On preperiodic points
of rational functions
defined over $\mathbb{F}_p(t)$

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Abstract. Let $P \in \mathbb{P}_1(\mathbb{Q})$ be a periodic point for a monic polynomial with coefficients in $\mathbb{Z}$. With elementary techniques one sees that the minimal periodicity of $P$ is at most 2. Recently we proved a generalization of this fact to the set of all rational functions defined over $\mathbb{Q}$ with good reduction everywhere (i.e. at any finite place of $\mathbb{Q}$). The set of monic polynomials with coefficients in $\mathbb{Z}$ can be characterized, up to conjugation by elements in $\text{PGL}_2(\mathbb{Z})$, as the set of all rational functions defined over $\mathbb{Q}$ with a totally ramified fixed point in $\mathbb{Q}$ and with good reduction everywhere. Let $p$ be a prime number and let $\mathbb{F}_p$ be the field with $p$ elements. In the present paper we consider rational functions defined over the rational global function field $\mathbb{F}_p(t)$ with good reduction at every finite place. We prove some bounds for the cardinality of orbits in $\mathbb{F}_p(t) \cup \{\infty\}$ for periodic and preperiodic points.

Keywords. preperiodic points, function fields.

1. Introduction

In arithmetic dynamic there is a great interest about periodic and preperiodic points of a rational function $\phi : \mathbb{P}_1 \to \mathbb{P}_1$. A point $P$ is said to be periodic for $\phi$ if there exists an integer $n > 0$ such that $\phi^n(P) = P$. The minimal number $n$ with the above properties is called minimal or primitive period. We say that $P$ is a preperiodic point for $\phi$ if its (forward) orbit $O_\phi(P) = \{\phi^n(P) \mid n \in \mathbb{N}\}$ contains a periodic point. In other words $P$ is preperiodic if its orbit $O_\phi(P)$ is finite. In this context an orbit is also called a cycle and its size is called the length of the cycle.

Let $p$ be a prime and, as usual, let $\mathbb{F}_p$ be the field with $p$ elements. We denote by $K$ a global field, i.e. a finite extension of the field of rational numbers $\mathbb{Q}$ or a finite extension of the field $\mathbb{F}_p(t)$. Let $D$ be the degree of $K$ over the base field (respectively $\mathbb{Q}$ in characteristic 0 and $\mathbb{F}_p(t)$ in positive characteristic). We denote by $\text{PrePer}(\phi, K)$ the set of $K$-rational preperiodic points for $\phi$. By considering the notion of height, one sees that the set $\text{PrePer}(\phi, K)$ is finite for any rational map $\phi : \mathbb{P}_1 \to \mathbb{P}_1$ defined over $K$ (see for example [13] or [5]). The finiteness of the set $\text{PrePer}(f, K)$ follows from [5, Theorem B.2.5, p.179] and [5, Theorem B.2.3, p.177] (even if these last theorems are formulated in the case of number fields, they have a similar statement in the function field case). Anyway, the bound deduced by those results depends strictly on the coefficients of the map $\phi$ (see also [13, Exercise 3.26 p.99]). So, during the last twenty years, many dynamists have searched for bounds that do not depend on the coefficients of $\phi$. In 1994 Morton and Silverman stated a conjecture known with the name “Uniform Boundedness Conjecture for Dynamical Systems”: for any number field $K$, the number of $K$-preperiodic points

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of a morphism \( \phi : \mathbb{P}_N \to \mathbb{P}_N \) of degree \( d \geq 2 \), defined over \( K \), is bounded by a number depending only on the integers \( d, N \) and \( D = [K : \mathbb{Q}] \). This conjecture is really interesting even for possible application on torsion points of abelian varieties. In fact, by considering the Lattès map associated to the multiplication by two map [2] over an elliptic curve \( E \), one sees that the Uniform Boundedness Conjecture for \( N = 1 \) and \( d = 4 \) implies Merel’s Theorem on torsion points of elliptic curves (see [6]). The Lattès map has degree 4 and its preperiodic points are in one-to-one correspondence with the torsion points of \( E/\{\pm 1\} \) (see [11]). So a proof of the conjecture for every \( N \), could provide an analogous of Merel’s Theorem for all abelian varieties. Anyway, it seems very hard to solve this conjecture, even for \( N = 1 \).

