecreasing and hence the theorem follows for $n = 2$.

Now let $n > 2$. The basic idea will be very similar to the two dimensional case. Let $T$ be the Steiner symmetrization along the $x_n$-axis and $S$ the Schwarz symmetrization perpendicular to the $x_n$-axis, i.e. $Sf = f^{*_n}$. Again, for each $j \in \mathbb{N}_m$ we consider the sequence $\{(TS\mathsf{R})^k \chi_{F_j}\}$, where $\mathsf{R} \in O(n)$ is any rotation that rotates the $x_n$-axis by $\frac{\pi}{2}$. Recalling the steps for $n = 2$, we have the following estimates

$$\|T\chi_{F_i} - T\chi_{F_j}\|_{L^2(\mathbb{R}^n)} \leqslant \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^n)}$$

and

$$\|S\chi_{F_i} - S\chi_{F_j}\|_{L^2(\mathbb{R}^n)} \leqslant \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^n)}$$

for $i, j \in \mathbb{N}_m$. Furthermore, the rotation part fulfills

$$\|\mathsf{R}\chi_{F_i} - \mathsf{R}\chi_{F_j}\|_{L^2(\mathbb{R}^n)} = \|\chi_{F_i} - \chi_{F_j}\|_{L^2(\mathbb{R}^n)} \quad \text{for } i, j \in \mathbb{N}_m.$$

Again, we can restrict ourselves to bounded sets. Using the analogous arguments as before we can deduce that the limiting sets $D_j$ for each $j \in \mathbb{N}_m$ are rotationally symmetric around the $x_n$-axis (also $\mathsf{R}D_j$ satisfies this property). From the induction step we already know that the respective cross sections are $(n-1)$-dimensional balls. Now we only need to deduce that each $D_j$ is a ball. For this consider $\eta \in C^\infty(\mathbb{R}^n)$ radial, such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. Additionally, let $\eta_\epsilon(x) := \epsilon^{-n}\eta(x/\epsilon)$ be the standard mollifier and consider $\chi_{\epsilon,j} := \eta_\epsilon * \chi_{D_j}$. From basic arguments we know that $\chi_{\epsilon,j}$ is smooth for each $j \in \mathbb{N}_m$ and that it converges strongly to $\chi_{D_j}$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0^+$ (see [31]). As $\eta$ is a radial function one deduces that $\chi_{\epsilon,j}$ has the same symmetry properties as $\chi_{D_j}$. Let $y_i := (x_1, \ldots, x_i)$ where $i \in \mathbb{N}_n$. Recalling that $D_j$ and $\mathsf{R}D_j$ are rotationally symmetric around the $x_n$-axis we find continuous functions $f$ and $g$, such that

$$\chi_{\epsilon,j}(x) = f_j(\sqrt{|y_{n-2}|^2 + x_{n-1}^2}, x_n) = g_j(\sqrt{|y_{n-2}|^2 + x_n^2}, x_{n-1}).$$

Now let's consider $x_n = 0$, then we have for each $|y_{n-2}| > 0$

$$g_j(|y_{n-2}|, x_{n-1}) = f_j(\sqrt{|y_{n-2}|^2 + x_{n-1}^2}, 0).$$

This argument can be done for different axes, i.e. $D_j$ and $\mathsf{R}D_j$ are rotationally symmetric around two perpendicular axes. But this readily implies that

$$\chi_{\epsilon,j}(x) = f_j(|x|, 0),$$

so $\chi_{\epsilon,j}$ is radial and hence $\chi_{D_j}$ is radial as well for each $j \in \mathbb{N}_m$. The rest of the argument to deduce the actual inequality follows analogously to the $n = 2$ case. $\qquad\square$

**Remark 3.2.8.** *The proof of Theorem 3.2.4 is basically done repeating Lemma 3.2.4 by chaining a rotation. As already stated, the hard part was going from $n = 1$ to $n = 2$, where we heavily used a simple density argument.*

*Additionally, one can easily check how the Riesz inequality from Theorem 3.2.3 is a special case of the Brascamp-Lieb-Luttinger inequality. Later, we will give a version of Theorem 3.2.4 for the Steiner symmetrization in codimension $n - 1$ (see Chapter 8).*

*Another possible idea for a proof of the classical Brascamp-Lieb-Luttinger inequality in Theorem 3.2.4 is to use Helly's selection theorem. This is discussed in [25] for the Riesz inequality.*

# Chapter 4

# Positive Definite Functions

In this Chapter we will talk about basic properties of positive definite and positive semi-definite functions in the sense of Bochner (e.g. see [31]). As before we give a gentle introduction with some simple results which lead to the goal of proving Bochner's theorem in Section 4.2. Our main references will be [33] and [31].

In particular, Bochner's theorem will give us a deeper understanding of the main results and their manifold ramifications in Chapter 6 and 8.

## 4.1 Preliminaries

A positive semi-definite function is basically an extension of the well known positive-definite matrices which were probably introduced in the first year of studying mathematics. Without further ado, the definition is as follows.

**Definition 4.1.1.** *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be a continuous function. We say* $f$ *is positive semi-definite if, for all* $N \in \mathbb{N}$, *all pairwise distinct* $x_1, \ldots, x_N \in \mathbb{R}^n$ *and all* $z_1, \ldots, z_N \in \mathbb{C}$, *the following quadratic form is nonnegative*

$$\sum_{j=1}^{N} \sum_{k=1}^{N} z_j \overline{z_k} f(x_j - x_k) \geqslant 0.$$

*We say that* $f$ *is positive definite if the quadratic form above is positive for all pairwise distinct* $z_1, ..., z_n \in \mathbb{C}\backslash\{0\}$.

For the first theorem we show some simple properties that will be very useful in the following results yet will be interesting on their own. For many of those we will include a short prove for the sake of completeness.

**Theorem 4.1.1.** *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be positive semi-definite. Then the following are satisfied.*

(1) $f(0) \geqslant 0$.

(2) $f(-x) = \overline{f(x)}$ *for all* $x \in R^n$.

(3) $|f(x)| \leqslant f(0)$ *for all* $x \in \mathbb{R}^n$.

(4) $f \equiv 0$ *if and only if* $f(0) = 0$.

(5) *A linear combination with nonnegative coefficients of positive semi-definite functions is still positive semi-definite, i.e. let* $(f_j)_{j \in \mathbb{N}_m}$ *be a sequence of positive semi-definite functions and let* $(b_j)_{j \in \mathbb{N}_m} \subseteq \mathbb{R}_{\geqslant 0}$ *then* $f := \sum_{j=1}^{m} b_j f_j$ *is positive semi-definite.*

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(6) *If one of the $f_j$'s in (5) is positive definite and $(b_j)_{j \in \mathbb{N}_m} \subseteq \mathbb{R}_{>0}$ then $f := \sum_{j=1}^{m} b_j f_j$ is also positive definite.*

(7) *A product of positive definite functions is still positive definite.*

*Proof.* The proof of the first statement already shows the idea how the others parts will work out. For *(1)* simply take $N = 1$, $z_1 = 1$ and $x_1 = 0$.

The second property follows by taking $N = 2$, $z_1 = 1$, $z_2 = q$ with any complex number $q \in \mathbb{C}$, $x_1 = 0$ and $x_2 = x$ for any $x \in \mathbb{R}^n \backslash \{0\}$. Upon using Definition 4.1.1 we find

$$q f(x) + \bar{q} f(-x) + (1 + |q|^2) f(0) \geqslant 0.$$

Henceforth taking $q = 1$ and $q = i$ respectively we have the following two inequalities

$$f(x) + f(-x) + 2f(0) \geqslant f(x) + f(-x) \geqslant 0$$

and

$$i(f(x) - f(-x)) + 2f(0) \geqslant i(f(x) - f(-x)) \geqslant 0,$$

which both follow from property *(1)* in the theorem. As a consequence $i(f(x) - f(-x)) \in \mathbb{R}$ and also $f(x) + f(-x) \in \mathbb{R}$. Simply splitting $f$ into real and imaginary parts, i.e. $f = f_1 + i f_2$, we finally see that

$$f(x) = \overline{f(-x)} \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\},$$

which implies property *(2)* by using property *(1)* once more. For the third property one takes $N = 2$, $x_1 = 0$, $x_2 = x$ for any $x \in \mathbb{R}^n \backslash \{0\}$, $z_1 = |f(x)|$ and $z_2 = -\overline{f(-x)}$. Hence Definition 4.1.1 and property *(2)* from above give

$$2|f(x)|^2 f(0) - 2|f(x)|^3 \geqslant 0,$$

which readily implies what we were looking for. Notice that property *(4)* is easily seen to follow from property *(3)*.

Properties *(5)* and *(6)* are obvious and a simple application of Definition 4.1.1.

For the last property we use a Schur decomposition (e.g. see [33]), i.e. if $A$ is a positive definite matrix then there exists a unitary matrix $U$, i.e. $U^H = U^{-1}$, where $U^H = \overline{U}^T$ is the Hermitian conjugate, such that $A = UDU^H$ with $D$ being a diagonal matrix filled with eigenvalues $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_N$. Note that this property will be important. Now let $f$ and $g$ be positive definite, hence using the remark above we can decompose the matrix $G_N := (g(x_i - x_j)_{(i,j) \in \mathbb{N}_N^2})$ coming from Definition 4.1.1 using the Schur decompositon. Then it readily implies that

$$g(x_i - x_j) = \sum_{k=1}^{N} u_{ik} \overline{u_{jk}} \lambda_k.$$

Following this idea we have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} z_i\overline{z_j}f(x_i - x_j)g(x_i - x_j) = \sum_{i=1}^{N}\sum_{j=1}^{N} z_i\overline{z_j}f(x_i - x_j)\sum_{k=1}^{N} u_{ik}\overline{u_{jk}}\lambda_k$$

$$= \sum_{k=1}^{N}\lambda_k \sum_{i=1}^{N}\sum_{j=1}^{N} z_i u_{ik}\overline{z_j u_{jk}}f(x_i - x_j)$$

$$\geqslant \lambda_1 \sum_{i=1}^{N}\sum_{j=1}^{N} z_i\overline{z_j}f(x_i - x_j)\sum_{k=1}^{N} u_{ik}\overline{u_{jk}}$$

$$= \lambda_1 \sum_{i=1}^{N} |z_i|^2 f(0).$$

This last part is nonnegative for every $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and hence upon assuming that $z \in \mathbb{C}^N \backslash \{0\}$ we even see positivity. This concludes the proof for property *(7)* in the theorem. $\qquad\square$

**Remark 4.1.2.** *For real-valued positive definite functions we can give another characterization. From Theorem 4.1.1 it is clear that such a function must be even. So upon taking $z_j = a_j + i b_j$ we see that*

$$\sum_{j=1}^{N}\sum_{k=1}^{N} z_j\overline{z_k}f(x_j - x_k) =: I_1 + I_2$$

*where*

$$I_1 = \sum_{j=1}^{N}\sum_{k=1}^{N} (a_j a_k + b_j b_k)f(x_j - x_k)$$

*and*

$$I_2 = i\sum_{j=1}^{N}\sum_{k=1}^{N} a_k b_j (f(x_j - x_k) - f(x_k - x_j)).$$

*But recalling that $f$ is even we find $I_2 = 0$. So a real-valued positive definite function can be characterized as follows:*

*Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then $f$ is positive definite if and only if $f$ is even and for all $N \in \mathbb{N}$, all pairwise distinct $x_1, \ldots, x_N \in \mathbb{R}^n$ and all $z_1, \ldots, z_N \in \mathbb{R}\backslash\{0\}$, the following quadratic form is positive*

$$\sum_{j=1}^{N}\sum_{k=1}^{N} z_j z_k f(x_j - x_k) > 0.$$

## 4.2   Bochner's Theorem

One of the most influential results in the whole theory of positive semi-definite functions is their connection to Fourier transforms.

Before going into details, let's give a simple introduction to the idea. Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ such that $\hat{f} \in L^1(\mathbb{R}^n)$. Upon using the inverse Fourier transform we have

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi \cdot x} d\xi.$$

Recalling the quadratic form in Definition 4.1.1 we easily see that

$$\sum_{j=1}^N \sum_{k=1}^N z_j z_k f(x_j - x_k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) \left| \sum_{j=1}^N z_j e^{i\xi \cdot x} \right|^2 d\xi.$$

So if $\hat{f}$ is nonnegative we find that

$$\sum_{j=1}^N \sum_{k=1}^N z_j z_k f(x_j - x_k) \geq 0,$$

which implies that $f$ is positive semi-definite. One simple remark is that we could have taken the Fourier transform instead of its inverse.

It is clear that this approach cannot work in a more general setting. For a better understanding we have to replace the measure. A first result is given by an integral characterization using test functions from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

**Lemma 4.2.1.** *Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous. Then $f$ is positive semi-definite if and only if $f$ is bounded and*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y)\varphi(x)\overline{\varphi(y)} dx dy \geq 0$$

*for all test functions $\varphi$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.*

*Proof.* To show the first part we use a classic argument. Suppose that $f$ is positive semi-definite, then by Theorem 4.1.1 we see that $f$ is bounded hence the integral

$$I(f; K) := \int_K \int_K f(x - y)\varphi(x)\overline{\varphi(y)} dx dy$$

is well defined for $K = \mathbb{R}^n$ and all test functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Using a standard approximation argument we see that for every $\epsilon > 0$ there exists a cube $C \subseteq \mathbb{R}^n$ with

$$\left| I(f; \mathbb{R}^d) - I(f; C) \right| < \frac{\epsilon}{2}.$$

The very definition of a Riemannian sum gives us

$$\left| I(f; W) - \sum_{j=1}^N \sum_{k=1}^N f(x_j - x_k)\varphi(x_j)\overline{\varphi(x_k)} q_j \overline{q_k} \right| < \frac{\epsilon}{2},$$

where $q = (q_1, \dots, q_N) \in \mathbb{C}^N$ is the properly chosen weight. This readily implies that

$$I(f; \mathbb{R}^n) + \epsilon > \sum_{j=1}^N \sum_{k=1}^N f(x_j - x_k)\varphi(x_j)\overline{\varphi(x_k)} q_j \overline{q_k},$$

so letting $\epsilon \to 0^+$ and using that $f$ is positive semi-definite imply the first half, i.e. $I(f; \mathbb{R}^n) \geqslant 0$.

For the second half we use a Fourier theory based argument and some approximation results with mollifiers. Assume that $f$ is bounded and $I(f; \mathbb{R}^n) \geqslant 0$ for all test functions. Let $\tilde{\varphi}(x) := \overline{\varphi(-x)}$, then we can rewrite the integral using a convolution in the following way

$$I(f; \mathbb{R}^n) = \int_{\mathbb{R}^n} f(x)(\varphi * \tilde{\varphi})(x) dx.$$

Let $(z_1, \ldots, z_N) \in \mathbb{C}^N$ and $(x_1, \ldots, x_N) \in (\mathbb{R}^n)^N$, then choose the following test function

$$\varphi_\epsilon(x) := \sum_{j=1}^N z_j \eta_{2\epsilon}(x - x_j),$$

where

$$\eta_\epsilon(x) := \frac{1}{(\pi \epsilon)^{n/2}} e^{-\frac{1}{\epsilon}|x|^2}.$$

From standard methods we deduce that the Fourier transform of $\varphi_\epsilon$ is given by

$$\mathcal{F}(\varphi_\epsilon)(\xi) = \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^N z_j e^{-i\xi \cdot x_j} e^{-\frac{\epsilon}{8}|\xi|^2},$$

and henceforth we have

$$\begin{aligned}
\mathcal{F}(\varphi_\epsilon * \widetilde{\varphi_\epsilon})(\xi) &= (2\pi)^{n/2} |\mathcal{F}(\varphi_\epsilon)(\xi)|^2 \\
&= (2\pi)^{-n/2} \left| \sum_{j=1}^N z_j e^{-i\xi \cdot x_j} \right|^2 e^{-\frac{\epsilon}{4}|\xi|^2} \\
&= \sum_{j=1}^N \sum_{k=1}^N z_j \overline{z_k} e^{-i\xi \cdot (x_j - x_k)} \mathcal{F}(\eta_\epsilon)(\xi) \\
&= \mathcal{F}\left( \sum_{j=1}^N \sum_{k=1}^N z_j \overline{z_k} \eta_\epsilon(\cdot - (x_j - x_k)) \right)(\xi).
\end{aligned}$$

Using basic harmonic analysis we deduce that

$$\begin{aligned}
\sum_{j=1}^N \sum_{k=1}^N z_j \overline{z_k} f(x_j - x_k) &= \sum_{j=1}^N \sum_{k=1}^N z_j \overline{z_k} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} f(x) \eta_\epsilon(x - (x_j - x_k)) dx \\
&= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} f(x)(\varphi_\epsilon * \widetilde{\varphi_\epsilon})(x) dx \\
&\geqslant 0.
\end{aligned}$$

This implies the second part of the lemma and finishes the proof. $\qquad \square$

As a second result we need some kind of generalization of Riesz' famous representation theorem (see [31] and [33]). Without a proof, the theorem goes as follows:

**Theorem 4.2.1** (Riesz' representation theorem)**.** *Let $\Omega$ be a locally compact metric space and $\mathcal{I}$ a linear, continuous and nonnegative functional on the space of continuous functions*

with compact support $C_c^0(\Omega)$. Then there exists a nonnegative Borel measure $\mu$ on $\Omega$ such that for all continuous functions $f \in C_c^0(\Omega)$ the following holds

$$\mathfrak{I}(f) = \int_\Omega f d\mu.$$

The following result is a generalization in the case of $\Omega = \mathbb{R}^n$. In this thesis this space is sufficient to work but could be generalized in a suitable way if needed.

**Theorem 4.2.2.** *Let $\mathfrak{I} : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ be a linear and nonnegative functional. Then $\mathfrak{I}$ has an extension to $C_c^0(\mathbb{R}^n)$ and there exists a nonnegative Borel measure $\mu$ such that for all $f \in C_c^0(\mathbb{R}^n)$ the following holds*

$$\mathfrak{I}(f) = \int_{\mathbb{R}^n} f d\mu.$$

*Proof.* First of all we show that the nonnegative linear functional $\mathfrak{I}$ is locally bounded. For this let $K$ be a compact subset of $\mathbb{R}^n$. We need to show that for all $f \in C_c^\infty(K)$ the following inequality holds

$$|\mathfrak{I}(f)| \leqslant C(K)\|f\|_{L^\infty(K)}.$$

Upon choosing $g \in C_c^\infty(\mathbb{R}^n)$ such that $g|_K \equiv 1$ and $g \geqslant 0$ we have for all real-valued $f \in C_c^\infty(\mathbb{R}^n)$ the following inequality

$$g\|f\|_{L^\infty(K)} \pm f \geqslant 0.$$

Then this implies that

$$|\mathfrak{I}(f)| \leqslant \mathfrak{I}(g)\|f\|_{L^\infty(K)}.$$

Now assume that $f$ is complex-valued. Upon multiplying $f$ by a complex phase such that $e^{i\vartheta}\mathfrak{I}(f) \in \mathbb{R}$ with some $\vartheta \in \mathbb{R}$, we conclude

$$\mathfrak{I}(\mathrm{Re}(e^{i\vartheta}f)) = e^{i\vartheta}\mathfrak{I}(f).$$

But this readily gives

$$|\mathfrak{I}(f)| = |e^{i\vartheta}\mathfrak{I}(f)| \leqslant \mathfrak{I}(g)\|f\|_{L^\infty(K)},$$

where we used the result from before. Hence we can choose $C(K) = \mathfrak{I}(g)$.

The second part will be proving the existence of the local extension of $\mathfrak{I}$. If the functional $\mathfrak{I}$ is restricted to $C_c^\infty(K)$ is has a unique extension to $C_c^0(K)$. Now, let $f \in C_0(K)$ and let $(f_j)_{j \in \mathbb{N}} \subseteq C_c^\infty(K)$ be a sequence such that $f_j$ converges uniformly to $f$, hence defines $\mathfrak{I}(f)$ as a limit of $\mathfrak{I}(f_j)$. Notice that the sequence $(f_j)_{j \in \mathbb{N}}$ ca be chosen as the convolution of $f$ with some suitable nonnegative mollifier and hence the extension $\mathfrak{I}$ to $C_c^0(K)$ is nonnegative.

Using Riesz' representation theorem from 4.2.1 we can find a unique Borel measure $\mu_K$ depending on the compact set $K$ which is defined on the Borel $\sigma$-algebra $\mathcal{B}(K)$ such that

$$\mathfrak{I}(f) = \int_K f d\mu_K \quad \text{for all } f \in C_c^0(K).$$

The measure $\mu_K$ is finite because the restriction $\mathfrak{I}|_{C_c^0(K)}$ is continuous.

The last step consists of defining a proper pre-measure on a suitable ring (see [25]). Consider $\mathcal{C}_n$, which is defined to be the ring of finite unions of semi-open cubes, so we use objects which are of the follwing form $C(a,b) := \{x \in \mathbb{R}^n |\ a_j \leqslant x_j < b_j \text{ for all } j \in \mathbb{N}_n\}$, where

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$a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. Now let $C \in \mathcal{C}_n$ and choose $j \in \mathbb{N}$ big enough such that $C$ lies in a closed ball around the origin of radius $j$ in $\mathbb{R}^n$, i.e. $C \subseteq B_j(0)$. Define the pre-measure $\beta$ on $\mathcal{C}_n$ as $\beta(C) := \mu_{K_j}(C)$. Since $\mu_{K_{j+1}}|_{\mathcal{B}(K_j)} = \mu_{K_j}$ we see that $\beta$ is well defined and independent of the choice of $j$. Recall that every pre-measure on a ring has an extension to a nonnegative measure on the corresponding $\sigma$-algebra (see [25]). In our case we can extend $\beta$ on $\mathcal{C}_n$ to a Borel measure $\mu$ on $\sigma(\mathcal{C}_n) = \mathcal{B}(\mathbb{R}^n)$, hence

$$\mathcal{I}(f) = \int_{\mathbb{R}^n} f d\mu,$$

for all $f \in C_c^0(\mathbb{R}^n)$. $\qquad\qquad\square$

Now we can start proving Bochner's theorem using Lemma 4.2.1 in conjunction with Theorem 4.2.2.

**Theorem 4.2.3** (Bochner's theorem). *Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous. $f$ is positive semi-definite if and only if there exists a nonnegative Borel measure $\mu$ on $\mathbb{R}^n$ such that*

$$f(x) = \mathcal{F}(\mu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot x} d\mu(\xi).$$

*Proof.* First assume that there exists a Borel measure $\mu$ on $\mathbb{R}^n$ such that

$$f(x) = \mathcal{F}(\mu)(x).$$

We claim that $f$ is positive semi-definite. Let $x_1, \ldots, x_N \in \mathbb{R}^n$ and assume $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, then the quadratic form from 4.1.1 reads

$$\sum_{j=1}^{N} \sum_{k=1}^{N} z_j \overline{z_k} f(x_j - x_k) = \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{N} \sum_{k=1}^{N} z_j \overline{z_k} \int_{\mathbb{R}^n} e^{-i\xi \cdot (x_j - x_k)} d\mu(\xi)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left| \sum_{j=1}^{N} z_j e^{-i\xi \cdot x_j} \right|^2 d\mu(\xi).$$

But the last expression is nonnegative, hence proves the claim by using the definition of a positive semi-definite function. Continuity of $f$ follows from the finiteness of the Borel measure $\mu$.

Now, suppose that $f$ is positive semi-definite. Let $\mathcal{I}$ be the distributional Fourier transform on $\mathcal{S}(\mathbb{R}^n)$, i.e. for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define

$$\mathcal{I}(\varphi) := \int_{\mathbb{R}^n} f(x) \mathcal{F}^{-1}(\varphi)(x) dx.$$

Assume that $\varphi = |\phi|^2$ for some $\phi \in \mathcal{S}(\mathbb{R}^n)$. Let $g := \mathcal{F}^{-1}(\phi)$ and additionally define $\tilde{g}(x) := \overline{g(-x)}$. Then

$$\mathcal{I}(\varphi) = \int_{\mathbb{R}^n} f(x) \mathcal{F}^{-1}(|\mathcal{F}(\phi)|^2)(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)(g * \tilde{g})(x) dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y) g(x) \overline{g(y)} dx dy$$

$$\geq 0,$$

which is a consequence of Lemma 4.2.1. Hence $\mathcal{I}$ is nonnegative for Schwartz functions $\varphi$ such that there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ with the property that $\varphi = |\phi|^2$. The next step is to extend this relation to all nonnegative smooth functions with compact support.

Consider $\varphi \in C_c^\infty(\mathbb{R}^n)$ nonnegative and let $\eta(x) = e^{-|x|^2/2}$ be a Gaussian, then consider $\varphi + \epsilon^2\eta$ for $\epsilon > 0$. It is easy too see that this is a positive Schwartz function. Hence we can define $\varphi_1 := \sqrt{\varphi + \epsilon^2\eta} \in C^\infty(\mathbb{R}^n)$. We can also show that $\varphi_1$ is a Schwartz function. This follows from the fact that $\varphi_1$ decays like $\epsilon\eta(\cdot/2)$ for large $x \in \mathbb{R}^n$. With the observation from above we find

$$0 \leqslant \mathcal{I}(|\varphi_1|^2) = \mathcal{I}(\varphi) + \epsilon^2\mathcal{I}(\eta).$$

Letting $\epsilon \to 0^+$ we have the extension we were looking for. Now $\mathcal{I}$ is a linear nonnegative functional on $C_c^\infty(\mathbb{R}^n)$ and so by Theorem 4.2.2 there exists a Borel measure $\mu$ such that for all $\varphi \in C_c^0(\mathbb{R}^n)$ the following holds

$$\mathcal{I}(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu.$$

The next part is a standard harmonic analysis method, nevertheless we will give some of the details. Let $\psi \in C^0(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$ be normed with $\|\psi\|_{L^1(\mathbb{R}^n)} = 1$ and nonnegative. Then define $\psi_\epsilon(x) := (1/\epsilon)^n\psi(\epsilon^{-1}x)$ for $x \in \mathbb{R}^n$. Additionally we need that $\mathcal{F}(\psi)$ is in $C_c^0(\mathbb{R}^n)$ and nonnegative. But this can easily be done in the following way:
Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be nonnegative and define $\psi := c\mathcal{F}^{-1}(\rho * \tilde{\rho})$, where the constant $c \in \mathbb{R}$ is chosen such that $\|\psi\|_{L^1(\mathbb{R}^n)} = 1$. This construction even yields that $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}(g) \in C_c^\infty(\mathbb{R}^n)$. Using the results above we easily see

$$\int_{\mathbb{R}^n} f(x)\psi_\epsilon(x)dx = \int_{\mathbb{R}^n} \mathcal{F}(\psi_\epsilon)(x)d\mu(x) = \int_{\mathbb{R}^n} \mathcal{F}(\psi)(\epsilon x)d\mu(x).$$

By Fatou's lemma we finally find

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^n} \lim_{\epsilon \to 0^+} \mathcal{F}(\psi)(x\epsilon)d\mu(x)$$
$$\leqslant \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \mathcal{F}(\psi)(x\epsilon)d\mu(x)$$
$$= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} f(x)\psi_\epsilon(x)dx$$
$$= f(0).$$

This follows because $f$ is bounded by Theorem 4.1.1. Therefore we see that the mass of $\mu$ is given by

$$\int_{\mathbb{R}^n} d\mu = (2\pi)^{n/2}f(0),$$

which implies the finiteness of the Borel measure. Henceforth for the final step we check the following limit

$$f(x) = \lim_{\epsilon \to 0+} (f * \psi_\epsilon)(x)$$
$$= \lim_{\epsilon \to 0+} \int_{\mathbb{R}^n} e^{-i\xi \cdot x}\mathcal{F}(\psi_\epsilon)(-\xi)d\xi$$
$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x}d\mu(\xi).$$

This readily implies what we were looking for. $\qquad\square$

**Remark 4.2.4.** *There are a few simpler results which follow from Bochner's theorem in 4.2.3 with a little work. One of those we will highlight in this remark is concerning the initial regularity of a positive definite function.*

*Assume that $f$ is positive definite and $f \in C^{2m}(\mathbb{R}^n)$ in some neighborhood around the origin. Upon recalling Bochner's theorem we find that for every testfunction $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(x)\varphi(x)dx = \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)d\mu(\xi)$$

*for some finite nonnegative Borel measure $\mu$. Now let $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$ where $\eta$ is a smooth function whose support is $B_1(0)$ and $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$ with $\epsilon > 0$. Then we have*

$$\int_{\mathbb{R}^n} \widehat{\eta_\epsilon}(\xi)(1 + |\xi|)^m d\mu(\xi) = \int_{\mathbb{R}^n} (1 - \Delta)^m f(x)\eta_\epsilon(x)dx,$$

*which converges to $(1 - \Delta)^m f(0)$ as $\epsilon \to 0^+$. As*

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^m d\mu(\xi) \leqslant C(1 - \Delta)^m f(0)$$

*we conclude that $f$ is of class $C^{2m}(\mathbb{R}^n)$ everywhere. This follows from Fatou's lemma. So we went from a local property to a global one by simply using positive definiteness of $f$.*

# Chapter 5

# A Guideline on Symmetry and Ground States

In this chapter we give an introduction to some symmetry properties of ground states. In Chapter 6 a full article on those problems will follow. Last but not least a section on the Hardy-Littlewood majorant property is included with a counterexample based in techniques in [24].

## 5.1 Linear Ground States

Let $P(D)$ be a self-adjoint, elliptic pseudo-differential operator of order $2s$ with constant coefficient, where $s > 0$. We consider consider the linear differential equation for $\psi \in H^s(\mathbb{R}^n)$

$$P(D)\psi + V\psi = E\psi,$$

where $E \in \mathbb{R}$ is some eigenvalue and $V : \mathbb{R}^n \to \mathbb{R}$ can be understood as a potential. For simplicity one may assume that $V$ is bounded. In the next part we need some growth estimates on $P(D)$, hence the following assumption is made.

**Assumption.** *Let $s > 0$. Using the Fourier representation, the pseudo-differential operator $P(D)$ is given by*

$$\mathcal{F}(P(D)\psi)(\xi) = p(\xi)\widehat{f}(\xi),$$

*where $p \in C^0(\mathbb{R}^n)$ satisfies the following growth estimates*

$$A|\xi|^{2s} + c \leqslant p(\xi) \leqslant B|\xi|^{2s} \quad \text{for all } \xi \in \mathbb{R}^n$$

*with proper constants $A > 0$, $B > 0$ and $c \in \mathbb{R}$.*

Now we can consider the following functional

$$\mathcal{W}(f) := \langle (P(D) + V)f, f \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^n)$. It's easy to see that the minimization problem given by

$$E_0 = \inf\{\mathcal{W}(f) |\ f \in H^s(\mathbb{R}^n) \text{ such that } \|f\|_{L^2(\mathbb{R}^n)} = 1\} > -\infty \tag{5.1.1}$$

is well-defined. If we assume that $\mathcal{L}^n\{|V| < \epsilon\} < +\infty$ for every $\epsilon > 0$ we find

$$E_0 \leqslant \inf_{\xi \in \mathbb{R}^n} p(\xi) = \inf \sigma_{\text{ess}}(H)$$

with $H = P(D) + V$. Notice that $H$ is defined via the quadratic form $\mathcal{W}$ and is self-adjoint. Recall that $\sigma_{\mathrm{ess}}(H)$ is the set of complex number $\lambda \in \mathbb{C}$ such that $I - \lambda H$ is not a Fredholm operator on the respective Hilbert space (see [31]).

If we assume the existence of a minimizer $\psi \in H^s(\mathbb{R}^n)$ it is easy to see that $\psi$ solves

$$P(D)\psi + V\psi = E_0\psi. \tag{5.1.2}$$

On the other side, any solution $\psi \in H^s(\mathbb{R}^n)\backslash\{0\}$ of (5.1.2) is a minimizer of $\mathcal{W}$ with $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$ up to some rescaling. We henceforth call $\psi$ a **linear ground state** if it satisfies either of those constraints.

The first example of such an operator is certainly the classical case with $P(D) = -\Delta$. Uniqueness of linear ground states (up to a constant) with respect to the Laplacian is a well known result and can be proven with different techniques (see [12]), for example with maximum principles. The fractional case with $P(D) = (-\Delta)^s$ with $s \in (0,1)$ can be done with similar methods. For more general operators it's not quite clear how things work out regarding the well-known results to tackle the minimization problem. Assuming that the potential $V$ has a negative Fourier transform we can state the following uniqueness result regarding a very general operator $P(D)$. Clearly the assumption on the Fourier symbol of $P(D)$ we made above will be crucial as is

$$E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi).$$

Before discussing some techniques used in the proof we recall the first main result from Chapter 6.

