The decomposition

group of a line

in the plane

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THE DECOMPOSITION GROUP OF A LINE IN THE PLANE

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Abstract. We show that the decomposition group of a line $L$ in the plane, i.e. the subgroup of plane birational transformations that send $L$ to itself birationally, is generated by its elements of degree 1 and one element of degree 2, and that it does not decompose as a non-trivial amalgamated product.

1. Introduction

We denote by $\text{Bir}(\mathbb{P}^2)$ the group of birational transformations of the projective plane $\mathbb{P}^2 = \text{Proj}(k[x, y, z])$, where $k$ is an algebraically closed field. Let $C \subset \mathbb{P}^2$ be a curve, and let

$$\text{Dec}(C) = \{ \varphi \in \text{Bir}(\mathbb{P}^2), \varphi(C) \subset C \text{ and } \varphi|_C : C \to C \text{ is birational} \}.$$ 

This group has been studied for curves of genus $\geq 1$ in [BPV2009], where it is linked to the classification of finite subgroups of $\text{Bir}(\mathbb{P}^2)$. It has a natural subgroup $\text{Ine}(C)$, the inertia group of $C$, consisting of elements that fix $C$, and Blanc, Pan and Vust give the following result: for any line $L \subset \mathbb{P}^2$, the action of $\text{Dec}(L)$ on $L$ induces a split exact sequence

$$0 \to \text{Ine}(L) \to \text{Dec}(L) \to \text{PGL}_2 = \text{Aut}(L) \to 0$$

and $\text{Ine}(L)$ is neither finite nor abelian and also it doesn’t leave any pencil of rational curves invariant [BPV2009, Proposition 4.1]. Further they ask the question whether $\text{Dec}(L)$ is generated by its elements of degree 1 and 2 [BPV2009, Question 4.1.2].

We give an affirmative answer to their question in the form of the following result, similar to the Noether-Castelnuovo theorem [Cas1901] which states that $\text{Bir}(\mathbb{P}^2)$ is generated by $\sigma : [x : y : z] \mapsto [yz : xz : xy]$ and $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3$.

Theorem 1. For any line $L \subset \mathbb{P}^2$, the group $\text{Dec}(L)$ is generated by $\text{Dec}(L) \cap \text{PGL}_3$ and any of its quadratic elements having three proper base points in $\mathbb{P}^2$.

The similarities between $\text{Dec}(L)$ and $\text{Bir}(\mathbb{P}^2)$ go further than this. Cornulier shows in [Cor2013] that $\text{Bir}(\mathbb{P}^2)$ cannot be written as an amalgamated product in any nontrivial way, and we modify his proof to obtain an analogous result for $\text{Dec}(L)$.

Theorem 2. The decomposition group $\text{Dec}(L)$ of a line $L \subset \mathbb{P}^2$ does not decompose as a non-trivial amalgam.

The article is organised as follows: in Section 2 we show that for any element of $\text{Dec}(L)$ we can find a decomposition in $\text{Bir}(\mathbb{P}^2)$ into quadratic maps such that the successive images of $L$ are curves (Proposition 2.6), i.e. the line is not contracted to a point at any
time. We then show in Section 3 that we can modify this decomposition, still in Bir(\(\mathbb{P}^2\)), into de Jonquières maps where all of the successive images of \(L\) have degree 1, i.e. they are lines. Finally we prove Theorem 1. Our main sources of inspiration for techniques and ideas in Section 3 have been [AC2002, \S 8.4, \S 8.5] and [Bla2012]. In Section 4 we prove Theorem 2 using ideas that are strongly inspired by [Cor2013].

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2. Avoiding to contract \(L\)

Given a birational map \(\rho : \mathbb{P}^2 \to \mathbb{P}^2\), the Noether-Castelnuovo theorem states that there is a decomposition \(\rho = \rho_m\rho_{m-1} \cdots \rho_1\) of \(\rho\) where each \(\rho_i\) is a quadratic map with three proper base points. This decomposition is far from unique, and the aim of this section is to show that if \(\rho \in \text{Dec}(L)\), we can choose the \(\rho_i\) so that none of the successive birational maps \((\rho_i \cdots \rho_1 : \mathbb{P}^2 \to \mathbb{P}^2)^{m_i}\) contracts \(L\) to a point. This is Proposition 2.6.

Given a birational map \(\varphi : X \to Y\) between smooth projective surfaces, and a curve \(C \subset X\) which is contracted by \(\varphi\), we denote by \(\pi_1 : Z_1 \to Y\) the blowup of the point \(\varphi(C) \in Y\). If \(C\) is contracted also by the birational map \(\pi_1^{-1}\varphi : X \to Z_1\), we denote by \(\pi_2 : Z_2 \to Z_1\) the blowup of \((\pi_1^{-1}\varphi)(C) \in Z_1\) and consider the birational map \((\pi_1\pi_2)^{-1}\varphi : X \to Z_2\). If this map too contracts \(C\), we denote by \(\pi_3 : Z_3 \to Z_2\) the blowup of the point onto which \(C\) is contracted. Repeating this procedure a finite number of times \(D \in \mathbb{N}\), we finally arrive at a variety \(Z := Z_D\) and a birational morphism \(\pi := \pi_1\pi_2 \cdots \pi_D : Z \to Y\) such that \((\pi^{-1}\varphi)\) does not contract \(C\). Then \((\pi^{-1}\varphi)|_C : C \to (\pi^{-1}\varphi)(C)\) is a birational map.

**Definition 2.1.** In the above situation, we denote by \(D(C, \varphi) \in \mathbb{N}\) the minimal number of blowups which are needed in order to not contract the curve \(C\) and we say that \(C\) is contracted \(D(C, \varphi)\) times by \(\varphi\). In particular, a curve \(C\) is sent to a curve by \(\varphi\) if and only if \(D(C, \varphi) = 0\).

**Remark 2.2.** The integer \(D(C, \varphi)\) can equivalently be defined as the order of vanishing of \(K_Z - \pi^*(K_Y)\) along \((\pi^{-1}\varphi)(C)\).

We recall the following well known fact, which will be used a number of times in the sequel.

**Lemma 2.3.** Let \(\varphi_1, \varphi_2 \in \text{Bir}(\mathbb{P}^2)\) be birational maps of degree 2 with proper base points \(p_1, p_2, p_3\) and \(q_1, q_2, q_3\) respectively. If \(\varphi_1\) and \(\varphi_2\) have (exactly) two common base points, say \(p_1 = q_1\) and \(p_2 = q_2\), then the composition \(\tau = \varphi_2\varphi_1^{-1}\) is quadratic. Furthermore the three base points of \(\tau\) are proper points of \(\mathbb{P}^2\) if and only if \(q_3\) is not on any of the lines joining two of the \(p_i\).

**Proof.** The lemma is proved by Figure 1, where squares and circles in \(\mathbb{P}^2\) denote the base points of \(\varphi_1\) and \(\varphi_2\) respectively. The crosses in \(\mathbb{P}^2\) denote the base points of \(\varphi_1^{-1}\) (corresponding to the lines in \(\mathbb{P}^2\)), and the conics in \(\mathbb{P}^2\) and \(\mathbb{P}^2\) denote the pullback of a general line \(\ell \in \mathbb{P}^2\).