Let \( R \) be the ring of algebraic integers of \( K \). Roughly speaking: we say that an endomorphism \( \phi : \mathbb{P}_1 \to \mathbb{P}_1 \) has (simple) good reduction at a place \( p \) if \( \phi \) can be written in the form \( \phi([x:y]) = [F(x,y), G(x,y)] \), where \( F(x,y) \) and \( G(x,y) \) are homogeneous polynomial of the same degree with coefficients in the local ring \( \mathcal{O}_R \) at \( p \) and such that their resultant \( \text{Res}(F,G) \) is a \( p \)-unit. In Section 3 we present more carefully the notion of good reduction.

The first author studied some problems linked to Uniform Boundedness Conjecture. In particular, when \( N = 1, K \) is a number field and \( \phi : \mathbb{P}_1 \to \mathbb{P}_1 \) is an endomorphism defined over \( K \), he proved in [3, Theorem 1] that the length of a cycle of a preperiodic point of \( \phi \) is bounded by a number depending only on the cardinality of the set of places of bad reduction of \( \phi \).

A similar result in the function field case was recently proved in [4]. Furthermore in the same paper there is a bound proved for number fields, that is slightly better than the one in [3].

**Theorem 1.1** (Theorem 1, [4]). Let \( K \) be a global field. Let \( S \) be a finite set of places of \( K \), containing all the archimedean ones, with cardinality \( |S| \geq 1 \). Let \( p \) be the characteristic of \( K \). Let \( D = [K : \mathbb{F}_p(t)] \) when \( p > 0 \), or \( D = [K : \mathbb{Q}] \) when \( p = 0 \). Then there exists a number \( \eta(p, D, |S|) \), depending only on \( p, D \) and \( |S| \), such that if \( P \in \mathbb{P}_1(K) \) is a preperiodic point for an endomorphism \( \phi \) defined over \( K \) with good reduction outside \( S \), then \( |O_K(P)| \leq \eta(p, D, |S|) \). We can choose

\[
\eta(0, D, |S|) = \max \left\{ (2^{16}|S|^8 + 3) [12|S||\log(5|S|)]^D, [12(|S| + 2) \log(5|S| + 5)]^D \right\}
\]

in zero characteristic and

\[
\eta(p, D, |S|) = (|p|S|)^{4D} \max \left\{ (|p|S|)^{2D}, p^{45|S| - 2} \right\}.
\]

in positive characteristic.

Observe that the bounds in Theorem 1.1 do not depend on the degree \( d \) of \( \phi \). As a consequence of that result, we could give the following bound for the cardinality of the set of \( K \)-rational preperiodic points for an endomorphism \( \phi \) of \( \mathbb{P}_1 \) defined over \( K \).

**Corollary 1.1.1** (Corollary 1.1, [4]). Let \( K \) be a global field. Let \( S \) be a finite set of places of \( K \) containing all the archimedean ones. Let \( p \) be the characteristic of \( K \). Let \( D \) be the degree of \( K \) over the rational function field \( \mathbb{F}_p(t) \), in the positive characteristic, and over \( \mathbb{Q}_p \) in the zero characteristic. Let \( d \geq 2 \) be an integer. Then there exists a number \( C = C(p, D, d, |S|) \), depending only on \( p, D, d \) and \( |S| \), such that for any endomorphism \( \phi \) of \( \mathbb{P}_1 \) of degree \( d \), defined over \( K \) and with good reduction outside \( S \), we have

\[
\#\text{PrePer}(\phi, \mathbb{P}_1(K)) \leq C(p, D, d, |S|).
\]

Theorem 1.1 extends to global fields and to preperiodic points the result proved by Morton and Silverman in [7, Corollary B]. The condition \( |S| \geq 1 \) in its statement is only
a technical one. In the case of number fields, we require that $S$ contains the archimedean places (i.e., the ones at infinity), then it is clear that the cardinality of $S$ is not zero. In the function field case any place is non archimedean. Recall that the place at infinity in the case $K = \mathbb{F}_p(t)$ is the one associated to the valuation given by the prime element $1/t$. When $K$ is a finite extension of $\mathbb{F}_p(t)$, the places at infinity of $K$ are the ones that extend the place of $\mathbb{F}_p(t)$ associated to $1/t$. The arguments used in the proof of Theorem 1.1 and Corollary 1.1.1 work also when $S$ does not contain all the places at infinity. Anyway, the most important situation is when all the ones at infinity are in $S$. For example, in order to have that any polynomial in $\mathbb{F}_p(t)$ is an $S$–integer, we have to put in $S$ all those places. Note that in the number field case the quantity $|S|$ depends also on the degree $D$ of the extension $K$ of $\mathbb{Q}$, because $S$ contains all archimedean places (whose amount depends on $D$).

