Relative Manin-Mumford for Abelian Varieties

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Abstract: *With an eye or two towards applications to Pell's equation and to Davenport's work on integration of algebraic functions, Umberto Zannier and I have recently characterised torsion points on a fixed algebraic curve in a fixed abelian scheme of dimension bigger than one (when all is defined over the algebraic numbers): there are at most finitely many points provided the natural obstacles are absent. I sketch the proof as well as the applications.*

A very simple problem of Manin-Mumford type is: find all roots of unity λ, μ with $\lambda + \mu = 1$. Here the solution is easy: we have $|\lambda| = |1 - \lambda| = 1$ and so in the complex plane λ lies on the intersection of two circles; in fact λ must be one of the two primitive sixth roots of unity (the picture doesn't work too well in positive characteristic, and indeed any non-zero element of any finite field is already a root of unity, so from now on we stick to zero characteristic). This result has something to do with the multiplicative group $\mathbf{G}_{\mathrm{m}}$, which can be regarded as $\mathbf{C}^*$. Actually with $\mathbf{G}_{\mathrm{m}}^2$ and the "line" inside it parametrized by $P = (\lambda, 1 - \lambda)$: we ask just that P is torsion.

Now it is easy to generalize, at least the problem, to other algebraic varieties in other commutative algebraic groups.

For example let E be the elliptic curve defined by $y^2 = x(x-1)(x-4)$. Asking for all complex λ such that the points

$$(2\lambda, \sqrt{2\lambda(2\lambda-1)(2\lambda-4)}), \quad (3\lambda, \sqrt{3\lambda(3\lambda-1)(3\lambda-4)}) \tag{1}$$

are both torsion amounts to asking for torsion points on a certain curve in the surface E^2. But here the solution is much more difficult (and it is not clear to me that one can find all λ explicitly as above).

It was Hindry [H] who solved the general problem with any algebraic variety in any commutative algebraic group G. The outcome for a curve in G is that it contains at most finitely many torsion points unless one of its components is a connected one-dimensional "torsion coset"; that is, a translate $P_0 + H$ of an algebraic subgroup H of G by a torsion point P_0. This H contains infinitely many torsion points and so $P_0 + H$ also.

Thus for $G = \mathbf{G}_{\mathrm{m}}^2$ the analogue for $\lambda\mu = 1$ of the problem above will not lead to finiteness, as the curve is such an H. Similarly $\lambda\mu = -1$ is $P_0 + H$ for $P_0 = (1, -1)$ with $2P_0 = 0$ (written additively).

More generally G can be $\mathbf{G}_{\mathrm{m}}^n$, E^n as in Habegger's talks in this volume, or an abelian variety A as in Orr's article, or products of these, or "twisted products" sitting inside an exact sequence

$$0 \to T \to G \to A \to 0 \tag{2}$$

where T is a power of $\mathbf{G}_{\mathrm{m}}$ or even a product $\mathbf{G}_{\mathrm{a}}^r \times \mathbf{G}_{\mathrm{m}}^s$ for the additive group $\mathbf{G}_{\mathrm{a}} = \mathbf{C}$. Here the twisting can be quite complicated and G can end up very far from just $T \times A$. It is classical that every commutative algebraic group over $\mathbf{C}$ has this form. We will see how several types turn up in applications.

The applications involve most naturally the "relative case", where G itself is allowed to vary in a family. Most of the current results allow only a single parameter here, and we already have parameters in the algebraic variety, so this had better stay a curve, essentially with a parameter λ, and G had better depend on no more than λ. An example like (1) involves the points

$$(2, \sqrt{4 - 2\lambda}), \quad (3, \sqrt{18 - 6\lambda}) \tag{3}$$

now on the elliptic curve E_λ defined by $y^2 = x(x-1)(x-\lambda)$; that is the famous Legendre family. Again we go to E_λ^2, where the square is the "fibre square" defined by the equations

$$y_1^2 = x_1(x_1 - 1)(x_1 - \lambda), \quad y_2^2 = x_2(x_2 - 1)(x_2 - \lambda)$$

with λ in common, and we get a curve defined by $x_1 = 2, x_2 = 3$. Then by [MZ1] there are at most finitely complex values of λ (now not $0, 1$ so that we have a genuine elliptic curve) such that both points in (3) are torsion. Here their effective determination may be a difficult problem in practice and even in principle.

