

**Solution of the Poisson Equation  
with a Thin Layer  
of Random Thickness**

M. Dambrine, I. Greff, H. Harbrecht, B. Puig

Institute of Mathematics  
University of Basel  
CH - 4051 Basel  
Switzerland

Preprint No. 2014-18  
December 2014

[www.math.unibas.ch](http://www.math.unibas.ch)

# SOLUTION OF THE POISSON EQUATION WITH A THIN LAYER OF RANDOM THICKNESS

M. DAMBRINE, I. GREFF, H. HARBRECHT, AND B. PUIG

ABSTRACT. The present article is dedicated to the numerical solution of the Poisson equation with a thin layer of different conductivity and of random thickness. We change the boundary condition to transform the boundary value problem given on a random domain into a boundary value problem on a fixed domain. The randomness is then contained in the coefficients of the new boundary condition. This thin coating can be expressed by a random Robin boundary condition which yields a third order accurate solution in the scale parameter of the layer's thickness. Based on the decay of the Karhunen-Loève expansion of the random fluctuations of the layer's thickness, we prove rates of decay of the derivatives of the random solution with respect to the stochastic variable. They are robust in the thickness parameter and enable the use of the quasi Monte-Carlo method or of the anisotropic stochastic collocation method for the computation of the boundary value problem's random solution. Numerical results validate our theoretical findings.

## 1. INTRODUCTION

**1.1. Problem formulation.** Many practical problems in engineering lead to boundary value problems for an unknown function. In this article, we consider uncertainties in the geometric definition of the domain motivated by tolerances in the fabrication processes or in a damaged boundary during the life of a mechanical device. Manufactured or damaged devices are close to a nominal geometry but differ of course from its mathematical definition. Since we are motivated by tolerances, we can make the crucial assumption of the *smallness of the random perturbations*. By identifying domains with their boundary, domains close to the nominal domain  $D$  can be seen as a perturbation in the normal direction of the nominal boundary  $\partial D$ .

The most common approach to study boundary value problems with stochastic inputs is the Monte-Carlo method, see e.g. [19] and the references therein. In many situations, this approach is easy to implement since it only requires a sufficiently large number of samples. However, for boundary value problems on random domains, each sample means a new domain and thus a new mesh, the building of new mass

---

*Key words and phrases.* Thin layer equation, random boundary value problems, random domains.

and stiffness matrices, etc. All these steps are mandatory to compute the quantity of interest. Therefore, the Monte-Carlo method is extremely costly and not so easy to implement in our context. This article is a contribution to the *development of a cheap method to solve boundary values problem in random domains*.

If one considers general deformations without assuming the smallness of the perturbation, one has to mention the work of Canuto and Kozubek [6] based on a fictitious domains approach and the work of Xiu and Tartakovsky [21] based on a mapping point of view, see also [7, 12] for related regularity results. In the same context of small random perturbations around a given domain, one has to mention the work of Harbrecht, Schneider and Schwab [14] based on the use of shape derivatives. Since shape derivatives are in general hard to compute, we will propose here an alternative approach based on approximated boundary conditions.

**1.2. Geometrical setting.** Let us make precise the geometrical situation: a given  $C^\infty$ -domain  $D$  is surrounded by a thin coating layer  $L_\varepsilon$ . The thickness  $h$  of the layer is a smooth, real-valued function which is defined on  $\partial D$ . We make the assumptions that the layer coats  $D$  everywhere and that its characteristic size is a small parameter  $\varepsilon > 0$  so that the layer  $L_\varepsilon$  is described as:

$$L_\varepsilon = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}) : 0 \leq t < \varepsilon h(\mathbf{x}), \mathbf{x} \in \partial D\}.$$

Moreover, there exist nonnegative real numbers  $0 < h_{\min} \leq h_{\max}$  such that

$$h_{\min} \leq h(\mathbf{x}) \leq h_{\max} \text{ for all } \mathbf{x} \in \partial D.$$

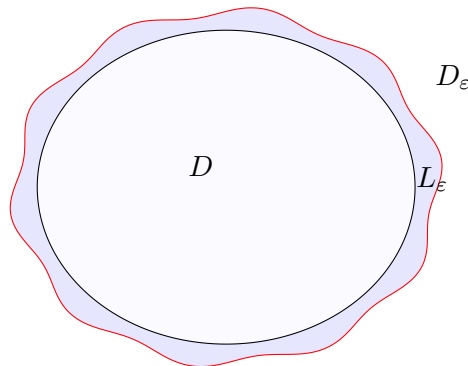


FIGURE 1.1. The geometrical setting – the domain  $D$  and the layer  $L_\varepsilon$ .

Now, we are interested in the numerical resolution of the following model boundary value problem posed in  $D_\varepsilon = D \cup L_\varepsilon$ : for a given function  $f \in L^2(D_\varepsilon)$ , find the

function  $u$  such that

$$(1.1) \quad \begin{cases} -\operatorname{div}(\sigma \nabla u) = f \text{ in } D_\varepsilon, \\ u = 0 \text{ on } \partial D_\varepsilon, \end{cases}$$

where the conductivity  $\sigma$  is piecewise constant, taking the value  $\sigma_0$  in  $D$  and 1 in the layer  $L_\varepsilon$ . This change in conductivity is motivated by the possibility to take into account surface treatments or surface damage.

**1.3. Asymptotic analysis.** In order to efficiently compute a numerical approximation  $u|_D$  of the solution to (1.1), a classical idea is to introduce impedance boundary conditions (see [4, 10], and derived works) to avoid the meshing of the thin layer. The strategy is thus the following: work only in  $D$  and search for a boundary condition on  $\partial D$  so that the solution of the new boundary value problem defined in  $D$  (that is without the thin layer) is a *good* approximation of the restriction to  $D$  of the solution of the real boundary value problem set in  $D_\varepsilon$ .

A heuristic way to derive such a condition is as follows. Consider a point  $\mathbf{x}$  on  $\partial D$ . The thickness  $\varepsilon h$  being small, Taylor's formula provides

$$u(\mathbf{x} + \varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})) \approx u(\mathbf{x}) + \varepsilon h(\mathbf{x})\partial_{\mathbf{n}}u(\mathbf{x})$$

up to a second order error. Since the point  $\mathbf{x} + \varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})$  lays on the boundary  $\partial D_\varepsilon$ , we get  $u(\mathbf{x} + \varepsilon h(\mathbf{x})\mathbf{n}(\mathbf{x})) = 0$ . Hence, a natural choice is to solve the Robin boundary value problem:

$$(1.2) \quad \begin{cases} -\sigma_0 \Delta u^{[1]} = f \text{ in } D, \\ u^{[1]} + \varepsilon h \sigma_0 \partial_{\mathbf{n}} u^{[1]} = 0 \text{ on } \partial D. \end{cases}$$

It is well-known in the literature [1, 5, 18, 20] that indeed this choice is pertinent. There is a constant  $C$ , independent of  $\varepsilon$ , such that

$$(1.3) \quad \|u - u^{[1]}\|_{\mathbf{H}^1(D)} \leq C\varepsilon^2.$$

A more precise but less intuitive approximated boundary value problem is

$$(1.4) \quad \begin{cases} -\sigma_0 \Delta u^{[2]} = f & \text{in } D, \\ \left(1 + \frac{\kappa h \varepsilon}{2}\right) u^{[2]} + \varepsilon \sigma_0 h \partial_{\mathbf{n}} u^{[2]} = \frac{\varepsilon^2 h^2}{2} f & \text{on } \partial D. \end{cases}$$

Of course, this approximation obviously needs additional regularity of the right hand side  $f$  to be well defined. This will be specified later in Subsection 2.2. Then, for the solution of this boundary value problem, there is another constant  $C$ , independent of  $\varepsilon$ , such that

$$(1.5) \quad \|u - u^{[2]}\|_{\mathbf{H}^1(D)} \leq C\varepsilon^3.$$

Proving the error estimates (1.3) and (1.5) requires a careful asymptotic analysis of problem (1.1) as already performed in the previously cited references. Note that the method has been also used for other boundary value conditions and other differential operators. Since we will consider families of random layers, a crucial point for our work is the dependency of the constant  $C$  in terms of the thickness  $h$ . Since this point is not specified in the aforementioned literature, we will present it in Section 2. Our result is that the constants  $C$  appearing in (1.3) and (1.5) depend only on the  $\mathcal{C}^1(\partial D)$ -norm of  $h$ . The proof of this crucial point requires to elaborate on asymptotic expansions of the solution of (1.1) in  $D_\varepsilon$ .

**1.4. Layers of random thickness.** Once the uniform error estimates (1.3) and (1.5) are obtained, we shall consider the situation that the layer's thickness  $h$  is random. To that end, let  $(\Omega, \Sigma, \mathbb{P})$  be a complete probability space and assume that  $h : \partial D \times \Omega \rightarrow \mathbb{R}$  is a process which satisfies the following assumptions:

**(UB)** Uniform boundedness: there exist two nonnegative real numbers  $h_{\min} \leq h_{\max}$  and  $q < 1$  such that the random field

$$h(\mathbf{x}, \omega) = \bar{h}(\mathbf{x}) + \tilde{h}(\mathbf{x}, \omega) \quad \text{with} \quad \bar{h}(\mathbf{x}) = \mathbb{E}(h(\mathbf{x}, \omega))$$

fulfills

$$(1.6) \quad 0 < h_{\min} \leq \bar{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\tilde{h}(\mathbf{x}, \omega)| \leq q|\bar{h}(\mathbf{x})|$$

for all  $\mathbf{x} \in D$  and for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ .

**(UR)** Uniform regularity: for all  $\omega$  in  $\Omega$ , the function  $\mathbf{x} \mapsto h(\mathbf{x}, \omega)$  is  $\mathcal{C}^1$ , that is, the stochastic process  $h$  belongs to the Bochner space  $L_{\mathbb{P}}^2(\Omega, \mathcal{C}^1(\partial D))$ .

