Sparse grid approximation of the Riccati operator for closed loop parabolic control problems with Dirichlet boundary control

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SPARSE GRID APPROXIMATION OF THE RICCATI OPERATOR FOR CLOSED LOOP PARABOLIC CONTROL PROBLEMS WITH DIRICHLET BOUNDARY CONTROL

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Abstract. We consider the sparse grid approximation of the Riccati operator P arising from closed loop parabolic control problems. In particular, we concentrate on the linear quadratic regulator (LQR) problems, i.e. we are looking for an optimal control u_{opt} in the linear state feedback form $u_{\mathrm{opt}}(t,\cdot) = Px(t,\cdot)$, where $x(t,\cdot)$ is the solution of the controlled partial differential equation (PDE) for a time point t. Under sufficient regularity assumptions, the Riccati operator P fulfills the algebraic Riccati equation (ARE)

$$AP + PA - PBB^\star P + Q = 0,$$

where A, B, and Q are linear operators associated to the LQR problem. By expressing P in terms of an integral kernel p, the weak form of the ARE leads to a non-linear partial integro-differential equation for the kernel p – the Riccati-IDE. We represent the kernel function as an element of a sparse grid space, which considerably reduces the cost to solve the Riccati IDE. Numerical results are given to validate the approach.

1. Introduction

Operator Riccati differential equaions play an important role in a number of different applications in engineering, physics, and mathematics. To give a few examples, we mention model reduction ([24, 17]), filtering ([25]), scattering theory ([33]), radiative transfer and the solution of two point boundary value problems via the theory of invariant embedding ([2]). A well-known application of the Riccati equation stems from the optimal control theory, in particular from the unconstrained linear quadratic (LQ) optimal control of parabolic partial differential equations, see e.g. [2, 5, 29, 32] and the references therein. In Section 2, we consider unconstrained LQ optimal control for infinite time horizon. In this case, the optimal control can be obtained by solving the algebraic Riccati equation (ARE). We refer to the solution of the ARE as Riccati operator P.

In order to obtain an approximation of the Riccati operator, we follow the approach presented in [8, 23]. Therein, the representation of P in terms of a kernel function $p(x,\xi)$ is considered:

$$(Pu)(x) = \int_\Omega p(x,\xi)u(\xi)\,\mathrm{d}\xi.$$

By this means, the solution of the ARE can be characterized via an integro-differential equation of Riccati type (Riccati-IDE) for the kernel $p(x,\xi)$. We present

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the derivation of the Riccati-IDE for the Dirichlet boundary control of the heat equation in Section 3.

The Riccati-IDE is a non-linear equation with a non-linearity in form of a quadratic term. A number of methods for the solution of non-linear equations, which have been studied for the ARE (see e.g. [3, 4, 28] for a survey), can similarly be implemented for the Riccati-IDE. In this article, we apply Newton's method as suggested in [27]. We describe this approach for the discretization of the Riccati-IDE with a standard finite elements method in Section 4.

As the Riccati operator P is a linear operator on the state space with domain Ω, the kernel $p(x,\xi)$ is defined on the product domain $\Omega\times\Omega$. Provided we use N degrees of freedom for the discretization of the state space, the discretization of the kernel by a regular tensor product approach $p(x,\xi)$ amounts to N^2 degrees of freedom. This leads in general to a cubic over-all complexity $\mathcal{O}(N^3)$ for the evaluation of the right-hand side and the computation of the gradient in the Newton's method.

The $\mathcal{O}(N^3)$-complexity is a major bottleneck in the numerical treatment of the LQ optimal control problems and large scale AREs. At least for $d=3$ spatial dimensions, the quadratic growth of the memory requirements makes the discretization in the regular tensor product space prohibitively expensive if not even impossible. This is an example of a more general problem known as curse of dimensionality. Different approaches, like e.g. multigrid methods [15] or $\mathcal{H}$-matrices [16] have been studied to overcome this drawback. In the present article, we discretize the Riccati-IDE in the *sparse* tensor product space – a numerical technique, which allows to overcome the curse of dimensionality to some extend. Thus, the kernel $p(x,\xi)$ is represented by only $\mathcal{O}(N\log N)$ degrees of freedom, which in turn improves the over-all complexity. We will introduce the sparse tensor product space and the corresponding discretization of the Riccati-IDE in Section 5.

In Section 6, we verify our approach by numerical experiments, in which convergence rates for the approximation of the Riccati kernel $p(x,\xi)$ as well as the computational complexity are considered. Finally, in Section 7, we state concluding remarks.

2. LQR Dirichlet boundary control

This section briefly describes the main ideas of the linear quadratic (LQ) optimal control of partial differential equations. A detailed discussion of this topic can be found e.g. in [5, 32, 37].

2.1. **Heat equation with Dirichlet boundary conditions.**

We consider the heat equation on the domain $\Omega\subset\mathbb{R}^d$ with Dirichlet boundary control

$$\left\{\begin{aligned}\frac{\partial}{\partial t}z(t,x)-\Delta z(t,x)&=0 && \text{in } \Omega\times(0,T],\\ z(0,x)&=z_0(x) && \text{for } x\in\Omega,\\ z(t,x)&=u(t) && (x,t)\in\Sigma=\Gamma\times[0,T],\end{aligned}\right. \tag{1}$$

where $\Gamma=\partial\Omega$, $z_0\in L^2(\Omega)$, and $u\in L^2(\Sigma)$ is a given control function. The existence and uniqueness of the solution to (1) in $L^2\big((0,T);\Omega\big)$ can be shown, e.g., by the method of transposition (cf. [32, Chapter III, Section 9]). Here, following [5, 9, 30], we will interpret (1) as an abstract differential equation. To this end, we first introduce some notation.

Let $\mathcal{H}, \mathcal{U}, \mathcal{Y}$ be Hilbert spaces of states, controls, and observations, respectively. In the particular case of Dirichlet control for the heat equation (1), we set

$$\mathcal{H} = L^2(\Omega), \quad \mathcal{U} = L^2(\Gamma), \quad \mathcal{Y} = \mathbb{R}.$$

The abstract differential equation corresponding to the system (1) reads

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = Az(t) + Bu(t), \quad t \in (0,T], \\ z(0) = z_0, \end{cases} \tag{2}$$

where

$$u \in L^2\big((0,T);\mathcal{U}\big), \ z_0 \in \mathcal{H}.$$

The derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is interpreted in a vector distributional sense, compare [5, pp. 87 and 202, 37, p. 117]. The linear operator A is defined by

$$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}, \quad v \mapsto Av = \Delta_x v, \tag{3}$$

where $\mathcal{D}(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega)$.

The definition of the control operator B is more involved. In general, for the boundary control of parabolic partial differential equations, B is considered to be a continuous linear operator from the control space $\mathcal{U}$ to $\mathcal{D}(A^\star)'$, whereat $A^\star$ is the adjoint operator of A, compare [5, p. 210, 11]). In fact, boundary control problems are defined by B being an element of $\mathcal{L}\big(\mathcal{U}, \mathcal{D}(A^\star)'\big)$ in contrast to distributed control problems, where we have $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. This viewpoint arises in the variational formulation as well as in the method of transposition for the Dirichlet boundary control of parabolic problems (cf. [5, Part II, Chapter 2] and [32, Chapter III]).

Another assumption is for the control operator to be of the form $B = (\lambda_0 - A)D$ (see [5, Part IV, Section 1] or [11]). Here, $\lambda_0 \in \rho(A)$ is an element of the resolvent set of A such that λ_0 is strictly larger than the type of semigroup generated by A. Note that B is of this form for parabolic Dirichlet boundary control problems as well as for parabolic Neumann boundary control problems.

In the case of the Dirichlet boundary control, the operator D is the Dirichlet mapping defined as an extension of the Green mapping $G : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ for the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases}$$

cf. [5, p. 436] or [35, p. 254]. In other words, we have

$$D \in \mathcal{L}(\mathcal{U}, \mathcal{H}), \quad v \mapsto Dv = w, \text{ where } \Delta w = 0 \text{ in } \Omega, \ w = v \text{ on } \Gamma. \tag{4}$$

A from (3) is a strictly negative self-adjoint operator in $L^2(\Omega)$ and therefore a generator of an analytic semigroup of negative type, cf. [5, p. 436]). By this means we can set $\lambda_0 = 0$, i.e. we obtain $B = -AD$.

With these observations regarding the control operator B we can rewrite the problem (1) as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = Az(t) - ADu(t), \quad t \in (0,T], \\ z(0) = z_0, \end{cases} \tag{5}$$

where $u \in L^2\big((0,T);\mathcal{U}\big)$, $z_0 \in \mathcal{H}$,D as in (4), and $A : \mathcal{H} \to \mathcal{D}(A^\star)'$ being an extension of (3). According to [5, Part II, Chapter 3], there exists a unique solution

$$z \in \left\{ v \in L^2\big((0,T);\mathcal{H}\big) : \frac{\mathrm{d}v}{\mathrm{d}t} \in L^2\big((0,T);\mathcal{D}(A^\star)'\big) \right\}$$

for abstract differential equations of the type (5).

