

**Multilevel quadrature for  
elliptic problems on  
random domains by the coupling  
of FEM and BEM**

H. Harbrecht and M. Schmidlin

Departement Mathematik und Informatik  
Fachbereich Mathematik  
Universität Basel  
CH-4051 Basel

Preprint No. 2018-03  
February 2018

[www.math.unibas.ch](http://www.math.unibas.ch)

# MULTILEVEL QUADRATURE FOR ELLIPTIC PROBLEMS ON RANDOM DOMAINS BY THE COUPLING OF FEM AND BEM

HELMUT HARBRECHT AND MARC SCHMIDLIN

ABSTRACT. Elliptic boundary value problems which are posed on a random domain can be mapped to a fixed, nominal domain. The randomness is thus transferred to the diffusion matrix and the loading. This domain mapping method is quite efficient for theory and practice, since only a single domain discretization is needed. Nonetheless, it is not useful for applying multilevel accelerated methods to efficiently deal with the random parameter. This issues from the fact that the domain discretization needs to be fine enough in order to avoid indefinite diffusion matrices. To overcome this obstruction, we are going to couple the finite element method with the boundary element method. In this article, we verify the required regularity with respect to the random perturbation field, derive the coupling formulation, and show by numerical results that the approach is feasible.

## 1. INTRODUCTION

Many practical problems in science and engineering lead to elliptic boundary value problems for an unknown function. Their numerical treatment by e.g. finite difference or finite element methods is in general well understood provided that the input parameters are given exactly. This, however, is often not the case in practical applications.

If a statistical description of the input data is available, one can mathematically describe data and solutions as random fields and aim at the computation of corresponding deterministic statistics of the unknown random solution. The present article is dedicated to the treatment of uncertainties in the description of the computational domain. Applications are, besides traditional engineering, for example uncertain domains which are derived from inverse methods such as tomography. In recent years, this situation has become of growing interest, see e.g. [6, 7, 20, 22, 25, 27, 28] and the references therein.

---

*2010 Mathematics Subject Classification.* 35R60, 65N30, 65N38.

*Key words and phrases.* Uncertainty quantification, random domain, regularity, multi-level method, FEM-BEM coupling.

The authors gratefully acknowledge the support from the Swiss National Science Foundation (Grant No. 205321\_169599).

In this article, we are going to focus on the so-called domain mapping method which has been introduced in [28] and rigorously analyzed in [7, 20], where analytic dependency of the solution on the random domain with regard to the energy norm has been verified. The key idea of the method is to map the boundary value problem

$$(1.1) \quad -\Delta_{\mathbf{x}} u[\omega] = f \text{ in } \mathfrak{D}[\omega], \quad u[\omega] = 0 \text{ on } \partial\mathfrak{D}[\omega],$$

which is posed on a random domain

$$\mathfrak{D}[\omega] := \mathbf{V}[\omega](D) \subset \mathbb{R}^d$$

onto a fixed, nominal domain  $D \subset \mathbb{R}^d$ . Thus, the randomness is transferred to the diffusion matrix and the loading of the boundary value problem

$$(1.2) \quad -\operatorname{div}_{\mathbf{x}}(\hat{\mathbf{A}}[\omega] \nabla_{\mathbf{x}} \hat{u}[\omega]) = \hat{f}[\omega] \text{ in } D, \quad \hat{u}[\omega] = 0 \text{ on } \partial D.$$

Herein, it holds

$$(1.3) \quad \hat{\mathbf{A}}[\omega] := (\mathbf{J}[\omega]^T \mathbf{J}[\omega])^{-1} \det \mathbf{J}[\omega]$$

and

$$(1.4) \quad \hat{f}[\omega] := (f \circ \mathbf{V}[\omega]) \det \mathbf{J}[\omega],$$

where  $\mathbf{J}[\omega]$  denotes the Jacobian of the field  $\mathbf{V}[\omega]: D \rightarrow \mathfrak{D}[\omega]$

$$(1.5) \quad \mathbf{J}[\omega](\mathbf{x}) := D_{\mathbf{x}} \mathbf{V}[\omega](\mathbf{x}).$$

and  $\hat{u}[\omega]$  is connected to  $u[\omega]$  by  $\hat{u}[\omega] := (u[\omega] \circ \mathbf{V}[\omega])$ .

In practical applications, it turns out that the discretization of the boundary value problem (1.2) needs to be very fine in order to ensure definiteness of the diffusion matrix field  $\hat{\mathbf{A}}[\omega]$  for almost all  $\omega \in \Omega$ . This fact unfortunately rules out multilevel acceleration techniques such as the multilevel Monte Carlo method, see [1, 8], or general multilevel quadrature methods, see [19], for efficiently dealing with the random parameter  $\omega$ .

Having in mind that the quantity of interest

$$(1.6) \quad \operatorname{QoI}(u) = \int_{\Omega} \mathcal{F}(u[\omega]) \, d\mathbb{P}[\omega]$$

is generally sought on a *deterministic* subdomain  $B \subset D$  where the field  $\mathbf{V}[\omega]$  coincides with the identity almost surely, we propose to couple finite element methods with boundary element methods for the spatial approximation. We thus apply finite elements on the subdomain  $B \subset D$  and treat the rest of the domain by a boundary element method. Hence, also large domain deformations can be handled on coarse discretizations. We present the resulting coupling formulation and then discuss the efficient solution by multilevel quadrature methods. Especially, we verify the required regularity with respect to the random perturbation field.

The rest of this article is organized as follows. Section 2 is dedicated to the mathematical formulation of the problem under consideration. The problem's regularity is studied in Section 3. Here, we provide estimates in stronger spatial norms which are needed for multilevel accelerated quadrature methods. The coupling of finite elements and boundary elements is the topic of Section 4. The multilevel quadrature method for the solution of the random boundary value problem is then introduced in Section 5. Numerical experiments are carried out in Section 6. Finally, we state concluding remarks in Section 7.

## 2. NOTATION AND MODEL PROBLEM

In the following, in order to avoid the repeated use of generic but unspecified constants, by  $C \lesssim D$  we mean that  $C$  can be bounded by a multiple of  $D$ , independently of parameters which  $C$  and  $D$  may depend on. Obviously,  $C \gtrsim D$  is defined as  $D \lesssim C$  and  $C \approx D$  as  $C \lesssim D$  and  $C \gtrsim D$ .

For a given Banach space  $\mathcal{X}$  and a complete measure space  $\mathcal{M}$  with measure  $\mu$  the space  $L_\mu^p(\mathcal{M}; \mathcal{X})$  for  $1 \leq p \leq \infty$  denotes the Bochner space, see [24], which contains all equivalence classes of strongly measurable functions  $v: \mathcal{M} \rightarrow \mathcal{X}$  with finite norm

$$\|v\|_{L_\mu^p(\mathcal{M}; \mathcal{X})} := \begin{cases} \left[ \int_{\mathcal{M}} \|v(x)\|_{\mathcal{X}}^p d\mu(x) \right]^{1/p}, & p < \infty, \\ \operatorname{ess\,sup}_{x \in \mathcal{M}} \|v(x)\|_{\mathcal{X}}, & p = \infty. \end{cases}$$

A function  $v: \mathcal{M} \rightarrow \mathcal{X}$  is strongly measurable if there exists a sequence of countably-valued measurable functions  $v_n: \mathcal{M} \rightarrow \mathcal{X}$ , such that for almost every  $m \in \mathcal{M}$  we have  $\lim_{n \rightarrow \infty} v_n(m) = v(m)$ . Note that, for finite measures  $\mu$ , we also have the usual inclusion  $L_\mu^p(\mathcal{M}; \mathcal{X}) \supset L_\mu^q(\mathcal{M}; \mathcal{X})$  for  $1 \leq p < q \leq \infty$ .

Subsequently, we will always equip  $\mathbb{R}^d$  with the norm  $\|\cdot\|_2$  induced by the canonical inner product  $\langle \cdot, \cdot \rangle$  and  $\mathbb{R}^{d \times d}$  with the induced norm  $\|\cdot\|_2$ .

Let  $\mathcal{X}, \mathcal{X}_1, \dots, \mathcal{X}_r$  and  $\mathcal{Y}$  be Banach spaces, then we denote the Banach space of bounded, linear maps from  $\mathcal{X}$  to  $\mathcal{Y}$  as  $\mathcal{B}(\mathcal{X}; \mathcal{Y})$ ; furthermore, we recursively define

$$\mathcal{B}(\mathcal{X}_1, \dots, \mathcal{X}_r; \mathcal{Y}) := \mathcal{B}(\mathcal{X}_1; \mathcal{B}(\mathcal{X}_2, \dots, \mathcal{X}_r; \mathcal{Y}))$$

and the special case

$$\mathcal{B}^0(\mathcal{X}; \mathcal{Y}) := \mathcal{Y} \quad \text{and} \quad \mathcal{B}^{r+1}(\mathcal{X}; \mathcal{Y}) := \mathcal{B}(\mathcal{X}; \mathcal{B}^r(\mathcal{X}; \mathcal{Y})).$$

For  $\mathbf{T} \in \mathcal{B}(\mathcal{X}_1, \dots, \mathcal{X}_r; \mathcal{Y})$  and  $\mathbf{v}_j \in \mathcal{X}_j$  we use the notation  $\mathbf{T}\mathbf{v}_1 \cdots \mathbf{v}_r := \mathbf{T}(\mathbf{v}_1, \dots, \mathbf{v}_r) \in \mathcal{Y}$ . Lastly, note that for the natural numbers  $\mathbb{N}$  denotes them including 0 and  $\mathbb{N}^*$  excluding 0.