Let us recall for the reader's convenience the definition of Bochner spaces. Consider a real number  $p \geq 1$ . Then, for a Banach space  $X$ , the Bochner space  $L_{\mathbb{P}}^p(\Omega, X)$  consists of all functions  $v : \Omega \rightarrow X$  whose norm

$$\|v\|_{L_{\mathbb{P}}^p(\Omega, X)} := \begin{cases} \left( \int_{\Omega} \|v(\cdot, \omega)\|_X^p d\mathbb{P}(\omega) \right)^{1/p}, & p < \infty, \\ \text{ess sup}_{\omega \in \Omega} \|v(\cdot, \omega)\|_X, & p = \infty, \end{cases}$$

is finite. If  $p = 2$  and  $X$  is a Hilbert space, then the Bochner space is isomorphic to the tensor product space  $L_{\mathbb{P}}^2(\Omega) \otimes X$ .

In the sequel, we tacitly assume that the right hand side  $f$  is defined in a sufficiently large hold-all  $\mathcal{D}$  such that  $f \in L^2(D_\varepsilon(\omega))$  holds for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . We then continue with the second order approximation of the solution to the thin layer equation. The reason for this is that it is more accurate without being computationally

much more demanding. We shall thus finally consider the following elliptic partial differential equation with random Robin boundary condition:

$$\left. \begin{aligned} -\sigma_0 \Delta u^{[2]}(\omega) &= f && \text{in } D \\ \partial_{\mathbf{n}} u^{[2]}(\omega) + \left[ \frac{1}{\varepsilon \sigma_0 h(\omega)} + \frac{\kappa}{2\sigma_0} \right] u^{[2]}(\omega) &= \frac{\varepsilon h(\omega)}{2\sigma_0} f && \text{on } \partial D \end{aligned} \right\} \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

**1.5. Road map.** The rest of this article is organized as follows. As mentioned above, the asymptotic analysis for the thin layer is performed in Section 2. Then, in Section 3, we investigate the existence of solutions to this Robin boundary value problem in case of a layer of random thickness and estimate the systematic error committed on the expectation and the variance by this solution. In order to solve the random Robin problem, we assume in Section 4 that the random fluctuations are given in the form of a Karhunen-Loève expansion. Our main result is then Theorem 4.2 which gives precise estimates on the derivatives of the random solution with respect to the stochastic variable. These estimates allow the use of quasi Monte-Carlo methods and anisotropic collocation schemes with convergence rates that are independent of the truncation length of the Karhunen-Loève expansion. Numerical experiments are performed in Section 5 to show the feasibility of our approach and to validate the theoretical findings. The article's conclusion is finally drawn in Section 6.

## 2. APPROXIMATED BOUNDARY CONDITIONS FOR SLOWLY VARYING LAYERS.

**2.1. Asymptotic expansion in  $D$  and in the layer  $L_\varepsilon$ .** Let us introduce the notation and needed objects. For ease of notation, we deal with the bidimensional case and assume that  $D$  is simply connected so that its boundary has a parametrization by the arclength  $s \mapsto \gamma(s)$  defined on the segment  $[0, |\partial D|]$  where  $|\partial D|$  is the perimeter of  $D$ . At the point  $\gamma(s)$ , the unit tangent vector  $\mathbf{t}(s)$  is  $\gamma'(s)$ , the curvature  $\kappa(s)$  is defined by the equality  $\mathbf{n}'(s) = \kappa(s)\mathbf{t}(s)$ . Within these notations, the boundary of  $D$  is parametrized by  $s \mapsto \gamma_h(s) = \gamma(s) + \varepsilon h(s)\mathbf{n}(s)$ . Of course, this parametrization is not by the arclength and the unit tangential and outward normal field are

$$\mathbf{t}_h(s) = \frac{(1 + \varepsilon h \kappa)\mathbf{t}(s) + \varepsilon h' \mathbf{n}(s)}{\sqrt{(1 + \varepsilon h \kappa)^2 + (\varepsilon h')^2}} \quad \text{and} \quad \mathbf{n}_h(s) = \frac{(1 + \varepsilon h \kappa)\mathbf{n}(s) - \varepsilon h' \mathbf{t}(s)}{\sqrt{(1 + \varepsilon h \kappa)^2 + (\varepsilon h')^2}}.$$

We drop the dependency in  $s$  for  $\kappa$  and  $h$  for ease of notation. Let  $\chi$  be the cut locus of  $\partial D$ . For a function  $u$  defined in  $D_\varepsilon$ , we consider  $\tilde{u} : [0, |\partial D|] \times (-\chi, \chi)$  be defined by  $\tilde{u}(s, t) = u(\mathbf{x})$ . In accordance with [5], the gradient and Laplace operator

are expressed in the curvilinear coordinates as:

$$\begin{aligned}\nabla &= \frac{1}{1+t\kappa} \frac{\partial}{\partial s} \mathbf{t}(s) + \frac{\partial}{\partial t} \mathbf{n}(s), \\ \Delta &= \frac{1}{1+t\kappa} \frac{\partial}{\partial s} \left( \frac{1}{1+t\kappa} \frac{\partial}{\partial s} \right) + \frac{\kappa}{1+t\kappa} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}.\end{aligned}$$

The usual strategy relies on an asymptotic expansion of  $u$  with respect to the scaling factor  $\varepsilon$  with a double ansatz one valid in  $D$  and the other in the layer  $L_\varepsilon$ . Namely, we postulate that there are real-valued functions  $u_{\text{int}}^k$  defined on  $D$  and  $u_{\text{ext}}^k$  defined on  $[0, |\partial D|] \times [0, 1]$  such that

$$(2.7) \quad \begin{cases} u(\mathbf{x}) = u_{\text{int}}(\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k u_{\text{int}}^k(\mathbf{x}) \text{ in } D, \\ u(\mathbf{x}) = u_{\text{ext}}(\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k u_{\text{ext}}^k \left( s, \frac{t}{\varepsilon h} \right) \text{ in } L_\varepsilon. \end{cases}$$

With respect to previous works, the anisotropy in the second curvilinear coordinate takes the variation of the thickness into account. With these ansätze at hand, we can reformulate the boundary value problem (1.1) as a transmission problem:

$$(2.8) \quad \begin{cases} -\sigma_0 \Delta u_{\text{int}} = f & \text{in } D, \\ -\Delta u_{\text{ext}} = f & \text{in } L_\varepsilon, \\ u_{\text{int}} = u_{\text{ext}} & \text{on } \partial L_\varepsilon, \\ \sigma_0 \partial_{\mathbf{n}} u_{\text{int}} = \partial_{\mathbf{n}} u_{\text{ext}} & \text{on } \partial L_\varepsilon. \end{cases}$$

In accordance with [5], the equation in the layer  $L_\varepsilon$  is

$$(2.9) \quad \mathcal{L}\tilde{u} = -(1+t\kappa)f \quad \text{with} \quad \mathcal{L} = \partial_s \left( \frac{1}{1+t\kappa} \partial_s \right) + \kappa \partial_t + (1+t\kappa) \partial_t^2.$$

In order to rewrite this equation with respect to anisotropic, curvilinear coordinates  $(s, \tau) = (s, t/(\varepsilon h))$ , corresponding to the ansätze (2.7), we compute:

$$\begin{aligned}\partial_t &= \frac{1}{\varepsilon h} \partial_\tau, \quad \kappa \partial_t = \frac{\kappa}{\varepsilon h} \partial_\tau, \\ \partial_t^2 &= \frac{1}{\varepsilon^2 h^2} \partial_\tau^2, \quad (1+t\kappa) \partial_t^2 = \frac{1}{\varepsilon^2 h^2} \partial_\tau^2 + \frac{\kappa \tau}{\varepsilon h} \partial_\tau^2, \\ \partial_s \left( \frac{1}{1+t\kappa} \partial_s \right) &= \sum_{n \geq 0} (-1)^n \varepsilon^n \tau^n \kappa^n h^n \partial_s^2 - \sum_{n \geq 1} (-1)^n \varepsilon^n \tau^n n (\kappa' \kappa^{n-1} h^n + \kappa^n h' h^{n-1}) \partial_s, \\ (1+t\kappa) f(., t) &= (1 + \varepsilon h \kappa \tau) f(., \varepsilon h \tau).\end{aligned}$$

When  $f$  is smooth in the layer, the right hand side in the layer can be expanded by a Taylor expansion

$$(1 + \varepsilon h \kappa \tau) f(., \varepsilon h \tau) = \sum_{n=0}^N \varepsilon^n f_n(., \tau) + \mathcal{O}(\varepsilon^{N+1}).$$

In particular, one has simply  $f_0(s, \tau) = f(s, 0)$ . As a consequence, the operator  $\mathcal{L}$  can be split in powers of the small parameter  $\varepsilon$  as  $\mathcal{L} = \sum_{n \geq -2} \varepsilon^n \mathcal{L}_n$  with

$$\mathcal{L}_{-2} = \frac{1}{h^2} \partial_\tau^2, \quad \mathcal{L}_{-1} = \frac{\kappa}{h} (\partial_\tau + \tau \partial_\tau^2), \quad \mathcal{L}_0 = \partial_s^2,$$

and, for  $n \geq 1$ , with

$$\mathcal{L}_n = (-1)^n \tau^n [\kappa^n h^n \partial_s^2 - n(\kappa' \kappa^{n-1} h^n + \kappa^n h' h^{n-1}) \partial_s].$$

