Numerical solution of the homogeneous Neumann boundary value problem on domains with a thin layer of random thickness

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Abstract

The present article is dedicated to the numerical solution of homogeneous Neumann boundary value problems on domains with a thin layer of different conductivity and of random thickness. By changing the boundary condition, the boundary value problem given on the random domain can be transformed into a boundary value problem on a fixed domain. The randomness is then contained in the coefficients of the new boundary condition. This thin coating can be expressed by a random Ventcell boundary condition and yields a second order accurate solution in the scale parameter ε of the layer's thickness. With the help of the Karhunen-Loève expansion, we transform this random boundary value problem into a deterministic, parametric one with a possibly high-dimensional parameter $\mathbf{y}$. Based on the decay of the random fluctuations of the layer's thickness, we prove rates of decay of the derivatives of the random solution with respect to this parameter $\mathbf{y}$ which are robust in the scale parameter ε. Numerical results validate our theoretical findings.

Keywords: Thin layer equation, boundary value problem, random domain.

1. Introduction

Many practical problems in engineering lead to boundary value problems for an unknown function. In this article, we consider uncertainties in the geometric definition of the domain motivated by tolerances in the manufacturing processes or in a damaged boundary during the life of a mechanical device. Manufactured or damaged devices are close to a nominal geometry but differ of course from its mathematical definition. Since we are motivated by tolerances, we can make the crucial assumption that the random perturbations are small. By identifying domains with their boundary, domains close to the nominal domain D can be seen as a perturbation in the normal direction of the nominal boundary ∂D.

The most common approach to study boundary value problems with stochastic inputs is the Monte-Carlo method, see e.g. [5, 14, 23] and the references therein. In many situations,

this approach is easy to implement since it only requires a sufficiently large number of samples. However, for boundary value problems on random domains, each sample means a new domain and thus a new mesh, the building of new mass and stiffness matrices, etc. All these steps are mandatory to compute the quantity of interest. Therefore, the Monte-Carlo method is extremely costly and not so easy to implement in our context. This article is dedicated to the development of a method for solving boundary value problems in random domains that requires only a single, fixed mesh. It is the second part of a work initiated in [11], where we have considered the case of Dirichlet boundary conditions. The modified boundary condition we obtain now is of Ventcell type, which involves second order derivatives in the tangential directions.

The treatment of boundary value problems on random domains is of high interest. There are several approaches for dealing with boundary value problems on random domains. Besides the fictitious domain approach considered in [7], one might essentially distinguish two approaches: the domain mapping method, cf. [8, 21, 18, 27, 26], and the perturbation method, cf. [16, 19], which is based on shape derivatives. They result from a description of the random domain either in Lagrangian coordinates or in Eulerian coordinates, see e.g. [24]. In this article, we consider Eulerian coordinates and propose a perturbation method based on approximate boundary conditions. This is especially motivated by the fact that shape derivatives are in general hard to compute.

This article is organized as follows. First, in Section 2, we present the preliminaries to deal with the problem under consideration. Then, in Section 3, we recall the theory for Ventcell's boundary value problem. In Section 4, we provide the asymptotic analysis required for the derivation of the deterministic thin layer equation. That is, we consider a thin layer, derive the artificial boundary condition and provide an error estimate that is uniform in both, the small parameter ε and the layer thickness h. In Section 5, we consider the case that the layer thickness h is a random field. We prove existence and uniqueness of the solution of such a problem. We then derive a high-dimensional, parametric boundary value problem and study the regularity of the parameter to solution map in Section 6. The regularity estimates obtained allow for methods which produce convergence rates that are independent of the dimension of the parameter space. In Section 7, we present numerical simulations which validate the proposed approach. Finally, we state concluding remark.

2. Impedance conditions for a slowly varying thin layer

2.1. Geometrical setting

We consider the following geometrical situation: A given $\mathcal{C}^\infty$ domain D is surrounded by a thin coating layer L_ε. We assume that I is an inclusion $I \subset D$ inside D with boundary Γ_D. The thickness h of the layer is a smooth, real valued function defined on the outer boundary Γ_V of D. We make the assumptions that the layer denoted by L_ε coats D on the outer boundary and that its characteristic size is a small parameter $0 < \varepsilon \leq \varepsilon_0$ so that the layer L_ε can be described as

$$L_\varepsilon = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}) \text{ with } 0 \leq t < \varepsilon h(\mathbf{x}) \text{ and } \mathbf{x} \in \Gamma_V\},$$

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see Figure 1 for an illustration of the geometrical situation.

We make the following assumptions on the function h:

(H1) h is continuously differentiable and there exists a non-negative number $d_{\max}$ such that $\|\nabla_\Gamma h(\mathbf{x})\| \leq d_{\max}$ for all $\mathbf{x} \in \Gamma_V$,

(H2) there exists nonnegative real numbers $0 < h_{\min} \leq h_{\max}$ such that $h_{\min} \leq h(\mathbf{x}) \leq h_{\max}$ for all $\mathbf{x} \in \Gamma_V$.
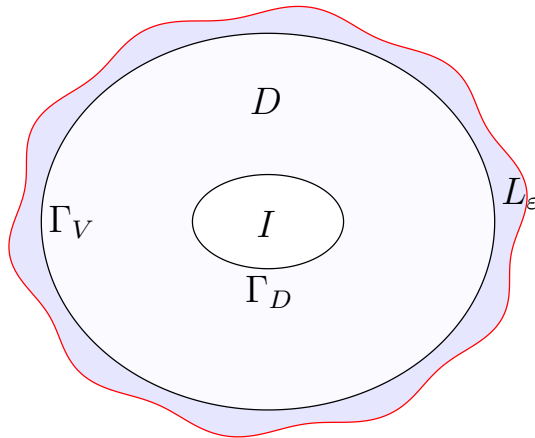

Figure 1: The geometrical setting – the domain D, the layer L_ε, the inner boundary Γ_D, and the outer boundary Γ_V.

2.2. Impedance boundary conditions

We are interested in the numerical resolution of the following model boundary value problem posed in $D_\varepsilon = D \cup L_\varepsilon$: for a given function $f \in \mathrm{L}^2(\mathbb{R}^2)$, find $u_\varepsilon \in \mathrm{H}^1_{\Gamma_D}(D_\varepsilon) = \{v \in \mathrm{H}^1(D_\varepsilon) : v = 0 \text{ on } \Gamma_D\}$ such that

$$\begin{cases} -\mathrm{div}(\sigma \nabla u_\varepsilon) = f & \text{in } D_\varepsilon \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial L_\varepsilon, \end{cases} \tag{1}$$

where the conductivity σ is piecewise constant and positiv taking the value σ_0 in D and 1 in the layer L_ε. In order to efficiently compute a numerical approximation for the solution of (1), a classical idea is to introduce impedance conditions (see [12], [4] and derived works) to avoid the meshing of the thin layer. The strategy is the following: work only in D and search for a boundary condition on Γ_V so that the solution of the new boundary value problem defined on D (i.e., the domain without the thin layer) is a *good* approximation of the restriction to D of the solution of the real boundary value problem set in D_ε.

3. On Ventcell's boundary value problem

We shall briefly recall some facts about boundary value problems with a Ventcell's boundary condition. To the end, let D be a domain in $\mathbb{R}^d$, $d = 2, 3$, with a smooth inclusion $I \subset D$. The exterior boundary of D is denoted by Γ_V and the interior boundary of D (i.e., the boundary of I) is denoted by Γ_D. We consider a function h which is defined on Γ_V such that there exist nonnegative real numbers $0 < h_{\min} \leq h_{\max}$:

$$\forall \mathbf{x} \in \Gamma_V, \ h_{\min} \leq h(\mathbf{x}) \leq h_{\max}.$$

Denoting the surface gradient by ∇_Γ and the tangential divergence by div_Γ, we shall then be concerned with the boundary value problem

$$\begin{cases} -\sigma_0 \Delta w = f & \text{in } D, \\ w = 0 & \text{on } \Gamma_D, \\ \sigma_0 \partial_{\mathbf{n}} w - \varepsilon \, \mathrm{div}_\Gamma(h \nabla_\Gamma w) = \varphi & \text{on } \Gamma_V. \end{cases} \tag{2}$$

In order to define the energy space of (2), we shall introduce the Sobolev space

$$\mathrm{H}^1_{\Gamma_D}(D) := \{ v \in \mathrm{H}^1(D) : v|_{\Gamma_D} = 0 \}.$$

Its dual will be denoted by $\widetilde{\mathrm{H}}^{-1}_{\Gamma_D}(D)$. Then, the natural space to search the solution of (2) is the space

$$\mathcal{H} := \left\{ v \in \mathrm{H}^1_{\Gamma_D}(D) : v|_{\Gamma_V} \in \mathrm{H}^1(\Gamma_V) \right\} \supset \mathrm{H}^{3/2}(D),$$

which we equip with the ε-dependent norm

$$\|v\| := \sqrt{\sigma_0 |v|^2_{\mathrm{H}^1(D)} + \varepsilon |v|^2_{\mathrm{H}^1(\Gamma_V)}}. \tag{3}$$

Notice that the right hand sides f and φ in (2) have to belong to $\widetilde{\mathrm{H}}^{-1}_{\Gamma_D}(D)$ and $\mathrm{H}^{-1}(\Gamma_V)$, respectively.

