Conformally Euclidean metrics on $\mathbb{R}^n$ with arbitrary total Q-curvature

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Abstract

We study the existence of solution to the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu}dx < \infty,$$

where $Q \geq 0$, $\kappa \in (0, \infty)$ and $n \geq 3$. Using ODE techniques Martinazzi for $n = 6$ and Huang-Ye for $n = 4m + 2$ proved the existence of solution to the above problem with $Q \equiv const > 0$ and for every $\kappa \in (0, \infty)$. We extend these results in every dimension $n \geq 5$, thus completely answering the problem opened by Martinazzi. Our approach also extends to the case in which Q is non-constant, and under some decay assumptions on Q we can also treat the cases $n = 3$ and 4.

1 Introduction

For a function $Q \in C^0(\mathbb{R}^n)$ we consider the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \quad \kappa := \int_{\mathbb{R}^n} Qe^{nu}dx < \infty, \tag{1}$$

where for n odd the non-local operator $(-\Delta)^{\frac{n}{2}}$ is defined in Definition 2.1.

Geometrically if u is a smooth solution of (1) then the conformal metric $g_u := e^{2u}|dx|^2$ ($|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) has the Q-curvature Q. Moreover, the total Q-curvature of the metric g_u is κ.

Solutions to (1) have been classified in terms of there asymptotic behavior at infinity, more precisely we have the following:

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Theorem A ([4, 5, 14, 16, 13, 10, 22]) *Let $n \geq 1$. Let u be a solution of*

$$(-\Delta)^{\frac{n}{2}} u = (n-1)! e^{nu} \quad in\ \mathbb{R}^n, \quad \kappa := (n-1)! \int_{\mathbb{R}^n} e^{nu} dx < \infty. \tag{2}$$

Then

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + P(x) + o(\log |x|), \ \ as\ |x| \to \infty, \tag{3}$$

where $\Lambda_1 := (n-1)!|S^n|$, $o(\log|x|)/\log|x| \to 0$ as $|x| \to \infty$ and P is a polynomial of degree at most $n-1$ and P is bounded from above. If $n \in \{3, 4\}$ then $\kappa \in (0, \Lambda_1]$ and $\kappa = \Lambda_1$ if and only if u is a spherical solution, that is,

$$u(x) = u_{\lambda, x_0}(x) := \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}, \tag{4}$$

for some $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. Moreover u is spherical if and only if P is constant (which is always the case when $n \in \{1, 2\}$).

Chang-Chen [2] showed the existence of non-spherical solutions to (2) in even dimension $n \geq 4$ for every $\kappa \in (0, \Lambda_1)$.

A partial converse to Theorem A has been proven in dimension 4 by Wei-Ye [21] and extended by Hyder-Martinazzi [12] for $n \geq 4$ even and Hyder [11] for $n \geq 3$.

Theorem B ([21, 12, 11]) *Let $n \geq 3$. Then for every $\kappa \in (0, \Lambda_1)$ and for every polynomial P with*

$$deg(P) \leq n - 1, \quad and\ P(x) \xrightarrow{|x| \to \infty} -\infty,$$

there exists a solution u to (2) having the asymptotic behavior given by (3).

Although the assumption $\kappa \in (0, \Lambda_1]$ is a necessary condition for the existence of solution to (2) for $n = 3$ and 4, it is possible to have a solution for $\kappa > \Lambda_1$ arbitrarily large in higher dimension as shown by Martinazzi [18] for $n = 6$. Huang-Ye [9] extended Martinazzi's result in arbitrary even dimension n of the form $n = 4m+2$ for some $m \geq 1$, proving that for every $\kappa \in (0, \infty)$ there exists a solution to (2). The case $n = 4m$ remained open.

The ideas in [18, 9] are based upon ODE theory. One considers only radial solutions so that the equation in (2) becomes an ODE, and the result is obtained by choosing suitable initial conditions and letting one of the parameters go to $+\infty$ (or $-\infty$). However, this technique does not work if the dimension n is a multiple of 4, and things get even worse in odd dimension since $(-\Delta)^{\frac{n}{2}}$ is nonlocal and ODE techniques cannot be used.

In this paper we extend the works of [18, 9] and completely solve the cases left open, namely we prove that when $n \geq 5$ Problem (2) has a solution for every $\kappa \in (0, \infty)$. In fact we do not need to assume that Q is constant, but only that it is radially symmetric with growth at infinity suitably controlled, or not even radially symmetric. Moreover, we are able to prescribe the asymptotic behavior of the solution u (as in (3)) up to a polynomial of degree 4 which cannot be prescribed and in particular it cannot be required to vanish when $\kappa \geq \Lambda_1$. This in turn, together with Theorem A, is consistent with the requirement $n \geq 5$, because only when $n \geq 5$ the asymptotic expansion of u at infinity admits polynomials of degree 4.

We prove the following two theorems.

Theorem 1.1 *Let $n \geq 5$ be an integer. Let P be a polynomial on $\mathbb{R}^n$ with degree at most $n-1$. Let $Q \in C^0(\mathbb{R}^n)$ be such that $Q(0) > 0$, $Q \geq 0$, Qe^{nP} is radially symmetric and*

$$\sup_{x \in \mathbb{R}^n} Q(x)e^{nP(x)} < \infty.$$

Then for every $\kappa > 0$ there exists a solution u to (1) such that

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + P(x) + c_1|x|^2 - c_2|x|^4 + o(1), \quad \text{as } |x| \to \infty,$$

for some $c_1, c_2 > 0$. In fact, there exists a radially symmetric function v on $\mathbb{R}^n$ and a constant c_v such that

$$v(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) + o(1), \quad \text{as } |x| \to \infty,$$

and

$$u = P + v + c_v - |x|^4, \quad x \in \mathbb{R}^n.$$

Taking $Q = (n-1)!$ and $P = 0$ in Theorem 1.1 one has the following corollary.

Corollary 1.2 *Let $n \geq 5$. Let $\kappa \in (0, \infty)$. Then there exists a radially symmetric solution u to (2) such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log |x| + c_1|x|^2 - c_2|x|^4 + o(1), \quad \text{as } |x| \to \infty,$$

for some $c_1, c_2 > 0$.