2.2. **Optimal control problem.** We introduce the following quadratic cost functional for the abstract differential equation (5)

$$J_\infty(u) := \int_0^\infty \Big\{ \|Cz(t)\|_{\mathcal{Y}}^2 + \|u(t)\|_{\mathcal{U}}^2 \Big\} \,\mathrm{d}t,$$

where $C \in \mathcal{L}(\mathcal{H},\mathcal{Y})$ is an observation operator. As we consider the case $T \to \infty$, further assumptions on the existence of a control $u \in L^2\big((0,\infty);\mathcal{U}\big)$ with $J_\infty(u) < \infty$ has to be made. Such a control is called admissible. If there exists an admissible control for each initial state z_0, the system (5) is called C-stabilizable, cf. [5, p. 517]. For C-stabilizable systems, we can consider the unconstrained (i.e. with respect to the control space) linear quadratic optimal control problem for the heat equation with Dirichlet boundary control

$$\left\{ \begin{array}{l} \displaystyle\min_{u \in L^2((0,\infty);\mathcal{U})} J_\infty(u) \\ \text{subject to system (5).} \end{array} \right. \tag{6}$$

The optimal control u_{opt} to the problem (6) is given by the feedback formula (cf. [5, Part V, Chapter 2, 13, 30, 32, Chapter III, Section 4])

$$u_{\mathrm{opt}}(t) = -B^\star P z_{\mathrm{opt}}(t),$$

where $B^\star$ is the adjoint of the control operator B, z_{opt} is the solution of the closed loop system (see e.g. [5, p. 518]) and P is the unique solution of the algebraic Riccati equation (ARE):

$$A^\star P + PA - PBB^\star P + C^\star C = 0. \tag{7}$$

It can be shown that P –the Riccati operator– is a positive, self-adjoint, and bounded operator on the state space $\mathcal{H}$.

If $A = A^\star$ holds, as in the case of the heat equation, (7) is equivalent to

$$AP + PA - PBB^\star P + C^\star C = 0. \tag{8}$$

By this result, we can proceed with solving the ARE (8) to obtain the solution to the optimization problem (6).

3. Riccati partial integro-differential equation

There are different approaches to the solution of equation (8) (see e.g. [8, 23, Chapters 3 and 4, 32, Chapter 3, 31, 34]). In this article, we concentrate on the representation of the Riccati operator in terms of a kernel function

$$\big[P\phi\big](x) = \int_\Omega p(x,\xi)\phi(\xi)\,\mathrm{d}\xi, \tag{9}$$

where in general $p(x,\xi)$ is a distribution on $\Omega \times \Omega$ (cf. [32, Chapter III, Section 5]). The existence of such a kernel is guaranteed by the Schwartz kernel theorem.

3.1. **Variational formulation.** Next, we want to combine (9) with the weak form of the ARE (8):

$$(10)\quad (A\phi, P\psi) + (P\phi, A\psi) - (B^\star P\phi, B^\star P\psi)_{\mathcal{U}} + (C^\star C\phi, \psi) = 0 \text{ for all } \phi, \psi \in \mathcal{D}(A).$$

For the sake of brevity, here and in the following, $(\cdot,\cdot)$ denotes the scalar product in the state space $\mathcal{H}$, while $(\cdot,\cdot)_{\mathcal{U}}$ denotes the scalar product in $\mathcal{U}$. In addition, we shall assume that $p \in H^1(\Omega\times\Omega)$. Then, for all $\varphi(x,\xi) = \phi(x)\psi(\xi)$ with $\phi, \psi \in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}(A\phi, P\psi) &= \int_\Omega \Delta\phi(x) \int_\Omega p(x,\xi)\psi(\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \int_\Omega\int_\Omega p(x,\xi)\Delta_x\phi(x)\psi(\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_{\Omega\times\Omega} p(x,\xi)\Delta_x\varphi(x,\xi)\,\mathrm{d}(x,\xi) = -\int_{\Omega\times\Omega} \nabla_x p(x,\xi)\nabla_x\varphi(x,\xi)\,\mathrm{d}(x,\xi),\end{aligned}$$

and likewise

$$\begin{aligned}(P\phi, A\psi) &= \int_\Omega\int_\Omega p(x,\xi)\phi(\xi)\,\mathrm{d}\xi\,\Delta\psi(x)\,\mathrm{d}x = \int_\Omega\int_\Omega p(x,\xi)\phi(x)\Delta_\xi\psi(\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_{\Omega\times\Omega} p(x,\xi)\Delta_\xi\varphi(x,\xi)\,\mathrm{d}(x,\xi) = -\int_{\Omega\times\Omega} \nabla_\xi p(x,\xi)\nabla_\xi\varphi(x,\xi)\,\mathrm{d}(x,\xi),\end{aligned}$$

where we used the relation $p(x,\xi) = p(\xi,x)$ which comes from P being self-adjoint. We thus deduce

$$(A\phi, P\psi) + (P\phi, A\psi) = -\int_{\Omega\times\Omega} \nabla p(x,\xi)\nabla\varphi(x,\xi)\,\mathrm{d}(x,\xi).$$

We proceed with the non-linear term. First, notice that it holds for arbitrary $\eta \in \mathcal{U}$ and $\psi \in H_0^1(\Omega)$

$$\begin{aligned}(B\eta,\psi) &= -\,(AD\eta,\psi) = -(D\eta, A\psi) = -\int_\Omega (D\eta)(x)\Delta\psi(x)\,\mathrm{d}x \\ &= -\int_\Gamma D\eta(x)\frac{\partial\psi}{\partial\nu}(x)\,\mathrm{d}\Gamma + \int_\Omega \nabla D\eta(x)\nabla\psi(x)\,\mathrm{d}x,\end{aligned}$$

which yields in view of the definition of D in (4)

$$(B\eta,\psi) = -\int_\Gamma \eta(x)\frac{\partial\psi}{\partial\nu}(x)\,\mathrm{d}\Gamma = \Big(\eta, -\frac{\partial\psi}{\partial\nu}\Big)_{\mathcal{U}} = (\eta, B^\star\psi)_{\mathcal{U}}.$$

Therefore, the operator $B^\star$ is given by

$$B^\star \in \mathcal{L}(\mathcal{D}(A^\star),\mathcal{U}), \quad v \mapsto B^\star v = -\frac{\partial v}{\partial\nu},$$

compare [5, pp. 189, 195] and [29, p. 181]).

We can now plug in $B^\star$ into the non-linear term of (10)

$$\begin{aligned}(B^\star P\phi, B^\star P\psi)_{\mathcal{U}} &= \int_\Gamma \frac{\partial}{\partial\nu_\zeta}\int_\Omega p(\zeta,x)\phi(x)\,\mathrm{d}x \cdot \frac{\partial}{\partial\nu_\zeta}\int_\Omega p(\zeta,\xi)\psi(\xi)\,\mathrm{d}\xi\,\mathrm{d}\Gamma_\zeta \\ &= \int_\Gamma \frac{\partial}{\partial\nu_\zeta}\int_\Omega p(x,\zeta)\phi(x)\,\mathrm{d}x \cdot \frac{\partial}{\partial\nu_\zeta}\int_\Omega p(\zeta,\xi)\psi(\xi)\,\mathrm{d}\xi\,\mathrm{d}\Gamma_\zeta.\end{aligned}$$

By applying Fubini's theorem, we conclude

$$\begin{aligned}(B^\star P\phi, B^\star P\psi)_{\mathcal{U}} &= \int_\Gamma \int_\Omega \frac{\partial p}{\partial \nu_\zeta}(x,\zeta)\phi(x)\,\mathrm{d}x \int_\Omega \frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi)\psi(\xi)\,\mathrm{d}\xi\,\mathrm{d}\Gamma_\zeta \\ &= \int_{\Omega\times\Omega}\int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta)\frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta\,\varphi(x,\xi)\,\mathrm{d}(x,\xi).\end{aligned}$$

Note that the boundary integral is well-defined if we assume that it holds $\partial p/\partial\nu_x \in L^2(\Gamma\times\Omega)$ and likewise $\partial p/\partial\nu_\xi \in L^2(\Omega\otimes\Gamma)$.

In order to complete the derivation in terms of kernel functions, we assume in accordance with [8] and [23, Chapter 3]) the operator $C : \mathcal{H} \to \mathcal{Y}$ to be of the form

$$C\phi = \int_\Omega c(x)\phi(x)\,\mathrm{d}x$$

with $c \in L^2(\Omega)$. By this means $C^\star C$ takes the form

$$\begin{aligned}(C^\star C\phi,\psi) = (C\phi, C\psi)_{\mathbb{R}^2} &= \int_\Omega c(x)\phi(x)\,\mathrm{d}x \int_\Omega c(\xi)\psi(\xi)\,\mathrm{d}\xi \\ &= \int_{\Omega\times\Omega} c(x)c(\xi)\phi(x)\psi(\xi)\,\mathrm{d}(x,\xi).\end{aligned}$$

We thus set

$$Q = C^\star C : \mathcal{H}\to\mathcal{H}, \quad v \mapsto Qv = \int_\Omega c(x)c(\xi)v(\xi)\,\mathrm{d}\xi = \int_\Omega q(x,\xi)v(\xi)\,\mathrm{d}\xi, \tag{11}$$

where $q(x,\xi) = c(x)c(\xi)$ is the kernel of Q.