We are lead to the following variational formulation: seek $u \in \mathcal{H}$ such that

$$\int_D \sigma_0 \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h \nabla_\Gamma u \cdot \nabla_\Gamma v \, \mathrm{d}o = \int_D f v \, \mathrm{d}\mathbf{x} + \int_{\Gamma_V} \varphi v \, \mathrm{d}o$$

for all $v \in \mathcal{H}$. Here, $\mathrm{d}o$ stands for the surface measure. The assumptions on h ensure that the bilinear form

$$a(v, w) = \int_D \sigma_0 \nabla v \cdot \nabla w \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h \nabla_\Gamma v \cdot \nabla_\Gamma w \, \mathrm{d}o$$

is coercive and continuous with constants that depends only on $h_{\min}$ and $h_{\max}$. Thus, Lax Milgram's theorem can be applied and we obtain the existence and uniqueness of a solution $u \in \mathcal{H}$ to (2), satisfying the stability estimate

$$\|u\| \leq C(h_{\min}, h_{\max}) \left\{ \|f\|_{\widetilde{\mathrm{H}}^{-1}_{\Gamma_D}(D)} + \|\varphi\|_{\mathrm{H}^{-1}(\Gamma_V)} \right\}. \tag{4}$$

4. On the approximated boundary conditions

In this section, we derive a boundary value problem with Ventcell's boundary condition that approximates a boundary value problem which has a thin layer with Neumann boundary conditions. The results we obtain are also valid for the three-dimensional situation. But in order to keep the arguments simple, we perform the explicit computations only for the two-dimensional case.

4.1. Asymptotic expansion in D and in the layer L_ε

4.1.1. Notation

Let us introduce the notations and the objects needed to derive the approximate boundary conditions. For ease of notation, we deal with the bidimensional case and assume that D is simply connected so that its boundary has a parametrization by the arclength $s \mapsto \boldsymbol{\vartheta}(s)$ defined on the segment $[0, |\Gamma_V|]$, where $|\Gamma_V|$ is the perimeter of D. At the point $\boldsymbol{\vartheta}(s)$, the unit tangent vector $\mathbf{t}(s)$ is $\boldsymbol{\vartheta}'(s)$ and the curvature $\kappa(s)$ is defined by the equality $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$.

With these notations, the boundary of D_ε can be parametrized by $s \mapsto \boldsymbol{\vartheta}_h(s) = \boldsymbol{\vartheta}(s) + \varepsilon h(s)\mathbf{n}(s)$. Of course, this curve is not parametrized by the arclength and the unit tangent and the outward normal fields are

$$\mathbf{t}_h(s) = \frac{(1 + \varepsilon h\kappa)\mathbf{t}(s) + \varepsilon h'\mathbf{n}(s)}{\sqrt{(1 + \varepsilon h\kappa)^2 + (\varepsilon h')^2}} \quad \text{and} \quad \mathbf{n}_h(s) = \frac{(1 + \varepsilon h\kappa)\mathbf{n}(s) - \varepsilon h'\mathbf{t}(s)}{\sqrt{(1 + \varepsilon h\kappa)^2 + (\varepsilon h')^2}}. \tag{5}$$

It is convenient to drop the dependency in s for κ and h in the sequel.

If Γ_V and h are smooth enough, then one can expand the fields $\mathbf{t}_h$ and $\mathbf{n}_h$ into a power series, such that there are functions α_k and β_k which depend on the arclength s, satisfying

$$\mathbf{t}_h = \left(\sum_{k=0}^\infty \alpha_k \varepsilon^k\right)\mathbf{t} + \left(\sum_{k=0}^\infty \beta_k \varepsilon^k\right)\mathbf{n} \quad \text{and} \quad \mathbf{n}_h = \left(\sum_{k=0}^\infty \alpha_k \varepsilon^k\right)\mathbf{n} - \left(\sum_{k=0}^\infty \beta_k \varepsilon^k\right)\mathbf{t}.$$

The first coefficients of these expansions are given as

$$\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{(h')^2}{2}, \quad \alpha_3 = \kappa h(h')^2 \ldots \tag{6}$$

and

$$\beta_0 = 0, \quad \beta_1 = h', \quad \beta_2 = -h'h\kappa, \quad \beta_3 = \kappa h^2 h' - \frac{1}{2}(h')^3 \ldots, \tag{7}$$

respectively.

Let χ be the cut locus of Γ_V. Then, each function u, defined in D, can locally be represented in curvilinear coordinates according to $\tilde{u}(s, t) = u(\mathbf{x})$, where $\tilde{u} : [0, |\Gamma_V|] \times (-\chi, \chi) \to \mathbb{R}$. In particular, the gradient and Laplace operator are expressed in curvilinear coordinates as

$$\nabla = \frac{1}{1 + t\kappa}\frac{\partial}{\partial s}\mathbf{t}(s) + \frac{\partial}{\partial t}\mathbf{n}(s) \tag{8}$$

and

$$\Delta = \frac{1}{1+t\kappa} \frac{\partial}{\partial s} \left(\frac{1}{1+t\kappa} \frac{\partial}{\partial s} \right) + \frac{\kappa}{1+t\kappa} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}. \tag{9}$$

The usual strategy to transfer the boundary conditions at the layer's boundary to the boundary Γ_V relies on an asymptotic expansion of u_ε with respect to the scaling factor ε by a double ansatz. One ansatz is valid in D and the other ansatz is valid in the layer L_ε. Namely, we postulate that there are real-valued functions u_{int}^k, defined on D, and u_{ext}^k, defined on $[0, |\Gamma_V|] \times [0, 1]$, such that

$$u_\varepsilon(\mathbf{x}) = u_{\text{int}}(\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k u_{\text{int}}^k(\mathbf{x}) \text{ in } D \quad \text{and} \quad u_\varepsilon(\mathbf{x}) = u_{\text{ext}}(\mathbf{x}) = \sum_{k=0}^{\infty} \varepsilon^k u_{\text{ext}}^k \left(s, \frac{t}{\varepsilon h} \right) \text{ in } L_\varepsilon. \tag{10}$$

Relative to previous works, the anisotropy in the second curvilinear coordinate takes the variation of the thickness into account. With these ansätze at hand, we can reformulate the boundary value problem (1) as a transmission problem:

$$\begin{cases} -\sigma_0 \Delta u_{\text{int}} = f & \text{in } D, \\ -\Delta u_{\text{ext}} = f & \text{in } L_\varepsilon, \\ u_{\text{int}} = 0 & \text{on } \Gamma_D, \\ u_{\text{int}} = u_{\text{ext}} & \text{on } \Gamma_V, \\ \sigma_0 \partial_{\mathbf{n}} u_{\text{int}} = \partial_{\mathbf{n}} u_{\text{ext}} & \text{on } \Gamma_V, \\ \partial_{\mathbf{n}} u_{\text{ext}} = 0 & \text{on } \partial L_\varepsilon. \end{cases} \tag{11}$$

In view of (9), the equation in the layer L_ε reads in curvilinear coordinates as

$$\mathcal{L}\tilde{u} = -(1+t\kappa)f \text{ with } \mathcal{L} = \frac{\partial}{\partial s} \left(\frac{1}{1+t\kappa} \partial_s \right) + \kappa \partial_t + (1+t\kappa)\partial_{tt}^2. \tag{12}$$

In order to write this equation in the anisotropic, curvilinear coordinates $(s, \tau) = (s, t/(\varepsilon h))$ in L_ε corresponding to the ansätze (10), we require the following relations:

$$\partial_t = \frac{1}{\varepsilon h} \partial_\tau, \quad \kappa \partial_t = \frac{\kappa}{\varepsilon h} \partial_\tau, \quad \partial_{tt}^2 = \frac{1}{\varepsilon^2 h^2} \partial_{\tau\tau}^2 \quad \text{and} \quad (1+t\kappa)\partial_{tt}^2 = \frac{1}{\varepsilon^2 h^2} \partial_{\tau\tau}^2 + \frac{\kappa\,\tau}{\varepsilon h} \partial_{\tau\tau}^2.$$

Moreover, with the help of the decomposition of $(1+t\kappa)^{-1}$ into a power series and the product rule, we obtain

$$\partial_s \left(\frac{1}{1+t\kappa} \partial_s \right) = \sum_{n\geq 0}(-1)^n \varepsilon^n \tau^n \kappa^n h^n \partial_{ss}^2 + \sum_{n\geq 0}(-1)^n \varepsilon^n \tau^n n(\kappa' \kappa^{n-1}h^n + \kappa^n h' h^{n-1})\partial_s.$$

The operator $\mathcal{L}$ can thus be split in powers of the small parameter ε according to

$$\mathcal{L} = \sum_{n\geq -2} \varepsilon^n \mathcal{L}_n \tag{13}$$

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with

$$\mathcal{L}_{-2} = \frac{1}{h^2} \partial^2_{\tau\tau}, \quad \mathcal{L}_{-1} = \frac{\kappa}{h} \left(\partial_\tau + \tau \, \partial^2_{\tau\tau} \right),$$

and, for general $n \geq 0$, with

$$\mathcal{L}_n = (-1)^n \tau^n \left[\kappa^n h^n \partial^2_{ss} + n(\kappa' \kappa^{n-1} h^n + \kappa^n h' h^{n-1}) \partial_s \right].$$

Especially, we find

$$\mathcal{L}_0 = \partial^2_{ss}, \quad \mathcal{L}_1 = -\tau \left[\kappa h \partial^2_{ss} + \partial_s(\kappa h) \partial_s \right] = -\tau \partial_s(\kappa h \partial_s).$$

We finally have to assume that the right hand side f is smooth in the layer, so that it can be expended in the layer in accordance with

$$(1 + \varepsilon \kappa h \tau) f(\cdot, \varepsilon h \tau) = \sum_{n=0}^{N} \varepsilon^n f_n(., \tau) + \mathcal{O}(\varepsilon^{N+1}). \tag{14}$$

In particular, it holds

$$f_0(s, \tau) = f(s, 0) \text{ and } f_1(s, \tau) = \kappa h \tau \big(f(s, 0) + \partial_{\mathbf{n}} f(s, 0) \big).$$

4.1.2. Derivations of a sequence of differential equations

Determination of the differential equation for u_{ext}^k. By plugging (13) and (14) into (12), we obtain first

$$\mathcal{L} u_{\text{ext}} = \varepsilon^{-2} \frac{1}{h^2} \partial^2_{\tau\tau} u_{\text{ext}}^0 + \varepsilon^{-1} \left(\frac{1}{h^2} \partial^2_{\tau\tau} u_{\text{ext}}^1 + \frac{\kappa}{h} \left(\partial_\tau u_{\text{ext}}^0 + \tau \partial^2_{\tau\tau} u_{\text{ext}}^0 \right) \right)$$

$$+ \sum_{n \geq 0} \varepsilon^n \left(\frac{1}{h^2} \partial^2_{\tau\tau} u_{\text{ext}}^{n+2} + \frac{\kappa}{h} \left(\partial_\tau u_{\text{ext}}^{n+1} + \tau \partial^2_{\tau\tau} u_{\text{ext}}^{n+1} \right) + \sum_{k=0}^{n} \mathcal{L}_{n-k} u_{\text{ext}}^k \right).$$