**Theorem 5.1.1.** *Let $n \geqslant 1$, $s > 0$ and $P(D)$ satisfies the assumption above. Furthermore assume that $V : \mathbb{R}^n \to \mathbb{R}$ has a Fourier transform $\widehat{V} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\widehat{V} < 0$ almost everywhere. Lastly we assume that $E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi)$ holds for 5.1.2. Then the following two statements can be made:*

(a) ***Uniqueness:*** *The linear ground state solution $\psi \in H^s(\mathbb{R}^n)$ is unique up to a constant phase. Moreover we have*

$$e^{i\theta}\widehat{\psi}(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n,$$

*where $\theta \in \mathbb{R}$ is a constant.*

(b) ***Symmetries:*** *Up to a constant phase the linear ground state $\psi$ has an even symmetry, i.e.*

$$\psi(-x) = \overline{\psi(x)} \quad \text{for a.e. } x \in \mathbb{R}^n.$$

*Additionally, if the symbol $p$ is even we conclude that $\psi : \mathbb{R}^n \to \mathbb{R}$ is real-valued.*

**Remark 5.1.2.** *It is quite clear that a linear ground state $\psi$ doesn't have to be real-valued in $x$-space at all. A nice example comes from the linearized problem for traveling solitary waves for dispersion generalized nonlinear Schrödinger equations, e.g. a solution $\psi \in H^s(\mathbb{R}^n)$ for equations of the form*

$$((-\Delta)^s + iv \cdot \nabla + V)\psi = E\psi,$$

*where $s \geqslant 1/2$ and $v \in R^n\backslash\{0\}$. If $s = 1/2$ we need $|v| < 1$. Still, under suitable assumptions on the potential $V$ one has the strict positivity of $e^{i\theta}\widehat{\psi}(\xi) > 0$ for all $\xi \in \mathbb{R}^n$.*

Recalling Bochner's theorem 4.2.3 and assuming that $\widehat{\psi} \in L^1(\mathbb{R}^n)$ then we can easily see that $\psi : \mathbb{R}^n \to \mathbb{C}$ is positive definite.

### 5.1.1 Outlines of the Proof

Before going into detail a little preparation of equation (5.1.2) has to be done. In the following we let $-\lambda = E = E_0$ and $\widehat{W} = -\widehat{V}$, where the Fourier transform is given by

$$\mathcal{F}(f)(\xi) \equiv \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi \cdot x}dx.$$

Notice that $p(\xi) + \lambda > 0$ for all $\xi \in \mathbb{R}^n$ by assumption. Hence we can write equation (5.1.2) in the following sense

$$\hat{\psi}(\xi) = \frac{1}{p(\xi) + \lambda}\widehat{W\psi}(\xi).$$

Clearly $\widehat{W\psi} = \widehat{W} * \hat{\psi}$ because $\widehat{W}$ is a $L^2$ function, additionally $\widehat{W} * \hat{\psi} \in C^0(\mathbb{R}^n)$. Now we make the following claim:

**Claim.** *The Fourier transform of the linear ground state $\psi$ is positive.*

*Proof Sketch.* First of all we prove that the symmetrization $\psi^\bullet = \mathcal{F}^{-1}(|\hat{\psi}|)$ is also a ground state. This can be achieved using the autocorrelation function of $\hat{\psi}$ given by $\Psi_{\hat{\psi}}(\xi) :=$ $\int_{\mathbb{R}^n} \hat{\psi}(\xi + \xi')\overline{\hat{\psi}(\xi')}d\xi'$ which implies

$$(\widehat{W} * \hat{\psi})(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

The positivity of $p(\cdot) + \lambda$ implies the result. $\qquad\square$

The next claim will seal the deal in proving Theorem 5.1.1. The proof heavily uses the structure of equation (5.1.2).

**Claim.** *There exists a constant $\theta \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}^n$ the following equality holds*

$$\hat{\psi}(\xi) = e^{i\theta}|\hat{\psi}(\xi)|.$$

*Proof Sketch.* Using the continuity of $\hat{\psi}$ and that $|\hat{\psi}| > 0$ we know there exists a continuous function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that

$$\hat{\psi}(\xi) = e^{i\theta(\xi)}|\hat{\psi}(\xi)| \quad \text{for all } \xi \in \mathbb{R}^n.$$

Knowing that both $\psi$ and $\psi^\bullet$ are linear ground states we deduce

$$(\widehat{W} * \Psi_{\hat{\psi}})(0) = (\widehat{W} * \Psi_{|\hat{\psi}|})(0).$$

Now the idea is to show that $\theta \equiv \text{const.}$, this implies the claim. $\qquad\square$

Finally, the proof of Theorem 5.1.1 follows with a good investigation of the last claim above.

## 5.2 Ground States for the Nonlinear Case

We now turn our attention to solutions $Q \in H^s(\mathbb{R}^n)$ of nonlinear elliptic PDEs of the following form

$$P(D)Q + \lambda Q - |Q|^{2\sigma}Q = 0, \tag{5.2.1}$$

with $s > 0$, $\sigma \in \mathbb{N}$ and $1 \leqslant \sigma < \sigma_*(n, s)$ where

$$\sigma_*(n, s) = \begin{cases} \frac{2n}{n-2s} & \text{for } s < \frac{n}{2} \\ +\infty & \text{for } s \geqslant \frac{n}{2} \end{cases}$$

is the critical exponent. In this thesis we only talk about the $H^s$-subcritical case. With further work one can also tackle the critical case when $\sigma = \sigma_*(n, s)$. As in Section 5.1 we need some assumptions on the pseudo-differential operator $P(D)$.

**Assumption.** *Let $s > 0$ be a real number. We assume that $P(D)$ is a pseudo-differential operator of order $2s$ with a Fourier symbol $p$ in the Hörmander class $S_{1,0}^{2s}$ that satisfies the following conditions:*

(1) **Real-valuedness:** *The symbol $p : \mathbb{R}^n \to \mathbb{R}$ given by $\mathcal{F}(P(D)u)(\xi) = p(\xi)\widehat{f}(\xi)$ is real-valued.*

(2) **Ellipticity Condition:** *There exist constants $c > 0$ and $R > 0$ such that*

$$p(\xi) \geqslant c|\xi|^{2s} \quad \text{for } |\xi| \geqslant R.$$

As a consequence of this assumption we easily see that $P(D) = P(D)^*$ is self-adjoint and bounded from below on $L^2(\mathbb{R}^n)$ with operator domain $H^{2s}(\mathbb{R}^n)$.

Regarding equation (5.2.1) we may say that $\lambda$ takes the position of a nonlinear eigenvalue. As in Section 5.1 we need some condition on $\lambda$. We assume that

$$\lambda > \inf_{\xi \in \mathbb{R}^n} p(\xi),$$

which implies that $-\lambda$ lies strictly below the essential spectrum of the operator $P(D)$. As an additional consequence we have an equivalence of norms in the sense that

$$\langle f, (P(D) + \lambda)f \rangle \simeq \|f\|_{H^s(\mathbb{R}^n)}^2.$$

Note the important fact that we have a Sobolev-type inequality given by

$$\|f\|_{L^{2\sigma+2}}^2 \leqslant C \langle f, (P(D) + \lambda)f \rangle, \tag{5.2.2}$$

where $C > 0$ denotes a suitable constant and $f \in H^s(\mathbb{R}^n)$. In the case of subcriticality, i.e. $\sigma = \sigma_*(n, s)$, we can use standard variational methods (see [25] and [12]) to deduce the existence of an optimal constant $C > 0$ in the inequality (5.2.2) as well as the existence of optimizers $Q \in H^s(\mathbb{R}^n)$. After a suitable linear rescaling $Q \to \alpha Q$ with a constant $\alpha$ one can show that $Q$ solves equation (5.2.1). This will be the definition of a ground state in the nonlinear case. Check the similarity with the given definition in Section 5.1.

**Definition.** *$Q \in H^s(\mathbb{R}^n) \backslash \{0\}$ is called a **ground state solution** if $Q$ solves equation (5.2.1) and optimizes inequality (5.2.2).*

We could also give the following characterization of ground states using an action functional. For this, let

$$\mathcal{A}(f) := \frac{1}{2} \langle f, (P(D) + \lambda)f \rangle - \frac{1}{2\sigma + 2} \|f\|_{L^{2\sigma+2}}^{2\sigma+2}$$

and define the set of ground state solutions by

$$\mathcal{G} := \{Q \in K \mid \mathcal{A}(Q) \leqslant \mathcal{A}(R) \text{ for all } R \in K\},$$

where $K := \{u \in H^s(\mathbb{R}^n) \backslash \{0\} \mid \mathcal{A}'(u) = 0\}$. Then the following lemma yields another form of defining a ground state.

**Lemma 5.2.1.** $Q \in H^s(\mathbb{R}^n)$ *is a ground state solution of equation (5.2.1) if and only if* $Q \in \mathcal{G}$.

Having the initial definitions out of the way we can start with the two main theorems. As a consequence of the real-valuedness of the symbol $p$ we notice the reflection-conjugation property given by

$$(P(D)f)(-x) = \overline{(P(D)f)(x)}.$$

A natural question would be whether all ground state solutions to equation (5.2.1) have such a symmetry property using their variational characterization. The following result gives an answer to this.

**Theorem 5.2.1.** *Let* $n \geqslant 1$, $s > 0$ *and* $\sigma \in \mathbb{N}$ *with* $1 \leqslant \sigma < \sigma_*(n,s)$. *Suppose that* $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ *is a ground state solution of equation (5.2.1) where* $\lambda$ *satisfies the before mentioned property. Assume that* $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ *for some* $a > 0$. *Then it holds that*

$$Q(x) = e^{i\alpha}Q^\bullet(x + x_0)$$

*with some constant* $\alpha \in \mathbb{R}$ *and shift constant* $x_0 \in \mathbb{R}^n$. *Additionally* $Q^\bullet$ *is a smooth, bounded and positive definite function in the sense of Bochner and thus satisfies the properties listed in Theorem 4.1.1. If in addition the Fourier symbol $p$ possesses an even symmetry the function* $Q^\bullet$ *has to be real-valued. Consequently any ground state $Q$ is real and even.*

**Remark.** *At first the condition that* $e^{a|\cdot|}Q$ *is in* $L^2(\mathbb{R}^n)$ *for some* $a > 0$ *might sound a little artificial, but without listing the argument one can show that there is a nice assumption which has to be made for $P(D)$ that ensures this condition is met quite easily. For more details on that see Chapter 6. Those analiticity conditions given therein ensure for example that operators of the form*

$$P(D) = c_k(-\Delta)^k + \sum_{\alpha \in \mathbb{N}^n,\ |\alpha| \leqslant m/2-1} c_\alpha(-i\partial_x)^\alpha$$

*with positive* $c_k > 0$, $k \geqslant 1$ *and real coefficients* $\alpha_k \in \mathbb{R}$ *satisfy all the properties needed.*

### 5.2.1 Outlines of the Proof

Let $Q$ be a ground state solution as in Theorem 5.2.1. Then consider the set

$$\Omega = \{\xi \in \mathbb{R}^n|\ |\widehat{Q}(\xi)| > 0\}.$$

Using standard Paley-Wiener arguments (see [31]) we find that $\widehat{Q}$ is analytic and hence $|\widehat{Q}|$ is continuous, hence $\Omega$ is open. This follows from the assumption that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$. It's absolutely important that $\Omega$ is connected, in Section 7.5 one finds a counterexample in nonconnected case. Now we claim the following:

**Claim.** *It holds that* $\Omega = \mathbb{R}^n$.

*Proof Sketch.* We recall that $Q^\bullet \in H^s(\mathbb{R}^n)$ is a ground state solution to equation 5.2.1 as well. Hence we may assume that $\widehat{Q} = |\widehat{Q}| \geqslant 0$ is nonnegative. Applying the Fourier transform to equation 5.2.1 yields

$$\widehat{Q}(\xi) = \frac{1}{p(\xi) + \lambda}(\widehat{Q} * \ldots * \widehat{Q})(\xi),$$

where $(\widehat{Q} * \ldots * \widehat{Q})(\cdot)$ is the $k$-fold convolution with $k = 2\sigma + 1 \in \mathbb{N}$. Then we can show that $\Omega$ equals its $k$-fold Minkowski sum (see Chapter 6 for details)

$$\Omega = \bigoplus_{m=1}^{k} \Omega.$$

Hence we have proven the claim if $0 \in \Omega$. Arguing by contraction and using that $\widehat{Q}$ is analytic yields the result we are looking for. $\qquad\square$

The next step in proving Theorem 5.2.1 is based on an argument in [24]. We recall the result therein and state it without a proof.

**Lemma 5.2.2** (Equality in the Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$). *Let $n \geqslant 1$ and $p \in 2\mathbb{N} \cup \{\infty\}$ with $p > 2$. Suppose that $f, g \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$ with $1/p + 1/p' = 1$ satisfy the majorant condition*

$$|\widehat{f}(\xi)| \leqslant \widehat{g}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

*Additionally, we assume that $\widehat{f}$ is continuous and that $\{|\widehat{f}| > 0\}$ is connected. Then equality*

$$\|f\|_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$$

*holds if and only if*

$$\widehat{f}(\xi) = e^{i(\alpha + \beta \cdot \xi)} \widehat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

*where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ are constants.*

The rest of the proof of Theorem 5.2.1 is based on applying Lemma 5.2.2 with $f = Q$ and $g = Q^\bullet$. The first part follows immediately. For the second part assume that $p(-\xi) = p(\xi)$ for all $\xi \in \mathbb{R}^n$. In this case we need a trick (see Chapter 6 for full details and references) to show that a ground state $Q \in H^s(\mathbb{R}^n)$ has to be real-valued up to a trivial constant complex phase, i.e. we claim that

$$e^{i\theta}Q(\xi) \in \mathbb{R} \quad \text{for all } \xi \in \mathbb{R}^n.$$

This can be shown by using a decomposition into real and imaginary part

$$Q = Q_R + iQ_I.$$

Without stating much details here, we can apply the trick from [14] (see Chapter 6 for details) to prove our claim.

## 5.3   Hardy-Littlewood Majorant problem

As already mentioned the proof of Theorem 5.2.1 is based on an argument developed in [24] (see Lemma 5.2.2). Therein is a counterexample in the case where $\Omega$ is not connected. In this section we will give a short introduction and a counterexample in the case where $p$ is not an even integer. This argument is based on an article by Mockenhaupt and Schlag (see [27]).

### 5.3.1 Introduction to the Upper Majorant Property

In the following we denote $\mathbb{T} : \mathbb{R}/2\pi\mathbb{Z}$ the one dimension torus or circle group. Hardy and Littlewood have investigated the following assumption (e.g. see [3]): $f$ is said to be majorant to $g$ if $|\widehat{g}| \leqslant \widehat{f}$. Clearly, this implies that $f$ is positive definite. The upper majorant property is the following statement:

Whenever $f \in L^p(\mathbb{T})$ is a majorant of $g \in L^p(\mathbb{T})$ then $\|g\|_{L^p(\mathbb{T})} \leqslant \|f\|_{L^p(\mathbb{T})}$.

Hardy and Littlewood proved this fact for all $p \in 2\mathbb{N}$. This is done using convolutions and the Parseval identity. In the case $p = 3$ they found a simple counterexample. Indeed, let $f = 1 + e_1 + e_3$ and $g = 1 - e_1 + e_3$, where $e_k(x) := e(kx)$ and $e(x) := e^{2\pi i x}$ for $x \in \mathbb{T}$. They concluded that $\|f\|_{L^3(\mathbb{T})} < \|g\|_{L^3(\mathbb{T})}$. Later on Boas showed the failure of the majorant property for the group $\mathbb{T}$ for any $p \notin 2\mathbb{N}$ (see [3]). The construction is very much based on the original argument and uses

$$f = 1 + re_1 + r^{k+2}e_{k+2} \quad \text{and} \quad f = 1 + re_1 - r^{k+2}e_{k+2},$$

where $r$ is sufficiently small and $2k < p < 2k + 2$.

In [27] Mockenhaupt and Schlag proved a failure of the upper majorant property with a much simpler example.

**Theorem 5.3.1.** *Suppose $p > 2$ is not an even integer, then there are trigonometric polynomials $q$ and $Q$ with coefficients in $\{0, 1, -1\}$ such that $|\widehat{q}(n)| = \widehat{Q}(n)$ and*

$$\|q\|_{L^p(\mathbb{T})} > (1 + C_p)\|Q\|_{L^p(\mathbb{T})},$$

*where $C_p > 0$ is a constant only dependent on $p$.*

**Remark 5.3.2.** *One can also generalize the concept of the upper majorant property to other groups $G$. For more details one can take a look at the following article [23].*

#### Failure of the Upper Majorant Property on the Real Line $\mathbb{R}$

In the following we will construct a counterexample for the upper majorant property on the real line $\mathbb{R}$ for $p$, where $p$ is not an even integer. The idea behind this is heavily dependent on Theorem 5.3.1 and uses a simple extension technique. The first theorem exemplifies the ideas in $L^1(\mathbb{R})$.

**Theorem 5.3.3.** *Let $g \in C^0(\mathbb{T})$ be a continuous function. Upon defining the following extension $g_\lambda(x) := \lambda^{1/2}g(x)e^{-\lambda x^2}$ for $\lambda > 0$ we conclude*

$$\|g_\lambda\|_{L^1(\mathbb{R})} \to C\|g\|_{L^1(\mathbb{S}^1)} \quad \text{as } \lambda \to 0^+,$$

*for some constant $C > 0$.*

*Proof.* We may assume that $g$ is nonnegative. Fixing $\delta > 0$ and using a density result we find a nonnegative periodic smooth function $g^\delta \in C^\infty(\mathbb{T})$ such that

$$\|g - g^\delta\|_{L^\infty(\mathbb{T})} < \delta.$$

Upon using the Fourier representation we can write $g^\delta$ in the following way

$$g^\delta(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \widehat{g_n^\delta} e^{inx}.$$

By analogy let $g_\lambda^\delta(x) := \lambda^{1/2} g^\delta(x) e^{-\lambda x^2}$ and so we have

$$\|g_\lambda^\delta\|_{L^1(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \widehat{g_n^\delta} \int_{\mathbb{R}} \lambda^{1/2} e^{-\lambda x^2} e^{inx} dx$$

$$= \sum_{n \in \mathbb{Z}} \widehat{g_n^\delta} \mathcal{F}(f_\lambda)(-n),$$

with $\mathcal{F}(f_\lambda)(\xi) = \frac{1}{\sqrt{2}} e^{-\frac{1}{4\lambda}\xi^2}$. Splitting the $L^1$-norm in two parts we finally have

$$\|g_\lambda^\delta\|_{L^1(\mathbb{R})} = \frac{\widehat{g_0^\delta}}{\sqrt{2}} + \sum_{n \in \mathbb{Z}, \ n \neq 0} \widehat{g_n^\delta} \mathcal{F}(f_\lambda)(-n),$$

using that $\|g_\lambda^\delta - g_\lambda\|_{L^1(\mathbb{R})} \leqslant C(\lambda) \|g^\delta - g\|_{L^\infty(\mathbb{T})}$ we conclude

$$\|g_\lambda^\delta\|_{L^1(\mathbb{R})} \to \frac{\widehat{g_0^\delta}}{\sqrt{2}} \quad \text{as } \lambda \to 0^+.$$

Furthermore it's easy to see that

$$\widehat{g_0^\delta} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g^\delta(x) dx = C \|g^\delta\|_{L^1(\mathbb{T})}$$

and so $\|g_\lambda^\delta\|_{L^1(\mathbb{R})} \to C \|g^\delta\|_{L^1(\mathbb{T})}$ as $\lambda \to 0^+$. Finally, using the uniform convergence of $g^\delta \to g$ as $\delta \to 0^+$ one can conclude the result. $\qquad\square$

For the next result let $N \in \mathbb{N}$ and $(c_n)_{|n| \leqslant N} \subseteq \mathbb{C}$ be a finite sequence of complex numbers. Let $P(x) = \sum_{|n| \leqslant N} c_n e^{inx}$ be a trigonometric polynomial then we define

$$P_\lambda(x) := \lambda^{\frac{1}{2p}} P(x) e^{-\frac{\lambda}{p} x^2} \quad \text{for } \lambda > 0 \text{ and } p \geqslant 1,$$

which extends our given polynomial $P$ from the torus group $\mathbb{T}$ to the real line $\mathbb{R}$ in a suitable way.

**Corollary 5.3.1.** *Let $P$ and $P_\lambda$ be defined as above, then for all $p \geqslant 1$ we have*

$$\|P_\lambda\|_{L^p(\mathbb{R})} \to C \|P\|_{L^p(\mathbb{S}^1)} \quad \text{as } \lambda \to 0^+,$$

*for some constant $C = C(p) > 0$ only depending on $p$.*

*Proof.* Following the construction of $P_\lambda$, the $L_p$-norm is given by

$$\|P_\lambda\|_{L^p(\mathbb{R})}^p = \lambda^{1/2} \int_{\mathbb{R}} |P(x)|^p e^{-\lambda x^2} dx.$$

Let $g(x) := |P(x)|^p$ and $g_\lambda(x) := \lambda^{1/2} |P(x)|^p e^{-\lambda x^2}$. Then by Theorem 5.3.3 we can conclude the result. $\qquad\square$

The follwing lemma compares the Fourier transform of two extension of trigonometric polynomials. This will be the final step in proving a failure of the upper majorant property on the real line $\mathbb{R}$ for $p$ which is not an even integer.

**Lemma 5.3.1.** *Let $P$ and $Q$ be trigonometric polynomials with their Fourier coefficients $\widehat{P_n}$ and $\widehat{Q_n}$. Let $P_\lambda$ and $Q_\lambda$ be their respective extensions to $\mathbb{R}$. Additionally assume that $|\widehat{P_n}| = \widehat{Q_n}$. Then we have $|\widehat{P_\lambda}| \leqslant \widehat{Q_\lambda}$.*

*Proof.* Let $P(x) = \sum_{|n| \leqslant N} \widehat{P_n} e^{inx}$. Then from a simple calculation we have

$$
\begin{aligned}
\widehat{P_\lambda}(\xi) &= \frac{\lambda^{\frac{1}{2p}}}{\sqrt{2\pi}} \int_{\mathbb{R}} P(x) e^{-\frac{\lambda}{p} x^2} e^{-i\xi x} dx \\
&= \frac{\lambda^{\frac{1}{2p}}}{\sqrt{2\pi}} \sum_{|n| \leqslant N} \widehat{P_n} \int_{\mathbb{R}} e^{-\frac{\lambda}{p} x^2} e^{-i(\xi-n)x} dx \\
&= C(\lambda, p) \sum_{|n| \leqslant N} \widehat{P_n} e^{-\frac{p(n-\xi)^2}{4\lambda}},
\end{aligned}
$$

where $C(\lambda, p) > 0$ is a constant only dependent on $\lambda$ and $p$. Hence we have

$$
|\widehat{P_\lambda}(\xi)| \leqslant C(\lambda, p) \sum_{|n| \leqslant N} \widehat{Q_n} e^{-\frac{p(n-\xi)^2}{4\lambda}} = \widehat{Q_\lambda}(\xi)
$$

for all $\lambda > 0$, $p \geqslant 1$ and $\xi \in \mathbb{C}$. $\qquad \square$

The last part will give a slight modification of Theorem 5.3.1.

**Theorem 5.3.4.** *Suppose $p > 2$ is not an even integer, then there are trigonometric polynomials $P$ and $Q$ with coefficients in $\{-1, 0, 1\}$ such that $|\widehat{P_n}| = \widehat{Q_n}$ and*

$$
\|P_\lambda\|_{L^p(\mathbb{R})} > (1 + C)\|Q_\lambda\|_{L^p(\mathbb{R})}
$$

*with*

$$
|\widehat{P_\lambda}| \leqslant \widehat{Q_\lambda},
$$

*where $P_\lambda$ (resp. $Q_\lambda$) is the extension to $\mathbb{R}$ and $C = C(p)$ is a constant only dependent on $p$.*

*Proof.* Following Theorem 5.3.1 we have

$$
\|P\|_{L^p(\mathbb{T})} > (1 + C)\|Q\|_{L^p(\mathbb{T})}.
$$

Recalling Corollary 5.3.1 and letting $\lambda$ be very small we can conclude the first part. The second part follows directly from Lemma 5.3.1. $\qquad \square$

# Chapter 6

# On Symmetry and Uniqueness of Ground States for Linear and Nonlinear Elliptic PDEs

This Chapter consists of an article which was written in collaboration with my mentor Enno Lenzmann and postdoc Jérémy Sok, who also works in the same research group. The original article is found in [5]. In the following pages the original article undergoes some small modifications due to formatting but the mathematical content is identical and proper citations are included as in [5]. Note that due to including the article in this thesis the reference numbers might be different compared to the original ones.

## 6.1 Introduction and Main Results

We study symmetry properties and uniqueness of ground states for linear and nonlinear elliptic PDEs posed on $\mathbb{R}^n$. In particular, we will be interested in a general class of problems (including higher-order PDEs) which cannot be studied by classical methods such as maximum principles or Polya-Szegö inequalities. Instead our approach here is based on Fourier methods together with a classification of the Hardy-Littlewood majorant problem in $\mathbb{R}^n$, which was recently obtained in [24].

Our results on the linear and nonlinear problems are presented in two separate subsections.

### 6.1.1 Linear Results

Let $s > 0$ be a real number. We consider ground states $\psi \in H^s(\mathbb{R}^n)$ of linear equations of the from

$$P(D)\psi + V\psi = E\psi, \tag{6.1.1}$$

where $E \in \mathbb{R}$ is the eigenvalue and $V : \mathbb{R}^n \to \mathbb{R}$ denotes a given potential. Here $P(D)$ stands for a self-adjoint, elliptic constant coefficient pseudo-differential operator of order $2s$. More precisely, we assume the following condition.

**Assumption 1.** *Let $s > 0$. The pseudo-differential operator $P(D)$ is given by*

$$(\widehat{P(D)f})(\xi) = p(\xi)\widehat{f}(\xi),$$

*with some continuous function $p : \mathbb{R}^n \to \mathbb{R}$ that satisfies the estimates*

$$A|\xi|^{2s} + c \leqslant p(\xi) \leqslant B|\xi|^{2s} \quad \text{for all } \xi \in \mathbb{R}^n$$

*with suitable constants $A > 0$, $B > 0$, and $c \in \mathbb{R}$.*

Let us now suppose that $P(D)$ satisfies Assumption 1. We assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a bounded potential[1]. Hence we can consider the well-defined minimization problem

$$E_0 = \inf\{\langle f, (P(D) + V)f\rangle : f \in H^s(\mathbb{R}^n), \|f\|_{L^2} = 1\} > -\infty. \tag{6.1.2}$$

Furthermore, if we assume that $V(x) \to 0$ as $|x| \to \infty$ in the sense that $\{|V(x)| < \varepsilon\}$ has finite Lebesgue measure for every $\varepsilon > 0$, it easy to see that

$$E_0 \leqslant \inf_{\xi \in \mathbb{R}^n} p(\xi) = \inf \sigma_{\mathrm{ess}}(H), \tag{6.1.3}$$

where $\sigma_{\mathrm{ess}}(H)$ denotes the essential spectrum of the self-adjoint operator $H = P(D) + V$ defined via the quadratic form appearing in (6.1.2). Provided a minimizer $\psi \in H^s(\mathbb{R}^n)$ for (6.1.2) exists, it is easy to see $\psi$ solves (6.1.1) with $E = E_0$. Conversely, any solution $\psi \in H^s(\mathbb{R}^n)\backslash\{0\}$ of (6.1.1) with $E = E_0$ is a minimizer of problem (6.1.2) up to a trivial rescaling to ensure the normalization condition $\|\psi\|_{L^2} = 1$. Following usual nomenclature in spectral theory of Schrödinger operators, we refer to such minimizing solutions $\psi \in H^s(\mathbb{R}^n)$ as **ground states** for the linear problem (6.1.1). To have a better contradisctinction for the nonlinear problems discussed below, we will also use the term **linear ground state** sometimes.

In the setting of Schrödinger operators when $P(D) = -\Delta$, we remark that uniqueness of ground states $\psi$ (up to a trivial multiplicative constant) is a classical result, which can be proven by an wide array of known methods such as maximum principles, Polya-Szegö principle, and Perron-Frobenius arguments involving the corresponding heat kernel $e^{t\Delta}$. Also, the fractional case for $P = (-\Delta)^s$ with $0 < s < 1$ can be readily tacked with such methods.

However, it is fair to say that the study of uniqueness of ground states of linear problems like (6.1.1) becomes quite elusive in the case of operators $P(D)$ with higher order $2s > 1$. In fact, uniqueness of ground states may fail in such cases. But in certain natural cases of interest (e.g. arising from linearizations around ground states of nonlinear PDEs), the potential $V$ does have the noteworthy property of having a negative Fourier transform $\widehat{V} < 0$ almost everywhere. As our first main result in this paper, we prove that ground states for (6.1.1) are in fact unique (up to a trivial constant) under this condition on $V$.

**Theorem 6.1.1** (Uniquenes of Linear Ground States)**.** *Let $n \geqslant 1$, $s > 0$, and suppose that $P(D)$ satisfies Assumption 1. Assume that $V : \mathbb{R}^n \to \mathbb{R}$ has a Fourier transform $\widehat{V} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\widehat{V}(\xi) < 0$ for almost every $\xi \in \mathbb{R}^n$. Finally, we suppose that $E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi)$ holds in (6.1.2). Then we have the following properties.*

(a) ***Uniqueness:*** *The ground state solution $\psi \in H^s(\mathbb{R}^n)$ for (6.1.1) is unique (up to a constant phase). Moreover, we have the strict positivity property of its Fourier transform*

$$e^{\mathrm{i}\theta}\widehat{\psi}(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n,$$

*where $\theta \in \mathbb{R}$ is a constant.*

---

[1]We could relax this condition to unbounded potentials $V \in L^\infty(\mathbb{R}^n) + L^p(\mathbb{R}^n)$ with $p > \max\{n/2s, 1\}$. For the sake of simplicity, we omit this generalization here.

(b) **Symmetries:** As a consequence of (i), the ground state $\psi(x)$ (is up to a constant phase) has the even symmetry

$$\psi(-x) = \overline{\psi(x)} \quad \text{for a. e. } x \in \mathbb{R}^n.$$

If, in addition, the symbol $p(-\xi) = p(\xi)$ is even, then $\psi : \mathbb{R}^n \to \mathbb{R}$ is real-valued (up to a constant phase).

**Remarks.** 1) Under some technical assumptions, we could also treat the non-generic case when $E_0 = \inf_{\xi \in \mathbb{R}^n} p(\xi) = \inf \sigma_{\mathrm{ess}}(H)$ coincides with the bottom of the essential spectrum of $H = P(D) + V$. However, we omit this discussion here.

2) Note that $V \in L^\infty$ by our assumption that $\widehat{V} \in L^1(\mathbb{R}^n)$. As mentioned above, we could relax our conditions to unbounded potentials $V$. But again in order to keep our focus on its simple main argument, we refrain from considering more general cases here.

3) In some sense, the result above yields a Perron-Frobenius type result (i. e. positivity and uniqueness of ground states) but when viewed in Fourier space. Of course, the ground state $\psi(x)$ may fail to be real-valued at all (let alone strictly positive) in $x$-space. In fact, a simple example arises in the linearized problem for traveling solitary waves for dispersion-generalized NLS, e. g., the linear ground state of $\psi \in H^s(\mathbb{R}^n)$ for equations of the form

$$((-\Delta)^s + \mathrm{i} v \cdot \nabla + V)\psi = E\psi$$

with $s \geqslant 1/2$ and $v \in \mathbb{R}^n \backslash \{0\}$ (and $|v| < 1$ when $s = 1/2$). It is easy to see that any non-trivial solutions $\psi \in H^s(\mathbb{R}^n)$ must be complex-valued due to the presence of the 'boost term' $\mathrm{i} v \cdot \nabla$. However, the result above shows that (under suitable assumptions on $V$), we always have the strict positivity $e^{\mathrm{i}\theta}\widehat{\psi}(\xi) > 0$ for all $\xi \in \mathbb{R}^n$.

4) If we additionally assume that $\widehat{\psi} \in L^1(\mathbb{R}^n)$ (or more generally $\widehat{\psi}$ is a finite positive measure on $\mathbb{R}^n$), then $\psi : \mathbb{R}^n \to \mathbb{C}$ is a positive definite function in the sense of Bochner. See also below.

5) Notice since $\widehat{V}$ and $V$ are both assumed to be real-valued, the potential $V(-x) = V(x)$ is an even function.

## 6.1.2  Nonlinear Results

We now turn to ground state solutions of nonlinear elliptic PDEs in $\mathbb{R}^n$ with pseudo-differential operators $P(D)$ of arbitrary order. As before, let $s > 0$ be a real number. We consider solutions $Q \in H^s(\mathbb{R}^n)$ of nonlinear elliptic PDEs of the form

$$P(D)Q + \lambda Q - |Q|^{2\sigma}Q = 0. \tag{6.1.4}$$

Here $\sigma > 0$ is a given number, which we later assume to be an integer, and $\lambda \in \mathbb{R}$ denotes a given parameter, which plays the role of a nonlinear eigenvalue. We opted to use the letters $Q$ and $\lambda$ instead of $\psi$ and $E$ above in order to keep the distinction between linear and nonlinear problems more clearly.

As before, we suppose that $P(D)$ denotes a pseudo-differential operator with constant coefficients defined in Fourier space as

$$\widehat{(P(D)u)}(\xi) = p(\xi)\widehat{u}(\xi). \tag{6.1.5}$$

For the nonlinear problem (8.1.3), we now impose the following conditions on $P(D)$, where $S_{1,0}^m$ with $m \in \mathbb{R}$ denotes the usual Hörmander class of symbols for pseudo-differential operators on $\mathbb{R}^n$.