Notice that the polynomial part of the solution u in Theorem 1.1 is not exactly the prescribed polynomial P (compare [21, 12, 11]). In general, without perturbing the polynomial part, it is not possible to find a solution for $\kappa \geq \Lambda_1$. For example, if P is non-increasing and non-constant then there is no solution u to (2) with $\kappa \geq \Lambda_1$ such that u has the asymptotic behavior (3) (see Lemma 3.6 below). This justifies the term $c_1|x|^2$ in Theorem 1.1. Then the additional term $-c_2|x|^4$ is also necessary to avoid that $u(x) \geq \frac{c_1}{2}|x|^2$ for x large, which would contrast with the condition $\kappa < \infty$, at least if Q does not decays fast enough at infinity. In the latter case, the term $-c_2|x|^4$ can be avoided, and one obtains an existence result also in dimension 3 and 4.

Theorem 1.3 *Let $n \geq 3$. Let $Q \in C^0_{rad}(\mathbb{R}^n)$ be such that $Q \geq 0$, $Q(0) > 0$ and*

$$\int_{\mathbb{R}^n} Q(x)e^{\lambda|x|^2}dx < \infty, \quad \text{for every } \lambda > 0, \quad \int_{B_1(x)} \frac{Q(y)}{|x-y|^{n-1}}dy \xrightarrow{|x|\to\infty} 0.$$

Then for every $\kappa > 0$ there exists a radially symmetric solution u to (1).

The decay assumption on Q in Theorem 1.3 is sharp in the sense that if $Qe^{\lambda|x|^2} \notin L^1(\mathbb{R}^n)$ for some $\lambda > 0$, then Problem (1) might not have a solution for every $\kappa > 0$. For instance, if $Q = e^{-\lambda|x|^2}$ for some $\lambda > 0$, then (1) with $n = 3, 4$ and $\kappa > \Lambda_1$ has no radially symmetric solution (see Lemma 3.5 below).

The proof of Theorem 1.1 is based on the Schauder fixed point theorem, and the main difficulty is to show that the "approximate solutions" are pre-compact (see in particular Lemma 2.2). We will do that using blow up analysis (see for instance [1, 7, 17, 19]). In general, if $\kappa \geq \Lambda_1$ one can expect blow up, but we will construct our approximate solutions carefully in a way that this does not happen. For instance in [21, 12] one looks for solutions of the form $u = P + v + c_v$ where v satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)}e^{n(v(y+c_v)}dy,$$

and c_v is a constant such that

$$\int_{\mathbb{R}^n} Qe^{n(P+v+c_v)}dx = \kappa.$$

With such a choice we would not be able to rule out blow-up. Instead, by looking for solutions of the form

$$u = P + v + P_v + c_v$$

where a posteriori $P_v = -|x|^4$, v satisfies

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{n(P(y)+P_v(y)+v(y)+c_v)}dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \quad (5)$$

and c_v is again a normalization constant, one can prove that the integral equation (5) enjoys sufficient compactness, essentially due to the term $\frac{1}{2n}|x|^2|\Delta v(0)|$ on the right-hand side. Indeed a sequence of (approximate) solutions v_k blowing up (for simplicity) at the origin, up to rescaling, leads to a sequence (η_k) of functions satisfying for every $R > 0$

$$\int_{B_R} |\Delta\eta_k - c_k|dx \leq CR^{n-2} + o(1)R^{n+2}, \quad o(1) \xrightarrow{k\to\infty} 0, \quad c_k > 0,$$

and converging to η_∞ solving (for simplicity here we ignore some cases)

$$(-\Delta)^{\frac{n}{2}}\eta_\infty = e^{n\eta_\infty} \quad \text{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^{n\eta_\infty}dx < \infty,$$

and

$$\int_{B_R} |\Delta\eta_\infty - c_\infty| dx \le C R^{n-2}, \quad c_\infty \ge 0, \tag{6}$$

where $c_\infty = 0$ corresponds to $\Delta\eta_\infty(0) = 0$ (see Sub-case 1.1 in Lemma 2.2 with $x_k = 0$). The estimate on $\|\Delta\eta_\infty\|_{L^1(B_R)}$ in (6) shows that the polynomial part P_∞ of η_∞ (as in (3)) has degree at most 2, and hence $\Delta P_\infty \le 0$ as P_∞ is bounded from above. Therefore, $c_\infty = 0 = \Delta P_\infty$, and in particular η_∞ is a spherical solution, that is, $\eta_\infty = u_{\lambda,x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where u_{λ,x_0} is given by (4). This leads to a contradiction as $\Delta\eta_\infty(0) = 0$ and $\Delta u_{\lambda,x_0} < 0$ in $\mathbb{R}^n$.

In this work we focus only on the case $Q \ge 0$ because the negative case has been relatively well understood. For instance by a simple application of maximum principle one can show that Problem (1) has no solution with $Q \equiv \mathit{const} < 0$, $n = 2$ and $\kappa > -\infty$, but when Q is non-constant, solutions do exist, as shown by Chanillo-Kiessling in [3] under suitable assumptions. Martinazzi [15] proved that in higher even dimension $n = 2m \ge 4$ Problem (1) with $Q \equiv \mathit{const} < 0$ has solutions for some κ, and it has been shown in [12] that actually for every $\kappa \in (-\infty, 0)$ and Q negative constant (1) has a solution. The same result has been recently extended to odd dimension $n \ge 3$ in [11].

2 Proof of Theorem 1.1

We consider the space

$$X := \left\{ v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty \right\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n} \left(\sum_{|\alpha|\le 3} (1+|x|)^{|\alpha|-4} |D^\alpha v(x)| + \sum_{3<|\alpha|\le n-1} |D^\alpha v(x)| \right).$$

For $v \in X$ we set

$$A_v := \max\left\{ 0, \sup_{|x|\ge 10} \frac{v(x) - v(0)}{|x|^4} \right\}, \quad P_v(x) := -|x|^4 - A_v|x|^4.$$

Then

$$v(x) + P_v(x) \le v(0) - |x|^4, \quad \text{for } |x| \ge 10.$$

Let c_v be the constant determined by

$$\int_{\mathbb{R}^n} K e^{n(v+c_v)} dx = \kappa, \quad K := Q e^{nP} e^{nP_v},$$

where the functions Q and P satisfy the hypothesis in Theorem 1.1. Since $Q(0) > 0$, without loss of generality we can also assume that $Q > 0$ in B_3. Then $u = P + P_v + v + c_v$ satisfies

$$(-\Delta)^{\frac{n}{2}} u = Q e^{nu}, \quad \kappa = \int_{\mathbb{R}^n} Q e^{nu} dx,$$

if and only if v satisfies

$$(-\Delta)^{\frac{n}{2}} v = K e^{n(v+c_v)}.$$

For odd integer n, the operator $(-\Delta)^{\frac{n}{2}}$ is defined as follows:

Definition 2.1 *Let n be an odd integer. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We say that u is a solution of*

$$(-\Delta)^{\frac{n}{2}} u = f \quad \text{in } R^n,$$

if $u \in W^{n-1,1}_{loc}(\mathbb{R}^n)$ and $\Delta^{\frac{n-1}{2}} u \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and for every test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} u (-\Delta)^{\frac{1}{2}} \varphi dx = \langle f, \varphi \rangle.$$

Here, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and the space $L_s(\mathbb{R}^n)$ is defined by

$$L_s(\mathbb{R}^n) := \left\{ u \in L^1_{loc}(\mathbb{R}^n) : \|u\|_{L_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\}, \quad s > 0.$$

For more details on fractional Laplacian we refer the reader to [6].