We compare the terms with the same order of ε to obtain the sequence of differential equations for the u_{ext}^k. It holds

$$\partial^2_{\tau\tau} u_{\text{ext}}^0 = 0, \quad \partial^2_{\tau\tau} u_{\text{ext}}^1 = -\kappa h \left(\partial_\tau u_{\text{ext}}^0 + \tau \partial^2_{\tau\tau} u_{\text{ext}}^0 \right), \tag{15}$$

and, for general $n \geq 0$,

$$\partial^2_{\tau\tau} u_{\text{ext}}^{n+2} = - \left(\kappa h \left(\partial_\tau u_{\text{ext}}^{n+1} + \tau \partial^2_{\tau\tau} u_{\text{ext}}^{n+1} \right) + h^2 \sum_{k=0}^{n} \mathcal{L}_{n-k} u_{\text{ext}}^k \right) - h^2 f_n. \tag{16}$$

In particular, we have

$$\partial^2_{\tau\tau} u_{\text{ext}}^2 = - \left(\kappa h (\partial_\tau u_{\text{ext}}^1 + \tau \partial^2_{\tau\tau} u_{\text{ext}}^1) + h^2 \mathcal{L}_0 u_{\text{ext}}^0 \right) - h^2 f_0, \tag{17}$$

$$\partial^2_{\tau\tau} u_{\text{ext}}^3 = - \left(\kappa h (\partial_\tau u_{\text{ext}}^2 + \tau \partial^2_{\tau\tau} u_{\text{ext}}^2) + h^2 (\mathcal{L}_0 u_{\text{ext}}^1 + \mathcal{L}_1 u_{\text{ext}}^0) \right) - h^2 f_1.$$

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Determination of the boundary conditions for u_{ext}^k. The boundary conditions on the boundary Γ_V are given by the transmission conditions stated in (11). They are obtained by setting $\tau = 0$ and read as

$$\sigma_0 \partial_{\mathbf{n}} u_{\text{int}} = \frac{1}{\varepsilon h} \partial_\tau u_{\text{ext}} \quad \Longrightarrow \quad \sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^{k-1} = \frac{1}{h} \partial_\tau u_{\text{ext}}^k, \tag{18}$$
$$u_{\text{int}} = u_{\text{ext}} \quad \Longrightarrow \quad u_{\text{int}}^k = u_{\text{ext}}^k.$$

On the outer boundary ∂L_ε, that is for $\tau = 1$, one has $\partial_{\mathbf{n}_h} u_{\text{ext}} = 0$ according to (11). Therefore, it holds $\nabla u_{\text{ext}} \cdot \mathbf{n}_h = 0$, which corresponds thanks to the expression of the normal field (5) and of the gradient (8) to

$$\left(\frac{1}{1 + \varepsilon h \tau \kappa} \frac{\partial}{\partial s} \left[\sum_{k=0}^{\infty} u_{\text{ext}}^k \varepsilon^k \right] \mathbf{t} + \frac{1}{\varepsilon h} \frac{\partial}{\partial \tau} \left[\sum_{k=0}^{\infty} u_{\text{ext}}^k \varepsilon^k \right] \mathbf{n} \right) \cdot \left[\left(\sum_{k=0}^{\infty} \alpha_k \varepsilon^k \right) \mathbf{n} - \left(\sum_{k=0}^{\infty} \beta_k \varepsilon^k \right) \mathbf{t} \right] = 0.$$

This is equivalent to

$$- \frac{1}{1 + \varepsilon h \tau \kappa} \left(\sum_{k=0}^{\infty} \frac{\partial u_{\text{ext}}^k}{\partial s} \varepsilon^k \right) \left(\sum_{k=0}^{\infty} \beta_k \varepsilon^k \right) + \frac{1}{\varepsilon h} \left[\sum_{k=0}^{\infty} \frac{\partial u_{\text{ext}}^k}{\partial \tau} \varepsilon^k \right] \left(\sum_{k=0}^{\infty} \alpha_k \varepsilon^k \right) = 0.$$

and finally to

$$\frac{1}{1 + \varepsilon h \tau \kappa} \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} \frac{\partial u_{\text{ext}}^k}{\partial s} \beta_\ell \right) \varepsilon^n = \frac{1}{\varepsilon h} \sum_{n=0}^{\infty} \left(\sum_{k+\ell=n} \frac{\partial u_{\text{ext}}^k}{\partial \tau} \alpha_\ell \right) \varepsilon^n.$$

We plug in the expressions of the coefficients $(\alpha_\ell, \beta_\ell)$ from (6), (7) and sort with respect to the powers ε^n on the left and on the right hand side of the last equation. The first four terms are given as follows.

- For $n = -1$, we obtain $\frac{1}{h} \frac{\partial u_{\text{ext}}^0}{\partial \tau} \alpha_0 = 0$. Hence, we conclude

$$\frac{\partial u_{\text{ext}}^0}{\partial \tau} = 0. \tag{19}$$

- For $n = 0$, we obtain $\frac{\partial u_{\text{ext}}^0}{\partial s} \beta_0 = \frac{1}{h} \left(\frac{\partial u_{\text{ext}}^1}{\partial \tau} \alpha_0 + \frac{\partial u_{\text{ext}}^0}{\partial \tau} \alpha_1 \right)$. Hence, we conclude

$$\frac{\partial u_{\text{ext}}^1}{\partial \tau} = 0. \tag{20}$$

- For $n = 1$, we obtain

$$\frac{\partial u_{\text{ext}}^1}{\partial s} \beta_0 + \frac{\partial u_{\text{ext}}^0}{\partial s} \beta_1 - h \tau \kappa \beta_0 \frac{\partial u_{\text{ext}}^0}{\partial s} = \frac{1}{h} \left(\frac{\partial u_{\text{ext}}^2}{\partial \tau} \alpha_0 + \frac{\partial u_{\text{ext}}^1}{\partial \tau} \alpha_1 + \frac{\partial u_{\text{ext}}^0}{\partial \tau} \alpha_2 \right).$$

Hence, we conclude

$$\frac{\partial}{\partial \tau} u_{\text{ext}}^2 = h h' \frac{\partial}{\partial s} u_{\text{ext}}^0 + \frac{1}{2} (h')^2 \frac{\partial}{\partial \tau} u_{\text{ext}}^0.$$

- For $n = 2$, we obtain

$$\frac{\partial u_{\text{ext}}^2}{\partial s}\beta_0 + \frac{\partial u_{\text{ext}}^1}{\partial s}\beta_1 + \frac{\partial u_{\text{ext}}^0}{\partial s}\beta_2 - h\tau\kappa\left(\frac{\partial u_{\text{ext}}^1}{\partial s}\beta_0 + \frac{\partial u_{\text{ext}}^0}{\partial s}\beta_1\right) + h^2\tau^2\kappa^2\frac{\partial}{\partial s}u_{\text{ext}}^0\beta_0$$

$$= \frac{1}{h}\left(\frac{\partial u_{\text{ext}}^3}{\partial \tau}\alpha_0 + \frac{\partial u_{\text{ext}}^2}{\partial \tau}\alpha_1 + \frac{\partial u_{\text{ext}}^1}{\partial \tau}\alpha_2 + \frac{\partial u_{\text{ext}}^0}{\partial \tau}\alpha_3\right).$$

Hence, we conclude

$$\frac{\partial}{\partial \tau}u_{\text{ext}}^3 = hh'\frac{\partial}{\partial s}u_{\text{ext}}^1 - 2h'h^2\kappa\frac{\partial}{\partial s}u_{\text{ext}}^0 - \alpha_2\frac{\partial}{\partial \tau}u_{\text{ext}}^1 - \alpha_3\frac{\partial}{\partial \tau}u_{\text{ext}}^0.$$

4.1.3. Computations of the first order terms

The resolution for u_{ext}^n and u_{int}^n is now iterative. For our purpose, we should make the first steps explicit.

Step 1: Partial resolution of order $n = 0$. Since $\partial_{\tau\tau}^2 u_{\text{ext}}^0 = 0$ in accordance with (15), the first function u_{ext}^0 should be affine in the variable τ with a derivative that vanishes at $\tau = 1$ due to (19). Therefore, u_{ext}^0 is constant in τ.

Step 2: Partial resolution of order $n = 1$. According to (15), the equation for u_{ext}^1 is

$$\partial_{\tau\tau}^2 u_{\text{ext}}^1 = -\kappa h\left(\partial_\tau u_{\text{ext}}^0 + \tau\partial_{\tau\tau}^2 u_{\text{ext}}^0\right) = 0.$$

In addition, we have the boundary condition

$$\partial_\tau u_{\text{ext}}^1(s, 1) = 0,$$

cf. (19). Hence, u_{ext}^1 is also constant in τ.

Step 3: Complete resolution of order $n = 0$. From the flux condition (18) at the interface, we get

$$\sigma_0 h\,\partial_\mathbf{n} u_{\text{int}}^0(s, 0) = \partial_\tau u_{\text{ext}}^1(s, 0) = 0.$$

Since u_{ext}^0 is constant in τ, it follows $\partial_\mathbf{n} u_{\text{ext}}^0 = \partial_\tau u_{\text{ext}}^0 = 0$. Thus, we conclude by inserting the ansatz (10) for u_{int} in the transmission problem (11) that u_{int}^0 is uniquely determined as the solution of

$$\begin{cases} -\sigma_0\Delta u_{\text{int}}^0 = f & \text{in } D, \\ u_{\text{int}}^0 = 0 & \text{on } \Gamma_D, \\ \sigma_0\partial_\mathbf{n} u_{\text{int}}^0 = 0 & \text{on } \Gamma_V. \end{cases}$$

Due to step 1, i.e., $u_{\text{ext}}^0(s, \tau) = u_{\text{ext}}^0(s, 0)$, the continuity at the interface (cf. (18)) leads to $u_{\text{ext}}^0(s, \tau) = u_{\text{int}}^0(s, 0)$.