Plugging in the ansätze (2.7), we obtain first

$$\begin{aligned} \mathcal{L}u_{\text{ext}} &= \varepsilon^{-2} \frac{1}{h^2} \partial_\tau^2 u_{\text{ext}}^0 + \varepsilon^{-1} \left( \frac{1}{h^2} \partial_\tau^2 u_{\text{ext}}^1 + \frac{\kappa}{h} (\partial_\tau u_{\text{ext}}^0 + \tau \partial_\tau^2 u_{\text{ext}}^0) \right) \\ &+ \sum_{n \geq 0} \varepsilon^n \left( \frac{1}{h^2} \partial_\tau^2 u_{\text{ext}}^{n+2} + \frac{\kappa}{h} (\partial_\tau u_{\text{ext}}^{n+1} + \tau \partial_\tau^2 u_{\text{ext}}^{n+1}) + \sum_{k=0}^n \mathcal{L}_{n-k} u_{\text{ext}}^k \right) \end{aligned}$$

and then, in view of (2.9), the sequence of differential equations for the functions  $u_{\text{ext}}^k$ :

$$\begin{aligned} \partial_\tau^2 u_{\text{ext}}^0 &= 0, \\ \partial_\tau^2 u_{\text{ext}}^1 &= -\kappa h (\partial_\tau u_{\text{ext}}^0 + \tau \partial_\tau^2 u_{\text{ext}}^0), \\ \partial_\tau^2 u_{\text{ext}}^{n+2} &= - \left( \kappa h (\partial_\tau u_{\text{ext}}^{n+1} + \tau \partial_\tau^2 u_{\text{ext}}^{n+1}) + h^2 \sum_{k=0}^n \mathcal{L}_{n-k} u_{\text{ext}}^k \right) - h^2 f_n \text{ for all } n \geq 0. \end{aligned}$$

The boundary conditions are given by the transmission conditions on  $\partial D$  stated in (2.8):

$$\begin{aligned} \sigma_0 \partial_{\mathbf{n}} u_{\text{int}} &= \partial_t u_{\text{ext}} = \frac{1}{\varepsilon h} \partial_\tau u_{\text{ext}} \Rightarrow \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^{k-1} = \frac{1}{h} \partial_\tau u_{\text{ext}}^k \\ u_{\text{int}} &= u_{\text{ext}} \Rightarrow u_{\text{int}}^k = u_{\text{ext}}^k \end{aligned}$$

The resolution is then iterative. Let us make the first terms explicit.

Order  $n = 0$ . It comes that  $u_{\text{int}}^0$  is the solution of  $-\Delta u = f$  in  $H_0^1(D)$ . Thus, the first function  $u_{\text{ext}}^0$  should be affine in the variable  $\tau$ , taking the value 0 for  $\tau = 0$  and  $\tau = 1$ . Hence, it follows  $u_{\text{ext}}^0 = 0$  and in particular

$$u_{\text{ext}}^0 = 0 \text{ on } \partial D.$$

Order  $n = 1$ . The equations for  $u_{\text{ext}}^1$  are

$$\begin{aligned} \partial_\tau^2 u_{\text{ext}}^1 &= -\kappa h (\partial_\tau u_{\text{ext}}^0 + \tau \partial_\tau^2 u_{\text{ext}}^0) = 0, \\ \partial_\tau u_{\text{ext}}^1(s, 0) &= \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0(s, 0), \\ u_{\text{ext}}^1(s, 1) &= 0. \end{aligned}$$



Consequently,  $u_{\text{ext}}^1$  is the affine (in  $\tau$ ) function  $u_{\text{ext}}^1(s, \tau) = (\tau - 1)\sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0(s, 0)$  so that

$$u_{\text{ext}}^1 = -\sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0 \text{ on } \partial D.$$

Thus,  $u_{\text{int}}^1$  solves  $\Delta u_{\text{int}}^1 = 0$  with  $u_{\text{int}}^1 = u_{\text{ext}}^1 = -\sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0$  on  $\partial D$ .

Order  $n = 2$ . Since  $u_{\text{ext}}^0 = 0$  and  $u_{\text{ext}}^1$  is affine in  $\tau$ , the equations for  $u_{\text{ext}}^2$  are

$$\begin{aligned} \partial_{\tau}^2 u_{\text{ext}}^2 &= -\kappa h \partial_{\tau} u_{\text{ext}}^1 - h^2 f_0 = -\kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^0(s, 0) - h^2 f_0, \\ \partial_{\tau} u_{\text{ext}}^2(s, 0) &= \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^1(s, 0), \\ u_{\text{ext}}^2(s, 1) &= 0. \end{aligned}$$

These equations for  $u_{\text{ext}}^2$  can still be solved analytically. Now, since  $f_0(s, \xi) = f(s, 0)$  is independent of  $\tau$ , the second order primitive integral of  $-f_0(s, \tau)$  which vanishes at 1 and which has a derivative that vanishes at 0 is  $f(s, 0)(1 - \tau^2)/2$ . Namely, it holds

$$u_{\text{ext}}^2(s, \tau) = \frac{1 - \tau^2}{2} \kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^0(s, 0) + (\tau - 1) \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^1(s, 0) + f(s, 0) \frac{1 - \tau^2}{2}.$$

Therefore, we obtain the following value of  $u_{\text{ext}}^2 = u_{\text{int}}^2$  on  $\partial D$ :

$$u_{\text{ext}}^2 = \frac{1}{2} \kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^0 - \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^1 + \frac{f(s, 0)}{2} \text{ on } \partial D.$$

**2.2. Derivation of the approximate boundary conditions.** Let us now derive the first and second order approximate boundary conditions. To that end, we introduce the partial sums  $u_{\text{int}}^{[1]} = u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1$  and  $u_{\text{int}}^{[2]} = u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1 + \varepsilon^2 u_{\text{int}}^2$ . By construction, we check that  $-\sigma_0 \Delta u_{\text{int}}^{[i]} = f$  in  $D$  for  $i = 1, 2$ .

Order  $n = 1$ . On  $\partial D$ , one has  $u_{\text{int}}^{[1]} = -\varepsilon \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0$  and it follows

$$u_{\text{int}}^{[1]} + \varepsilon \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^{[1]} = \varepsilon^2 \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^1.$$

We therefore introduce  $v_{\varepsilon}^{[1]}$  as the solution of the following Robin boundary value problem:

$$\begin{aligned} -\sigma_0 \Delta v &= f \text{ in } D, \\ v + \varepsilon \sigma_0 h \partial_{\mathbf{n}} v &= 0 \text{ on } \partial D. \end{aligned}$$

Notice that  $u_{\text{int}}^{[1]}$  and  $v_{\varepsilon}^{[1]}$  solve two distinct boundary value problems differing from a second order term (in  $\varepsilon$ ) in the boundary condition.

Order  $n = 2$ . On the boundary  $\partial D$ , one has

$$\begin{aligned} u_{\text{int}}^{[2]} &= -\varepsilon\sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^0 + \varepsilon^2 \left( \frac{1}{2} \kappa h^2 \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^0 - \sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^1 + \frac{h^2}{2} f \right) \\ &= -\varepsilon\sigma_0 h \left( 1 - \frac{\kappa h \varepsilon}{2} \right) \partial_{\mathbf{n}} u_{\text{int}}^0 - \varepsilon^2 \sigma_0 h \left( 1 - \frac{\kappa h \varepsilon}{2} \right) \partial_{\mathbf{n}} u_{\text{int}}^1 \\ &\quad + \frac{\varepsilon^2 h^2}{2} f - \frac{\kappa \varepsilon^3}{2} \sigma_0 h^2 \partial_{\mathbf{n}} u_{\text{int}}^1. \end{aligned}$$

Therefore, using the Taylor expansion  $(1-x)^{-1} = 1+x + \mathcal{O}(x^2)$ , we obtain

$$\left( 1 + \frac{\kappa h \varepsilon}{2} \right) u_{\text{int}}^{[2]} + \varepsilon\sigma_0 h \partial_{\mathbf{n}} u_{\text{int}}^{[2]} - \frac{\varepsilon^2 h^2}{2} f = \mathcal{O}(\varepsilon^3).$$

We therefore introduce  $v_\varepsilon^{[2]}$  the solution of the Robin boundary value problem (1.4) that we recall for convenience:

$$\begin{aligned} -\sigma_0 \Delta v &= f && \text{in } D, \\ \left( 1 + \frac{\kappa h \varepsilon}{2} \right) v + \varepsilon\sigma_0 h \partial_{\mathbf{n}} v &= \frac{\varepsilon^2 h^2}{2} f && \text{on } \partial D. \end{aligned}$$

Notice that  $u_{\text{int}}^{[2]}$  and  $v_\varepsilon^{[2]}$  solve two distinct boundary value problems differing from a third order term in the boundary condition.

**2.3. Error estimates and dependency in the layer thickness  $\varepsilon$ .** We proceed in two steps: first, to obtain an error estimate for the remainders in the asymptotic expansion of  $u_\varepsilon$  then in a second step, to obtain an asymptotic expansion of  $u_{\text{int}}^{[i]}$  for  $i = 1, 2$ . Both steps are adopted from the classical proofs in the case of constant thickness layers. We therefore explain the main lines of the proof without entering into the details.