If \(q_3\) is not on any of the three lines, the base points of \(\tau\) are \(E_1, E_2, \varphi_1(q_3)\). If \(q_3\) is on one of the three lines, then the base points of \(\tau\) are \(E_1, E_2\) and a point infinitely close to the \(E_i\), which corresponds to the line that \(q_3\) is on. \(\Box\)
The following lemma describes how the number of times that a line is contracted changes when composing with a quadratic transformation of $\mathbb{P}^2$ with three proper base points.

**Lemma 2.4.** Let $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and let $\varphi': \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic birational map with base points $q_1, q_2, q_3 \in \mathbb{P}^2$. For $1 \leq i < j \leq 3$ we denote by $\ell_{ij} \subset \mathbb{P}^2$ the line which joins the base points $q_i$ and $q_j$. If $D(L, \varphi) = k \geq 1$, we have

$$D(L, \varphi') = \begin{cases} k + 1 & \text{if } \varphi(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \text{Bp}(\varphi), \\ k & \text{if } \varphi(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}, \\ k & \text{if } \varphi(L) = q_i \text{ for some } i, \text{ and } (\varphi' \varphi)(L) \in \text{Bp}(\varphi^{-1}), \\ k - 1 & \text{if } \varphi(L) = q_i \text{ for some } i, \text{ and } (\varphi' \varphi)(L) \notin \text{Bp}(\varphi^{-1}). \end{cases}$$

**Proof.** We consider the minimal resolutions of $\varphi'$; in Figures 2-5, the filled black dots denote the successive images of $L$, i.e. $\varphi(L)$, $(\varphi^{-1} \varphi)(L)$ and $(\varphi^{-1} \varphi)(L)$ respectively.

We argue by Figure 2 and 3 in the case where $\varphi(L)$ does not coincide with any of the base points of $\varphi$. If $\varphi(L) \in \ell_{ij}$ for some $i, j$, then $D(L, \varphi') = D(L, \varphi) + 1$, since $\ell_{ij}$ is contracted by $\varphi$. Otherwise, the number of times $L$ is contracted does not change. Suppose that $\varphi(L) = q_i$ for some $i$. If $D(L, \varphi) = 1$, we have $(\varphi^{-1} \varphi)(L) = E_i$, and then

**Figure 1.** The composition of $\varphi_1$ and $\varphi_2$ in Lemma 2.3

**Figure 2.** $D(L, \varphi') = k + 1$; $\varphi(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \text{Bp}(\varphi)$.

**Figure 3.** $D(L, \varphi') = k$; $\varphi(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}$. 

The following lemma describes how the number of times that a line is contracted changes when composing with a quadratic transformation of $\mathbb{P}^2$ with three proper base points.
clearly $D(L, \varphi \rho) = 0$ since $E_i$ is not contracted by $\eta$. If $D(L, \rho) \geq 2$ we argue by the Figures 4 and 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{\(D(L, \varphi \rho) = k; \rho(L) = q_i\) and \((\rho \varphi)(L) \in \text{Bp}(\varphi^{-1})\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{\(D(L, \varphi \rho) = k - 1; \rho(L) = q_i\) and \((\rho \varphi)(L) \notin \text{Bp}(\varphi^{-1})\).}
\end{figure}

**Remark 2.5.** If $D(L, \rho) \geq 2$, then the point $(\pi^{-1} \rho)(L)$ in the first neighbourhood of $\rho(L)$ defines a tangent direction at $\rho(L) \in \mathbb{P}^2$. If we take $\varphi$ as in Lemma 2.4 with $q_i \in \text{Bp}(\varphi)$ for some $i$, then this tangent direction coincides with the direction of one of $\ell_{ij}, \ell_{ik}$ if and only if $(\rho \varphi)(L) \in \text{Bp}(\varphi^{-1})$.

**Proposition 2.6.** For any given element $\rho \in \text{Dec}(L)$, there is a decomposition of $\rho$ into quadratic maps $\rho = \rho_m \ldots \rho_1$ with three proper base points such that none of the successive compositions $(\rho_i \ldots \rho_1)_{i=1}^m$ contract $L$ to a point.

**Proof.** Let $\rho = \rho_m \ldots \rho_1$ be a decomposition of $\rho$ into quadratic maps with only proper base points. We can assume that $d := \max\{D(L, \rho_j \ldots \rho_1) \mid 1 \leq j \leq m\} > 0$, otherwise we are done. Let $n := \max\{j \mid D(L, \rho_j \ldots \rho_1) = d\}$. We denote the base points of $\rho_n^{-1}$ and $\rho_{n+1}$ by $p_1, p_2, p_3$ and $q_1, q_2, q_3$ respectively.

We first look at the case where $D(L, \rho_{n-1} \ldots \rho_1) = D(L, \rho_{n+1} \ldots \rho_1) = d - 1$. Then composition with $\rho_n$ and $\rho_{n+1}$ fall under Cases 1 and 4 of Lemma 2.4, so both $\rho_n^{-1}$ and $\rho_{n+1}$ have a base point at $(\rho_n \ldots \rho_1)(L) \in \mathbb{P}^2$. We may assume that this point is $p_1 = q_1$, as in Figure 6. Interchanging the roles of $q_2$ and $q_3$ if necessary, we may assume that $p_1, p_2, q_2$ are not collinear. Let $r \in \mathbb{P}^2$ be a general point, and let $c_1$ and $c_2$ denote quadratic maps with base points $[p_1, p_2, r]$ and $[p_1, q_2, r]$ respectively; then the maps $\tau_1, \tau_2, \tau_3$ (defined by the commutative diagram in Figure 6) are quadratic with three proper base points in $\mathbb{P}^2$. Note that $D(L, \tau_i \ldots \tau_1 \rho_{n-1} \ldots \rho_1) = d - 1$ for $i = 1, 2, 3$. Thus we obtained a new decomposition of $\rho$ into quadratic maps with three proper base points

$$\rho = \rho_m \ldots \rho_{n+2} \tau_3 \tau_2 \tau_1 \rho_{n-1} \ldots \rho_1,$$

where the number of instances where $L$ is contracted $d$ times has decreased by 1.