Step 4: Partial resolution of order $n = 2$. Since u_{ext}^1 is constant in τ, the differential equation (17) for u_{ext}^2 reads

$$\partial_{\tau\tau}^2 u_{\text{ext}}^2 = -\kappa h\left(\partial_\tau u_{\text{ext}}^1 + \tau\partial_{\tau\tau}^2 u_{\text{ext}}^1\right) - h^2\partial_{ss}^2 u_{\text{ext}}^0 - h^2 f_0 = -h^2\left(\partial_{ss}^2 u_{\text{ext}}^0 + f_0\right).$$

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The boundary condition at the outer boundary is

$$\frac{\partial}{\partial \tau} u_{\text{ext}}^2(s,1) = hh' \frac{\partial}{\partial s} u_{\text{ext}}^0(s,1) + \frac{1}{2}(h')^2 \frac{\partial}{\partial \tau} u_{\text{ext}}^0(s,1) = hh' \frac{\partial}{\partial s} u_{\text{ext}}^0(s,1),$$

since u_{ext}^0 is constant in τ. We still can find an analytic expression for u_{ext}^2. There exists a constant C_0 still undetermined such that

$$
\begin{aligned}
u_{\text{ext}}^2(s,\tau) = &-\frac{1}{2} h^2(s) \left[\partial_{ss}^2 u_{\text{ext}}^0(s,0) + f_0(s) \right] \tau^2 \\
&+ \left[h^2(s) \left[\partial_{ss}^2 u_{\text{ext}}^0(s,0) + f_0(s) \right] + h(s) h'(s) \partial_s u_{\text{int}}^0(s,0) \right] \tau + C_0.
\end{aligned}
$$

Step 5: Complete resolution of order $n = 1$. From the flux condition (18) at the interface, we get

$$
\begin{aligned}
\sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^1(s,0) &= \frac{1}{h} \partial_\tau u_{\text{ext}}^2(s,0) \\
&= h(s) \left[\partial_{ss}^2 u_{\text{ext}}^0(s,0) + f_0(s) \right] + h'(s) \partial_s u_{\text{int}}^0(s,0) \\
&= h(s) f_0(s) + \partial_s \left[h(s) \partial_s u_{\text{int}}^0(s,0) \right].
\end{aligned}
$$

This determines u_{int}^1 as the solution of

$$
\begin{cases}
-\sigma_0 \Delta u_{\text{int}}^1 = 0 & \text{in } D, \\
u_{\text{int}}^1 = 0 & \text{on } \Gamma_D, \\
\sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^1 = h(s) f_0(s) + \partial_s \left[h(s) \partial_s u_{\text{int}}^0(s,0) \right] & \text{on } \Gamma_V.
\end{cases}
\tag{21}
$$

4.2. Derivation of the approximated boundary conditions.

In order to derive the approximated boundary conditions, we introduce the partial sum $u_{\text{int}}^{[1]} = u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1$. By construction, we check that $-\sigma_0 \Delta u_{\text{int}}^{[1]} = f$ in D and $u_{\text{int}}^{[1]} = 0$ on Γ_D. On the outer boundary Γ_V, one has

$$\sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^{[1]}(s,0) = \varepsilon \left(h(s) f_0(s) + \partial_s \left[h(s) \partial_s u_{\text{int}}^0(s,0) \right] \right),$$

so that it follows

$$\sigma_0 \partial_{\mathbf{n}} u_{\text{int}}^{[1]}(s,0) - \varepsilon \partial_s \left[h(s) \partial_s u_{\text{int}}^{[1]}(s,0) \right] = \varepsilon h(s) f_0(s) + \varepsilon^2 \partial_s \left[h(s) \partial_s u_{\text{int}}^1(s,0) \right].$$

Finally, $u_{\text{int}}^{[1]}$ is solution of

$$
\begin{cases}
-\sigma_0 \Delta u = f & \text{in } D, \\
u = 0 & \text{on } \Gamma_D, \\
\sigma_0 \partial_{\mathbf{n}} u(s,0) - \varepsilon \partial_s \left[h(s) \partial_s u(s,0) \right] = \varepsilon h(s) f_0(s) + \varepsilon^2 \partial_s \left[h(s) \partial_s u_{\text{int}}^1(s,0) \right] & \text{on } \Gamma_V.
\end{cases}
\tag{22}
$$

Therefore, one introduces the approximate first order problem:

$$\begin{cases} -\sigma_0 \Delta u = f & \text{in } D, \\ u = 0 & \text{on } \Gamma_D, \\ \sigma_0 \partial_{\mathbf{n}} u - \varepsilon \partial_s \left[h(s) \partial_s u \right] = \varepsilon h(s) f_0(s) & \text{on } \Gamma_V. \end{cases} \qquad (23)$$

It corresponds in three dimensions to a boundary value problem with Ventcells boundary conditions, as introduced in Section 3:

$$\begin{cases} -\sigma_0 \Delta u = f & \text{in } D, \\ u = 0 & \text{on } \Gamma_D, \\ \sigma_0 \partial_{\mathbf{n}} u - \varepsilon \text{div}_\Gamma \left[h(s) \nabla_\Gamma u \right] = \varepsilon h(s) f_0(s) & \text{on } \Gamma_V. \end{cases} \qquad (24)$$

One can continue the analysis and write down higher order approximated boundary conditions. However, they become more and more complex and from the second order involves the curvature and its derivative (technically coming from the $\mathcal{L}_1$ operator). If one does not have a parametric model for the surface and only dispose of a mesh of the computational domain, it is not reasonable to use these high order models. Hence, we will study only the first order model for Neumann boundary conditions.

4.3. On the approximation error

In order to estimate the error made by using the artificial boundary condition, we proceed in two steps. First, we obtain an error estimate for the remainders $r_\varepsilon^1 := u_\varepsilon - u_{\text{int}}^{[1]}$ in the truncated asymptotic expansions of the solution u_ε of (1). Then, in a second step, we compute the difference between $u_{\text{int}}^{[1]}$ the solution of (22) and the solution of the approximate first order problems (23) and (24), respectively. Both steps are adapted from the proofs of [6] in the case of layers with constant thickness, i.e. if $h(\mathbf{x})$ takes a constant value, and of our work [11] in the case of Dirichlet boundary conditions with varying thickness. We therefore explain the main lines of the proof without entering into the details. Notice that from now on, in this section, the term "uniform" means uniform with respect to both ε and h.

Theorem 4.1. *There is a constant C that depends only of D, $h_{\min}$, $h_{\max}$ and $d_{\max}$ such that the committed error satisfies*

$$\|u_\varepsilon - u\|_{\mathrm{H}^1(D)} \leq C\varepsilon^2.$$

Here, u_ε solves (1) and u solves the approximated problem (23).

Proof. The idea is to split the error into two parts

$$\|u_\varepsilon - u\|_{\mathrm{H}^1(D)} \leq \|u_\varepsilon - (u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1)\|_{\mathrm{H}^1(D)} + \|u - (u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1)\|.$$

In Step 1, we estimate the first term on the right hand side while, in step 2, we estimate the second term on the right hand side, both of which estimates are of the desired order.

Step 1: Estimation of the truncation error. To estimate the truncation error for u_ε, the first step is to write a precise error estimate for the remainder r_ε^N in the asymptotic expansion of u_{ext} and u_{int}. The reminder r_ε^N is piecewise defined as

$$r_\varepsilon^N = r_{\varepsilon,\text{int}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{\text{int}}^k \text{ in } D \quad \text{and} \quad r_\varepsilon^N = r_{\varepsilon,\text{ext}}^N = u_\varepsilon - \sum_{k=0}^N \varepsilon^k u_{\text{ext}}^k \text{ in } L_\varepsilon.$$

These remainders satisfy the boundary value problems

$$\begin{cases} \sigma_0 \Delta r_{\varepsilon,\text{int}}^N = 0 & \text{in } D, \\ \Delta r_{\varepsilon,\text{ext}}^N = f_N & \text{in } L_\varepsilon, \\ \sigma_0 \partial_{\mathbf{n}} r_{\varepsilon,\text{int}}^N = \partial_{\mathbf{n}} r_{\varepsilon,\text{ext}}^N + g_N & \text{on } \Gamma_V, \\ r_{\varepsilon,\text{int}}^N = 0 & \text{on } \Gamma_D, \\ r_{\varepsilon,\text{int}}^N = r_{\varepsilon,\text{ext}}^N & \text{on } \Gamma_V, \\ \partial_{\mathbf{n}} r_{\varepsilon,\text{ext}}^N = 0 & \text{on } \partial D_\varepsilon, \end{cases} \tag{25}$$

with $f_N = \mathcal{O}(\varepsilon^{N-1})$ and $g_N = \mathcal{O}(\varepsilon^N)$ by construction. Precisely, f_N is given by (14) and the normal derivatives satisfy:

$$\sigma_0 \partial_{\mathbf{n}} r_{\varepsilon,\text{int}}^N - \partial_{\mathbf{n}} r_{\varepsilon,\text{ext}}^N = \sum_{n=0}^N \varepsilon^n \left[\sigma_0 \partial_{\mathbf{n}} u_{\text{int},n} - \frac{1}{\varepsilon h} \partial_t u_{\text{ext},n} \right] = \sigma_0 \varepsilon^N \partial_{\mathbf{n}} u_{\text{int},N}.$$