Let  $\kappa \in \mathbb{N}^*$  and  $d \in \mathbb{N}^*$ ;  $D \subset \mathbb{R}^d$  denote the *reference domain* with boundary  $\partial D$  that is of class  $C^{\kappa,1}$  — when  $\kappa = 1$  then we also consider the case where  $D$  is a bounded and convex domain with Lipschitz continuous boundary — and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a separable, complete probability space with  $\sigma$ -field  $\mathcal{F} \subset 2^\Omega$  and probability measure  $\mathbb{P}$ . Furthermore, let

$$\mathbf{V} \in L_{\mathbb{P}}^\infty(\Omega; C^{\kappa,1}(\overline{D}; \mathbb{R}^d))$$

be the *random domain mapping*. Moreover, we require that, for  $\mathbb{P}$ -almost any  $\omega$ ,  $\mathbf{V}[\omega]: D \rightarrow \mathfrak{D}[\omega]$  is a  $C^1$ -diffeomorphism that fulfils the *uniformity condition*

$$\|\mathbf{V}[\omega]\|_{C^1(\overline{D}; \overline{\mathfrak{D}[\omega]})}, \quad \|\mathbf{V}[\omega]^{-1}\|_{C^1(\overline{\mathfrak{D}[\omega]}; \overline{D})} \leq C$$

for a  $C \in (0, \infty)$  independent of  $\omega$ . Finally, we require that we have a *hold-all domain*  $\mathcal{D}$  that satisfies  $\mathfrak{D}[\omega] \subset \mathcal{D}$  for  $\mathbb{P}$ -almost any  $\omega \in \Omega$  and consider  $f \in C^\omega(\mathcal{D})$ .

Note that while we restrict ourselves to the Poisson equation here to simplify the analysis, the extension of the regularity result to an operator  $\operatorname{div}_{\mathbf{x}} \mathbf{A} \nabla_{\mathbf{x}}$ , with an  $\mathbf{A} \in C^\omega(\mathcal{D}; \mathbb{R}^{d \times d})$  and  $\mathbf{A}$  fulfilling an ellipticity condition is straightforward.

Now, since for  $\mathbb{P}$ -almost any  $\omega \in \Omega$  we have a  $C^1$ -diffeomorphism from  $D \rightarrow \mathfrak{D}[\omega]$  we can use the one-to-one correspondence to pull back the model problem onto the reference domain  $D$  instead of considering it on the actual domain realisations  $\mathfrak{D}[\omega]$ . According to the chain rule, we then have for  $v \in C^1(\mathfrak{D}[\omega])$  that

$$(\nabla_{\mathbf{x}} v) \circ \mathbf{V}[\omega] = (\mathbf{J}[\omega])^{-\top} \nabla_{\mathbf{x}}(v \circ \mathbf{V}[\omega]).$$

Now, with (1.3) and (1.4) this leads us to the following formulation of our model problem (1.2) on the reference domain, cf. [20]:

$$(2.1) \quad \left\{ \begin{array}{l} \text{Find } \hat{u} \in L_{\mathbb{P}}^\infty(\Omega; H_0^1(D)) \text{ such that} \\ \int_D \langle \hat{\mathbf{A}}[\omega](\mathbf{x}) \nabla_{\mathbf{x}} \hat{u}[\omega](\mathbf{x}), \nabla_{\mathbf{x}} \hat{v}(\mathbf{x}) \rangle \, d\mathbf{x} = \int_D \hat{f}[\omega](\mathbf{x}) \hat{v}(\mathbf{x}) \, d\mathbf{x} \\ \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega \text{ and all } \hat{v} \in H_0^1(D). \end{array} \right.$$

Note, especially, that by the uniformity condition we have that

$$(2.2) \quad \underline{\sigma} \leq \operatorname{ess\,inf}_{\omega \in \Omega} \operatorname{ess\,inf}_{\mathbf{x} \in D} \sigma_{\min}(\mathbf{J}[\omega](\mathbf{x})) \leq \operatorname{ess\,sup}_{\omega \in \Omega} \operatorname{ess\,sup}_{\mathbf{x} \in D} \sigma_{\max}(\mathbf{J}[\omega](\mathbf{x})) \leq \bar{\sigma}$$

for some constants  $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$ . Without loss of generality, we assume  $\underline{\sigma} \leq 1 \leq \bar{\sigma}$ .

From here on, we assume that the spatial variable  $\mathbf{x}$  and the stochastic parameter  $\omega$  of the random field have been separated by the Karhunen-Loève expansion of  $\mathbf{V}$  coming from the mean field  $\mathbb{E}[\mathbf{V}]$  and the covariance  $\operatorname{Cov}[\mathbf{V}]$  yielding a parametrised expansion

$$(2.3) \quad \mathbf{V}[\mathbf{y}](\mathbf{x}) = \mathbb{E}[\mathbf{V}](\mathbf{x}) + \sum_{k=1}^{\infty} \sigma_k \boldsymbol{\psi}_k(\mathbf{x}) y_k,$$

where  $\mathbf{y} = (y_k)_{k \in \mathbb{N}^*} \in \square := [-1, 1]^{\mathbb{N}^*}$  is a sequence of uncorrelated random variables, see e.g. [20]; we denote the pushforward measure of  $\mathbb{P}$  onto  $\square$  as  $\mathbb{P}_{\mathbf{y}}$ . Thus, we then also view all randomness as being parametrised by  $\mathbf{y}$ , i.e.  $\omega$ ,  $\Omega$  and  $\mathbb{P}$  are replaced by  $\mathbf{y}$ ,  $\square$  and  $\mathbb{P}_{\mathbf{y}}$ .

We now impose some common assumptions, which make the Karhunen-Loève expansion computationally feasible.

**Assumption 2.1.** (1) *The random variables  $(y_k)_{k \in \mathbb{N}^*}$  are independent and identically distributed. Moreover, they are uniformly distributed on  $[-1, 1]$ .*

(2) *We assume that the  $\boldsymbol{\psi}_k$  are elements of  $W^{\kappa+1, \infty}(D; \mathbb{R}^d)$  and that the sequence  $\boldsymbol{\gamma} = (\gamma_k)_{k \in \mathbb{N}}$ , given by*

$$\gamma_k := \|\sigma_k \boldsymbol{\psi}_k\|_{W^{\kappa+1, \infty}(D; \mathbb{R}^d)},$$

*is at least in  $\ell^1(\mathbb{N})$ , where we have defined  $\boldsymbol{\psi}_0 := \mathbb{E}[\mathbf{V}]$  and  $\sigma_0 := 1$ . Furthermore, we define*

$$c_{\boldsymbol{\gamma}} = \max\{\|\boldsymbol{\gamma}\|_{\ell^1(\mathbb{N})}, 1\}.$$

For the following regularity estimates, we assume that the vector field  $\mathbf{V}$  is given by a finite rank Karhunen-Loève expansion, i.e.

$$\mathbf{V}[\mathbf{y}](\mathbf{x}) = \sigma_0 \boldsymbol{\psi}_0(\mathbf{x}) + \sum_{k=1}^M \sigma_k \boldsymbol{\psi}_k(\mathbf{x}) y_k,$$

where  $\square := [-1, 1]^M$ . We note that the regularity estimates however will not depend on the rank  $M$ . If necessary, a finite rank can be attained by appropriate truncation.

### 3. REGULARITY

**3.1. Precursory remarks.** In this subsection, we introduce some norms, lemmata and corollaries from [21], which will then be used in the following subsections to discuss the regularity of the diffusion coefficient and the solution.

For the Sobolev–Bochner spaces  $W^{\eta, p}(D; \mathcal{X})$  with  $\eta \in \mathbb{N}$  and  $1 \leq p \leq \infty$ , we introduce the norms given by

$$\|v\|_{\eta, p, D; \mathcal{X}} := \|v\|_{W^{\eta, p}(D; \mathcal{X})} := \sum_{|\boldsymbol{\alpha}| \leq \eta} \frac{1}{\boldsymbol{\alpha}!} \|\partial_{\mathbf{x}}^{\boldsymbol{\alpha}} v\|_{p, D; \mathcal{X}},$$

for  $v \in W^{\eta, p}(D; \mathcal{X})$  where  $\mathcal{X}$  is a Banach space with norm  $\|\cdot\|_{\mathcal{X}}$  and where we make use of the shorthand

$$\|\cdot\|_{p, D; \mathcal{X}} := \|\cdot\|_{L^p(D; \mathcal{X})}.$$

We also introduce the shorthand notations

$$\|\cdot\|_{p,D;\mathcal{X}} := \|\cdot\|_{L_{\mathbb{P}_y}^\infty(\square; L^p(D;\mathcal{X}))} \quad \text{and} \quad \|\cdot\|_{\eta,p,D;\mathcal{X}} := \|\cdot\|_{L_{\mathbb{P}_y}^\infty(\square; W^{\eta,p}(D;\mathcal{X}))}$$

We may omit the specification of the Banach space  $\mathcal{X}$ , for example when  $\mathcal{X}$  is the space  $\mathbb{R}$ ,  $\mathbb{R}^d$  or  $\mathbb{R}^{d \times d}$ .