Even when the cardinality of $S$ is small, the bounds in Theorem 1.1 is quite big. This is a consequence of our searching for some uniform bounds (depending only on $p, D, |S|$). The bound $C(p, D, d, |S|)$ in Corollary 1.1.1 can be effectively given, but in this case too the bound is big, even for small values of the parameters $p, D, d, |S|$. For a much smaller bound see for instance the one proved by Benedetto in [1] for the case where $\phi$ is a polynomial.

In the more general case when $\phi$ is a rational function with good reduction outside a finite $S$, the bound in Theorem 1.1 can be significantly improved for some particular sets $S$. For example if $K = \mathbb{Q}$ and $S$ contains only the place at infinity, then we have the following bounds (see [4]):

- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a periodic point for $\phi$ with minimal period $n$, then $n \leq 3$.
- If $P \in \mathbb{P}_1(\mathbb{Q})$ is a preperiodic point for $\phi$, then $|O_\phi(P)| \leq 12$.

Here we prove some analogous bounds when $K = \mathbb{F}_p(t)$.

**Theorem 1.2.** Let $\phi: \mathbb{F}_p \to \mathbb{F}_p$ of degree $d \geq 2$ defined over $\mathbb{F}_p(t)$ with good reduction at every finite place. If $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a periodic point for $\phi$ with minimal period $n$, then

- $n \leq 3$ if $p = 2$;
- $n \leq 72$ if $p = 3$;
- $n \leq (p^3 - 1)p$ if $p \geq 5$.

More generally if $P \in \mathbb{P}_1(\mathbb{F}_p(t))$ is a preperiodic point for $\phi$ we have

- $|O_\phi(P)| \leq 9$ if $p = 2$;
- $|O_\phi(P)| \leq 288$ if $p = 3$;
- $|O_\phi(P)| \leq (p + 1)(p^2 - 1)p$ if $p \geq 5$.

Observe that the bounds do not depend on the degree of $\phi$ but they depend only on the characteristic $p$. In the proof we will use some ideas already written in [2], [3] and [4]. The original idea of using $S$–integer theorems in the context of the arithmetic of dynamical systems is due to Narkiewicz [9].

2. **Valuations, $S$–integers and $S$–units**

We adopt the present notation: let $K$ be a global field and $v_p$ a valuation on $K$ associated to a non archimedean place $\mathfrak{p}$. Let $K_\mathfrak{p} = \{x \in K \mid v_\mathfrak{p}(x) \geq 1\}$ be the local ring of $K$ at $\mathfrak{p}$.

Recall that we can associate an absolute value to any valuation $v_{\mathfrak{p}}$, or more precisely a place $\mathfrak{p}$ that is a class of absolute values (see [5] and [12] for a reference about this topic). With $K = \mathbb{F}_p(t)$, all places are exactly the ones associated either to a monic irreducible polynomial in $\mathbb{F}_p[t]$ or to the place at infinity given by the valuation $v_{\infty}(f(x)/g(x) = \deg(g(x)) - \deg(f(x))$, that is the valuation associated to $1/t$.

In an arbitrary finite extension $K$ of $\mathbb{F}_p(t)$ the valuations of $K$ are the ones that extend the valuations of $\mathbb{F}_p(t)$. We shall call places at infinity the ones that extend the above valuation.
v_\infty \text{ on } \mathbb{F}_p(t). \) The other ones will be called finite places. The situation is similar to the one in number fields. The non archimedean places in \( \mathbb{Q} \) are the ones associated to the valuations at any prime \( p \) of \( \mathbb{Z} \). But there is also a place that is not non–archimedean, the one associated to the usual absolute value on \( \mathbb{Q} \). With an arbitrary number field \( K \) we call archimedean places all the ones that extend to \( K \) the place given by the absolute value on \( \mathbb{Q} \).

From now on \( S \) will be a finite fixed set of places of \( K \). We shall denote by

\[
R_S := \{ x \in K \mid v_p(x) \geq 0 \text{ for every prime } p \notin S \}
\]

the ring of \( S \)-integers and by

\[
R^*_S := \{ x \in K^* \mid v_p(x) = 0 \text{ for every prime } p \notin S \}
\]

the group of \( S \)-units.