Therefore, since $C_0^\infty(\Omega\times\Omega)$ is dense in $H_0^1(\Omega\times\Omega)$, the kernel p solves the following variational problem

(12)

$$\begin{aligned}&\int_{\Omega\times\Omega} \nabla p(x,\xi)\nabla\varphi(x,\xi)\,\mathrm{d}(x,\xi) + \int_{\Omega\times\Omega}\int_\Gamma \frac{\partial p}{\partial\nu_\zeta}(\zeta,x)\frac{\partial p}{\partial\nu_\zeta}(\xi,\zeta)\,\mathrm{d}\Gamma_\zeta\varphi(x,\xi)\,\mathrm{d}(x,\xi) \\ &\qquad = \int_{\Omega\times\Omega} q(x,\xi)\varphi(x,\xi)\,\mathrm{d}(x,\xi) \text{ for all } \varphi \in H_0^1(\Omega\times\Omega).\end{aligned}$$

3.2. **Boundary conditions.** In order to derive the boundary conditions for $p(x,\xi)$, we follow [5, p. 520]). To this end, we note first that

$$P \in \mathcal{L}\Big(\mathcal{H}, \mathcal{D}\big((-A)^{1-\alpha}\big)\Big), \tag{13}$$

where for A as in (3) we can choose $\alpha \in (0,1/4)$. Furthermore, it holds

$$\mathcal{D}\big((-A)^{1-\alpha}\big) = \begin{cases} H^{2(1-\alpha)}(\Omega), & \text{if } \alpha \in (3/4,1), \\ \big\{u \in H^{2(1-\alpha)}(\Omega) : u = 0 \text{ on } \partial\Omega\big\}, & \text{if } \alpha \in (0,3/4). \end{cases} \tag{14}$$

Therefore, we deduce from (13) and (14) that

$$\text{for all } v \in \mathcal{H} : Pv \in \Big\{u \in H^{2(1-\alpha)}(\Omega) : u = 0 \text{ on } \partial\Omega\Big\}, \text{ where } \alpha \in (0,1/4). \tag{15}$$

We next assume that there exists a part $\widetilde{\Gamma}\times\widetilde{\Omega} \subset \partial(\Omega\times\Omega)$ of the boundary such that $p(x,\xi) \neq 0$ for almost all $(x,\xi) \in \widetilde{\Gamma}\times\widetilde{\Omega}$. Then, taking some $v \in \mathcal{H}$ with $\widetilde{\Omega} \subset \operatorname{supp} v$, we have

$$u(x) = \int_\Omega p(x,\xi)v(\xi)\,\mathrm{d}\xi > 0$$

for almost all $x \in \widetilde{\Gamma}$, which is a contradiction to (15). Hence, with the symmetry of $p(x,\xi)$, we conclude

$$\begin{cases} p(x,\xi) = 0, & x \in \Gamma, \quad \xi \in \Omega, \\ p(x,\xi) = 0, & x \in \Omega, \quad \xi \in \Gamma, \end{cases}$$

compare also [32, p. 158].

We therefore arrive at the following result.

Theorem 3.1. *The kernel $p \in V$ for the Riccati operator associated with the Dirichlet boundary control of the heat equation* (2)*, where*

$$V := \left\{ f \in H_0^1(\Omega \times \Omega) : \frac{\partial f}{\partial \nu_x} \in L^2(\Gamma \times \Omega) \text{ and } \frac{\partial f}{\partial \nu_\xi} \in L^2(\Omega \times \Gamma) \right\},$$

is the weak solution of the following integro-differential equation (IDE) of Riccati type:

$$\begin{cases} -\Delta p(x,\xi) + \displaystyle\int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta) \frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi) \, \mathrm{d}\Gamma_\zeta = q(x,\xi) & \text{in } \Omega \times \Omega, \ \Gamma = \partial\Omega \\ p(x,\xi) = 0 & \text{on } \partial(\Omega \times \Omega). \end{cases} \tag{16}$$

In [32, Chapter III, Section 5], several results of this type are derived, in particular for distributed control (i.e. $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$) and Neumann boundary control. [23, Chapter 3] considers the Riccati-PDE for Robin boundary control. The Riccati-PDE for Dirichlet boundary control in the one-dimensional situation can be found in [8]. There, the results are based on a stronger regularity of the kernel $p(x,\xi)$, i.e. $p \in C(\Omega \times \Omega)$, compare [26] for one-dimensional problems.

4. Finite element discretization

In this section, we derive a discrete version of the Riccati-IDE (16) by means of a Galerkin discretization by finite elements. To this end, we consider the *full* tensor product discretization of functions defined on the product domain $\Omega \times \Omega$.

4.1. Tensor product approximation.

Let Z be a Hilbert space with

$$Z \otimes Z \subset V,$$

where $\otimes$ denotes the algebraic tensor product, cf. [19, p. 52]. The closure can be taken with respect to an appropriate norm. Furthermore, suppose we are given a finite dimensional subspace $Z_J \subset Z$. We define the full tensor product space $V_{J,J}$ via

$$V_{J,J} \coloneqq Z_J \otimes Z_J. \tag{17}$$

If $\Phi_J \coloneqq \{\phi_j\}_{j=1}^{N_J}$ is a basis of Z_J, i.e. $N_J = \dim Z_J$, then

$$\Phi_J \otimes \Phi_J = \{\phi_{j_1} \otimes \phi_{j_2}\}_{j_1,j_2=1}^{N_J}$$

is a basis of $V_{J,J}$. Thus, we obtain $\dim V_{J,J} = N_J^2$.

The finite dimensional subspace Z_J might be given by the span of globally continuous, piecewise linear ansatz functions defined with respect to a triangulation or tetrahedralization of Ω, respectively. Thus, the tensor product space $V_{J,J}$ would be spanned by products of those functions, compare Section 5 for details.

Next, we want to discuss the discretization of Riccati-IDE (16) with respect to the full tensor product space $V_{J,J}$. We make the following ansatz

$$p(x,\xi) = \sum_{j_1,j_2=1}^{N_J} p_{j_1,j_2}\phi_{j_1}(x)\phi_{j_2}(\xi) \in V_{J,J} \tag{18}$$

for the discretization of the kernel function in the space $V_{J,J}$ and write

$$p_{J,J} \coloneqq \big[p_{1,1}, p_{1,2}, \dots, p_{N_J,N_J}\big]^\intercal$$

for the coefficient vector of the Riccati kernel.

4.2. **Linear part and right-hand side.** First, let us consider the linear part of (12), i.e., the evaluation of

$$\int_{\Omega\times\Omega} \nabla p(x,\xi)\nabla\varphi(x,\xi)\,\mathrm{d}(x,\xi) \text{ for all } \varphi(x,\xi) = \phi_{k_1}(x)\phi_{k_2}(\xi), \tag{19}$$

which corresponds to the weak formulation of Laplace operator on the product domain $\Omega\times\Omega$. Denoting by $A_J = \big[a_{k,\ell}\big]_{k,\ell=1}^{N_J}$ and $E_J = \big[e_{k,\ell}\big]_{k,\ell=1}^{N_J}$ the stiffness and mass matrices with respect to the ansatz space Z_J, respectively, i.e.

$$a_{k,\ell} = \int_\Omega \nabla\phi_k(x)\nabla\phi_\ell(x)\,\mathrm{d}x, \quad e_{k,\ell} = \int_\Omega \phi_k(x)\phi_\ell(x)\,\mathrm{d}x, \tag{20}$$

we obtain the following discrete representation for (19):

$$(A_J \otimes E_J + E_J \otimes A_J)\, p_{J,J}.$$

Since the right-hand side is a rank-1 function, compare (11), it can simply be computed in accordance with

$$Q_J = q_J \otimes q_J \text{ where } q_J = [q_k]_{k=1}^{N_J} \text{ and } q_k = \int_\Omega c(x)\phi_k(x)\,\mathrm{d}x.$$

4.3. **Nonlinear part.** The nonlinear part of the Riccati equation (16) reads

$$\int_{\Omega\times\Omega}\int_\Gamma \frac{\partial p}{\partial\nu_\zeta}(x,\zeta)\frac{\partial p}{\partial\nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta\,\varphi(x,\xi)\,\mathrm{d}(x,\xi) \quad \text{for all } \varphi(x,\xi) = \phi_{k_1}(x)\phi_{k_2}(\xi). \tag{21}$$

We plug in the ansatz (18) for the Riccati kernel $p(x,\xi)$ into (21) and consider first the integral over the boundary Γ. We find

$$\begin{aligned}&\int_\Gamma \frac{\partial p}{\partial\nu_\zeta}(x,\zeta)\frac{\partial p}{\partial\nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta \\ &\quad= \sum_{i_1,j_2=1}^{N_J} \phi_{i_1}(x)\phi_{j_2}(\xi)\sum_{i_2=1}^{N_J} p_{i_1,i_2}\sum_{j_1=1}^{N_J} p_{j_1,j_2}\int_\Gamma \frac{\partial\phi_{i_2}}{\partial\nu_\zeta}(\zeta)\frac{\partial\phi_{j_1}}{\partial\nu_\zeta}(\zeta)\,\mathrm{d}\Gamma_\zeta.\end{aligned}$$

Hence, defining the matrix $B_J = \big[B_{k,\ell}\big]_{k,\ell=1}^{N_J} \in \mathbb{R}^{N_J\times N_J}$ with

$$b_{k,\ell} \coloneqq \int_\Gamma \frac{\partial\phi_k}{\partial\nu_\zeta}(\zeta)\frac{\partial\phi_\ell}{\partial\nu_\zeta}(\zeta)\,\mathrm{d}\Gamma_\zeta$$

and setting

$$p_{\bullet,\ell} \coloneqq \big[p_{1,\ell},\dots,p_{N_J,\ell}\big]^\intercal \text{ and } p_{\ell,\bullet} \coloneqq \big[p_{\ell,1},\dots,p_{\ell,N_J}\big]^\intercal \text{ for } \ell = 1,\dots,N_J,$$

we conclude

$$\int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta)\frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta = \sum_{i_1,j_2=1}^{N_J} \phi_{i_1}(x)\phi_{j_2}(\xi)\, p^T_{i_1,\bullet} B_J p_{\bullet,j_2}. \tag{22}$$

With this intermediate result, we can investigate the evaluation of the full expression (21):

$$\begin{aligned}\int_{\Omega\times\Omega}\int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta)\frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta\,\varphi(x,\xi)\,\mathrm{d}(x,\xi)\\ = \sum_{i_1,j_2=1}^{N_J} p^T_{i_1,\bullet} B_J p_{\bullet,j_2}\int_{\Omega\times\Omega}\phi_{i_1}(x)\phi_{j_2}(\xi)\phi_{k_1}(x)\phi_{k_2}(\xi)\,\mathrm{d}(x,\xi).\end{aligned}$$