In order to derive the variational formulation of (25), we define the bilinear form a_ε on $\mathrm{H}^1_{\Gamma_D}(D_\varepsilon)$ by

$$a_\varepsilon(v, w) := \sigma_0 \int_D \nabla v \cdot \nabla w \, \mathrm{d}\mathbf{x} + \int_{L_\varepsilon} \nabla v \cdot \nabla w \, \mathrm{d}\mathbf{x}$$

and the linear form F_ε^N on $\mathrm{H}^1_{\Gamma_D}(D_\varepsilon)$ by

$$\langle F_\varepsilon^N, v \rangle := \int_{\Gamma_V} g_N v \, \mathrm{d}o - \int_{L_\varepsilon} f_N v \, \mathrm{d}\mathbf{x}.$$

Thus, (25) has the variational formulation

$$\forall v \in \mathrm{H}^1_{\Gamma_D}(D_\varepsilon), \ a_\varepsilon(r_\varepsilon^N, v) = F_\varepsilon^N(v).$$

In [11, Section 2–3], we have shown the uniform coercivity and continuity of the bilinear forms a_ε and there is a constant K such that $|\langle F_\varepsilon^N, v \rangle| \leq K \|v\|_{\mathrm{H}^1(L_\varepsilon)}$. Consequently, Lax Milgram's theorem can be applied and combining the uniform coercivity of a_ε with respect to ε and the continuity of F_ε^N, we check that there is a constant C, independent of $\varepsilon \leq \varepsilon_0$ and of h, such that

$$\|r_\varepsilon^N\|_{\mathrm{H}^1(D_\varepsilon)} \leq C\varepsilon^{N-1} \tag{26}$$

since $f_N = \mathcal{O}(\varepsilon^{N-1})$ and $g_N = \mathcal{O}(\varepsilon^N)$. Finally, we split the remainder of order N according to

$$r_\varepsilon^N = r_\varepsilon^{N+2} + \varepsilon^{N+1} u_\varepsilon^{N+1} + \varepsilon^{N+2} u_\varepsilon^{N+2},$$

where u_ε^k is u_{int}^k in D and u_{ext}^k in the layer. The triangle inequality and (26), applied to the remainder r_ε^{N+2} of order $N+2$, give

$$\|r_\varepsilon^N\|_{\mathrm{H}^1(D_\varepsilon)} \leq \|r_\varepsilon^{N+2}\|_{\mathrm{H}^1(D_\varepsilon)} + \varepsilon^{N+1}\|u_\varepsilon^{N+1}\|_{\mathrm{H}^1(D_\varepsilon)} + \varepsilon^{N+2}\|u_\varepsilon^{N+2}\|_{\mathrm{H}^1(D_\varepsilon)} \leq C\varepsilon^{N+1}.$$

We thus immediately get

$$\|r_{\varepsilon,\text{int}}^N\|_{\mathrm{H}^1(D)} \leq C\varepsilon^{N+1}. \tag{27}$$

Note that the constant C depends of the truncation order N.

Step 2: Estimation of the difference. For u being the solution of (23), we shall estimate the difference $r_2 = u - (u_{\text{int}}^0 + \varepsilon u_{\text{int}}^1) = u - u_{\text{int}}^{[1]}$. It obviously solves the boundary value problem

$$\begin{cases} -\sigma_0 \Delta r_2 = 0 & \text{in } D, \\ r_2 = 0 & \text{on } \Gamma_D, \\ \sigma_0 \partial_{\mathbf{n}} r_2 - \varepsilon \partial_s(h \partial_s r_2) = \varphi & \text{on } \Gamma_V, \end{cases} \tag{28}$$

where the right hand side $\varphi = -\varepsilon^2 \partial_s(h \partial_s u_{\text{int}}^1)$. By the *a priori* estimate (4) for Ventcell's boundary value problem, we get $\|r_2\| \leq \varepsilon^2 C(h_{\min}, h_{\max}) \|\partial_s(h \partial_s u_{\text{int}}^1)\|_{\mathrm{H}^{-1}(\Gamma_V)}$. Since

$$\langle v, \partial_s(h \partial_s u_{\text{int}}^1) \rangle_{\mathrm{H}^1(\Gamma_V) \times \mathrm{H}^{-1}(\Gamma_V)} = \int_{\Gamma_V} h \partial_s v \partial_s u_{\text{int}}^1 \, do \leq h_{\max} \|u_{\text{int}}^1\|_{\mathrm{H}^1(\Gamma_V)} \|v\|_{\mathrm{H}^1(\Gamma_V)}$$

for all $v \in \mathrm{H}^1(\Gamma_V)$, we obtain $\|r_2\| \leq \varepsilon^2 C(h_{\min}, h_{\max}, d_{\max}) h_{\max} \|u_{\text{int}}^1\|_{\mathrm{H}^1(\Gamma_V)}$. Standard regularity theory for the Neumann boundary value problem (21), satisfied by u_{int}^1, provides the ε-independent bound

$$\|u_{\text{int}}^1\|_{\mathrm{H}^1(\Gamma_V)} \leq C\Big\{ h_{\max} \|f_0\|_{\mathrm{L}^2(\Gamma_V)} + \|h\|_{\mathrm{W}^{1,\infty}(\Gamma_V)} \|u_{\text{int}}^0\|_{\mathrm{H}^2(\Gamma_V)} \Big\}.$$

$\square$

5. Randomly varying thin layer

From now on, we shall consider the situation that the layer's thickness h is random. To that end, let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space and assume that $h : \Gamma_V \times \Omega \to \mathbb{R}$ is a random field which satisfies the following assumptions:

(UB) Uniform boundedness: there exist two nonnegative real numbers $h_{\min} \leq h_{\max}$ and a real $0 \leq q < 1$ such that the random field

$$h(\mathbf{x}, \omega) = \overline{h}(\mathbf{x}) + \widetilde{h}(\mathbf{x}, \omega) \quad \text{with} \quad \overline{h}(\mathbf{x}) = \mathbb{E}\big(h(\mathbf{x}, \omega)\big)$$

satisfies

$$0 < h_{\min} \leq \overline{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\widetilde{h}(\mathbf{x}, \omega)| \leq q\overline{h}(\mathbf{x}) \tag{29}$$

for all $\mathbf{x} \in \Gamma_V$ and for $\mathbb{P}$-almost all $\omega \in \Omega$.

(UR) Uniform regularity: the function $\mathbf{x} \mapsto h(\mathbf{x}, \omega)$ is uniformly bounded in $\mathcal{C}^1$ for all ω in Ω, that is, the random field h belongs to the Bochner space $\mathrm{L}_\mathbb{P}^\infty(\Omega, \mathcal{C}^1(\Gamma_V))$.

Let us recall for the reader's convenience the definition of Bochner spaces. Consider a real number $p \geq 1$. Then, for a Banach space X, the Bochner space $\mathrm{L}_\mathbb{P}^p(\Omega, \mathrm{X})$ consists of all functions $v : \Omega \to \mathrm{X}$ whose norm

$$\|v\|_{\mathrm{L}_\mathbb{P}^p(\Omega,\mathrm{X})} := \begin{cases} \left(\int_\Omega \|v(\cdot, \omega)\|_\mathrm{X}^p \, \mathrm{d}\mathbb{P}(\omega) \right)^{1/p}, & p < \infty, \\ \mathrm{ess\,sup}_{\omega \in \Omega} \|v(\cdot, \omega)\|_\mathrm{X}, & p = \infty, \end{cases}$$

is finite. If $p = 2$ and X is a Hilbert space, then the Bochner space is isomorphic to the tensor product space $\mathrm{L}_\mathbb{P}^2(\Omega) \otimes \mathrm{X}$.

Since the layer's thickness is random, the approximate first order boundary value problem (23) becomes the following partial differential equation with random boundary data:

$$\begin{cases} -\sigma_0 \Delta u(\omega) = f & \text{in } D \\ u(\omega) = 0 & \text{on } \Gamma_D, \\ \sigma_0 \partial_\mathbf{n} u(\omega) - \varepsilon \partial_s \left[h(\omega) \partial_s u(\omega) \right] = \varepsilon h(\omega) f_0 & \text{on } \Gamma_V. \end{cases}$$

To obtain its variational formulation, we multiply this equation with an arbitrary test function from $L_\mathbb{P}^2(\Omega, \mathcal{H})$: seek $u \in \mathrm{L}_\mathbb{P}^2(\Omega, \mathcal{H})$ such that

$$\int_\Omega \left\{ \sigma_0 \int_D \nabla u(\omega) \cdot \nabla v(\omega) \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\omega) \partial_s u(\omega) \partial_s v(\omega) \, \mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega)$$
$$= \int_\Omega \left\{ \int_D f v(\omega) \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\omega) f_0 v(\omega) \, \mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega) \tag{30}$$

holds for all $v \in \mathrm{L}_\mathbb{P}^2(\Omega, \mathcal{H})$.

Theorem 5.1. *Under the conditions* **(UB)** *given by* (29), *there exists for all $\varepsilon \leq \varepsilon_0$ a unique solution u in $\mathrm{L}_\mathbb{P}^2(\Omega, \mathcal{H})$ to the variational formulation* (30). *In particular, we have the stability estimate*

$$\sqrt{\int_\Omega \|u(\omega)\|^2 \, \mathrm{d}\mathbb{P}(\omega)} \leq C \left\{ \|f\|_{\widetilde{\mathrm{H}}_{\Gamma_D}^{-1}(D)} + \|f_0\|_{H^{-1}(\Gamma_V)} \right\} \tag{31}$$

uniformly as ε tends to 0.