The following lemma, corollary and theorem are now from [21]:

**Lemma 3.1.** *Let  $\eta \in \mathbb{N}^*$ ,  $1 \leq p \leq \infty$ . For  $\mathbf{v} \in W^{\eta,p}(D; \mathbb{R}^d)$  we have that  $\operatorname{div}_{\mathbf{x}} \mathbf{v} \in W^{\eta-1,p}(D; \mathbb{R})$  with*

$$\|\operatorname{div}_{\mathbf{x}} \mathbf{v}\|_{\eta-1,p,D} \leq \eta d \|\mathbf{v}\|_{\eta,p,D}$$

and  $D_{\mathbf{x}} \mathbf{v} \in W^{\eta-1,p}(D; \mathbb{R}^{d \times d})$  with

$$\|D_{\mathbf{x}} \mathbf{v}\|_{\eta-1,p,D} \leq \eta d \|\mathbf{v}\|_{\eta,p,D}.$$

For  $v \in W^{\eta,p}(D)$  we have that  $\nabla_{\mathbf{x}} v \in W^{\eta-1,p}(D; \mathbb{R}^d)$

$$\|\nabla_{\mathbf{x}} v\|_{\eta-1,p,D} \leq \eta d \|v\|_{\eta,p,D}.$$

**Corollary 3.2.** *Let  $\eta, \in \mathbb{N}$ ,  $1 \leq p_1, \dots, p_r \leq \infty$ ,  $\boldsymbol{\nu} \in \mathbb{N}^M$ ,  $\mathcal{X}_1, \dots, \mathcal{X}_r$  and  $\mathcal{Y}$  be Banach spaces and*

$$\mathbf{M} \in \mathcal{B}(\mathcal{X}_1, \dots, \mathcal{X}_r; \mathcal{Y}), \quad \partial_{\mathbf{y}}^\alpha \mathbf{v}_j \in L_{\mathbb{P}_y}^\infty(\square; W^{\eta,p_j}(D; \mathcal{X}_j))$$

for all  $\boldsymbol{\alpha} \leq \boldsymbol{\nu}$  with  $q = (p_1^{-1} + \dots + p_r^{-1})^{-1} \geq 1$ . Then, we have

$$\partial_{\mathbf{y}}^\alpha (\mathbf{M} \mathbf{v}_1 \cdots \mathbf{v}_r) \in L_{\mathbb{P}_y}^\infty(\square; W^{\eta,p}(D; \mathcal{Y}))$$

with

$$\begin{aligned} & \|\partial_{\mathbf{y}}^\alpha (\mathbf{M} \mathbf{v}_1 \cdots \mathbf{v}_r)\|_{\eta,q,D;\mathcal{Y}} \\ & \leq \|\mathbf{M}\|_{\mathcal{B}(\mathcal{X}_1, \dots, \mathcal{X}_r; \mathcal{Y})} \sum_{\boldsymbol{\beta}_1 + \dots + \boldsymbol{\beta}_r = \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r} \prod_{j=1}^r \|\partial_{\mathbf{y}}^{\boldsymbol{\beta}_j} \mathbf{v}_j\|_{\eta,p_j,D;\mathcal{X}_j} \end{aligned}$$

for all  $\boldsymbol{\alpha} \leq \boldsymbol{\nu}$ .

**Theorem 3.3.** *Let  $\eta \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $\boldsymbol{\nu} \in \mathbb{N}^M$ ,  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces,*

$$\mathbf{v}: \square \rightarrow W^{\eta,\infty}(D; \mathcal{X}),$$

$X \subset \mathcal{X}$  be open with  $\operatorname{img}_{\square} \operatorname{img}_D \mathbf{v} \subset X$ ,  $\mathbf{W}: X \rightarrow \mathcal{Y}$  be  $\eta + |\boldsymbol{\nu}|$ -times Fréchet differentiable and, for  $\boldsymbol{\alpha} \leq \boldsymbol{\nu}$  and  $0 \leq t \leq \eta + |\boldsymbol{\nu}|$ ,

$$\partial_{\mathbf{y}}^\alpha \mathbf{v} \in L_{\mathbb{P}_y}^\infty(\square; W^{\eta,\infty}(D; \mathcal{X})), \quad D^t \mathbf{W} \circ \mathbf{v} \in L_{\mathbb{P}_y}^\infty(\square; L^p(D; \mathcal{B}^t(\mathcal{X}; \mathcal{Y}))).$$

Then, we have

$$\partial_{\mathbf{y}}^\alpha (\mathbf{W} \circ \mathbf{v}) \in L_{\mathbb{P}_y}^\infty(\square; W^{\eta,p}(D; \mathcal{Y}))$$

with

$$\|\mathbf{W} \circ \mathbf{v}\|_{\eta,p,D;\mathcal{Y}} \leq \sum_{r=0}^{\eta} \frac{1}{r!} \|D^r \mathbf{W} \circ \mathbf{v}\|_{p,D;\mathcal{B}^r(\mathcal{X};\mathcal{Y})} \|\mathbf{v}\|_{\eta,\infty,D;\mathcal{X}}^r$$

and, for  $\boldsymbol{\alpha} \neq \mathbf{0}$ ,

$$\begin{aligned} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\mathbf{W} \circ \mathbf{v})\|_{\eta,p,D;\mathcal{Y}} &\leq \boldsymbol{\alpha}! \sum_{s=1}^{|\boldsymbol{\alpha}|} \frac{1}{s!} \left( \sum_{r=0}^{\eta} \frac{1}{r!} \|D^{r+s} \mathbf{W} \circ \mathbf{v}\|_{p,D;\mathcal{B}^{r+s}(\mathcal{X};\mathcal{Y})} \|\mathbf{v}\|_{\eta,\infty,D;\mathcal{X}}^r \right) \\ &\quad \sum_{P(\boldsymbol{\alpha},s)} \prod_{j=1}^s \frac{1}{\beta_j!} \|\partial_{\mathbf{y}}^{\beta_j} \mathbf{v}\|_{\eta,\infty,D;\mathcal{X}}. \end{aligned}$$

**3.2. Parametric regularity of the diffusion coefficient.** We now provide regularity estimates for the different terms that make up the diffusion coefficient and the right hand side, based on the decay of the expansion of  $\mathbf{V}$  as per Assumption 2.1.

**Lemma 3.4.** *We have for all  $\boldsymbol{\alpha} \in \mathbb{N}^M$  that*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathbf{V}\|_{\kappa+1,\infty,D} \leq k_{\mathbf{V}} \gamma^{\boldsymbol{\alpha}} \quad \text{and} \quad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathbf{J}\|_{\kappa,\infty,D} \leq k_{\mathbf{J}} \gamma^{\boldsymbol{\alpha}},$$

where  $k_{\mathbf{V}} := c_{\gamma}$  and  $k_{\mathbf{J}} := (\kappa + 1)dc_{\gamma}$

*Proof.* By definition we have that  $\mathbf{J}[\mathbf{y}] = D_{\mathbf{x}} \mathbf{V}[\mathbf{y}]$  and so it follows that

$$\mathbf{V}[\mathbf{y}] = \sigma_0 \boldsymbol{\psi}_0 + \sum_{k=1}^M \sigma_k \boldsymbol{\psi}_k y_k \quad \text{and} \quad \mathbf{J}[\mathbf{y}] = \sigma_0 D_{\mathbf{x}} \boldsymbol{\psi}_0 + \sum_{k=1}^M \sigma_k D_{\mathbf{x}} \boldsymbol{\psi}_k y_k.$$

From this we can derive that first order derivatives are given by

$$\partial_{y_i} \mathbf{V}[\mathbf{y}] = \sigma_i \boldsymbol{\psi}_i \quad \text{and} \quad \partial_{y_i} \mathbf{J}[\mathbf{y}] = \sigma_i D_{\mathbf{x}} \boldsymbol{\psi}_i$$

and all higher derivatives vanish.

Thus, we obviously have

$$\|\mathbf{V}\|_{\kappa+1,\infty,D} \leq \|\sigma_0 \boldsymbol{\psi}_0\|_{\kappa+1,\infty,D} + \sum_{k=1}^M \|\sigma_k \boldsymbol{\psi}_k\|_{\kappa+1,\infty,D} \leq c_{\gamma}$$

and

$$\begin{aligned} \|\mathbf{J}\|_{\kappa,\infty,D} &\leq \|\sigma_0 D_{\mathbf{x}} \boldsymbol{\psi}_0\|_{\kappa,\infty,D} + \sum_{k=1}^M \|\sigma_k D_{\mathbf{x}} \boldsymbol{\psi}_k\|_{\kappa,\infty,D} \\ &\leq (\kappa + 1)d \|\sigma_0 \boldsymbol{\psi}_0\|_{\kappa+1,\infty,D} + \sum_{k=1}^M (\kappa + 1)d \|\sigma_k \boldsymbol{\psi}_k\|_{\kappa+1,\infty,D} \leq (\kappa + 1)dc_{\gamma} \end{aligned}$$

as well as

$$\|\partial_{y_i} \mathbf{V}\|_{\kappa+1,\infty,D} \leq \|\sigma_i \boldsymbol{\psi}_i\|_{\kappa+1,\infty,D} \leq \gamma_i$$

and

$$\|\partial_{y_i} \mathbf{J}\|_{\kappa, \infty, D} \leq \|\sigma_i D_{\mathbf{x}} \boldsymbol{\psi}_i\|_{\kappa, \infty, D} \leq (\kappa + 1)d \|\sigma_i \boldsymbol{\psi}_i\|_{\kappa+1, \infty, D} \leq (\kappa + 1)d\gamma_i.$$

Clearly, this implies the assertion.  $\square$

**Lemma 3.5.** *Let us define*

$$\mathbf{C}[\mathbf{y}](\mathbf{x}) := (\mathbf{J}[\mathbf{y}](\mathbf{x}))^{-1} \quad \text{and} \quad b[\mathbf{y}](\mathbf{x}) := \det(\mathbf{J}[\mathbf{y}](\mathbf{x})).$$