Now assume instead that $D(L, \rho_{n-1} \ldots \rho_1) = d$ and $D(L, \rho_{n+1} \ldots \rho_1) = d - 1$. Then composition with $\rho_{n+1}$ falls under Case 4 of Lemma 2.4, so $(\rho_n \ldots \rho_1)(L)$ is a base point of $\rho_{n+1}$, which we may assume to be $q_1$. Furthermore composition with $\rho_n$ falls under
we conclude by induction. Dratic maps with three proper base points $p$ from the tangent direction corresponding to $(\cdot \cdot \cdot)$ are quadratic with three proper base points and be general points and define quadratic maps that it corresponds to the direction of the line through $p$ and $\cdot \cdot \cdot$. Note also that renumbering the three directions at $\cdot \cdot \cdot$ is needed if $\cdot \cdot \cdot$. Only for $\cdot \cdot \cdot$ proper base points respectively as $\cdot \cdot \cdot$. With base points $\cdot \cdot \cdot$, we are in Case 4 of Lemma $\cdot \cdot \cdot$ and obtain $D(L, c_{\cdot \cdot \cdot}, r, \cdot \cdot \cdot) = D(L, \cdot \cdot \cdot, r, \cdot \cdot \cdot) - 1$. Let $r, s \in \mathbb{P}^2$ be two general points and define $c_1, c_2, c_3$ with three proper base points respectively as $\cdot \cdot \cdot$. Note that the corresponding maps $\cdot \cdot \cdot$, defined in an analogous way as in Figure 6, are quadratic with three proper base points. Note also that $D(L, c_{\cdot \cdot \cdot}) = D(L, \cdot \cdot \cdot) - 1$ for $i = 2, 3, 4$. Only for $i = 4$ this is not immediately clear, so suppose that this is not the case, i.e. $D(L, c_4 \cdot \cdot \cdot) = D(L, \cdot \cdot \cdot)$. It follows that $D(L, \cdot \cdot \cdot) \geq 2$ and that the tangent direction corresponding to $(\cdot \cdot \cdot)(L)$ is given by the line through $q_1$ and $q_2$, but this is not possible by the assumption that $D(L, \cdot \cdot \cdot) = d - 1$.

In the second case we have $p_1 = q_1$ and the tangent direction at $p_1 = q_1$ corresponding to $(\cdot \cdot \cdot)(L)$ is the direction either of the line through $p_1$ and $p_3$ or the line through $p_1$ and $p_3$ (see Figure 4). By interchanging the roles of $p_2$ and $p_3$ if necessary, we may assume that it corresponds to the direction of the line through $p_1$ and $p_3$. Interchanging the roles of $q_2$ and $q_3$ if necessary, we may assume that $p_1, q_2, p_3$ are not collinear. Let $r, s \in \mathbb{P}^2$ be general points and define quadratic maps $c_1, c_2, c_3$ with three proper base points respectively by $\cdot \cdot \cdot$, $\cdot \cdot \cdot$, $\cdot \cdot \cdot$. Then the corresponding maps $\cdot \cdot \cdot$ are quadratic with three proper base points and $D(L, c_{\cdot \cdot \cdot}) = D(L, \cdot \cdot \cdot) - 1$ for $i = 1, 2, 3$. The latter holds for $c_3$ since the direction given by $p_1$ and $p_3$ is different from the tangent direction corresponding to $(\cdot \cdot \cdot)(L)$, and for $c_3$ it follows from the assumption that the image of $L$ is contracted $d - 1$ times by $(\cdot \cdot \cdot)$ and that $p_1, q_2, p_3$ are not collinear.

Both in the first and second case, we again arrive at a new decomposition into quadratic maps with three proper base points $\rho = \rho_n \cdot \cdot \cdot \cdot \rho_1 (j \in \{4, 5\})$, where the number of instances where $L$ is contracted $d$ times has decreased by 1, and we conclude by induction. \qed
3. Avoiding to send \( L \) to a curve of degree higher than 1.

By Proposition 2.6, any element \( \rho \in \text{Dec}(L) \) can be decomposed as

\[
\rho = \rho_n \ldots \rho_1
\]

where each \( \rho_j \) is quadratic with three proper base points, and all of the successive images \( ((\rho_1 \ldots \rho_1)(L))_{i=1}^n \) of \( L \) are curves. The aim of this section is to show that the \( \rho_j \) even can be chosen so that all of these curves have degree 1. That is, we find a decomposition of \( \rho \) into quadratic maps such that all the successive images of \( L \) are lines. This means in particular that \( \text{Dec}(L) \) is generated by its elements of degree 1 and 2.

**Definition 3.1.** A birational transformation of \( \mathbb{P}^2 \) is called de Jonquières if it preserves the pencil of lines passing through \([1 : 0 : 0]\) \( \in \mathbb{P}^2 \). These transformations form a subgroup of \( \text{Bir}(\mathbb{P}^2) \) which we denote by \( \mathcal{J} \).

**Remark 3.2.** In [AC2002], a de Jonquières map is defined by the slightly less restrictive property that it sends a pencil of lines to a pencil of lines. Given a map with this property, we can always obtain an element in \( \mathcal{J} \) by composing from left and right with elements of \( \text{PGL}_3 \).

For a curve \( C \subset \mathbb{P}^2 \) and a point \( p \) in \( \mathbb{P}^2 \) or infinitely near, we denote by \( m_C(p) \) the multiplicity of \( C \) in \( p \). If it is clear from context which curve we are referring to, we will use the notation \( m(p) \).

**Lemma 3.3.** Let \( \varphi \in \mathcal{J} \) be of degree \( e \geq 2 \), and \( C \subset \mathbb{P}^2 \) a curve of degree \( d \). Suppose that

\[
\deg(\varphi(C)) \leq d.
\]

Then there exist two base points \( q_1, q_2 \) of \( \varphi \) different from \([1 : 0 : 0]\) such that

\[
m_C([1 : 0 : 0]) + m_C(q_1) + m_C(q_2) \geq d.
\]

This inequality can be made strict in case \( \deg(\varphi(C)) < d \), with a completely analogous proof.

**Proof.** Since \( \varphi \in \mathcal{J} \) is of degree \( e \), it has exactly \( 2e - 1 \) base points \( r_0 := [1 : 0 : 0], r_1, \ldots, r_{2e-2} \) of multiplicity \( e - 1, 1, \ldots, 1 \) respectively. Then

\[
d \geq \deg(\varphi(C)) = ed - (e - 1)m_C(r_0) - \sum_{i=1}^{e-1}(m_C(r_{2i-1}) + m_C(r_{2i}))
\]

\[
=d + \sum_{i=1}^{e-1}(d - m_C(r_0) - m_C(r_{2i-1}) - m_C(r_{2i}))
\]

Hence there exist \( i_0 \) such that \( d \leq m_C(r_0) + m_C(r_{2i_0-1}) + m_C(r_{2i_0}) \).

**Remark 3.4.** Note also that we can choose the points \( q_1, q_2 \) such that \( q_1 \) either is a proper point in \( \mathbb{P}^2 \) or in the first neighbourhood of \([1 : 0 : 0]\), and that \( q_2 \) either is proper point of \( \mathbb{P}^2 \) or is in the first neighbourhood of \([1 : 0 : 0]\) or \( q_1 \).

**Remark 3.5.** A quadratic map sends a pencil of lines through one of its base points to a pencil of lines, and we conclude from Proposition 2.6 and Remark 3.2 that there exists maps \( \alpha_1, \ldots, \alpha_{m+1} \in \text{PGL}_3 \) and \( \rho_i \in \mathcal{J} \setminus \text{PGL}_3 \) such that

\[
\rho = \alpha_{m+1} \rho_m \alpha_m \rho_{m-1} \alpha_{m-1} \ldots \alpha_2 \rho_1 \alpha_1
\]

and such that all of the successive images of \( L \) with respect to this decomposition are curves.
The following proposition is an analogue of the classical Castelnuovo’s Theorem stating that any map in Bir($\mathbb{P}^2$) is a product of de Jonquières maps.