Proof. By introducing the bilinear form

$$a : \mathrm{L}_\mathbb{P}^2(\Omega, \mathcal{H}) \times \mathrm{L}_\mathbb{P}^2(\Omega, \mathcal{H}) \to \mathbb{R},$$
$$a(v, w) := \int_\Omega \left\{ \sigma_0 \int_D \nabla v \cdot \nabla w \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h \partial_s v \partial_s w \, \mathrm{d}o \right\} \mathrm{d}\mathbb{P}(\omega)$$

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and the linear form

$$\ell : \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathcal{H}) \to \mathbb{R}, \quad \ell(v) := \int_\Omega \left[\int_D fv \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h f_0 v \, \mathrm{d}o \right] \mathrm{d}\mathbb{P}(\omega),$$

the variational formulation (30) is equivalent to the problem:

$$\text{seek } u \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathcal{H}) \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in \mathrm{L}^2_{\mathbb{P}}(\Omega, \mathcal{H}). \tag{32}$$

In view of (29), it holds that

$$0 < (1-q)h_{\min} \le (1-q)\overline{h}(\mathbf{x}) \le |h(\mathbf{x}, \omega)| \le (1+q)\overline{h}(\mathbf{x}) \le (1+q)h_{\max}$$

for all $\mathbf{x} \in D$ and for $\mathbb{P}$-almost all $\omega \in \Omega$. This bound ensures the uniform ellipticity and boundedness of the bilinear form $a(\cdot, \cdot)$:

$$\min\left\{1, (1-q)h_{\min}\right\} \int_\Omega \|v(\omega)\|^2 \, \mathrm{d}\mathbb{P}(\omega) \le a(v,v),$$

$$|a(v,w)| \le \max\left\{1, (1+q)h_{\max}\right\} \sqrt{\int_\Omega \|v(\omega)\|^2 \, \mathrm{d}\mathbb{P}(\omega)} \sqrt{\int_\Omega \|w(\omega)\|^2 \, \mathrm{d}\mathbb{P}(\omega)}.$$

In addition, the linear form $\ell(\cdot)$ satisfies

$$|\ell(v)| \le \|f\|_{\widetilde{\mathrm{H}}^{-1}_{\Gamma_D}(D)} \sqrt{\int_\Omega \|v(\omega)\|^2_{\mathrm{H}^1(D)} \, \mathrm{d}\mathbb{P}(\omega)}$$

$$+ \varepsilon(1+q)h_{\max}\|f\|_{\mathrm{H}^{-1}(\Gamma_V)} \sqrt{\int_\Omega \|v(\omega)\|^2_{\mathrm{H}^1(\Gamma_V)} \, \mathrm{d}\mathbb{P}(\omega)}$$

$$\le c_f \left\{ \|f\|_{\widetilde{\mathrm{H}}^{-1}_{\Gamma_D}(D)} + \|f_0\|_{\mathrm{H}^{-1}(\Gamma_V)} \right\} \sqrt{\int_\Omega \|v(\omega)\|^2 \, \mathrm{d}\mathbb{P}(\omega)}.$$

Note that we used in the last step Poincaré Friedrichs' inequality and $\varepsilon \le \varepsilon_0$. According to the Lax Milgram's theorem, we conclude thus the desired result. $\square$

This theorem implies the well-posedness of the thin layer equation (30) with random thickness. In particular, we conclude that

$$\|u_\varepsilon(\omega) - u(\omega)\|_{\mathrm{H}^1(D)} \le C\varepsilon^2 \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{33}$$

where the constant $C > 0$ is independent of ω. Hence, the random solution $u(\omega)$ of (30) is in the Bochner space $\mathrm{L}^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))$, satisfying the error estimate

$$\|u_\varepsilon - u\|_{L^2_{\mathbb{P}}(\Omega, \mathrm{H}^1(D))} \le C\varepsilon^2. \tag{34}$$

Therefore, we derive the following proposition.

Proposition 5.2. *The random solution* $u \in L^2_{\mathbb{P}}(\Omega, H_1(D))$ *satisfies the error estimates*

$$\|\mathbb{E}(u_\varepsilon) - \mathbb{E}(u)\|_{H^1(D)} \leq C\varepsilon^2, \quad \|\mathbb{V}(u_\varepsilon) - \mathbb{V}(u)\|_{W^{1,1}(D)} \leq C\varepsilon^3.$$

Proof. The first assertion follows immediately from (33). For the second assertion, one has to consider the difference of the covariances

$$\begin{aligned}
\mathrm{Cov}(u_\varepsilon)(\mathbf{x}, \mathbf{x}') &= \mathbb{E}\Big(\big[u_\varepsilon(\mathbf{x}) - \mathbb{E}(u_\varepsilon(\mathbf{x}))\big]\big[u_\varepsilon(\mathbf{x}') - \mathbb{E}(u_\varepsilon(\mathbf{x}'))\big]\Big), \\
&= \mathbb{E}\Big(\big[u_\varepsilon - \mathbb{E}(u_\varepsilon)\big] \otimes \big[u_\varepsilon - \mathbb{E}(u_\varepsilon)\big]\Big)\Big|_{(\mathbf{x}, \mathbf{x}')}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}(u)(\mathbf{x}, \mathbf{x}') &= \mathbb{E}\Big(\big[u(\mathbf{x}) - \mathbb{E}(u(\mathbf{x}))\big]\big[u(\mathbf{x}') - \mathbb{E}(u(\mathbf{x}'))\big]\Big) \\
&= \mathbb{E}\Big(\big[u - \mathbb{E}(u)\big] \otimes \big[u - \mathbb{E}(u)\big]\Big)\Big|_{(\mathbf{x}, \mathbf{x}')}
\end{aligned}$$

in the space $H^1_{\mathrm{mix}}(D \times D) := H^1(D) \otimes H^1(D)$. Due to

$$u(\omega) - \mathbb{E}(u) = \big[u(\omega) - u_\varepsilon(\omega)\big] + \big[u_\varepsilon(\omega) - \mathbb{E}(u_\varepsilon)\big] + \mathbb{E}\big(u_\varepsilon - u\big),$$

using linearity and symmetry, we find

$$\begin{aligned}
\|\mathrm{Cov}(u_\varepsilon) &- \mathrm{Cov}(u)\|_{H^1_{\mathrm{mix}}(D \times D)} \\
&\leq \Big\|\mathbb{E}\Big(\big[u - u_\varepsilon\big] \otimes \big[u - u_\varepsilon\big]\Big)\Big\|_{H^1_{\mathrm{mix}}(D \times D)} \\
&\quad + 2\Big\|\mathbb{E}\Big(\big[u - u_\varepsilon\big] \otimes \big[u_\varepsilon - \mathbb{E}(u_\varepsilon)\big]\Big)\Big\|_{H^1_{\mathrm{mix}}(D \times D)} \\
&\quad + \Big\|\mathbb{E}\big(u - u_\varepsilon\big) \otimes \mathbb{E}\big(u - u_\varepsilon\big)\Big\|_{H^1_{\mathrm{mix}}(D \times D)}.
\end{aligned}$$

Here, in view of (33), the first term and the last term on the right hand side of this estimate are of order $\mathcal{O}(\varepsilon^4)$. The second term is of order $\mathcal{O}(\varepsilon^3)$ since we only know that $\|u_\varepsilon(\omega) - \mathbb{E}(u_\varepsilon)\|_{H^1(D)} = \mathcal{O}(\varepsilon)$ which follows by a linearization in terms of the local shape derivative, see [15] for the details. Hence, we arrive at

$$\|\mathrm{Cov}(u_\varepsilon) - \mathrm{Cov}(u)\|_{H^1_{\mathrm{mix}}(D \times D)} \leq C\varepsilon^3.$$

Taking the trace $\mathbf{x} = \mathbf{x}'$ gives finally the desired result. $\qquad\square$

6. Regularity of the random solution

From now on, we shall assume that the random fluctuations $(\mathbf{x}, \omega) \mapsto \widetilde{h}(\mathbf{x}, \omega)$ are given by a possibly infinite Karhunen-Loève expansion, that is

$$\widetilde{h}(\mathbf{x}, \omega) = \sum_{k=1}^{m} h_k(\mathbf{x}) Y_k(\omega) \tag{35}$$

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where the coefficient functions $\{h_k(\mathbf{x})\}$ are pairwise orthonormal in $\mathrm{L}^2(\Gamma_V)$ and the random variables $\{Y_k(\omega)\}$ are assumed to be independently and uniformly distributed in $[-1/2, 1/2]$. Although a finite Karhunen-Loève expansion is assumed here, we shall derive estimates which are independent of the number of terms m. This means, the situation of $m \to \infty$ shall be covered by the following theory. To that end, we have to assume that

$$\gamma_k := \|h_k\|_{\mathrm{L}^\infty(\Gamma_V)} < \infty \quad \text{for all } k \in \{1, 2, \ldots, m\} \tag{36}$$

and that the sequence $\{\gamma_k\}$ is always summable as $m \to \infty$, i.e. $\sum_{k=1}^m \gamma_k \leq c_\gamma$ independently of m.

The assumption that the random variables $\{Y_k\}_k$ are stochastically independent implies that the pushforward measure $\mathbb{P}_{\mathbf{Y}} := \mathbb{P} \circ \mathbf{Y}^{-1}$ with respect to the measurable mapping

$$\mathbf{Y} : \Omega \to \square := [-1/2, 1/2]^m, \quad \omega \mapsto \mathbf{Y}(\omega) := \big(Y_1(\omega), \ldots, Y_m(\omega)\big)$$

is given by the joint density function 1. With this representation at hand, we can reformulate the stochastic problem (30) as a parametric, deterministic problem where, for ease of notation, we take the same function names as before. To that end, we replace the space $\mathrm{L}^2_\mathbb{P}(\Omega)$ by $\mathrm{L}^2(\square)$ and substitute the random variables Y_k by the coordinates $y_k \in [-1/2, 1/2]$. Then, we have to seek $u \in \mathrm{L}^2(\square, \mathcal{H})$ such that

$$\int_\square \bigg\{ \sigma_0 \int_D \nabla u(\mathbf{x}, \mathbf{y}) \nabla v(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{x}$$
$$+ \varepsilon \int_{\Gamma_V} h(\mathbf{x}, \mathbf{y}) \partial_s u(\mathbf{x}, \mathbf{y}) \partial_s v(\mathbf{x}, \mathbf{y}) \, \mathrm{d}o_\mathbf{x} \bigg\} \mathrm{d}\mathbf{y} \tag{37}$$
$$= \int_\square \bigg\{ \int_D f(\mathbf{x}) v(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\mathbf{x}, \mathbf{y}) f_0(\mathbf{x}) v(\mathbf{x}, \mathbf{y}) \, \mathrm{d}o_\mathbf{x} \bigg\} \mathrm{d}\mathbf{y}$$

holds for all $v \in \mathrm{L}^2(\square, \mathcal{H})$. Herein, the function $h(\mathbf{x}, \mathbf{y})$ is affine in the stochastic parameter $\mathbf{y}$:

$$h(\mathbf{x}, \mathbf{y}) = \overline{h}(\mathbf{x}) + \widetilde{h}(\mathbf{x}, \mathbf{y}) = \overline{h}(\mathbf{x}) + \sum_{k=1}^m h_k(\mathbf{x}) y_k. \tag{38}$$

In particular, the solvability condition (29) is equivalent to

$$0 < h_{\min} \leq \overline{h}(\mathbf{x}) \leq h_{\max} \quad \text{and} \quad |\widetilde{h}(\mathbf{x}, \mathbf{y})| \leq q\overline{h}(\mathbf{x}) \text{ for some } 0 \leq q < 1 \tag{39}$$

for all $\mathbf{x} \in \Gamma_V$ and $\mathbf{y} \in \square$.