To estimate the truncation error for  $u_\varepsilon$ , the first step is to write a precise error estimate for the remainder  $r_\varepsilon^N$  in the asymptotic expansion of  $u_{\text{ext}}$  and  $u_{\text{int}}$ , defined in  $D$  and  $L_\varepsilon$  as

$$r_{\varepsilon,\text{int}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{\text{int}}^k \quad \text{and} \quad r_{\varepsilon,\text{ext}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{\text{ext}}^k.$$

These remainders satisfy the boundary values problem:

$$\begin{aligned} \sigma_0 \Delta r_{\varepsilon,\text{int}}^N &= 0 && \text{in } D, \\ \Delta r_{\varepsilon,\text{ext}}^N &= \mathcal{O}(\varepsilon^{N-1}) && \text{in } L_\varepsilon, \\ \sigma_0 \partial_{\mathbf{n}} r_{\varepsilon,\text{int}}^N &= \partial_{\mathbf{n}} r_{\varepsilon,\text{ext}}^N + \mathcal{O}(\varepsilon^N) && \text{on } \partial D, \\ r_{\varepsilon,\text{int}}^N &= r_{\varepsilon,\text{ext}}^N && \text{on } \partial D, \\ r_{\varepsilon,\text{ext}}^N &= 0 && \text{on } \partial D_\varepsilon. \end{aligned}$$

Since, for  $\varepsilon \leq \varepsilon_0$ , the family  $D_\varepsilon$  is uniformly contained in a fixed domain, a uniform Poincaré inequality holds in  $H_0^1(D_\varepsilon)$  which provides the uniform coercivity of the bilinear forms  $a_\varepsilon$  defined on  $H_0^1(D_\varepsilon)$  by

$$a_\varepsilon(u, v) = \sigma_0 \int_D \nabla u \nabla v \, d\mathbf{x} + \int_{L_\varepsilon} \nabla u \nabla v \, d\mathbf{x}.$$

Now, a classical *a priori* estimate provides: there is constant  $C$  independent of  $\varepsilon \leq \varepsilon_0$  such that  $\|r_\varepsilon^N\|_{H_0^1(D_\varepsilon)} \leq C\varepsilon^{N-1}$  and the usual trick of spitting the remainder of order  $N$  as  $r_\varepsilon^N = r_\varepsilon^{N+2} + \varepsilon^{N+1}u^{N+1} + \varepsilon^{N+2}u^{N+2}$  provides in fact  $\|r_\varepsilon^N\|_{H_0^1(D_\varepsilon)} \leq C\varepsilon^N$ . We then immediately get

$$(2.10) \quad \|r_{\varepsilon, \text{int}}^N\|_{H_0^1(D)} \leq C\varepsilon^N.$$

Of course, this constant  $C$  depends of the truncation order  $N$ .

Let us now consider the Robin boundary problems for  $u_{\text{int}}^{[i]}$  for  $i = 1, 2$ . We compute asymptotic expansions of  $u_{\text{int}}^{[i]}$ . Thanks to the ansatz

$$u_{\text{int}}^{[i]} = \sum_{k=0}^{\infty} \varepsilon^k u_{\text{int},k}^{[i]},$$

we obtain recursion formulae that provide the same  $i$  first order terms:  $u_{\text{int},k}^{[i]} = u_{\text{int},k}^k$  for  $k \leq i$ . The previous error estimate allows to conclude that

$$\|u_{\text{int}} - u_{\text{int}}^{[i]}\|_{H^1(D)} \leq C\varepsilon^{i+1} \quad \text{for } i = 1, 2.$$

Notice that these errors estimates are optimal since the next terms differ.

### 3. RANDOMLY VARYING THIN LAYERS AND RANDOM ROBIN BOUNDARY CONDITION

We consider the following elliptic partial differential equation with random Robin boundary condition:

$$(3.11) \quad \left. \begin{aligned} -\sigma_0 \Delta u^{[2]}(\omega) &= f && \text{in } D \\ \partial_{\mathbf{n}} u^{[2]}(\omega) + \frac{1}{\varepsilon \sigma_0} \left[ \frac{1}{h(\omega)} + \frac{\varepsilon}{2} \kappa \right] u^{[2]}(\omega) &= \frac{\varepsilon h(\omega)}{2\sigma_0} f && \text{on } \partial D \end{aligned} \right\} \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

To obtain the variational formulation, we test (3.11) with an arbitrary test function from  $L_{\mathbb{P}}^2(\Omega, H^1(D))$ : seek  $u^{[2]} \in L_{\mathbb{P}}^2(\Omega, H^1(D))$  such that

$$(3.12) \quad \begin{aligned} & \int_{\Omega} \left\{ \sigma_0 \int_D \nabla u^{[2]}(\omega) \nabla v(\omega) \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} \left[ \frac{1}{h(\omega)} + \frac{\varepsilon}{2} \kappa \right] u^{[2]}(\omega) v(\omega) \, d\sigma \right\} d\mathbb{P}(\omega) \\ &= \int_{\Omega} \left\{ \int_D f v(\omega) \, d\mathbf{x} + \int_{\partial D} \frac{\varepsilon}{2\sigma_0} f v(\omega) \, d\sigma \right\} d\mathbb{P}(\omega) \end{aligned}$$

holds for all  $v \in L_{\mathbb{P}}^2(\Omega, H^1(D))$ .

**Theorem 3.1.** *Under the conditions (1.6), there exists a unique solution  $u^{[2]}$  in  $L^2_{\mathbb{P}}(\Omega, H^1(D))$  to the variational formulation (3.12) provided that  $\varepsilon$  is so small such that*

$$(3.13) \quad |\varepsilon\kappa(\mathbf{x})| \leq \frac{1}{h_{\max}} \text{ for all } \mathbf{x} \in \partial D.$$

*In particular, introducing the spatial energy norm*

$$(3.14) \quad \|u(\omega)\| := \sqrt{\sigma_0 |u(\omega)|^2_{H^1(D)} + \frac{1}{\varepsilon\sigma_0} \|\gamma(u(\omega))\|^2_{L^2(\partial D)}}$$

*where  $\gamma : H^1(D) \rightarrow L^2(\partial D)$  is the trace operator, we have the stability estimate*

$$(3.15) \quad \sqrt{\int_{\Omega} \|u(\omega)\|^2 d\mathbb{P}} \leq C \{ \|f\|_{\tilde{H}^{-1}(D)} + \|\gamma(f)\|_{L^2(\partial D)} \}$$

*uniformly as  $\varepsilon$  tends to 0, where  $\|\cdot\|_{\tilde{H}^{-1}(D)}$  denotes as usual the dual norm to  $\|\cdot\|_{H^1(D)}$ .*

*Proof.* By introducing the bilinear form

$$a : L^2_{\mathbb{P}}(\Omega, H^1(D)) \times L^2_{\mathbb{P}}(\Omega, H^1(D)) \rightarrow \mathbb{R},$$

$$a(u, v) := \int_{\Omega} \left\{ \sigma_0 \int_D \nabla u \nabla v d\mathbf{x} + \frac{1}{\varepsilon\sigma_0} \int_{\partial D} \left[ \frac{1}{h(\omega)} + \frac{\varepsilon}{2} \kappa \right] uv d\sigma \right\} d\mathbb{P}(\omega)$$

and the linear form

$$\ell : L^2_{\mathbb{P}}(\Omega, H^1(D)) \rightarrow \mathbb{R}, \quad \ell(v) := \int_{\Omega} \left[ \int_D f v d\mathbf{x} + \int_{\partial D} \frac{\varepsilon h(\omega)}{2\sigma_0} f v d\sigma \right] d\mathbb{P}(\omega),$$

the variational formulation (3.12) is equivalent to the problem:

$$(3.16) \quad \text{seek } u^{[2]} \in L^2_{\mathbb{P}}(\Omega, H^1(D)) \text{ such that}$$

$$a(u^{[2]}, v) = \ell(v) \text{ for all } v \in L^2_{\mathbb{P}}(\Omega, H^1(D)).$$

In view of (1.6), it holds that

$$0 < (1 - q)h_{\min} \leq (1 - q)\bar{h}(\mathbf{x}) \leq h(\mathbf{x}, \omega) \leq (1 + q)\bar{h}(\mathbf{x}) \leq (1 + q)h_{\max}$$

and hence with (3.13) that

$$(3.17) \quad \frac{1}{2(1 + q)h_{\max}} \leq \frac{1 + \frac{\varepsilon}{2}\kappa(\mathbf{x})h(\mathbf{x}, \omega)}{h(\mathbf{x}, \omega)} \leq \frac{3}{2(1 - q)h_{\min}}$$

for all  $\mathbf{x} \in D$  and for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . Thus, the bilinear form  $a(\cdot, \cdot)$  is uniformly elliptic and bounded:

$$\min \left\{ 1, \frac{1}{2(1 + q)h_{\max}} \right\} \int_{\Omega} \|u(\omega)\|^2 d\mathbb{P}(\omega) \leq a(u, u),$$

$$|a(u, v)| \leq \max \left\{ 1, \frac{3}{2(1 - q)h_{\min}} \right\} \sqrt{\int_{\Omega} \|u(\omega)\|^2 d\mathbb{P}(\omega)} \sqrt{\int_{\Omega} \|v(\omega)\|^2 d\mathbb{P}(\omega)}.$$

In addition, the linear form  $\ell(\cdot)$  satisfies

$$|\ell(v)| \leq \|f\|_{\tilde{H}^{-1}(D)} \sqrt{\int_{\Omega} \|v(\omega)\|_{H^1(D)}^2 d\mathbb{P}(\omega)} \\ + \frac{\varepsilon}{2\sigma_0} (1+q) h_{\max} \|\gamma(f)\|_{L^2(\partial D)} \sqrt{\int_{\Omega} \|\gamma(v(\omega))\|_{L^2(\partial D)}^2 d\mathbb{P}(\omega)}$$

provided that  $f \in L^2(D) \cap L^2(\partial D)$ . Since the energy norm (3.14) is equivalent to the  $H^1(D)$ -norm with

$$\underline{c} \min \left\{ \sqrt{\sigma_0}, \frac{1}{\sqrt{\varepsilon\sigma_0}} \right\} \|u(\omega)\|_{H^1(\Omega)} \leq \|u(\omega)\| \leq \bar{c} \max \left\{ \sqrt{\sigma_0}, \frac{1}{\sqrt{\varepsilon\sigma_0}} \right\} \|u(\omega)\|_{H^1(\Omega)},$$

we can set

$$c_f := \frac{1}{\underline{c}} \max \left\{ \frac{1}{\sqrt{\sigma_0}}, \sqrt{\varepsilon\sigma_0} \right\} \|f\|_{\tilde{H}^{-1}(D)} + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}} (1+q) h_{\max} \|\gamma(f)\|_{L^2(\partial D)} < \infty$$

to arrive at the continuity of the linear form  $\ell(\cdot)$ :

$$|\ell(v)| \leq c_f \sqrt{\int_{\Omega} \|v(\omega)\|^2 d\mathbb{P}(\omega)}.$$

Herein, the constant  $c_f$  does not depend on the layer's thickness  $\varepsilon$  any more provided that  $\varepsilon \leq \varepsilon_0$ . According to the theorem of Lax-Milgram, we conclude thus the desired result.  $\square$

This theorem implies the well-posedness of the thin layer equation with random thickness. In particular, as an immediate consequence of our analysis in Subsection 2.3, we conclude that the random solution  $u^{[2]} \in L^2_{\mathbb{P}}(\Omega, H^1(D))$  of (3.12) satisfies the error estimates

$$(3.18) \quad \|u - u^{[2]}\|_{L^2_{\mathbb{P}}(\Omega, H^1(D))} \leq C\varepsilon^3.$$

Therefore, we derive the following proposition.