**Proposition 3.6.** Let $\rho \in \text{Dec}(L)$. Then there exists $\rho_i \in \mathcal{J} \setminus \text{PGL}_3$ and $\alpha_i \in \text{PGL}_3$ such that $\rho = \alpha_{m+1} \rho_m \alpha_m \rho_{m-1} \alpha_{m-1} \ldots \alpha_2 \rho_1 \alpha_1$ and all of the successive images of $L$ are lines.

**Proof.** Start with a decomposition $\rho = \alpha_{m+1} \rho_m \alpha_m \rho_{m-1} \alpha_{m-1} \ldots \alpha_2 \rho_1 \alpha_1$ as in Remark 3.5.

Denote $C_i := (\rho_i \alpha_i \cdots \rho_1 \alpha_1)(L) \subset \mathbb{P}^2$, $d_i := \deg(C_i)$ and let

$$D := \max\{d_i \mid i = 1, \ldots, m\}, \quad n := \max\{i \mid D = d_i\}, \quad k := \sum_{i=1}^n(\deg\rho_i - 1).$$

We use induction on the lexicographically ordered pair $(D, k)$.

We may assume that $D > 1$, otherwise our goal is already achieved. We may also assume that $\alpha_{n+1} \notin \mathcal{J}$, otherwise the pair $(D, k)$ decreases as we replace the three maps $\rho_{n+1}, \alpha_{n+1}, \rho_n$ by their composition $\rho_{n+1} \alpha_{n+1} \rho_n \in \mathcal{J}$. Indeed, either $D$ decreases, or $D$ stays the same while $k$ decreases at least by $\deg\rho_n - 1$. Using Lemma 3.3, we find simple base points $p_1, p_2$ of $\rho_{n+1}^{-1}$ and simple base points $\tilde{q}_1, \tilde{q}_2$ of $\rho_{n+1}^{-1}$, all different from $p_0 := [1 : 0 : 0]$, such that

$$m_{C_n}(p_0) + m_{C_n}(p_1) + m_{C_n}(p_2) \geq D$$

and

$$m_{\alpha_{n+1}(C_n)}(p_0) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_1) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_2) > D.$$ 

We choose $p_1, p_2, \tilde{q}_1, \tilde{q}_2$ as in Remark 3.4. By slight abuse of notation, we denote by $q_0 = \alpha_{n+1}^{-1}(p_0)$, $q_1 = \alpha_{n+1}^{-1}(\tilde{q}_1)$ and $q_2 = \alpha_{n+1}^{-1}(\tilde{q}_2)$ respectively the (proper or infinitely near) points in $\mathbb{P}^2$ that correspond to $p_0, \tilde{q}_1,$ and $\tilde{q}_2$ under the isomorphism $\alpha_{n+1}^{-1}$. Note that $p_0$ and $q_0$ are two distinct points of $\mathbb{P}^2$ since $\alpha_{n+1} \notin \mathcal{J}$. We number the points so that $m(p_1) \geq m(p_2)$, $m(q_1) \geq m(q_2)$ and so that if $p_i$ (resp. $q_i$) is infinitely near $p_j$ (resp. $q_j$), then $j < i$.

We study two cases separately depending on the multiplicities of the base points.

Case (a): $m(q_0) \geq m(q_1)$ and $m(p_0) \geq m(p_1)$. Then we find two quadratic maps $\tau', \tau \in \mathcal{J}$ and $\beta \in \text{PGL}_3$ so that $\rho_{n+1} \alpha_{n+1} \rho_n = (\rho_{n+1} \tau_1)^{-1} \beta(\tau \rho_n)$ and so that the pair $(D, k)$ is reduced as we replace the sequence $\rho_{n+1}, \alpha_{n+1}, \rho_n$ by $(\rho_{n+1} \tau_1^{-1}, \beta, \tau \rho_n)$. The procedure goes as follows.

If possible we choose a point $r \in \{p_1, q_1\} \setminus \{p_0, q_0\}$. Should this set be empty, i.e. $p_0 = q_1$ and $p_1 = q_0$, we choose $r = q_2$ instead. The ordering of the points implies that the point $r$ is either a proper point in $\mathbb{P}^2$ or in the first neighbourhood of $p_0$ or $q_0$. Furthermore, the assumption implies that $m(p_0) + m(q_0) + m(r) > D$, so $r$ is not on the line passing through $p_0$ and $q_0$. In particular, there exists a quadratic map $\tau \in \mathcal{J}$ with base points $p_0, q_0, r$; then

$$\deg(\tau(C_n)) = 2D - m(p_0) - m(q_0) - m(r) < D.$$ 

Choose $\beta \in \text{PGL}_3$ so that the quadratic map $\tau' := \beta \tau(\alpha_{n+1})^{-1}$ in the below commutative diagram is de Jonquières – this is possible since $\tau$ has $q_0$ as a base point. This decreases the pair $(D, k)$. 

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Case (b): $m(p_0) < m(p_1)$. Let $\tau$ be a quadratic de Jonquières map with base points $p_0, p_1, p_2$. This is possible since our assumption implies that $p_1$ is a proper base point and because $p_0, p_1, p_2$ are base points of $\rho_n^{-1}$ of multiplicity $\deg \rho_n = 1, 1, 1$ respectively and hence not collinear. Choose $\beta_1 \in \text{PGL}_3$ which exchanges $p_0$ and $p_1$, let $\gamma = \alpha_{n+1} \beta_1^{-1}$ and choose $\beta_2 \in \text{PGL}_3$ so that $\gamma := \beta_2 \beta_1^{-1} \in \mathcal{J}$. The latter is possible since $\beta_1^{-1}(p_0) = p_1$ is a base point of $\tau$, and we have the following diagram.

Since $\deg(\tau \rho_n) = \deg \rho_n - 1$, the pair $(D, k)$ stays unchanged as we replace the sequence $(\alpha_{n+1}, \rho_n)$ in the decomposition of $\rho$ by the sequence $(\gamma, (\gamma')^{-1}, \beta_2, \tau \rho_n)$. In the new decomposition of $\rho$ the maps $(\gamma')^{-1}$ and $\gamma$ play the roles that $\rho_n$ and $\alpha_{n+1}$ respectively played in the previous decomposition. In the squared $\mathbb{P}^2$, we have

$$m(p_0) = m(\beta_1(p_1)) > m(\beta_1(p_0)) = m(p_1).$$

Define $q'_0 := \gamma^{-1}(p_0), q'_1 := \gamma^{-1}(q_1), q'_2 := \gamma^{-1}(q_2)$, and note that $q'_0 = \beta_1(q_0), q'_1 = \beta_1(q_1)$ and $q'_2 = \beta_1(q_2)$. In the new decomposition these points play the roles that $q_0, q_1, q_2$ played in the previous decomposition.

If $m(q'_0) \geq m(q'_1)$, we continue as in case (a) with the points $p_0, p_1, \beta_1(p_2)$ and $q'_0, q'_1, q'_2$.