Then, we have for all  $\boldsymbol{\alpha} \in \mathbb{N}^M$  that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathbf{C}\|_{\kappa, \infty, D} \leq |\boldsymbol{\alpha}|! k_{\mathbf{C}} c_{\mathbf{C}}^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}} \quad \text{and} \quad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} b\|_{\kappa, \infty, D} \leq |\boldsymbol{\alpha}|! k_b c_b^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}},$$

where

$$k_{\mathbf{C}} := \frac{\kappa + 1}{\underline{\sigma}} \left( \frac{2k_{\mathbf{J}}}{\underline{\sigma}} \right)^{\kappa}, \quad c_{\mathbf{C}} := \frac{2k_{\mathbf{J}}}{\underline{\sigma} \log 2}, \quad k_b := (\kappa + 1)d! \bar{\sigma}^d (2k_{\mathbf{J}})^{\kappa} \quad \text{and} \quad c_b := \frac{2k_{\mathbf{J}}}{\log 2}.$$

*Proof.* We can consider that  $\mathbf{C}[\mathbf{y}](\mathbf{x}) = \mathbf{v}(\mathbf{J}[\mathbf{y}](\mathbf{x}))$ , where  $\mathbf{v}(\mathbf{X}) = \mathbf{X}^{-1}$ . Clearly, the  $t$ -th Fréchet derivative of  $\mathbf{v}$  is given by

$$D^t \mathbf{v}(\mathbf{X}) \mathbf{H}_1 \cdots \mathbf{H}_t = (-1)^t \sum_{\sigma \in S_t} \mathbf{X}^{-1} \prod_{j=1}^t (\mathbf{H}_{\sigma(j)} \mathbf{X}^{-1}) = (-1)^t \sum_{\sigma \in S_t} \mathbf{v}(\mathbf{X}) \prod_{j=1}^t (\mathbf{H}_{\sigma(j)} \mathbf{v}(\mathbf{X})).$$

Thus, for  $\mathbf{X}$  with  $\underline{\sigma} \leq \sigma_{\min}(\mathbf{X})$ , we have

$$\|D^t \mathbf{v}(\mathbf{X})\|_{\mathcal{B}^t(\mathbb{R}^{d \times d}; \mathbb{R}^{d \times d})} \leq t! \|\mathbf{v}(\mathbf{X})\|_2^{t+1} \leq \frac{t!}{\underline{\sigma}^{t+1}},$$

which using Theorem 3.3 leads to the result for  $\boldsymbol{\alpha} = \mathbf{0}$ :

$$\|\mathbf{v}(\mathbf{J})\|_{\kappa, \infty, D} \leq \sum_{r=0}^{\kappa} \frac{1}{r!} \frac{r!}{\underline{\sigma}^{r+1}} (k_{\mathbf{J}} \gamma^0)^r = \sum_{r=0}^{\kappa} \frac{k_{\mathbf{J}}^r}{\underline{\sigma}^{r+1}} \leq \frac{\kappa + 1}{\underline{\sigma}} \left( \frac{k_{\mathbf{J}}}{\underline{\sigma}} \right)^{\kappa}.$$

Using Theorem 3.3 for  $\boldsymbol{\alpha} \neq \mathbf{0}$  yields

$$\begin{aligned} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} (\mathbf{v}(\mathbf{J}))\|_{\kappa, \infty, D} &\leq |\boldsymbol{\alpha}|! \sum_{s=1}^{|\boldsymbol{\alpha}|} \frac{1}{s!} \left( \sum_{r=0}^{\kappa} \frac{1}{r!} \frac{(r+s)!}{\underline{\sigma}^{r+s+1}} (k_{\mathbf{J}} \gamma^0)^r \right) \sum_{P(\boldsymbol{\alpha}, s)} \prod_{j=1}^s \frac{1}{\beta_j!} k_{\mathbf{J}} \gamma^{\beta_j} \\ &\leq |\boldsymbol{\alpha}|! \gamma^{\boldsymbol{\alpha}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \frac{2^s}{\underline{\sigma}^s} k_{\mathbf{J}}^s \left( \sum_{r=0}^{\kappa} \frac{2^r k_{\mathbf{J}}^r}{\underline{\sigma}^{r+1}} \right) \sum_{P(\boldsymbol{\alpha}, s)} \prod_{j=1}^s \frac{1}{\beta_j!} \\ &\leq \gamma^{\boldsymbol{\alpha}} \frac{\kappa + 1}{\underline{\sigma}} \left( \frac{2k_{\mathbf{J}}}{\underline{\sigma}} \right)^{\kappa} \left( \frac{2k_{\mathbf{J}}}{\underline{\sigma}} \right)^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|! \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{P(\boldsymbol{\alpha}, s)} \prod_{j=1}^s \frac{1}{\beta_j!} \\ &\leq \gamma^{\boldsymbol{\alpha}} \frac{\kappa + 1}{\underline{\sigma}} \left( \frac{2k_{\mathbf{J}}}{\underline{\sigma}} \right)^{\kappa} \left( \frac{2k_{\mathbf{J}}}{\underline{\sigma}} \right)^{|\boldsymbol{\alpha}|} \frac{|\boldsymbol{\alpha}|!}{(\log 2)^{|\boldsymbol{\alpha}|}}, \end{aligned}$$

where we have used that  $\frac{(r+s)!}{r!s!} \leq 2^{r+s}$  and the bound

$$\boldsymbol{\alpha}! \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{P(\boldsymbol{\alpha},s)} \prod_{j=1}^s \frac{1}{\boldsymbol{\beta}_j!} \leq \frac{|\boldsymbol{\alpha}|!}{(\log 2)^{|\boldsymbol{\alpha}|}},$$

which follows from [2], see [21]. Clearly, these two bounds imply the assertion for  $\mathbf{C}$ .

The situation for  $b[\mathbf{y}](\mathbf{x}) = \det(\mathbf{J}[\mathbf{y}](\mathbf{x}))$  is mainly analogously. However, the  $t$ -th Fréchet derivative of  $\det$  is given by

$$D^t \det(\mathbf{X}) \mathbf{H}_1 \cdots \mathbf{H}_t = \sum_{\substack{1 \leq i_1, \dots, i_t \leq d \\ \text{p.w. unequal}}} \det(\mathbf{X}_{[i_1, \mathbf{H}_1], \dots, [i_t, \mathbf{H}_t]});$$

here  $\mathbf{X}_{[i_1, \mathbf{H}_1], \dots, [i_t, \mathbf{H}_t]}$  denotes the matrix  $\mathbf{X}$  whose  $i_k$ -th column is replaced by the  $i_k$ -th column of the matrix  $\mathbf{H}_k$  for all  $k$  from 1 to  $t$ . Now, since we can bound a determinant by the product of the norms of its columns, i.e.

$$\left| \det \left( \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_d \end{bmatrix} \right) \right| \leq \prod_{j=1}^d \|\mathbf{z}_j\|_2,$$

and since we know that

$$\|\mathbf{z}_j\|_2 = \left\| \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_d \end{bmatrix} \mathbf{e}_j \right\|_2 \leq \left\| \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_d \end{bmatrix} \right\|_2 \|\mathbf{e}_j\|_2 = \left\| \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_d \end{bmatrix} \right\|_2.$$

it follows that, for  $\mathbf{X}$  with  $\sigma_{\max}(\mathbf{X}) \leq \bar{\sigma}$ ,

$$\|D^t \det(\mathbf{X})\|_{\mathcal{B}^t(\mathbb{R}^{d \times d}; \mathbb{R})} \leq d! \|\mathbf{X}\|_2^d \leq d! \bar{\sigma}^d \leq t! d! \bar{\sigma}^d.$$

The rest of the arguments yielding the assertion for  $b$  are the same as before for  $\mathbf{C}$ .  $\square$

**Lemma 3.6.** *Define*

$$g[\mathbf{y}](\mathbf{x}) := f(\mathbf{V}[\mathbf{y}](\mathbf{x})).$$

*Then, we have for all  $\boldsymbol{\alpha} \in \mathbb{N}^M$  that*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} g\|_{\kappa-1, 2, D} \leq |\boldsymbol{\alpha}|! k_g c_g^{|\boldsymbol{\alpha}|} \boldsymbol{\gamma}^{\boldsymbol{\alpha}},$$

*where*

$$k_g := \kappa \sqrt{\frac{|\mathcal{D}|}{\underline{\sigma}^d}} k_f (2c_f k_{\mathbf{V}})^{\kappa-1} \quad \text{and} \quad c_g := \frac{2c_f k_{\mathbf{V}}}{\log 2}.$$

*Proof.* As  $f \in C^\omega(\mathcal{D})$  we know that

$$\|D^t f\|_{\infty, \mathcal{D}; \mathcal{B}^t(\mathbb{R}^{d \times d}; \mathbb{R})} \leq t! k_f c_f^t$$

for some constants  $c_f, k_f \geq 1$  and all  $t \in \mathbb{N}$ . This leads to the estimate

$$\begin{aligned}
\|D^t f \circ \mathbf{V}\|_{2,D;\mathcal{B}^t(\mathbb{R}^{d \times d};\mathbb{R})} &= \operatorname{ess\,sup}_{\mathbf{y} \in \square} \|D^t f \circ \mathbf{V}[\mathbf{y}]\|_{2,D;\mathcal{B}^t(\mathbb{R}^{d \times d};\mathbb{R})} \\
&\leq \operatorname{ess\,sup}_{\mathbf{y} \in \square} \|D^t f\|_{2,\mathcal{D}[\mathbf{y}];\mathcal{B}^t(\mathbb{R}^{d \times d};\mathbb{R})} \sqrt{\underline{\sigma}^{-d}} \\
&\leq \operatorname{ess\,sup}_{\mathbf{y} \in \square} \|D^t f\|_{\infty,\mathcal{D}[\mathbf{y}];\mathcal{B}^t(\mathbb{R}^{d \times d};\mathbb{R})} \sqrt{|\mathcal{D}[\mathbf{y}]| \underline{\sigma}^{-d}} \\
&\leq \|D^t f\|_{\infty,\mathcal{D};\mathcal{B}^t(\mathbb{R}^{d \times d};\mathbb{R})} \sqrt{|\mathcal{D}| \underline{\sigma}^{-d}} \\
&\leq t! \sqrt{|\mathcal{D}| \underline{\sigma}^{-d}} k_f c_f^t.
\end{aligned}$$