**Assumption 2.** *Let $s > 0$ be a real number. We suppose that $P(D)$ is a pseudo-differential operator of order $2s$ with a symbol $p(\xi) \in S_{1,0}^{2s}$ that satisfies the following conditions.*

(i) **Real-Valuedness:** *The symbol $p : \mathbb{R}^n \to \mathbb{R}$ is real-valued.*

(ii) **Ellipticity Condition:** *There exist constants $c > 0$ and $R > 0$ such that*

$$p(\xi) \geqslant c|\xi|^{2s} \quad for \quad |\xi| \geqslant R.$$

For the rest of this subsection, we will always assume that $P(D)$ satisfies Assumption 2. As a consequence, the operator $P(D) = P(D)^*$ is self-adjoint and bounded below on $L^2(\mathbb{R}^n)$ with operator domain $H^{2s}(\mathbb{R}^n)$. Furthermore, we assume the eigenvalue parameter $\lambda \in \mathbb{R}$ in (8.1.3) satisfies the condition

$$\lambda > \inf_{\xi \in \mathbb{R}^n} p(\xi), \tag{6.1.6}$$

which is equivalent to saying that $-\lambda$ lies strictly below the essential spectrum $\sigma_{\mathrm{ess}}(P(D))$. As a direct consequence, we obtain the norm equivalence

$$\langle f, (P(D) + \lambda)f \rangle \simeq \|f\|_{H^s}^2,$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f} g$ denotes the standard scalar product on $L^2(\mathbb{R}^n)$. Likewise, we introduce the critical exponent $\sigma_*(n,s)$ (which is not necessarily an integer) given by

$$\sigma_*(n,s) = \begin{cases} \dfrac{2s}{n-2s} & \text{for } s < \frac{n}{2}, \\ +\infty & \text{for } s \geqslant \frac{n}{2}. \end{cases}$$

Thus exponents $\sigma < \sigma_*(n,s)$ correspond to the *$H^s$-subcritical case*, which is the situation we shall consider in this paper[2]. Note that we have the Sobolev-type inequality

$$\|f\|_{L^{2\sigma+2}}^2 \leqslant C\langle f, (P(D) + \lambda)f \rangle \tag{6.1.7}$$

for any $f \in H^s(\mathbb{R}^n)$, where $C > 0$ denotes a suitable constant. Due to the subcriticality $\sigma < \sigma_*(n,s)$, standard variational methods yield existence of an optimal constant $C > 0$ as well as the existence of optimizers $Q \in H^s(\mathbb{R}^n)$ for (6.1.7), which are easily seen to solve (8.1.3) after a suitable rescaling $Q \mapsto \alpha Q$ with some constant $\alpha$. In fact, we relate this fact to our definition of ground state solutions for (8.1.3) as follows.

**Definition 6.1.1.** *With the notation and assumptions above, we say that $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ is a **ground state solution** if $Q$ solves equation (8.1.3) and optimizes inequality (6.1.7).*

Equivalently, as shown in Lemma 6.2.3 below, we obtain that $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ is a ground state solution for (8.1.3) if and only if $Q$ minimizes the action functional

$$\mathcal{A}(f) = \frac{1}{2}\langle f, (P(D) + \lambda)f \rangle - \frac{1}{2\sigma+2}\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \tag{6.1.8}$$

among all its non-trivial critical points. Thus the set of ground state solutions is given by

$$\mathcal{G} = \{Q \in K : \mathcal{A}(Q) \leqslant \mathcal{A}(R) \text{ for all } R \in K\}, \tag{6.1.9}$$

where $K = \{u \in H^s(\mathbb{R}^n)\backslash\{0\} : \mathcal{A}'(u) = 0\}$.

---

[2]To avoid technicalities, we shall omit the discussion of the critical case $\sigma = \sigma_*(n,s)$ in this paper.

We now turn to the question of *symmetries* for ground states solutions for (8.1.3). As consequence of the real-valuedness of the symbol $p(\xi)$, we notice the reflection-conjugation property

$$(P(D)f)(-x) = \overline{(P(D)f)(x)}. \tag{6.1.10}$$

Based on this observation, we may ask whether all ground state solutions $Q$ 'inherit' this symmetry property by their variational characterization. In fact, we will prove the following result in this paper when the exponent $\sigma \in \mathbb{N}$ is an integer.

**Theorem 6.1.2** (Symmetry for Nonlinear Ground States). *Let $n \geqslant 1$, $s > 0$, and $\sigma \in \mathbb{N}$ with $1 \leqslant \sigma < \sigma_*(s,n)$. Suppose $Q \in H^s(\mathbb{R}^n)\backslash\{0\}$ is a ground state solution of (8.1.3) where $\lambda \in \mathbb{R}$ satisfies (6.1.6). Finally, we assume that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$. Then it holds that*

$$Q(x) = e^{i\alpha}Q^{\bullet}(x + x_0)$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Here $Q^{\bullet} : \mathbb{R}^d \to \mathbb{C}$ is a smooth, bounded, and positive definite function in the sense of Bochner. As a consequence, it holds that*

$$Q^{\bullet}(-x) = \overline{Q^{\bullet}(x)} \quad and \quad Q^{\bullet}(0) \geqslant |Q^{\bullet}(x)| \quad for\ all\ x \in \mathbb{R}^n.$$

*If, in addition, the operator $P(D)$ has an even symbol $p(\xi) = p(-\xi)$, the function $Q^{\bullet}$ must be real-valued (up to a trivial constant complex phase). Consequently, any ground state $Q$ for (8.1.3) is real and even, i.e., we have $Q(-x) = Q(x)$ for all $x \in \mathbb{R}^n$.*

**Remarks.** *1) In Theorem 6.1.3 below, we shall give an analyticity condition on $P(D)$ that ensures the exponential decay property $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$. In particular, it applies to operators of the form*

$$P(D) = c_k(-\Delta)^k + \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant m/2-1} c_\alpha(-i\partial_x)^\alpha$$

*with positive $c_k > 0$, $k \geqslant 1$, and real arbitrary coefficients $c_\alpha \in \mathbb{R}$. For example, we could take $P(D) = \Delta^2 - \mu\Delta$ with any $\mu \in \mathbb{R}$. Another important class is given by the pseudo-differential operators*

$$P(D) = (1 - \Delta)^s \quad for\ any\ s > 0.$$

*2) The proof of Theorem 6.1.2 will be based on the recent characterization [24] of the case of equality in **Hardy-Littlewood majorant problem** in $\mathbb{R}^n$. Here the topological property that the set $\Omega = \{\xi \in \mathbb{R}^n : |\widehat{Q}(\xi)| > 0\}$ is connected in $\mathbb{R}^n$ will enter in an essential way.*

*3) The function $Q^{\bullet} : \mathbb{R}^n \to \mathbb{C}$ will be obtained by taking the absolute value on the Fourier side, i.e., we set $Q^{\bullet} = \mathcal{F}^{-1}(|\mathcal{F}Q|)$. See Section 6.2 for more details.*

*4) If the symbol $p = p(|\xi|)$ is radially symmetric and strictly increasing in $|\xi|$, then we actually can show that $Q = Q^{\sharp}$ holds (up to tranlation and complex phase), where $Q^{\sharp}$ denotes the symmetric-decreasing Fourier rearrangement of $Q$. See [24].*

Next, we turn to the question whether (not necessarily ground state) solutions $Q \in H^s(\mathbb{R}^n)$ of (8.1.3) satisfy the exponential decay estimate that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$, which is a condition imposed in Theorem 6.1.2 above. In fact, we can adapt an analytic continuation argument originally developed to study exponential decay of eigenfunctions of Schrödinger operators due to Combes and Thomas [11], building upon O'Connors work [29]. Here is a list of sufficient conditions on $P(D)$ to carry out such an argument in our case.

**Assumption 3.** *Suppose $P(D)$ has a symbol $p(\xi)$ which has an analytic continuation to the strip $T_\delta = \{z \in \mathbb{C}^n : |\operatorname{Im} z| < \delta\}$ with some $\delta > 0$. Moreover, we assume the following conditions.*

(i) *For each $\kappa \in T_\delta$, there exist constant $\gamma \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ such that*

$$|\arg(p(\xi + \kappa) - \gamma)| \leqslant \theta \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

(ii) *For each $\kappa \in T_\delta$, there exist constants $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$ such that*

$$a_1 |\xi|^{2s} - b_1 \leqslant \operatorname{Re}(p(\xi + \kappa)) \leqslant a_2 |\xi|^{2s} + b_2 \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

**Remark.** *It is elementary to check that any polynomial $p(\xi) = \sum_{|\alpha| \leqslant m} c_\alpha \xi^\alpha$ with coefficients $c_\alpha \in \mathbb{R}$ and $\inf_{\xi \in \mathbb{R}^n} p(\xi) > -\infty$ satisfies the above conditions (with $m = 2s$). In particular, operators of the form*

$$P(D) = \Delta^2 - \mu\Delta + \mathrm{i}v \cdot \nabla \quad \text{with} \quad \mu \in \mathbb{R}, v \in \mathbb{R}^n$$

*fall under the scope of Assumption 3. Also, one can verify that the same is true for operators $P(D) = (1 - \Delta)^s$ with $s > 0$.*

We can now state the following result, which established the assumed exponential decay $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$ appearing in Theorem 6.1.2 above.

**Theorem 6.1.3** (Exponential Decay)**.** *Let $n, s$, and $\sigma$ be as in Theorem 6.1.2. If $P(D)$ satisfies Assumption 3, then any solution $Q \in H^s(\mathbb{R}^n)$ of (8.1.3) satisfies $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$. As a consequence, the conclusions of Theorem 6.1.2 hold true.*

**Remark.** *For an in-depth analysis of exponential decay of eigenfunctions of $P(D) + V$ with polynomial symbol $p(\xi)$, we refer to the recent work [?]. However, for our purposes here, it is sufficient to obtain a 'coarse' exponential decay estimate saying that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$.*

### 6.1.3  Strategy of the Proofs

Let us briefly describe the strategy behind the proofs of our main results. The idea to prove Theorems 6.1.1 and 6.1.2 is based on taking absolute values of the Fourier transform. That is, for a given function $f \in L^2(\mathbb{R}^n)$, we define

$$f^\bullet = \mathcal{F}^{-1}(|\mathcal{F}f|). \tag{6.1.11}$$

By Plancherel's identity, we immediately find that $\|f^\bullet\|_{L^2} = \|f\|_{L^2}$ and $\langle f^\bullet, P(D)f^\bullet \rangle = \langle f, P(D)f \rangle$. Moreover, for potentials $V : \mathbb{R}^n \to \mathbb{R}$ as in Theorem 6.1.1 as well as for integers $\sigma \in \mathbb{N}$ with $1 \leqslant \sigma < \sigma_*(s, n)$, we readily obtain the inequalities[3]

$$\langle f^\bullet, Vf^\bullet \rangle \leqslant \langle f, Vf \rangle \quad \text{and} \quad \|f\|_{L^{2\sigma+2}} \leqslant \|f^\bullet\|_{L^{2\sigma+2}} \tag{6.1.12}$$

for any $f \in H^s(\mathbb{R}^n)$. Thus if $\psi \in H^s(\mathbb{R}^n)$ and $Q \in H^s(\mathbb{R}^n)$ are ground states for (6.1.1) and (8.1.3), respectively, so are the functions $\psi^\bullet$ and $Q^\bullet$. Therefore, the conclusions of Theorems 6.1.1 and 6.1.2 will follow once we can show that the F

$$\widehat{\psi}(\xi) = e^{\mathrm{i}\theta}|\widehat{\psi}(\xi)| \quad \text{and} \quad \widehat{Q}(\xi) = e^{\mathrm{i}(\alpha + \beta \cdot \xi)}|\widehat{Q}(\xi)| \tag{6.1.13}$$

---

[3]See also the remark following Lemma 6.2.1 for the case of non-integer $\sigma$.

with some constants $\theta, \alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. We remark that $\widehat{\psi}$ and $\widehat{Q}$ are easily seen to be continuous functions in our setting.

In terms of harmonic analysis, we are faced to solve a *phase retrieval problem*, i.e., given the modulus of the Fourier transform of a function, we try reconstruct its phase by exploiting some additional facts. For the linear problem (6.1.1), this is an elementary task provided that the potential $V$ satisfies the hypothesis of Theorem 6.1.1. Not surprisingly, the nonlinear problem (8.1.3) is harder to analyze. Here, a rigidity result for the so-called *Hardy–Littlewood majorant problem* in $\mathbb{R}^n$ (recently obtained in [24]) enters in an essential way; see also Lemma 8.6.4 below. In order to apply this result, we must verify the topological property that

$$\Omega = \{\xi \in \mathbb{R}^n : |\widehat{Q}(\xi)| > 0\} \tag{6.1.14}$$

is a *connected* set in $\mathbb{R}^n$. To prove this fact (where indeed we show that $\Omega = \mathbb{R}^n$ holds in our case), we will make use of analyticity argument: By standard Payler–Wiener arguments, the exponential decay $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$ will ensure that $\widehat{Q}(\xi)$ is analytic in some complex strip around $\mathbb{R}^n$. The analyticity of $\widehat{Q}$ together with the fact $Q$ solves (8.1.3) will then yield the desired result.

Finally, we recall from above that the proof of Theorem 6.1.3 is based on a strategy for deriving exponential decay for $N$-body Schrödinger operators due to Combes and Thomas [11] based on O'Connor's lemma [29].

## 6.2   Preliminaries

### 6.2.1   Fourier Inequalities and Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$

For a function $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform by

$$(\mathcal{F}f)(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi\mathrm{i}x\cdot\xi}\,d\xi, \tag{6.2.1}$$

with the usual extension to $f \in L^2(\mathbb{R}^n)$ by density. For $f \in L^2(\mathbb{R}^n)$ given, we recall that the function $f^\bullet \in L^2(\mathbb{R}^n)$ is obtained by taking the absolute value on the Fourier side, i.e., we set

$$f^\bullet = \mathcal{F}^{-1}(|\mathcal{F}f|). \tag{6.2.2}$$

From Plancherel's identity it is clear that $\|f\|_{L^2} = \|f^\bullet\|_{L^2}$ holds. We record some further elementary properties of this operation.

**Lemma 6.2.1.** *Let $n \geq 1$, $s > 0$, and $\sigma \in \mathbb{N}$ with $\sigma < \sigma_*(s, n)$.*

(i) *For any $f \in H^s(\mathbb{R}^n)$, we have*

$$\langle f^\bullet, P(D)f^\bullet \rangle = \langle f, P(D)f \rangle \quad and \quad \|f\|_{L^{2\sigma+2}} \leq \|f^\bullet\|_{L^{2\sigma+2}}.$$

(ii) *For any $f \in L^2(\mathbb{R}^n)$, it holds that $f^\bullet(-x) = \overline{f^\bullet(x)}$ for a.e. $x \in \mathbb{R}^n$.*

(iii) *If $f \in L^2(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then $f^\bullet : \mathbb{R}^n \to \mathbb{C}$ is a continuous and bounded function which is **positive definite** in the sense that for any points $x_1, \ldots, x_N \in \mathbb{R}^n$ the matrix $[f^\bullet(x_k - x_l)]_{1 \leq k, l \leq N}$ is positive semi-definite, i.e.,*

$$\sum_{k,l=1}^N f^\bullet(x_k - x_l)\overline{v}_k v_l \geq 0 \quad for\ all\ v \in \mathbb{C}^N.$$

*In particular, the inequality $f^\bullet(0) \geq |f^\bullet(x)|$ holds for all $x \in \mathbb{R}^n$.*

**Remark.** *The inequality $\|f\|_{L^{2\sigma+2}} \leqslant \|f^\bullet\|_{L^{2\sigma+2}}$ for integer $\sigma \in \mathbb{N}$ is a consequence of the so-called **upper majorant property (UMP)** for $L^p$-norms with $p \in 2\mathbb{N} \cup \{\infty\}$. That is, for such $p$ and $f, g \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$ we have the implication*

$$|\widehat{f}(\xi)| \leqslant \widehat{g}(\xi) \text{ for a. e. } \xi \in \mathbb{R}^n \quad \Longrightarrow \quad \|f\|_{L^p} \leqslant \|g\|_{L^p}.$$

*On the other hand, it is well-known that (UMP) fails for $L^p$-norms when $p \notin 2\mathbb{N} \cup \{\infty\}$. Indeed, the known counterexamples (see e. g. [3, 26, 27]) show the failure of (UMP) in the torus case, i. e., for $L^p(\mathbb{T})$. But these examples can be easily transferred to the real line case as follows. Suppose $p > 2$ is not an even integer. Then, as shown in [27], there exist trigonometric polynomials $q$ and $Q$ with Fourier coefficients $|\widehat{q}(n)| = \widehat{Q}(n)$ for all $n \in \mathbb{Z}$ satisfying $\|q\|_{L^p(\mathbb{T})} > \|Q\|_{L^p(\mathbb{T})}$. We can lift this example to Fourier transform in $\mathbb{R}$ by considering the Schwartz functions*

$$q_\lambda(x) = \lambda^{\frac{1}{2p}} q(x) e^{-\lambda x^2}, \quad Q_\lambda(x) = \lambda^{\frac{1}{2p}} Q(x) e^{-\lambda x^2}$$

*with $\lambda > 0$. It is elementary to check that $\|q_\lambda\|_{L^p(\mathbb{R})} \to \|q\|_{L^p(\mathbb{T})}$ and $\|Q_\lambda\|_{L^p(\mathbb{R})} \to \|Q\|_{L^p(\mathbb{T})}$ as $\lambda \to 0^+$. Furthermore, we readily check for the Fourier transforms $|\widehat{q}_\lambda(\xi)| \leqslant \widehat{Q}_\lambda(\xi)$ for all $\xi \in \mathbb{R}^n$. Thus by taking $\lambda > 0$ sufficiently small, we see that (UMP) fails for $L^p(\mathbb{R})$ with non-even integer $p$.*

*Proof.* First, it is evident that $\langle f, P(D)f \rangle = \int_{\mathbb{R}^n} p(\xi)|\widehat{f}(\xi)|^2 \, d\xi = \langle f^\bullet, P(D)f^\bullet \rangle$. Next, let $p = 2\sigma + 2$ with $\sigma \in \mathbb{N}$ with $\sigma < \sigma_*(s, n)$. By Hölder's inequality, we note that $f \in H^s(\mathbb{R}^n)$ implies that $f \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$, i. e. we have $\widehat{f} \in L^{p'}(\mathbb{R}^n)$, where $p' = \frac{2\sigma+2}{2\sigma+1}$ denotes the dual exponent of $p = 2\sigma + 2$. Thus we can apply to conclude

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} = (\widehat{f} * \overline{\widehat{f}} * \ldots * \widehat{f} * \overline{\widehat{f}})(0)$$

with $2\sigma + 1$ convolutions on the right-hand side. With the use of the autocorrelation function

$$\Psi_{\widehat{f}}(\xi) = (\widehat{f} * \overline{\widehat{f}})(\xi) = (\widehat{f} * \overline{\widehat{f}(-\cdot)})(\xi) = \int_{\mathbb{R}^n} \widehat{f}(\xi + \xi')\overline{\widehat{f}(\xi')} \, d\xi',$$

we can write

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} = (\Psi_{\widehat{f}} * \ldots * \Psi_{\widehat{f}})(0),$$

where the number of convolutions is equal to $\sigma$. Since $|\Psi_{\widehat{f}}|(\xi) \leqslant \Psi_{|\widehat{f}|}(\xi)$, we deduce

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant (\Psi_{|\widehat{f}|} * \ldots * \Psi_{|\widehat{f}|})(0) = \|f^\bullet\|_{L^{2\sigma+2}}^{2\sigma+2},$$

which completes the proof of item (i).

The proof of (ii) is a direct consequence of the fact that $\widehat{f^\bullet} = |\widehat{f}|$ is real-valued. Furthermore, item (iii) is a classical fact using that $\widehat{f^\bullet} = |\widehat{f}| \geqslant 0$ is non-negative and assuming that $\widehat{f^\bullet} \in L^1(\mathbb{R}^n)$ (or more generally $\widehat{f^\bullet}$ is a finite measure on $\mathbb{R}^n$); see, e. g., for a discussion of positive-definite functions and Bochner's theorem. $\square$

As a next essential fact we recall from [24] the following **rigidity result**.

**Lemma 6.2.2** (Equality in the Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$). *Let $n \geqslant 1$ and $p \in 2\mathbb{N} \cup \{\infty\}$ with $p > 2$. Suppose that $f, g \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$ with $1/p + 1/p' = 1$ satisfy the majorant condition*

$$|\widehat{f}(\xi)| \leqslant \widehat{g}(\xi) \quad \text{for a. e. } \xi \in \mathbb{R}^n.$$

*In addition, we assume that $\widehat{f}$ is continuous and that $\{\xi \in \mathbb{R}^n : |\widehat{f}(\xi)| > 0\}$ is a connected set. Then equality*

$$\|f\|_{L^p} = \|g\|_{L^p}$$

*holds if and only if*

$$\widehat{f}(\xi) = e^{i(\alpha + \beta \cdot \xi)}\widehat{g}(\xi) \quad \textit{for all } \xi \in \mathbb{R}^n,$$

*with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$.*

**Remark.** *The connectedness of the set $\Omega \subset \mathbb{R}^n$ is essential. See also [24] for a counterexample when $\Omega$ is not connected. However, as we will show below, the set $\Omega = \{\xi \in \mathbb{R}^n : |\widehat{Q}(\xi)| > 0\}$ will turn out to be connected (in fact, we show $\Omega = \mathbb{R}^n$ holds) for the ground states $Q$ of (8.1.3) in the setting considered in this paper.*

### 6.2.2 Smoothness and Exponential Decay of $Q$

Recall that we always suppose that $P(D)$ satisfies Assumptions 2.

**Proposition 6.2.1.** *Let $n \geqslant 1$, $s > 0$, and $\sigma \in \mathbb{N}$ with $1 \leqslant \sigma < \sigma_*(n,s)$. Then any solution $Q \in H^s(\mathbb{R}^n)$ satisfies $Q \in H^\infty(\mathbb{R}^n) = \bigcap_{k \geqslant 0} H^k(\mathbb{R}^n)$.*

*Proof.* This follows from Sobolev embeddings and regularity theory for pseudo-differential operators. For the reader's convenience, we give the details. By picking a sufficiently large constant $\mu > 0$, we can assume that $p(\xi) + \mu \gtrsim \langle \xi \rangle^{2s}$ holds. Hence $Q \in H^s(\mathbb{R}^n)$ solves

$$(P(D) + \mu)Q = (Q\overline{Q})^\sigma Q + (\mu - \lambda)Q. \tag{6.2.3}$$

Indeed, let us first suppose that $Q \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $(P(D) + \mu)Q = (Q\overline{Q})^\sigma Q + (\mu - \lambda)Q \in H^s \cap L^\infty(\mathbb{R}^n)$ holds, since $\sigma$ is an integer and $H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ forms an algebra. Now since $p(\xi) + \mu \gtrsim \langle \xi \rangle^{2s}$, we have that $(P(D) + \mu)^{-1}$ belongs to class $S_{1,0}^{-2s}$. Therefore $(P(D) + \mu)^{-1} : H^m(\mathbb{R}^n) \to H^{m+2s}(\mathbb{R}^n)$ for any $m \in \mathbb{R}$ and we deduce that $Q \in H^\infty(\mathbb{R}^n) = \cap_{k \geqslant 0} H^k(\mathbb{R}^n)$ by iterating the equation (6.2.3).

It remains to show that $Q \in L^\infty(\mathbb{R}^n)$ follows from our assumptions. If $s > n/2$, this is clearly true by Sobolev embeddings. For $0 < s \leqslant n/2$, we need to bootstrap the equation by using the mapping properties of the inverse $(P(D) + \mu)^{-1}$. Indeed, we note that $|Q|^{2\sigma}Q \in L^{\frac{p_*}{2\sigma+1}}(\mathbb{R}^n)$ with $p_* = 2n/(n-2s)$ by the Sobolev embedding $H^s(\mathbb{R}^n) \subset L^{p*}(\mathbb{R}^n)$. Since $(P(D) + \mu)^{-1} : H^{m,p}(\mathbb{R}^n) \to H^{m+2s,p}(\mathbb{R}^n)$ for any $m \in \mathbb{R}$ and $1 < p < \infty$, we deduce that $Q \in H^{2s, \frac{p_*}{2\sigma+1}}(\mathbb{R}^n)$, which is a gain of regularity for $Q$. We can proceed this argument to obtain after finitely many steps that $Q \in H^{m,p}(\mathbb{R}^n)$ with $m > n/p$, which yields that $Q \in L^\infty(\mathbb{R}^n)$ by Sobolev embeddings. $\qquad \square$

### 6.2.3 On the Notion of Ground State Solutions

As remarked in the introduction, we have the following simple fact, where we assume $n, s, \sigma$, and $\lambda$ satisfy the assumptions of Theorem 6.1.2. Recall the definition of the set $\mathcal{G}$ in (6.1.9).

**Lemma 6.2.3.** *$Q \in H^s(\mathbb{R}^n)$ is a ground state solution of (8.1.3) if and only if $Q \in \mathcal{G}$.*

*Proof.* Let $Q, R \in H^s(\mathbb{R}^n)$ be two non-trivial solutions of (8.1.3). By integrating the equation (8.1.3) against $\overline{Q}$ and $\overline{R}$, we find

$$\langle Q, (P(D) + \lambda)Q \rangle = \|Q\|_{L^{2\sigma+2}}^{2\sigma+2}, \quad \langle R, (P(D) + \lambda)R \rangle = \|R\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{6.2.4}$$

As a consequence, we get

$$\mathcal{A}(Q) = \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|Q\|_{L^{2\sigma+2}}^{2\sigma+2}, \quad \mathcal{A}(R) = \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|R\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

Hence we have the equivalence

$$\mathcal{A}(Q) \leqslant \mathcal{A}(R) \quad \Longleftrightarrow \quad \|Q\|_{L^{2\sigma+2}} \leqslant \|R\|_{L^{2\sigma+2}}.$$

Next, let $C > 0$ denote the optimal constant for (6.1.7). From (6.2.4) we obtain the bounds

$$\|Q\|_{L^{2\sigma+2}}^{2\sigma} \geqslant \frac{1}{C}, \quad \|R\|_{L^{2\sigma+2}}^{2\sigma} \geqslant \frac{1}{C},$$

where equality occurs if and only if $Q$ and $R$ are optimizers for (6.1.7), respectively.

Suppose now that $Q$ is a ground state solution, which means an optimizer for (6.1.7) by definition. Then we must have $\|R\|_{L^{2\sigma+2}} \geqslant \|Q\|_{L^{2\sigma+2}}$. This show that $Q \in \mathcal{G}$.

On the other hand, let us assume that $Q \in \mathcal{G}$. To show that $Q$ must optimize (6.1.7), we argue by contradiction as follows. Suppose $Q$ is not an optimizer. Then $\|Q\|_{L^{2\sigma+2}} > C^{-1}$. But by taking $R$ to be an optimizer, we deduce that $C^{-1} = \|R\|_{L^{2\sigma+2}} < \|Q\|_{L^{2\sigma+2}}$, which contradicts that we must have $\mathcal{A}(Q) \leqslant \mathcal{A}(R)$. $\qquad \square$

## 6.3 Proof of Theorem 6.1.1

Let $\psi \in H^s(\mathbb{R}^n)$ be a ground state for (6.1.1) with $E = E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi)$. If we set $\lambda = -E$, we can write (6.1.1) in Fourier space as

$$\widehat{\psi}(\xi) = \frac{1}{p(\xi) + \lambda}(\widehat{W} * \psi)(\xi), \quad \text{with } \widehat{W} = -\widehat{V}. \tag{6.3.1}$$

Note that $\widehat{W} \in L^2(\mathbb{R}^n)$ by assumption and hence $\widehat{(W\psi)} = \widehat{W} * \widehat{\psi}$ and, moreover, this is a continuous function because it is the convolution of two $L^2$-functions. Since $p(\xi) + \lambda > 0$ is also continuous by assumption on $p$, we deduce that the Fourier transform $\widehat{\psi}(\xi)$ is a continuous function from (6.3.1).

Next, we claim that

$$|\widehat{\psi}(\xi)| > 0 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{6.3.2}$$

To see this, we first note that

$$\psi^\bullet = \mathcal{F}^{-1}(|\widehat{\psi}|)$$

is also a ground state solution for (6.1.1). Indeed, in view of $\widehat{V}(\xi) < 0$ almost everywhere, we can argue as in the proof of Lemma 6.2.1 to conclude

$$\langle \psi, V\psi \rangle = (\widehat{V} * \Psi_{\widehat{\psi}})(0) \geqslant (\widehat{V} * \Psi_{|\widehat{\psi}|})(0) = \langle \psi^\bullet, V\psi^\bullet \rangle,$$

where $\Psi_g(\xi) = \int_{\mathbb{R}^n} g(\xi + \xi')\overline{g(\xi')}\, d\xi$ denotes the autocorrelation function of $g$. Thus from Lemma 6.2.1 (i) we readily find that

$$\langle \psi^\bullet, (P(D) + V)\psi^\bullet \rangle \leqslant \langle \psi, (P(D) + V)\psi \rangle,$$

whence $\psi^\bullet$ is also a ground state, since we trivially have $\|\psi^\bullet\|_{L^2} = \|\psi\|_{L^2}$.

Therefore, in order to show (6.3.2), we can assume that $\widehat{\psi}(\xi) = |\widehat{\psi}(\xi)| \geqslant 0$ is non-negative. But from the assumption that $\widehat{W} = -\widehat{V} > 0$ almost everywhere we deduce that $(\widehat{W} * \widehat{\psi})(\xi) > 0$ for all $\xi \in \mathbb{R}^n$. By the positivity $p(\xi) + \lambda > 0$, we immediately deduce that (6.3.2) holds from (6.3.1).

Next, we establish the following result.

**Proposition 6.3.1.** *There exists a constant $\theta \in \mathbb{R}$ such that*

$$\widehat{\psi}(\xi) = e^{i\theta}|\widehat{\psi}(\xi)| \quad \text{for all } \xi \in \mathbb{R}^n.$$

*Proof of Proposition 6.3.1.* By the continuity of $\widehat{\psi}$ and the fact that $|\widehat{\psi}(\xi)| > 0$ for all $\xi \in \mathbb{R}^n$, there exists a continuous function $\vartheta : \mathbb{R}^n \to \mathbb{R}$ such that

$$\widehat{\psi}(\xi) = e^{i\vartheta(\xi)}|\widehat{\psi}(\xi)| \quad \text{for all } \xi \in \mathbb{R}^n. \tag{6.3.3}$$

Since $\psi$ and $\psi^\bullet$ are both ground states for (6.1.1), we must have equality

$$(\widehat{W} * \Psi_{\widehat{\psi}})(0) = (\widehat{W} * \Psi_{|\widehat{\psi}|})(0), \tag{6.3.4}$$

with the autocorrelation function $\Psi_g(\xi) = \int_{\mathbb{R}^n} g(\xi+\eta)\overline{g(\eta)}\,d\eta$. In view of (6.4.5), we conclude

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{W}(\xi) e^{i\{\vartheta(-\xi+\eta)-\vartheta(\eta)\}} |\widehat{\psi}(\xi+\eta)||\widehat{\psi}(\eta)|\,d\xi\,d\eta = \int_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{W}(\xi)|\widehat{\psi}(\xi+\eta)||\widehat{\psi}(\eta)|\,d\xi\,d\eta.$$

Since $W(\xi)|\widehat{\psi}(\xi+\eta)||\widehat{\psi}(\eta)| > 0$ for all $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, we deduce that

$$\vartheta(-\xi+\eta) - \vartheta(\eta) \in 2\pi\mathbb{Z} \quad \text{for all } (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By the continuity of $\vartheta$, the difference above must be locally constant. Since $\mathbb{R}^n \times \mathbb{R}^n$ is connected, we infer that

$$\vartheta(-\xi+\eta) - \vartheta(\eta) = c \quad \text{for all } (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{6.3.5}$$

with some constant $c \in 2\pi\mathbb{Z}$. But by choosing $\xi = 0$, we see that $c = 0$ is the only possibility. From the functional equation (6.3.5) with $c = 0$ we readily deduce that $\vartheta(-\xi) = \vartheta(0)$ for all $\xi \in \mathbb{R}^n$. Hence $\vartheta$ is a constant function and by taking $\theta = \vartheta(0) \in \mathbb{R}$, we complete the proof of Proposition 6.3.1. $\qquad\square$

By applying Proposition 6.3.1, we complete the proof of Theorem 6.1.1 part (i).

The symmetry property in part (ii) directly follows from the fact that $e^{i\theta}\widehat{\psi}(\xi) > 0$ together with the elementary property $f(-x) = \overline{f(x)}$ holds a.e. for $f \in L^2(\mathbb{R}^n)$ whenever $\widehat{f}(\xi)$ is real-valued. Finally, let us suppose that $p(-\xi) = p(\xi)$ is even. Then $H = P(D) + V$ is real operator, i.e., we have $\text{Re}\,(Hf) = H\text{Re}\,f$. In particular, we thus choose any eigenfunction of $H$ to be real-valued and, in particular, this applies to the ground state $\psi$.