In various works Umberto Zannier and I have treated any curve in any parametrized abelian variety A_λ of "relative dimension" at least two, sometimes with the proviso that everything is defined over the field $\overline{\mathbf{Q}}$ of all algebraic numbers. We get finiteness with a similar condition about torsion cosets, now interpreted schemewise or more intuitively "identically in λ". For example in E_λ^2 above these are defined essentially by the vanishing of non-trivial integral linear combinations $n_1(x_1, y_1) + n_2(x_2, y_2)$. It is not difficult to show that this is impossible for (3). On the other hand

$$2(0,0) = 2(1,0) = 2(\lambda,0) = 4(\sqrt{\lambda}, i(\lambda - \sqrt{\lambda})) = 0$$

so it is not quite easy.

A start has been made on more general G_λ. First with Bertrand and Pillay we have considered

$$1 \to \mathbf{G}_{\mathrm{m}} \to G_\lambda \to E_\lambda \to 0 \qquad (4)$$

and

$$1 \to \mathbf{G}_{\mathrm{m}} \to G_\lambda \to E \to 0. \qquad (5)$$

with E not depending on λ. Bertrand [Be] had already given a surprising counterexample in (5): in rather special situations it is possible to construct what he calls a "Ribet curve" having infinitely many torsion points, even though it is not a torsion coset. We then checked in [BMPZ] that this happens only for Ribet curves.

And Harry Schmidt in Basle [Sc] has done

$$0 \to \mathbf{G}_{\mathrm{a}} \to G_\lambda \to E_\lambda \to 0 \qquad (6)$$

where it is reassuring to find that there are no counterexamples.

We give a short proof sketch of the result for (3). As in Habegger's talks, it hinges on the analytic representation of an elliptic curve as a quotient of $\mathbf{C}$ by a lattice, as in the general strategy of Zannier expounded in [PZ]. For E_λ this lattice Ω_λ depends on λ, and in fact one can take a basis of periods

$$\Omega_\lambda = \mathbf{Z} f_\lambda + \mathbf{Z} g_\lambda$$

where f_λ is hypergeometric $\pi F(\frac{1}{2}, \frac{1}{2}, 1; \lambda)$ and $g_\lambda = i f_{1-\lambda}$. The points (3) correspond in $\mathbf{C}$ to "elliptic logarithms" u_λ, v_λ, say; and since $\mathbf{C} = \mathbf{R} f_\lambda + \mathbf{R} g_\lambda$ there are real functions $p_\lambda, q_\lambda, r_\lambda, s_\lambda$ with

$$u_\lambda = p_\lambda f_\lambda + q_\lambda g_\lambda, \quad v_\lambda = r_\lambda f_\lambda + s_\lambda g_\lambda.$$

As λ moves, the locus of $z_\lambda = (p_\lambda, q_\lambda, r_\lambda, s_\lambda)$ has real dimension two in $\mathbf{R}^4$ and is in fact a (sub-)analytic surface Z. The torsion in $\mathbf{C}$ is $\mathbf{Q} f_\lambda + \mathbf{Q} g_\lambda$, and so our particular λ gives a point of $Z \cap \mathbf{Q}^4$. It is even in $Z \cap \frac{1}{N}\mathbf{Z}^4$ if the torsion order divides N. Such points cannot be very numerous: in Wilkie's talks we saw that the cardinality

$$|Z^{trans} \cap \frac{1}{N}\mathbf{Z}^4 \cap K| \;\leq\; c(\epsilon, Z, K)N^\epsilon$$

for a certain subset Z^{trans} of Z and any compact K (maybe this could be eliminated using o-minimality) and any $\epsilon > 0$. See also Pila [Pil] and Pila-Wilkie [PW]. It may be very difficult to write down $c(\epsilon, Z, K)$ in an effective way.