From here on the proof is analogous to the one for Lemma 3.5.  $\square$

**Lemma 3.7.** *Define*

$$\mathbf{B}[\mathbf{y}](\mathbf{x}) := \mathbf{C}[\mathbf{y}](\mathbf{x}) \mathbf{C}[\mathbf{y}](\mathbf{x})^\top.$$

Then, we have for all  $\boldsymbol{\alpha} \in \mathbb{N}^M$  that

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \mathbf{B}\|_{\kappa,\infty,D} \leq |\boldsymbol{\alpha}|! k_{\mathbf{B}} c_{\mathbf{B}}^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}},$$

where

$$k_{\mathbf{B}} := k_{\mathbf{C}}^2 \quad \text{and} \quad c_{\mathbf{B}} := 2c_{\mathbf{C}}.$$

*Proof.* By applying Corollary 3.2 we arrive at

$$\begin{aligned}
\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} (\mathbf{C} \mathbf{C}^\top)\|_{\kappa,\infty,D} &\leq \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \|\partial_{\mathbf{y}}^{\boldsymbol{\beta}} \mathbf{C}\|_{\kappa,\infty,D} \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \mathbf{C}\|_{\kappa,\infty,D} \\
&\leq \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\boldsymbol{\beta}|! k_{\mathbf{C}} c_{\mathbf{C}}^{|\boldsymbol{\beta}|} \gamma^{\boldsymbol{\beta}} |\boldsymbol{\alpha}-\boldsymbol{\beta}|! k_{\mathbf{C}} c_{\mathbf{C}}^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|} \gamma^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\
&= k_{\mathbf{C}}^2 c_{\mathbf{C}}^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\boldsymbol{\beta}|! |\boldsymbol{\alpha}-\boldsymbol{\beta}|!
\end{aligned}$$

The following identity with its obvious bound, see [21],

$$\sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} |\boldsymbol{\alpha}-\boldsymbol{\beta}|! |\boldsymbol{\beta}|! = (|\boldsymbol{\alpha}|+1)! \leq 2^{|\boldsymbol{\alpha}|} |\boldsymbol{\alpha}|!,$$

lead to the assertion.  $\square$

**Lemma 3.8.** *We have for all  $\boldsymbol{\alpha} \in \mathbb{N}^M$  that*

$$\|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \hat{\mathbf{A}}\|_{\kappa,\infty,D} \leq |\boldsymbol{\alpha}|! k_{\hat{\mathbf{A}}} c_{\hat{\mathbf{A}}}^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}} \quad \text{and} \quad \|\partial_{\mathbf{y}}^{\boldsymbol{\alpha}} \hat{f}\|_{\kappa-1,2,D} \leq |\boldsymbol{\alpha}|! k_{\hat{f}} c_{\hat{f}}^{|\boldsymbol{\alpha}|} \gamma^{\boldsymbol{\alpha}},$$

where

$$k_{\hat{\mathbf{A}}} := k_{\mathbf{B}} k_b, \quad c_{\hat{\mathbf{A}}} := 2 \max\{c_{\mathbf{B}}, c_b\}, \quad k_{\hat{f}} := k_g k_b \quad \text{and} \quad c_{\hat{f}} := 2 \max\{c_g, c_b\}.$$

*Proof.* The arguments are analogous to the ones for Lemma 3.7.  $\square$

**3.3. Parametric regularity of the solution.** For this subsection, we require an elliptic regularity result, which we state as an assumption:

**Assumption 3.9.** Let  $\mathcal{R}_\kappa$  be a Banach space with norm  $\|\cdot\|_{\mathcal{R}_\kappa}$  such that, for all

$$\mathbf{A} \in W^{\kappa, \infty}(D; \mathbb{R}^{d \times d}) \cap \mathcal{R}_\kappa$$

that fulfil (2.2), we have that the problem of solving

$$\left( \mathbf{A} \nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} v \right)_{L^2(D; \mathbb{R}^d)} = (h, v)_{L^2(D)}$$

for any  $h \in H^{\kappa-1}(D)$  has a unique solution  $u \in H_0^1(D)$ , which also lies in  $H^{\kappa+1}(D)$ , with

$$\|u\|_{\kappa+1, 2, D} \leq C_{\kappa, er}(D, \|\mathbf{A}\|_{\mathcal{R}_\kappa}) \|h\|_{\kappa-1, 2, D},$$

where  $C_{\kappa, er}$  only depends on  $D$  and continuously on  $\|\mathbf{A}\|_{\mathcal{R}_\kappa}$ .

Such an elliptic regularity estimate for example is known for  $\kappa = 1$ , when the domain is convex and bounded and  $\mathcal{R}_\kappa = C^{0,1}(\bar{D}; \mathbb{R}^{d \times d})$ , see [14, Propositions 3.2.1.2 and 3.1.3.1]. The elliptic regularity estimate is also known to hold for  $\kappa \geq 1$  and  $d = 2$ , when the domain's boundary is smooth and  $\mathcal{R}_\kappa = W^{\kappa, \infty}(D; \mathbb{R}^{d \times d})$ , see [5].

We assume from here on that  $\hat{\mathbf{A}}$  also lies in  $L_{\mathbb{P}_{\mathbf{y}}}^\infty(\square; \mathcal{R}_\kappa)$ . This assumption then directly implies the following result.

**Lemma 3.10.** The unique solution  $\hat{u} \in L_{\mathbb{P}_{\mathbf{y}}}^\infty(\square; H_0^1(D))$  of (2.1) fulfils  $\hat{u} \in L_{\mathbb{P}_{\mathbf{y}}}^\infty(\square; H^{\kappa+1}(D))$ , with

$$\|\hat{u}\|_{\kappa+1, 2, D} \leq C_{\kappa, er} \|f\|_{\kappa-1, 2, D},$$

where

$$C_{\kappa, er} := \max_{0 \leq s \leq \|\hat{\mathbf{A}}\|_{L_{\mathbb{P}_{\mathbf{y}}}^\infty(\square; \mathcal{R}_\kappa)}} C_{\kappa, er}(D, s).$$

Moreover, this higher spatial regularity also carries over to the derivatives  $\partial_{\mathbf{y}}^\alpha \hat{u}$ .

**Theorem 3.11.** For almost every  $\mathbf{y} \in \square$ , the derivatives of the solution  $\hat{u}$  of (2.1) satisfy

$$\begin{aligned} \|\partial_{\mathbf{y}}^\alpha \hat{u}\|_{\kappa+1, 2, D} &\leq |\alpha|! \max\{2, 3C_{er}\kappa^2 d^2 k_{\hat{\mathbf{A}}}, 3C_{er} k_{\hat{f}}\} \\ &\quad \left( \max\{2, 3C_{er}\kappa^2 d^2 k_{\hat{\mathbf{A}}}, 3C_{er} k_{\hat{f}}\} \max\{c_{\hat{f}}, c_{\hat{\mathbf{A}}}\} \right)^{|\alpha|} \gamma^\alpha. \end{aligned}$$

*Proof.* By differentiation of the variational formulation (2.1) with respect to  $\mathbf{y}$  we arrive, for arbitrary  $v \in H_0^1(D)$ , at

$$\left( \partial_{\mathbf{y}}^\alpha (\hat{\mathbf{A}} \nabla_{\mathbf{x}} \hat{u}), \nabla_{\mathbf{x}} \hat{v} \right)_{L^2(D; \mathbb{R}^d)} = \left( \partial_{\mathbf{y}}^\alpha \hat{f}, \hat{v} \right)_{L^2(D; \mathbb{R})}.$$

Applying the Leibniz rule on the left-hand side yields

$$\left( \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \partial_{\mathbf{y}}^{\beta} \nabla_{\mathbf{x}} \hat{u}, \nabla_{\mathbf{x}} \hat{v} \right)_{L^2(D; \mathbb{R}^d)} = \left( \partial_{\mathbf{y}}^{\alpha} \hat{f}, \hat{v} \right)_{L^2(D; \mathbb{R})}.$$

Then, by rearranging and using the linearity of the gradient, we find

$$\begin{aligned} \left( \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\alpha} \hat{u}, \nabla_{\mathbf{x}} \hat{v} \right)_{L^2(D; \mathbb{R}^d)} &= \left( \sum_{\beta < \alpha} \binom{\alpha}{\beta} \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u}, \nabla_{\mathbf{x}} \hat{v} \right)_{L^2(D; \mathbb{R}^d)} \\ &\quad + \left( \partial_{\mathbf{y}}^{\alpha} \hat{f}, \hat{v} \right)_{L^2(D; \mathbb{R})}. \end{aligned}$$