The proof of Theorem 6.1.1 is now complete. $\qquad\square$

## 6.4   Proof of Theorem 6.1.2

Let $Q \in H^s(\mathbb{R}^n)$ be a ground state solution as in Theorem 6.1.2. We define the set

$$\Omega = \{\xi \in \mathbb{R}^n : |\widehat{Q}(\xi)| > 0\}. \tag{6.4.1}$$

This is an open set in $\mathbb{R}$, since the function $|\widehat{Q}| : \mathbb{R}^n \to \mathbb{C}$ is continuous due to analyticity of $\widehat{Q}(\xi)$ is analytic by our assumption $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for some $a > 0$ and using standard Paley–Wiener arguments.

**Lemma 6.4.1.** *It holds that $\Omega = \mathbb{R}^n$.*

**Remark.** *For non-ground state solutions $Q \in H^s(\mathbb{R}^n)$ of (8.1.3), we expect that $\widehat{Q}$ vanishes at certain points. In fact, we expect that the set $\{|\widehat{Q}(\xi)| > 0\}$ is not connected for non-ground state solutions $Q$.*

*Proof.* In view of Lemma 6.2.1, we remark that $Q^\bullet \in H^s(\mathbb{R}^n)$ is also a ground state solution for (8.1.3). Hence we can assume that $\widehat{Q} = |\widehat{Q}| \geqslant 0$ is non-negative without loss of generality. Next, by applying the Fourier transform to (8.1.3) and using that $\sigma \in \mathbb{N}$ is an integer, we get

$$\widehat{Q}(\xi) = \frac{1}{p(\xi) + \lambda}(\widehat{Q} * \ldots * \widehat{Q})(\xi) \tag{6.4.2}$$

with $k = 2\sigma + 1 \in \mathbb{N}$ convolutions appearing on the right-hand side. From this identity and Lemma 8.4.1 and iteration, we deduce that $\Omega \subset \mathbb{R}^n$ must be identical to its $k$-**fold Minkowski sum**, i.e.,

$$\Omega = \bigoplus_{m=1}^{k} \Omega \equiv \{\xi_1 + \ldots + \xi_k : \xi_m \in \Omega \text{ for } m = 1, \ldots, k\}. \tag{6.4.3}$$

For the moment, let us now suppose that

$$0 \in \Omega. \tag{6.4.4}$$

Since $\Omega$ is open, this implies that $B_r(0) \subset \Omega$ for some $r > 0$. By (6.4.3), this implies that

$$\bigoplus_{m=1}^{k} B_r(0) \subset \Omega.$$

On the other hand, we readily see that $B_{2r}(0) \subset B_r(0) \oplus B_r(0) \subset \oplus_{m=1}^{k} B_r(0)$. Iterating this argument, we conclude that

$$B_{Nr}(0) \subset \Omega \quad \text{for all} \quad N \in \mathbb{N},$$

whence it follows that $\Omega = \mathbb{R}^n$ must hold.

Thus it remains to show that (6.4.4) is true. We argue by contradiction as follows. Suppose that $0 \notin \Omega$ and define the function $F : \mathbb{R}^n \to \mathbb{R}$ by setting

$$F(\xi) = \widehat{Q}((k-1)\xi)\widehat{Q}(-\xi)$$

However, we must have

$$F(\xi) \equiv 0.$$

Indeed, if $F(\xi_*) \neq 0$ for some $\xi_* \in \mathbb{R}^n$ then $(k-1)\xi_* \in \Omega$ and $-\xi_* \in \Omega$. This implies that $0 = (k-1)\xi_* - \sum_{m}^{k-1} \xi_* \in \oplus_{m=1}^{k} \Omega$ so that $0 \in \Omega$ by (6.4.3). Thus $0 \notin \Omega$ implies that $F(\xi) \equiv 0$ vanishes identically. Since $\widehat{Q}((k-1)\xi) \not\equiv 0$, this yields that the function $\widehat{Q}(-\xi)$ must vanish on some non-empty open set in $\mathbb{R}^n$. By the (real) analyticity of $\widehat{Q} : \mathbb{R}^n \to \mathbb{R}$ this implies $\widehat{Q} \equiv 0$ on $\mathbb{R}^n$. But this is a contradiction.

Thus we have shown that (6.4.4) holds, which completes the proof. $\square$

With the result of Lemma 6.4.1 at hand, we are ready to finish the proof of Theorem 6.1.2. Indeed, if $Q \in H^s(\mathbb{R}^n)$ is a ground state solution, we must necessarily have the equality

$$\|Q\|_{L^{2\sigma+2}} = \|Q^\bullet\|_{L^{2\sigma+2}}.$$

But we can apply Lemma 8.6.4 with $f = Q$ and $g = Q^\bullet$ to conclude that $\widehat{Q} = e^{i(\alpha+\beta\cdot\xi)}|\widehat{Q}(\xi)|$ for all $\xi$ with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. Hence we find

$$Q(x) = e^{i\alpha}Q^\bullet(x + x_0)$$

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with the constant $x_0 = -\frac{1}{2\pi}\beta \in \mathbb{R}^n$. The asserted properties of $Q^\bullet$ now follow from Lemma 6.2.1 together with the fact that $\widehat{Q^\bullet} \in L^1(\mathbb{R}^n)$, since we have $(1 + |\xi|)^m \widehat{Q} \in L^2(\mathbb{R}^n)$ for $m > n/2$ by Proposition 6.2.1.

Finally, let us additionally assume that the symbol

$$p(-\xi) = p(\xi)$$

is even. In this case, we can adapt a trick from [14] (see also Lemma 6.6.1) to show that any ground state $Q \in H^s(\mathbb{R}^n)$ must be real-valued up to a trivial constant complex phase, i.e., we claim that

$$e^{i\theta}Q(x) \in \mathbb{R} \quad \text{for all} \quad \xi \in \mathbb{R}^n \tag{6.4.5}$$

with some constant $\theta \in \mathbb{R}$. To prove this, we decompose

$$Q = Q_R + iQ_I$$

into real and imaginary part. If either $Q_R \equiv 0$ or $Q_I \equiv 0$, then there is nothing is left to prove. Hence we assume that both parts are non-trivial. From Lemma 6.6.1 we obtain

$$\langle Q, (P(D) + \lambda)Q \rangle = \langle Q_R, (P(D) + \lambda)Q_R \rangle + \langle Q_I, (P(D) + \lambda)Q_I \rangle =: D_R + D_I, \tag{6.4.6}$$

$$\|Q\|_{L^{2\sigma+2}}^2 \leqslant \|Q_R\|_{L^{2\sigma+2}}^2 + \|Q_I\|_{L^{2\sigma+2}}^2 =: N_R + N_I. \tag{6.4.7}$$

Now let $C > 0$ denote the optimal constant for (6.1.7). Since $Q$ is an optimizer, we deduce

$$C = \frac{\|Q\|_{L^{2\sigma+2}}^2}{\langle f, (P(D) + \lambda)f \rangle} \leqslant \frac{N_R + N_I}{D_R + D_M} \leqslant \max\left(\frac{N_R}{D_R}, \frac{N_M}{D_M}\right) \leqslant C.$$

This shows that we must have equality in (6.4.7), which by Lemma 6.6.1 and $Q_R \not\equiv 0 \not\equiv Q_I$ implies that there is some constant $\alpha > 0$ such that $Q_I^2 = \alpha^2 Q_R^2$. We want to establish $Q_I = \pm\alpha Q_R$. To do so, we apply Lemma 6.6.1 now to the decomposition

$$Q = e^{i\pi/4}Q_a + ie^{i\pi/4}Q_b$$

with real-valued functions $Q_a$ and $Q_b$. In fact, an elementary computation shows that $Q_a = \frac{i}{\sqrt{2}}(Q_R + Q_I)$ and $Q_b = \frac{1}{\sqrt{2}}(-Q_R + Q_I)$. We still have $|Q(x)|^2 = Q_a(x)^2 + Q_b(x)^2$ and also $\langle Q, (P(D) + \lambda)Q \rangle = \langle Q_a, (P(D) + \lambda Q_a \rangle + \langle Q_b, (P(D) + \lambda)Q_b \rangle$ by using that $p(-\xi) = p(\xi)$ is even. Now if $Q_a \equiv 0$, then we are done since $Q_I = -Q_R$ in this case. If $Q_a \not\equiv 0$, we obtain $Q_b^2 = \beta^2 Q_a^2$ with some constant $\beta > 0$. Note that $\beta^2 \neq 1$ because otherwise this would imply $Q_R Q_I \equiv 0$ (which would yield $Q \equiv 0$ from using $Q_I^2 = \alpha^2 Q_R^2$). In summary, we conclude

$$Q_I^2 = \alpha^2 Q_R^2 \quad \text{and} \quad \frac{1}{2}(1 + \alpha^2)(1 - \beta^2)Q_R^2 = (1 + \beta^2)Q_R Q_I.$$

But this implies that $Q_I = \pm\alpha Q_R$, which proves that (6.4.5) is true.

The proof of Theorem 6.1.2 is now complete. □

## 6.5 Proof of Theorem 6.1.3

We will adapt an elegant idea due Combes and Thomas [11] who proved exponential decay of eigenfunctions for ($N$-body) Schrödinger operators by an analytic continuation argument, which is based on O'Connor's lemma (see Lemma **??** below) together with standard analytic perturbation theory (see [21, 31]).

We define the operator $H = P(D) + V$ with $V = -|Q|^{2\sigma}$ acting on $L^2(\mathbb{R}^n)$. Note that $V \in L^\infty(\mathbb{R}^n)$ is bounded by Proposition 6.2.1. Hence, by standard theory, the operator $H$ is self-adjoint with operator domain $H^{2s}(\mathbb{R}^n)$. In particular, we see that $Q$ is an $L^2$-eigenfunction of $H$ satisfying

$$HQ = -\lambda Q.$$

Since $V(x) \to 0$ as $|x| \to \infty$, we have $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(P(D)) = \inf_{\xi \in \mathbb{R}^n} p(\xi)$. By our assumption (6.1.6), we see that the eigenvalue $-\lambda$ lies strictly below the essential spectrum of $H$.

We shall now implement an analytic continuation argument to show that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ must hold for some sufficiently small $a > 0$. To do so, we adapt an argument due to Combes and Thomas as follows. For real $\kappa \in \mathbb{R}^n$, we can define the unitary operators

$$(U(\kappa)f)(x) = e^{2\pi i \kappa \cdot x} f(x)$$

acting on $L^2(\mathbb{R}^n)$. Likewise, we consider the family of unitarily equivalent operators

$$H(\kappa) = U(\kappa)HU(\kappa)^{-1}.$$

We readily find that

$$U(\kappa)P(D)U(\kappa)^{-1} = P_\kappa(D), \quad U(\kappa)VU(\kappa)^{-1} = V,$$

where $P_\kappa(D)$ has the shifted symbol $p(\xi + \kappa)$.

Now, by standard Paley-Wiener theory, we note that if $U(\kappa)Q$ has an analytic continuation for $|\text{Im}\,\kappa| < \delta$ then $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for all $0 < a < \delta$, which would finish the proof. To see that $U(\kappa)Q$ can be analytically continued if $|\text{Im}\,\kappa| < \delta$ for some $\delta > 0$, we prove that $H(\kappa)$ is an *analytic family of type (B)* on the complex strip $T_\delta$. We use an form argument. For any $\kappa \in T_\delta$, we can define the quadratic form

$$\mathsf{q}(\kappa)[f,f] = \int_{\mathbb{R}^n} p(\xi + \kappa)|\hat{f}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^n} V|f|^2 \, dx \quad \text{for} \quad f \in H^s(\mathbb{R}^n). \tag{6.5.1}$$

We claim that $\{\mathsf{q}(\kappa)\}_{\kappa \in T_\delta}$ is an analytic family of quadratic forms of type (b) with form domain $H^s(\mathbb{R}^n)$ (in the nomenclature of [31]). That is, we have the following properties.

(1) For each $\kappa \in T_\delta$, the form $\mathsf{q}(\kappa)$ is closed and strictly $m$-sectorial with domain $H^s(\mathbb{R}^n)$.

(2) For each $f \in H^s(\mathbb{R}^n)$, the function $\kappa \mapsto \mathsf{q}(\kappa)[f,f]$ is analytic in $\kappa \in T_\delta$.

Indeed, by Assumption 3 item (i), we see that $\mathsf{q}$ is strictly $m$-sectorial (see [31] for the relevant definition). To show that $\mathsf{q}(\kappa)$ is closed on the domain $H^s(\mathbb{R}^n)$, it suffices to show that its real part $\text{Re}(\mathsf{q})(\kappa)$ is closed, i.e., if $f_n \in H^s(\mathbb{R}^n)$ with $f_n \to f$ in $L^2(\mathbb{R}^n)$ and $\text{Re}(\mathsf{q})(\kappa)[f_n - f_m, f_n - f_m] \to 0$ as $m, n \to \infty$ then $f \in H^s(\mathbb{R}^n)$. But this later claim easily from property (ii) in Assumption 3. This shows (1) above. Finally, we note that (2) obviously holds by our analyticity assumption on the symbol $p$. From the fact that $\mathsf{q}(\kappa)$ is an analytic family of form of type (b) it follows that the set of associated operators $\{H(\kappa)\}_{\kappa \in T_\delta}$ defines an analytic family of operators of type (B).

Now, by standard perturbation theory, any discrete eigenvalue $E(\kappa_0)$ of $H(\kappa_0)$ moves analytically for $\kappa$ close to $\kappa_0$. But if $\text{Im}(\kappa - \kappa_0) = 0$, we have that $E(\kappa) = E(\kappa_0)$ since the operators $H(\kappa)$ and $H(\kappa_0)$ are unitarily equivalent in this case. Hence $E(\kappa)$ is constant and remains an eigenvalue as long as it stays away from $\sigma_{\text{ess}}(H(\kappa))$.

Now we recall that $Q$ is an eigenfunction of $H = H(0)$ with the discrete eigenvalue $E = -\lambda \in \sigma_{\text{disc}}(H)$. By standard perturbation theory [21, 31], we find that $E(\kappa) \in \sigma_{\text{disc}}(H(\kappa))$

provided that $|\kappa| \leqslant b$ with some sufficiently small number $b > 0$. Since the operators $H(\kappa) = H(\mathrm{i}\,\mathrm{Im}\,\kappa)$ are unitarily equivalent, we see

$$\sigma_{\mathrm{disc}}(H(\kappa)) = \sigma_{\mathrm{disc}}(H(\mathrm{i}\,\mathrm{Im}\kappa)).$$

Thus we deduce that $E \in \sigma_{\mathrm{disc}}(H(\kappa))$ for all $\kappa$ with $|\mathrm{Im}\,\kappa| < b$. Hence it follows from standard perturbation theory that the finite rank projections

$$P(\kappa) = \frac{1}{2\pi\mathrm{i}} \oint_{|E-z|=r} (z - H(\kappa))^{-1}\, dz$$

with some small constant $r > 0$ are analytic in the strip $T_b = \{\kappa \in \mathbb{C}^n : |\mathrm{Im}\,\kappa| < b\}$. We now apply O'Connor's lemma to conclude that $U(\kappa)Q$ has an analytic continuation to the strip $T_b$, which shows that $e^{a|\cdot|}Q \in L^2(\mathbb{R}^n)$ for all $0 < a < b$.

The proof of Theorem 6.1.3 is now complete. $\qquad\square$

## 6.6  Auxiliary Results

**Lemma 6.6.1.** *Suppose $P(D)$ satisfies Assumption 1 with some $s > 0$ and its multiplier $p(-\xi) = p(\xi)$ is an even function and let $\lambda \in \mathbb{R}$. Let $f \in H^s(\mathbb{R}^n)$ with $f : \mathbb{R}^n \to \mathbb{C}$ be of the form*

$$f(x) = e^{\mathrm{i}\vartheta} f_R(x) + \mathrm{i}e^{\mathrm{i}\vartheta} f_I(x)$$

*with some constant $\vartheta \in \mathbb{R}$ and real-valued functions $f_R, f_I : \mathbb{R}^n \to \mathbb{R}$. Then we have*

$$\langle f, (P(D) + \lambda)f \rangle = \langle f_R, (P(D) + \lambda)f_R \rangle + \langle f_I, (P(D) + \lambda)f_I \rangle.$$

*Moreover, if $f \in L^q(\mathbb{R}^n)$ for some $2 < q < \infty$ then*

$$\|f\|_{L^q}^2 \leqslant \|f_R\|_{L^q}^2 + \|f_I\|_{L^q}^2,$$

*where equality holds if and only if $f_I = 0$ or $f_R^2 = \mu^2 f_I^2$ with some constant $\mu \geqslant 0$.*

*Proof.* By subtracting the constant $\lambda$ from $p(\xi)$, we can assume without loss of generality that $\lambda = 0$ holds. Since $f_R, f_I : \mathbb{R}^n \to \mathbb{R}$ are real-valued, their Fourier transforms satisfy $\widehat{f_R}(-\xi) = \overline{\widehat{f_R}(\xi)}$ and $\widehat{f_I}(-\xi) = \overline{\widehat{f_I}(\xi)}$. Using that $p(-\xi) = p(\xi)$ is even and $|e^{\mathrm{i}\vartheta}z| = |z|$ for all $z \in \mathbb{C}$, we calculate

$$\langle f, P(D)f \rangle = \int_{\mathbb{R}^n} p(\xi)|\widehat{f_R}(\xi) + \mathrm{i}\widehat{f_I}(\xi)|^2\, d\xi = \int_{\mathbb{R}^n} |\widehat{f_R}(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} p(\xi)|\widehat{f_I}(\xi)|^2\, d\xi$$

$$+ \mathrm{i}\int_{\mathbb{R}^n} p(\xi)\left[\overline{\widehat{f_R}(\xi)}\widehat{f_I}(\xi) - \widehat{f_R}(\xi)\overline{\widehat{f_I}(\xi)}\right]\, d\xi = \langle f_R, P(D)f_R \rangle + \langle f_I, P(D)f_I \rangle,$$

as claimed.

Assume now that $f \in L^q(\mathbb{R}^n)$ for some $2 < q < \infty$. From the triangle inequality for the $L^{q/2}$-norm we find

$$\|f\|_{L^q}^2 = \||f_R|^2 + |f_I|^2\|_{L^{q/2}} \leqslant \||f_R|^2\|_{L^{q/2}} + \||f_I|^2\|_{L^{q/2}} = \|f_R\|_{L^q}^2 + \|f_I\|_{L^q}^2.$$

By the strict convexity of the $L^{q/2}$-norm for $2 < q < \infty$, we have equality if and only if $f_I = 0$ or $f_R^2 = \mu^2 f_I^2$ for some constant $\mu \geqslant 0$. $\qquad\square$

**Lemma 6.6.2.** *Let $f, g \in \mathbb{R}^n \to [0, \infty)$ be two non-negative and continuous functions. Assume that their convolution*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y$$

*has finite values for all $x \in \mathbb{R}^n$. Then it holds that*

$$\{x \in \mathbb{R}^n : f * g > 0\} = \{x \in \mathbb{R}^n : f > 0\} \oplus \{x \in \mathbb{R}^n : g > 0\}.$$

*where $A \oplus B = \{a + b : a \in A, b \in B\}$ denotes the Minkowski sum of two sets $A, B \subset \mathbb{R}^n$.*

**Remark.** *We could also allow that $(f * g)(x) = +\infty$ for some $x \in \mathbb{R}^n$ and the result remains valid. But since we apply this lemma iteratively in the proof of Theorem 6.1.2, we assume that $(f * g)(x) < +\infty$ for all $x \in \mathbb{R}^n$.*

*Proof.* The proof is elementary. For the reader's convenience, we give the details.

Let us write $\Omega_f = \{f > 0\}$, $\Omega_g = \{g > 0\}$ and $\Omega_{f*g} = \{f * g > 0\}$. We suppose that both $f \not\equiv 0$ and $g \not\equiv 0$, since otherwise the claimed result trivially follows.

First, we show that $\Omega_f \oplus \Omega_g \subset \Omega_{f*g}$. Let $x = x_1 + x_2$ with $x_1 \in \Omega_f$ and $x_2 \in \Omega_g$. By continuity of $f$ and $g$, there exists some $\varepsilon > 0$ such that $f > 0$ on $B_\varepsilon(x_1)$ and $g > 0$ on $B_\varepsilon(x_2)$. Thus, by using that $f \geqslant 0$ and $g \geqslant 0$ on all of $\mathbb{R}^n$, we get

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y \geqslant \int_{B_\varepsilon(x_2)} f(x_1 + x_2 - y) g(y) \, \mathrm{d}y > 0,$$

since $x_1 + x_2 - y \in B_\varepsilon(x_1)$ when $y \in B_\varepsilon(x_2)$. This shows that $\Omega_f \oplus \Omega_g \subset \Omega_{f*g}$.

Next, we prove that $\Omega_{f*g} \subset \Omega_f \oplus \Omega_g$ holds. Indeed, for every $x \in \mathbb{R}^n$, we can write

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y = \int_{(\{x\} - \Omega_f) \cap \Omega_g} f(x - y) g(y) \, \mathrm{d}y,$$

since $f(x - \cdot) \equiv 0$ on $\mathbb{R}^n \backslash (\{x\} \ominus \Omega_f))$[4] and $g \equiv 0$ on $\mathbb{R}^n \backslash \Omega_g$. However, if $x \notin \Omega_f \oplus \Omega_g$ then $(\{x\} \ominus \Omega_f) \cap \Omega_g = \varnothing$. Thus $(f * g)(x) = 0$ for any $x \notin \Omega_f \oplus \Omega_g$, whence it follows that the inclusion $\Omega_{f*g} \subset \Omega_f \oplus \Omega_g$ is valid. $\square$

**Lemma 6.6.3** (O'Connor's lemma [29])**.** *Let $H$ be a Hilbert space and suppose $U(\kappa)$ that are unitary operators on $H$ parametrized by $\kappa \in \mathbb{R}^n$. Let $P$ be a finite-rank projection on $H$ such that that $P(\kappa) = U(\kappa) P U(\kappa)^{-1}$ has an analytic continuation to $D = \{z \in \mathbb{C}^n : |\mathrm{Im}\, z| < a\}$ for some $a > 0$. Then any $f \in \mathrm{ran}\, P$ has an analytic continuation from $D \cap \mathbb{R}$ to $D$ given by $f(\kappa) = U(\kappa) f$.*

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[4]We denote $A \ominus B = \{a - b : a \in A, b \in B\}$ for subsets $A$ and $B$ in $\mathbb{R}^n$.

# Chapter 7

# A Guideline on Symmetry of Traveling Solitary Waves

In this chapter we introduce the concept of a boosted ground state and analyze symmetry properties of traveling solitary waves. Similar to Chapter 5 this serves as a guideline for reading [5]. In Chapter 8 a full article on those problems will follow. In addition a short numerical section is included for the sake of a visualization.

## 7.1 Assumptions and Setup

We consider the following class of nonlinear Schrödinger equations given by

$$i\partial_t u = P(D)u - |u|^{2\sigma}u, \tag{7.1.1}$$

where $u : [0, T) \times \mathbb{R}^n \to \mathbb{C}$ and $P(D)$ denotes a self-adjoint and constant coefficient pseudo-differential operator defined on the Fourier side as

$$\widehat{P(D)u}(\xi) = p(\xi)\widehat{u}(\xi).$$

Let us start with the some assumptions on the operator $P(D)$.

**Assumption.** *The Fourier symbol $p$ of the operator $P(D)$ is real-valued, continuous and satisfies the following growth assumption*

$$A|\xi|^{2s} + c \leqslant p(\xi) \leqslant B|\xi|^{2s} \quad \text{for all } \xi \in \mathbb{R}^n, \tag{7.1.2}$$

*where $s \geqslant \frac{1}{2}$, $A > 0$, $B > 0$ and $c \in \mathbb{R}$.*

With the given assumption on the Fourier symbol of $P(D)$ it is easy to see that the following norm equivalence holds true

$$\|u\|_{H^s(\mathbb{R}^n)}^2 \simeq \langle u, (P(D) + \lambda)u \rangle = \int_{\mathbb{R}^n} (p(\xi) + \lambda)|\widehat{u}(\xi)|^2 d\xi$$

with $\lambda > 0$ sufficiently large. Moreover, with the realy number $s \geqslant \frac{1}{2}$ we define the following exponent

$$\sigma_*(n, s) := \begin{cases} \frac{2s}{n-2s} & \text{if } s < \frac{n}{2}, \\ +\infty & \text{if } s \geqslant \frac{n}{2}. \end{cases}$$

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We say that the range $1 \leqslant \sigma < \sigma_*$ corresponds to the energy-subcritical case for equation 7.1.1. We will mostly focus on this regime and give little comments on the energy-critical case $\sigma = \sigma_*$.

In the following we are interested in **traveling solitary waves** for equation (7.1.1). For this, consider the following ansatz

$$u(t, x) = e^{i\omega t} Q_{\omega,v}(x - vt)$$

with frequency $\omega \in \mathbb{R}$ and velocity $v \in \mathbb{R}^n$. Plugging this into equation (7.1.1) we see that $Q_{\omega,v} \in H^s(\mathbb{R}^n)$ has to be a weak solution of

$$P(D)Q_{\omega,v} + iv \cdot Q_{\omega,v} + \omega Q_{\omega,v} - |Q_{\omega,v}|^{2\sigma} Q_{\omega,v} = 0. \tag{7.1.3}$$

**Remark 7.1.1.** *For a brief moment, let $P(D) = -\Delta$, which corresponds to the classical nonlinear Schrödinger equation. Then there exists a gauge transform, called a Galilean transform in this context, which transforms equation (7.1.3) from general $v \in \mathbb{R}^n$ to vanishing velocity $v = 0$. Clearly, consider*

$$Q(x) \mapsto e^{\frac{i}{2} v \cdot x} Q(x).$$

*Now the analysis of (7.1.3) reduces to*

$$-\Delta Q + \omega_v Q - |Q|^{2\sigma} Q = 0 \quad \text{with } \omega_v = \omega - \frac{1}{2}|v|^2.$$

*Another interesting point is that the Galilean transform preserves the $L^2$-norm, i.e. it is a unitary transform on $L^2(\mathbb{R}^n)$.*

*Up to now, for general dispersion operators $P(D)$ such a boost transform is not known.*

## 7.2 Existence Result

The first section is dealing with the basic question of existence of solutions to equation (7.1.3). To prove this result we need a suitable variational setting. Given $v \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ we define a Weinstein-type functional as follows

$$\mathcal{J}_{v,\omega,\sigma}(u) := \frac{\langle u, (P_v(D) + \omega)u \rangle^{\sigma+1}}{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}},$$

where $u \in H^s(\mathbb{R}^n)$ with $u \not\equiv 0$. Additionally we used

$$P_v(D) = P(D) + iv \cdot \nabla$$

for a shorter notation. Recalling the assumption on the symbol $p$ of $P(D)$ we find that

$$\Sigma_v := \inf_{\xi \in \mathbb{R}^n} p_v(\xi) > -\infty,$$

proved $s > 1/2$ and $v \in \mathbb{R}^n$ or $|v| < A$ for the case $s = \frac{1}{2}$. The following result shows the existence of a minimizer for $\mathcal{J}_{v,\omega,\sigma}$.

**Theorem 7.2.1.** *Let $n \geqslant 1$, $v \in \mathbb{R}^n$ and suppose that $P(D)$ satisfies assumption 7.1.2 with $s \geqslant \frac{1}{2}$ and $A > 0$. If $s = \frac{1}{2}$ we assume that $|v| < A$. Then, for $0 < \sigma < \sigma_*$ and $\omega > -\Sigma_v$, every minimizing sequence for $\mathcal{J}_{v,\omega,\sigma}$ is relatively compact in $H^s(\mathbb{R}^n)$ up to translations in $\mathbb{R}^n$. Hence there exists a minimizer $Q_{v,\omega} \in H^s(\mathbb{R}^n) \backslash \{0\}$, i.e.*

$$\mathcal{J}_{v,\omega,\sigma}(Q_{v,\omega}) = \inf_{u \in H^s(\mathbb{R}^n)\backslash\{0\}} \mathcal{J}_{v,\omega,\sigma}(u).$$

### 7.2.1 Outlines of the Proof

We will follow a technique from Dominik Himmelsbach's PhD thesis (see [18]) and adapt a proof therein. Suppose that $P(D)$ satisfies assumption 7.1.2. Additionally take all the proper assumptions in Theorem 7.2.1 for granted.

Recalling that $P_v(D) = P(D) + iv \cdot \nabla$ we can define

$$\|u\|_{\omega,\mathbf{v}} := \langle u, (P_{\mathbf{v}}(D) + \omega)u \rangle^{1/2} = \left( \int_{\mathbb{R}^n} (p(\xi) - \mathbf{v} \cdot \xi + \omega)|\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

In the following let

$$\mathcal{J}^*_{v,\omega,p} := \inf \left\{ \mathcal{J}_{v,\omega,p}(u) \mid u \in H^s(\mathbb{R}^n), \ u \not\equiv 0 \right\},$$

using a Sobolev-type inequality it's easy to see that $\mathcal{J}^*_{v,\omega,p} > 0$ is strictly positive. Next we need to prove the following claim:

**Claim.** *Let $(u_j)_{j\in\mathbb{N}} \subseteq H^s(\mathbb{R}^n) \backslash \{0\}$ be a minimizing sequence for $\mathcal{J}_{v,\omega,\sigma}$. Then $(u_j)_{j\in\mathbb{N}}$ has a non-zero weak limit in $H^s(\mathbb{R}^n)$, up to spatial translations and passing to a subsequence.*

This is certainly not a trivial result and we need to invoke the following to lemmas given in the appendix of [6].

**Lemma** (pqr Lemma; see [15])**.** *Let $(\Omega, \Sigma, \mu)$ be a measure space. Let $1 \leqslant p < q < r \leqslant \infty$ and let $C_p, C_q, C_r > 0$ be positive constants. Then there exist constants $\eta, c > 0$ such that, for any measurable function $f \in L^p_\mu(\Omega) \cap L^r_\mu(\Omega)$ satisfying*

$$\|f\|^p_{L^p_\mu} \leqslant C_p, \quad \|f\|^q_{L^q_\mu} \geqslant C_q, \quad \|f\|^r_{L^r_\mu} \leqslant C_r,$$

*it holds that*

$$d_f(\eta) := \mu(\{x \in \Omega; \ |f(x)| > \eta\}) \geqslant c.$$

*The constant $\eta > 0$ only depends on $p, q, C_p, C_q$ and the constant $c > 0$ only depends on $p, q, r, C_p, C_q, C_r$.*

**Lemma** (Compactness modulo translations in $\dot{H}^s(\mathbb{R}^n)$; see [1])**.** *Let $s > 0$, $1 < p < \infty$ and $(u_j)_{j\in\mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ be a sequence with*

$$\sup_{j\in\mathbb{N}} \left( \|u_j\|_{\dot{H}^s} + \|u_j\|_{L^p} \right) < \infty,$$

*and, for some $\eta, c > 0$ (with $|\cdot|$ being Lebesgue measure)*

$$\inf_{j\in\mathbb{N}} |\{x \in \mathbb{R}^n; \ |u_j(x)| > \eta\}| \geqslant c.$$

*Then there exists a sequence of vectors $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ such that the translated sequence $u_j(x+x_j)$ has a subsequence that converges weakly in $\dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to a nonzero function $u \not\equiv 0$.*

Last but not least one needs to check the following claim:

**Claim.** *The limit obtain from the claim above is indeed an optimizer for $\mathcal{J}_{v,\omega,\sigma}$.*

The proof of this claim is done using the Brézis-Lieb refinement of Fatou's Lemma and some normalization. For a detailed proof of this fact we refer the reader to Chapter 8.

Recalling Chapter 5 and the definition of a ground state (see Definition 5.2) we will refer to minimizers of the functional $\mathcal{J}_{v,\omega,\sigma}$ as **boosted ground states**. Additionally, solutions $u(t,x) = e^{it\omega}Q_{v,\omega}(x-vt)$ to equation (7.1.1) will be called **ground sate traveling solitary waves**. It's not hard to check that any boosted ground state $Q_{v,\omega} \in H^s(\mathbb{R}^n)$ satisfies equation (7.1.3) up to a proper linear rescaling with a positive constant.

## 7.3   Fourier Rearrangements for $n \geqslant 2$

The following section will give a gentle introduction to one of the main results given in Chapter 8. In order to prove a symmetry results on boosted ground states for the Weinstein-type functional $\mathcal{J}_{v,\omega,\sigma}$ we will develop techniques based on a recent result in [24]. Recall that the Fourier rearrangement is defined as

$$u^\sharp := \mathcal{F}^{-1}\left\{(\mathcal{F}u)^*\right\} \quad \text{for } u \in L^2(\mathbb{R}^n) \text{ with } n \geqslant 1,$$

where $f^*$ denotes the classical symmetric-decreasing rearrangement introduced in Chapter 3. For a complete introduction to this topic we refer to [24].