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) K(y) e^{n(v(y)+c_v)} dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|,$$

where $\gamma_n := \frac{(n-1)!}{2}|S^n|$.

Lemma 2.1 *Let v solve $tT(v) = v$ for some $0 < t \le 1$. Then*

$$v(x) = \frac{t}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) K(y) e^{n(v(y)+c_v)} dy + \frac{t}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \tag{7}$$

$\Delta v(0) < 0$, and $v(x) \to -\infty$ as $|x| \to \infty$. Moreover,

$$\sup_{x \in B_1^c} v(x) \le \inf_{x \in B_1} v(x),$$

and in particular $A_v = 0$.

Proof. Since v satisfies $tT(v) = v$, (7) follows from the definition of T. Differentiating under integral sign, from (7) one can get $\Delta v(0) < t|\Delta v(0)|$, which implies that $\Delta v(0) < 0$. The remaining part of the lemma follows from the fact that

$$\Delta v(x) < \frac{t}{2n}|\Delta v(0)|\Delta(|x|^2 - |x|^4), \quad x \in \mathbb{R}^n, \tag{8}$$

and the integral representation of radially symmetric functions given by

$$v(\xi) - v(\bar{\xi}) = \int_{\bar{\xi}}^{\xi} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta v(x) dx dr, \quad 0 \le \bar{\xi} < \xi, \quad \omega_{n-1} := |S^{n-1}|. \tag{9}$$

□

Lemma 2.2 *Let $(v,t) \in X \times (0,1]$ satisfy $v = tT(v)$. Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{\frac{1}{8}}} w \le C, \quad w := v + c_v + \frac{1}{n}\log t.$$

Proof. Let us assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ such that $\max_{\bar{B}_{\frac{1}{8}}} w_k =: w_k(\theta_k) \to \infty$.

If θ_k is a point of local maxima of w_k then we set $x_k = \theta_k$. Otherwise, we can choose $x_k \in B_{\frac{1}{4}}$ such that x_k is a point of local maxima of w_k and $w_k(x_k) \ge w_k(x)$ for every $x \in B_{|x_k|}$. This follows from the fact that

$$\inf_{B_{\frac{1}{4}} \setminus B_{\frac{1}{8}}} w_k \not\to \infty,$$

which is a consequence of

$$\int_{\mathbb{R}^n} K e^{nw_k} dx = t_k \kappa \le \kappa, \quad K > 0 \text{ on } B_3.$$

We set $\mu_k := e^{-w_k(x_k)}$. We distinguish the following cases.

Case 1 Up to a subsequence $t_k \mu_k^2 |\Delta v_k(0)| \to c_0 \in [0,\infty)$.

We set

$$\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k) = w_k(x_k + \mu_k x) - w_k(x_k).$$

Notice that by (7) we have for some dimensional constant C_1

$$\begin{aligned}\Delta \eta_k(x) &= \mu_k^2 \Delta v_k(x_k + \mu_k x) \\ &= C_1 \frac{\mu_k^2}{\gamma_n} \int_{\mathbb{R}^n} \frac{K(y) e^{nw_k(y)}}{|x_k + \mu_k x - y|^2} dy + t_k \mu_k^2 \left(1 - \frac{4(n+2)}{2n} |x_k + \mu_k x|^2\right) |\Delta v_k(0)|,\end{aligned}$$

so that

$$\begin{aligned}&\int_{B_R} \left| \Delta \eta_k(x) - t_k \mu_k^2 |\Delta v_k(0)| \left(1 - \frac{2(n+2)}{n}|x_k|^2\right)\right| dx \\ &\le \frac{C_1}{\gamma_n} \int_{\mathbb{R}^n} K(y) e^{nw_k(y)} \int_{B_R} \frac{\mu_k^2 dx}{|x_k + \mu_k x - y|^2} dy + C t_k \mu_k^2 |\Delta v_k(0)| \int_{B_R} (\mu_k |x_k \cdot x| + \mu_k^2 |x|^2) dx \\ &\le \frac{C_1}{\gamma_n} t_k \kappa \int_{B_R} \frac{1}{|x|^2} dx + C t_k \mu_k^2 |\Delta v_k(0)| \int_{B_R} (\mu_k |x| + \mu_k^2 |x|^2) dx \\ &\le C \kappa t_k R^{n-2} + C t_k \mu_k^2 |\Delta v_k(0)| \left(\mu_k R^{n+1} + \mu_k^2 R^{n+2}\right). \end{aligned} \tag{10}$$

The function η_k satisfies

$$(-\Delta)^{\frac{n}{2}} \eta_k(x) = K(x_k + \mu_k x) e^{n\eta_k(x)} \quad \text{in } \mathbb{R}^n, \quad \eta_k(0) = 0.$$

Moreover, $\eta_k \le C(R)$ on B_R. This follows easily if $|x_k| \le \frac{1}{9}$ as in that case $\eta_k \le 0$ on B_R for $k \ge k_0(R)$. On the other hand, for $\frac{1}{9} < |x_k| \le \frac{1}{4}$ one can use Lemma 2.4 (below). Therefore, by Lemma A.3 (and Lemmas 2.6, 2.7 if n is odd), up to a subsequence, $\eta_k \to \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$ where η satisfies

$$(-\Delta)^{\frac{n}{2}}\eta = K(x_\infty)e^{n\eta} \quad \text{in } \mathbb{R}^n, \quad K(x_\infty)\int_{\mathbb{R}^n} e^{n\eta}dx \le t_\infty\kappa < \infty,\, K(x_\infty) > 0,$$

where (up to a subsequence) $t_k \to t_\infty$ and $x_k \to x_\infty$. Notice that $t_\infty \in (0,1]$, $x_\infty \in \bar{B}_{\frac{1}{4}}$ and for every $R > 0$, by (10)

$$\int_{B_R} |\Delta\eta - c_0c_1|\, dx \le CR^{n-2}, \quad c_1 =: 1 - \frac{2(n+2)}{n}|x_\infty|^2 > 0. \tag{11}$$

Hence by Theorem A we have

$$\eta(x) = P_0(x) - \alpha \log|x| + o(\log|x|), \quad \text{as } |x| \to \infty,$$

where P_0 is a polynomial of degree at most $n-1$, P_0 is bounded from above and α is a positive constant. In fact, by (11)

$$\int_{B_R} |\Delta P_0(x) - c_0c_1|dx \le CR^{n-2}, \quad \text{for every } R > 0.$$

Hence P_0 is a constant. This implies that η is a spherical solution and in particular $\Delta\eta < 0$ on $\mathbb{R}^n$, and therefore, again by (11), we have $c_0 = 0$.