Setting

$$r_{i_1,i_2} \coloneqq p^T_{i_1,\bullet} B_J p_{\bullet,i_2},$$

we can interpret this term as multiplication of the vector

$$r_{J,J} \coloneqq \left[r_{1,1},\dots,r_{N_J,1},r_{1,2},\dots,r_{N_J,N_J}\right]^\intercal$$

with the row corresponding to the test function $\phi_{k_1}\otimes\phi_{k_2}$ of the matrix $E_J\otimes E_J$, where E_J is the mass matrix as defined in (20). Thus, the discretization of the non-linear part leads to

$$(E_J\otimes E_J) r_{J,J}. \tag{23}$$

A slightly different representation in terms of matrices can be obtained by setting

$$P_J \coloneqq \left[p_{k,\ell}\right]_{k,\ell=1}^{N_J} \in \mathbb{R}^{N_J\times N_J}.$$

We first can write

$$\left[p^T_{i_1,\bullet} B_J p_{\bullet,j_2}\right]_{i_1,j_2=1}^{N_J} = P_J B_J P_J, \tag{24}$$

and, in view of (23), we get

$$\left[\int_{\Omega\times\Omega}\int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta)\frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta\,\phi_{k_1}(x)\phi_{k_2}(\xi)\,\mathrm{d}(x,\xi)\right]_{k_1,k_2=1}^{N_J} = E_J P_J B_J P_J E_J.$$

This expression corresponds to the usual discretization of the ARE.

Theorem 4.1. *The computational cost of evaluating the Riccati-IDE discretized by the finite element method are of the order* $\mathcal{O}(N_J^2 N_J^{\frac{d-1}{d}})$.

Proof. The computational cost are dominated by the evaluation of the quadratic term. Here, we have to evaluate a matrix product $P_J B_J P_J$, whereat P_J is a dense matrix having N_J^2-matrix coefficients. The matrix B_J consists of integrals of normal derivatives on the boundary Γ of which only $\mathcal{O}(N_J^{\frac{d-1}{d}})$ do not vanish. Due to locality of the finite element basis and of the normal derivative operator, B_J has $\mathcal{O}(N_J^{\frac{d-1}{d}})$ non-zero entries. Making use of this observation we can evaluate the inner part of (24) with complexity $\mathcal{O}(N_J^{\frac{d-1}{d}})$ for each fixed i_1, j_2. Therefore, the over-all cost amount to $\mathcal{O}(N_J^2 N_J^{\frac{d-1}{d}})$. □

4.4. **Newton's method.** The Riccati-IDE is a non-linear equation with quadratic non-linearity. Thus, to find a solution, we have to apply some iterative scheme. To this end, we use Newton's method as suggested in e.g. [27].

We first introduce the following notation to simplify the presentation. The linear part of the Riccati-IDE (16) is given by the Laplace operator on $\Omega \times \Omega$. We set

$$\mathcal{R}_L : p \mapsto \left[\varphi \mapsto \int_{\Omega\times\Omega} \nabla p(x,\xi) \nabla \varphi(x,\xi) \,\mathrm{d}(x,\xi) \right]. \tag{25}$$

The quadratic part is

$$\mathcal{R}_{NL} : p \mapsto \left[\varphi \mapsto \int_{\Omega\times\Omega} \int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta) \frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi) \,\mathrm{d}\Gamma_\zeta \, \varphi(x,\xi) \,\mathrm{d}(x,\xi) \right].$$

Finally, the right-hand side can be written as

$$\mathcal{Q} : q \mapsto \left[\varphi \mapsto \int_{\Omega\times\Omega} q(x,\xi) \varphi(x,\xi) \,\mathrm{d}(x,\xi) \right].$$

With these operators at hand, we can write the Riccati-IDE as

$$\mathcal{R}_L(p) - \mathcal{R}_{NL}(p) + \mathcal{Q} = 0.$$

Applying the Newton's method to this equation results in

$$D(\mathcal{R}_L - \mathcal{R}_{NL})[p^{(i)}]\big(p^{(i+1)} - p^{(i)}\big) = -\big(\mathcal{R}_L(p^{(i)}) - \mathcal{R}_{NL}(p^{(i)}) + \mathcal{Q}\big),$$

where D denotes the Fréchet derivative and i the iteration index of the Newton's method.

The Fréchet derivative of a linear operator is the operator itself, i.e. we obtain

$$D\mathcal{R}_L[g](h) = \mathcal{R}_L(h),$$

while the Fréchet derivative of the non-linear part is given by

$$D\mathcal{R}_{NL}[g](h) = \left[\varphi \mapsto \int_{\Omega\times\Omega} \int_\Gamma \frac{\partial g}{\partial \nu_\zeta}(x,\zeta) \frac{\partial h}{\partial \nu_\zeta}(\zeta,\xi) + \frac{\partial h}{\partial \nu_\zeta}(x,\zeta) \frac{\partial g}{\partial \nu_\zeta}(\zeta,\xi) \,\mathrm{d}\Gamma_\zeta \, \varphi(x,\xi) \,\mathrm{d}(x,\xi) \right].$$

Therefore, for Newton's method in the i-th iteration, we seek the new iterate $p^{(i+1)}$ such that

$$\big(\mathcal{R}_L - D\mathcal{R}_{NL}[p^{(i)}]\big)(p^{(i+1)}) = -\big(\mathcal{R}_{NL}(p^{(i)}) + \mathcal{Q}\big), \quad i = 1, 2, \dots. \tag{26}$$

The discrete version of Newton's method (26) is the Sylvester type equation of the form

$$(EP_J^{(i)} B_J - A_J) P_J^{(i+1)} E_J + E_J P_J^{(i+1)} (B_J P_J^{(i)} E_J - A_J) = E_J P_J^{(i)} B_J P_J^{(i)} E_J + Q_J.$$

Notice that, in accordance with Theorem 4.1, each iteration of Newton's method can be realized within $\mathcal{O}(N_J^2 N_J^{\frac{d-1}{d}})$ cost if an optimal preconditioner like the multigrid method is used. Therefore, the over-all cost are $\mathcal{O}(N_{\mathrm{iter}} N_J^2 N_J^{\frac{d-1}{d}})$, where N_{iter} denotes the number of iterations used by Newton's method.

5. Sparse grid discretization

Sparse grids are a numerical discretization approach, which is especially of interest for high dimensional problems. In this section, we intend to discretize and evaluate the Riccati-IDE (19) in a sparse grid space. A detailed presentation and introduction to sparse grids can be found in [1, 7, 12, 14, 36], see also [6, 18, p. 260, 19, p. 280, 20, 21, 22]. This section recalls the main ideas, where the representation follows [40].

5.1. **Discretization by sparse grids.** As in Section 4, we consider Hilbert spaces Z and V with $Z \otimes Z \subset V$. Suppose we are given a nested sequence of finite dimensional subspaces Z_j of Z, that is

$$Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_J \subset Z.$$

We are going to construct a finite dimensional subspace of V, which will be our ansatz respectively test space later, upon the spaces Z_j. In accordance with [7, 14, 21], let us introduce hierarchical difference spaces W_j of dimension $N_j = \dim W_j$ via

$$W_j \coloneqq Z_j \ominus Z_{j-1},$$

where we set $Z_{-1} \coloneqq \{0\}$. We shall assume that N_j behaves like an increasing geometric sequence, which is for example the case if the sequence $\{Z_j\}$ is constructed from dyadic subdivisions of a given coarse grid triangulation or tetrahedralization of the underlying domain.

For the multi-index $\mathbf{j} = (j_1, j_2)$, let $W_{\mathbf{j}} = W_{j_1,j_2}$ denote the tensor product of two spaces W_{j_1} and W_{j_2}

$$W_{\mathbf{j}} \coloneqq W_{j_1} \otimes W_{j_2} = \left(Z_{j_1} \ominus Z_{j_1-1}\right) \otimes \left(Z_{j_2} \ominus Z_{j_2-1}\right),$$

where it obviously holds $N_{\mathbf{j}} \coloneqq \dim W_{\mathbf{j}} = N_{j_1} N_{j_2}$. With these spaces at hand, we can write the full tensor product space $V_{J,J}$ from (17) also as a direct sum of spaces $W_{\mathbf{j}}$

$$V_{J,J} = \bigoplus_{0 \leqslant j_1, j_2 \leqslant J} W_{j_1,j_2} = \bigoplus_{0 \leqslant \|\mathbf{j}\|_\infty \leqslant J} W_{\mathbf{j}}. \tag{27}$$

The idea of a *sparse grid* is to consider now only those basis function in the space $V_{J,J}$, which have a large contribution to the representation of an interpolated function $f \in V$, cf. [7, 14]. We denote the sparse grid function space with $\widehat{V}_{J,J}$ and give the following formal definition

$$\widehat{V}_{J,J} \coloneqq \bigoplus_{0 \leqslant j_1 + j_2 \leqslant J} W_{j_1,j_2} = \bigoplus_{0 \leqslant \|\mathbf{j}\|_1 \leqslant J} W_{\mathbf{j}}. \tag{28}$$

From the representation (28) we infer that $\widehat{V}_{J,J}$ consists only of hierarchical difference spaces with $j_1 + j_2 \leqslant J$. This construction leads to the relation

$$\widehat{N}_{J,J} \coloneqq \dim \widehat{V}_{J,J} = \mathcal{O}(N_J \log N_J).$$

In general, for sparse grids on m-fold tensor product spaces, there holds $\widehat{N}_{J,J} \coloneqq \dim \widehat{V}_{J,J} = \mathcal{O}(N_J \log N_J^{m-1})$ while essentially no approximation power is lost provided that the function to be approximated exhibits extra smoothness in terms of bounded mixed derivatives. In other words, the exponential dependency is only in the $\log N_J$ factor, which substantially reduces the dimension of of the sparse grid space compared to the full grid.