Clearly, for non-zero velocities $v$ the boost term breaks the radial symmetry in general. In such a case, all rearrangement operations that give a spherical symmetric function (for example $\sharp$) can't be applied properly to our problem. Indeed, under a proper assumption on the dispersion operator $P(D)$ we are still able to make a conclusion concerning cylindrical symmetries of minimizers with respect to the direction of the velocity term $v$.

**Definition 7.3.1.** *We say that $f : \mathbb{R}^n \to \mathbb{C}$ is **cylindrically symmetric** with respect to a direction $\mathbf{e} \in \mathbb{S}^{n-1}$ if we have*

$$(f \circ \mathsf{R})(y) = f(y) \quad \text{for a.e. } y \in \mathbb{R}^n \text{ and all } \mathsf{R} \in \mathrm{O}(n) \text{ with } \mathsf{R}\mathbf{e} = \mathbf{e}.$$

Now the main idea lies behind the following decomposition. If $f$ is cylindrically symmetric then we can write

$$f(y) = f(y_\parallel, |y_\perp|)$$

where $y_\perp$ is perpendicular to $\mathbf{e} \in \mathbb{S}^{n-1}$ for given dimension $n \geqslant 2$. In that spirit, we introduce a new notion of rearrangement in Fourier space given by

$$u^{\sharp_\mathbf{e}} := \mathcal{F}^{-1}\left\{(\mathcal{F}u)^{*_\mathbf{e}}\right\} \quad \text{for } u \in L^2(\mathbb{R}^n) \text{ with } n \geqslant 2,$$

where $u^{*_\mathbf{e}} : \mathbb{R}^n \to \mathbb{R}_+$ denotes the **Steiner symmetrization in $(n-1)$ codimensions** with respect to $\mathbf{e} \in \mathbb{S}^{n-1}$, which is obtained by symmetric-decreasing rearrangements in $n-1$-dimensional planes perpendicular to the direction $\mathbf{e}$. This rearrangement will be called the **Fourier Steiner symmetrization in $n-1$ codimensions**. For a much more detailed description see Chapter 8 and [24].

Having such a symmetry calls out for $P(D)$ to have another property besides the assumption in 7.1.2.

**Assumption.** *In addition to [7.1.2](#) the operator $P(D)$ has a cylindrically symmetric multiplier function $p : \mathbb{R}^n \to \mathbb{R}$ with respect to some direction $\mathbf{e} \in \mathbb{S}^{n-1}$. Additionally the map*

$$|\xi_\perp| \mapsto p(\xi_\|, |\xi_\perp|)$$

*is strictly increasing.*

Without further ado, with such an assumption on $P(D)$ and techniques from [24] we can prove the following theorem.

**Theorem 7.3.1** (Symmetry of Boosted Ground States for $n \geqslant 2$). *Let $n \geqslant 2$ and suppose $P(D)$ satisfies the assumptions above with some $s \geqslant \frac{1}{2}$ and $\mathbf{e} \in \mathbb{S}^{n-1}$. Furthermore, let $\mathbf{v} = |v|\mathbf{e} \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ satisfy the hypotheses in Theorem [7.2.1](#) and assume $\sigma \in \mathbb{N}$ is an integer with $0 < \sigma < \sigma_*(n, s)$.*

*Then any boosted ground state $Q_{\omega,v} \in H^s(\mathbb{R}^n)$ is of the form*

$$Q_{\omega,v}(x) = e^{\mathrm{i}\alpha} Q^{\sharp \mathbf{e}}(x + x_0)$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. As a consequence, any such $Q_{\omega,v}$ satisfies (up to a translation and phase) the following two symmetry properties for almost every $x \in \mathbb{R}^n$.*

(i) *$Q_{\omega,v}$ is cylindrically symmetric with respect to $v \in \mathbb{R}^n$,*

(ii) *$Q_{\omega,v}$ has a conjugation symmetry, i.e.*

$$Q_{\omega,v}(x) = \overline{Q_{\omega,v}(-x)}.$$

### 7.3.1 Outlines of the Proof

As with every new operation, one should start with the basic properties. For this we basically follow the same path as in Chapter [3](#). For a quick breakdown yet still giving a good peak we state some of those properties without a sketch of a proof.

**Lemma 7.3.1.** *Let $n \geqslant 2$, $\mathbf{e} \in \mathbb{S}^{n-1}$, and $u \in L^2(\mathbb{R}^n)$. Then the following properties hold.*

(i) *$\|u^{\sharp \mathbf{e}}\|_{L^2} = \|u\|_{L^2}$.*

(ii) *$u^{\sharp \mathbf{e}}$ is cylindrically symmetric with respect to $\mathbf{e}$, i.e., for every matrix $\mathsf{R} \in \mathrm{O}(n)$ with $\mathsf{R}\mathbf{e} = \mathbf{e}$ it holds that*
$$u^{\sharp \mathbf{e}}(x) = u^{\sharp \mathbf{e}}(\mathsf{R}x) \quad \text{for a. e. } x \in \mathbb{R}^n.$$

(iii) *If in addition $\hat{u} \in L^1(\mathbb{R}^n)$, then $u^{\sharp \mathbf{e}}$ is a continuous and **positive definite** function in the sense of Bochner, i.e., we have*

$$\sum_{k,l=1}^m u^{\sharp \mathbf{e}}(x_k - x_l)\overline{z}_k z_l \geqslant 0$$

*for all integers $m \geqslant 1$ and $x_1, \ldots, x_m \in \mathbb{R}^n$ and $z \in \mathbb{C}^N$. In particular, it holds that*

$$u^{\sharp \mathbf{e}}(0) \geqslant |u^{\sharp \mathbf{e}}(x)| \quad \text{for all } x \in \mathbb{R}^n.$$

With such properties in mind we can start proving many of the main inequalities, for example a version of the Brascamp-Lieb-Luttinger inequality (see Theorem 3.2.4) which involves the Fourier Steiner symmetrization instead of a classical symmetric-decreasing re-arrangement. But those results will not be addressed in this chapter and we forward the reader to Chapter 8 for a complete breakdown of those results.

Additionally we note that a clear path in proving Theorem 7.3.1 is already outlined in [24]. We will closely follow this structure and start by recalling the main lemma given in Lemma 5.2.2. In order to apply this results, the following claim needs to hold.

**Claim.** *Let $Q_{v,\omega}$ be a boosted ground state, then $\{|\widehat{Q_{v,\omega}}| > 0\}$ is connected.*

Luckily for us, we can generalize the idea behind such a proof and can state a much more general result. Note that the idea behind the next lemma still comes from observations we made for equation (7.1.3) where the model case was $P(D) = (-\Delta)^s$. The following result has a very topological flavor to it and shows how the $n$-fold Minkowski sum and connectedness of sets hold together.

**Lemma 7.3.2.** *Let $n \geqslant 2$ and suppose $\ell \geqslant 2$ is an integer. Let $f \in L^{\ell/(\ell-1)}(\mathbb{R}^n) \geqslant 0$ be a continuous nonnegative function with $f = f^{*e}$ with some $\mathbf{e} \in \mathbb{S}^{n-1}$ and assume $f$ satisfies an equation of the form*

$$f(x) = h(x) \left( f * \ldots * f \right)(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{7.3.1}$$

*with $\ell$ factors in the convolution product on the left side and $h : \mathbb{R}^n \to (0, +\infty)$ is some continuous positive function. Then the set $\{f > 0\} \subset \mathbb{R}^n$ is connected.*

*Proof Sketch.* First of all, let $\Omega = \{f > 0\}$ and assume that $\Omega$ is not empty. Then for any set $X \subseteq \mathbb{R}^n$ and $L \in \mathbb{N}$ we define the $L$-fold Minkowski sum of $X$ with itself as

$$S_L(X) := \bigoplus_{j=1}^{L} X.$$

Clearly, with using a proper rotation $R \in O(n)$ we can assume without loss of generality that $e = e_1$, which denotes the unit vector in $x_1$-direction. Using that

$$\{(f * \ldots * f) > 0\} = \Omega$$

we find

$$S_{n\ell}(\Omega) = \Omega \quad \text{for any } n \in \mathbb{N}.$$

Recalling that for any $x_1\mathbb{R}$ fixed the sets $\{x' \in \mathbb{R}^{n-1} | f(x_1, x') > 0\}$ are open balls centered at the origin. Due to the cylindrical symmetry of $f$ we can find the following map

$$\mathbb{R} \to [0, +\infty], \quad x_1 \mapsto \rho(x_1)$$

such that $B_{\mathbb{R}^{n-1}}(0, \rho(x_1)) = \{x' \in \mathbb{R}^{n-1} | f(x_1, x') > 0\}$. Hence we can assume that

$$\Omega \cap \{x_1 \geqslant 0\} \neq \varnothing.$$

So one of the two cases must occur:

**(A)** $\Omega \backslash \{x_1 \geqslant 0\} = \varnothing$, or

**(B)** $\Omega \backslash \{x_1 \geqslant 0\} \neq \varnothing$.

The proof that in both statements $\Omega$ is connected is rather convoluted and will be omitted in this chapter. For a detailed discussion we refer to Chapter 8. $\qquad\square$

The rest is the claim then simply follows from the following two facts. First, we can now show that

$$\Omega = \{\widehat{Q}^{*_1} > 0\} = \{|\widehat{Q}| > 0\}$$

is connected by Lemma 7.3.2.

Finally, since both $Q$ and $Q^{*_e}$ are boosted ground states we have equality in $L^p$-norm. The rest of the proof of Theorem 7.3.1 follows easily from applying Lemma 7.3.2.

**Remark.** *The main properties in proving that $\Omega \subseteq \mathbb{R}^n$ is connected are the following two:*

*(1) $\Omega = \bigoplus_{j=1}^N \Omega$ for some $N \in \mathbb{N}$, and*

*(2) $\Omega$ is open and not empty.*

*In the case of $n = 1$ we can directly show that $\Omega$ is connected and can give a nice characterization of this set as well (see Section 7.4). For $n > 1$ this is still a conjecture.*

## 7.4 Fourier Rearrangement for $n = 1$

Clearly, in the case of one dimension the concept of $\sharp_e$ doesn't make any sense. As seen in Chapter 6 we define

$$f^{\bullet} = \mathcal{F}^{-1}(|\mathcal{F}(f)|) \quad \text{for } f \in L^2(\mathbb{R}).$$

Showing a conjugation symmetry for boosted ground states is possible, but again requires $\{|\widehat{Q_{v,\omega}}| > 0\}$ to be connected. A detailed approach using analyticity is given in Chapter 6. In the following we consider the case where $P(D) = \sqrt{-\Delta}$ which is referred to as the half-wave operator. Clearly, the symbol for $P(D)$ is not analytic anymore and hence doesn't fall in the same category as symbols considered in Chapter 6. To be more clear, we consider traveling solitary waves to the one dimension half-wave equations given by

$$i\partial_t u = \sqrt{-\Delta}u - |u|^{2\sigma}u, \tag{7.4.1}$$

where $u : [0,T) \times \mathbb{R} \to \mathbb{C}$ and $\sigma \in \mathbb{N}$ is an integer. We remark that $\sigma_* = +\infty$ in this case, hence no upper bound for $\sigma$ is needed. The existence of boosted ground states for $n = 1$ is covered by Theorem 7.2.1, if $|v| < 1$ and $\omega > 0$. With similar assumptions we can prove the following symmetry result.

**Theorem 7.4.1** (Conjugation Symmetry for $n = 1$)**.** *Let $n = 1$ and suppose the hypotheses of Theorem 7.2.1 are satisfied. Moreover, we assume $\sigma \in \mathbb{N}$ is an integer. Then any boosted ground state $Q_{\omega,v} \in H^{\frac{1}{2}}(\mathbb{R})$ is of the form*

$$Q_{\omega,v}(x) = e^{i\alpha}Q_{\omega,v}^{\bullet}(x + x_0) \quad \text{for a. e. } x \in \mathbb{R},$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. In particular, any such $Q_{\omega,v} \in H^{\frac{1}{2}}(\mathbb{R})$ satisfies (up to translation and phase) the following conjugation symmetry for a. e. $x \in \mathbb{R}$*

$$Q_{\omega,v}(x) = \overline{Q_{\omega,v}(-x)}.$$

### 7.4.1 Outlines of the Proof

As in Section 7.3 we want to be able to use Lemma 5.2.2. In order to do this, we need to prove that $\Omega = \{|\widehat{Q_{v,\omega}}| > 0\}$ is connected. But in the case for $n = 1$ this is much easier with the following lemma.

**Lemma 7.4.1.** *Suppose $\Omega \subset \mathbb{R}$ is an open and non-empty set such that*

$$\Omega = \bigoplus_{k=1}^{m} \Omega$$

*for some integer $m \geqslant 2$. Then it holds that*

$$\Omega \in \{\mathbb{R}_{>0}, \mathbb{R}_{<0}, \mathbb{R}\}.$$

The proof of Theorem 7.4.1 is now a simple application of the lemma above.

## 7.5 Counterexample in the Case of Non-Connectedness

We construct an example to show the necessity of the topological assumption in most of the main theorems concerning the connectedness of $\Omega = \{|\widehat{f}| > 0\}$.

Let $y_1 = (10, 0) \in \mathbb{R}^2$ be a given point with $y_2 = -y_1$ and let $U = B_1(y_1) \cup B_1(y_2)$. We choose a function $\psi \in C_c^\infty(\mathbb{R}^2)$ with $\psi \geqslant 0$, $\mathrm{supp}(\psi) \subset U$ with $\psi|_{B_1(y_1)} \not\equiv 0$ and $\psi|_{B_1(y_2)} \not\equiv 0$. Additionally we assume $\psi$ to be cylindrically symmetric, i.e. $\forall (x_1, x_2) \in \mathbb{R}^2$ : $\psi(x_1, x_2) = \psi(x_1, -x_2)$, and $\psi$ to be non-increasing in the second variable, i.e. for each $(x_1, x_2), (x_1, x_3) \in \mathbb{R}^2$ with $|x_2| \leqslant |x_3|$ one has $\psi(x_1, x_2) \geqslant \psi(x_1, x_3)$.

Now we pick numbers $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$ and choose $\vartheta \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\vartheta|_{B_1(y_1)} \equiv \alpha$ and $\vartheta|_{B_1(y_2)} \equiv \beta$. Moreover, we define the function $\tilde{\psi} \in C_c^\infty(\mathbb{R}^2)$ as

$$\tilde{\psi}(\xi) = e^{i\vartheta(\xi)} \psi(\xi).$$

Using the properties of $\psi$ we readily find that the Fourier Steiner rearrangement in 1 codimension is given by

$$\tilde{\psi}^{*_1} = \psi.$$

Next, we consider $f := \mathcal{F}^{-1}(\tilde{\psi})$. By construction we have $f^{\sharp_1} = \mathcal{F}^{-1}(\psi)$ and the function $\widehat{f}$ is continuous and the set $\Omega = \{|\widehat{f}| > 0\} = \{\psi > 0\}$ is not connected in $\mathbb{R}^2$. We now claim that

$$\|f\|_{L^4} = \|f^{\sharp_1}\|_{L^4}.$$

By recalling the proof in [24] and taking the notation therein it is sufficient to show that the following equality holds

$$\Theta(\boldsymbol{\eta}, \boldsymbol{\xi}) = \sum_{k=1}^{2} \vartheta(\eta_k + \xi_k) - \vartheta(\xi_k) = 0 \text{ for all } (\boldsymbol{\eta}, \boldsymbol{\xi}) \in S.$$

To see this, one remembers that $(\boldsymbol{\eta}, \boldsymbol{\xi}) = ((\eta_1, \xi_1), (\eta_2, \xi_2)) \in S$ implies that

$$\eta_1 + \eta_2 = 0 \text{ and } (\eta_k + \xi_k, \xi_k) \in \Omega \times \Omega \subset U \times U \text{ for } k = 1, 2.$$

Therefore we find that $((\eta, -\eta), (\xi_1, \xi_2)) \in S$ if and only if $(\eta + \xi_1, -\eta + \xi_2) \in B_1(y_1) \times B_1(y_2)$ or $(\eta + \xi_1, -\eta + \xi_2) \in B_1(y_2) \times B_1(y_1)$.

Hence we conclude that for all $(\boldsymbol{\eta}, \boldsymbol{\xi}) \in S$ we have

$$\sum_{k=1}^{2} \vartheta(\eta_k + \xi_k) - \vartheta(\xi_k) = 0,$$

where one uses that $\vartheta|_{B_1(y_1)} \equiv \alpha$, $\vartheta|_{B_1(y_2)} \equiv \beta$ and the structure of the set $S$.

Hence we can conclude that $\|f\|_{L^4} = \|f^{\sharp_1}\|_{L^4}$ and

$$\mathcal{F}(f) = e^{i\vartheta}\mathcal{F}(f)^{*_1},$$

where the phase function $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ does not need to be affine in general.

## 7.6 Spectral Renormalization Method

In this section we will introduce the spectral renormalization method (see [12]). A rigorous proof will not be included and we refer to [12] for further details. The main idea of the spectral renormalization theorem is a fixed point argument on the Fourier side of the equation instead of working in $x$-space.

In the following consider equations of the form

$$((-\Delta)^s - iv \cdot \nabla + \alpha)Q - |Q|^{2\sigma}Q = 0, \tag{7.6.1}$$

where the parameters are given as in Section 7.1 (see Theorem 7.2.1). Upon taking the Fourier transform one easily sees that

$$\mathcal{Q}(\xi) = \frac{\mathcal{F}(|Q|^{2\sigma}Q)(\xi)}{|\xi|^{2s} - v \cdot \xi + \alpha}. \tag{7.6.2}$$

Thus a first idea is to use a fix point iteration of the following form

$$\begin{cases} Q_{(0)}(x) & = e^{-|x|^2} \\ Q_{(j+1)}(x) & = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(|Q_{(j)}|^{2\sigma}Q_{(j)})(\xi)}{|\xi|^{2s} - v \cdot \xi + \alpha}\right)(x) & \text{for } j \in \{0, 1, \ldots\}. \end{cases} \tag{7.6.3}$$

Note that the initial guess $Q_{(0)}(x)$ is a function and not a scalar, certainly other choices could have been made but not all of them converge to a proper nontrivial solution. For example, take $Q_{(0)} \equiv 0$. The next result shows the convergence with a bad initial guess which is given by a scalar multiple of a nontrivial solution to equation (7.6.1).

**Lemma 7.6.1.** *Let $Q$ be a nontrivial solution to equation (7.6.1). Consider $Q_{(0)} := cQ$ to be the initial guess for the scheme in (7.6.3). Then the following holds*

$$\lim_{j \to \infty} Q_{(j)} = \begin{cases} 0 & ; & \text{if } c \in (0, 1) \\ +\infty & ; & \text{if } c \in (1, +\infty). \end{cases} \tag{7.6.4}$$

*Proof.* Clearly, using equation (7.6.2) we see that

$$\mathcal{F}(Q_{(1)})(\xi) = c^{2\sigma+1}\mathcal{F}(Q)(\xi)$$

for all $\xi \in \mathbb{R}^n$. Hence by induction we find

$$Q_{(j)} = c^{(2\sigma+1)^j}Q$$

for each $j \in \{0, 1, \ldots\}$. The result readily follows from that observation. $\qquad\square$

In order to prevent such a behavior we do the following. Multiplying equation (7.6.2) by the complex conjugate $\overline{\mathcal{F}(Q)}$ and integrating yields

$$\mathcal{L}(Q) = \mathcal{R}(Q),$$

where the functionals $\mathcal{L}$ and $\mathcal{R}$ are given by

$$\mathcal{L}(Q) := \|\mathcal{F}(Q)\|^2_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \mathcal{R}(Q) := \int_{\mathbb{R}^n} \frac{\mathcal{F}(|Q|^{2\sigma}Q)\overline{\mathcal{F}(Q)}(\xi)}{|\xi|^{2x} - v \cdot \xi + \alpha} d\xi$$

Clearly, $Q_{(j)}$ does not need to satisfy $\mathcal{L}(Q) = \mathcal{R}(Q)$ in general. Hence we choose

$$Q_{(j+1/2)} := c_j Q_{(j)},$$

where $c_j$ is determined by the fact that $Q_{(j+1/2)}$ satisfied the said equation. To be more precise, consider

$$\mathcal{L}_j(Q) := \mathcal{L}(Q_{(j)}) \quad \text{and} \quad \mathcal{R}_j(Q) := \mathcal{R}(Q_{(j)}),$$

then $c_j$ is chosen such that

$$\mathcal{L}(c_j Q_{(j)}) = \mathcal{R}(c_j Q_{(j)}).$$

Hence we see that

$$|c_j|^2 \mathcal{L}_j(Q) = |c_j|^{2\sigma+2} \mathcal{R}_j(Q),$$

which implies that for $c_j$ exist three seperate solutions. If $c_j = 0$ then $Q_j$ converges to 0. If $c_j = +\infty$ then $Q_j$ diverges. If $c_j = |\mathcal{L}_j(Q)/\mathcal{R}_j(Q)|$ we are on the good side. Applying the iteration scheme to $Q_{(j+1/2)}$ we end up with

$$\mathcal{F}(|Q_{(j+1/2)}|^{2\sigma}Q_{(j+1/2)}) = c_j^{2\sigma+1}\mathcal{F}(|Q_{(j)}|^{2\sigma}Q_{(j)})$$

and therefore we define the spectral renormalization scheme for equation (7.6.1) as follows.

**Numerical Scheme.** *Let $Q_{(0)}(x) = e^{-|x|^2}$ for $x \in \mathbb{R}^n$ be the initial iteration function. Then we define the following numerical scheme*

$$\mathcal{F}(Q_{(j+1)}) = \left|\frac{\mathcal{L}_j(Q)}{\mathcal{R}_j(Q)}\right|^{\frac{2\sigma+1}{2\sigma}} \frac{\mathcal{F}(|Q_{(j)}|^{2\sigma}Q_{(j)})}{|\xi|^{2s} - v \cdot \xi + \alpha}.$$

*This scheme is called the* **spectral renormalization method** *for equation (7.6.1).*

### 7.6.1 Visualization for $n = 1$

In the following we give an example code for such a scheme in the language of Matlab. A code for the plot is already included and shows the analytical solution to

$$(-\Delta + 1)Q - |Q|^{2\sigma}Q = 0$$

for $n = 1$ next to the numerical one. Clearly, a code for $n = 2$ is very similar and simply uses some other function already included in Matlab. For the sake of clarity only the case $n = 1$ will be included.

**Matlab Code**

```matlab
function [] = Spectral_renormalization_method(alpha, velocity, s,
    sigma)

    % Defining the interval used for the evaluation
    Nx=100;
    Xmax = 10;
    dx = 2*Xmax/Nx;
    x = [-Xmax:dx:Xmax-dx];

    % Defining the vector for the Fourier space
    dk = pi/Xmax;
    k = fftshift([-Nx/2:Nx/2-1]*dk);
    beta = 1 + 1/(2*sigma);

    % Input function to start the iteration
    R0 = exp(-x.^2);
    Rn = R0;

    % Conditions on the loop to end
    m = 1;
    thresh = 1e-6;
    max_iter = 100;
    error = 1;

    while(error > thresh && m < max_iter)
        % Spectral renormalization algorithm
        Rn_hat = fft(Rn);
        NL_hat = fft(abs(Rn).^(2*sigma).*Rn);
        psi = abs(k).^(2*s) - velocity*k + alpha;
        SL = dk*sum(conj(Rn_hat).*Rn_hat);
        SR = dk*sum(conj(Rn_hat).*NL_hat./(psi));
        Rn_hat = (SL/SR)^beta*NL_hat./(psi);
        Rn = ifft(Rn_hat);
        error = abs(SL/SR-1);
        m = m+1;
    end

    % plotting the analytic solution for alpha=1, velocity = 0, s
        =1,
    % sigma=1/2 and the numerical solution with the specified
        parameters.
    R_analytic = (sigma + 1).^(1/(2*sigma))*(1/sqrt(velocity.^2 +
        alpha))^(-1/sigma)*cosh((2*sigma)/2.*sqrt(velocity.^2 +
        alpha).*x).^(-1/sigma);
    plot(x, real(Rn),"-",x, R_analytic,"--");
    xlabel("x");
    ylabel("R^{(0)}");
    legend("numerical solution", "analytic solution for s=1");
end
```

**Visualization of a Solution to a Boosted Biharmonic NLS**

In the following we show a picture for a biharmonic NLS with boost term $v$ given by

$$(\Delta^2 + vx + 1)Q(x) - |Q(x)|^2 Q(x) = 0. \tag{7.6.5}$$

A good reference for the study of biharmonic NLS is in Fibich's book (see [12]). To compute the solution to this equation we use the spectral renormalization method given above for the case $s = 2$, $\alpha = 1$, $v = 1/2$ and $\sigma = 1$. The real part of the solution will be symmetric with respect to the origin, this is shown in the following picture on the next page.
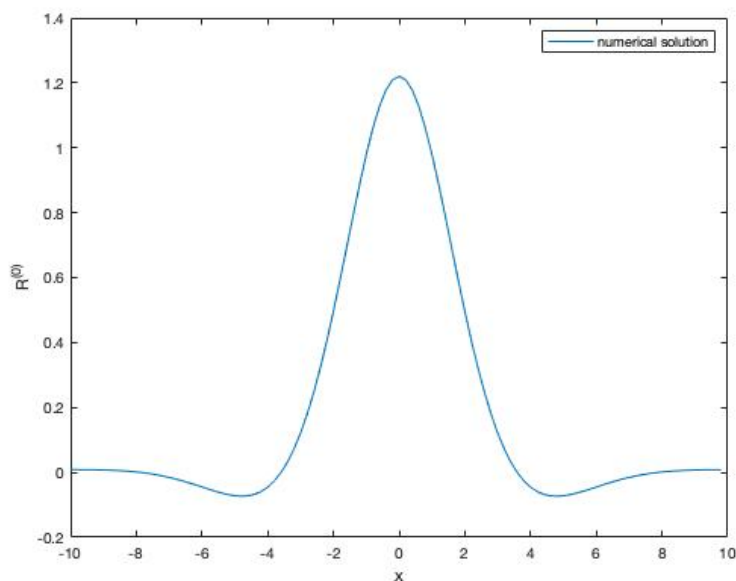


Figure 7.1: Solution to biharmonic NLS with $s = 2$, $\alpha = 1$, $v = 1/2$ and $\sigma = 1$.

The interesting part is actually what the boost term $v$ is doing on the Fourier side. Clearly, on an intuitive level one might think that it "shifts" the solution to one side in the sense that the maximum is shifted to the right if $v > 0$.

In the next figure we investigate equation 7.6.5 for different types of $v$ (note that in the spectral renormalization scheme $v$ is called velocity) and let $x_{\max} = 40$ for the scheme. The following three pictures show the development for $v = 0$, $v = 1/4$ and $v = 1/2$ on the Fourier side for the real part of $\widehat{Q}$. To exemplify how severe this shift is a horizontal line through the origin is added.
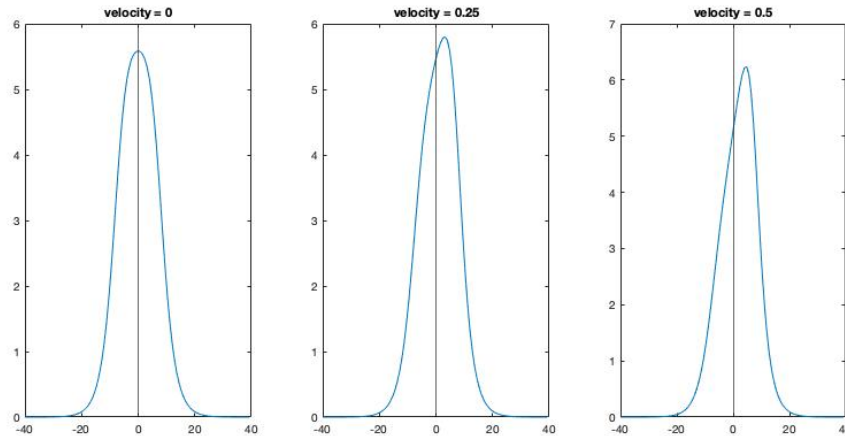
Figure 7.2: Development of a solution to 7.6.5 for $v = 0$, $v = 1/4$ and $v = 1/2$.

**Visualisation of the Solution to Equation (7.6.5) as a Movie**

In the following movie we show the conjugation symmetry of solutions to equation (7.6.5) and how they shift with different values of $v$. Note that the actual mathematical backgroup to this is fully discussed in Chapter 6 and Chapter 8.

In the following code we choose the values from $v = 0$ up to $v = 0.99$ and apply a spectral renormalization method to compute a numerical solution for each velocity. To show the change of the solution more clearly a fixed plot for the zero-velocity case is included for the real part of the solution $Q$.

**Matlab Codes**

```
1   clear; close all;
2   Nx=100;
3   Xmax = 10;
4   dx = 2*Xmax/Nx;
5   x = [-Xmax:dx:Xmax-dx];
6   frames=1:500;
7
8   Initial = Spectral_renormalization_method(1,0,2,1, Xmax, Nx);
9   vidfile = VideoWriter('biharmonic_movie');
10  open(vidfile);
11  for j = frames
12      z=Spectral_renormalization_method(1,(j-1)/length(frames),2,1,
            Xmax, Nx);
13      subplot(1,2,1)
14      plot(x,z(1,[1:end]),"-",x,Initial(1,[1:end]),"-")
15      xline(0);
16      yline(0);
17      ylim([-0.5,1.5]);
18      title("Real Part of the Solution")
19      subplot(1,2,2)
20      plot(x,z(2,[1:end]),"-")
```

```matlab
21          xline(0);
22          yline(0);
23          ylim([-0.5,1.5]);
24          title("Imaginary Part of the Solution")
25          drawnow
26          F(j) = getframe(gcf);
27          writeVideo(vidfile,F(j));
28     end
29     close(vidfile)
30
31     function Solution = Spectral_renormalization_method(alpha,
            velocity, s, sigma, Xmax, Nx)
32
33          % Defining the interval used for the evaluation
34          dx = 2*Xmax/Nx;
35          x = [-Xmax:dx:Xmax-dx];
36
37          % Defining the vector for the Fourier space
38          dk = pi/Xmax;
39          k = fftshift([-Nx/2:Nx/2-1]*dk);
40          beta = 1 + 1/(2*sigma);
41
42          % Input function to start the iteration
43          R0 = exp(-x.^2);
44          Rn = R0;
45
46          % Conditions on the loop to end
47          m = 1;
48          thresh = 1e-6;
49          max_iter = 100;
50          error = 1;
51
52          while(error > thresh && m < max_iter)
53              % Spectral renormalization algorithm
54              Rn_hat = fft(Rn);
55              NL_hat = fft(abs(Rn).^(2*sigma).*Rn);
56              psi = abs(k).^(2*s) - velocity*k + alpha;
57              SL = dk*sum(conj(Rn_hat).*Rn_hat);
58              SR = dk*sum(conj(Rn_hat).*NL_hat./(psi));
59              Rn_hat = (SL/SR)^beta*NL_hat./(psi);
60              Rn = ifft(Rn_hat);
61              error = abs(SL/SR-1);
62              m = m+1;
63          end
64
65          % Solution  x-space
66          Solution_real = real(Rn);
67          Solution_imag = imag(Rn);
68          Solution = [Solution_real; Solution_imag];
69     end
```

# Chapter 8

# On Symmetry for Traveling Solitary Waves for Dispersion Generalized NLS

This Chapter consists of an article which was written in collaboration with my mentor Enno Lenzmann, former research group member Armin Schikorra (now professor at the University of Pittsburgh) and postdoc Jérémy Sok, who also works in the same research group. The original article is found in [6]. In the following pages the original article undergoes some small modifications due to formatting but the mathematical content is identical and proper citations are included as in [6]. Note that due to including the article in this thesis the reference numbers might be different compared to the original ones.

## 8.1 Introduction and Main Results

The aim of the present chapter is to derive symmetry results for traveling solitary waves for nonlinear dispersive equations of nonlinear Schrödinger (NLS) type. As a model case in space dimension $n \geqslant 1$, we consider equations of the form

$$i\partial_t u = P(D)u - |u|^{2\sigma}u \tag{gNLS}$$

for functions $u : [0, T) \times \mathbb{R}^n \to \mathbb{C}$. Here $P(D)$ denotes a self-adjoint and constant coefficient (pseudo-)differential operator defined by multiplication in Fourier space as

$$(\widehat{P(D)u})(\xi) = p(\xi)\hat{u}(\xi), \tag{8.1.1}$$

where suitable assumptions on the multiplier $p(\xi)$ will be stated below. In fact, the class of allowed symbols $p(\xi)$ will be rather broad including e. g. fractional and polyharmonic NLS, higher-order NLS with mixed dispersions, half-wave and square-root Klein-Gordon equations (see, e. g., [?, 4, 7, 12, 13, 20, 22]) and also Subsection 5.1 below.