We consider the following sub-cases.

Sub-case 1.1 There exists $M > 0$ such that $\frac{|x_k|}{\mu_k} \le M$.

We set $y_k := -\frac{x_k}{\mu_k}$. Then (up to a subsequence) $y_k \to y_\infty \in B_{M+1}$. Therefore,

$$\Delta\eta(y_\infty) = \lim_{k\to\infty} \Delta\eta_k(y_k) = \lim_{k\to\infty} \mu_k^2\Delta v_k(0) = \frac{c_0}{t_\infty} = 0,$$

a contradiction as $\Delta\eta < 0$ on $\mathbb{R}^n$.

Sub-case 1.2 Up to a subsequence $\frac{|x_k|}{\mu_k} \to \infty$.

For any $N \in \mathbb{N}$ we can choose $\xi_{1,k}, \ldots, \xi_{N,k} \in \mathbb{R}^n$ such that $|\xi_{i,k}| = |x_k|$ for all $i = 1, \ldots, N$ and the balls $B_{2\mu_k}(\xi_{i,k})$'s are disjoint for k large enough. Since v_k's are radially symmetric, the functions $\eta_{i,k} := v_k(\xi_{i,k} + \mu_k x) - v_k(\xi_{i,k}) \to \eta_i = \eta$ in $C^{m-1}_{loc}(\mathbb{R}^n)$. Therefore,

$$\lim_{k\to\infty}\int_{B_1} e^{n(v_k+c_{v_k})}dx \ge N \lim_{k\to\infty}\int_{B_{\mu_k}(\xi_{1,k})} e^{n(v_k+c_{v_k})}dx = N\frac{1}{t_\infty}\int_{B_1} e^{n\eta}dx.$$

This contradicts to the fact that

$$\int_{B_1} Ke^{n(v_k+c_{v_k})}dx \le \kappa, \quad K > 0 \text{ on } B_3.$$

Case 2 Up to a subsequence $t_k\mu_k^2|\Delta v_k(0)| \to \infty$.

We choose $\rho_k > 0$ such that $t_k\rho_k^2\mu_k^2|\Delta v_k(0)| = 1$. We set

$$\psi_k(x) = v_k(x_k + \rho_k\mu_k x) - v_k(x_k).$$

Then one can get (similar to (10))

$$\begin{aligned}&\int_{B_R}\left|\Delta\psi_k(x) - \left(1 - \frac{2(n+2)}{n}|x_k|^2\right)\right| dx \\ &\le C_1\int_{\mathbb{R}^n} K(y)e^{nw_k(y)}\int_{B_R}\frac{\rho_k^2\mu_k^2}{|x_k + \mu_k\rho_k x - y|^2}dxdy + C_2\mu_k\rho_k\int_{B_R}(|x| + \mu_k\rho_k|x|^2)dx \\ &\xrightarrow{k\to\infty} 0,\end{aligned}$$

thanks to Lemma 2.5 (below). Moreover, together with Lemma 2.4, ψ_k satisfies

$$(-\Delta)^{\frac{n}{2}}\psi_k = o(1) \quad \text{in } B_R,\ \psi_k(0) = 0,\ \psi_k \le C(R) \text{ on } B_R.$$

Hence, by Lemma A.3 (and Lemma 2.6 if n is odd), up to a subsequence $\psi_k \to \psi$ in $C^{n-1}_{loc}(\mathbb{R}^n)$. Then ψ must satisfy

$$\int_{B_1}|\Delta\psi - c_0|dx = 0, \quad c_0 := 1 - \frac{2(n+2)}{n}|x_\infty|^2 > 0,$$

where (up to a subsequence) $x_k \to x_\infty$. This shows that $\Delta\psi(0) = c_0 > 0$, which is a contradiction as

$$\Delta\psi(0) = \lim_{k\to\infty}\Delta\psi_k(0) = \lim_{k\to\infty}\rho_k^2\mu_k^2\Delta v_k(x_k) \le 0.$$

Here, $\Delta v_k(x_k) \le 0$ follows from the fact that x_k is a point of local maxima of v_k. □

A consequence of the local uniform upper bounds of w is the following global uniform upper bounds:

Lemma 2.3 *There exists a constant $C > 0$ such that for all $(v,t) \in X \times (0,1]$ with $v = tT(v)$ we have $|\Delta v(0)| \le C$ and*

$$v(x) + c_v + \frac{1}{n}\log t \le C, \quad \text{on } \mathbb{R}^n.$$

Proof. By Lemma 2.2 we have

$$\sup_{B_{\frac{1}{8}}} w := \sup_{B_{\frac{1}{8}}}\left(v + c_v + \frac{1}{n}\log t\right) \le C.$$

Differentiating under integral sign from (7) we obtain

$$\begin{aligned}|\Delta v(0)| &\le C\int_{B_{\frac{1}{8}}}\frac{1}{|y|^2}K(y)e^{nw(y)}dy + C\int_{B^c_{\frac{1}{8}}}\frac{1}{|y|^2}K(y)e^{nw(y)}dy\\ &\le C\sup_{B_{\frac{1}{8}}}K\int_{B_\varepsilon}\frac{1}{|y|^2}dy + C\int_{B^c_{\frac{1}{8}}}Ke^{nw}dy\\ &\le C(\varepsilon,\kappa,K).\end{aligned}$$

By (8) we get

$$\Delta v(x) \le t|\Delta v(0)| \le C, \quad x\in\mathbb{R}^n,$$

and hence, together with (9)

$$w(x) = w(0) + \int_0^{|x|}\frac{1}{\omega_{n-1}r^{n-1}}\int_{B_r}\Delta v(y)dydr \le w(0) + C|x|^2 \le C, \quad x\in B_2.$$