**Proposition 3.2.** *The random solution  $u^{[2]} \in L^2_{\mathbb{P}}(\Omega, H^1(D))$  satisfies the error estimates*

$$\|\mathbb{E}(u) - \mathbb{E}(u^{[2]})\|_{H^1(D)} \leq C\varepsilon^3, \quad \|\mathbb{V}(u) - \mathbb{V}(u^{[2]})\|_{W^{1,1}(D)} \leq C\varepsilon^4.$$

*Proof.* The first assertion follows by taking the expectation in the estimate (3.18).

For the second assertion, one has to consider the difference of the covariances

$$\text{Cov}(u)(\mathbf{x}, \mathbf{x}') = \mathbb{E} \left( [u(\mathbf{x}, \omega) - \mathbb{E}(u(\mathbf{x}, \omega))] [u(\mathbf{x}', \omega) - \mathbb{E}(u(\mathbf{x}', \omega))] \right), \\ \text{Cov}(u^{[2]})(\mathbf{x}, \mathbf{x}') = \mathbb{E} \left( [u^{[2]}(\mathbf{x}, \omega) - \mathbb{E}(u^{[2]}(\mathbf{x}, \omega))] [u^{[2]}(\mathbf{x}', \omega) - \mathbb{E}(u^{[2]}(\mathbf{x}', \omega))] \right)$$

in the space  $H_{\text{mix}}^1(D \times D) := H^1(D) \otimes H^1(D)$ . Due to

$$u^{[2]}(\omega) - \mathbb{E}(u^{[2]}(\omega)) = [u^{[2]}(\omega) - u(\omega)] + [u(\omega) - \mathbb{E}(u(\omega))] + [\mathbb{E}(u(\omega) - u^{[2]}(\omega))],$$

we find

$$\begin{aligned} & \|\text{Cov}(u) - \text{Cov}(u^{[2]})\|_{H_{\text{mix}}^1(D \times D)} \\ & \leq \left\| \mathbb{E} \left( [u^{[2]}(\omega) - u(\omega)] \otimes [u^{[2]}(\omega) - u(\omega)] \right) \right\|_{H_{\text{mix}}^1(D \times D)} \\ & \quad + 2 \left\| \mathbb{E} \left( [u^{[2]}(\omega) - u(\omega)] \otimes [u(\omega) - \mathbb{E}(u(\omega))] \right) \right\|_{H_{\text{mix}}^1(D \times D)} \\ & \quad + 3 \left\| \mathbb{E} (u^{[2]}(\omega) - u(\omega)) \otimes \mathbb{E} (u^{[2]}(\omega) - u(\omega)) \right\|_{H_{\text{mix}}^1(D \times D)}. \end{aligned}$$

By using the fact that the first term and the last term on the right hand side of this estimate are of order  $\mathcal{O}(\varepsilon^6)$ . The second term is of order  $\mathcal{O}(\varepsilon^4)$  since we only know that  $\|u(\omega) - \mathbb{E}(u(\omega))\|_{H^1(D)} = \mathcal{O}(\varepsilon)$  which follows by a linearization in terms of the local shape derivative, see [14] for the details. Hence, we arrive at

$$\|\text{Cov}(u) - \text{Cov}(u^{[2]})\|_{H_{\text{mix}}^1(D \times D)} \leq C\varepsilon^4.$$

Taking the trace  $\mathbf{x} = \mathbf{x}'$  gives finally the desired result.  $\square$

#### 4. REGULARITY OF THE RANDOM SOLUTION

For ease of notation, we will drop the suffix of the solution  $u^{[2]}$ , i.e., we will denote the solution to (4.21) only by  $u$ . Moreover, we shall assume that the random fluctuations  $\tilde{h}(\omega)$  are given by a possibly infinite Karhunen-Loève expansion, that is

$$(4.19) \quad \tilde{h}(\mathbf{x}, \omega) = \sum_{k=1}^m h_k(\mathbf{x}) Y_k(\omega)$$

where the coefficient functions  $\{h_k(\mathbf{x})\}$  are pairwise orthonormal in  $L^2(D)$  and the random variables  $\{Y_k(\omega)\}$  are assumed to be independently and uniformly distributed in  $[-1/2, 1/2]$ . Although a finite Karhunen-Loève expansion is assumed here, we shall derive estimates which are independent of the truncation length  $m$ . This means, the situation of  $m \rightarrow \infty$  shall be covered by the following theory. To that end, we have to assume that

$$(4.20) \quad \gamma_k := \|h_k\|_{L^\infty(D)} < \infty \quad \text{for all } k \in \{1, 2, \dots, m\}$$

and that the sequence  $\{\gamma_k\}$  is always summable as  $m \rightarrow \infty$ , i.e.,  $\sum_{k=1}^m \gamma_k \leq c_\gamma$  independently of  $m$ .

The assumption that the random variables  $\{Y_k(\omega)\}_k$  are stochastically independent implies that the pushforward measure  $\mathbb{P}_{\mathbf{Y}} := \mathbb{P} \circ \mathbf{Y}^{-1}$  with respect to the measurable

mapping

$$\mathbf{Y} : \Omega \rightarrow \square := [-1/2, 1/2]^m, \quad \omega \mapsto \mathbf{Y}(\omega) := (Y_1(\omega), \dots, Y_m(\omega))$$

is given by the joint density function 1. With this representation at hand, we can reformulate the stochastic problem (3.11) as a parametric, deterministic problem. To that end, we replace the space  $L^2_{\mathbb{P}}(\Omega)$  by  $L^2(\square)$  and substitute the random variables  $Y_k$  by the coordinates  $y_k \in [-1/2, 1/2]$ . Then, we have to seek  $u \in L^2(\square, H^1(D))$  such that

$$(4.21) \quad \int_{\square} \left\{ \sigma_0 \int_D \nabla u(\mathbf{x}, \mathbf{y}) \nabla v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} \left[ \frac{1}{h(\mathbf{x}, \mathbf{y})} + \frac{\varepsilon}{2} \kappa(\mathbf{x}) \right] u(\mathbf{x}, \mathbf{y}) v(\mathbf{x}, \mathbf{y}) \, d\sigma_{\mathbf{x}} \right\} d\mathbf{y} \\ = \int_{\square} \left\{ \int_D f(\mathbf{x}) v(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} + \int_{\partial D} \frac{\varepsilon h(\mathbf{x}, \mathbf{y})}{2\sigma_0} f(\mathbf{x}) \, d\sigma_{\mathbf{x}} \right\} d\mathbf{y}$$

holds for all  $v \in L^2(\square, H^1(D))$ . Herein, the function  $h(\mathbf{x}, \mathbf{y})$  is affine in the random variable  $\mathbf{y}$ :

$$(4.22) \quad h(\mathbf{x}, \mathbf{y}) = \bar{h}(\mathbf{x}) + \sum_{k=1}^{\infty} h_k(\mathbf{x}) y_k \quad \text{with} \quad \bar{h}(\mathbf{x}) := \mathbb{E}(h(\mathbf{x})).$$

The solvability condition (1.6) is equivalent to

$$(4.23) \quad 0 < h_{\min} \leq \bar{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\tilde{h}(\mathbf{x}, \mathbf{y})| \leq q \bar{h}(\mathbf{x}) \quad \text{for some } 0 \leq q < 1$$

for all  $\mathbf{x} \in D$  and  $\mathbf{y} \in \square$ .

The next lemma is concerned with the decay of the derivatives of the function

$$(4.24) \quad g(\mathbf{x}, \mathbf{y}) = \bar{g}(\mathbf{x}) + \tilde{g}(\mathbf{x}, \mathbf{y}) := \frac{1}{h(\mathbf{x}, \mathbf{y})} + \frac{\varepsilon}{2} \kappa(\mathbf{x})$$

with

$$(4.25) \quad \bar{g}(\mathbf{x}) := \frac{1}{\bar{h}(\mathbf{x})} + \frac{\varepsilon}{2} \kappa(\mathbf{x}) \neq \mathbb{E}(g(\mathbf{x}, \mathbf{y}))$$

with respect to the random variable  $\mathbf{y}$ .

**Lemma 4.1.** *Let (3.13), (4.20), (4.22), and (4.23) hold. Then, the derivatives of  $g(\mathbf{y})$  satisfy*

$$\|\partial_{\mathbf{y}}^{\alpha} g(\mathbf{y})\|_{L^{\infty}(D)} \leq \frac{|\alpha|!}{((1-q)h_{\min})^{|\alpha|+1}} \gamma^{\alpha}$$

for all  $\alpha \in \mathbb{N}^m$  and for all  $\mathbf{y} \in \square$ . Here,  $\gamma^{\alpha}$  has to be understood as the product  $\prod_{k=1}^m \gamma_k^{\alpha_k}$ .