Let us first start with some informal remarks. Due to the focusing nature of the non-linearity in (gNLS), we expect the existence of solitary waves $u(t, x) = e^{it\omega}Q(x)$. In fact, by the translational invariance exhibited by the problem at hand, we expect that **traveling solitary waves** exist, which by definition are solutions of the form

$$u(t, x) = e^{i\omega t}Q_{\omega, \mathbf{v}}(x - \mathbf{v}t) \tag{8.1.2}$$

with some non-trivial profile $Q : \mathbb{R}^n \to \mathbb{C}$ depending on the given parameters $\omega \in \mathbb{R}$ (frequency) and $\mathbf{v} \in \mathbb{R}^n$ (velocity). However, except for the important but special case of classical NLS when $P(D) = -\Delta$ and its Galilean invariance (see (8.1.4) below), there is no known *boost symmetry*, which transforms a solitary wave at rest with $\mathbf{v} = 0$ into a traveling solitary wave with $\mathbf{v} \neq 0$ for a general NLS-type equation like (gNLS). More importantly, in the absence of an explicit boost transform, the symmetries of the profile function $Q_{\omega,\mathbf{v}}$ remain elusive in general. Yet, by inspecting the known explicit case when $P(D) = -\Delta$, we may conjecture that the following symmetries are also present in the general case: Up to spacial translation and complex phase, i.e., replacing $Q_{\omega,\mathbf{v}}$ by $e^{i\theta} Q_{\omega,\mathbf{v}}(\cdot + x_0)$ with constants $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, we have that:

(S1) $Q_{\omega,\mathbf{v}}$ is **cylindrically symmetric with respect to** $\mathbf{v} \in \mathbb{R}^n$, $n \geqslant 2$, i.e., we have

$$Q_{\omega,\mathbf{v}}(x) = Q_{\omega,\mathbf{v}}(\mathsf{R}x) \quad \text{for all } \mathsf{R} \in \mathrm{O}(n) \text{ with } \mathsf{R}\mathbf{v} = \mathbf{v}.$$

(S2) We have the **conjugation symmetry** given by

$$Q_{\omega,\mathbf{v}}(x) = \overline{Q_{\omega,\mathbf{v}}(-x)}.$$

That is, $\operatorname{Re} Q_{\omega,\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ is an even function, whereas $\operatorname{Im} Q_{\omega,\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ is an odd function.

As our main results below, we will establish the symmetry properties (S1) and (S2) for so-called **boosted ground states** $Q_{\omega,\mathbf{v}}$ which are by definition obtained as optimizers for a certain variational problem. In fact, we will show that (under suitable assumptions) that all such boosted ground state *must* satisfy (S1) and (S2). Our arguments will be based on rearrangement techniques (Steiner symmetrizations) performed in Fourier space. The core of our argument to obtain such a sharp symmetry result will be based on a topological property of the set $\{\xi \in \mathbb{R}^n : |\widehat{Q}_{\omega,\mathbf{v}}(\xi)| > 0\}$ combined with a recent rigidity result [24] obtained for the *Hardy–Littlewood majorant problem* in $\mathbb{R}^n$. A more detailed sketch of the proof will be given below.

### 8.1.1 Setup of the Problem

Let us formulate the assumptions needed for our result. We impose the following conditions on the operator $P(D)$ in (gNLS).

**Assumption 4.** *The operator $P(D)$ has a real-valued and continuous symbol $p : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following bounds*

$$A|\xi|^{2s} + c \leqslant p(\xi) \leqslant B|\xi|^{2s} \quad \text{for all } \xi \in \mathbb{R}^n,$$

*with some constants $s \geqslant \frac{1}{2}, A > 0, B > 0$, and $c \in \mathbb{R}$.*

Let us assume that $P(D)$ satisfies the assumption above. We readily deduce the norm equivalence

$$\|u\|_{H^s}^2 = \|(1-\Delta)^{s/2} u\|_{L^2}^2 \simeq \langle u, (P(D) + \lambda) u \rangle = \int_{\mathbb{R}^n} (p(\xi) + \lambda) |\widehat{u}(\xi)|^2 \, d\xi,$$

where $\lambda > 0$ is a sufficiently large constant. Moreover, we notice that the problem (gNLS) exhibits (formally at least) conservation of energy and $L^2$-mass, which are given by

$$E[u] = \frac{1}{2} \langle u, P(D) u \rangle - \frac{1}{2\sigma + 2} \|u\|_{L^{2\sigma+2}}^{2\sigma+2}, \quad M[u] = \|u\|_{L^2}^2.$$

Furthermore, with the real number $s \geqslant \frac{1}{2}$ as in Assumption 4, we define the following exponent (not necessarily an integer number) given by

$$\sigma_*(s, n) := \begin{cases} \dfrac{2s}{n - 2s} & \text{if } s < n/2, \\ +\infty & \text{if } s \geqslant n/2, \end{cases}$$

which marks the threshold of energy-criticality for exponents, i.e., the range $1 \leqslant \sigma < \sigma_*$ corresponds to the energy-subcritical case for problem (gNLS). In fact, we will focus on the range in the rest of this paper with some marginal comments on the energy-critical case $\sigma = \sigma_*$ (which of course can occur only if $s < n/2$).

We are interested in traveling solitary waves with finite energy for the model problem (gNLS). By plugging the ansatz (8.1.2) into (gNLS), we readily find that the profile $Q_{\mathbf{v},\omega} \in H^s(\mathbb{R}^n)$ has to be a weak solution of the nonlinear equation

$$P(D)Q_{\omega,\mathbf{v}} + iv \cdot \nabla Q_{\omega,\mathbf{v}} + \omega Q_{\omega,\mathbf{v}} - |Q_{\omega,\mathbf{v}}|^{2\sigma}Q_{\omega,\mathbf{v}} = 0. \tag{8.1.3}$$

As briefly mentioned above, there exists a well-known 'gauge transform' (corresponding to Galilean boosts in physical terms) for the classical Schrödinger, where we can reduce the general case $\mathbf{v} \in \mathbb{R}^n$ to vanishing velocity $\mathbf{v} = 0$. More precisely, if we consider (gNLS) with $P(D) = -\Delta$, the Galilean boost transform given by

$$Q(x) \mapsto e^{\frac{i}{2}\mathbf{v} \cdot x}Q(x) \tag{8.1.4}$$

reduces the analysis of (8.1.3) to the study of the nonlinear equation

$$-\Delta Q + \omega_{\mathbf{v}}Q - |Q|^{2\sigma}Q = 0 \quad \text{with} \quad \omega_{\mathbf{v}} = \omega - \frac{1}{2}|\mathbf{v}|^2, \tag{8.1.5}$$

where the boost term $iv \cdot \nabla$ has been gauged away. An important feature of the Galilean transform (8.1.4) is that preserves the $L^2$-norm $\|Q_{\mathbf{v}}\|_{L^2} = \|Q\|_{L^2}$; in fact, it is a unitary transform on $L^2(\mathbb{R}^n)$.

However, for general dispersion operators $P(D) \neq -\Delta$, no such explicit boost transform in the spirit (8.1.4) is known to exist. Therefore, an alternative approach is needed to deal with more general $P(D)$ in both respects concerning existence and symmetries of non-trivial profiles $Q_{\mathbf{v}}$.

### 8.1.2 Existence of Traveling Solitary Waves

We first recall an existence result from [18] for non-trivial solutions $Q_{v,\omega} \in H^s(\mathbb{R}^n)$ of (??). To construct these solutions, we introduce a suitable variational setting as follows. For given $\mathbf{v} \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ (satisfying some conditions below), we define the Weinstein-type functional of the form

$$\mathcal{J}_{v,\omega,p}(u) := \frac{\langle u, (P_{\mathbf{v}}(D) + \omega)u \rangle^{\sigma+1}}{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}} \tag{8.1.6}$$

where $u \in H^s(\mathbb{R}^n)$ with $u \not\equiv 0$. Here and in what follows, we set

$$P_{\mathbf{v}}(D) := P(D) + i\mathbf{v} \cdot \nabla, \tag{8.1.7}$$

which has the multiplier $p_{\mathbf{v}}(\xi) = p(\xi) - \mathbf{v} \cdot \xi$. Recalling that $P(D)$ satisfies Assumption 1 with some $s \geqslant \frac{1}{2}$ and $A > 0$, it is straightforward to check that

$$\Sigma_{\mathbf{v}} := \inf_{\xi \in \mathbb{R}^n} p_{\mathbf{v}}(\xi) = \inf_{\xi \in \mathbb{R}^n} \{p(\xi) - \mathbf{v} \cdot \xi\} > -\infty, \tag{8.1.8}$$

provided that either $s > \frac{1}{2}$ and $\mathbf{v} \in \mathbb{R}^n$ arbitrary or $|\mathbf{v}| \leqslant A$ in the special case $s = \frac{1}{2}$. We have the following existence result.

**Theorem 8.1.1** (Existence of Boosted Ground States [18])**.** *Let $n \geqslant 1$, $\mathbf{v} \in \mathbb{R}^n$, and suppose that $P(D)$ satisfies Assumption 1 with some constants $s \geqslant \frac{1}{2}$ and $A > 0$, where if $s = 1/2$, we also assume that $|\mathbf{v}| < A$ holds.*

*Then, for $0 < \sigma < \sigma_*$ and $\omega > -\Sigma_{\mathbf{v}}$, every minimizing sequence for $\mathcal{J}_{v,\omega,p}$ is relatively compact in $H^s(\mathbb{R}^n)$ up to translations in $\mathbb{R}^n$. In particular, there exists some minimizer $Q_{\mathbf{v},\omega} \in H^s(\mathbb{R}^n)\backslash\{0\}$, i. e.,*

$$\mathcal{J}_{v,\omega,p}(Q_{\mathbf{v},\omega}) = \inf_{u \in H^s(\mathbb{R}^n)\backslash\{0\}} \mathcal{J}_{v,\omega,p}(u),$$

*and $Q_{\mathbf{v},\omega}$ solves the profile equation (8.1.3).*

**Remarks.** *1) Note that for the borderline case when $s = \frac{1}{2}$ and $|v| = A$ we still have that the $\inf_{0 \neq f \in H^s(\mathbb{R}^n)} \mathcal{J}_{v,\omega,p}(f) > -\infty$, but we do not expect this infimum to be attained. For such non-existence result for the (important) special case of the half-wave equations when $P(D) = \sqrt{-\Delta}$ and $|v| \geqslant 1$, we refer to [2].*

*2) Clearly, the variational ansatz using the functional $\mathcal{J}_{v,\omega,p}$ will break down if $P(D)$ satisfies the bounds in Assumption 1 with some $0 < s < 1/2$. In this case, the boost term $\mathrm{i}v \cdot \nabla$ cannot be treated as a perturbation of $P(D)$. In this case, we conjecture that the profile equation has only trivial solutions in $H^{1/2}(\mathbb{R}^n)$.*

*3) The infimum $\Sigma_{\mathbf{v}}$ defined in (8.1.8) corresponds to the bottom of the essential spectrum of the self-adjoint operator $P_v(D)$ acting on $L^2(\mathbb{R}^n)$ with domain $H^{2s}(\mathbb{R}^n)$. For the specific choices $P(D) = (-\Delta)^s$ and $P(D) = (-\Delta + 1)^s$, the number $\Sigma_{\mathbf{v}}$ can be explicitly calculated using the Legendre transform of the convex maps $\xi \mapsto |\xi|^{2s}$ and $\xi \mapsto (|\xi|^{2s} + 1)^s$, respectively. For details on this, we refer to [18].*

*4) See also [19, 22, 28], where the existence of boosted ground states for NLS type equations were shown by concentration-compactness methods for fractional NLS when $P(D) = (-\Delta)^s$ in the range $s \in [\frac{1}{2}, 1)$.*

From now on, we will refer to minimizers of the functional $\mathcal{J}_{v,\omega,p}$ as **boosted ground states**. Correspondingly, the solutions $u(t, x) = e^{\mathrm{i}t\omega}Q_{v,\omega}(x - \mathbf{v}t)$ will be called **ground state traveling solitary waves.** It is easy to check that any such boosted ground state $Q_{s,v} \in H^s(\mathbb{R}^n)$ satisfies the profile equation (8.1.3) after a suitable rescaling $Q_{s,v} \mapsto \alpha Q_{s,v}$ with some constant $\alpha > 0$.

### 8.1.3 Cylindrical and Conjugation Symmetry for $n \geqslant 2$

We now turn to our first main symmetry result, which establishes necessary symmetry properties of minimizers for the Weinstein-type functional $\mathcal{J}_{v,\omega,p}$ in space dimensions $n \geqslant 2$, under suitable assumptions on $P(D)$ and for integer $\sigma \in \mathbb{N}$.

In order to prove a symmetry results for minimizers of $\mathcal{J}_{v,\omega,p}$, we will further develop the Fourier symmetrization method recently introduced in [24]. The main idea there is to use symmetric-decreasing rearrangement in Fourier space. In fact, this approach proves to be a useful substitute for standard rearrangement techniques in $x$-space, which are easily seen to fail for a large class of (e. g. higher-order) operators (such as $P(D) = \Delta^2$) or operators with non-radially symmetric Fourier symbols such as $P_{\mathbf{v}}(D)$ above.

From [24] we recall the notion of Fourier rearrangement which is defined as

$$u^\sharp := \mathcal{F}^{-1}\{(\mathcal{F}u)^*\} \quad \text{for } u \in L^2(\mathbb{R}^n) \text{ with } n \geqslant 1, \tag{8.1.9}$$

where $f^*$ denotes the symmetric-decreasing rearrangement of a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ vanishing at infinity. For a non-zero velocities, the presence of the boost term $\mathrm{i}\mathbf{v} \cdot \nabla$ breaks radially symmetry in general. In this case, all rearrangement operations that yield spherically symmetric functions (such as $\sharp$ defined above) cannot be applied to the

minimization problem for $\mathcal{J}_{v,\omega,p}(f)$. However, under a suitable assumption on $P(D)$, we still expect to be able to show cylindrical symmetry of minimizers with respect to the direction given by the vector $\mathbf{v} \neq 0$. Thus we introduce the following notion: We say that $f : \mathbb{R}^n \to \mathbb{C}$ is **cylindrically symmetric** with respect to a direction $\mathbf{e} \in \mathbb{S}^{n-1}$ if we have

$$f(Ry) = f(y) \quad \text{for a.\,e. } y \in \mathbb{R}^n \text{ and all } \mathsf{R} \in \mathsf{O}(n) \text{ with } \mathsf{R}\mathbf{e} = \mathbf{e}. \tag{8.1.10}$$

For such functions $f$, we will employ some abuse of notation by writing

$$f = f(y_\|, |y_\perp|),$$

where we decompose $y \in \mathbb{R}^n$ as $y = y_\| + y_\perp$ with $y_\perp$ perpendicular to $\mathbf{e} \in \mathbb{S}^{n-1}$. For dimensions $n \geqslant 2$, we now introduce the following rearrangement operation defined as

$$u^{\sharp \mathbf{e}} := \mathcal{F}^{-1}\left\{(\mathcal{F}u)^{*\mathbf{e}}\right\} \quad \text{for } u \in L^2(\mathbb{R}^n) \text{ with } n \geqslant 2, \tag{8.1.11}$$

where $f^{*\mathbf{e}} : \mathbb{R}^n \to \mathbb{R}_+$ denotes the **Steiner symmetrization in $n-1$ codimensions** with respect to a direction $\mathbf{e} \in \mathbb{S}^{n-1}$, which is obtained by symmetric-decreasing rearrangements in $n-1$-dimensional planes perpendicular to $\mathbf{e}$; see Section 8.3 below for a precise definition. It is elementary to check that $f^{\sharp \mathbf{e}}$ is cylindrically symmetric with respect to $\mathbf{e}$.

We now formulate the following assumption for $P(D)$.

**Assumption 5.** *The operator $P(D)$ has a multiplier function $p : \mathbb{R}^n \to \mathbb{R}$ which is cylindrically symmetric with respect to some direction $\mathbf{e} \in \mathbb{S}^{n-1}$. Moreover, the map*

$$|\xi_\perp| \mapsto p(\xi_\|, |\xi_\perp|)$$

*is strictly increasing.*

We have the following general symmetry result.

**Theorem 8.1.2** (Symmetry of Boosted Ground States for $n \geqslant 2$). *Let $n \geqslant 2$ and suppose $P(D)$ satisfies Assumptions 4 and 5 with some $s \geqslant \frac{1}{2}$ and $\mathbf{e} \in \mathbb{S}^{n-1}$. Furthermore, let $\mathbf{v} = |\mathbf{v}|\mathbf{e} \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$ satisfy the hypotheses in Theorem 8.1.1 and assume $\sigma \in \mathbb{N}$ is an integer with $0 < \sigma < \sigma_*(n, s)$.*

*Then any boosted ground state $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R}^n)$ is of the form*

$$Q_{\omega,\mathbf{v}}(x) = e^{i\alpha} Q^{\sharp \mathbf{e}}(x + x_0)$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. As a consequence, any such $Q_{\omega,\mathbf{v}}$ satisfies (up to a translation and phase) the symmetry properties (P1) and (P2) for almost every $x \in \mathbb{R}^n$.*

**Remark.** *Since the Fourier transform $\widehat{(Q_{\omega,\mathbf{v}}^{\sharp \mathbf{e}})} = |\widehat{Q}_{\omega,\mathbf{v}}|^{*\mathbf{e}} \geqslant 0$ is nonnegative, we conclude that any boosted ground state $Q_{\omega,\mathbf{v}}$ is a positive-definite function in the sense of Bochner, provided we also assume that $\widehat{Q}_{\omega,\mathbf{v}} \in L^1(\mathbb{R}^n)$ (or more generally a finite Borel measure on $\mathbb{R}^n$). In many examples of interest, it is easy to check that indeed $\widehat{Q}_{\omega,\mathbf{v}} \in L^1(\mathbb{R}^n)$ holds. Recall that a continuous function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be positive-definite in the sense of Bochner if for any collections of points $x_1, \ldots, x_m \in \mathbb{R}^n$ we have*

$$\sum_{k,l=1}^{m} f(x_k - x_l)\overline{z}_k z_l \geqslant 0 \quad \text{for all } \mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m,$$

*i.\,e., the complex matrix $[f(x_k - x_k)]_{1 \leqslant k,l \leqslant m}$ is positive semi-definite. As a direct consequence, we find that*

$$f(0) \geqslant |f(x)| \quad \text{for all } x \in \mathbb{R}^n.$$

*We refer to [31] for a discussion of positive-definite functions.*

First, we briefly sketch the main line of argumentation for proving Theorem 8.1.2. Using the fact that $\sigma \in \mathbb{N}$ is an integer and by applying the *Brascamp–Lieb–Luttinger inequality* (a.k.a. multilinear Riesz-Sobolev inequality) in Fourier space, we deduce that any boosted ground state $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R}^n)$ satisfies

$$\mathcal{J}_{v,\omega,p}(Q^{\sharp_\mathbf{e}}_{\omega,\mathbf{v}}) \leqslant \mathcal{J}_{v,\omega,p}(Q_{\omega,\mathbf{v}}). \tag{8.1.12}$$

In particular, we see that $Q^{\sharp_\mathbf{e}}_{\omega,\mathbf{v}}$ is also a boosted ground state. More importantly, we find that equality in (8.1.12) holds if and only if

$$|\widehat{Q}_{\omega,\mathbf{v}}(\xi)| = |\widehat{Q}_{\omega,\mathbf{v}}(\xi)|^{*_\mathbf{e}} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{8.1.13}$$

This fixes the modulus of the Fourier transform $\widehat{Q}_{\omega,\mathbf{v}}$, whereas its phase appears is yet completely undetermined. However, the conclusion of Theorem 8.1.2 will follow once we show

$$\widehat{Q}_{\omega,\mathbf{v}}(\xi) = e^{\mathrm{i}(\alpha + \beta \cdot \xi)} |\widehat{Q}_{\omega,\mathbf{v}}(\xi)|^{*_\mathbf{e}} \tag{8.1.14}$$

with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. In fact, such a "rigidity result" about the phase function (i. e. being just an affine function on $\mathbb{R}^n$) can be deduce from the recent result in [24] on the Hardy-Littlewood majorant problem in $\mathbb{R}^n$, provided we know that the open set

$$\Omega = \{\xi \in \mathbb{R}^n : |\widehat{Q}_{\omega,\mathbf{v}}(\xi)| > 0\} \tag{8.1.15}$$

is connected. Establishing this topological fact is the crux of this paper. We remark that in [24] where the symmetric-decreasing (Schwarz) symmetrization in $\mathbb{R}^n$ was used, we always have that $\Omega$ is either an open ball or all of $\mathbb{R}^n$; in particular, the set $\Omega$ is connected. However, for the Steiner symmetrization in $n-1$ codimensions needed to define $\sharp_\mathbf{e}$ it is far from clear that the $\Omega$ is a connected set. Indeed, it is not hard to construct explicit examples of functions $f$ on $\mathbb{R}^n$ such that $|f| = |f|^{*_\mathbf{e}}$ such that $\{|f| > 0\}$ is not connected.

To eventually show that $\Omega$ above is in fact connected in our case, we will exploit the equation (8.1.5) in Fourier space. As a consequence, we find that $\Omega$ must be equal to its **$m$-fold Minkowski sum** with the integer $m = 2\sigma + 1$, i. e., we have

$$\Omega = \bigoplus_{k=1}^m \Omega := \{y_1 + \ldots + y_m : y_k \in \Omega, \ 1 \leqslant k \leqslant m\}. \tag{8.1.16}$$

The key step is now to establish the connectedness of $\Omega \subset \mathbb{R}^n$ from this information. Surprisingly, we did not succeed in finding a general argument to conclude that any open (non-empty) set $\Omega \subset \mathbb{R}^n$ that satisfies (8.1.16) is necessarily connected. However, by additionally using the cylindrical symmetry of $\Omega$, we are able to conclude that the sets $\Omega$ in question are indeed connected. See also the specific argument for the proof of Theorem 8.1.3 below addressing the one-dimensional case $\Omega \subset \mathbb{R}$.

### 8.1.4 Conjugation Symmetry for $n = 1$

In one space dimension, the concept of the symmetrization operation $\sharp_\mathbf{e}$ becomes void. Still, we expect the conjugation symmetry (P2) to hold for boosted ground states in the one-dimensional case. To this end, we define the following operation

$$f^\bullet = \mathcal{F}^{-1}\left\{|\mathcal{F}f|\right\} \quad \text{for } f \in L^2(\mathbb{R}^n).$$

We may still ask whether the boosted ground states $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R})$ as given by Theorem 8.1.1 always obey that

$$Q_{\omega,\mathbf{v}} = e^{\mathrm{i}\alpha} Q^\bullet_{\omega,\mathbf{v}}(x + x_0) \quad \text{for almost every } x \in \mathbb{R}^n,$$

with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. As already mentioned for the proof of Theorem 8.1.2 above, the key ingredient needed to be shown is that $\{|\widehat{Q}_{\omega,\mathbf{v}}| > 0\}$ is a connected set. Luckily, by exploiting the one-dimensionality of the problem, we can show that must have $\Omega \in \{\mathbb{R}_{>0}, \mathbb{R}_{<0}, \mathbb{R}\}$, whence it follows that $\Omega$ is connected.

**Theorem 8.1.3** (Conjugation Symmetry for $n = 1$). *Let $n = 1$ and suppose the hypotheses of Theorem 8.1.1 are satisfied. Moreover, we assume $\sigma \in \mathbb{N}$ is an integer. Then any boosted ground state $Q_{\omega,\mathbf{v}} \in H^{\frac{1}{2}}(\mathbb{R})$ is of the form*

$$Q_{\omega,\mathbf{v}}(x) = e^{\mathrm{i}\alpha} Q_{\omega,\mathbf{v}}^\bullet(x + x_0) \quad \text{for a. e. } x \in \mathbb{R},$$

*with some constants $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. In particular, any such $Q_{\omega,\mathbf{v}} \in H^{\frac{1}{2}}(\mathbb{R})$ satisfies (up to translation and phase) the conjugation symmetry (P2) for a. e. $x \in \mathbb{R}$.*

**Remarks.** *1) As in Theorem 8.1.2 above, we actually obtain that $Q_{\omega,\mathbf{v}}$ has non-negative Fourier transform. In particular, if $\widehat{Q}_{\omega,\mathbf{v}} \in L^1(\mathbb{R})$, we see that $Q_{\omega,\mathbf{v}}$ (up to translation and phase) is a positive-definite function in the sense of Bochner.*

*2) For a conjugation symmetry result in general dimensions $n \geqslant 1$, we refer to our companion paper [5], where an analyticity condition on the Fourier symbol $p(\xi)$ is imposed in order to be able to deal with $n \geqslant 2$.*

### 8.1.5 Examples

We list some essential examples, where we can deduce symmetries of boosted ground states for the following equation of the form (gNLS).

- **Fourth-order/biharmonic NLS** of the form

  $$\mathrm{i}\partial_t u = \Delta^2 u + \mu \Delta u - |u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

  where $\mu \in \mathbb{R}$ and integer $\sigma \in \mathbb{N}$ with $1 \leqslant \sigma < \infty$ if $1 \leqslant n \leqslant 4$ and $1 \leqslant \sigma < \frac{4}{n-4}$ if $n \geqslant 5$.

- **Fractional NLS** of the form

  $$\mathrm{i}\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

  with $s > 0$ and integers $\sigma \in \mathbb{N}$ such that $1 \leqslant \sigma < \sigma_*(s, n)$.

- **Half-Wave and Square-Root Klein-Gordon equations** of the form

  $$\mathrm{i}\partial_t u = \sqrt{-\Delta + m^2}\, u - |u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

  with $m \geqslant 0$ and arbitrary integer $\sigma \in \mathbb{N}$.

Finally, we also remark that the Fourier symmetrization techniques in this paper seem to be ready-made to be generalized to anisotropic NLS type equations, where the order of derivatives may depend on the spatial direction. For instance, we could study symmetries of boosted ground states for the focusing **half-wave-Schrödinger type equations** of the form

$$\mathrm{i}\partial_t u = \Delta_x u - \gamma\sqrt{-\Delta_y} u - |u|^{2\sigma} u, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}_x^k \times \mathbb{R}_y^l$$

with parameter $\gamma > 0$ and suitable integers $\sigma \in \mathbb{N}$. However, the relevant Sobolev space now becomes of the form

$$X = \{u \in L^2(\mathbb{R}^{k+l}) : \int_{\mathbb{R}^{k+l}} (|\xi|^2 + |\eta|)|\widehat{u}(\xi, \eta)|^2 \, d\xi \, d\eta < \infty\},$$

where $\widehat{u}(\xi, \eta)$ with $(\xi, \eta) \in \mathbb{R}^k \times \mathbb{R}^l$ denotes the Fourier transform of $u$ in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$.

## 8.2 Existence of Traveling Solitary Waves

This section is devoted to the proof of Theorem 8.1.1 by following the arguments in [18]. Instead of concentration-compactness methods, we shall follow a different approach by adapting the techniques in [1] based on a general compactness lemma in $\dot{H}^s$ for general $s > 0$ (originally due to E. Lieb for the case $s = 1$).

### 8.2.1 Proof of Theorem 8.1.1

We follow [18] adapted to our setting here. Suppose that $P(D)$ satisfies Assumption 4 with constants $s \geqslant \frac{1}{2}$, $A, B > 0$. Let $\mathbf{v} \in \mathbb{R}^n$ with be given, where we additionally assume $|\mathbf{v}| < A$ if $s = \frac{1}{2}$. Finally, we impose that $\omega > -\Sigma_{\mathbf{v}}$ with $\Sigma_{\mathbf{v}}$ defined in (8.1.8). Recalling that $P_{\mathbf{v}}(D) = P(D) + i\mathbf{v} \cdot \nabla$, we can define the norm

$$\|u\|_{\omega,\mathbf{v}} := \langle u, (P_{\mathbf{v}}(D) + \omega)u \rangle^{1/2} = \left( \int_{\mathbb{R}^n} (p(\xi) - \mathbf{v} \cdot \xi + \omega)|\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

It is elementary to see that we have the norm equivalence

$$\|u\|_{\omega,\mathbf{v}} \sim_{A,B,\mathbf{v},\omega} \|u\|_{H^s}.$$

Note that the functional $\mathcal{J}_{v,\omega,p}$ can be written as

$$\mathcal{J}_{v,\omega,p}(u) = \frac{\|u\|_{\omega,\mathbf{v}}^{2\sigma+2}}{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}}.$$

In what follows, we shall use $X \lesssim Y$ to mean that $X \leqslant CY$ with some constant $C > 0$ that only depends on $s, n, A, B, \sigma, \omega$. We set

$$\mathcal{J}_{v,\omega,p}^* := \inf \{ \mathcal{J}_{v,\omega,p}(u) \mid u \in H^s(\mathbb{R}^n), \ u \not\equiv 0 \}$$

Since $0 < \sigma < \sigma_*(n,s)$, we obtain the Sobolev-type inequality

$$\|u\|_{L^{2\sigma+2}} \lesssim \|u\|_{H^s} \lesssim \|u\|_{\omega,\mathbf{v}},$$

which shows that $\mathcal{J}_{v,\omega,p}^* > 0$ is strictly positive.

Suppose that $(u_j) \subset H^s(\mathbb{R}^n) \backslash \{0\}$ is a minimizing sequence, i.e., we have $\mathcal{J}_{v,\omega,p}(u_j) \to \mathcal{J}_{v,\omega,p}^*$ as $j \to \infty$. By scaling properties, we can assume without loss of generality that $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j \in \mathbb{N}$. Obviously, we find that $\sup_j \|u_j\|_{\omega,\mathbf{v}} \lesssim 1$. Hence the sequence $(u_j)$ is bounded in $H^s(\mathbb{R}^n)$.

Next, we show that $(u_j)$ has a non-zero weak limit in $H^s(\mathbb{R}^n)$, up to spatial translations and passing to a subsequence. To prove this claim, let us first assume that $s \neq n/2$ holds and therefore we have the continuous embedding $H^s(\mathbb{R}^n) \subset L^{2\sigma_*+2}(\mathbb{R}^n)$. Now we choose a number $r \in (2\sigma + 2, 2\sigma_* + 2)$. By Hölder's and Sobolev's inequality, we have

$$\|u_j\|_{L^r} \leqslant \|u_j\|_{L^{2\sigma+2}}^{\theta} \|u_j\|_{L^{2\sigma_*+2}}^{1-\theta} \lesssim \|u_j\|_{L^{2\sigma+2}}^{\theta} \|u_j\|_{H^s}^{1-\theta}, \tag{8.2.1}$$

with $\frac{\theta}{2\sigma+2} + \frac{1-\theta}{2\sigma_*+2} = \frac{1}{r}$. Since $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j$ and $\|u_j\|_{L^2} \leqslant \|u_j\|_{H^s} \lesssim 1$, we deduce from (8.2.1) that there exist constants $\alpha, \beta, \gamma > 0$ such that

$$\|u_j\|_{L^2} \leqslant \alpha, \quad \|u_j\|_{L^{2\sigma+2}} \geqslant \beta, \quad \|u_j\|_{L^r} \leqslant \gamma$$

holds for all $j \in \mathbb{N}$. In the borderline case $s = n/2$, we also deduce the existence of such constants $\alpha, \beta, \gamma > 0$, where we just have to replace $2\sigma_* + 2$ above by any number $q \in (2\sigma + 2, \infty)$ and use that $H^s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ holds. We omit the details.

Next, by invoking the Lemma 8.6.1, we deduce that

$$\inf_{j \in \mathbb{N}} |\{x \in \mathbb{R}^n \mid |u_j(x)| > \eta\}| \geqslant c$$

with some strictly positive constants $\eta, c > 0$, where $|\cdot|$ denotes the $n$-dimensional Lebesgue measure. Thus we can apply Lemma 8.6.2 to conclude (after passing to a subsequence if necessary) that there exists a sequence of translations $(x_j)$ in $\mathbb{R}^n$ and some non-zero function $u \in H^s(\mathbb{R}^n) \backslash \{0\}$ such that

$$u_j(\cdot + x_j) \rightharpoonup u \text{ in } H^s(\mathbb{R}^n). \tag{8.2.2}$$

Next, we show that the weak limit $u \not\equiv 0$ is indeed an optimizer for $\mathcal{J}_{v,\omega,p}$ and that $u_j \to u$ strongly in $H^s(\mathbb{R}^n)$. By the translational invariance of $\mathcal{J}_{v,\omega,p}$, we can assume that $x_j = 0$ for all $j$. Moreover, since the sequence $(u_j)$ is bounded in $H^s(\mathbb{R}^n)$, we can also assume pointwise convergence $u_j(x) \to u(x)$ almost everywhere. Recalling that $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j$, the Brézis-Lieb refinement of Fatou's lemma yields that

$$\|u_j - u\|_{L^{2\sigma+2}}^{2\sigma+2} + \|u\|_{L^{2\sigma+2}}^{2\sigma+2} = 1 + o(1).$$

Furthermore, from $\mathcal{J}_{v,\omega,p}(u_j) \to \mathcal{J}_{v,\omega,p}^*$ together with $\|u_j\|_{L^{2\sigma+2}} = 1$ for all $j$ we conclude that

$$\|u_j\|_{\omega,\mathbf{v}}^2 \to (\mathcal{J}_{v,\omega,p}^*)^{\frac{1}{\sigma+1}}.$$

On the other hand, since $u_j \rightharpoonup u$ in $H^s(\mathbb{R}^n)$ and writing $H = P_v(D) + \omega$ so that $\langle f, Hf \rangle = \|f\|_{\omega,\mathbf{v}}^2$ for all $f \in H^s(\mathbb{R}^n)$, we readily find that

$$\langle u_j - u, H(u_j - u) \rangle + \langle u, Hu \rangle = (\mathcal{J}_{v,\omega,p}^*)^{\frac{1}{\sigma+1}} + o(1)$$

by using elementary properties of the $L^2$-inner product. In summary, we thus deduce

$$\mathcal{J}_{v,\omega,p}^* \left\{ \|u_j - u\|_{L^{2\sigma+2}}^{2\sigma+2} + \|u\|_{L^{2\sigma+2}}^{2\sigma+2} + o(1) \right\} = \mathcal{J}_{v,\omega,p}^*$$
$$= \{\langle u_j - u, H(u_j - u) \rangle + \langle u, Hu \rangle\}^{\sigma+1}$$
$$\geqslant \langle u_j - u, H(u_j - u) \rangle^{\sigma+1} + \langle u, Hu \rangle^{\sigma+1} + o(1)$$
$$\geqslant \mathcal{J}_{v,\omega,p}^* \|u_j - u\|_{L^{2\sigma+2}}^{2\sigma+2} + \langle u, Hu \rangle^{\sigma+1} + o(1).$$

In the first inequality above, we used the elementary inequality $(x+y)^q \geqslant x^q + y^q$ for $x, y \geqslant 0$ and $q \geqslant 1$. Passing to the limit $j \to \infty$ and using that $u \not\equiv 0$, we obtain

$$\mathcal{J}_{v,\omega,p}^* \geqslant \frac{\langle u, Hu \rangle^{\sigma+1}}{\|u\|_{L^{2\sigma+2}}^{2\sigma+2}} = \mathcal{J}_{v,\omega,p}(u),$$

which shows that $u \in H^s(\mathbb{R}^n) \backslash \{0\}$ must be a minimizer. Also, we remark that we must have $\langle u_j - u, H(u_j - u) \rangle = \|u_j - u\|_{\omega,\mathbf{v}}^2 \to 0$ as $j \to \infty$, since equality must hold everywhere. This shows that in fact $u_j \to u$ strongly in $H^s(\mathbb{R}^n)$ due to the equivalence of norms $\|\cdot\|_{H^s} \sim \|\cdot\|_{\omega,\mathbf{v}}$.