As usual let \( \overline{\mathbb{F}}_p \) be the algebraic closure of \( \mathbb{F}_p \). The case when \( S = \emptyset \) is trivial because if so, then the ring of \( S \)-integers is already finite; more precisely \( R_S = R^*_S = K^* \cap \overline{\mathbb{F}}_p \).

Therefore in what follows we consider \( S \neq \emptyset \).

In any case we have that \( K^* \cap \overline{\mathbb{F}}_p \) is contained in \( R^*_S \). Recall that the group \( R^*_S / K^* \cap \overline{\mathbb{F}}_p \) has finite rank equal to \( |S| - 1 \) (e.g. see [10, Proposition 14.2 p.243]). Thus, since \( K \cap \overline{\mathbb{F}}_p \) is a finite field, we have that \( R^*_S \) has rank equal to \( |S| \).

\section{3. Good reduction}

We shall state the notion of good reduction following the presentation given in [11] and in [4].

\textbf{Definition 3.0.1.} Let \( \Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1 \) be a rational map defined over \( K \), of the form

\[
\Phi([X : Y]) = [F(X, Y) : G(X, Y)]
\]

where \( F, G \in K[X, Y] \) are coprime homogeneous polynomials of the same degree. We say that \( \Phi \) is in \( p \)-\textit{reduced form} if the coefficients of \( F \) and \( G \) are in \( R_p[X, Y] \) and at least one of them is a \( p \)-unit (i.e. a unit in \( R_p \)).

Recall that \( R_p \) is a principal local ring. Hence, up to multiplying the polynomials \( F \) and \( G \) by a suitable non-zero element of \( K \), we can always find a \( p \)-reduced form for each rational map. We may now give the following definition.

\textbf{Definition 3.0.2.} Let \( \Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1 \) be a rational map defined over \( K \). Suppose that the morphism \( \Phi([X : Y]) = [F(X, Y) : G(X, Y)] \) is written in \( p \)-reduced form. The \textit{reduced map} \( \Phi_p : \mathbb{P}_1^{(p)} \rightarrow \mathbb{P}_1^{(p)} \) is defined by \( [F_p(X, Y) : G_p(X, Y)] \), where \( F_p \) and \( G_p \) are the polynomials obtained from \( F \) and \( G \) by reducing their coefficients modulo \( p \).

With the above definitions we give the following one:

\textbf{Definition 3.0.3.} A rational map \( \Phi : \mathbb{P}_1 \rightarrow \mathbb{P}_1 \), defined over \( K \), \textit{has good reduction} at \( p \) if \( \deg \Phi = \deg \Phi_p \). Otherwise we say that it has bad reduction at \( p \). Given a set \( S \) of places of \( K \) containing all the archimedean ones. We say that \( \Phi \) has good reduction outside \( S \) if it has good reduction at any place \( p \notin S \).

Note that the above definition of good reduction is equivalent to ask that the homogeneous resultant of the polynomial \( F \) and \( G \) is invertible in \( R_p \), where we are assuming that \( \Phi([X : Y]) = [F(X, Y) : G(X, Y)] \) is written in \( p \)-reduced form.
4. Divisibility Arguments

We define the $p$-adic logarithmic distance as follows (see also [8]). The definition is independent of the choice of the homogeneous coordinates.

**Definition 4.0.1.** Let $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2]$ be two distinct points in $\mathbb{P}_1(K)$. We denote by

$$
\delta_v(P_1, P_2) = v_p(x_1y_2 - x_2y_1) - \min\{v_p(x_1), v_p(y_1)\} - \min\{v_p(x_2), v_p(y_2)\}
$$

the $p$-adic logarithmic distance.

The divisibility arguments, that we shall use to produce the $S$–unit equation useful to prove our bounds, are obtained starting from the following two facts:

**Proposition 4.0.1.** [8, Proposition 5.1]

$$
\delta_v(P_1, P_2) \geq \min\{\delta_v(P_1, P_3), \delta_v(P_2, P_3)\}
$$

for all $P_1, P_2, P_3 \in \mathbb{P}_1(K)$.