The lemma follows from Lemma 2.1. □

Proof of Theorem 1.1 Let $v\in X$ be a solution of $v = tT(v)$ for some $0 < t \le 1$. Then $A_v = 0$ and $|\Delta v(0)| \le C$, thanks to Lemmas 2.1 and 2.3. Hence, for $0 \le |\beta| \le n-1$

$$\begin{aligned}|D^\beta v(x)| &\le C\int_{\mathbb{R}^n}\left|D^\beta\log\left(\frac{1}{|x-y|}\right)\right|K(y)e^{n(v(y)+c_v+\frac{1}{n}\log t)}dy + C|D^\beta(|x|^2-|x|^4)|\\ &\le C\int_{\mathbb{R}^n}\left|D^\beta\log\left(\frac{1}{|x-y|}\right)\right|e^{-|y|^4}dy + C|D^\beta(|x|^2-|x|^4)|,\end{aligned}$$

where in the second inequality we have used that

$$v(x) + c_v + \frac{1}{n}\log t \le C, \qquad C \text{ is independent of } v \text{ and } t,$$

which follows from Lemma 2.3. Now as in Lemma 2.8 one can show that

$$\|v\|_X \le M,$$

and therefore, by Lemma A.1, the operator T has a fixed point (say) v. Then

$$u = P + v + c_v - |x|^4,$$

is a solution to the Problem (1) and u has the asymptotic behavior given by

$$u(x) = P(x) - \frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) - |x|^4 + c_v + o(1), \quad \text{as } |x|\to\infty.$$

This completes the proof of Theorem 1.1. □

Now we give a proof of the technical lemmas used in the proof of Lemma 2.2.

Lemma 2.4 *Let $\varepsilon > 0$. Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) or (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1 \setminus B_\varepsilon$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. Then*

$$v_k(x_k + x) - v_k(x_k) \leq C(n, \varepsilon)|x|^2 t_k |\Delta v_k(0)|, \quad x \in B_1.$$

Proof. If $|x_k + x| \leq |x_k|$ then $v_k(x_k + x) - v_k(x_k) \leq 0$ as $v_k(x_k) \geq v_k(y)$ for every $y \in B_{|x_k|}$. For $|x_k| < |x_k + x|$, setting $a = a(k, x) := x_k + x$, and together with (9) we obtain

$$\begin{aligned} v_k(x_k + x) - v_k(x_k) &= \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r \setminus B_{|x_k|}} \Delta v_k(x) dx dr \\ &\leq \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_{|a|} \setminus B_{|x_k|}} t_k |\Delta v_k(0)| dx d\rho \\ &\leq C(n) t_k |\Delta v_k(0)| (|B_{|a|}| - |B_{|x_k|}|) \left(\frac{1}{|x_k|^{n-2}} - \frac{1}{|a|^{n-2}} \right) \\ &\leq C(n, \varepsilon) t_k |x|^2 |\Delta v_k(0)|, \end{aligned}$$

where in the first equality we have used that

$$0 = v_k'(x_k) = \frac{1}{\omega_{n-1}|x_k|^{n-1}} \int_{B_{|x_k|}} \Delta v_k dx.$$

Hence we have the lemma. □

Lemma 2.5 *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| \leq C$ and $\rho_k \mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} dx dy =: \lim_{k \to \infty} I_k = 0.$$

Proof. In order to prove the lemma we fix $R > 0$ (large). We split B_{R_0} into

$$A_1(R, y) := \{x \in B_{R_0} : |x_k + \rho_k \mu_k x - y| > R \rho_k \mu_k\}, \quad A_2(R, y) := B_{R_0} \setminus A_1(R, y).$$

Then we can write $I_k = I_{1,k} + I_{2,k}$, where

$$I_{i,k} := \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{A_i(R,y)} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2} dx dy, \quad i = 1, 2.$$

Changing the variable $y \mapsto x_k + \rho_k \mu_k y$ and by Fubini's theorem one gets

$$\begin{aligned} I_{2,k} &= \rho_k^n \int_{B_{R_0}} \int_{\mathbb{R}^n} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2} \chi_{|x-y| \leq R} dy dx \\ &\leq \rho_k^n \int_{B_{R_0}} \int_{B_{R+R_0}} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2} dy dx \\ &\leq C(n, \varepsilon) \left(\sup_{B_{R+R_0+1}} K e^{n \eta_k} \right) (R + R_0)^n R_0^{n-2} \rho_k^n, \end{aligned}$$

where $\eta_k(y) := w_k(x_k + \rho_k\mu_k y) - w_k(x_k)$. If $x_k \to 0$ then $\eta_k \le 0$ on B_{R+R_0+1} for k large. Otherwise, for k large $\rho_k\mu_k y \in B_1$ for every $y \in B_{R+R_0+1}$ and hence, by Lemma 2.4

$$\eta_k(y) = v_k(x_k + \rho_k\mu_k y) - v_k(x_k) \le C|\rho_k\mu_k y|^2 t_k|\Delta v_k(0)| \le C(R, R_0).$$

Therefore,

$$\lim_{k\to\infty} I_{2,k} = 0.$$

Using the definition of c_v we bound

$$I_{1,k} \le \frac{|B_{R_0}|}{R^2}\int_{\mathbb{R}^n} K(y)e^{nw_k(y)}dy \le C(n,\kappa,R_0)\frac{1}{R^2}.$$

Since $R > 0$ is arbitrary, we conclude the lemma. □

We need the following two lemmas only for n odd.