We proceed analogously to Section 4 and discretize the Riccati kernel in the sparse grid space $\widehat{V}_{J,J}$. To this end, we assume the space Z_J to be spanned by some hierarchical basis $\Phi_J \coloneqq \{\phi_i\}_{i=1}^{N_J}$, i.e. the spaces Z_j are spanned by subsets of Φ_J. Let us denote by $\delta(j) \subset \{1,\dots,N_J\}$ the index set of the basis functions which span the difference space W_j, i.e.

$$W_j = \operatorname{span}\{\phi_i \in \Phi_J : i \in \delta(j)\}.$$

In what follows, we will include for sake of clearness of representation the level in the notation, i.e., we will write $\phi_{j,k}$ instead of ϕ_k for $k \in \delta(j)$.

Furthermore, for $W_{\mathbf{j}} = W_{j_1} \otimes W_{j_2}$ we set $\delta(\mathbf{j}) \coloneqq \delta(j_1) \times \delta(j_2)$. Thus, the ansatz $\widehat{p}$ for the Riccati kernel reads

$$\widehat{p}(x,\xi) = \sum_{\|\mathbf{j}\|_1 \leqslant J} \sum_{\mathbf{k} \in \delta(\mathbf{j})} p_{\mathbf{j},\mathbf{k}} \varphi_{\mathbf{j},\mathbf{k}}(x,\xi) \in \widehat{V}_{J,J}, \tag{29}$$

where we abbreviated $\varphi_{\mathbf{j},\mathbf{k}} = \phi_{j_1,k_1} \otimes \phi_{j_2,k_2} \in W_{\mathbf{j}}$. The vector $\widehat{p}_{J,J} \in \mathbb{R}^{\widehat{N}_{J,J}}$ of coefficients takes the form

$$\widehat{p}_{J,J} \coloneqq \big[p_{\mathbf{j}}\big]_{\|\mathbf{j}\|_1 \leqslant J},$$

where $p_{\mathbf{j}} \in \mathbb{R}^{N_{\mathbf{j}}}$ are the coefficients vectors corresponding to the spaces $W_{\mathbf{j}}$, i.e.

$$p_{\mathbf{j}} \coloneqq \big[p_{\mathbf{j},\mathbf{k}}\big]_{\mathbf{k} \in \delta(\mathbf{j})}.$$

5.2. **Linear part.** First, we shall be concerned with the linear part of the Riccati-IDE (16). It is possible to consider the application of a linear operator L on V to a function from the sparse grid space $\widehat{V}_{J,J}$ in a rather abstract setting. Note that the spaces Z_j do not even need to be nested. The only additional assumption, besides (28), is of the operator L to be a tensor product operator. This means that L must be an extension from $Z \otimes Z$ to V of a tensor product of operators S_1, S_2 acting on Z, i.e. $L|_{Z\otimes Z} = S_1 \otimes S_2$, see e.g. [19, p. 72] for tensor product operators. This extension is unique if $Z \otimes Z$ is a dense subspace of V and $S_1 \otimes S_2$ is continuous on $Z \otimes Z$, see [18, p. 122, 39, p. 48]. Together with the definition (28), the assumption on L of being a tensor product operator leads to a block tensor structure of the discretization matrix of L with respect to the space $\widehat{V}_{J,J}$, compare [40]. The block tensor structure, in turn, can be utlized for fast matrix–vector multiplication.

Provided the operators S_1, S_2 can be evaluated with linear complexity $\mathcal{O}(N_j)$ on the spaces Z_j, the product operator L can be applied to an element of $\widehat{V}_{J,J}$ with $\mathcal{O}(N_J \log N_J)$ operations, cf. [40]. The algorithm for the evaluation of the matrix-vector product in the space $\widehat{V}_{J,J}$ is called UniDir, cf. [7, 40]. Algorithms which employ similar techniques have been developed in [20, 21, 22].

The linear part of the Riccati-IDE (16) is given by the Laplace-operator on $\Omega \times \Omega$

$$\Delta : H_0^1(\Omega\times\Omega) \to H^{-1}(\Omega\times\Omega), \quad p \mapsto \left[\varphi \mapsto \int_{\Omega\times\Omega} \nabla p(x,\xi) \nabla \varphi(x,\xi)\,\mathrm{d}(x,\xi)\right].$$

Here we have

$$H_0^1(\Omega) \otimes H_0^1(\Omega) \subset H_0^1(\Omega\times\Omega),$$

and $H_0^1(\Omega) \otimes H_0^1(\Omega)$ is dense in $H_0^1(\Omega\times\Omega)$, cf. [19, p. 103]. This means that we can write

$$\Delta|_{H_0^1(\Omega)\otimes H_0^1(\Omega)} = \Delta_x \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_\xi : H_0^1(\Omega) \otimes H_0^1(\Omega) \to H^{-1}(\Omega\times\Omega),$$

with

$$\Delta_x, \Delta_\xi : H_0^1(\Omega) \to H^{-1}(\Omega), \quad v \mapsto \left[w \mapsto \int_\Omega \nabla v \nabla w \,\mathrm{d}x \right],$$

and

$$\mathrm{Id} : H_0^1(\Omega) \to H^{-1}(\Omega), \quad v \mapsto \left[w \mapsto \int_\Omega v w \,\mathrm{d}x \right], \tag{30}$$

(see also Section 4.2), and Δ is a sum of tensor product operators, as required by the UniDir algorithm.

Besides the tensor product structure, we have to ensure that Δ_x, Δ_ξ, and Id can be evaluated with linear complexity on the discretization spaces Z_j. There are different examples of appropriate sequences $\{Z_j\}$ and corresponding bases Φ_j, like e.g. hierachical bases ([7]), wavelets ([10]), multilevel frames ([21, 40]), or polynomials of different degrees ([1, 7]). In this article, we will take Z_j spanned by the hierarchical basis of standard hat functions (cf. [7, 21]). We provide an exact definition in Section 5.3.

Thus, albeit the discretization of the term (19) with sparse grids leads to a matrix which is not sparse (cf. [21, 40]), the product

$$\left(\widehat{A_J \otimes E_J} + \widehat{E_J \otimes A_J} \right) \widehat{p}_{J,J}$$

can be computed with complexity $\mathcal{O}(N_J \log N_J)$, i.e. linear in the number of degrees of freedom of the sparse grid.

5.3. **Nonlinear part.** In general, for the evaluation of the non-linear part (21) of the Riccati-IDE, we have to consider the evaluation of a non-linear operator

$$\mathcal{R}_{NL} : p \mapsto \left[\varphi \mapsto \int_{\Omega\times\Omega} \int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta) \frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi) \,\mathrm{d}\Gamma_\zeta \, \varphi(x,\xi) \,\mathrm{d}(x,\xi) \right].$$

We can apply some quadrature rule to calculate the boundary integral, i.e. we approximate the operator

$$p(x,\xi) \mapsto \int_\Gamma \frac{\partial p}{\partial \nu_\zeta}(x,\zeta) \frac{\partial p}{\partial \nu_\zeta}(\zeta,\xi) \,\mathrm{d}\Gamma_\zeta$$

by

$$p(x,\xi) \mapsto \sum_{s=1}^{S} w_s \frac{\partial p}{\partial \nu_\zeta}(x,\zeta_s) \frac{\partial p}{\partial \nu_\zeta}(\zeta_s,\xi) =: \sum_{s=1}^{S} \widetilde{p}_s(x) \widetilde{p}_s(\xi) =: r(x,\xi), \tag{31}$$

where $r \in H_0^1(\Omega) \otimes H_0^1(\Omega)$ is a finite sum of tensor products. Note that, e.g. for the discretization of $p(x,\xi)$ by hat functions, the integral can be evaluated exactly by an appropriate quadrature rule. However, the discrete output associated with the function r lives on the full tensor product space.

Let us consider the evaluation of (31) for the ansatz (29)

$$\begin{aligned} \frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,\zeta_s) &= \frac{\partial}{\partial \nu_\zeta} \sum_{\|\mathbf{j}\|_1 \leqslant J} \sum_{\mathbf{k} \in \delta(\mathbf{j})} p_{\mathbf{j},\mathbf{k}} \varphi_{\mathbf{j},\mathbf{k}}(x,\zeta_s) \\ &= \sum_{j_1 \leqslant J} \sum_{k_1 \in \delta(j_1)} \phi_{j_1,k_1}(x) \sum_{j_2 \leqslant J - j_1} \sum_{k_2 \in \delta(j_2)} p_{\mathbf{j},\mathbf{k}} \frac{\partial \phi_{j_2,k_2}}{\partial \nu_\zeta}(\zeta_s). \end{aligned} \tag{32}$$

We set

$$\widetilde{p}_{j_1,k_1}(\zeta_s) \coloneqq \sum_{j_2 \leqslant J-j_1} \sum_{k_2 \in \delta(j_2)} p_{\mathbf{j},\mathbf{k}} \frac{\partial \phi_{j_2,k_2}}{\partial \nu_\zeta}(\zeta_s), \quad j_1 = 1,\dots,J,\ k_1 \in \delta(j_1).$$

and

$$\widetilde{p}_{j_1}(\zeta_s) \coloneqq \big[\widetilde{p}_{j_1,k_1}(\zeta_s)\big]_{k_1 \in \delta(j_1)} \in \mathbb{R}^{N_{j_1}}.$$

The expressions for $\widetilde{p}_{j_1,k_1}$ can be calculated by a point evaluation of the derivatives of the ansatz functions. Due to the hierarchical sorting of the basis, this operation is of the complexity $\mathcal{O}(J-j_1)$. The overall complexity for the evaluation of (32) is therefore

$$\sum_{j_1 \leqslant J} \#\delta(j_1) \cdot (J-j_1) \lesssim \sum_{j_1 \leqslant J} 2^{j_1 d} J = \mathcal{O}(N_J \log N_J)$$

and, hence, $\mathcal{O}(SN_J \log N_J)$ for the complete quadrature of the boundary integral (31).