**Proposition 4.0.2.** [8, Proposition 5.2] Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ be a morphism defined over $K$ with good reduction at a place $v$. Then for any $P, Q \in \mathbb{P}(K)$ we have

$$
\delta_v(\phi(P), \phi(Q)) \geq \delta_v(P, Q).
$$

As a direct application of the previous propositions we have the following one.

**Proposition 4.0.3.** [8, Proposition 6.1] Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ be a morphism defined over $K$ with good reduction at a place $v$. Let $P \in \mathbb{P}(K)$ be a periodic point for $\phi$ with minimal period $n$. Then

- $\delta_v(\phi^i(P), \phi^j(P)) = \delta_v(\phi^{i+j}(P), \phi^{i+j}(P))$ for every $i, j, k \in \mathbb{Z}$.
- Let $i, j \in \mathbb{N}$ such that $\gcd(i - j, n) = 1$. Then $\delta_v(\phi^i(P), \phi^j(P)) = \delta_v(\phi(P), P)$.

5. Proof of Theorem 1.2

We first recall the following statements.

**Theorem 5.1** (Morton and Silverman [8], Zieve [14]). Let $K, \wp, p$ be as above. Let $\Phi$ be an endomorphism of $\mathbb{P}_1$ of degree at least two defined over $K$ with good reduction at $\wp$. Let $P \in \mathbb{P}_1(K)$ be a periodic point for $\Phi$ with minimal period $n$. Let $m$ be the primitive period of the reduction of $P$ modulo $\wp$ and $r$ the multiplicative period of $(\Phi^m)'(P)$ in $k(\wp)$. Then one of the following three conditions holds

- (i) $n = m$;
- (ii) $n = mr$;
- (iii) $n = p^e mr$, for some $e \geq 1$.

In the notation of Theorem 5.1, if $(\Phi^m)'(P) \equiv 0$ modulo $\wp$, then we set $r = \infty$. Thus, if $P$ is a periodic point, then the cases (ii) and (iii) are not possible with $r = \infty$.

**Proposition 5.1.1.** [8, Proposition 5.2] Let $\phi: \mathbb{P}_1 \to \mathbb{P}_1$ be a morphism defined over $K$ with good reduction at a place $\wp$. Then for any $P, Q \in \mathbb{P}(K)$ we have

$$
\delta_v(\phi(P), \phi(Q)) \geq \delta_v(P, Q).
$$

**Lemma 5.1.1.** Let

$$
P = P_{m+1} \mapsto P_{m+2} \mapsto \ldots \mapsto P_{-1} \mapsto P_0 = [0 : 1] \mapsto [0 : 1].
$$

be an orbit for an endomorphism $\phi$ defined over $K$ with good reduction outside $S$. For any $a, b$ integers such that $0 < a < b \leq m - 1$ and $\wp \notin S$, it holds
a) \( \delta_\phi(P_{-b}, P_0) \leq \delta_\phi(P_{-a}, P_0); \)

b) \( \delta_\phi(P_{-b}, P_{-a}) = \delta_\phi(P_{-b}, P_0). \)

Proof a) It follows directly from Proposition 5.1.1.

b) By Proposition 4.0.1 and part a) we have

\[
\delta_\phi(P_{-b}, P_{-a}) \geq \min \{ \delta_\phi(P_{-b}, P_0), \delta_\phi(P_{-a}, P_0) \} = \delta_\phi(P_{-b}, P_0).
\]

Let \( r \) be the largest positive integer such that \(-b + r(b - a) < 0.\) Then

\[
\delta_\phi(P_{-b}, P_0) \geq \min \{ \delta_\phi(P_{-b}, P_{-a}), \delta_\phi(P_{-a}, P_{-2a}), \ldots, \delta_\phi(P_{-b+r(b-a)}, P_0) \}
\]

\[
= \delta_\phi(P_{-b}, P_{-a}).
\]

The inequality is obtained by applying Proposition 4.0.1 several times. \( \Box \)

**Lemma 5.1.2** (Lemma 3.2 [4]). Let \( K \) be a function field of degree \( D \) over \( \mathbb{F}_p(t) \) and \( S \) a non empty finite set of places of \( K. \) Let \( P_i \in \mathbb{F}_i(K) \) with \( i \in \{0, \ldots, n-1\} \) be \( n \) distinct points such that

\[
(4) \quad \delta_\phi(P_0, P_1) = \delta_\phi(P_i, P_j), \text{ for each distinct } 0 \leq i, j \leq n-1 \text{ and for each } p \notin S.
\]