Using Green's identity, we can then write

$$\begin{aligned} \left( \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\alpha} \hat{u}, \nabla_{\mathbf{x}} \hat{v} \right)_{L^2(D; \mathbb{R}^d)} &= \left( \sum_{\beta < \alpha} \binom{\alpha}{\beta} \operatorname{div}_{\mathbf{x}} \left( \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u} \right) + \partial_{\mathbf{y}}^{\alpha} \hat{f}, \hat{v} \right)_{L^2(D; \mathbb{R})}. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} \|\partial_{\mathbf{y}}^{\alpha} \hat{u}\|_{\kappa+1,2,D} &\leq C_{er} \left\| \partial_{\mathbf{y}}^{\alpha} \hat{f} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \operatorname{div}_{\mathbf{x}} \left( \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u} \right) \right\|_{\kappa-1,2,D} \\ &\leq C_{er} \left( \|\partial_{\mathbf{y}}^{\alpha} \hat{f}\|_{\kappa-1,2,D} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left\| \operatorname{div}_{\mathbf{x}} \left( \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u} \right) \right\|_{\kappa-1,2,D} \right) \\ &\leq C_{er} \left( \|\partial_{\mathbf{y}}^{\alpha} \hat{f}\|_{\kappa-1,2,D} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \kappa d \left\| \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u} \right\|_{\kappa,2,D} \right) \\ &\leq C_{er} \left( \|\partial_{\mathbf{y}}^{\alpha} \hat{f}\|_{\kappa-1,2,D} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \kappa d \left\| \partial_{\mathbf{y}}^{\alpha-\beta} \hat{\mathbf{A}} \right\|_{\kappa,\infty,D} \left\| \nabla_{\mathbf{x}} \partial_{\mathbf{y}}^{\beta} \hat{u} \right\|_{\kappa,2,D} \right) \\ &\leq C_{er} \left( |\alpha|! k_{\hat{f}} c_{\hat{f}}^{|\alpha|} \gamma^{\alpha} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} |\alpha - \beta|! \kappa^2 d^2 k_{\hat{\mathbf{A}}} c_{\hat{\mathbf{A}}}^{|\alpha-\beta|} \gamma^{\alpha-\beta} \left\| \partial_{\mathbf{y}}^{\beta} \hat{u} \right\|_{\kappa+1,2,D} \right) \end{aligned}$$

from which we derive

$$\|\partial_{\mathbf{y}}^{\alpha} \hat{u}\|_{\kappa+1,2,D} \leq \frac{k}{3} |\alpha|! c^{|\alpha|} \gamma^{\alpha} + \frac{k}{3} \sum_{\beta < \alpha} \binom{\alpha}{\beta} |\alpha - \beta|! c^{|\alpha-\beta|} \gamma^{\alpha-\beta} \|\partial_{\mathbf{y}}^{\beta} \hat{u}\|_{\kappa+1,2,D},$$

where

$$k := \max \left\{ 2, 3C_{er} \kappa^2 d^2 k_{\hat{\mathbf{A}}}, 3C_{er} k_{\hat{f}} \right\}$$

and

$$c := \max \left\{ c_{\hat{f}}, c_{\hat{\mathbf{A}}} \right\}$$

We note that, by definition of  $k$ , we have  $k \geq 2$  and furthermore, because of Lemma 3.10, we also have that  $\|\hat{u}\|_{\kappa+1,2,D} \leq C_{er} k_{\hat{f}} \leq k$ , which means that the

assertion is true for  $|\alpha| = 0$ . Thus, we can use an induction over  $|\alpha|$  to prove the hypothesis

$$\|\partial_{\mathbf{y}}^{\alpha} \hat{u}\|_{\kappa+1,2,D} \leq |\alpha|! \gamma^{\alpha} k (kc)^{|\alpha|}$$

for  $|\alpha| > 0$ .

Let the assertions hold for all  $\alpha$ , which satisfy  $|\alpha| \leq n - 1$  for some  $n \geq 1$ . Then, we know for all  $\alpha$  with  $|\alpha| = n$  that

$$\begin{aligned} \|\partial_{\mathbf{y}}^{\alpha} \hat{u}\|_{\kappa+1,2,D} &\leq \frac{k}{3} |\alpha|! c^{|\alpha|} \gamma^{\alpha} + \frac{k}{3} \sum_{\beta < \alpha} \binom{\alpha}{\beta} |\alpha - \beta|! c^{|\alpha - \beta|} \gamma^{\alpha - \beta} \|\partial_{\mathbf{y}}^{\beta} \hat{u}\|_{\kappa+1,2,D} \\ &\leq \frac{k}{3} |\alpha|! c^{|\alpha|} \gamma^{\alpha} + \frac{k}{3} \sum_{\beta < \alpha} \binom{\alpha}{\beta} |\alpha - \beta|! c^{|\alpha - \beta|} \gamma^{\alpha - \beta} |\beta|! \gamma^{\beta} k (kc)^{|\beta|} \\ &\leq \frac{k}{3} |\alpha|! c^{|\alpha|} \gamma^{\alpha} + \frac{k}{3} \gamma^{\alpha} c^{|\alpha|} k \sum_{j=0}^{n-1} k^j \sum_{\substack{\beta < \alpha \\ |\beta|=j}} \binom{\alpha}{\beta} |\alpha - \beta|! |\beta|!. \end{aligned}$$

Making use of the combinatorial identity

$$\sum_{\substack{\beta < \alpha \\ |\beta|=j}} \binom{\alpha}{\beta} |\alpha - \beta|! |\beta|! = |\alpha|!,$$

see [21], yields

$$k \sum_{j=0}^{n-1} k^j \sum_{\substack{\beta < \alpha \\ |\beta|=j}} \binom{\alpha}{\beta} |\alpha - \beta|! |\beta|! = |\alpha|! k \sum_{j=0}^{n-1} k^j = |\alpha|! k \frac{k^{|\alpha|}}{k-1} \leq |\alpha|! 2k^{|\alpha|},$$

as  $k \geq 2$  implies that  $2(k-1) \geq k$ . Finally, we arrive at

$$\|\partial_{\mathbf{y}}^{\alpha} \hat{u}\|_{\kappa+1,2} \leq \frac{k}{3} |\alpha|! c^{|\alpha|} \gamma^{\alpha} + \frac{k}{3} \gamma^{\alpha} c^{|\alpha|} |\alpha|! 2k^{|\alpha|} \leq |\alpha|! k (kc)^{|\alpha|} \gamma^{\alpha}$$

which completes the proof.  $\square$

## 4. THE COUPLING OF FEM AND BEM

**4.1. Newton potential.** For sake of simplicity in representation, we shall restrict ourselves the deterministic boundary value problem

$$(4.1) \quad -\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \Gamma := \partial D,$$

i.e., the domain  $D$  is assumed to be fixed. Of course, when applying a sampling method for (1.1), the underlying domains are always different. Finite element methods suffer then from generating a suitable triangulation for each new domain. Hence,

we shall reformulate the boundary value problem as two coupled problems involving only boundary integral equations on the free boundary. In order to resolve the inhomogeneity in (4.1), we introduce a Newton potential  $\mathcal{N}_f$  which satisfies

$$(4.2) \quad -\Delta \mathcal{N}_f = f \quad \text{in } \tilde{D}.$$

Here,  $\tilde{D}$  is a sufficiently large domain containing  $\mathfrak{D}[\omega]$  almost surely.

The Newton potential is supposed to be explicitly known like in our numerical example (see Section 6) or computed with sufficiently high accuracy. Especially, since the domain  $\tilde{D}$  can be chosen fairly simple, one can apply finite elements based on tensor products of higher order spline functions (in  $[-R, R]^d$ ) or dual reciprocity methods. Notice that the Newton potential has to be computed only once in advance.

By making the ansatz

$$(4.3) \quad u = \mathcal{N}_f + \tilde{u}$$

and setting  $\tilde{g} := g - \mathcal{N}_f$ , we arrive at the problem of seeking a harmonic function  $\tilde{u}$  which solves the following Dirichlet problem for the Laplacian

$$(4.4) \quad \Delta \tilde{u} = 0 \text{ in } D, \quad \tilde{u} = \tilde{g} \text{ on } \Gamma.$$

Now, we are able to apply the coupling of finite elements and boundary elements.

**4.2. Reformulation as a coupled problem.** For the subdomain  $B \subset D$ , we set  $\Sigma := \partial B$ , see Figure 1 for an illustration. The normal vectors  $\mathbf{n}$  at  $\Gamma$  and  $\Sigma$  are assumed to point into  $D \setminus \overline{B}$ . We shall split (4.4) in two coupled boundary value problems in accordance with

$$(4.5) \quad \begin{aligned} \Delta \tilde{u} &= 0 && \text{in } B, \\ \Delta \tilde{u} &= 0 && \text{in } D \setminus \overline{B}, \\ \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{z} \in B}} \tilde{u}(\mathbf{z}) &= \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{z} \in D \setminus \overline{B}}} \tilde{u}(\mathbf{z}) && \text{for all } \mathbf{x} \in \Sigma, \\ \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{z} \in B}} \frac{\partial \tilde{u}}{\partial \mathbf{n}}(\mathbf{z}) &= \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{z} \in D \setminus \overline{B}}} \frac{\partial \tilde{u}}{\partial \mathbf{n}}(\mathbf{z}) && \text{for all } \mathbf{x} \in \Sigma, \\ \tilde{u} &= \tilde{g} && \text{on } \Gamma. \end{aligned}$$

In order to derive suitable boundary integral equations for the problem in  $D \setminus \overline{B}$ , we define the single layer operator  $\mathcal{V}_{\Phi\Psi}$ , the double layer operator  $\mathcal{K}_{\Phi\Psi}$  and its adjoint  $\mathcal{K}_{\Psi\Phi}^*$ , and the hypersingular operator  $\mathcal{W}_{\Phi\Psi}$  with respect to the boundaries

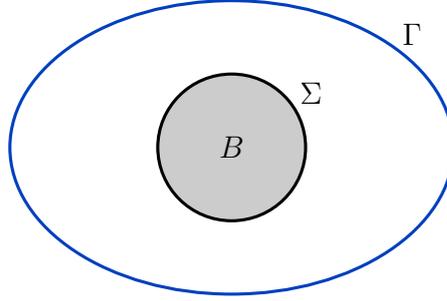


FIGURE 1. The domain  $D$ , the subdomain  $B$ , and the boundaries  $\Gamma = \partial D$  and  $\Sigma = \partial B$ .