In the next step, we would compute the linear combination

$$\sum_{s=1}^{S} \widetilde{p}_s(x)\widetilde{p}_s(\xi),$$

where

$$\widetilde{p}_s(x) = \sum_{j_1 \leqslant J} \sum_{k_1 \in \delta(j_1)} \widetilde{p}_{j_1,k_1} \phi_{j_1,k_1}(x),$$
$$\widetilde{p}_s(\xi) = \sum_{j_2 \leqslant J} \sum_{k_2 \in \delta(j_2)} \widetilde{p}_{j_2,k_2} \phi_{j_2,k_2}(\xi),$$

are full grid functions on Ω. The evaluation of the tensor products

$$\widetilde{p}_s(x)\widetilde{p}_s(\xi), \quad s = 1,\dots,S,$$

is of complexity $\mathcal{O}(N_J^2)$ each. However, we can first apply the operator $\mathrm{Id} \otimes \mathrm{Id}$ with the test space is $\widehat{V}_{J,J}$ and the ansatz space $V_{J,J}$. In this case, utilizing the tensor product structure

$$(\mathrm{Id} \otimes \mathrm{Id})\, \widetilde{p}_s(x)\widetilde{p}_s(\xi) = \big(\mathrm{Id}\, \widetilde{p}_s(x)\big) \otimes \big(\mathrm{Id}\, \widetilde{p}_s(\xi)\big),$$

we need $\mathcal{O}(N_J \log N_J)$ operations for the evaluation. This means that we obtain the overall complexity of $\mathcal{O}(SN_J \log N_J)$ for the evaluation of the nonlinear part $\mathcal{R}_{NL}$ of the Riccati equation with sparse grids ansatz. In view of $S \sim N_J^{\frac{d-1}{d}}$, since we integrate only over the boundary Γ of Ω, we end up with the computational complexity $\mathcal{O}\big(N_J N_J^{\frac{d-1}{d}} \log N_J\big)$ for evaluating the Riccati-IDE. This means that we save essentially one order in N_J in both, memory requirement and computation time, compared to the traditional finite element discretization from the previous section:

Theorem 5.1. *The computational cost of evaluating the Riccati-IDE discretized by the sparse grid method are of the order $\mathcal{O}(N_J N_J^{\frac{d-1}{d}} \log N_J)$ while the memory requirement is of the order $\mathcal{O}(N_J \log N_J)$.*

Remark 5.2. *1. The realization of Newton's method based on the above algorithms is straightforward, compare Section 4.4. Especially, the over-all cost for computing the optimal kernel function $\widehat{p}$ are $\mathcal{O}(N_{iter}N_J N_J^{\frac{d-1}{d}} \log N)$, where N_{iter} denotes the number of iterations used by Newton's method.*

2. The bottle-neck of the presented sparse grid discretization is the evaluation of the non-linear term $\mathcal{R}_{NL}$, which does not scale linearly. A much more involved algorithm is able to evaluate $\mathcal{R}_{NL}$ in complexity $\mathcal{O}\big(N_J N_J^{\frac{d-1}{2d}}\big)$. This is still not of linear complexity but is essentially the square root of the cost the finite element method has.

3. The discretization of the Riccati-IDE has been performed in an exact way, meaning that we compute the exact Galerkin system. Instead, one could also evaluate $\mathcal{R}_{NL}$ in an approximate way, reducing the over-all complexity further. This would introduce a consistency error which, however, would not matter if it is of the same order than the discretization error.

6. Numerical results

We shall present numerical results of the sparse grid discretization of the Riccati-IDE (16). To this end, we assume that $\Omega = (0,1) \subset \mathbb{R}$ is one-dimensional. Hence, $\Omega \times \Omega = (0,1)^2$ is the unit square.

6.1. **Discretization space.**

The sparse grid spaces $\widehat{V}_{J,J}$ on $(0,1)^2$ are constructed by piecewise linear hat functions (see e.g. [7]). Given a multi-index $\mathbf{j} = (j_1, j_2)$, we write

$$h_{\mathbf{j}} = (h_{j_1}, h_{j_2}) = (2^{j_1}, 2^{j_2})$$

for the tuple of mesh parameters, which represents the spatial resolution in every dimension. We define the following points

$$x_{\mathbf{j},\mathbf{k}} = (x_{j_1,k_1}, x_{j_2,k_2}), \quad x_{j_t,k_t} = k_t \cdot h_{j_t}, \quad t = 1,2,$$

and consider the mesh, i.e., a collection of spatial points, $\widehat{\Omega}_{J,J} \subset \Omega$

$$\widehat{\Omega}_{J,J} \coloneqq \left\{ x_{\mathbf{j},\mathbf{k}} \in \mathbb{R}^2 : \|\mathbf{j}\|_1 \leqslant J,\ \mathbf{k} \in \delta(j) \right\}.$$

The multi-index $\mathbf{j}$ is also referred to as level and $\mathbf{k}$ defines the position of the point $x_{\mathbf{j},\mathbf{k}}$.

Next, we introduce a set of basis functions defined on Ω which span the discrete spaces Z_j. Let us start with the standard linear hat function

$$\phi(x) = \max\{1 - |x|, 0\}, \quad x \in \mathbb{R}.$$

For level j and index k, we define by translation and dilatetation

$$\phi_{j,k}(x) \coloneqq \phi\left(\frac{x - k \cdot h_j}{h_j}\right) = \phi(2^j x - k).$$

The basis functions on $\Omega \times \Omega$ are the piecewise bilinear hat functions

$$\phi_{\mathbf{j},\mathbf{k}}(\mathbf{x}) \coloneqq \phi_{j_1,k_1}(x_1)\phi_{j_2,k_2}(x_2),$$

where $\mathbf{x} = (x_1, x_2) \in \Omega \times \Omega$.

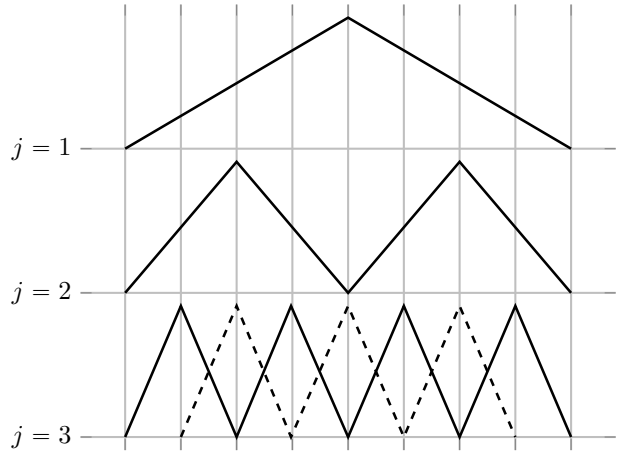

FIGURE 1. Piecewise linear hierarchical basis for levels $j = 1, 2, 3$ (solid) and nodal point basis (dashed).

Due to the homogeneous boundary conditions of the Riccati-IDE (16) for the Dirichlet boundary control case, we omit functions which are not zero in $\partial\Omega = \{0, 1\}$ and consider one-dimensional discrete space

$$Z_j := \operatorname{span}\big\{\phi_{j,k} : k = 1, \dots, 2^j\big\}.$$

The spaces W_j, W_{j_1,j_2}, $\widehat{V}_{J,J}$ are constructed as in Section 5. Figure 1 shows the one-dimensional linear hat functions corresponding to the hierarchical difference spaces W_1, W_2 and W_3 as well as a nodal point basis of level 3.

6.2. **About the quadratic term of the Riccati-IDE.** The evaluation of $\mathcal{R}_{NL}$ simplifies considerably for a one-dimensional domain $\Omega = (0, 1) \subset \mathbb{R}$. In this case, we obtain

$$\int_\Gamma \frac{\partial}{\partial\nu_\zeta} p(x,\zeta) \frac{\partial}{\partial\nu_\zeta} p(\zeta,\xi)\,\mathrm{d}\Gamma_\zeta = \frac{\partial}{\partial\nu_\zeta} p(x,0) \frac{\partial}{\partial\nu_\zeta} p(0,\xi) + \frac{\partial}{\partial\nu_\zeta} p(x,1) \frac{\partial}{\partial\nu_\zeta} p(1,\xi),$$

where the notation

$$\frac{\partial}{\partial\nu_\zeta} p(x,c) := \left(\frac{\partial}{\partial\nu_\zeta} p(x,\zeta)\right)\Big|_{\zeta=c}, \quad \frac{\partial}{\partial\nu_\zeta} p(c,\xi) := \left(\frac{\partial}{\partial\nu_\zeta} p(\zeta,\xi)\right)\Big|_{\zeta=c}$$

is used with $c \in \{0, 1\} = \partial\Omega$. In other words, the scalar product on the space of controls $\mathcal{U}$ is reduced to a finite sum. This means that $\mathcal{U}$ is a finite dimensional space, in particular $\dim \mathcal{U} = 2$.