*Proof.* Due to (4.22), we conclude

$$(4.26) \quad \partial_{y_k}^m h(\mathbf{x}, \mathbf{y}) = \begin{cases} h_k(\mathbf{x}), & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases}$$

It thus follows by the chain rule that

$$\partial_{\mathbf{y}}^{\alpha} g(\mathbf{x}, \mathbf{y}) = (-1)^{|\alpha|} |\alpha|! \frac{\prod_{k=1}^m (h_k(\mathbf{x}))^{\alpha_k}}{(h(\mathbf{x}, \mathbf{y}))^{|\alpha|+1}} \quad \text{for all } \alpha \in \mathbb{N}^m.$$

Hence, because of

$$h(\mathbf{x}, \mathbf{y}) \geq (1 - q)\bar{h}(\mathbf{x}) \geq (1 - q)h_{\min} > 0$$

for all  $\mathbf{x} \in D$  and for all  $\mathbf{y} \in \square$ , cf. (4.23), we derive the assertion with (3.13) and (4.20) for all  $\alpha \in \mathbb{N}^m$ :

$$\|\partial_{\mathbf{y}}^{\alpha} g(\mathbf{y})\|_{L^{\infty}(D)} \leq |\alpha|! \frac{\prod_{k=1}^m \|h_k\|_{L^{\infty}(D)}^{\alpha_k}}{((1 - q)h_{\min})^{|\alpha|+1}} = \frac{|\alpha|!}{((1 - q)h_{\min})^{|\alpha|+1}} \gamma^{\alpha}.$$

□

With the help of this lemma, we can prove the main theorem in this section which provides estimates on the derivatives of the random solution with respect to the stochastic variable. These estimates are robust in the scale parameter of the layer's thickness and show the analytic dependence of the random solution on the stochastic variable  $\mathbf{y}$ .

**Theorem 4.2.** *Under the assumptions of Lemma 4.1, the derivatives of the solution  $u \in L^2(\square, H^1(D))$  to (4.21) satisfy the pointwise estimate*

$$(4.27) \quad \|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\| \leq c_f |\alpha|! \left( \frac{c_u}{(1 - q)h_{\min}} \right)^{|\alpha|} \gamma^{\alpha}$$

for all  $\mathbf{y} \in \square$  and  $\alpha \in \mathbb{N}^m$ . Herein, the constant  $c_f$  depends only on  $\|f\|_{\tilde{H}^{-1}(D)}$ ,  $\|\gamma(f)\|_{L^2(\partial D)}$  and  $\sigma_0$ , but not on the layer thickness  $\varepsilon$ , while the constant  $c_u$  is given by

$$c_u = 2 \max \left\{ \frac{1}{(1 - q)h_{\min}}, \frac{2(1 + q)h_{\max}}{(1 - q)h_{\min}} \right\} \geq 2.$$

*Proof.* For  $|\alpha| = 0$ , the assertion follows immediately from Theorem 3.1. For  $|\alpha| > 0$ , we shall have a look at the parametrized problem (4.21) which, for given  $\mathbf{y} \in \square$ , implies the identity

$$\sigma_0 \int_D \nabla u(\mathbf{y}) \nabla v \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} g(\mathbf{y}) u(\mathbf{y}) v \, d\sigma = \int_D f v \, d\mathbf{x} + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h(\mathbf{y}) f v \, d\sigma$$



for all  $v \in H^1(D)$ , where the function  $g$  is given as in (4.24) and (4.25). Thus, in view of the Leibniz rule, differentiation with respect to  $\mathbf{y}$  on both sides of this equality leads

$$(4.28) \quad \begin{aligned} \sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\alpha} u(\mathbf{y}) \nabla v \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \partial_{\mathbf{y}}^{\alpha - \alpha'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y}) v \, d\sigma \\ = \frac{\varepsilon}{2\sigma_0} \int_{\partial D} \partial_{\mathbf{y}}^{\alpha} h(\mathbf{y}) f v \, d\sigma. \end{aligned}$$

Here, due to (4.26), the term on the right hand side vanishes if  $|\alpha| > 1$ . Hence, we shall first consider the case  $|\alpha| = 1$  where we obtain

$$\begin{aligned} \sigma_0 \int_D \nabla \partial_{y_k} u(\mathbf{y}) \nabla v \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} g(\mathbf{y}) \partial_{y_k} u(\mathbf{y}) v \, d\sigma \\ = -\frac{1}{\varepsilon \sigma_0} \int_{\partial D} \partial_{y_k} g(\mathbf{y}) u(\mathbf{y}) v \, d\sigma + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h_k f v \, d\sigma. \end{aligned}$$

In view of (3.14) and (3.17), the special choice  $v = \partial_{y_k} u(\mathbf{y})$  yields

$$\begin{aligned} \min \left\{ 1, \frac{1}{2(1+q)h_{\max}} \right\} \|\partial_{y_k} u(\mathbf{y})\|^2 \\ = -\frac{1}{\varepsilon \sigma_0} \int_{\partial D} \partial_{y_k} g(\mathbf{y}) u(\mathbf{y}) \partial_{y_k} u(\mathbf{y}) \, d\sigma + \frac{\varepsilon}{2\sigma_0} \int_{\partial D} h_k f \partial_{y_k} u(\mathbf{y}) \, d\sigma \\ \leq \|\partial_{y_k} g(\mathbf{y})\|_{L^\infty(\partial D)} \|u(\mathbf{y})\| \|\partial_{y_k} u(\mathbf{y})\| + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}} \|h_k\|_{L^\infty(\partial D)} \|\gamma(f)\|_{L^2(\partial D)} \|\partial_{y_k} u(\mathbf{y})\|. \end{aligned}$$

By possibly increasing  $c_f$ , this leads to the assertion in the case  $|\alpha| = 1$ :

$$\begin{aligned} \|\partial_{y_k} u(\mathbf{y})\| &\leq \frac{c_u}{2(1-q)h_{\min}} \gamma_k \|u(\mathbf{y})\| + \frac{\varepsilon^{3/2}}{2\sqrt{\sigma_0}} \max\{1, 2(1+q)h_{\max}\} \gamma_k \|\gamma(f)\|_{L^2(\partial D)} \\ &\leq c_f \frac{c_u}{(1-q)h_{\min}} \gamma_k. \end{aligned}$$

Finally, we consider the case of arbitrary multiindices  $|\alpha| > 1$  where we rewrite (4.28) in accordance with

$$\begin{aligned} \sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\alpha} u(\mathbf{y}) \nabla v \, d\mathbf{x} + \frac{1}{\varepsilon \sigma_0} \int_{\partial D} g(\mathbf{y}) \partial_{\mathbf{y}}^{\alpha} u(\mathbf{y}) v \, d\sigma \\ = -\frac{1}{\varepsilon \sigma_0} \int_{\partial D} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \partial_{\mathbf{y}}^{\alpha - \alpha'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y}) v \, d\sigma. \end{aligned}$$

We use again (3.14), (3.17) and the special choice  $v = \partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})$  to conclude

$$\begin{aligned} & \min \left\{ 1, \frac{1}{2(1+q)h_{\max}} \right\} \|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\|^2 \\ & \leq -\frac{1}{\varepsilon\sigma_0} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \int_{\partial D} \partial_{\mathbf{y}}^{\alpha-\alpha'} g(\mathbf{y}) \partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y}) \partial_{\mathbf{y}}^{\alpha} u(\mathbf{y}) \, d\sigma \\ & \leq \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \|\partial_{\mathbf{y}}^{\alpha-\alpha'} g(\mathbf{y})\|_{L^{\infty}(\partial D)} \|\partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y})\| \|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\|, \end{aligned}$$

which means that

$$\|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\| \leq \max\{1, 2(1+q)h_{\max}\} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \|\partial_{\mathbf{y}}^{\alpha-\alpha'} g(\mathbf{y})\|_{L^{\infty}(\Gamma)} \|\partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y})\|.$$

Inserting the result of Lemma 4.1 on the decay of the derivatives of  $g$ , we arrive at

$$\|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\| \leq \frac{c_u}{2} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \frac{|\alpha - \alpha'|!}{((1-q)h_{\min})^{|\alpha-\alpha'|}} \gamma^{\alpha-\alpha'} \|\partial_{\mathbf{y}}^{\alpha'} u(\mathbf{y})\|.$$

By induction, we may further estimate this expression according to

$$(4.29) \quad \|\partial_{\mathbf{y}}^{\alpha} u(\mathbf{y})\| \leq c_f \frac{c_u}{2} \gamma^{\alpha} \left( \frac{1}{(1-q)h_{\min}} \right)^{|\alpha|} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha - \alpha'|! |\alpha'|! c_u^{|\alpha-\alpha'|}.$$

We find that

$$\sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha'|! |\alpha - \alpha'|! c_u^{|\alpha-\alpha'|} = \sum_{j=0}^{|\alpha|-1} j!(|\alpha-j)! c_u^{|\alpha|-j} \sum_{\substack{\alpha \neq \alpha' \leq \alpha \\ |\alpha'|=j}} \binom{\alpha}{\alpha'}.$$

By employing the combinatorial identity

$$\sum_{\substack{\alpha \neq \alpha' \leq \alpha \\ |\alpha'|=j}} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{j},$$

we thus obtain

$$\begin{aligned} \sum_{\alpha \neq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha'|! |\alpha - \alpha'|! c_u^{|\alpha-\alpha'|} &= \sum_{j=0}^{|\alpha|-1} j!(|\alpha-j)! \binom{|\alpha|}{j} c_u^{|\alpha|-j} = |\alpha|! \sum_{j=0}^{|\alpha|-1} c_u^{|\alpha|-j} \\ &= |\alpha|! \frac{c_u^{|\alpha|} - 1}{c_u - 1}. \end{aligned}$$

Since  $c_u \geq 2$ , it holds that

$$\frac{c_u c_u^{|\alpha|} - 1}{2 c_u - 1} \leq c_u^{|\alpha|}.$$

Inserting the latter two estimates into (4.29) implies finally the desired assertion for arbitrary  $|\alpha| \geq 1$ .  $\square$

The decay estimate (4.27) coincides with the one which is obtained in case of a diffusion problem with uniformly elliptic random coefficient, see e.g. [2, 8]. It is sufficient to conclude that the solution  $u$  admits an analytic extension into the complex plane with respect to each particular direction  $y_k$  (see [2]). In fact, there exists even an analytic extension with respect to the variable to an appropriately chosen Bernstein ellipse (see [3, 9]). As a consequence, anisotropic stochastic collocation schemes [2, 17] become applicable. Moreover, besides the Monte-Carlo method, also the quasi Monte-Carlo method produces convergence rates which are independent of the stochastic dimension  $m$  provided that it holds  $\gamma_k \lesssim k^{-2-\delta}$  for some  $\delta > 0$ , see [13, 15].