Here it is possible to show that $Z^{trans} = Z$; this is a concealed algebraic independence result as in Pila's talks, for which the hard (Hodge-theoretical) work was done by André [An].

We deal with K using bounded height: more later. We get at most $c(\epsilon)N^\epsilon$ points z_λ. An easy argument with a faint flavour of zero-estimates of the type used in transcendence theory leads to at most $c(\epsilon)N^\epsilon$ values λ. But right now we don't know any upper bound for N.

In fact it is easy to see that these values λ all lie in $\overline{\mathbf{Q}}$; further any given λ yields $D = [\mathbf{Q}(\lambda) : \mathbf{Q}]$ in all by conjugation λ^σ (compare also Habegger's talks). We deduce

$$D \leq c(\epsilon)N^\epsilon. \tag{7}$$

But there are also lower bounds for D. If we go back to the original problem of $\lambda, 1-\lambda$, now with say λ of exact order N_1 then of course $D = \phi(N_1)$ for the Euler function, and this is classically known to be at least $c_1(\epsilon)N_1^{1-\epsilon}$ (now all constants are assumed positive). For our problem the analogue is that if $(2, \sqrt{4-2\lambda})$ in (3) has exact order N_1 then a famous Theorem of Serre [Se1] implies even

$$D \geq C_1 N_1^2 \tag{8}$$

(which is classical in the case of complex multiplication); but here the elliptic curve depends on λ and therefore so does C_1. Furthermore it is not so easy to calculate this dependence. The work [MW2] (based on transcendence among other things) applies only if N_1 is prime, an assumption we cannot afford. It was extended to arbitrary N_1 by Zywina [Zy], but only for an elliptic curve defined over $\mathbf{Q}$, which we also cannot assume here. Very recently Lombardo [L] has extended the field of definition to $\overline{\mathbf{Q}}$; in a first version the dependence on D was not quite good enough for application here, but he has since fixed this. There is also a dependence on the absolute height $h(\lambda)$ of λ. Fortunately a Theorem of Silverman [Si] implies that this height is bounded above by an absolute constant. Combining everything leads to $D \geq cN_1^\delta$ now with c absolute. Here δ is less than 10^{-10}.

For effectivity purposes it will probably be very convenient to have a bigger δ. This arises from a more direct application of the transcendence methods, starting with an exponent smaller than 2 in (8). Then the very precise version [Davi2] due to David yields $D \geq cN_1^{1/2}$.

Similarly $D \geq cN_2^{1/2}$ for the exact order N_2 of $(3, \sqrt{18-6\lambda})$ in (3). Taking N now as the exact order, we have $N = \mathrm{lcm}(N_1, N_2) \leq N_1 N_2$, and it follows that $D \geq cN^{1/4}$. Comparing this with (7), we see that it suffices to choose $\epsilon = \frac{1}{5}$ to get an absolute bound for $D = [\mathbf{Q}(\lambda) : \mathbf{Q}]$. Combined with the absolute bound on $h(\lambda)$, this gives by a well-known result of Northcott (see below) the required finiteness.

As height bounds were not much mentioned in the other talks, we sketch here a proof that

$$h(\lambda) \leq 6$$

for the absolute height

$$h(\lambda) \;=\; \frac{1}{D} \log \left(|a_0| \prod_\sigma \max\{1, |\lambda^\sigma|\} \right), \tag{9}$$

where $a_0 \lambda^D + \cdots = 0$ is the minimal equation for λ over $\mathbf{Z}$. All we use is $N_1 P_1 = 0$ for $P_1 = (2, \sqrt{4-2\lambda})$, where the value of $N_1 \geq 1$ is now irrelevant.