The operator $\mathcal{R}_{NL}$ can be represented in terms of tensor product operators, i.e., for an elementary tensor $p = p_1 \otimes p_2$, we have

$$\begin{aligned}\mathcal{R}_{NL}(p) &= \int_{\Omega\times\Omega} \left(\frac{\partial p}{\partial \nu_\zeta}(x,0)\frac{\partial p}{\partial \nu_\zeta}(0,\xi) + \frac{\partial p}{\partial \nu_\zeta}(x,1)\frac{\partial p}{\partial \nu_\zeta}(1,\xi) \right) \mathrm{d}(x,\xi) \\ &= \left(\frac{\partial p_2}{\partial \nu_\zeta}(0)\int_\Omega p_1(x)\ \mathrm{d}x\right)\left(\frac{\partial p_1}{\partial \nu_\zeta}(0)\int_\Omega p_2(\xi)\ \mathrm{d}\xi\right) \\ &\quad + \left(\frac{\partial p_2}{\partial \nu_\zeta}(1)\int_\Omega p_1(x)\ \mathrm{d}x\right)\left(\frac{\partial p_1}{\partial \nu_\zeta}(1)\int_\Omega p_2(\xi)\ \mathrm{d}\xi\right).\end{aligned} \tag{33}$$

We proceed with the discretization of (33) and consider exemplary the term stemming from the evaluation of the normal derivative in the point 0:

$$\begin{aligned}\frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,0) &= \frac{\partial}{\partial \nu_\zeta} \sum_{\|j\|_1 \leqslant J} \sum_{\mathbf{k}\in\delta(\mathbf{j})} p_{k_1,k_2}\phi_{k_1}(x)\phi_{k_2}(0) \\ &= \sum_{\|\mathbf{j}\|_1\leqslant J} \sum_{k_1\in\delta(j_1)} \phi_{k_1}(x) \sum_{k_2\in\delta(j_2)} p_{k_1,k_2}\frac{\partial \phi_{k_2}}{\partial \nu_\zeta}(0) \\ &= \sum_{j_1\leqslant J} \sum_{k_1\in\delta(j_1)} \phi_{k_1}(x) \sum_{j_2=1}^{j_1} \sum_{k_2\in\delta(j_2)} p_{k_1,k_2}\frac{\partial \phi_{k_2}}{\partial \nu_\zeta}(0).\end{aligned}$$

This computation will give thus regular grid functions:

$$\frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,0), \frac{\partial \widehat{p}}{\partial \nu_\zeta}(0,\xi) \in Z_J.$$

Likewise, we can obtain

$$\frac{\partial}{\partial \nu_\zeta}\widehat{p}(x,1), \frac{\partial}{\partial \nu_\zeta}\widehat{p}(1,\xi) \in Z_J$$

by the evaluation of the normal derivative in the point 1.

In the second step, we have to apply the identity operator to the tensor products

$$\frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,0) \otimes \frac{\partial \widehat{p}}{\partial \nu_\zeta}(0,\xi), \quad \frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,1) \otimes \frac{\partial \widehat{p}}{\partial \nu_\zeta}(1,\xi),$$

i.e., we compute

$$\mathcal{R}_{NL}(p) = (\mathrm{Id}\otimes\mathrm{Id})\left(\frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,0) \otimes \frac{\partial \widehat{p}}{\partial \nu_\zeta}(0,\xi) + \frac{\partial \widehat{p}}{\partial \nu_\zeta}(x,1) \otimes \frac{\partial \widehat{p}}{\partial \nu_\zeta}(1,\xi) \right),$$

which can be done with complexity $\mathcal{O}(N_J \log N_J)$. Hence, the overall complexity is of $\mathcal{O}(N_J \log N_J)$, which is consistent to the estimate obtained in Theorem 5.1 for $d = 1$.

6.3. **Computation time.** First of all, we shall confirm the expected linear complexity of $\mathcal{O}(N_J \log N_J)$ by measuring computation times. To this end, the algorithm for the solution of the Riccati equation is implemented based on a general sparse grids library SG++, see [36, 38]. We measure the computation times with `boost::timer` as an estimation for complexity. In particular, we consider the mean value over 20,000 measurements of the computation time for the evaluation of the quadratic part $\mathcal{R}_{NL}(p)$ from (33).

Figure 2 shows the binary logarithm of the measured time against the level J of the sparse grid. Here, we observe a nearly linear growth of computation time

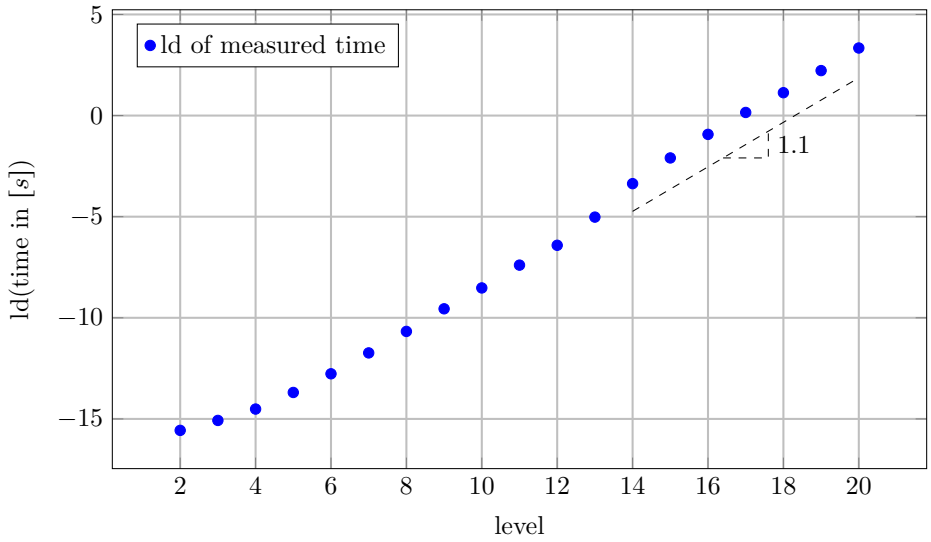

FIGURE 2. Measurement of the computation time for the quadratic term of the Riccati equation.

with increasing level. A more detailed information is presented in Table 1. Therein, besides the measured time, we calculate the complexity rates

$$\eta_i(t) = \mathrm{ld}\left(\frac{t_i}{t_{i-1}}\right) = \mathrm{ld}\left(\frac{2^i i}{2^{i-1}(i-1)}\right) = 1 + \mathrm{ld}\left(\frac{i}{i-1}\right) \overset{i\to\infty}{\longrightarrow} 1,$$

where t_i is the computation time on level i, and ld denotes the binary logarithm.

TABLE 1. Mean value over 20,000 measurements of the computational time and corresponding complexity rates $\eta_i(t) = \mathrm{ld}(t_i/t_{i-1})$ for the quadratic term of the Riccati-IDE.

level	time[s]	$\eta_i(t)$	level	time[s]	$\eta_i(t)$	level	time[s]	$\eta_i(t)$
2	2.061_{-5}	$\star$	9	1.329_{-3}	1.12	16	5.249_{-1}	1.17
3	2.898_{-5}	0.49	10	2.718_{-3}	1.03	17	1.112_{0}	1.08
4	4.281_{-5}	0.56	11	5.942_{-3}	1.13	18	2.191_{0}	0.98
5	7.557_{-5}	0.82	12	1.172_{-2}	0.98	19	4.672_{0}	1.09
6	1.433_{-4}	0.92	13	3.081_{-2}	1.39	20	1.013_{1}	1.12
7	2.932_{-4}	1.03	14	9.700_{-2}	1.65	$\star$	$\star$	$\star$
8	6.123_{-4}	1.06	15	2.341_{-1}	1.27			

6.4. **Experiment 1.** In the first numerical experiment, we consider the following function $q(x,\xi)$

$$q(x,\xi) = (1-|2x-1|)(1-|2\xi-1|),$$

which is illustrated in Figure 3, besides the approximation of the corresponding Riccati kernel.

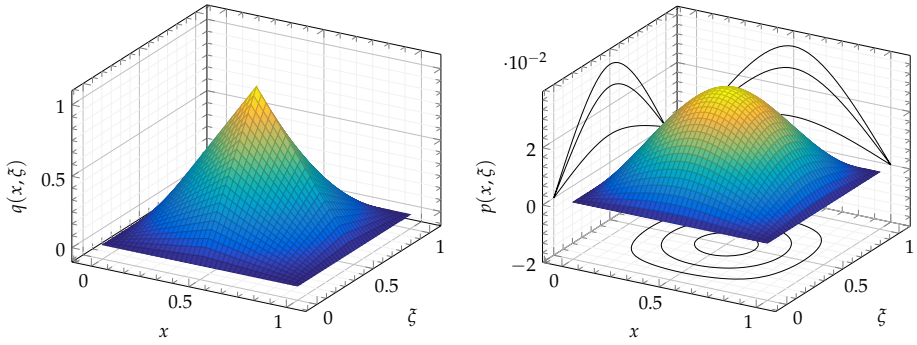

FIGURE 3. Right-hand side $q(x,\xi)$ (left) and an approximation of the associated Riccati kernel $p(x,\xi)$ (right) for the first numerical experiment.