If $m(q'_0) < m(q'_1)$, we replace the sequence $(\rho_{n+1}, \gamma)$ by a new sequence such that, similar to case (a), the roles of $q'_0$ and $q'_1$ are exchanged, and we will do this without touching $p_0, p_1, \beta(b_2)$. The replacement will not change $(D, k)$ and we can apply case (a) to the new sequence.

As $m(q'_0) < m(q'_1)$, the point $q'_1$ is a proper point of $\mathbb{P}^2$. Analogously to the previous case, there exists $\sigma \in \mathcal{J}$ with base points $\gamma(q'_0) = p_0, \gamma(q'_1) = q_1, \gamma(q'_2) = q_2$, and there exists $\delta_1 \in \text{PGL}_3$ which exchanges $p_0$ and $q_1$. Since $\delta_1^{-1}(p_0) = q_1$ is a base point of $\sigma$, there furthermore exists $\delta_2 \in \text{PGL}_3$ such that $\sigma' := \delta_2 \sigma \delta_1^{-1} \in \mathcal{J}$. Let $\gamma_2 := \delta_1 \gamma$. 

\[\text{Diagram here}\]
Replacing the sequence \((\rho_{n+1}, \gamma)\) with \((\rho_{n+1} \sigma^{-1}, \delta_2^{-1}, \sigma', \delta_1 \gamma)\) does not change the pair \((D, k)\). The latest position with the highest degree is still the squared \(\mathbb{P}^2\) but in the new sequence we have

\[
m(\gamma_2^{-1}(p_0)) = m(\beta_1(q_1)) > m(\beta_1(q_0)) = m(\gamma_2^{-1}(\delta_1(q_1)))
\]

Since \(p_0, p_1, \beta_1(p_2)\) were undisturbed, the inequality \(m(p_0) > m(p_1)\) still holds, and we proceed as in case (a).

In this proof, we have used several different quadratic maps \(\tau, \tau', \sigma, \sigma'\). Note that none of these can contract \(C\) (or an image of \(C\)), since quadratic maps only can contract curves of degree 1.

**Remark 3.7.** Suppose that \(\rho \in \mathcal{J}\) preserves a line \(L\). Then the Noether-equalities imply that \(L\) passes either through \([1 : 0 : 0]\) and no other base points of \(\rho\), or that it passes through exactly \(\deg \rho - 1\) simple base points of \(\rho\) and not through \([1 : 0 : 0]\).

**Lemma 3.8.** Let \(\rho \in \mathcal{J}\) be of degree \(\geq 2\) and let \(L\) be a line passing through exactly \(\deg \rho - 1\) simple base points of \(\rho\) and not through \([1 : 0 : 0]\). Then there exist \(\rho_1, \ldots, \rho_i \in \mathcal{J}\) of degree 2 such that \(\rho = \rho_{i} \cdots \rho_1\) and the successive images of \(L\) are lines.

**Proof.** Note that the curve \(\rho(L)\) is a line not passing through \(\rho(L)\). Call \(p_0 := [1 : 0 : 0], p_1, \ldots, p_{2d-2}\) the base points of \(\rho\). Without loss of generality, we can assume that \(p_1, \ldots, p_{2d-1}\) are the simple base points of \(\rho\) that are contained in \(L\) and that \(p_1\) is a proper base point in \(\mathbb{P}^2\). We do induction on the degree of \(\rho\).

If there is no simple proper base point \(p_i, \ i \geq d\), of \(\rho\) in \(\mathbb{P}^2\) that is not on \(L\), choose a general point \(r \in \mathbb{P}^2\). There exists a quadratic transformation \(\tau \in \mathcal{J}\) with base points \(p_0, p_1, r\). The transformation \(\rho \tau^{-1} \in \mathcal{J}\) is of degree \(\deg \rho\) and sends the line \(\tau(L)\) (which does not contain \([1 : 0 : 0]\)) onto the line \(\rho(L)\). The point \(\rho(r) \in \mathbb{P}^2\) is a base point of \((\rho \tau^{-1})^{-1}\) not on the line \(\rho(L)\).

So, we can assume that there exists a proper base point of \(\rho\) in \(\mathbb{P}^2\) that is not on \(L\), lets call it \(p_d\). The points \(p_0, p_1, p_d\) are not collinear (because of their multiplicities), hence there exists \(\tau \in \mathcal{J}\) of degree 2 with base points \(p_0, p_1, p_d\). The map \(\rho \tau^{-1} \in \mathcal{J}\) is of degree \(\deg \rho - 1\) and \(\tau(L)\) is a line passing through exactly \(\deg \rho - 2\) simple base points of \(\rho \tau^{-1}\) and not through \([1 : 0 : 0]\). \(\square\)

**Lemma 3.9.** Let \(\rho \in \mathcal{J}\) be of degree \(\geq 2\) and let \(L\) be a line passing through \([1 : 0 : 0]\) and no other base points of \(\rho\). Then there exist \(\rho_1, \ldots, \rho_m \in \mathcal{J}\) of degree 2 such that \(\rho = \rho_m \cdots \rho_1\) and the successive images of \(L\) are lines.

**Proof.** Note that the curve \(\rho(L)\) is a line passing through \([1 : 0 : 0]\). We use induction on the degree of \(\rho\).

Assume that \(\rho\) has no simple proper base points, i.e. all simple base points are infinitely near \(p_0 := [1 : 0 : 0]\). There exists a base point \(p_1\) of \(\rho\) in the first neighbourhood of \(p_0\). Choose a general point \(q \in \mathbb{P}^2\). There exists \(\tau \in \mathcal{J}\) quadratic with base points...
The map $\rho^{-1} \in J$ is of degree $\deg \rho$ and $\tau(L)$ is a line passing through the base point $p_0$ of $\rho^{-1}$ of multiplicity $\deg \rho - 1$ and through no other base points of $\rho^{-1}$. Moreover, the point $\rho(q)$ is a (simple proper) base point of $\tau^{-1}$. Therefore, $\tau^{-1}$ has a simple proper base point in $\mathbb{P}^2$ and sends the line $\rho(L)$ onto the line $\tau(L)$, both of which pass through $p_0$ and no other base points.

So, we can assume that $\rho$ has at least one simple proper base point $p_1$. Let $p_2$ be a base point of $\rho$ that is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of $p_0$ or $p_1$. Because of their multiplicities, the points $p_0, p_1, p_2$ are not collinear. Hence there exists $\tau \in J$ quadratic with base points $p_0, p_1, p_2$. The map $\rho^{-1}$ is a map of degree $\deg \rho - 1$ and $\tau(L)$ is a line passing through $p_0$ and no other base points.

**Lemma 3.10.** Let $\rho \in J$ be a map of degree 2 that sends a line $L$ onto a line. Then there exist quadratic maps $\rho_1, \ldots, \rho_n \in J$ with only proper base points such that

$$\rho = \rho_n \cdots \rho_1,$$

and the successive images of $L$ are lines.