Theorem 6.1. *The derivatives of the solution $u \in \mathrm{L}^2(\square, \mathcal{H})$ to (37) satisfy the pointwise estimate*

$$\|\partial_\mathbf{y}^{\boldsymbol{\alpha}} u(\mathbf{y})\| \leq c_f |\boldsymbol{\alpha}|! c_u^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \tag{40}$$

for all $\mathbf{y} \in \square$ and $\boldsymbol{\alpha} \in \mathbb{N}^m$. Herein, the constant c_f does only depend on $\|f\|_{\widetilde{\mathrm{H}}_{\Gamma_D}^{-1}(D)}$ and $\|f\|_{H^{-1}(\Gamma_V)}$, but not on the layer thickness ε, while the constant c_u is given by

$$c_u = \max\bigg\{ 1, \frac{1}{(1-q)h_{\min}} \bigg\} \geq 1.$$

Here, $\boldsymbol{\gamma}^{\boldsymbol{\alpha}}$ has to be understood as the product $\prod_{k=1}^m \gamma_k^{\alpha_k}$.

Proof. For $|\boldsymbol{\alpha}| = 0$, the assertion follows by straightforward modification of the proof of Theorem 5.1. For $|\boldsymbol{\alpha}| > 0$, we shall have a look at the parametrized problem (37) which, for given $\mathbf{y} \in \square$, implies the identity

$$\sigma_0 \int_D \nabla u(\mathbf{y}) \cdot \nabla v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\mathbf{y}) \partial_s u(\mathbf{y}) \partial_s v \, \mathrm{d}o = \int_D f v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\mathbf{y}) f_0 v \, \mathrm{d}o$$

for all $v \in \mathcal{H}$. Thus, in view of the Leibniz rule, differentiation with respect to $\mathbf{y}$ on both sides of this equality leads to

$$\sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) \cdot \nabla v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} \sum_{\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\alpha}'} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}'} h(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}'} \partial_s u(\mathbf{y}) \partial_s v \, \mathrm{d}o$$
$$= \varepsilon \int_{\Gamma_V} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} h(\mathbf{y}) f_0 v \, \mathrm{d}o. \tag{41}$$

Due to (38), we find

$$\partial_{y_k}^m h(\mathbf{x}, \mathbf{y}) = \begin{cases} h_k(\mathbf{x}), & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases} \tag{42}$$

Hence, in (41), the higher order derivatives of h vanish. For $|\boldsymbol{\alpha}| = 1$, we thus arrive at

$$\sigma_0 \int_D \nabla \partial_{y_k} u(\mathbf{y}) \cdot \nabla v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\mathbf{y}) \partial_s \partial_{y_k} u(\mathbf{y}) \partial_s v \, \mathrm{d}o$$
$$= -\varepsilon \int_{\Gamma_V} h_k \partial_s u(\mathbf{y}) \partial_s v \, \mathrm{d}o + \varepsilon \int_{\Gamma_V} h_k f_0 v \, \mathrm{d}o.$$

In view of (3), the special choice $v = \partial_{y_k} u(\mathbf{y})$ yields

$$\min \left\{1, (1-q) h_{\min}\right\} \|\partial_{y_k} u(\mathbf{y})\|^2$$
$$\leq -\varepsilon \int_{\Gamma_V} h_k \partial_s u(\mathbf{y}) \partial_s \partial_{y_k} u(\mathbf{y}) \, \mathrm{d}o + \varepsilon \int_{\Gamma_V} h_k f_0 \partial_{y_k} u(\mathbf{y}) \, \mathrm{d}o$$
$$\leq \|h_k\|_{\mathrm{L}^\infty(\Gamma_V)} \|u(\mathbf{y})\| \|\partial_{y_k} u(\mathbf{y})\| + \varepsilon \|h_k\|_{\mathrm{L}^\infty(\Gamma_V)} \|f_0\|_{\mathrm{H}^{-1}(\Gamma_V)} \|\partial_{y_k} u(\mathbf{y})\|_{H^1(\Gamma_V)}$$
$$\leq \|h_k\|_{\mathrm{L}^\infty(\Gamma_V)} \|\partial_{y_k} u(\mathbf{y})\| \left\{c_1 \|u(\mathbf{y})\| + c_2\right\}.$$

Note that the constant c_1 depends on the Poincaré Friedrichs inequality but is independent of ε since $\varepsilon \leq \varepsilon_0$. By possibly increasing c_f, this leads to the assertion in the case $|\boldsymbol{\alpha}| = 1$:

$$\|\partial_{y_k} u(\mathbf{y})\| \leq \gamma_k \max \left\{1, \frac{1}{(1-q) h_{\min}}\right\} \left\{c_1 \|u(\mathbf{y})\| + c_2\right\} \leq c_f c_u \gamma_k.$$

Next, we consider the case of arbitrary multiindices $|\boldsymbol{\alpha}| > 1$, where we rewrite (41) in accordance with

$$\sigma_0 \int_D \nabla \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) \cdot \nabla v \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\Gamma_V} h(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y}) v \, \mathrm{d}o$$
$$= -\varepsilon \int_{\Gamma_V} \sum_{k=1}^m \alpha_k \partial_{y_k} h(\mathbf{y}) \partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\mathbf{e}_k} \partial_s u(\mathbf{y}) \partial_s v \, \mathrm{d}o.$$

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Here, $\mathbf{e}_k$ denotes the k-th unit vector, where we use the convention that the summand disappears if $\alpha_k = 0$. We consider again (3), (42) and the special choice $v = \partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})$ and conclude

$$\min\left\{1, (1-q)h_{\min}\right\}\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})\|^2 \leq \sum_{k=1}^{m} \alpha_k \|h_k\|_{\mathrm{L}^\infty(\Gamma_V)} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\mathbf{e}_k} u(\mathbf{y})\| \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})\|.$$

By induction, we may further estimate this expression according to

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} u(\mathbf{y})\| \leq c_f \max\left\{1, \frac{1}{(1-q)h_{\min}}\right\}(|\boldsymbol{\alpha}|-1)! c_u^{|\boldsymbol{\alpha}|-1} \boldsymbol{\gamma}^{\boldsymbol{\alpha}} \sum_{k=1}^{m} \alpha_k$$
$$= c_f |\boldsymbol{\alpha}|! c_u^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}}.$$

This is the desired assertion for arbitrary $|\boldsymbol{\alpha}| \geq 1$. $\qquad\square$

The decay estimate (40) coincides with the one which is obtained in the case of a diffusion problem with uniformly elliptic random coefficient, see e.g. [2, 9]. It is sufficient to conclude that the solution u admits an analytic extension into the complex plane with respect to each particular direction y_k (see [2]). In fact, there exists even an analytic extension with respect to the variable $\mathbf{y}$ to an appropriately chosen Bernstein ellipse (see [3, 10]). As a consequence, several approaches are available to deal with the possibly very high-dimensional stochastic parameter. For example, the anisotropic stochastic collocation method [2, 22] can be applied. Moreover, besides the Monte-Carlo method, also the quasi Monte-Carlo method produces convergence rates which are essentially independent of the stochastic dimension m provided that it uniformly holds $\gamma_k \lesssim k^{-2-\delta}$, $k = 1, 2, \ldots$, for some $\delta > 0$, see [20, 25]. In our numerical examples, we will employ a quasi Monte-Carlo method since it is easy to implement.

7. Numerical experiments

For numerical illustration of our results, we consider the Poisson equation

$$\begin{cases} -\Delta u_\varepsilon(\omega) = 1 & \text{in } D_\varepsilon(\omega), \\ u_\varepsilon(\omega) = 0 & \text{on } \Gamma_D, \\ \partial_{\mathbf{n}} u_\varepsilon(\omega) = 0 & \text{on } \Gamma_N(\omega). \end{cases} \tag{43}$$

The random domain $D_\varepsilon(\omega)$ is a square with a randomly perturbed circular hole. Precisely, the deterministic exterior boundary of $D_\varepsilon(\omega)$ is the Dirichlet boundary Γ_D and given by $\partial\big((-1,1)^2\big)$. The random interior boundary of $D_\varepsilon(\omega)$ is the Neumann boundary $\Gamma_N(\omega)$ and given by

$$\Gamma_N(\omega) = \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = \frac{1}{2} - \varepsilon h(\varphi, \omega)\right\}. \tag{44}$$

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Here, $0 \le \varphi < 2\pi$ is the polar angle of a given point $\mathbf{x} \in \partial D$ and

$$h(\varphi, \omega) = 1 + \frac{1}{8} \sum_{k=0}^{5} \{a_k(\omega) \cos(k\varphi) + b_k(\omega) \sin(k\varphi)\}$$

with $a_k, b_k \in [-1/2, 1/2]$ being independent and uniformly distributed random variables. Notice that we have $\overline{h}(\varphi) = \mathbb{E}\big(h(\varphi, \omega)\big) \equiv 1$ and $0.5 \le h(\varphi, \omega) \le 1.5$ for all $0 \le \varphi < 2\pi$ and $\omega \in \Omega$. In the following, we will consider in (44) the specific choices $\varepsilon = 0.02, 0.04, \ldots, 0.2$ to examine the asymptotic estimates given in Proposition 5.2.