Lemma 2.6 *Let $n \ge 5$. Let v be given by (7). For any $r > 0$ and $\xi \in \mathbb{R}^n$ we set*

$$w(x) = v(rx + \xi), \quad x \in \mathbb{R}^n.$$

Then there exists $C > 0$ (independent of v, t, r, ξ) such that for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = n-1$ we have $\|D^\alpha w\|_{L_{\frac{1}{2}}(\mathbb{R}^n)} \le Ct(1 + r^4|\Delta v(0)|)$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ (independent of r, ξ and t) such that

$$\int_{B_R^c} \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}dx < \varepsilon t(1 + r^4|\Delta v(0)|), \quad |\alpha| = n-1.$$

Proof. Differentiating under integral sign we obtain

$$|D^\alpha w(x)| \le Ct\int_{\mathbb{R}^n} \frac{r^{n-1}}{|rx+\xi-y|^{n-1}}f(y)dy + Ctr^4|\Delta v(0)|, \quad f(y) := K(y)e^{n(v(y)+c_v)}.$$

If $n > 5$ then the above inequality is true without the term $Ctr^4|\Delta v(0)|$. Using a change of variable $y \mapsto \xi + ry$, we get

$$\begin{aligned}&\int_\Omega \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}dx \\ &\le Ctr^n\int_{\mathbb{R}^n} f(\xi+ry)\int_\Omega \frac{1}{|x-y|^{n-1}}\frac{1}{1+|x|^{n+1}}dxdy + Ctr^4|\Delta v(0)|\int_\Omega \frac{dx}{1+|x|^{n+1}}.\end{aligned}$$

The lemma follows by taking $\Omega = \mathbb{R}^n$ or B_R^c. □

Lemma 2.7 *Let $\eta_k \to \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$. We assume that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R^c} \frac{|\Delta^{\frac{n-1}{2}} \eta_k(x)|}{1+|x|^{n+1}} dx < \varepsilon, \quad \textit{for } k = 1, 2, \ldots . \tag{12}$$

We further assume that

$$(-\Delta)^{\frac{n}{2}} \eta_k = K(x_k + \mu_k x) e^{n\eta_k} \quad \textit{in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |K(x_k + \mu_k x)| e^{n\eta_k(x)} dx \leq C,$$

where $x_k \to x_\infty$, $\mu_k \to 0$, K is a continuous function and $K(x_\infty) > 0$. Then $e^{n\eta} \in L^1(\mathbb{R}^n)$ and η satisfies

$$(-\Delta)^{\frac{n}{2}} \eta = K(x_\infty) e^{n\eta} \quad \textit{in } \mathbb{R}^n.$$

Proof. First notice that $\Delta^{\frac{n-1}{2}} \eta_k \to \Delta^{\frac{n-1}{2}} \eta$ in $L_{\frac{1}{2}}(\mathbb{R}^n)$, thanks to (12) and the convergence $\eta_k \to \eta$ in $C^{n-1}_{loc}(\mathbb{R}^n)$.

We claim that η satisfies $(-\Delta)^{\frac{n}{2}} \eta = K(x_\infty) e^{n\eta}$ in $\mathbb{R}^n$ in the sense of distribution.

In order to prove the claim we let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} K(x_k + \mu_k x) e^{n\eta_k(x)} \varphi(x) dx = \int_{\mathbb{R}^n} K(x_\infty) e^{n\eta(x)} \varphi(x) dx,$$

and

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} \eta_k (-\Delta)^{\frac{1}{2}} \varphi dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{n-1}{2}} \eta (-\Delta)^{\frac{1}{2}} \varphi dx.$$

We conclude the claim.

To complete the lemma first notice that $e^{n\eta} \in L^1(\mathbb{R}^n)$, which follows from the fact that for any $R > 0$

$$\int_{B_R} e^{n\eta} dx = \lim_{k \to \infty} \int_{B_R} e^{n\eta_k} dx = \lim_{k \to \infty} \int_{B_R} \frac{K(x_k + \mu_k x)}{K(x_\infty)} e^{n\eta_k(x)} dx \leq \frac{C}{K(x_\infty)}.$$

We fix a function $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on B_1. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we set $\varphi_k(x) = \varphi(x)\psi(\frac{x}{k})$. The lemma follows by taking $k \to \infty$, thanks to the previous claim. $\square$

Lemma 2.8 *The operator $T : X \to X$ is compact.*

Proof. Let v_k be a bounded sequence in X. Then (up to a subsequence) $\{v_k(0)\}$, $\{\Delta v_k(0)\}$, $\{A_{v_k}\}$ and $\{c_{v_k}\}$ are convergent sequences. Therefore, $|\Delta v_k(0)|(|x|^2 - |x|^4)$ converges to some function in X. To conclude the lemma, it is sufficient to show that up to a subsequence $\{f_k\}$ converges in X, where f_k is defined by

$$f_k(x) = \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{nP(y)} e^{nP_{v_k}(y)} e^{n(v_k(y) + c_{v_k})} dy.$$

Differentiating under integral sign one gets

$$\begin{aligned}|D^\beta f_k(x)| &\le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}dy, \quad 0<|\beta|\le n-1\\ &\le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}e^{-|y|^4}dy\\ &\le C,\end{aligned}$$

where the second inequality follows from the uniform bounds

$$|v_k(0)|\le C,\ |c_{v_k}|\le C,\ Qe^{nP}\le C, \text{ and } v_k(x)+P_{v_k}(x)\le v_k(0)-|x|^4. \tag{13}$$

Indeed, for $0<|\beta|\le n-1$

$$\lim_{R\to\infty}\sup_k\sup_{x\in B_R^c}|D^\beta f_k(x)|=0,$$

and for every $0<s<1$ we have $\|D^{n-1}f_k\|_{C^{0,s}(B_R)}\le C(R,s)$. Finally, using (13) we bound

$$|f_k(x)|\le C\int_{\mathbb{R}^n}|\log|x-y||e^{-|y|^4}dy\le C\log(2+|x|).$$

Thus, up to a subsequence, $f_k\to f$ in $C^{n-1}_{loc}(\mathbb{R}^n)$ for some $f\in C^{n-1}(R^n)$, and the global uniform estimates of f_k and $D^\beta f_k$ would imply that $f_k\to f$ in X. □

3 Proof of Theorem 1.3

We consider the space

$$X:=\left\{v\in C^{n-1}(\mathbb{R}^n): v \text{ is radially symmetric}, \|v\|_X<\infty\right\},$$

where

$$\|v\|_X:=\sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|\le 1}(1+|x|)^{|\alpha|-2}|D^\alpha v(x)|+\sum_{1<|\alpha|\le n-1}|D^\alpha v(x)|\right).$$

For $v\in X$, let c_v be the constant determined by

$$\int_{\mathbb{R}^n}Qe^{n(v+c_v)}dy=\kappa,$$

where Q satisfies the hypothesis in Theorem 1.3. Without loss of generality we can assume that $Q>0$ on B_3.

We define an operator $T:X\to X$ given by $T(v)=\bar v$, where

$$\bar v(x)=\frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)Q(y)e^{n(v(y)+c_v)}dy+\frac{1}{2n}|\Delta v(0)||x|^2.$$

As in Lemma 2.8 one can show that the operator T is compact.

Proof of the following two lemmas is similar to Lemmas 2.1 and 2.5 respectively.