**Proof.** Suppose first that exactly two of the three base points of $\rho$ are proper. We number the base points so that $p_1, p_2 \in \mathbb{P}^2$ and so that $p_3$ is in the first neighbourhood of $p_1$, and denote by $\ell_1 \subset \mathbb{P}^2$ the line through $p_1$ which has the tangent direction defined by $p_3$. Choose a general point $r \in \mathbb{P}^2$, and define a quadratic map $\rho_1$ with three base points $p_1, p_2, r \in \mathbb{P}^2$. A minimal resolution of $\rho$ is given by $\pi$ and $\eta$ as in Figure 7; it is obtained by blowing up, in order, $p_1, p_2, p_3$, and then contracting in order $\tilde{\ell}_2 := \eta^{-1}(\ell_2), \tilde{\ell}_1 := \eta^{-1}(\ell_1)$ and the exceptional divisor corresponding to $p_3$.

By looking at the pull back of a general line in $\mathbb{P}^2$ with respect to $\rho_2 := \rho_1\rho^{-1}$, we see that this map has three proper base points $E_{p_1}, \rho(r), \pi_*(\tilde{\ell}_1)$. This gives us a decomposition of the desired form: $\rho = \rho_2^{-1}\rho_1$. Note that since $\rho$ sends the line $L$ onto a line, $L$ has to pass through exactly one of the base points of $\rho$, and this base point has to be proper. Thus $L$ is sent to a line by $\rho_1$. Using the diagram in Figure 7, we can see that this line is further sent by $\rho_2^{-1}$ to a line through $E_{p_1}$ if $L$ passes through $p_1$ and a line through $\pi_*(\tilde{\ell}_1)$ if $L$ passes through $p_2$.

If $[1:0:0]$ is the only proper base point of $\rho$, we reduce to the first case as follows. Denote by $q$ the base point in the first neighbourhood of $[1:0:0]$ and choose a general point $r \in \mathbb{P}^2$. Let $\rho_1$ be a quadratic map with base points $[1:0:0], q, r$, and let $\rho_2 := \rho_1\rho^{-1}$. If we denote the base points of $\rho^{-1}$ by $q_1, q_2, q_3$ so that $q_1$ is the proper base point and $q_2$ the base point in the first neighbourhood of $q_1$, then the base points of $\rho_2$ are $q_1, q_2, \rho(r)$, i.e. it has exactly two proper base points.
It is also clear that $\rho_1$ sends $L$ to a line, which is further sent by $\rho_2^{-1}$ to a line through $q_1$. Thus we can apply the first part of this proof to each of $\rho_2^{-1}$ and $\rho_1$ in $\rho = \rho_2^{-1}\rho_1$, and thus get a decomposition of the desired form.

\[\square\]

**Theorem 1.** For any line $L$, the group $\text{Dec}(L)$ is generated by $\text{Dec}(L) \cap \text{PGL}_3$ and any of its quadratic elements having three proper base points in $\mathbb{P}^2$.

**Proof.** By conjugating with an appropriate automorphism of $\mathbb{P}^2$, we can assume that $L$ is given by $x = y$. Note that the standard quadratic involution $\sigma : [x : y : z] \mapsto [yz : xz : xy]$ is contained in $\text{Dec}(L)$. It follows from Proposition 3.6, Remark 3.7, and Lemmata 3.8, 3.9 and 3.10 that every element $\rho \in \text{Dec}(L)$ has a composition $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\cdots\alpha_2\rho_1\alpha_1$, where $\alpha_i \in \text{PGL}_3$ and $\rho_i \in \mathcal{A}$ are quadratic with only proper base points in $\mathbb{P}^2$ such that the successive images of $L$ are lines. By composing $\rho_i$ from the left and the right with linear maps, we obtain a decomposition

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\cdots\alpha_2\rho_1\alpha_1$$

where $\alpha_i \in \text{PGL}_3 \cap \text{Dec}(L)$ and $\rho_i \in \text{Dec}(L)$ are of degree 2 with only proper base points in $\mathbb{P}^2$. It therefore suffices to show that for any quadratic element $\rho \in \text{Dec}(L)$ having three proper base points in $\mathbb{P}^2$ there exist $\alpha, \beta \in \text{Dec}(L) \cap \text{PGL}_3$ such that $\sigma = \beta\rho\alpha$.

By Remark 3.7, for any quadratic element of $\text{Dec}(L)$ the line $L$ passes through exactly one of its base points in $\mathbb{P}^2$.

Let $q_1 = [0 : 0 : 1], q_2 = [0 : 1 : 0], q_3 = [1 : 0 : 0]$. They are the base points of $\sigma$, and $\sigma$ sends the pencil of lines through $q_i$ onto itself. Furthermore, $q_i \in L$ but $q_2, q_3 \notin L$. Let $s := [1 : 1 : 1] \in L$. Remark that $\sigma(s) = s$ and that no three of $q_1, q_2, q_3, s$ are collinear.

Let $\rho \in \text{Dec}(L)$ be another quadratic map having three proper base points in $\mathbb{P}^2$. Let $p_1, p_2, p_3$ (resp. $p'_1, p'_2, p'_3$) be its base points (resp. the ones of $\rho^{-1}$). Say $L$ passes through $p_1$ and $\rho$ sends the pencil of lines through $p_i$ onto the pencil of lines through $p'_i$, $i = 1, 2, 3$. Pick a point $r \in L \setminus \{p_i\}$, not collinear with $p_2, p_3$. Then no three of $p_1, p_2, p_3, r$ (resp. $p'_1, p'_2, p'_3, \rho(r)$) are collinear. In particular, there exist $\alpha, \beta \in \text{PGL}_3$ such that

$$\alpha : \begin{cases} q_i \mapsto p_i \\ s \mapsto r \end{cases}, \quad \beta : \begin{cases} p'_i \mapsto q_i \\ \rho(r) \mapsto s \end{cases}$$

Note that $\alpha, \beta \in \text{Dec}(L) \cap \text{PGL}_3$. Furthermore, the quadratic maps $\sigma, \rho' := \beta\rho\alpha \in \text{Dec}(L)$ and their inverse all have the same base points (namely $q_1, q_2, q_3$) and both $\sigma, \rho'$ send the pencil through $q_i$ onto itself. Since moreover $\rho'(s) = \sigma(s) = s$, we have $\sigma = \rho'$.

\[\square\]

4. $\text{Dec}(L)$ is not an amalgam

Just like Bir($\mathbb{P}^2$), its subgroup $\text{Dec}(L)$ is generated by its linear elements and one quadratic element (Theorem 1). In [Cor2013, Corollary A.2], it is shown that Bir($\mathbb{P}^2$) is not an amalgamated product. In this section we adjust the proof to our situation and prove that the same statement holds for $\text{Dec}(L)$.

The notion of being an amalgamated product is closely related to actions on trees, or, in this case, $\mathbb{R}$-trees.

**Definition and Lemma 4.1.** A real tree, or $\mathbb{R}$-tree, can be defined in the following three equivalent ways [Cis2001]:

1. A geodesic space which is 0-hyperbolic in the sense of Gromov.
2. A uniquely geodesic metric space for which $[a, c] \subset [a, b] \cup [b, c]$ for all $a, b, c$. 
A geodesic metric space with no subspace homeomorphic to the circle. We say that a real tree is a complete real tree if it is complete as a metric space.