For any algebraic λ and $P = (x, y)$ on E_λ with algebraic x, y we can reasonably define $h(P) = h(x)$ (but in Habegger's talks it was $h(x)/2$), as y is determined by x and λ. For example $h(P_1) = \log 2$. The Néron-Tate height $\hat{h}(P)$ is defined for example by

$$\hat{h}(P) = \lim_{k \to \infty} \frac{h(2^k P)}{4^k},$$

and $|\hat{h}(P) - h(P)|$ is bounded above independently of P. Explicit bounds for Weierstrass elliptic curves are practically classical, but I calculated for Legendre

$$|\hat{h}(P) - h(P)| \;\leq\; \frac{5}{3} h(\lambda) + c \tag{10}$$

with explicit c absolute.

Further $\hat{h}(P) = 0$ if and only if P is torsion.

We look at $4P_1$ on E_λ. With the help of classical duplication formulae we find

$$4P_1 = \left(\frac{A(\lambda)}{B(\lambda)}, *\right)$$

with A, B in $\mathbf{Z}[t]$ of degrees 8 and 7 respectively. In fact

$$A = t^8 - 160t^7 + 7104t^6 - 57344t^5 + 206336t^4 - 401408t^3 + 442368t^2 - 262144t + 65536,$$

$$B_4 = -288t^7 + 3648t^6 - 17408t^5 + 38912t^4 - 40960t^3 + 16384t^2.$$

Now $h(\lambda^8) = 8h(\lambda)$ (not transparent from (9) by the way) and similarly one can show, after a bit of effort with resultants, that

$$h(4P_1) \geq 8h(\lambda) - c'$$

with c' absolute.

On the other hand (10) gives

$$h(4P_1) \;\leq\; \hat{h}(4P_1) + \frac{5}{3}h(\lambda) + c \;=\; \frac{5}{3}h(\lambda) + c$$

so

$$h(\lambda) \;\leq\; \frac{c + c'}{19/3} \;<\; 6$$

with some extra computation.

Here the Northcott result just mentioned becomes clear: if $h(\lambda)$ and D are both bounded above, then so are $|a_0|$ and the $|\lambda^\sigma|$ in (9), and then so are the absolute values of the coefficients in the minimal polynomial $a_0 \prod_\sigma (t - \lambda^\sigma)$.

This completes the sketch for (3) and E_λ^2. The general curve in E_λ^2 was treated in [MZ2], and more general products like $E_\lambda \times E_{-\lambda}$ in [MZ3]. For A_λ as in [MZ4] and [MZ5] there are several extra technicalities. The results of André and Silverman apply also to the abelian situation. But despite the enormous advances by Serre in [Se2], still the extensions of [Se1] seem to be less clear-cut, even for powers of a fixed prime, let alone effective. But once more a transcendence approach succeeds, and we use David's result in [Davi1] pre-dating [Davi2]. In fact this result seems to require that the value λ is such that A_λ is simple. At first sight this looks like a problem. At second sight one suspects that such λ are probably rare, possibly controlled by conjectures of André-Oort-Pink-Zilber type (see [Pin] and [Zi] for example). At third sight one realizes that such conjectures are not yet proved. But finally by going back into the proof in [Davi1] to winkle out the "obstruction subgroup" in the zero-estimate, one sees that some easy tricks from the geometry of numbers (as in [MW1] for example) suffice. One ends up with $D \geq cN^{\delta}$ with a ridiculously small exponent δ (depending on the dimension of A_λ). One could also use [MW3] to factorize the non-simple A_λ, but then the exponent would be even smaller. Probably recent work of Gaudron and Rémond [GR] would give more reasonable values.

Now for the applications of these results, denoted by (I) and (II) below.