Finally, we note that an elementary calculation shows that any minimizer $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R}^n) \backslash \{0\}$ for $\mathcal{J}_{v,\omega,p}$ with $\|Q_{v,\omega}\|_{L^{2\sigma+2}} = 1$ satisfies the corresponding Euler-Lagrange equation

$$P_{\mathbf{v}}(D)Q_{\omega,\mathbf{v}} + \omega Q_{v,\omega} - (\mathcal{J}_{v,\omega,p}^*)^{\frac{1}{\sigma+1}} |Q_{v,\omega}|^{2\sigma} Q_{v,\omega} = 0. \tag{8.2.3}$$

After a rescaling $Q_{\omega,\mathbf{v}} \mapsto \alpha Q_{\omega,\mathbf{v}}$ with a suitable constant $\alpha > 0$, we find that $Q_{\omega,\mathbf{v}}$ solves (8.1.5). This completes the proof of Theorem 8.1.1. $\qquad\square$

## 8.3 Rearrangements in Fourier Space

In this section, we recall and introduce some notions needed to prove Theorems 8.1.2 and 8.1.3.

### 8.3.1 Preliminaries

We start by recalling some standard definitions in rearrangement techniques. Let $\mu_k$ denote the Lebesgue measure in dimension $k \geqslant 1$. For a Borel set $A \subset \mathbb{R}^k$, we denote by $A^*$ its symmetric rearrangement defined as the open ball $B_R(0)$ centered at the origin whose Lebesgue measure equals that of $A$, i. e., we set

$$A^* = \{x \in \mathbb{R}^k : |x| < R\} \quad \text{such that } V_k R^k = \mu_k(A),$$

where $V_k = \mu_k(B_1(0))$ is the volume of the unit ball in $\mathbb{R}^k$. Next, let $u : \mathbb{R}^k \to \mathbb{C}$ be measurable function that vaniihes at infinity, which means that $\mu_k(\{x \in \mathbb{R}^k : |u(x)| > t\})$ is finite for all $t > 0$. We recall that the **symmetric-decreasing rearrangement** of $u$ is defined as the nonnegative function $u : \mathbb{R}^k \to \mathbb{R}_+$ by setting

$$u^*(x) = \int_0^\infty \chi_{\{|u|>t\}^*}(x) \, \mathrm{d}t,$$

where $\chi_B$ denotes characteristic function of a the set $B \subset \mathbb{R}^k$.

Let us now take $n \geqslant 2$ dimensions and decompose $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. Accordingly, we write elements $x \in \mathbb{R}^n$ often as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. For a measurable (Borel) function $u : \mathbb{R}^n \to \mathbb{C}$ vanishing at infinity, we define its **Steiner symmetrization in $n-1$ codimensions**[1]. as the function $u^{*_1} : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}_+$ given by

$$u^{*_1}(x_1, x') := u(x_1, \cdot)^*(x'),$$

where $*$ on the right side denotes the symmetric-decreasing rearrangement of the function $x' \mapsto u(x_1, x')$ in $\mathbb{R}^{n-1}$ for each $x_1 \in \mathbb{R}$ fixed. Of course, the rearrangement operator $*^1$ can be easily generalized to arbitrary coordinate directions. More precisely, given a unit vector $\mathbf{e} \in \mathbb{S}^{n-1}$, we pick a matrix $\mathsf{R} \in \mathrm{O}(n)$ such that $\mathsf{R}\mathbf{e} = \mathbf{e}_1 = (1, 0, \dots, 0)$ and let $(\mathsf{R}u)(x) := f(\mathsf{R}^{-1}x)$ denote the action of $\mathsf{R}$ on functions $u : \mathbb{R}^n \to \mathbb{C}$. We can then define the Steiner symmetrization in $n-1$-dimensions with respect to $\mathbf{e}$ as the nonnegative function $u^{*_e} : \mathbb{R}^n \to \mathbb{R}_+$ that is given by

$$u^{*_\mathbf{e}} := \mathsf{R}^{-1}((\mathsf{R}u)^{*_1}).$$

Recalling the definition in [24], we define the **Fourier rearrangement** of a function $u \in L^2(\mathbb{R}^n)$ to be given by

$$u^\sharp := \mathcal{F}^{-1}\left\{(\mathcal{F}(u))^*\right\}, \tag{8.3.1}$$

where $*$ denotes the symmetric-decreasing rearrangement in $\mathbb{R}^n$ and $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}u(\xi) \equiv \widehat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)e^{-i\xi \cdot x} \, \mathrm{d}x, \tag{8.3.2}$$

defined for $u \in L^1(\mathbb{R}^n)$ and extended to $u \in L^2(\mathbb{R}^n)$ by density. Finally, we come to the main technical tool used in this paper. Given a direction $\mathbf{e} \in \mathbb{S}^{n-1}$ and $u \in L^2(\mathbb{R}^n)$, we define its **Fourier Steiner rearrangement in $n-1$ codimensions** by setting

$$u^{\sharp_\mathbf{e}} := \mathcal{F}^{-1}\left\{\mathcal{F}(u)^{*_\mathbf{e}}\right\}. \tag{8.3.3}$$

---

[1]We follow the nomenclature in [10].

By a suitable rotation of coordinates in $\mathbb{R}^n$, it will often suffice to consider the case $\mathbf{e} = \mathbf{e}_1 = (1, 0, \ldots, 0)$ and likewise we simply write

$$u^{\sharp_1} := \mathcal{F}^{-1}\left\{ (\mathcal{F}(u)^{*_1} \right\}. \tag{8.3.4}$$

Next, we collect some basic properties of the operation $\sharp_{\mathbf{e}}$ as follows.

**Lemma 8.3.1.** *Let $n \geqslant 2$, $\mathbf{e} \in \mathbb{S}^{n-1}$, and $u \in L^2(\mathbb{R}^n)$. Then the following properties hold.*

(i) $\|u^{\sharp_{\mathbf{e}}}\|_{L^2} = \|u\|_{L^2}$.

(ii) $u^{\sharp_{\mathbf{e}}}$ *is cylindrically symmetric with respect to* $\mathbf{e}$, *i. e., for every matrix* $\mathsf{R} \in \mathrm{O}(n)$ *with* $\mathsf{R}\mathbf{e} = \mathbf{e}$ *it holds that*
$$u^{\sharp_{\mathbf{e}}}(x) = u^{\sharp_{\mathbf{e}}}(\mathsf{R}x) \quad \text{for a. e. } x \in \mathbb{R}^n.$$

(iii) *If in addition* $\widehat{u} \in L^1(\mathbb{R}^n)$, *then* $u^{\sharp_{\mathbf{e}}}$ *is a continuous and **positive definite** function in the sense of Bochner, i. e., we have*

$$\sum_{k,l=1}^{m} u^{\sharp_{\mathbf{e}}}(x_k - x_l) \overline{z}_k z_l \geqslant 0$$

*for all integers* $m \geqslant 1$ *and* $x_1, \ldots, x_m \in \mathbb{R}^n$ *and* $z \in \mathbb{C}^N$. *In particular, it holds that*

$$u^{\sharp_{\mathbf{e}}}(0) \geqslant |u^{\sharp_{\mathbf{e}}}(x)| \quad \text{for all } x \in \mathbb{R}^n.$$

**Remark.** *Note that item (iv) says in particular that* $u^{\sharp_{\mathbf{e}}}(0)$ *is a real number. However, the values* $u^{\sharp_{\mathbf{e}}}(x)$ *can be complex numbers for* $x \neq 0$ *in general.*

*Proof.* Without loss of generality we can assume that $\mathbf{e} = \mathbf{e}_1 = (1, 0, \ldots, 0)$.

Item (i) follows from elementary arguments. Indeed, by Fubini's theorem, we find for any $f \in L^2(\mathbb{R}^n)$ that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} |f(x_1, x_2, \ldots, x_n)|^2 \, dx_2 \ldots dx_n \right) dx_1$$

$$= \int_{\mathbb{R}} \left( |f(x_1, x_2, \ldots, x_n)|^{*_1}|^2 \, dx_2 \ldots dx_n \right) dx_1 = \|f^{*_1}\|_{L^2}^2,$$

where used the equimeasurability of the functions $f(x_1, \ldots)$ and $f(x_1, \ldots)^{*_1}$ on $\mathbb{R}^{n-1}$ for every $x_1 \in \mathbb{R}$ fixed. By Plancherel's identity, we conclude that (i) is true.

Likewise, we see that (ii) holds true by elementary properties of the Fourier transform. Finally, we mention that (iii) follows from the fact that $\widehat{u^{\sharp_1}} = (\widehat{u}(\xi))^{*_1} \geqslant 0$ is non-negative and classical arguments for positive-definite functions; see, e. g., [31]. $\qquad \square$

### 8.3.2 Rearrangement Inequalities: Steiner meets Fourier

Recall that the operator $P(D)$ is defined as $\widehat{(P(D)u)}(\xi) = p(\xi)\widehat{u}(\xi)$ through its real-valued multiplier $p : \mathbb{R}^n \to \mathbb{R}$. Furthermore, we recall that for the given velocity $\mathbf{v} \in \mathbb{R}^n$ we define the operator

$$P_{\mathbf{v}}(D) = P(D) + i\mathbf{v} \cdot \nabla,$$

which has the Fourier symbol $p_{\mathbf{v}}(\xi) = p(\xi) - \mathbf{v} \cdot \xi$.

**Lemma 8.3.2.** *Let $n \geqslant 2$. Suppose that $P(D)$ satisfies Assumptions 4 and 5 with some $s \geqslant 1/2$. Let $\mathbf{e} \in \mathbb{S}^{n-1}$ be some direction and assume that $\mathbf{v} \in \mathbb{R}^n$ is parallel to $\mathbf{e}$. Then it holds that*

$$\langle u^{\sharp \mathbf{e}}, P_{\mathbf{v}}(D)u^{\sharp \mathbf{e}}\rangle \leqslant \langle u, P_{\mathbf{v}}(D)u\rangle \quad \text{for all } u \in H^s(\mathbb{R}^n).$$

*Moreover, we have equality if and only if $|\hat{u}(\xi)| = (\hat{u}(\xi))^{*\mathbf{e}}$ for almost every $\xi \in \mathbb{R}^n$.*

*Proof.* By a suitable rotation in $\mathbb{R}^n$, we can assume without loss of generality that $\mathbf{e} = \mathbf{e}_1 = (1, 0, \ldots, 0)$ holds and thus $\mathbf{v} = (|\mathbf{v}|, 0, \ldots, 0)$. As before, we decompose $\xi \in \mathbb{R}^n$ as $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. With some slight abuse of notation we can write $p(\xi) = p(\xi_1, |\xi'|)$ and $p_{\mathbf{v}}(\xi) = p_{\mathbf{v}}(\xi_1, |\xi'|) = p(\xi_1, |\xi'|) - |\mathbf{v}|\xi_1$.

We adapt the following arguments in [24] to our setting here.

**Step 1.** Suppose $A \subset \mathbb{R}^{n-1}$ is a measurable set with finite Lebesgue measure $\mu_{n-1}(A) < \infty$ in $n-1$ dimensions. For notational simplicity, we shall simply write $\mu$ instead of $\mu_{n-1}$ in the following. Let $A^*$ denote its symmetric-decreasing rearrangement in $\mathbb{R}^{n-1}$, i.e., the set $A^* = B_R(0) \subset \mathbb{R}^{n-1}$ is the open ball centered at the origin with measure $\mu(A^*) = \mu(A)$. We claim that the following inequality holds

$$\int_{A^*} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' \leqslant \int_A p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' \tag{8.3.5}$$

for any $\xi_1 \in \mathbb{R}$. Indeed, we have $\mu(A \backslash A^*) = \mu(A) - \mu(A \cap A^*)$ and $\mu(A^* \backslash A) = \mu(A^*) - \mu(A \cap A^*)$. Since $\mu(A) = \mu(A^*)$, we deduce that $\mu(A \backslash A^*) = \mu(A^* \backslash A)$. Next we recall that $|\xi'| \mapsto p(\xi_1, |\xi'|)$ is strictly increasing for all $\xi_1 \in \mathbb{R}$ fixed. Hence the map $|\xi'| \mapsto p_{\mathbf{v}}(\xi_1, |\xi'|) = p(\xi'1, |\xi'|) - |\mathbf{v}|\xi_1$ is strictly increasing as well. Since $|\xi'| \geqslant R$ for $\xi' \in A \backslash A^*$ and $|\xi'| < R$ for $\xi' \in A^* \backslash A$, this implies that

$$\int_{A^* \backslash A} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' \leqslant \int_{A^* \backslash A} p_{\mathbf{v}}(\xi_1, R) \, d\xi' = p_{\mathbf{v}}(\xi_1, R)\mu(A^* \backslash A)$$

$$= p_{\mathbf{v}}(\xi_1, R)\mu(A \backslash A^*) = \int_{A \backslash A^*} p_{\mathbf{v}}(\xi_1, R) \, d\xi' \leqslant \int_{A \backslash A^*} p_{\mathbf{v}}(\xi_1, \xi') \, d\xi'. \tag{8.3.6}$$

Therefore we conclude

$$\int_{A^*} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' = \int_{A^* \backslash A} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' + \int_{A^* \cap A} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi$$

$$\leqslant \int_{A \backslash A^*} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' + \int_{A^* \cap A} p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi = \int_A p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi',$$

which proves (8.3.5).

**Step 2.** Now let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a nonnegative measurable function vanishing at infinity. We claim that

$$\int_{\mathbb{R}^n} f^{*_1}(\xi)p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi \leqslant \int_{\mathbb{R}^n} f(\xi)p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi, \tag{8.3.7}$$

where $f^{*_1}$ denotes the Steiner rearrangement in $n-1$ codimensions. To show the claimed inequality, we note that $f(\xi) = \int_0^\infty \chi_{\{f>t\}}(\xi) \, dt$ by the layer cake representation and accordingly we have $f^{*_1}(\xi) = \int_0^\infty \chi_{\{f>t\}*_1}(\xi) \, dt$. Thus, by applying Fubini's theorem, we need to show that

$$\int_0^\infty \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} \chi_{\{f>t\}*_1}(\xi_1, \xi')p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' \right) d\xi_1 \right) dt \leqslant$$

$$\int_0^\infty \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} \chi_{\{f>t\}}(\xi_1, \xi')p_{\mathbf{v}}(\xi_1, |\xi'|) \, d\xi' \right) d\xi_1 \right) dt.$$

If we use (8.3.5) with the sets $B_{\xi_1} = \{\xi' \in \mathbb{R}^{n-1} : f(\xi_1, \xi') > t\} \subset \mathbb{R}^{n-1}$ with $\xi_1 \in \mathbb{R}$, the definition of $*_1$ implies that

$$\int_{\mathbb{R}^{n-1}} \chi_{\{f>t\}*_1}(\xi_1, \xi') p_{\mathbf{v}}(\xi_1, |\xi'|) \, \mathrm{d}\xi' \leqslant \int_{\mathbb{R}^{n-1}} \chi_{\{f>t\}}(\xi_1, \xi') p_{\mathbf{v}}(\xi_1, |\xi'|) \, \mathrm{d}\xi'$$

for any $\xi_1 \in \mathbb{R}$. By integrating this inequality over $\xi_1$ and $t$, we arrive at the desired inequality stated in (8.3.7).

**Step 3.** By Plancherel's theorem and the definition of $u^{\sharp_1}$, the claimed inequality is equivalent to

$$\int_{\mathbb{R}^n} p_{\mathbf{v}}(\xi) |(\widehat{u}(\xi))^{*_1}|^2 \, \mathrm{d}\xi \leqslant \int_{\mathbb{R}^n} p_{\mathbf{v}}(\xi) |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

We now define the nonnegative function $f : \mathbb{R}^n \to \mathbb{R}_+$ with $f(\xi) = |\widehat{u}(\xi)|^2$. Clearly, $f$ is measurable and vanishes at infinity. Furthermore, we note that $f^{*_1}(\xi) = (|\widehat{u}(\xi)|^2)^{*_1} = |(\widehat{u}(\xi))^{*_1}|^2$, where the last equality follows from basic properties of the rearrangement $*_1$. By applying (8.3.7), we obtain the claimed inequality stated in Lemma 8.3.2.

**Step 3.** Finally, we suppose that equality $\langle u^{\sharp_1}, P_{\mathbf{v}}(D) u^{\sharp_1} \rangle = \langle u, P_{\mathbf{v}}(D) u \rangle$ holds. Since $|\xi'| \mapsto p_{\mathbf{v}}(\xi_1, |\xi'|)$ is strictly increasing, equality holds in (8.3.6) if and only if $\mu(A \backslash A^*) = 0$. Since $\mu(A) = \mu(A^*)$, this means that the sets $A$ and $A^*$ coincide (up to a set of measure zero). Therefore, by using the layer-cake representation for $f = |\widehat{u}|^2$ in (8.3.7), we deduce the equality $f(\xi) = f^{*_1}(\xi)$ for almost every $\xi \in \mathbb{R}^n$, which is equivalent to $|\widehat{u}(\xi)| = (\widehat{u}(\xi))^{*_1}$ almost everywhere.

The proof of Lemma 8.3.2 is now complete. $\qquad\square$

Next, we turn to a rearrangement inequality for $L^p$-norms. By arguing along the lines in [**?**], we can prove the following result.

**Lemma 8.3.3.** *Let $n \geqslant 2$, $p \in 2\mathbb{N} \cup \{\infty\}$, and $\mathbf{e} \in \mathbb{S}^{n-1}$. Then for all $u \in L^2(\mathbb{R}^n) \cap \mathcal{F}(L^{p'}(\mathbb{R}^n))$ with $1/p + 1/p' = 1$, we have $u^{\sharp_{\mathbf{e}}} \in L^2(\mathbb{R}^n) \cap \mathcal{F}(L^{p'}(\mathbb{R}^n))$ and $\|u\|_{L^p} \leqslant \|u^{\sharp_{\mathbf{e}}}\|_{L^p}$.*

As a technical ingredient needed for the proof of Lemma 8.3.3, we need the following result concerning multiple convolutions in $\mathbb{R}^n$, which is a consequence of the classical *Brascamp-Lieb-Luttinger inequality*; see Lemma 8.6.3 below.

**Proposition 8.3.1.** *Let $n \geqslant 2$, $\mathbf{e} \in \mathbb{S}^{n-1}$, and $m \geqslant 2$. For any non-negative measurable functions $u_1, u_2, \ldots, u_m : \mathbb{R}^n \to \mathbb{R}_+$ vanishing at infinity, we have*

$$(u_1 * \ldots * u_m)(0) \leqslant (u_1^{*_{\mathbf{e}}} * \ldots * u_m^{*_{\mathbf{e}}})(0).$$

*Proof.* Without loss of generality we can assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. A calculation using Fubini's theorem yields

$$(u_1 * \cdots * u_m)(0)$$
$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[ u_1(y_1^1, \cdot), \ldots, u_{m-1}(y_1^{m-1}, \cdot), u_m\left( -\sum_{i=1}^{m-1} y_1^i, \cdot \right) \right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1}. \qquad (8.3.8)$$

Here $I_{n-1}$ is defined according to (8.6.1) with $B$ as the $(m-1) \times m$-matrix given by

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & \big| & -1 \\ 0 & 1 & \cdots & 0 & \big| & -1 \\ \vdots & & \ddots & \vdots & \big| & \vdots \\ 0 & 0 & \cdots & 1 & \big| & -1 \end{pmatrix}$$

and the matrix in the left block is the $(m-1) \times (m-1)$-unit matrix. By applying Lemma 8.6.3 with $d = n - 1$ and recalling the definition of $*_1$, we deduce that

$$(u_1 * \cdots * u_m)(0)$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[u_1(y_1^1, \cdot), \ldots, u_{m-1}(y_1^{m-1}, \cdot), u_m\left(-\sum_{i=1}^{m-1} y_1^i, \cdot\right)\right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1}$$

$$\leqslant \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[u_1(y_1^1, \cdot)^*, \ldots, u_{m-1}(y_1^{m-1}, \cdot)^*, u_m\left(-\sum_{i=1}^{m-1} y_1^i, \cdot\right)^*\right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1}$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} I_{n-1}\left[u_1^{*_1}(y_1^1, \cdot), \ldots, u_{m-1}^{*_1}(y_1^{m-1}, \cdot)^*, u_m^{*_1}\left(-\sum_{i=1}^{m-1} y_1^i, \cdot\right)\right] \mathrm{d}y_1^1 \cdots \mathrm{d}y_1^{m-1}$$

$$= (u_1^{*_1} * \cdots * u_m^{*_1})(0),$$

where the last equality again follows from applying Fubini's theorem. $\qquad\square$

*Proof of Lemma 8.3.3.* Without loss of generality we can assume that $\mathbf{e} = \mathbf{e}_1 = (1, 0, \ldots, 0)$. The case of $p = 2$ is clear. Let us assume $p = 2m$ with some integer $m \geqslant 2$ so that the corresponding dual exponent is given by $p' = \frac{2m}{2m-1}$. Since $u \in L^p(\mathbb{R}^n) \cap \mathcal{F}(L^{p'}(\mathbb{R}^n))$, we can apply the version of the convolution lemma in [24] to conclude

$$\|u\|_{L^p}^p = \mathcal{F}(|u|^{2m})(0) = (\widehat{u} * \widehat{\widehat{u}} * \cdots * \widehat{u} * \widehat{\widehat{u}})(0), \tag{8.3.9}$$

where the number of convolutions on the right-hand side equals $2m - 1$. By Proposition 8.3.1, we obtain that

$$(\widehat{u} * \widehat{\widehat{u}} * \cdots * \widehat{u} * \widehat{\widehat{u}})(0) \leqslant (\widehat{u}^{*_1} * (\widehat{\widehat{u}})^{*_1} * \cdots * (\widehat{u})^{*_1} * (\widehat{\widehat{u}})^{*_1})(0) = \mathcal{F}(|u^{\sharp_1}|^{2m})(0) = \|u^{\sharp_1}\|_{L^p}^p,$$

where we also used the fact that $\mathcal{F}(\overline{u^{\sharp_1}}) = \mathcal{F}(\overline{u})^{*_1}$ and the definition of $\sharp_1$.

Finally, let us take $p = \infty$ and thus $p' = 1$. We find, by using Fubini's theorem,

$$\|u\|_\infty \leqslant \int_{\mathbb{R}^d} |\widehat{u}(\xi)| \, d\xi = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |\widehat{u}(\xi_1, \xi')| \, d\xi'\right) d\xi_1$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \widehat{u}^{\sharp_1}(\xi_1, \xi') \, d\xi'\right) d\xi_1 = u^{\sharp_1}(0).$$

Since $\|u^{\sharp_1}\|_{L^\infty} = u^{\sharp_1}(0)$ holds by Lemma 8.3.1 (iii), we complete the proof. $\qquad\square$

## 8.4 Proof of Theorem 8.1.2

We divide the proof of Theorem 8.1.2 into two parts as follows. First, as the essential key point, we show that $\{\xi \in \mathbb{R}^n : |\widehat{Q}_{\omega,\mathbf{v}}(\xi)| > 0\}$ is a connected set in $\mathbb{R}^n$. This fact then enables us to apply the recent rigidity result [24] for the Hardy-Littlewood majorant problem in $\mathbb{R}^n$ to conclude the proof.

### 8.4.1 Connectedness of the Set $\{|\widehat{Q}_{\omega,\mathbf{v}}| > 0\}$

We start with with some notational preliminaries. Given two sets $X, Y \subset \mathbb{R}^n$, we shall use

$$X \oplus Y = \{x + y : x \in X, \ y \in Y\}$$

to denote their **Minkowski sum**. Likewise, we denote their Minkowski difference by

$$X \ominus Y = \{x - y : x \in X, \ y \in Y\}.$$

Furthermore, for a function $f : \mathbb{R}^n \to \mathbb{R}$ we use the short-hand notation

$$\{f > 0\} = \{x \in \mathbb{R}^n : f(x) > 0\}$$

throughout the following.

**Lemma 8.4.1.** *Let $f, g \in \mathbb{R}^n \to [0, \infty)$ be two non-negative and continuous functions. Assume that their convolution*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y$$

*has finite values for all $x \in \mathbb{R}^n$. Then it holds that*

$$\{f * g > 0\} = \{f > 0\} \oplus \{g > 0\}.$$

*Proof.* The proof is elementary. For the reader's convenience, we give the details. Let us write $\Omega_f = \{f > 0\}$, $\Omega_g = \{g > 0\}$ and $\Omega_{f*g} = \{f * g > 0\}$. We suppose that both $f \not\equiv 0$ and $g \not\equiv 0$, since otherwise the claimed result trivially follows.

First, we show that $\Omega_f \oplus \Omega_g \subset \Omega_{f*g}$. Let $x = x_1 + x_2$ with $x_1 \in \Omega_f$ and $x_2 \in \Omega_g$. By the continuity of $f$ and $g$, there exists some $\varepsilon > 0$ such that $f > 0$ on $B_\varepsilon(x_1)$ and $g > 0$ on $B_\varepsilon(x_2)$. Thus, by using that $f \geqslant 0$ and $g \geqslant 0$ on all of $\mathbb{R}^n$, we get

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y \geqslant \int_{B_\varepsilon(x_2)} f(x_1 + x_2 - y) g(y) \, \mathrm{d}y > 0,$$

since $x_1 + x_2 - y \in B_\varepsilon(x_1)$ when $y \in B_\varepsilon(x_2)$. This shows that $\Omega_f \oplus \Omega_g \subset \Omega_{f*g}$.

Next, we prove that $\Omega_{f*g} \subset \Omega_f \oplus \Omega_g$ holds. Indeed, for every $x \in \mathbb{R}^n$, we can write

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) \, \mathrm{d}y = \int_{(\{x\} \ominus \Omega_f) \cap \Omega_g} f(x - y) g(y) \, \mathrm{d}y,$$

since $f(x - \cdot) \equiv 0$ on $\mathbb{R}^n \backslash (\{x\} \ominus \Omega_f)$ and $g \equiv 0$ on $\mathbb{R}^n \backslash \Omega_g$. However, if $x \notin \Omega_f \oplus \Omega_g$ then $(\{x\} \ominus \Omega_f) \cap \Omega_g = \varnothing$. Thus $(f * g)(x) = 0$ for any $x \notin \Omega_f \oplus \Omega_g$, whence it follows that the inclusion $\Omega_{f*g} \subset \Omega_f \oplus \Omega_g$ is valid. $\qquad\square$

Next, we establish the following key result in order to prove Theorem 8.1.2.

**Lemma 8.4.2.** *Let $n \geqslant 2$ and suppose $\ell \geqslant 2$ is an integer. Let $f \in L^{\ell/(\ell-1)}(\mathbb{R}^n) \geqslant 0$ be a continuous nonnegative function with $f = f^{*\mathbf{e}}$ with some $\mathbf{e} \in \mathbb{S}^{n-1}$ and assume $f$ satisfies an equation of the form*

$$f(x) = h(x) \left( f * \ldots * f \right)(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{8.4.1}$$

*with $\ell$ factors in the convolution product on the left side and $h : \mathbb{R}^n \to (0, +\infty)$ is some continuous positive function. Then the set $\{f > 0\} \subset \mathbb{R}^n$ is connected.*

*Proof.* We divide the proof into the following steps. Without loss of generality we can assume that $\mathbf{e} = \mathbf{e}_1$ is the unit vector pointing in the $x_1$-direction.

**Step 1: Preliminaries.** For a set $X \subset \mathbb{R}^n$ and $L \in \mathbb{N}$, we use

$$S_L(X) = \bigoplus_{k=1}^{L} X$$

to denote the $L$-fold Minkowski sum of $X$ with itself. Let us define the set

$$\Omega = \{x \in \mathbb{R}^n : f(x) > 0\}. \tag{8.4.2}$$

We assume that $\Omega \neq \varnothing$, since otherwise the result is trivially true. Next, we will denote by $\pi_1(\Omega) \subset \mathbb{R} \simeq \mathbb{R} \times \{0\}$ the projection of $\Omega$ onto the $x_1$-axis, i.e., we have

$$\pi_1(\Omega) = \{x_1 \in \mathbb{R} : \exists x' \in \mathbb{R}^{n-1} \text{ with } (x_1, x') \in \Omega\}.$$

From (8.4.1) we clearly see that $\{(f * \ldots * f) > 0\} = \Omega$. By Lemma 8.4.1 and iteration, this implies the equality of sets

$$S_{\nu\ell}(\Omega) = \Omega \quad \text{for any } \nu \in \mathbb{N}.$$

Furthermore, we see that for every subset $A \subset \Omega$ this implies

$$S_{\nu\ell}(A) \subset S_{\nu\ell}(\Omega) \subset \Omega \quad \text{for all } \nu \in \mathbb{N}. \tag{8.4.3}$$

Next, we recall that, for any $x_1 \in \mathbb{R}$ fixed, the sets $\{x' \in \mathbb{R}^{n-1} : f(x_1, x') > 0\}$ are open balls in $\mathbb{R}^{n-1}$ centered at the origin, due to the fact that $f = f^{\star_1} \geqslant 0$, which implies that the map $x' \mapsto f(x_1, x')$ is radially symmetric in $\mathbb{R}^{n-1}$ and non-increasing in $|x'|$. Thus there exists a map

$$\mathbb{R} \to [0, +\infty], \quad x_1 \mapsto \rho(x_1)$$

such that

$$B_{\mathbb{R}^{n-1}}(0, \rho(x_1)) = \{x' \in \mathbb{R}^{n-1} : f(x_1, x') > 0\}.$$

As usual, we use the convention that $B_{\mathbb{R}^{n-1}}(0, +\infty) = \mathbb{R}^{n-1}$ and $B_{\mathbb{R}^{n-1}}(0, 0) = \varnothing$. By the continuity of $f$, the map $x_1 \mapsto \rho(x_1)$ is continuous on $\pi_1\Omega$. Next, by replacing $f(x)$ with $f(-x)$ if necessary, we can henceforth assume that

$$\Omega \cap \{x_1 \geqslant 0\} \neq \varnothing.$$

Clearly, one of the two following cases must occur:

**(A)** $\Omega \backslash \{x_1 \geqslant 0\} = \varnothing.$

**(B)** $\Omega \backslash \{x_1 \geqslant 0\} \neq \varnothing.$

Next, we will treat the cases **(A)** and **(B)** separately as follows.

**Step 2: Discussion of Case (A).** In this case, we must have the inclusion $\Omega \subset \{x_1 \geqslant 0\}$. Let us define the nonnegative number

$$x_+ := \inf\{x_1 > 0 : \rho(x_1) > 0\}. \tag{8.4.4}$$

Thus we have $\Omega \subset \{x_1 \geqslant x_+\}$. But we note that $S_\ell(\Omega) \subset \{x_1 \geqslant \ell x_+\}$ by elementary properties of the Minkowski sum. On the other hand, we also have $S_\ell(\Omega) = \Omega$ from above. Thus $\{x_1 \geqslant x_+\} \subset \{x_1 \geqslant \ell x_+\}$. Since $\ell \geqslant 2$ and $x_+ \geqslant 0$, we deduce that

$$x_+ = 0. \tag{8.4.5}$$

Next, we claim that

$$\rho(x_1) > 0 \text{ and } \rho(x_1) \text{ is non-decreasing for all } x_1 > 0. \tag{8.4.6}$$

We first show that $\rho > 0$ and that $\rho$ is non-decreasing on a dense subset in $\mathbb{R}_+$. By a density argument, it then follows that $\rho(x_1) > 0$ for all $x_1 > 0$.