Then \( n \leq (p|S|)^{2D}. \)

Since \( \mathbb{F}_p(t) \) is a principal ideal domain, every point in \( \mathbb{F}_r(\mathbb{F}_p(t)) \) can be written in \( S \)-coprime coordinates, i. e., for each \( P \in \mathbb{F}_r(\mathbb{F}_p(t)) \) we may write \( P = [a : b] \) with \( a, b \in R_i \) and \( \min \{v_i(a), v_i(b)\} = 0, \) for each \( p \notin S. \) We say that \([a : b]\) are \( S \)-coprime coordinates for \( P. \)

**Proof of Theorem 1.2** We use the same notation of Theorem 5.1. Assume that \( S \) contains only the place at infinity. Case \( p = 2. \) Let \( P \in \mathbb{F}_1(\mathbb{F}_p(t)) \) be a periodic point for \( \phi. \) Without loss of generality we can suppose that \( P = [0 : 1]. \) Observe that \( m \) is bounded by 3 and \( r = 1. \) By Theorem 5.1, we have \( n = m \cdot 2^r, \) with \( e \) a non negative integer number. Up to considering the \( m \)-th iterate of \( \phi, \) we may assume that the minimal periodicity of \( P \) is \( 2^e. \) So now suppose that \( n = 2^e, \) with \( e \geq 2. \) Consider the following 4 points of the cycle:

\[
[0 : 1] \mapsto [x_1 : y_1] \mapsto [x_2 : y_2] \mapsto [x_3 : y_3] \ldots
\]

where the points \([x_i : y_i]\) are written \( S \)-coprime integral coordinates for all \( i \in \{1, \ldots, n-1\}. \) By applying Proposition 4.0.3 we have \( \delta_\phi([0 : 1], P_1) = \delta_\phi([0 : 1], P_3), \) i. e. \( x_3 = x_1, \) because of \( R_3 \subset R_1. \) From \( \delta_\phi([0 : 1], P_1) = \delta_\phi(P_1, P_2) \) we deduce

\[
y_2 = \frac{x_2}{x_1} y_1 + 1.
\]

Furthermore, by Proposition 4.0.3 we have \( \delta_\phi([0 : 1], P_1) = \delta_\phi(P_2, P_3). \) Since \( x_3 = x_1, \) then

\[
y_3 x_2 - x_3 y_2 = x_1.
\]

This last equality combined with (5) provides \( y_3 = y_1, \) implying \([x_1 : y_1] = [x_3 : y_3].\)

Thus \( e \leq 1 \) and \( n \in \{1, 2, 3, 6\}. \) The next step is to prove that \( n \neq 6. \) If \( n = 6, \) with few calculations one sees that the cycle has the following form.

\[
[0 : 1] \mapsto [x_1 : y_1] \mapsto [A_2 x_1 : y_2] \mapsto [A_3 x_1 : y_3] \mapsto [A_2 x_1 : y_4] \mapsto [x_1 : y_5] \mapsto [0 : 1].
\]
where $A_2, A_3 \in \mathbb{R}_5$ and everything is written in $S$-coprime integral coordinates. We may apply Proposition 4.0.3, then by considering the $p$-adic distances $\delta_p(P_2, P_3)$ for all indexes $2 \leq i \leq 5$ for every place $p$, we obtain that there exists some $S$-units $u_j$ such that

$$y_2 = A_2y_1 + u_2; \quad y_3 = A_3y_1 + A_2u_3; \quad y_4 = A_2y_1 + A_3u_4; \quad y_5 = y_1 + A_2u_5.$$  

Since $R_5^2 = \{1\}$, we have that the identities in (8) become

$$y_2 = A_2y_1 + 1; \quad y_3 = A_3y_1 + A_2; \quad y_4 = A_2y_1 + A_3; \quad y_5 = y_1 + A_2$$

where $A_2, A_3$ are non zero elements in $\mathbb{F}_p[r]$. By considering the $p$-adic distance $\delta_p(P_2, P_3)$ for each finite place $p$, from Proposition 4.0.3 we obtain that

$$v_p(A_2x_1) = \delta_p(P_2, P_3) = v_p(A_2(x_1(2A_3y_1 + A_2) - A_2x_1(2A_3y_1 + 1))) = v_p(A_2A_3x_1 - A_2x_1),$$

i.e. $A_2x_1 = A_3x_1 - x_1$ (because $R_5^2 = \{1\}$). Then $A_2A_3x_1 = 0$ that contradicts $n = 6$. Thus $n \leq 3$.