(I) All know that Pell's equation

$$x^2 - dy^2 = 1, \quad y \neq 0$$

is solvable over $\mathbf{Z}$ provided $d > 0$ in $\mathbf{Z}$ is not a square. Moving to the polynomial ring $\mathbf{C}[t]$, by now a knee-jerk reaction, especially in view of "abcology", we consider

$$X^2 - DY^2 = 1, \quad Y \neq 0 \tag{11}$$

with D in $\mathbf{C}[t]$ not a square, surely easier. But in fact it is much more difficult to describe the set of D for which there is solvability over $\mathbf{C}[t]$. One can easily see that the degree m of D must be even. Now we proceed systematically.

$\underline{m = 2}$: there is always solvability. Thus for $D = at^2 + bt + c$ we can take

$$X = \frac{2at + b}{\sqrt{b^2 - 4ac}}, \quad Y = \frac{2\sqrt{a}}{\sqrt{b^2 - 4ac}}.$$

$\underline{m = 4}$: there is not always solvability. For example not for $D = t^4 + t + 1$. And in the family $D = t^4 + t + \lambda$ we have solvability exactly when λ lies in a certain countable subset of $\mathbf{C}$. In fact precisely when the point $(0, 1)$ is torsion on $y^2 = x^3 - 4\lambda x + 1$. This is essentially classical (Abel [Ab], Chebychev [C1],[C2]). In fact the set is infinite (which is not classical - for several proofs see [Za] pages 92,93 for example), as one might guess from its element (with its six conjugates)

$$\lambda = \frac{\sqrt[3]{2\sqrt{2} - 2}}{2}$$

where

$$X = \frac{(4 - 32\lambda^3)t^5 - (4\lambda - 16\lambda^4)t^4 + 4\lambda^2 t^3 + (3 - 28\lambda^3)t^2 - 8\lambda^4 t + 8\lambda^5}{32\lambda^8},$$

$$Y = \frac{(4-32\lambda^3)t^3 - (4\lambda - 16\lambda^4)t^2 + 4\lambda^2 t + (1-12\lambda^3)}{32\lambda^8}.$$

$\underline{m=6}$: there is rarely solvability. For example there are only finitely many λ in $\mathbf{C}$ for which solvability holds for $D = t^6 + t + \lambda$. This is proved in [MZ4] using the general result on A_λ described above. In fact here A_λ is the Jacobian of the hyperelliptic curve $s^2 = t^6 + t + \lambda$ of genus 2, or better a complete non-singular model

$$s^2 = t_3^2 + t_0 t_1 + \lambda t_0^2, \quad t_0 t_2 = t_1^2, \quad t_0 t_3 = t_1 t_2.$$

The curve inside A_λ is the locus of the divisor $\Delta_\lambda = \infty_\lambda^+ - \infty_\lambda^-$ as λ varies, where $\infty_\lambda^\pm$ are the two places at infinity. When (11) holds we write it as $f_\lambda^+ f_\lambda^- = 1$ with the functions $f_\lambda^\pm = X \pm sY$ to see that their divisors are multiples of Δ_λ thus giving a torsion point.

There are actually some λ; for example $\lambda = 0$ with

$$X = 2t^5 + 1, \quad Y = 2t^2.$$

$\underline{m \geq 8}$: even rarer. For example with the family $D = d_0(\lambda)t^m + \cdots + d_m(\lambda)$ in $\overline{\mathbf{Q}}[\lambda][t]$, say for safety identically squarefree, there is solvability for infinitely many λ in $\overline{\mathbf{Q}}$ only if the analogous Jacobian, now an abelian variety of dimension $\frac{m-2}{2} \geq 3$, contains an elliptic curve. This also follows from the A_λ result in [MZ5].