$\Phi, \Psi \in \{\Gamma, \Sigma\}$  by

$$\left. \begin{aligned} (\mathcal{V}_{\Phi\Psi}v)(\mathbf{x}) &:= \int_{\Phi} G(\mathbf{x}, \mathbf{z})v(\mathbf{z}) \, d\sigma_{\mathbf{z}}, \\ (\mathcal{K}_{\Phi\Psi}v)(\mathbf{x}) &:= \int_{\Phi} \frac{\partial G(\mathbf{x}, \mathbf{z})}{\partial \mathbf{n}(\mathbf{z})} v(\mathbf{z}) \, d\sigma_{\mathbf{z}}, \\ (\mathcal{K}_{\Phi\Psi}^*v)(\mathbf{x}) &:= \int_{\Phi} \frac{\partial}{\partial \mathbf{n}(\mathbf{x})} G(\mathbf{x}, \mathbf{z})v(\mathbf{z}) \, d\sigma_{\mathbf{z}}, \\ (\mathcal{W}_{\Phi\Psi}v)(\mathbf{x}) &:= -\frac{\partial}{\partial \mathbf{n}(\mathbf{x})} \int_{\Phi} \frac{\partial G(\mathbf{x}, \mathbf{z})}{\partial \mathbf{n}(\mathbf{z})} v(\mathbf{z}) \, d\sigma_{\mathbf{z}}, \end{aligned} \right\} \mathbf{x} \in \Psi.$$

Here,  $G(\mathbf{x}, \mathbf{z})$  denotes the fundamental solution of the Laplacian which is given by

$$G(\mathbf{x}, \mathbf{z}) = \begin{cases} -\frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{z}\|, & d = 2, \\ \frac{1}{4\pi \|\mathbf{x} - \mathbf{z}\|}, & d = 3. \end{cases}$$

By introducing the variables  $\sigma_{\Sigma} := (\partial \tilde{u} / \partial \mathbf{n})|_{\Sigma}$  and  $\sigma_{\Gamma} := (\partial \tilde{u} / \partial \mathbf{n})|_{\Gamma}$ , the coupled system (4.5) yields the following nonlocal boundary value problem: Find  $(\tilde{u}, \sigma_{\Sigma}, \sigma_{\Gamma})$  such that

$$(4.6) \quad \begin{aligned} \Delta \tilde{u} &= 0 && \text{in } B, \\ \Delta \tilde{u} &= 0 && \text{on } \Omega \setminus \bar{B}, \\ -\mathcal{W}_{\Sigma\Sigma} \tilde{u} - \mathcal{W}_{\Gamma\Sigma} \tilde{g} + \left(\frac{1}{2} - \mathcal{K}_{\Sigma\Sigma}^*\right) \sigma_{\Sigma} - \mathcal{K}_{\Gamma\Sigma}^* \sigma_{\Gamma} &= \sigma_{\Sigma} && \text{on } \Sigma, \\ \left(\frac{1}{2} - \mathcal{K}_{\Sigma\Sigma}\right) \tilde{u} - \mathcal{K}_{\Gamma\Sigma} \tilde{g} + \mathcal{V}_{\Sigma\Sigma} \sigma_{\Sigma} + \mathcal{V}_{\Gamma\Sigma} \sigma_{\Gamma} &= 0 && \text{on } \Sigma, \\ -\mathcal{K}_{\Sigma\Gamma} \tilde{u} + \left(\frac{1}{2} - \mathcal{K}_{\Gamma\Gamma}\right) \tilde{g} + \mathcal{V}_{\Sigma\Gamma} \sigma_{\Sigma} + \mathcal{V}_{\Gamma\Gamma} \sigma_{\Gamma} &= 0 && \text{on } \Gamma. \end{aligned}$$

This system is the so-called *two integral formulation*, which is equivalent to our original model problem (4.4), see for example [9, 16].

**4.3. Variational formulation.** We next introduce the product space  $\mathcal{H} := H^1(B) \times H^{-1/2}(\Sigma) \times H^{-1/2}(\Gamma)$ , equipped by the product norm

$$\|(v, \sigma_\Sigma, \sigma_\Gamma)\|_{\mathcal{H}}^2 := \|v\|_{H^1(B)}^2 + \|\sigma_\Sigma\|_{H^{-1/2}(\Sigma)}^2 + \|\sigma_\Gamma\|_{H^{-1/2}(\Gamma)}^2.$$

Further, let  $a : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ , be the bilinear form defined by

$$(4.7) \quad a((v, \sigma_\Sigma, \sigma_\Gamma), (w, \lambda_\Sigma, \lambda_\Gamma)) = \int_B \langle \nabla v, \nabla w \rangle \, d\mathbf{x} + \left( \begin{bmatrix} w \\ \lambda_\Sigma \\ \lambda_\Gamma \end{bmatrix}, \begin{bmatrix} \mathcal{W}_{\Sigma\Sigma} & \mathcal{K}_{\Sigma\Sigma}^* - 1/2 & \mathcal{K}_{\Gamma\Sigma}^* \\ 1/2 - \mathcal{K}_{\Sigma\Sigma} & \mathcal{V}_{\Sigma\Sigma} & \mathcal{V}_{\Gamma\Sigma} \\ -\mathcal{K}_{\Sigma\Gamma} & \mathcal{V}_{\Sigma\Gamma} & \mathcal{V}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} v \\ \sigma_\Sigma \\ \sigma_\Gamma \end{bmatrix} \right)_{L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Gamma)}.$$

For sake of simplicity in representation, we omitted the trace operator in expressions like  $(w, \mathcal{W}_{\Sigma\Sigma} v)_{L^2(\Sigma)}$  etc.

Introducing the linear functional  $F : \mathcal{H} \rightarrow \mathbb{R}$ ,

$$F(w, \lambda_\Sigma, \lambda_\Gamma) = (f, w)_{L^2(B)} + \left( \begin{bmatrix} w \\ \lambda_\Sigma \\ \lambda_\Gamma \end{bmatrix}, \begin{bmatrix} -\mathcal{W}_{\Gamma\Sigma} \\ \mathcal{K}_{\Gamma\Sigma} \\ \mathcal{K}_{\Gamma\Gamma} - 1/2 \end{bmatrix} \tilde{g} \right)_{L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Gamma)},$$

the variational formulation is given by: Seek  $(\tilde{u}, \sigma_\Sigma, \sigma_\Gamma) \in \mathcal{H}$  such that

$$(4.8) \quad a((\tilde{u}, \sigma_\Sigma, \sigma_\Gamma), (w, \lambda_\Sigma, \lambda_\Gamma)) = F(w, \lambda_\Sigma, \lambda_\Gamma)$$

for all  $(w, \lambda_\Sigma, \lambda_\Gamma) \in \mathcal{H}$ . In accordance with [12], the variational formulation (4.8) admits a unique solution  $(\tilde{u}, \sigma_\Sigma, \sigma_\Gamma) \in \mathcal{H}$  for all  $F \in \mathcal{H}'$ , provided that  $D$  has a conformal radius which is smaller than one if  $d = 2$ .

**4.4. Galerkin discretization.** Since the variational formulation is stable without further restrictions, the discretization is along the lines of [17]. We first introduce a uniform triangulation of  $B$  which in turn induces a uniform triangulation of  $\Sigma$ . Moreover, we introduce a uniform triangulation of the free boundary  $\Gamma$ , which we suppose to have the same mesh size as the triangulation of the domain  $B$ . For the FEM part, we consider continuous, piecewise linear ansatz functions  $\{\varphi_k^B : k \in \Delta^B\}$  with respect to the given domain mesh. For the BEM part, we employ piecewise constant ansatz functions  $\{\psi_k^\Phi : k \in \nabla^\Phi\}$  on the respective triangulations of the boundaries  $\Phi \in \{\Sigma, \Gamma\}$ .

For sake of simplicity in representation, we set  $\varphi_k^\Sigma := \varphi_k^B|_\Sigma$  for all  $k \in \Delta^B$ . Note that most of these functions vanish except for those with nonzero trace which coincide with continuous, piecewise linear ansatz functions on  $\Sigma$ . Finally, we shall introduce the set of continuous, piecewise linear ansatz functions on the triangulation of  $\Gamma$ , which we denote by  $\{\varphi_k^\Gamma : k \in \Delta^\Gamma\}$ , where  $|\Delta^\Gamma| \sim |\nabla^\Gamma|$ .