Lemma 3.1 *Let v solve $tT(v) = v$ for some $0 < t \le 1$. Then $\Delta v(0) < 0$, and*

$$v(x) = \frac{t}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{n(v(y)+c_v)} dy + \frac{t}{2n}|\Delta v(0)||x|^2. \tag{14}$$

Lemma 3.2 *Let $(v_k, t_k) \in X \times (0,1]$ satisfy (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $\rho_k^2 t_k \mu_k^2 |\Delta v_k(0)| \le C$ and $\rho_k \mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} Q(y) e^{nw_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k\mu_k x - y|^2} dx dy = 0.$$

Now we prove a similar local uniform upper bounds as in Lemma 2.2.

Lemma 3.3 *Let $(v,t) \in X \times (0,1]$ satisfy (14). Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{\frac{1}{8}}} w \le C, \quad w := v + c_v + \frac{1}{n}\log t.$$

Proof. The proof is very similar to Lemma 2.2. Here we briefly sketch the proof.

We assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence of (v_k, t_k) and a sequence of points x_k in $B_{\frac{1}{4}}$ such that

$$w_k(x_k) \to \infty, \quad w_k \le w_k(x_k) \text{ on } B_{|x_k|}, \quad x_k \text{ is a point of local maxima of } v_k.$$

We set $\mu_k := e^{-w_k(x_k)}$ and we distinguish following cases.

Case 1 Up to a subsequence $t_k\mu_k^2|\Delta v_k(0)| \to c_0 \in [0,\infty)$.

We set $\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k)$. Then we have

$$\int_{B_R} |\Delta \eta_k - t_k\mu_k^2|\Delta v_k(0)||dx \le C t_k R^{n-2}.$$

Now one can proceed exactly as in Case 1 in Lemma 2.2.

Case 2 Up to a subsequence $t_k\mu_k^2|\Delta v_k(0)| \to \infty$.

We set $\psi_k(x) = v_k(x_k + \rho_k\mu_k x) - v_k(x_k)$ where ρ_k is determined by $t_k\rho_k^2\mu_k^2|\Delta v_k(0)| = 1$. Then by Lemma 3.2

$$\int_{B_R} |\Delta\psi_k - 1| dx = o(1), \quad \text{as } k \to \infty.$$

Similar to Case 2 in Lemma 2.2 one can get a contradiction. ☐

With the help of Lemma 3.3 we prove

Lemma 3.4 *There exists a constant $M > 0$ such that for all $(v,t) \in X \times (0,1]$ satisfying (14) we have $\|v\| \le M$.*

Proof. Let $(v,t) \in X \times (0,1]$ satisfies (14). We set $w := v + c_v + \frac{1}{n}\log t$.

First we show that $|\Delta v(0)| \leq C$ for some $C > 0$ independent of v and t. Indeed, differentiating under integral sign, from (14), and together with Lemma 3.3, we get

$$\begin{aligned}|\Delta v(0)|(1+t) &\leq C\int_{\mathbb{R}^n} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy \\ &= C\int_{B_{\frac{1}{8}}} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy + C\int_{B^c_{\frac{1}{8}}} \frac{1}{|y|^2} Q(y) e^{nw(y)} dy \\ &\leq C\int_{B_{\frac{1}{8}}} \frac{1}{|y|^2} Q(y) dy + C\kappa \\ &\leq C.\end{aligned}$$

Hence $|\Delta v(0)| \leq C$.

We define a function $\xi(x) := v(x) - \frac{t}{2n}|\Delta v(0)||x|^2$. Then ξ is monotone decreasing on $(0,\infty)$, which follows from the fact that $\Delta\xi \leq 0$. Therefore,

$$\begin{aligned}w(x) &= \xi(x) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\ &\leq \xi(\frac{1}{8}) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\ &\leq w(\frac{1}{8}) + \frac{t}{2n}|\Delta v(0)||x|^2.\end{aligned}$$

Hence, $w(x) \leq \lambda(1+|x|^2)$ on $\mathbb{R}^n$ for some $\lambda > 0$ independent of v and t. Using this in (14) one can show that

$$|v(x)| \leq C\log(2+|x|) + C|x|^2,$$

and differentiating under integral sign, from (14)

$$|D^\beta v(x)| \leq C\int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y) e^{\lambda(1+|y|^2)} dy + C|D^\beta |x|^2|, \quad 0 < |\beta| \leq n-1.$$

The lemma follows easily. □

Proof of Theorem 1.3 By Schauder fixed point theorem (see Lemma A.1), the operator T has a fixed point, thanks to Lemma 3.4. Let v be a fixed point of T. Then $u = v + c_v$ is a solution of (1).

This finishes the proof of Theorem 1.3. □

Now we prove the non existence results stated in the introduction.

Lemma 3.5 *Let $n \in \{3,4\}$. Let $Q \in C^1_{rad}(\mathbb{R}^n)$ be monotone decreasing. We assume that*

$$Q(x) = \delta e^{-\lambda|x|^2} \quad \textit{for some } \delta > 0 \textit{ and } \lambda > 0,$$

or

$$Q(x) = e^{\xi(x)}, \quad |x\cdot \nabla Q(x)| \leq C, \quad \frac{\xi(x)}{|x|^2} \xrightarrow{|x|\to\infty} 0.$$

Then there is no radially symmetric solution to (1) *with* $\kappa > \Lambda_1$.

Proof. We assume by contradiction that there is a solution u to (1) with $\kappa > \Lambda_1$, where Q satisfies the hypothesis of the lemma.

We set

$$v(x) := \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) Q(y)e^{nu(y)}dy, \quad h := u - v.$$

Then $v(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + o(\log|x|)$ as $|x| \to \infty$. Notice that h is radially symmetric and $(-\Delta)^{\frac{n}{2}}h = 0$ on $\mathbb{R}^n$. Therefore, $h(x) = c_1 + c_2|x|^2$ for some $c_1, c_2 \in \mathbb{R}$. This follows easily if $n = 4$. For $n = 3$, first notice that $\Delta h \in L_{\frac{1}{2}}(\mathbb{R}^3)$. Hence, by [13, Lemma 15] $\Delta h \equiv const$. Now radial symmetry of h implies that $h(x) = c_1 + c_2|x|^2$.