## 5. NUMERICAL RESULTS

**5.1. The Poisson equation on a random domain.** In our first example, we will solve the Poisson equation

$$(5.30) \quad -\Delta u(\omega) = 4 \text{ in } D_\varepsilon(\omega), \quad u(\omega) = 0 \text{ on } \partial D_\varepsilon(\omega),$$

on the randomly perturbed unit disc. The treatment of boundary value problems on random domains is of high interest, see e.g. [6, 7, 12, 14, 16, 21, 22] and the references therein. One has to distinguish between two descriptions of the random solution, namely the Lagrangian and the Eulerian point of view. Our setup leads to the Eulerian specification which means that the physical position of the points stays fixed and only the domain's boundary is random.

To solve the random boundary value problem (5.30) within our framework, we choose  $\sigma_0 = 1$  and split the domain  $D_\varepsilon(\omega)$  according to

$$(5.31) \quad D_\varepsilon(\omega) = D \cup L_\varepsilon(\omega) \quad \text{where} \quad L_\varepsilon(\omega) = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}) : 0 \leq t < \varepsilon h(\mathbf{x}, \omega), \mathbf{x} \in \partial D\}.$$

To validate our approach for problem (5.30), (5.31), we should compare it with a Monte-Carlo approach and with the shape derivative approach from [14]. In particular, we shall consider different values of the perturbation parameter  $\varepsilon$  to examine the asymptotic estimates given in Proposition 3.2. To that end, we choose the reference domain  $D$  as the disc of radius  $1 - \varepsilon$  which leads to a layer of constant thickness  $\varepsilon$  in the mean. This means that  $\bar{h}(\mathbf{x}) = \mathbb{E}(h(\mathbf{x}, \omega)) \equiv 1$ . The particular shape of the random layer under consideration is given as

$$h(\varphi, \omega) = 1 + \frac{1}{8} \sum_{k=0}^5 \{a_k(\omega) \cos(k\varphi) + b_k(\omega) \sin(k\varphi)\},$$

where  $0 \leq \varphi < 2\pi$  is the polar angle of a given point  $\mathbf{x} \in \partial D$  and  $a_k, b_k \in [-1/2, 1/2]$  are independent and uniformly distributed random variables. Notice that we have  $0.5 \leq h(\varphi, \omega) \leq 1.5$  for all  $0 \leq \varphi < 2\pi$  and  $\omega \in \Omega$ .

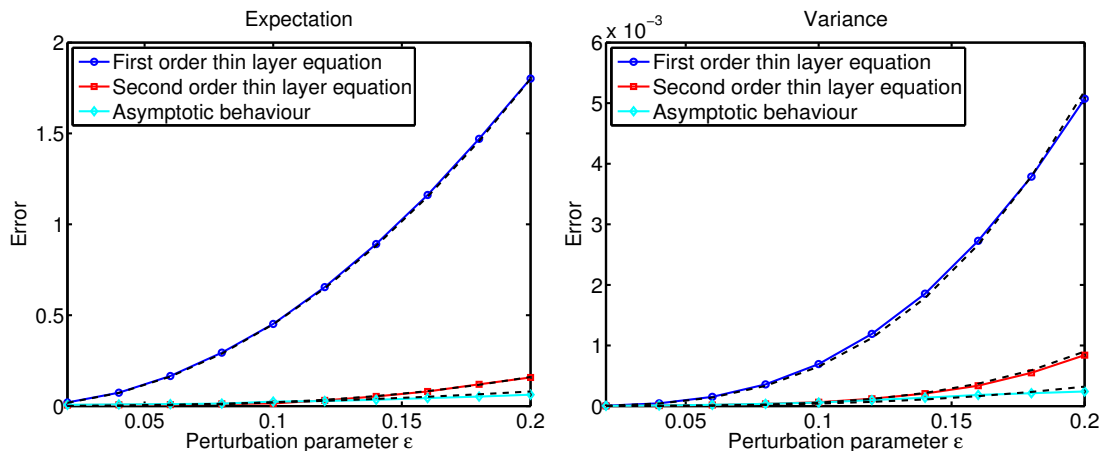


FIGURE 5.2. Absolute  $\ell^2$ -error of the discrete expectation (left) and the discrete variance (right) of the random solution versus the perturbation parameter  $\varepsilon$ .

The spatial discretization we use for the numerical simulation consists of about 4000 continuous, piecewise linear, triangular finite elements. We furthermore apply the quasi Monte-Carlo method for the stochastic discretization. This means that we directly compute the random solution's mean

$$\mathbb{E}(u^{[2]}(\mathbf{x}, \mathbf{y})) = \int_{\square} u^{[2]}(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}$$

and variance

$$\mathbb{V}(u^{[2]}(\mathbf{x}, \mathbf{y})) = \int_{\square} (u^{[2]}(\mathbf{x}, \mathbf{y}))^2 \, d\mathbf{y} - \mathbb{E}^2(u^{[2]}(\mathbf{x}, \mathbf{y}))$$

with the help of 10000 Halton points.

We compute the solution for  $\varepsilon = 0.02, 0.04, \dots, 0.2$  with the proposed approach, with the Monte-Carlo method, and with the shape derivative approach. The solution of the Monte-Carlo method is based on the mean of five runs with 10000 samples per run. All solutions are restricted onto the *fixed* disc  $K$  of radius 0.8 ( $K$  coincides with the domain  $D$  if  $\varepsilon = 0.2$ ) which is contained in the reference domain  $D$  for all values of  $\varepsilon$ . This is done by re-interpolating the solution onto a mesh on  $K$  consisting of about 2000 triangles and ensures that the shape derivative approach approximates the random solution's expectation with the order  $\mathcal{O}(\varepsilon^2)$  and its variance with the order  $\mathcal{O}(\varepsilon^3)$ , see [14] for the details. For the present approach, Proposition 3.2 predicts a convergence of order  $\mathcal{O}(\varepsilon^3)$  for the expectation and of order  $\mathcal{O}(\varepsilon^4)$  for the

variance. This asymptotic behaviour is indeed observed in the numerical results as seen in Figure 5.2 where we plotted the discrete  $\ell^2$ -errors of the expectation and variance. Although the approximation order of the expectation is higher for the proposed approach than for the shape derivative approach, the approximation of both, the expectation and variance, is somewhat less accurate if  $\varepsilon > 0.1$ . Nevertheless, for  $\varepsilon = 0.2$ , the relative error of the expectation and of the variance is only about 0.5% and 5%, respectively. Notice finally that also the errors for the less accurate thin layer equation (1.2) are presented in these plots. It is clearly seen that it produces much less accurate approximations which are only second order accurate in  $\varepsilon$  in case of the expectation and only third order accurate in  $\varepsilon$  in case of the variance.

**5.2. The Poisson equation with thin random layer.** Our second example is dedicated to the thin layer equation for the unit disc. In average, the layer is assumed to be constant and of thickness  $\varepsilon = 0.01$ . The conductivity is 1 in the layer and  $\sigma_0 = 10$  in the unit disc. The inhomogeneity is chosen like before as  $f \equiv 4$ . We use again about 4000 continuous, piecewise linear, triangular finite elements for the spatial discretization and a quasi Monte-Carlo method for the stochastic discretization.

The random boundary fluctuations  $\tilde{h}(\mathbf{x}, \omega)$  under consideration are given according to (4.19) where we either prescribe the exponential covariance kernel  $\exp(-\|\mathbf{x} - \mathbf{x}'\|)/16$ , which is quite rough, or the Gaussian covariance kernel  $\exp(-\|\mathbf{x} - \mathbf{x}'\|^2)/16$ , which is quite smooth. The Karhunen-Loève expansions of these covariance kernels are computed with the help of the pivoted Cholesky decomposition as proposed in [11]. We obtain  $m = 126$  terms for the exponential covariance and  $m = 12$  terms for the Gaussian covariance if we demand the truncation error  $10^{-3}$ . It turns out that a point on the random boundary varies at most  $\pm 0.0065$  around the nominal interface in case of the exponential variance kernel while it varies at most  $\pm 0.0035$  in case of the Gaussian covariance kernel.

The results of our computations are found in Figure 5.3 for the exponential covariance kernel and in Figure 5.4 for the Gaussian covariance kernel. The expectations look quite similar for both covariance kernels. Nevertheless, we observe clear differences between the variances. Namely, compared to the variance in case of the exponential covariance kernel, the variance is more concentrated near the boundary and also smaller in the interior of the domain in case of the Gaussian covariance kernel.

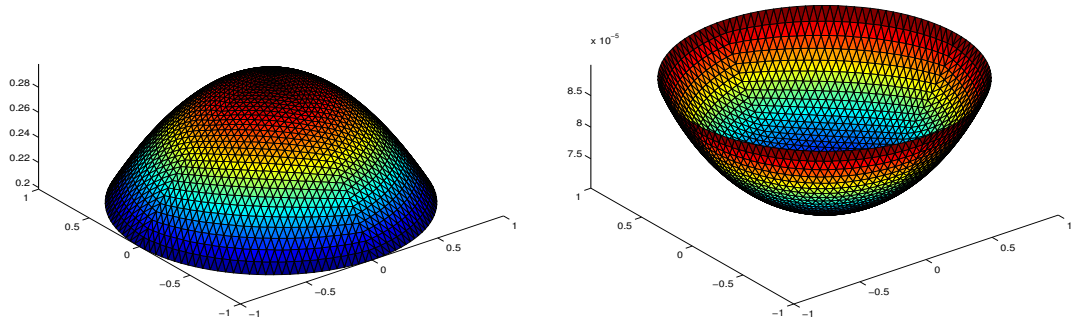


FIGURE 5.3. Expectation (left) and variance (right) of the random solution in case of a unit disc with random layer of average thickness  $\varepsilon = 0.01$  and random perturbations with exponential correlation.