To treat problem (43) within our framework, we choose $\sigma_0 = 1$ and split the domain $D_\varepsilon(\omega)$ according to

$$D_\varepsilon(\omega) = D \cup L_\varepsilon(\omega) \quad \text{where} \quad L_\varepsilon(\omega) = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}) : 0 \le t < \varepsilon h(\mathbf{x}, \omega), \ \mathbf{x} \in \Gamma_V\}. \quad (45)$$

where Γ_V is the interior boundary of D given by

$$\Gamma_V = \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = \frac{1}{2}\right\}.$$

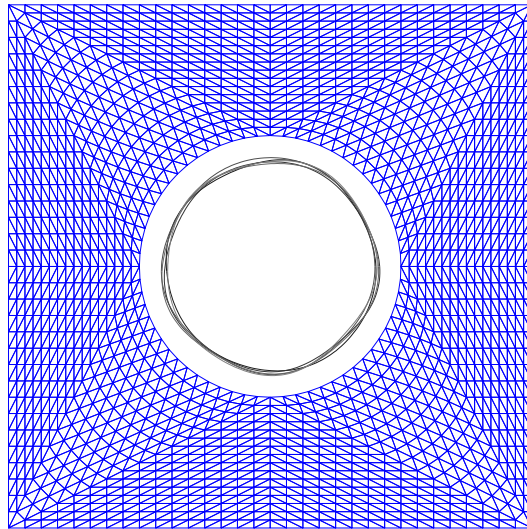We refer to Figure 2 for an illustration of the present setup.

Figure 2: The reference domain D with different realizations of the random Neumann boundary for $\varepsilon = 0.1$.

We shall approximately solve problem (43), (45) by employing the parametrized problem (37) derived in this article. To that end, we will compute the random solution's expectation

$$\mathbb{E}(u)(\mathbf{x}) = \int_\square u(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in D, \quad (46)$$

and variance

$$\mathbb{V}(u)(\mathbf{x}) = \int_{\square} \big[u(\mathbf{x}, \mathbf{y}) - \mathbb{E}(u)(\mathbf{x})\big]^2 \, \mathrm{d}\mathbf{y}, \quad \mathbf{x} \in D, \tag{47}$$

by the quasi Monte-Carlo method based on 10000 Halton points, see [13]. Note that, for $\varepsilon = 0.1$, the random solution's approximate expectation and variance are depicted in Figure 3.
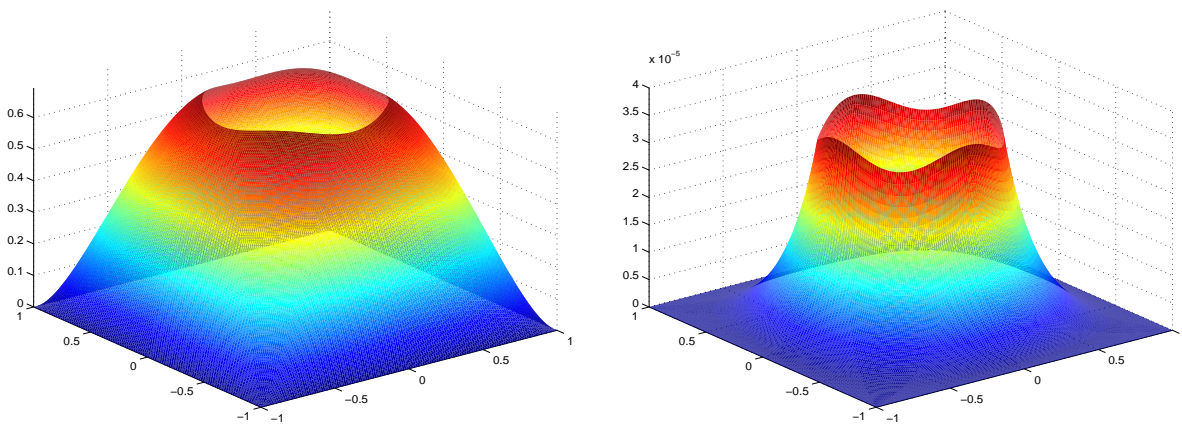

Figure 3: The approximate expectation (left) and the approximate variance (right) of the random solution for $\varepsilon = 0.1$.

In order to quantify the modelling error, we compare the mean (46) and the variance (47) with the related quantities $\mathbb{E}(u_\varepsilon)$ and $\mathbb{V}(u_\varepsilon)$ for the original problem (43), (45). This reference solution is also determined by a quasi Monte-Carlo method based on 10000 Halton points, where each sample corresponds to a new domain and thus to a new mesh, cf. Figure 5. All samples are restricted to the domain D which is done by re-interpolating the sample onto the mesh on D which consists of about 50000 triangles.

For both approaches under consideration, the spatial discretization consists of about 25000 continuous, piecewise linear, triangular finite elements. The numerical results are found in Figure 4, where we plotted the deviation (the ℓ^2-errors in the nodal values of the mesh on D) of the approximate expectation and variance from the reference solution. Proposition 5.2 predicts a modelling error of order $\mathcal{O}(\varepsilon^2)$ for the expectation (46) and a modelling error of order $\mathcal{O}(\varepsilon^3)$ for the variance (47). This asymptotic behaviour is indeed observed in Figure 4 (the respective asymptotics is indicated by the dashed lines).

8. Conclusion

In the present article, we considered a homogeneous Neumann boundary value problem on a domain with a thin layer of random thickness. This problem is approximated by
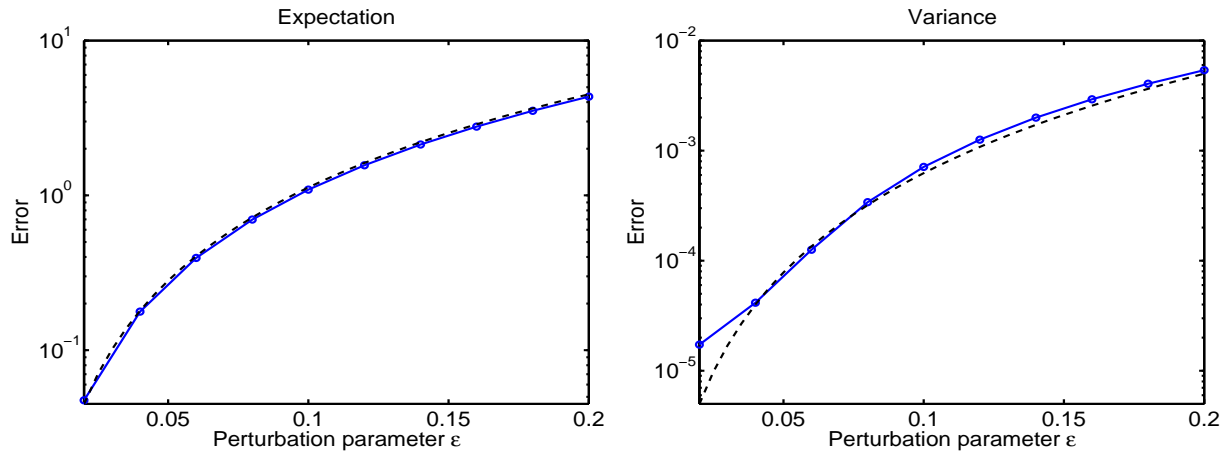
Figure 4: Error of the approximate expectation (left) and the approximate variance (right) of the random solution versus the perturbation parameter ε.

a boundary value problem on a fixed domain but with Ventcell's boundary condition. The approximation error is controlled by means of Proposition 5.2. We proved that the random solution of the approximate boundary value problem depends analytically on the layer thickness. This enables the construction of algorithms which are independent of the stochastic dimension. By numerical experiments, we validated the theoretical findings.

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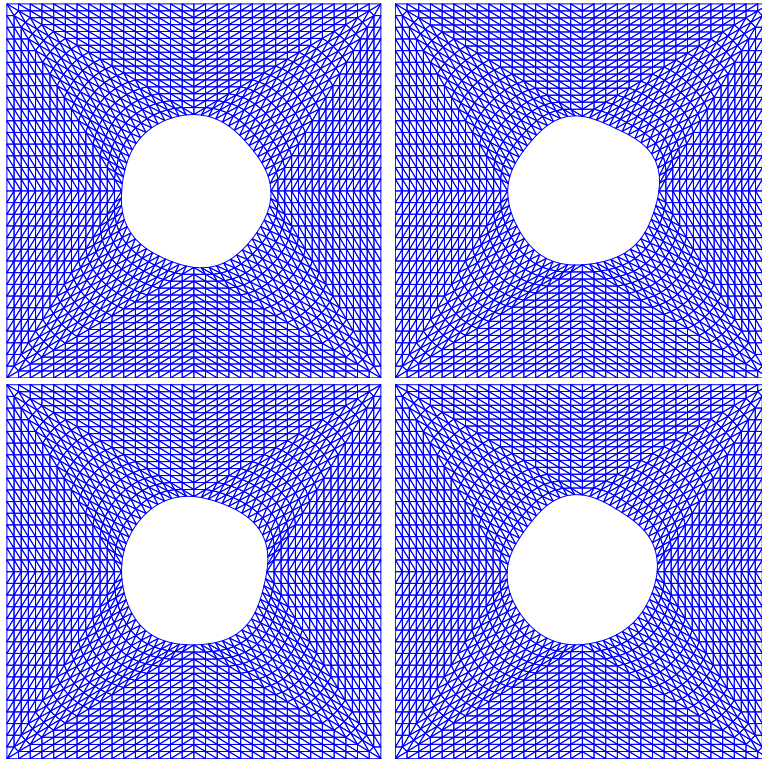[5] R.E. Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numer.*, 7:1–49, 1998.

Figure 5: Monte Carlo samples for $\varepsilon = 0.1$.

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