From a Pohozaev type identity in [22, Theorem 2.1] we get

$$\frac{\kappa}{\gamma_n}\left(\frac{\kappa}{\gamma_n} - 2\right) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} (x\cdot \nabla K(x))\, e^{nv(x)}dx, \quad K := Qe^{nh}. \tag{15}$$

Since $\kappa > \Lambda_1 = 2\gamma_n$, from (15) we deduce that $x\cdot \nabla K(x) > 0$ for some $x \in \mathbb{R}^n$. This implies that for $Q = \delta e^{-\lambda|x|^2}$ we must have $nc_2 - \lambda > 0$, which contradicts to the fact that $Qe^{nu} \in L^1(\mathbb{R}^n)$. For $Q = e^{\xi}$, using that $Qe^{nu} \in L^1(\mathbb{R}^n)$ and that $\xi(x) = o(|x|^2)$ at infinity, one has $c_2 \leq 0$. Therefore, $x\cdot \nabla K(x) \leq 0$ in $\mathbb{R}^n$, a contradiction. ☐

Proof of the following lemma is similar to Lemma 3.5.

Lemma 3.6 *Let* $\kappa \geq \Lambda_1$. *Let* P *be a non-constant and non-increasing radially symmetric polynomial of degree at most* $n-1$. *Then there is no solution* u *to* (2) *(with* $n \geq 3$*) such that* u *has the asymptotic behavior given by*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + o(\log|x|), \quad \text{as } |x| \to \infty.$$

A Appendix

Lemma A.1 (Theorem 11.3 in [8]) *Let* T *be a compact mapping of a Banach space* X *into itself, and suppose that there exists a constant* M *such that*

$$\|x\|_X < M$$

for all $x \in X$ *and* $t \in (0,1]$ *satisfying* $tTx = x$. *Then* T *has a fixed point.*

Lemma A.2 ([16]) *Let $\Delta^m h = 0$ in $B_{4R} \subset \mathbb{R}^n$. For any $x \in B_R$ and $0 < r < R - |x|$ we have*

$$\frac{1}{|B_r|}\int_{B_r(x)} h(z)dz = \sum_{i=0}^{m-1} c_i r^{2i}\Delta^i h(x), \tag{16}$$

where

$$c_0 = 1, \quad c_i = c(i,n) > 0, \quad \text{for } i \geq 1.$$

Moreover, for every $k \geq 0$ there exists $C = C(k,R) > 0$ such that

$$\|h\|_{C^k(B_R)} \leq C\|h\|_{L^1(B_{4R})}. \tag{17}$$

Lemma A.3 *Let $R > 0$ and $B_R \subset \mathbb{R}^n$. Let $u_k \in C^{n-1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (\frac{1}{2}, 1)$ be such that*

$$u_k(0) = 0, \quad \|u_k^+\|_{L^\infty(B_R)} \leq C, \quad \|(-\Delta)^{\frac{n}{2}} u_k\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta u_k| dx \leq C.$$

If n is an odd integer, we also assume that $\|\Delta^{\frac{n-1}{2}} u_k\|_{L_{\frac{1}{2}}(\mathbb{R}^n)} \leq C$. Then (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{\frac{R}{8}})$.

Proof. First we prove the lemma for n even.

We write $u_k = w_k + h_k$ where

$$\begin{cases} (-\Delta)^{\frac{n}{2}} w_k = (-\Delta)^{\frac{n}{2}} u_k & \text{in } B_R \\ \Delta^j w_k = 0, \quad \text{on } \partial B_R, & j = 0, 1, \ldots, \frac{n-2}{2}. \end{cases}$$

Then by standard elliptic estimates, w_k's are uniformly bounded in $C^{n-1,\beta}(B_R)$. Therefore,

$$|h_k(0)| \leq C, \quad \|h_k^+\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta h_k| dx \leq C.$$

Since h_k's are $\frac{n}{2}$-harmonic, Δh_k's are $(\frac{n}{2} - 1)$-harmonic in B_R, and by (17) we obtain

$$\|\Delta h_k\|_{C^n(B_{\frac{R}{4}})} \leq C\|\Delta h_k\|_{L^1(B_R)} \leq C.$$

Using the identity (16) we bound

$$\begin{aligned} \frac{1}{|B_R|}\int_{B_R(0)} h_k^-(z)dz &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)dz - \frac{1}{|B_R|}\int_{B_R(0)} h_k(z)dz \\ &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)dz - h_k(0) - \sum_{i=1}^{m-1} c_i R^{2i}\Delta^i h_k(0) \\ &\leq C, \end{aligned}$$

and hence
$$\int_{B_R} |h_k(z)|dz = \int_{B_R} h_k^+(z)dz + \int_{B_R} h_k^-(z)dz \leq C.$$
Again by (17) we obtain

$$\|h_k\|_{C^n(B_{\frac{R}{4}})} \leq C\|h_k\|_{L^1(B_R)} \leq C.$$

Thus, u_k's are uniformly bounded in $C^{n-1,\beta}(B_{\frac{R}{4}})$ and (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{\frac{R}{4}})$ for some $u \in C^{n-1}(B_{\frac{R}{4}})$.

It remains to prove the lemma for n odd.

If n is odd then $\frac{n-1}{2}$ is an integer. We split $\Delta^{\frac{n-1}{2}} u_k = w_k + h_k$ where

$$\begin{cases} (-\Delta)^{\frac{1}{2}} w_k = (-\Delta)^{\frac{1}{2}} \Delta^{\frac{n-1}{2}} u_k & \text{in } B_R \\ w_k = 0 \quad \text{in } B_R^c. \end{cases}$$

Then by Lemmas A.4 and A.5 one has $\|\Delta^{\frac{n-1}{2}} u_k\|_{C^{\frac{1}{2}}(B_{\frac{R}{2}})} \leq C$. Now one can proceed as in the case of even integer. □

Lemma A.4 ([13]) *Let $u \in L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0,1)$ and $(-\Delta)^\sigma u = 0$ in B_{2R}. Then for every $k \in \mathbb{N}$*

$$\|\nabla^k u\|_{C^0(B_R)} \leq C(n,\sigma,k)\frac{1}{R^k}\left(R^{2\sigma}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|u(x)|}{|x|^{n+2\sigma}}dx + \frac{\|u\|_{L^1(B_{2R})}}{R^n}\right)$$

where $\alpha \in (0,1)$ and k is an nonnegative integer.

Lemma A.5 ([20]) *Let $\sigma \in (0,1)$. Let u be a solution of*

$$\begin{cases} (-\Delta)^\sigma u = f & \text{in } B_R \\ u = 0 \quad \text{in } B_R^c \end{cases}$$

Then
$$\|u\|_{C^\sigma(\mathbb{R}^n)} \leq C(R,\sigma)\|f\|_{L^\infty(B_R)}.$$

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