Indeed, since we have $x_+ = 0$, there exists a sequence of positive reals $\varepsilon_n > 0$ in $\mathbb{R}_{>0}$ such that $\varepsilon_n \to 0$ and $\rho(\varepsilon_n) > 0$ for all $n$. Without loss of generality, we can henceforth assume that the sequence $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ is strictly decreasing. By continuity of $\rho$, we

find $0 < \delta_n < \varepsilon_n$ such that $\rho(t) \geqslant \frac{1}{2}\rho(\varepsilon_n)$ whenever $|t - \varepsilon_n| < \delta_n$. Hence we can find a sequence of cylindrical sets

$$\mathcal{C}_n := \; ]\varepsilon_n - \delta_n, \varepsilon_n + \delta_n[ \; \times \, B_{\mathbb{R}^{n-1}}\left(0, \frac{1}{2}\rho(\varepsilon_n)\right) \subset \Omega \qquad (8.4.7)$$

If we now apply (8.4.3), we see that $S_{\nu\ell}(\mathcal{C}_n) \subset \Omega$ for any integer $\nu \in \mathbb{N}$. Let us now define the set

$$D := \Big\{ \sum_{n=1}^{+\infty} p_n \varepsilon_n : p_n \in \mathbb{N}, \; \sum_{n=1}^{+\infty} p_n \in \ell\mathbb{N} \cup \{1\} \Big\},$$

which is dense in $\mathbb{R}_{>0}$. Since we have $\rho(\varepsilon_n) > 0$ and (8.4.7) for all $n$, we get

$$\rho(x_1) > 0 \text{ for all } x_1 \in D. \qquad (8.4.8)$$

Next, we show that the function $\rho$ is non-decreasing on $\mathbb{R}_{>0}$. To show this, we first establish that

$$\rho(x_1) \leqslant \rho(y_1) \text{ for all } x_1 \in D \text{ and } y_1 \geqslant x_1. \qquad (8.4.9)$$

Indeed, let $x_1 \in D$ be fixed and take $a \in (0,1)$. Since $\rho(x_1) > 0$, we can pick a point $(x_1, x') \in \text{supp} f$ with some $x' \in \mathbb{R}^{n-1}$ with $|x'| = \rho(x_1)$, where we assume that $\rho(x_1) < +\infty$ for the moment. Since $\{(x_1, ax')\} \cup \mathcal{C}_n \subset \Omega$, we deduce from (8.4.3) together with elementary properties of the Minkowski sum of set that

$$(x_1 + (\ell - 1)(\varepsilon_n - \delta_n), x_1 + (\ell - 1)(\varepsilon_n + \delta_n), ax') \in \Omega \quad \text{for all } n \in \mathbb{N}.$$

By taking the limit $a \to 1^-$, we get that for all $n \in \mathbb{N}$,

$$\rho(x_1) \leqslant \rho(y_1) \text{ when } x_1 + (\ell - 1)(\varepsilon_n - \delta_n) < y_1 < x_1 + (\ell - 1)(\varepsilon_n + \delta_n).$$

Iterating the process, we get that for all

$$\rho(x_1) \leqslant \rho(y_1) \text{ when } x_1 + M(\ell - 1)(\varepsilon_n - \delta_n) < y_1 < x_1 + M(\ell - 1)(\varepsilon_n + \delta_n) \qquad (8.4.10)$$

for arbitiray $n \in \mathbb{N}$ and $M \in \mathbb{N}$. We show (8.4.9) by a contradiction argument. Assume the existence of some real $y_1 > x_1$ with $0 \leqslant \rho(y_1) < \rho(x_1)$. By continuity of $f$, this inequality holds in a small neighbourhood $(y_1 - \delta, y_1 + \delta)$. However since $\varepsilon_n \to 0$ as $n \to +\infty$, there exist $n, M \in \mathbb{N}$ such that $x_1 + M(\ell - 1)\varepsilon_n \in (y_1 - \delta, y_1 + \delta)$, contradicting (8.4.10). The case of infinite $\rho(x_1) = +\infty$ follows from a straightforward modification of the arguments above. This completes the proof of (8.4.9).

Since the set $D$ is dense in $\mathbb{R}_{>0}$, the monotonicity (8.4.9) together with the continuity of $\rho$ on $\pi_1(\Omega)$ and the positivity of $\rho$ on $D$ imply that

1. $\rho > 0$ on $\mathbb{R}_{>0}$ (hence $\pi_1(\Omega) = \mathbb{R}_{>0}$), and

2. the function $x_1 \in \mathbb{R}_{>0} \mapsto \rho(x_1)$ is continuous and non-decreasing.

These last two facts then imply that $x_1 \in \mathbb{R}_{>0} \mapsto \rho(x_1)$ is increasing on $\mathbb{R}_{>0}$.

In summary, we have found that the set $\Omega = \{f > 0\}$ must be of the form

$$\Omega = \bigcup_{x_1 \in [0,\infty)} \{x_1\} \times B_{\mathbb{R}^{n-1}}(0, \rho(x_1)), \qquad (8.4.11)$$

where $0 < \rho(x_1) \leqslant +\infty$ for all $x_1 > 0$. Clearly, the set $\Omega \subset \mathbb{R}^n$ is connected.

**Step 3: Disussion of Case (B).** In this case, we have $\Omega \cap \{x_1 \geqslant 0\} \neq \varnothing$ and $\Omega \cap \{x_1 < 0\} \neq \varnothing$. Hence we can define the two nonnegative numbers $x_+, x_- \geqslant 0$ by setting

$$x_+ = \inf\{x_1 > 0 : \rho(x_1) > 0\} \quad \text{and} \quad x_- = -\sup\{x_1 < 0 : \rho(x_1) > 0\}.$$

We claim that

$$x_+ = x_- = 0. \tag{8.4.12}$$

We first show that equality $x_+ = x_-$ must hold. By replacing $f(x)$ by $f(-x)$ if necessary, we may assume $x_+ \geqslant x_-$ in what follows. By construction, there exist decreasing sequences $\varepsilon_n > 0$ and $\eta_n > 0$ with $\varepsilon_n \to 0$ and $\eta_n \to 0$ such that

$$\rho(x_+ + \varepsilon_n) > 0 \quad \text{and} \quad \rho(-x_- - \eta_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Arguing in a similar way as in the previous step, we can find cylindrical sets

$$\mathcal{C}_{+,n} = ]x_+ + \varepsilon_n - \delta_n, x_+ + \varepsilon_n + \delta_n[ \times B_{\mathbb{R}^{n-1}}\left(0, \frac{1}{2}\rho(x_+ + \varepsilon_n)\right) \subset \Omega,$$

$$\mathcal{C}_{-,n} = ]-x_- - \eta_n - \delta_n, -x_- - \eta_n + \delta_n[ \times B_{\mathbb{R}^{n-1}}\left(0, \frac{1}{2}\rho(-x_- - \eta_n)\right) \subset \Omega,$$

with sufficiently small numbers $0 < \delta_n < \min\{\varepsilon_n, \eta_n\} \to 0$ as $n \to \infty$.

Let us now assume that $x_- > 0$ is strictly positive and thus $x_+ > 0$ as well. We define the number

$$\varkappa = \frac{x_-}{x_+} \in \,]0, 1].$$

Let $N_1, N_2 \in \mathbb{N}$ be integers to be chosen below. Since $S_{N_1\ell}(\mathcal{C}_{+,n}) \subset \Omega$ and $S_{N_2\ell}(\mathcal{C}_{-,n}) \subset \Omega$, we deduce that

$$A = \{(N_1\ell(x_+ + r_n), 0), (-N_2\ell(x_- + r_n), 0)\} \subset \Omega$$

with some small numbers $r_n > 0$ with $r_n \to 0$. Using $S_{2\ell}(A) \subset \Omega$, we obtain

$$(N_1\ell^2(x_+ + r_n) - N_2\ell^2(x_- + r_n), 0) \in \Omega. \tag{8.4.13}$$

However, we now claim that

$$-x_- < N_1\ell^2(x_+ + r_n) - N_2\ell^2(x_- + r_n) < x_+ \tag{8.4.14}$$

for a suitable choice of integers $N_1, N_2 \in \mathbb{N}\backslash\{0\}$ and sufficiently large $n \gg 1$. But (8.4.14) and (8.4.13) then imply that $\rho(x_1) > 0$ for some $-x_- < x_1 < x_+$, which would contradict the definition of either $x_+ > 0$ or $x_- > 0$. To show the claimed inequality (8.4.14), we note that we can find integers $N_1, N_2 \in \mathbb{N}\backslash\{0\}$ such that

$$0 \leqslant N_1\ell^2 - N_2\ell^2\varkappa < 1. \tag{8.4.15}$$

Indeed, assume that $\varkappa = p/q \in ]0, 1]$ is a rational number. Then the choice of $N_1 = p$ and $N_2 = q$ yields $N_1\ell^2 - N_2\ell^2\varkappa = 0$. Next, we suppose that $\varkappa \in \,]0, 1]\backslash\mathbb{Q}$ is irrational. By the pigeonhole principle, we can find integers $N_1, N_2 \geqslant 1$ with the property

$$0 < N_1 - N_2\varkappa < \frac{1}{\ell^2},$$

which also yields (8.4.15).

Therefore, by a suitable choice of integers $N_1, N_2$, we see that (8.4.14) follows from

$$-\varkappa < (N_1 - N_2)\ell^2 \frac{r_n}{x_+} < 1,$$

which is true if $n \gg 1$ is sufficiently large, as we have $r_n \to 0^+$. This completes our proof that the case $x_- > 0$ cannot occur.

Finally, let us assume that $x_- = 0$. In this case, we can argue as in Step 2 to conclude that strict positivity $\rho(x_1) > 0$ holds for all $x_1 < 0$. Therefore,

$$(-y, 0) \in \Omega \quad \text{for all } y > 0.$$

Suppose now that $x_+ > 0$ holds. Recall that there is a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that $\rho(x_+ + \varepsilon_n) > 0$ for all $n$ and hence

$$(x_+ + \varepsilon_n, 0) \in \Omega.$$

Applying Lemma 8.4.1 to $f$ and $g = (f * \ldots * f)$ with $\ell - 1$ factors, we deduce that

$$(x_+ + \varepsilon_n - (\ell - 1)y, 0) \in \Omega \quad \text{for all } n \in \mathbb{N} \text{ and } y > 0.$$

But if $x_+ > 0$ this implies that there is $0 < x_1 < x_+$ with $\rho(x_1) > 0$, which contradicts the definition of $x_+$. Thus $x_- = 0$ implies that $x_+ = x_- = 0$.

Having established that $x_+ = x_- = 0$, we can argue as in Step 2 above to conclude

$$\Omega = \bigcup_{x_1 \in \mathbb{R}} \{x_1\} \times B_{\mathbb{R}^{n-1}}(0, \rho(x_1))$$

with $\rho(x_1) > 0$ for all $x_1 \neq 0$. Thus $(y, 0) \in \Omega$ for all $y \neq 0$. Applying Lemma 8.4.1 with $f = f$ and $g = (f * \ldots * f)$ ($\ell - 1$ times), we deduce

$$(x_1 + (\ell - 1)y, 0) \in \Omega \quad \text{for any } x_1 \neq 0 \text{ and } y \neq 0.$$

But this show that $(0, 0) \in \Omega$ and hence $\rho(0) > 0$. By continuity of $f$, there exists an open ball $B_\varepsilon(0) \subset \Omega$ with some $\varepsilon > 0$. Since $S_{\nu\ell}(B_\varepsilon(0)) \subset \Omega$ for any $\nu \in \mathbb{N}$ and $\bigcup_{\nu \in \mathbb{N}} S_{\ell\nu}(B_\varepsilon(0)) = \mathbb{R}^n$, we conclude in fact that $\Omega = \mathbb{R}^n$, which evidently shows that $\Omega$ is connected. $\qquad \square$

### 8.4.2 Completing the Proof of Theorem 8.1.2

Let $Q = Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R}^n)$ be a boosted ground state as in Theorem 8.1.2.

It is elementary to check that $|Q|^{2\sigma}Q \in L^1(\mathbb{R}^n)$ using that $\sigma \in (1, \sigma_*)$. Hence by (8.1.5) and taking the Fourier transform, we conclude that $\widehat{Q}(\xi) = \frac{1}{p_{\mathbf{v}}(\xi)+\omega}(\widehat{|Q|^{2\sigma}Q})(\xi)$ is a continuous function due to the assumed continuity of $p(\xi)$. Next, by Lemma 8.3.2 and 8.3.3, we conclude that $Q^{\sharp_1}$ is also a boosted ground state and it must hold that

$$|\widehat{Q}(\xi)| = (\widehat{Q}(\xi))^{*\mathbf{e}} \quad \text{for all } \xi \in \mathbb{R}^n.$$

By writing the equation (8.1.3) in Fourier space, we find that the set

$$\Omega = \{\widehat{Q}^{*_1} > 0\} = \{|\widehat{Q}(\xi)| > 0\}$$

is a connected set in $\mathbb{R}^n$ by using Lemma 8.4.2 with $f = |\widehat{Q}|^{*_1}$ and $h = (p_{\mathbf{v}}(\xi) + \omega)^{-1}$.

Finally, since $Q$ and $Q^{*\mathbf{e}}$ are both boosted ground states, we must also have the equality $\|Q\|_{L^p} = \|Q^{*\mathbf{e}}\|_{L^p}$. We can now invoke Lemma 8.6.4 to deduce that

$$\widehat{Q}(\xi) = e^{i(\alpha + \beta \cdot \xi)}\widehat{Q^{*\mathbf{e}}}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. Hence it follows that $Q(x) = e^{i\alpha}Q^{\sharp_1}(x + x_0)$ for almost every $x \in \mathbb{R}^n$, where $\alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ are some constants.

The proof of Theorem 8.1.2 is now complete. $\qquad \square$

## 8.5 Proof of Theorem 8.1.3

Let the hypotheses of Theorem 8.1.3 be satisfied and suppose $Q = Q_{\omega, \mathbf{v}} \in H^s(\mathbb{R})$ is a boosted ground state. As before, we consider the set

$$\Omega = \{\xi \in \mathbb{R} : |\widehat{Q}(\xi)| > 0\}.$$

Similarly as in the proof of Theorem 8.1.2, we conclude that $\widehat{Q}$ is a continuous function (and hence $\Omega$ is open). Moreover, it is elementary to see that (using that $\sigma \in \mathbb{N}$)

$$\langle Q^\bullet, P(D) Q^\bullet \rangle \leqslant \langle Q, P(D) Q \rangle \quad \text{and} \quad \|Q\|_{L^{2\sigma+2}} \leqslant \|Q^\bullet\|_{L^{2\sigma+2}},$$

see [5][Lemma 2.1]. Hence we conclude that $Q^\bullet \in H^s(\mathbb{R})$ is also a boosted ground state with $\|Q\|_{L^2} = \|Q^\bullet\|_{L^2}$. Furthermore, by arguing in the same way as in the proof of Theorem 8.1.2, we deduce that

$$\Omega = \bigoplus_{k=1}^{2\sigma+1} \Omega, \tag{8.5.1}$$

which means that the set $\Omega \subset \mathbb{R}$ is identical to its $(2\sigma + 1)$-fold Minkowski sum. Using the one-dimensionality of the problem, we can now prove the following auxiliary result.

**Lemma 8.5.1.** *Suppose $\Omega \subset \mathbb{R}$ is an open and non-empty set such that*

$$\Omega = \bigoplus_{k=1}^m \Omega$$

*for some integer $m \geqslant 2$. Then it holds that*

$$\Omega \in \{\mathbb{R}_{>0}, \mathbb{R}_{<0}, \mathbb{R}\}.$$

**Remark.** *For higher dimensions $\Omega \subset \mathbb{R}^n$ when $n \geqslant 2$, we conjecture that $\Omega$ is always a connected set.*

*Proof.* We split the proof into the following steps.

**Step 1.** Let us first suppose that $\Omega \subset \mathbb{R}_{\geqslant 0}$ holds. We claim that we necessarily have

$$\Omega = \mathbb{R}_{>0}. \tag{8.5.2}$$

To see this, we first show that

$$\inf \Omega = 0. \tag{8.5.3}$$

Indeed, let us denote $x_* = \inf \Omega \geqslant 0$. For every $\varepsilon > 0$, we can find $x \in \Omega$ such that $x_* \leqslant x < x + \varepsilon$. Since $\Omega = \oplus_{k=1}^m \Omega$ we can also find $x_1, \ldots, x_m \in \Omega$ such that $x = \sum_{k=1}^m x_k$ and, of course, we have $x_k \geqslant x_*$ for $k = 1, \ldots, m$. Thus we conclude

$$mx \leqslant \sum_{k=1}^m x_k = x < x_* + \varepsilon.$$

Therefore we find that

$$(m-1)x_* < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that (8.5.3) holds.

Next, we show that $\Omega$ is connected (and hence it is an open interval since we are in one dimension). We argue by contradiction. Suppose $\Omega$ is not connected, i.e., we can find

$x, y \in \Omega$ with $x < y$ and some $b \in (x, y)$ such that $b \notin \Omega$. Moreover, since $\Omega$ is open, we can always arrange that $b$ is chosen such that

$$(x, b) \subset \Omega \quad \text{and} \quad b \notin \Omega. \tag{8.5.4}$$

Recalling that $\inf \Omega = 0$ we can now find some $c \in \Omega$ with $0 < c < \frac{b-x}{m-1}$. Hence it follows

$$x + (m-1)c < b \quad \text{and} \quad b + (m-1)c > b. \tag{8.5.5}$$

Thus there exists $d \in (x, b) \subset \Omega$ with $d + (m-1)c = b$. Since $\Omega = \oplus_{k=1}^{m} \Omega$, we deduce from this that we have $b \in \Omega$ as well. But this is a contradiction. Hence the open set $\Omega \subset \mathbb{R}$ is connected, i.e., we have

$$\Omega = (\inf \Omega, \sup \Omega) = (0, \sup \Omega)$$

since $\inf \Omega = 0$. From the assumed Minkowski-sum property of $\Omega$ it is easy to see that $\sup \Omega = +\infty$. Thus we conclude $\Omega = (0, +\infty) = \mathbb{R}_{>0}$, provided that $\Omega \subset \mathbb{R}_{\geqslant 0}$ holds. Likewise, we can show that $\Omega = \mathbb{R}_{<0}$ whenever $\Omega \subset \mathbb{R}_{\leqslant 0}$.

**Step 2.** It remains to discuss the case when both $\Omega \cap \mathbb{R}_{\geqslant 0} \neq \varnothing$ and $\Omega \cap \mathbb{R}_{\leqslant 0} \neq \varnothing$. In this case, we first claim that there exist numbers $\underline{y} < 0$ and $\overline{y} > 0$ such that

$$(-\infty, \underline{y}) \cup (\overline{y}, +\infty) \subset \Omega. \tag{8.5.6}$$

Indeed, by assumption on $\Omega$, exist real numbers $y_- < 0$ and $y_+ > 0$ such that $y_-, y_+ \in \Omega$. Since $\Omega$ is open, we find $B_\varepsilon(y_-) \subset \Omega$ and $B_\varepsilon(y_+) \subset \Omega$ for some $\varepsilon > 0$. Let us introduce the integer $m = 2\sigma + 1 \geqslant 2$. From the elementary fact $B_{r_1}(x_1) \oplus B_{r_2}(x_2) = B_{r_1+r_2}(x_1 + x_2)$ for the Minkowski sum of two open balls together with (8.5.1), we deduce

$$\bigoplus_{k=1}^{m} B_\varepsilon(y_+) = B_{m\varepsilon}(my_+) \subset \Omega.$$

Using this fact inductively and (8.5.1), we obtain a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ with $I_n \subset \Omega$ that are given by the recursion formula

$$\begin{cases} I_{n+1} = B_{(m-1)\varepsilon}((m-1)y_+) \oplus I_n & \text{for } n \geqslant 1, \\ I_1 = B_\varepsilon(y_+). \end{cases}$$

Hence we have

$$\begin{aligned} I_{n+1} &= B_{(m-1)\varepsilon}((m-1)y_+) \oplus B_\varepsilon(y_+) \oplus_{k=1}^{m-1} B_{(m-1)\varepsilon}((m-1)y_+) \\ &= B_{(n(m-1)+1)\varepsilon}((n(m-1)+1)y_+). \end{aligned}$$

Now we claim that

$$I_{n+1} \cap I_{n+2} \neq \varnothing \quad \text{for } n \geqslant n_0, \tag{8.5.7}$$

where $n_0 \geqslant 1$ is sufficiently large. This is true if

$$((n+1)(m-1)+1)y_+ - ((n+1)(m-1)+1)\varepsilon \leqslant (n(m-1)+1)y_+ + (n(m-1)+1)\varepsilon,$$

which in turn is equivalent to

$$\left(2n + 1 + \frac{2}{m-1}\right)\varepsilon \geqslant y_+.$$

Evidently, this holds if $n \geqslant n_0$ with some sufficiently large integer $n_0 \in \mathbb{N}$.

By (8.5.7), we deduce that $I = \cup_{n \geqslant N} I_{n+1} \subset \Omega$ is an (open) interval and it is elementary to check that $\sup I = +\infty$. Hence we conclude that $I = (\overline{y}, +\infty) \subset \Omega$ for some $\overline{y} > 0$. Likewise, we show that $(-\infty, \underline{y}) \subset \Omega$ for some $\underline{y} < 0$. This proves (8.5.6).

Finally, we define $c = \max\{\overline{y}, -\underline{y}\} > 0$. Since $c \in [\overline{y}, \infty) \subset \Omega$ and $-(m-1)c \in (-\infty, \underline{y}) \subset \Omega$, we conclude from (8.5.1) that $0 \in \Omega$ since $0 = c - (m-1)c = c - \sum_{k=1}^{m-1} c \in \oplus_{k=1}^{m} \Omega$. Since $\Omega$ is open, we deduce that $B_r(0) \subset \Omega$ for some $r > 0$. By (8.5.1) and iteration, we conclude that

$$B_{Nmr}(0) \subset \Omega \quad \text{for any } N \in \mathbb{N},$$

which readily implies that

$$\Omega = \mathbb{R}.$$

The proof of Lemma 8.5.1 is now complete. $\qquad\square$

With Lemma 8.5.1 at hand, we can now finish the proof of Theorem 8.1.3 as follows. Since we must have equality $\|Q_{\omega,\mathbf{v}}\|_{L^{2\sigma+2}} = \|Q_{\omega,\mathbf{v}}^{\bullet}\|_{L^{2\sigma+2}}$ for any boosted ground state $Q_{\omega,\mathbf{v}} \in H^s(\mathbb{R})$, we deduce from Lemma 8.6.4 below that the conclusion of Theorem 8.1.3 holds. $\qquad\square$

## 8.6 Some Technical Results

**Lemma 8.6.1** (pqr Lemma; see [15])**.** *Let $(\Omega, \Sigma, \mu)$ be a measure space. Let $1 \leqslant p < q < r \leqslant \infty$ and let $C_p, C_q, C_r > 0$ be positive constants. Then there exist constants $\eta, c > 0$ such that, for any measurable function $f \in L_\mu^p(\Omega) \cap L_\mu^r(\Omega)$ satisfying*

$$\|f\|_{L_\mu^p}^p \leqslant C_p, \quad \|f\|_{L_\mu^q}^q \geqslant C_q, \quad \|f\|_{L_\mu^r}^r \leqslant C_r,$$

*it holds that*

$$d_f(\eta) := \mu(\{x \in \Omega; \ |f(x)| > \eta\}) \geqslant c.$$

*The constant $\eta > 0$ only depends on $p, q, C_p, C_q$ and the constant $c > 0$ only depends on $p, q, r, C_p, C_q, C_r$.*

*Proof.* See [15, Lemma 2.1]. $\qquad\square$

**Lemma 8.6.2** (Compactness modulo translations in $\dot{H}^s(\mathbb{R}^n)$; see [1])**.** *Let $s > 0$, $1 < p < \infty$ and $(u_j)_{j \in \mathbb{N}} \subset \dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ be a sequence with*

$$\sup_{j \in \mathbb{N}} \left( \|u_j\|_{\dot{H}^s} + \|u_j\|_{L^p} \right) < \infty,$$

*and, for some $\eta, c > 0$ (with $|\cdot|$ being Lebesgue measure)*

$$\inf_{j \in \mathbb{N}} |\{x \in \mathbb{R}^n; \ |u_j(x)| > \eta\}| \geqslant c.$$

*Then there exists a sequence of vectors $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^n$ such that the translated sequence $u_j(x + x_j)$ has a subsequence that converges weakly in $\dot{H}^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to a nonzero function $u \not\equiv 0$.*

*Proof.* See [1, Lemma 2.1]. $\qquad\square$

**Lemma 8.6.3** (Brascamp–Lieb–Luttinger Inequality)**.** *Let $d \geqslant 1$ and $m \geqslant 2$ be integers. Suppose that $u_1, u_2, \ldots, u_m : \mathbb{R}^d \to \mathbb{R}_+$ are nonnegative measurable functions vanishing at infinity. Let $1 \leqslant k \leqslant m$ and $B = [b_{ij}]$ be a given $k \times m$ matrix (with $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$). If we define*

$$I_d[u_1, \ldots, u_m] := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{m} u_j \left( \sum_{i=1}^{k} b_{ij} y^i \right) \mathrm{d}y^1 \cdots \mathrm{d}y^k, \tag{8.6.1}$$

*then it holds that*

$$I_d[u_1, \ldots, u_m] \leqslant I_d[u_1^*, \ldots, u_m^*],$$

*where $*$ denotes the symmetric-decreasing rearrangement in $\mathbb{R}^d$.*

We recall from [24] the following result.

**Lemma 8.6.4** (Equality in the Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$)**.** *Let $n \geqslant 1$ and $p \in 2\mathbb{N} \cup \{\infty\}$ with $p > 2$. Suppose that $f, g \in \mathcal{F}(L^{p'}(\mathbb{R}^n))$ with $1/p + 1/p' = 1$ satisfy the majorant condition*

$$|\widehat{f}(\xi)| \leqslant \widehat{g}(\xi) \quad \text{for a. e. } \xi \in \mathbb{R}^n.$$

*In addition, we assume that $\widehat{f}$ is continuous and that $\{\xi \in \mathbb{R}^n : |\widehat{f}(\xi)| > 0\}$ is a connected set. Then equality*

$$\|f\|_{L^p} = \|g\|_{L^p}$$

*holds if and only if*

$$\widehat{f}(\xi) = e^{\mathrm{i}(\alpha + \beta \cdot \xi)}\widehat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

*with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$.*

# Bibliography

[1] Jacopo Bellazzini, Rupert L. Frank and Nicola Visciglia, *Maximizers for Gagliardo-Nirenberg inequalities and related non-local problems*, Math. Ann. **360** (2014), no. 3-4, 653–673.

[2] Jacopo Bellazzini, Vladimir Georgiev, Enno Lenzmann, and Nicola Visciglia, *On Traveling Solitary Waves and Absence of Small Data Scattering for Nonlinear Half-Wave Equations*, Comm. Math. Phys. (2019), in press, https://doi.org/10.1007/s00220-019-03374-y.

[3] R. P. Boas, Jr., *Majorant problems for trigonometric series.* J. Analyse Math. **10** (1962/63), 253–271.

[4] Denis Bonheure, Jean-Baptiste Casteras, Edeson Moreira dos Santos, and Robson Nascimento, *Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation*, SIAM J. Math. Anal. **50** (2018), no. 5, 5027–5071.

[5] Lars Bugiera, Enno Lenzmann, Armin Schikorra, and Jérémy Sok, *On Symmetry and Uniqueness of ground state for linear and nonlinear elliptic PDEs*, Preprint (2019).

[6] Lars Bugiera, Enno Lenzmann, and Jérémy Sok, *On Symmetry for Traveling Solitary Waves for Dispersion Generalized NLS*, Preprint (2019).

[7] Thomas Boulenger, Dominik Himmelsbach, and Enno Lenzmann, *Blowup for fractional NLS*, J. Funct. Anal. **271** (2016), no. 9, 2569–2603.

[8] Herm J. Brascamp, Elliott H. Lieb and Joaquin M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Functional Analysis **17** (1974), 227–237.

[9] Haïm Brézis and Elliot Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.

[10] Guiseppe M. Capriani, *The Steiner rearrangement in any codimension*, Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 517–548.

[11] J. M. Combes and L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators.* Comm. Math. Phys. **34** (1973), 251–270.

[12] Gadi Fibich, *The nonlinear Schrödinger equation*, Applied Mathematical Sciences, 192, Springer, Cham, 2015.

[13] Gadi Fibich, Boaz Ilan, and George Papanicolaou, *Self-focusing with fourth-order dispersion*, SIAM J. Appl. Math. **62** (2002), no. 4, 1437–1462.

[14] R. L. Frank, E. H. Lieb, and J. Sabin, *Maximizers for the Stein-Tomas inequality.* Geom. Funct. Anal. **26** (2016), no. 4, 1095–1134.

[15] Jürg Fröhlich, Elliott H. Lieb, and Michael Loss, *Stability of Coulomb systems with magnetic fields. I. The one-electron atom*, Comm. Math. Phys. **104** (1986), no. 2, 251–270.

[16] Patrick Gérard, Enno Lenzmann, Oana Pocovnicu, and Pierre Raphaël, *A two-soliton with transient turbulent regime for the cubic half-wave equation on the real line*, Annals of PDE **4** (2018), Art. 7, 166pp.

[17] I. Herbst and E. Skibsted, *Decay of eigenfunctions of elliptic PDE's, I.* Adv. Math. **270** (2015), 138–180.

[18] Dominik Himmelsbach, *Blowup, solitary waves and scattering for the fractional nonlinear Schrödinger equation*, PhD Thesis (2017), University of Basel, http://edoc.unibas.ch/diss/DissB_12432.

[19] Younghun Hong and Yannick Sire, *On Fractional Schrödinger Equations in Sobolev Spaces*, Commun. Pure Appl. Anal. **14** (2015), no. 6, 2265–2282.

[20] V. I. Karpman and A. G. Shagalov, *Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion*, Phys. D **144** (2000), no. 1-2, 194–210.

[21] T. Kato, *Perturbation theory for linear operators.* Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[22] Joachim Krieger, Enno Lenzmann and Pierre Raphaël, *Nondispersive solutions to the $L^2$-critical Half-Wave Equation*, Arch. Rational Mech. Anal. **209** (2013), 61–129.

[23] E. T. Y. Lee and Gen-Ichirô Sunouchi, *On the Majorant Properties in $L^p(G)$.* Tôhoku Math. Journ. **31** (1979), 41–48.

[24] E. Lenzmann and J. Sok, *A sharp rearrangement principle in Fourier space and symmetry results for PDEs with arbitrary order*, Preprint (2018), arXiv:1805.06294.

[25] Elliott H. Lieb and Michael Loss, *Analysis* (Second edition), American Mathematical Society, Providence, RI, 2001.

[26] J. E. Littlewood, *On the inequalities between functions f and f\*.* J. London Math. Soc. **35** (1960), 352–365.

[27] G. Mockenhaupt and W. Schlag, *On the Hardy-Littlewood majorant problem for random sets.* J. Funct. Anal. **256** (2009), no. 4, 1189–1237.

[28] Ivan Naumkin and Pierre Raphaël, *Om small traveling waves to the mass critical fractional NLS*, Calc. Var. Partial Differential Equations **57** (2018), no. 3, 36pp.

[29] A. J. O'Connor, *Exponential decay of bound state wave functions.* Comm. Math. Phys. **32** (1973), 319–340.

[30] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York-London, 1978.

[31] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York-London, 1975.

[32] A.-M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2009.

[33] H. Wendland, *Scattered Data Approximation*, Cambridge University Press, Cambridge, 2005.

# Curriculum Vitae

## Lars Bugiera

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### Personal Details

Gender: Male
Date of birth: 19th of September, 1986
Place of birth: Rheinfelden, Switzerland
Present Citizenship: Swiss

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### Education

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|---|---|
| 9/1993–7/1998 | Primary School Magden |
| 9/1998–7/2001 | Realschule Magden |
| 9/2001–7/2003 | Sekundarschule Magden |
| 9/2003–5/2005 | Bezirksschule Rheinfelden |
| 9/2005–6/2008 | Georg-Büchner Gymnasium Rheinfelden (Baden) |
| 9/2008–6/2012 | Undergraduate Studies in Mathmatics at the University of Basel, Switzerland |
| 6/2011–10/2014 | M.A. in Mathematics at the University of Basel, Switzerland |
| | Specialization: PDE theory and commutative algebra |
| | Thesis: *The Fractional Burgers Equation*; Supervisor: Prof. Dr. G. Crippa and Prof. Dr. E. Lenzmann. |
| 11/2014–05/2019 | Ph.D. student at the University of Basel, Switzerland. |
| | Supervisor: Prof. Dr. Enno Lenzmann |
| 26th of June 2019 | Doctoral Colloquium |