The configuration of the numerical experiment is as follows. The tolerance on the ℓ_2-norm of the residuum for the Newton method is set to 10^{-12}. We obtain a reference solution on a regular grid with $12,801 \times 12,801$ points and measure the error on a regular grid of 201×201 points. The error is measured with the following L_2 respectively L_∞ estimators

$$e_2^p := \frac{\|p_{\text{ref}} - p_{\text{sg}}\|_2}{\sqrt{|X_{\text{eval}}|}}, \quad e_\infty^p := \|p_{\text{ref}} - p_{\text{sg}}\|_\infty.$$

The theoretical convergence for a sparse grids approximation of functions with bounded second mixed derivatives, i.e. elements of $H^2(\Omega) \otimes H^2(\Omega)$, is $\mathcal{O}(2^{-2J}J)$, whereby J denotes the level of discretization, as described in Section 5. In the first numerical experiment, we observe nearly this rate.

In Figure 4, logarithms of both error estimators are plotted against the level. Detailed information on error as well as convergence rates is given in Table 2. Expected theoretical value for the convergence rates is

$$\rho_i(e) = \operatorname{ld}\left(\frac{e_{i-1}}{e_i}\right) = \operatorname{ld}\left(2^2\left(1-\frac{1}{i}\right)\right) = 2 + \operatorname{ld}\left(1-\frac{1}{i}\right) \overset{i\to\infty}{\longrightarrow} 2,$$

where e_i is the error for level i.

TABLE 2. Estimations e_2^p of the $L^2(\Omega)$ and e_∞^p of the $L^\infty(\Omega)$ errors and the corresponding convergence rates $\rho_i(e) = \operatorname{ld}(e_{i-1}/e_i)$ for the first numerical experiment.

level	e_2^p	$\rho_i(e_2)$	e_∞^p	$\rho_i(e_\infty)$	level	e_2^p	$\rho_i(e_2)$	e_∞^p	$\rho_i(e_\infty)$
2	9.49_{-4}	$\star$	2.94_{-3}	$\star$	8	2.58_{-7}	1.99	2.25_{-6}	1.59
3	2.49_{-4}	1.93	1.03_{-3}	1.51	9	6.52_{-8}	1.99	7.33_{-7}	1.62
4	6.35_{-5}	1.97	3.21_{-4}	1.68	10	1.65_{-8}	1.98	2.31_{-7}	1.67
5	1.61_{-5}	1.98	9.76_{-5}	1.72	11	4.21_{-9}	1.97	7.06_{-8}	1.71
6	4.05_{-6}	1.99	3.06_{-5}	1.67	12	1.11_{-9}	1.92	2.13_{-8}	1.73
7	1.02_{-6}	1.99	6.77_{-6}	2.18	$\star$	$\star$	$\star$	$\star$	$\star$

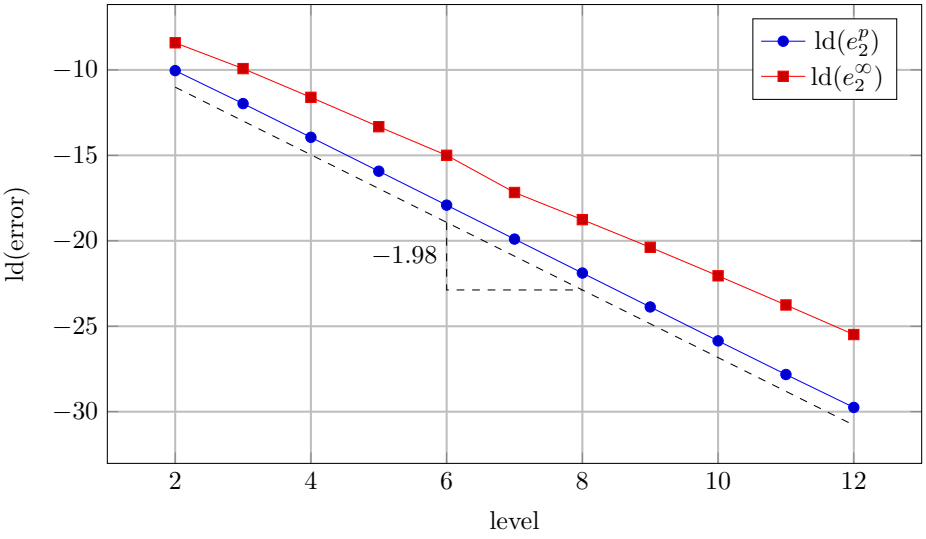

FIGURE 4. Estimations e_2^p of $L^2(\Omega \times \Omega)$ and e_∞^p of $L^\infty(\Omega \times \Omega)$ errors for the first numerical experiment.

6.5. **Experiment 2.** In the second example, the right-hand side $q(x, \xi)$ is given by

$$q(x, \xi) = \chi_{[\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]}(x, \xi).$$

Figure 5 shows the function $q(x, \xi)$ as well as the approximation of the associated Riccati kernel.

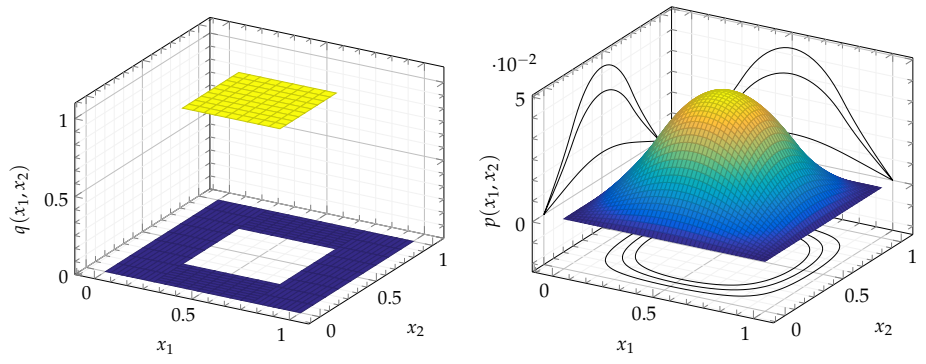

FIGURE 5
Right-hand side $q(x, \xi)$ (left) and an approximation of the associated Riccati kernel $p(x, \xi)$ (right) for the second numerical experiment.

In this experiment, we use the same setting as in the first one. As the plot in Figure 6 illustrates, we do not achieve the convergence rate of 2 in this case, which indicates that the Riccati kernel is not a function with bounded mixed second derivatives. Again, detailed numbers for errors and associated convergence rates ρ are presented in Table 3.

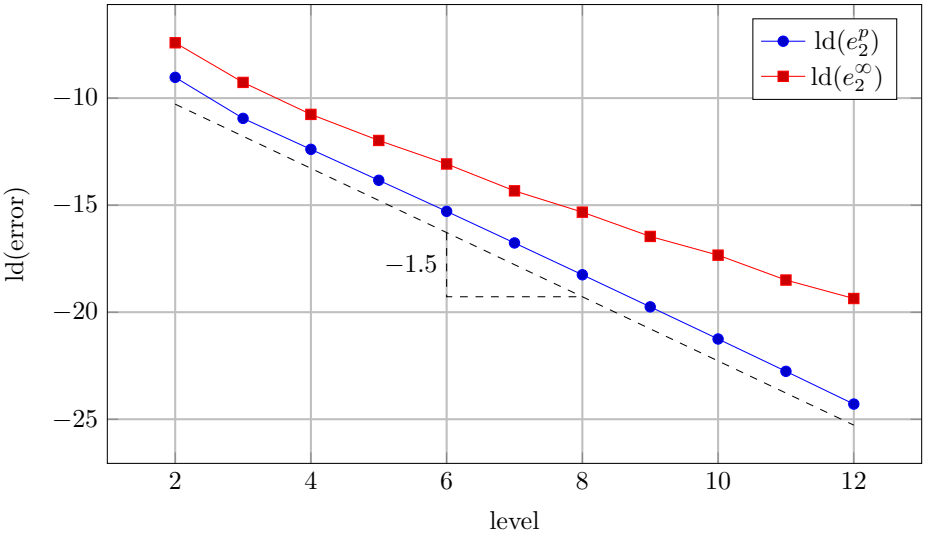

FIGURE 6. Estimations e_2^p of $L^2(\Omega \times \Omega)$ and e_∞^p of $L^\infty(\Omega \times \Omega)$ errors for the second numerical experiment.

TABLE 3. Estimations e_2^p of the $L^2(\Omega)$ and e_∞^p of the $L^\infty(\Omega)$ errors and the corresponding convergence rates $\rho_i(e) = \mathrm{ld}(e_{i-1}/e_i)$ for the second numerical experiment.

level	e_2^p	$\rho_i(e_2)$	e_∞^p	$\rho_i(e_\infty)$	level	e_2^p	$\rho_i(e_2)$	e_∞^p	$\rho_i(e_\infty)$
2	1.91_{-3}	$\star$	5.85_{-3}	$\star$	8	3.20_{-6}	1.49	2.43_{-5}	0.99
3	5.06_{-4}	1.92	1.62_{-3}	1.85	9	1.13_{-6}	1.5	1.11_{-5}	1.13
4	1.86_{-4}	1.45	5.76_{-4}	1.49	10	4.00_{-7}	1.5	6.04_{-6}	0.87
5	6.83_{-5}	1.45	2.48_{-4}	1.22	11	1.41_{-7}	1.51	2.70_{-6}	1.16
6	2.50_{-5}	1.45	1.16_{-4}	1.1	12	4.87_{-8}	1.53	1.49_{-6}	0.86
7	8.97_{-6}	1.48	4.85_{-5}	1.26	$\star$	$\star$	$\star$	$\star$	$\star$

7. CONCLUSION

In the present article, we considered the numerical solution of the algebraic Riccati equation by means of sparse grids. To that end, we did not start with the algebraic Riccati equation but with its continuous counterpart – the Riccati-IDE. This partial differential equation has then been discretized by the Galerkin method with sparse grid ansatz spaces. We have shown that both, memory requirements and computation times, are reduced considerably in comparison with a tensor-product finite element discretization. Nonetheless, future research has to be focus on further speeding-up the computational process.

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