Every ordinary tree can be seen as a real tree by endowing it with the usual metric but not every real tree is isometric to an simplicial tree (endowed with the usual metric) [Cis2001, §2.2, Proposition 2.5, Example].

**Definition 4.2.** A group $G$ has the property $(F\mathbb{R})_\infty$ if for every isometric action of $G$ on a complete real tree, every element has a fixed point.

We summarize the discussion in [Cor2013, before Remark A.3] in the following result.

**Lemma 4.3.** If a group $G$ has property $(F\mathbb{R})_\infty$, it does not decompose as non-trivial amalgam.

We will devote the rest of this section to proving Proposition 4.4 and thereby showing that $\text{Dec}(L)$ is not an amalgam.

**Proposition 4.4.** The decomposition group $\text{Dec}(L)$ has property $(F\mathbb{R})_\infty$.

By convention, from now on, $T$ will denote a complete real tree and all actions on $T$ are assumed to be isometric.

**Definition 4.5.** Let $T$ be a complete real tree.

1. A ray in $T$ is a geodesic embedding $(x_t)_{t \geq 0}$ of the half-line.
2. An end in $T$ is an equivalence class of rays, where we say that two rays $x$ and $y$ are equivalent if there exists $t, t' \in \mathbb{R}$ such that $\{x_s; s \geq t\} = \{y_s; s' \geq t'\}$.
3. Let $G$ be a group of isometries of $T$ and $\omega$ an end in $T$ represented by a ray $(x_t)_{t \geq 0}$. The group $G$ stably fixes the end $\omega$ if for every $g \in G$ there exists $t_0 := t_0(g)$ such that $g$ fixes $x_t$ for all $t \geq t_0$.

**Remark 4.6.** [Cor2013, Lemma A.9] For a group $G$, property $(F\mathbb{R})_\infty$ is equivalent to each of the following statements:

1. For every isometric action of $G$ on a complete real tree, every finitely generated subgroup has a fixed point.
2. Every isometric action of $G$ on a complete real tree has a fixed point or stably fixes an end.

**Definition 4.7.** For a line $L \subset \mathbb{P}^2$, define $A_L := \text{PGL}_3 \cap \text{Dec}(L)$. If $L$ is given by the equation $f = 0$, we also use the notation $A_{\{f = 0\}}$.

**Lemma 4.8.** For any line $L \subset \mathbb{P}^2$ the group $A_L$ has property $(F\mathbb{R})_\infty$.

**Proof.** Since for two lines $L$ and $L'$ the groups $\text{Dec}(L)$ and $\text{Dec}(L')$ are conjugate, it is enough to prove the lemma for one line, say the line given by $x = 0$. Note that $A = (a_{ij})_{1 \leq i, j \leq 3} \in \text{PGL}_3$ is in $A_{\{x = 0\}}$ if and only if $a_{12} = a_{13} = 0$.

Let $A_{\{x = 0\}}$ act on $T$ and let $F \subset A_{\{x = 0\}}$ be a finite subset. The elements of $F$ can be written as a product of elementary matrices contained in $A_{\{x = 0\}}$; let $R$ be the (finitely generated) subring of $k$ generated by all entries of the elementary matrices contained in $A_{\{x = 0\}}$ that are needed to obtain the elements in $F$. Then $F$ is contained in $\text{EL}_3(R)$, the subgroup of $\text{SL}_3(R)$ generated by elementary matrices. By the Shalom-Vaserstein theorem (see [EJZ010, Theorem 1.1]), $\text{EL}_3(R)$ has Kazhdan’s property $(T)$ and in particular (as $\text{EL}_3(R)$ is countable) has a fixed point in $T$ [Wat1982, Theorem 2], so $F$ has a fixed point in $T$. It follows that the subgroup of $A_{\{x = 0\}}$ generated by $F$ has a fixed point [Ser1977, §1.6.5, Corollary 3]. In particular, by Remark 4.6 (1), $A_{\{x = 0\}}$ has property $(F\mathbb{R})_\infty$. 

\[\square\]
From now on, we fix \( L \) to be the line given by \( x = y \). It is enough to prove Proposition 4.4 for this line since \( \text{Dec}(L) \) and \( \text{Dec}(L') \) are conjugate groups (by linear elements) for all lines \( L \) and \( L' \). As before, we denote the standard quadratic involution by \( \sigma \in \text{Bir}(\mathbb{P}^2) \); with our choice of \( L \), it is contained in \( \text{Dec}(L) \).

Let \( \mathcal{D}_L \subset \text{PGL}_3 \) be the subgroup of diagonal matrices that send \( L \) onto \( L' \), i.e.

\[
\mathcal{D}_L := \{ \text{diag}(s, s, t) \; s, t \in \mathbb{C}^* \} \subset \text{PGL}_3.
\]

**Lemma 4.9.** We have \( \langle \mathcal{D}_L, \mu_1, \mu_2, P \rangle = \mathcal{A}_L \), with the three involutions

\[
\mu_1 := \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L, \quad \mu_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \in \mathcal{A}_L, \quad \text{and } P := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L.
\]

**Proof.** Given any \( \lambda \in \mathbb{C}^* \), the matrices

\[
A_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix}, \quad B_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix}, \quad \text{and } C_\lambda := \begin{bmatrix} 0 & 1 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

belong to \( \langle \mathcal{D}_L, \mu_1, \mu_2, P \rangle \). Indeed, we have \( A_\lambda = \text{diag}(-\lambda^{-1}, -\lambda^{-1}, 1) \cdot \mu_2 \cdot \text{diag}(\lambda, \lambda, 1) \), \( B_\lambda = P A_\lambda P \), and \( C_\lambda = \text{diag}(1, 1, \lambda^{-1}) \cdot \mu_1 \cdot \text{diag}(-1, -1, \lambda) \).

Left multiplication by these corresponds to three types of row operations on matrices in \( \text{PGL}_3 \) and right multiplication corresponds in the same way to three types of column operations. We denote them respectively by \( r_1, r_2, r_3, c_1, c_2, c_3 \), and we write \( d \) for multiplication by an element in \( \mathcal{D}_L \).

Let \( A = (a_{ij})_{1 \leq i, j \leq 3} \in \text{PGL}_3 \) be a matrix which is in \( \mathcal{A}_L \), i.e. such that \( a_{13} = a_{23} \) and \( a_{11} + a_{12} = a_{21} + a_{22} \). We proceed as follows, using only the above mentioned operations.

\[
A = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow{d} \begin{bmatrix} * & * & * \\ * & * & 1 \\ * & * & 1 \end{bmatrix} \xrightarrow{r_3} \begin{bmatrix} * & * & 1 \\ 0 & 0 & 1 \\ -y & -z & 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

In the first step \( (d) \) we assume that \( a_{33} \neq 0 \) – this can always be achieved by performing a row operation of type \( r_1 \) on \( A \) if necessary. In the second step \( (r_3) \), we use that \( a_{13} = a_{23} \). The entries on place \((2, 1)\) and \((2, 2)\) after the second step are denoted by \( y \) and \( z \) respectively. In the fifth step \( (d) \), we use that the entry on place \((1, 1)\) is nonzero; this follows from the assumption \( a_{11} + a_{12} = a_{21} + a_{22} \) and that \( A \) is invertible.