Suppose now that $P$ is a preperiodic point. Without loss of generalities we can assume that the orbit of $P$ has the following shape:

$$P = P_{-m+1} \mapsto P_{-m+2} \mapsto \ldots \mapsto P_{-1} \mapsto P_0 = [0 : 1] \mapsto [0 : 1].$$

Indeed it is sufficient to take in consideration a suitable iterate $\phi^n$ (with $n \geq 3$), so that the orbit of the point $P$, with respect the iterate $\phi^n$, contains a fixed point $P_0$. By a suitable conjugation by an element of $\text{PGL}_2(\mathbb{R}_5)$, we may assume that $P_0 = [0 : 1]$.

For all $1 \leq j \leq m - 1$, let $P_j = [x_j : y_j]$ be written in $S$-coprime integral coordinates. By Lemma 5.1.1, for every $1 \leq i < j \leq m - 1$ there exists $T_{ij} \in R_5$ such that $x_i = T_{ij}x_j$. Consider the $p$-adic distance between the points $P_{-1}$ and $P_j$. Again by Lemma 5.1.1, we have

$$\delta_p(P_{-1}, P_j) = v_p(x_jy_j - x_1y_1/T_{ij}) = v_p(x_j/T_{ij}),$$

for all $p \not\in S$. Then, there exists $u_j \in R_5^2$ such that $x_j = (y_1 + u_j)/T_{ij}$, for all $p \not\in S$. Thus, there exists $u_j \in R_5^2$ such that $[x_{-1}, y_{-1}] = [x_1, y_1 + u_j]$. Since $R_5^2 = \{1\}$, then $P_j = [x_1 : y_1 + 1].$ So the length of the orbit (9) is at most 3. We get the bound 9 for the cardinality of the orbit of $P$.

**Case** $p > 2$.

Since $D = 1$ and $|S| = 1$, then the bound for the number of consecutive points as in Lemma 5.1.2 can be chosen equal to $p^2$. By Theorem 5.1 the minimal periodicity $n$ for a periodic point $P \in \mathbb{P}_1(\mathbb{Q})$ for the map $\phi$ is of the form $n = mr^p$ where $m \leq p + 1$, $r \leq p - 1$ and $e$ is a non negative integer.

Let us assume that $e \geq 2$. Since $p > 2$, by Proposition 4.0.3, for every $k \in \{0, 1, 2, \ldots, p^2 - 2\}$ and $i \in \{2, \ldots, p - 1\}$, we have that $\delta_k(P_0, P_i) = \delta_k(P_0, P_k, P_{k+1})$, for any $p \not\in S$. Then $P_{k, P_{k+1}} = [x_1, y_{k+1}].$ Furthermore $\delta_k(P_0, P_i) = \delta_k(P_0, P_{k, P_{k+1}})$ implying that there exists an element $u_{k, P_{k+1}} \in R_5^2$ such that

$$P_{k, P_{k+1}} = [x_1 : y_1 + u_{k, P_{k+1}}].$$

Since $R_5^2$ has $p - 1$ elements and there are $(p^2 - 2 + 1)(p - 2)$ numbers of the shape $k \cdot p + i$ as above, we have $(p^2 - 2 + 1)(p - 2) \leq p - 1$. Thus $e = 2$ and $p = 3$.

Then $n \leq 72$ if $p = 3$ and $n \leq (p^2 - 1)p$ if $p \geq 5$. For the more general case when $P$ is preperiodic, consider the same arguments used in the case when $p = 2$, showing $[x_{-1}, y_{-1}] = [x_1, y_1 + u_j],$ with $u_j \in R_5^2$. Thus, the orbit of a point $P \in \mathbb{P}_1(\mathbb{Q})$ containing $P_0 \in \mathbb{P}_1(\mathbb{Q})$, as in (9), has length at most $|R_5^2| + 2 = p + 1$. The bound in the preperiodic case is then 288 for $p = 3$ and $(p + 1)(p^2 - 1)p$ for $p \geq 5$. \qed
With similar proofs, we can get analogous bounds for every finite extension $K$ of $\mathbb{F}_p(t)$. The bounds of Theorem 1.2, with $K = \mathbb{F}_p(t)$, are especially interesting, for they are very small.

**References**


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