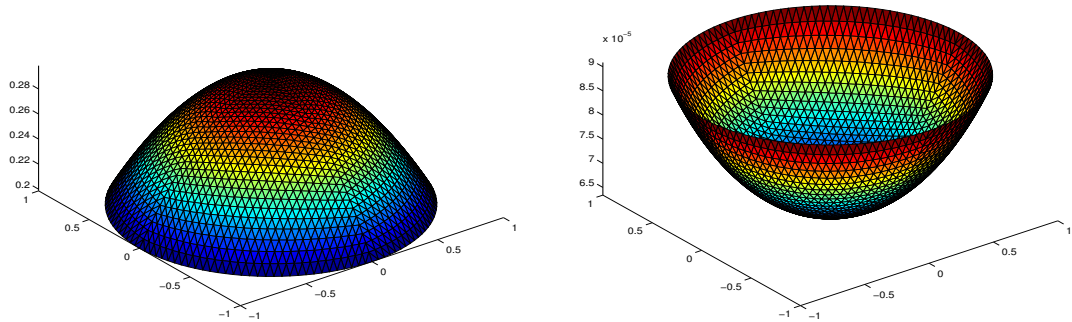


FIGURE 5.4. Expectation (left) and variance (right) of the random solution in case of a unit disc with random layer of average thickness  $\varepsilon = 0.01$  and random perturbations with Gaussian correlation.

## 6. CONCLUSION

In the present article, we considered the Dirichlet boundary value problem for the Poisson equation where the domain is surrounded by a randomly varying thin layer of size  $\mathcal{O}(\varepsilon)$  which has a different conductivity. We replaced this Dirichlet boundary value problem which is stated on a random domain by a random Robin boundary value problem which is stated on a deterministic domain. Decay estimates for the derivatives of its random solution with respect to the stochastic variable are provided which are required to build appropriate solution schemes. Numerical experiments were given to validate and quantify the approach.

## REFERENCES

- [1] B. Aslanyurek, H. Haddar, and H. Sahinturk. Generalized impedance boundary conditions for thin dielectric coatings with variable thickness. *Wave Motion*, 48(7):680–699, 2011.
- [2] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.*, 45(3):1005–1034, 2007.
- [3] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Mod. Methods Appl. Sci.*, 22(9):1250023, 2012.
- [4] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.
- [5] G. Caloz, M. Costabel, M. Dauge, and G. Vial. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptot. Anal.*, 50(1-2):121–173, 2006.
- [6] C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. *Numer. Math.*, 107(2):257–293, 2007.
- [7] J.E. Castrillon-Candas, F. Nobile, and R. Tempone. Analytic regularity and collocation approximation for PDEs with random domain deformations. ArXiv e-print #1312.7845, 2013.
- [8] A. Cohen, R. DeVore and C. Schwab. Convergence rates of best  $N$ -term Galerkin approximations for a class of elliptic sPDEs. *J. Found. Comp. Math.*, 10(6):615–646, 2010.
- [9] A. Cohen, R. DeVore and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs. *Anal. Appl.*, 9(1):11–47, 2010.
- [10] B. Engquist and J.-C. Nédélec. Effective boundary conditions for acoustic and electromagnetic scattering in thin layers. Technical Report RI 278, CMAP – Ecole Polytechnique, 1993.
- [11] H. Harbrecht, M. Peters, and R. Schneider. On the low-rank approximation by the pivoted Cholesky decomposition. *Appl. Numer. Math.*, 62(4):428–440, 2012.
- [12] H. Harbrecht, M. Peters, and M. Siebenmorgen. Numerical solution of elliptic diffusion problems on random domains. Preprint 2014-08, Mathematisches Institut, Universität Basel, Switzerland, 2014.
- [13] H. Harbrecht, M. Peters, and M. Siebenmorgen. Tractability of the Quasi-Monte Carlo quadrature with Halton points for elliptic PDEs with random diffusion. Preprint 2013-28, Mathematisches Institut, Universität Basel, Switzerland, 2013.
- [14] H. Harbrecht, R. Schneider, and C. Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numer. Math.*, 109(3):385–414, 2008.

- [15] F.Y. Kuo and C. Schwab and I.H. Sloan. Quasi-Monte Carlo methods for high dimensional integration. The standard (weighted Hilbert space) setting and beyond. *ANZIAM J.*, 53(1):1–37, 2011.
- [16] P.S. Mohan, P.B. Nair, and A.J. Keane. Stochastic projection schemes for deterministic linear elliptic partial differential equations on random domains. *Int. J. Numer. Meth. Eng.*, 85(7):874–895, 2011.
- [17] F. Nobile, R. Tempone, and C.G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.* 46(5):2309–2345, 2008.
- [18] C. Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Math. Methods Appl. Sci.*, 32(4):435–453, 2009.
- [19] P.E. Protter. Stochastic Integration and Differential Equations. 2nd edition, Springer, Berlin, 2004.
- [20] K. Schmidt and S. Tordeux. Asymptotic modelling of conductive thin sheets. *Z. Angew. Math. Phys.*, 61(4):603–626, 2010.
- [21] D. Xiu and D.M. Tartakovsky. Numerical methods for differential equations in random domains. *SIAM J. Sci. Comput.*, 28(3):1167–1185, 2006.
- [22] D. Xiu and D.M. Tartakovsky. Stochastic analysis of transport in tubes with rough walls. *J. Comput. Phys.*, 217(1):248–259, 2006.

M. DAMBRINE, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR, AVENUE DE L'UNIVERSITÉ, 64000 PAU, FRANCE.

*E-mail address:* marc.dambrine@univ-pau.fr

I. GREFF, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR, AVENUE DE L'UNIVERSITÉ, 64000 PAU, FRANCE.

*E-mail address:* isabelle.greff@univ-pau.fr

H. HARBRECHT, MATHEMATISCHES INSTITUT, UNIVERSITÄT BASEL, RHEINSPRUNG 21, 4051 BASEL, SCHWEIZ.

*E-mail address:* helmut.harbrecht@unibas.ch

B. PUIG, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR, AVENUE DE L'UNIVERSITÉ, 64000 PAU, FRANCE.

*E-mail address:* benedicte.puig@univ-pau.fr



## LATEST PREPRINTS

No.	Author: Title
2014-01	<b>Helmut Harbrecht, Michael Peters, Markus Siebenmorgen</b> <i>Efficient Approximation of Random Fields for Numerical Applications</i>
2014-02	<b>Ali Hyder, Luca Martinazzi</b> <i>Conformal Metrics on <math>R^{2m}</math> with Constant <math>Q</math>-Curvature, Prescribed Volume and Asymptotic Behavior</i>
2014-03	<b>Jürgen Dölz, Helmut Harbrecht, and Michael Peters</b> <i>H-Matrix Accelerated Second Moment Analysis for Potentials with Rough Correlation</i>
2014-04	<b>Tianling Jin, Ali Maalaoui, Luca Martinazzi, Jingang Xiong</b> <i>Existence and Asymptotics for Solutions of a Non-Local <math>Q</math>-Curvature Equation in Dimension Three</i>
2014-05	<b>Marcus J. Grote, Michaela Mehlin, Teodora Mitkova</b> <i>Runge-Kutta Based Explicit Local Time-Stepping Methods for Wave Propagation</i>
2014-06	<b>Hanspeter Kraft, Andriy Rejeta</b> <i>Automorphisms of the Lie Algebra of Vector Fields on Affine <math>n</math>-Space</i>
2014-07	<b>Jérémy Blanc, Alberto Calabri</b> <i>On Degenerations of Plane Cremona Transformations</i>
2014-08	<b>Helmut Harbrecht, Michael Peters, Markus Siebenmorgen</b> <i>Numerical Solution of Elliptic Diffusion Problems on Random Domains</i>
2014-09	<b>H. Harbrecht, W.L. Wendland, and N. Zorii</b> <i>Rapid Solution of Minimal Riesz Energy Problems</i>
2014-10	<b>Julien Diaz, Marcus J. Grote</b> <i>Multi-Level Explicit Local Time-Stepping Methods for Second-Order Wave Equations</i>
2014-11	<b>J. Dölz, H. Harbrecht, Ch. Schwab</b> <i>Covariance Regularity and H-Matrix Approximation for Rough Random Fields</i>
2014-12	<b>Giovanni Alberti, Gianluca Crippa, Anna L. Mazzucato</b> <i>Exponential Self-Similar Mixing and Loss of Regularity for Continuity Equations</i>

## LATEST PREPRINTS

- No.**      **Author:** *Title*
- 2014-13    **M. Dambrine, H. Harbrecht, B. Puig**  
*Computing Quantities of Interest for Random Domains with Second Order Shape Sensitivity Analysis*
- 2014-14    **Monica Bugeanu, Helmut Harbrecht**  
*A Second Order Convergent Trial Method for free Boundary Problems in Three Dimensions*
- 2014-15    **David Masser**  
*Relative Manin-Mumford for Abelian Varieties*
- 2014-16    **D. W. Masser, G. Wüstholz**  
*Polarization Estimates for Abelian Varieties*
- 2014-17    **H. Derksen, D. W. Masser**  
*Linear Equations over Multiplicative Groups, Recurrences, and Mixing II*
- 2014-18    **M. Dambrine, I. Greff, H. Harbrecht, B. Puig**  
*Solution of the Poisson Equation with a Thin Layer of Random Thickness*