**Lemma 4.10.** Suppose that \( \text{Dec}(L) \) acts on \( T \) so that \( \mathcal{A}_L \) has no fixed points. Then \( \text{Dec}(L) \) stably fixes an end.

**Proof.** Since \( \mathcal{A}_L \) has property \( (\text{FR}) \infty \) and has no fixed points, it stably fixes an end (Remark 4.6 (2)). Observe that this fixed end is unique: if \( \mathcal{A}_L \) stably fixes two different ends \( \omega_1, \omega_2 \), then \( \mathcal{A}_L \) pointwise fixes the line joining the two ends and has therefore fixed points (this uses that the only isometries on \( \mathbb{R} \) are translations and reflections [Cis2001, §1.2, Lemma 2.1]).
Let \( \omega \), represented by the ray \( (x_t)_{t \geq 0} \), be the unique end which is stably fixed by \( \mathcal{A}_L \) and define \( C := \langle D_L, P \rangle \). Being a subgroup of \( \mathcal{A}_L \), \( C \) obviously also stably fixes \( \omega \). Note that the end \( \sigma \omega \) is stably fixed by \( \sigma \mathcal{A}_L \sigma^{-1} \). In particular, since \( \sigma C \sigma^{-1} = C \), the end \( \sigma \omega \) is also stably fixed by \( C \). If \( \sigma \omega = \omega \), then \( \omega \) is stably fixed by \( \sigma \) and by Theorem 1, \( \omega \) is stably fixed by \( \text{Dec}(L) \). Otherwise, let \( l \) be the line joining \( \omega \) and \( \sigma \omega \). Since \( C \) stably fixes \( \omega \) and \( \sigma \omega \), it stably fixes both ends of \( l \). In particular, the line \( l \) is pointwise fixed by \( C \). Since \( \mu_1, \mu_2 \in \mathcal{A}_L \), \( \mu_1, \mu_2 \) stably fix the end \( \omega \) and therefore, \( x_i \) is fixed by \( \mu_1, \mu_2 \) for \( t \geq t_0 \) for some \( t_0 \), and hence, by Lemma 4.9, \( x_i \) is fixed by all of \( \mathcal{A}_L \) for \( t \geq t_0 \), contradicting the assumption.

**Proof of Proposition 4.4.** Recall that \( \mu_1, \mu_2 \in \mathcal{A}_L \) and note that \( \sigma \mu_1 \) has order 3 and that \( \sigma \mu_2 \) has order 6. It follows that

\[
\sigma = (\mu_1 \sigma) \mu_1 (\mu_1 \sigma)^{-1}
\]

By Theorem 1, \( \text{Dec}(L) \) is generated by \( \sigma \) and \( \mathcal{A}_L \). It follows that \( \mathcal{A}_L := \mathcal{A}_L \) and \( \mathcal{A}_2 := \sigma \mathcal{A}_L \sigma \) generate \( \text{Dec}(L) \).

Consider an action of \( \text{Dec}(L) \) on \( T \). It induces an action of \( \mathcal{A}_L \), which has property \((\mathbb{F}_\infty, \infty)\) by Lemma 4.8 (i.e. \( \mathcal{A}_L \) has a fixed point or stably fixes an end by Remark 4.6 (2)). If \( \mathcal{A}_L \) has no fixed point, Lemma 4.10 implies that \( \text{Dec}(L) \) stably fixes an end, and then we are done.

Assume that \( \mathcal{A}_L \) has a fixed point. We conclude the proof by showing that in this case, even \( \text{Dec}(L) \) has a fixed point.

For \( i = 1, 2 \), let \( T_i \) be the set of fixed points of \( \mathcal{A}_i \). The two trees are exchanged by \( \sigma \). If \( T_1 \cap T_2 \neq \emptyset \), \( \text{Dec}(L) \) has a fixed point since \( \langle \mathcal{A}_1, \mathcal{A}_2 \rangle = \text{Dec}(L) \). Let us consider the case where \( T_1 \) and \( T_2 \) are disjoint.

Let \( \mathcal{S} := [x_1, x_2], x_i \in T_i \), be the minimal segment joining the two trees and \( s > 0 \) its length. Let \( C := \langle D_L, P \rangle \). Then \( \mathcal{S} \) is pointwise fixed by \( C \subset \mathcal{A}_1 \cap \mathcal{A}_2 \) and reversed by \( \sigma \). For \( i = 1, 2 \), the image of \( \mathcal{S} \) by \( \mu_i \) is a segment \( \mu_i(\mathcal{S}) = [x_1, \mu_i x_2] \). By Lemma 4.9, \( \langle C, \mu_1, \mu_2 \rangle = \mathcal{A}_1 \), so it follows that for \( i = 1 \) or \( i = 2 \), we have \( \mu_i(\mathcal{S}) \cap \mathcal{S} = \{x_1\} \).

Otherwise, because \( T \) is a tree and \( \mathcal{A}_1 \) acts by isometries, both \( \mu_1, \mu_2 \) fix \( \mathcal{S} \) pointwise and so \( \mathcal{A}_1 \) fixes \( \mathcal{S} \) pointwise and in particular it fixes \( x_2 \) – this would contradict \( T_1 \cap T_2 = \emptyset \).

Choose an element \( i \in \{1, 2\} \) such that \( \mu_i(\mathcal{S}) \cap \mathcal{S} = \{x_1\} \).

Finally we arrive at a contradiction by computing \( d(x_1, (\sigma \mu_i)^k x_1) \) in two different ways. On the one hand we see that this distance is \( sk \), on the other hand we have \( (\sigma \mu_i)^l = 1 \). More generally, we show that

\[
d(\sigma \mu_i)^k x_1, (\sigma \mu_i)^l x_1) = |k - l| s
\]

for all \( k, l \). Since we are on a real tree, it suffices to show this for \( k, l \) with \( |k - l| \leq 2 \) (cf. [Cor2013, Lemma A.4]). By translation, we only have to check it for \( l = 0, k = 1, 2 \).

For \( k = 1 \), we have \( d(\sigma \mu_i x_1, x_1) = d(\sigma x_1, x_1) = d(x_2, x_1) = s \). For \( k = 2 \), the segment \( \mu_i(\mathcal{S}) = [x_1, \mu_i x_2] \) intersects \( \mathcal{S} \) only at \( x_1 \). In particular, \( d(\mu_i x_2, x_2) = 2s \) and hence

\[
d(\sigma \mu_i \sigma x_1, x_1) = d(\mu_i \sigma x_1, x_1) = d(\mu_i x_2, x_2) = 2s
\]

It follows that \( T_1 \) and \( T_2 \) cannot be disjoint, and we are done. \( \square \)

**References**


THE DECOMPOSITION GROUP OF A LINE IN THE PLANE


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