Incidentally, if we want to go beyond squarefree, then we can use the result of [BMPZ] on multiplicative extensions (4). Thus for $D = t^2(t^4 + t + \lambda)$ we get at most a finite set, despite the infinite set for $t^4 + t + \lambda$. And also for $D = t^3(t^3 + t + \lambda)$ using the additive extensions (6).

(II) This concerns the old problem of "integrating in elementary terms" (see for example the article [R] by Risch). By the way, the integration may be elementary but it need not be easy (just as for some proofs), as a wonderful example

$$\int \frac{\sqrt{1+t^4}}{1-t^4}\,dt \;=\; -\frac{1}{4}\sqrt{2}\log\left(\frac{\sqrt{2}t - \sqrt{1+t^4}}{1-t^2}\right) - \frac{i}{4}\sqrt{2}\log\left(\frac{i\sqrt{2}t + \sqrt{1+t^4}}{1+t^2}\right) \tag{12}$$

due to Euler shows. Not only can my Maple (version 9) not do the integration on the left-hand side, but it cannot even check the result by differentiating the right-hand side. Actually Euler's version was

$$\frac{1}{4}\sqrt{2}\log\left(\frac{\sqrt{2}t + \sqrt{1+t^4}}{1-t^2}\right) + \frac{1}{4}\sqrt{2}\arcsin\left(\frac{\sqrt{2}t}{1+t^2}\right)$$

thus staying over $\mathbf{R}$.

We give some more examples in the above hyperelliptic context.

$\underline{m=2}$: now $\int \frac{dt}{\sqrt{D}}$ is always integrable - see any engineer's handbook of indefinite integrals.

$\underline{m=4}$: now $\int \frac{dt}{\sqrt{t^4+t+\lambda}}$ is integrable if and only if $256\lambda^3 = 27$; then we can reduce it to

$$\int \frac{dt}{(t+4\lambda/3)\sqrt{t^2+\mu t+\nu}}$$

and run to the handbook.

$\underline{m=6}$: now the same methods show that $\int \frac{dt}{\sqrt{t^6+t+\lambda}}$ is integrable only if $46656\lambda^5 = 3125$, and with a bit more effort never.

However up to now all that is in fact relatively easy, and not at the same level as Pell. But

$$\int \frac{dt}{t\sqrt{t^4+t+\lambda}}$$

is integrable if and only if λ lies in a certain finite set. Oddly enough the proof does not use multiplicative extensions as for $t^2(t^4+t+\lambda)$ above but rather Schmidt's result [Sc] on additive extensions (6). Incidentally, he has made such results effective, using among other things a version [Ma] of the original result of Bombieri-Pila [BP] obtained as in Wilkie's talks with the Siegel Lemma). For example he shows that there are at most $e^{e^{e^{e^5}}}$ complex values of λ for which

$$\int \frac{dt}{(t-2)\sqrt{t(t-1)(t-\lambda)}}$$

is integrable. This is related to (3): thus integrability implies that $(2, \sqrt{4-2\lambda})$ is torsion on E_λ. But the converse fails, so we cannot deduce infinitely many λ. In fact we get a torsion point even on a suitable G_λ as in (6), so we may conclude finiteness.

And also by [MZ5]

$$\int \frac{dt}{t\sqrt{t^6+t+\lambda}}$$

is integrable at most on a finite set; but no-one knows how to make this effective. Here we use A_λ as above, but now with the locus of $\Gamma_\lambda = P_\lambda^+ - P_\lambda^-$, where $P_\lambda^\pm = (0, \pm\sqrt{\lambda})$; this time the torsion property arises from a classical criterion of Liouville, which implies that the integral, if elementary, must involve a single $\log g_\lambda$ (as opposed to (12) with a pair).

Now differentiation (without Maple) gives $dt/(ts) = cdg_\lambda/g_\lambda$ for c in $\mathbf{C}$, from which we see that the divisor of g_λ is a multiple of Γ_λ.

These examples support an assertion of James Davenport [Dave] from 1981.

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