Then, introducing the system matrices

$$(4.9) \quad \begin{aligned} \mathbf{A} &= \left[ (\nabla \varphi_{k'}^B, \varphi_k^B)_{L^2(B)} \right]_{k,k'}, & \mathbf{W}_{\Phi\Psi} &= \left[ (\mathcal{W}_{\Phi\Psi} \varphi_{k'}^\Phi, \varphi_k^\Psi)_{L^2(\Psi)} \right]_{k,k'}, \\ \mathbf{B}_\Phi &= \left[ \frac{1}{2} (\varphi_{k'}^\Phi, \psi_{j,k}^\Phi)_{L^2(\Phi)} \right]_{k,k'}, & \mathbf{K}_{\Phi\Psi} &= \left[ (\mathcal{K}_{\Phi\Psi} \varphi_{k'}^\Phi, \varphi_k^\Psi)_{L^2(\Psi)} \right]_{k,k'}, \\ \mathbf{G}_\Phi &= \left[ (\varphi_{k'}^\Phi, \varphi_{j,k}^\Phi)_{L^2(\Phi)} \right]_{k,k'}, & \mathbf{V}_{\Phi\Psi} &= \left[ (\mathcal{V}_{\Phi\Psi} \varphi_{k'}^\Phi, \varphi_{j,k}^\Psi)_{L^2(\Psi)} \right]_{k,k'}, \end{aligned}$$

where again  $\Phi, \Psi \in \{\Sigma, \Gamma\}$ , and the data vector

$$\mathbf{g} = \left[ (\tilde{g}, \varphi_k^\Gamma)_{L^2(\Gamma)} \right]_k,$$

we obtain the following linear system of equations

$$(4.10) \quad \begin{bmatrix} \mathbf{A} + \mathbf{W}_{\Sigma\Sigma} & \mathbf{K}_{\Sigma\Sigma}^T - \mathbf{B}_\Sigma^T & \mathbf{K}_{\Sigma\Gamma}^T \\ \mathbf{B}_\Sigma - \mathbf{K}_{\Sigma\Sigma} & \mathbf{V}_{\Sigma\Sigma} & \mathbf{V}_{\Sigma\Gamma} \\ -\mathbf{K}_{\Sigma\Gamma} & \mathbf{V}_{\Sigma\Gamma} & \mathbf{V}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\sigma}_\Sigma \\ \boldsymbol{\sigma}_\Gamma \end{bmatrix} = \begin{bmatrix} -\mathbf{W}_{\Sigma\Sigma} \\ \mathbf{K}_{\Gamma\Sigma} \\ \mathbf{K}_{\Gamma\Gamma} - \mathbf{B}_\Gamma \end{bmatrix} \mathbf{G}_\Gamma^{-1} \mathbf{g}.$$

We mention that  $\mathbf{G}_\Gamma^{-1} \mathbf{g}$  corresponds to the  $L^2(\Gamma)$ -orthogonal projection of the given Dirichlet data  $\tilde{g} \in H^{1/2}(\Gamma)$  onto the space of the continuous, piecewise linear ansatz functions on  $\Gamma$ . That way, we can also apply fast boundary element techniques to the boundary integral operators on the right hand side of the system (4.10) of linear equations.

The present discretization yields the following error estimate, see [12].

**Proposition 4.1.** *Let  $h$  denote the mesh size of the triangulations of  $B$  and  $\Gamma$ , respectively. We denote the solution of (4.8) by  $(\tilde{u}, \sigma_\Sigma, \sigma_\Gamma)$  and the Galerkin solution by  $(\tilde{u}_h, \sigma_{\Sigma,h}, \sigma_{\Gamma,h})$ , respectively. Then, we have the error estimate*

$$\begin{aligned} & \|(\tilde{u}, \sigma_\Sigma, \sigma_\Gamma) - (\tilde{u}_h, \sigma_{\Sigma,h}, \sigma_{\Gamma,h})\|_{H^1(B) \times H^{-1/2}(\Sigma) \times H^{-1/2}(\Gamma)} \\ & \lesssim h \left( \|(\tilde{u}, \sigma_\Sigma, \sigma_\Gamma)\|_{H^2(B) \times H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)} \right) \end{aligned}$$

uniformly in  $h$ .

**4.5. Multilevel based solution of the coupling formulation.** We shall encounter some issues on the efficient multilevel based solution of the system (4.10) of linear equations. The complexity is governed by the BEM part since the boundary element matrices are densely populated. Following [17, 18], we apply wavelet matrix compression to reduce this complexity such that the over-all complexity is governed by the FEM part. On the other hand, according to [18, 23], the Bramble-Pasciak-CG (see [3]) provides an efficient and robust iterative solver for the above saddle point system. Combining a nested iteration with the BPX preconditioner (see [4]) for the FEM part and a wavelet preconditioning (see [10, 26]) for the BEM part, we derive an asymptotical optimal solver for the above system, see [18] for the details. We

refer the reader to [18] for the details of the implementation of a similar coupling formulation.

## 5. MULTILEVEL QUADRATURE METHOD

The crucial idea of the multilevel quadrature to compute the quantity of interest (1.6) is to combine an appropriate sequence of quadrature rules for the stochastic variable with the multilevel discretization in the spatial variable. To that end, we first parametrize the quantity of interest by using (2.3) over the cube  $\square = [-1, 1]^{\mathbb{N}^*}$  and compute

$$(5.1) \quad \text{QoI}(u) = \int_{\square} \mathcal{F}(u(\cdot)) \, d\mathbb{P}_{\mathbf{y}} \approx \sum_{\ell=0}^L \mathbf{Q}_{L-\ell} \left( \mathcal{F}(u_{\ell}(\cdot)) - \mathcal{F}(u_{\ell-1}(\cdot)) \right),$$

where  $u_{\ell-1} := 0$ . Herein, for the spatial approximation, we shall use the multilevel representation from Subsection 4.5 to compute the Galerkin solution  $u_{\ell} \in H^1(B)$  on level  $\ell$  that corresponds to the step size  $h_{\ell} = 2^{-\ell}$ . For the approximation in the stochastic variable  $\mathbf{y}$ , we shall thus provide a sequence of quadrature formulae  $\{\mathbf{Q}_{\ell}\}$  for the integral

$$\int_{\square} v(\mathbf{x}, \mathbf{y}) \, d\mathbb{P}_{\mathbf{y}}$$

of the form

$$\mathbf{Q}_{\ell} v = \sum_{i=1}^{N_{\ell}} \omega_{\ell,i} v(\cdot, \boldsymbol{\xi}_{\ell,i}).$$

For our purposes, we assume that the number of points  $N_{\ell}$  of the quadrature formula  $\mathbf{Q}_{\ell}$  is chosen such that the corresponding accuracy is

$$(5.2) \quad \varepsilon_{\ell} = 2^{-\ell}, \quad \ell = 0, 1, \dots, L.$$

Since the multilevel quadrature can be interpreted as a sparse-grid approximation, cf. [19], it is known that mixed regularity results of the integrand have to be provided as derived in Section 3, compare [11, 13, 19] for example. Since the mapping  $u : \square \rightarrow H^{\kappa+1}(B)$  is analytic, we can especially apply the quasi-Monte Carlo method, the Gaussian quadrature, or the sparse grid quadrature, see e.g. [13]. Especially, in case of  $H^2$ -regularity ( $\kappa = 1$ ) and  $\mathcal{F} = u|_B$ , i.e.,  $\text{QoI}(u) = \mathbb{E}(u|_B)$ , we obtain then the error estimate

$$(5.3) \quad \left\| \mathbb{E}(u) - \sum_{\ell=0}^L \mathbf{Q}_{L-\ell} \left( u_{\ell}(\cdot) - u_{\ell-1}(\cdot) \right) \right\|_{H^1(B)} = \mathcal{O}(h_L).$$

Notice that the computational complexity of the multilevel quadrature (5.1) is considerably reduced compared to a standard single-level quadrature method which has the same accuracy, see e.g. [1, 8, 19].

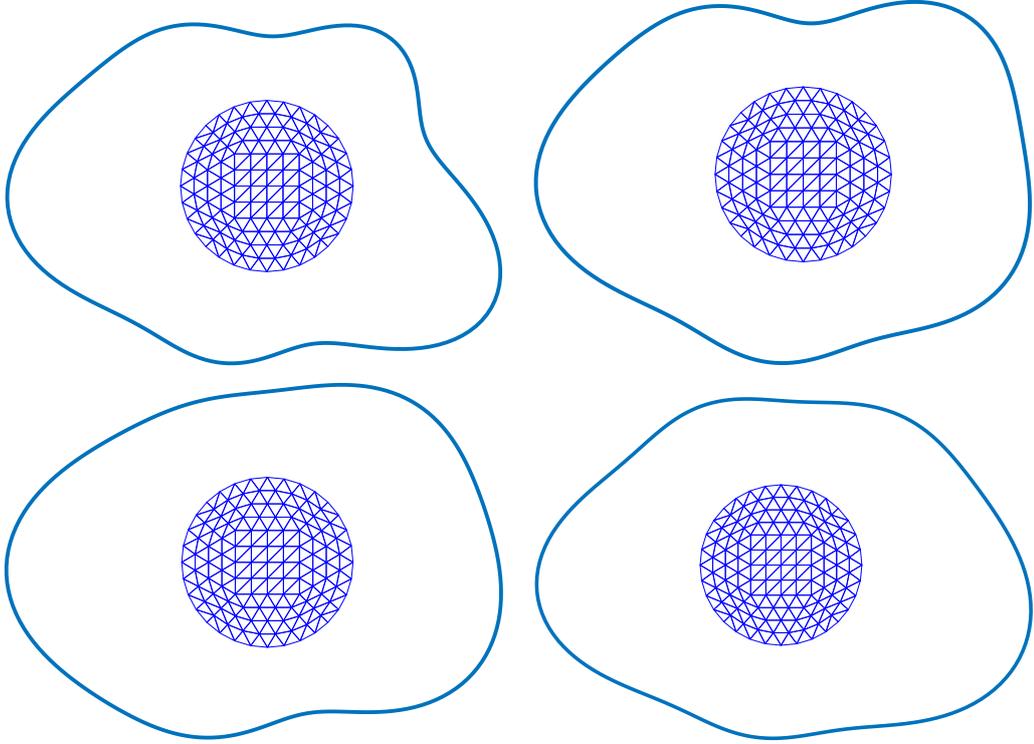


FIGURE 2. Four samples of the random domain with finite element triangulation of  $B$  on refinement level 2.

**Remark 5.1.** *By choosing the accuracy of the quadrature in accordance with  $\varepsilon_\ell = 4^{-\ell}$  for  $\ell = 0, 1, \dots, L$  instead of (5.2), the application of the Aubin-Nitsche trick in Proposition 4.1 implies the  $L^2$ -error estimate*

$$(5.4) \quad \left\| \mathbb{E}(u) - \sum_{\ell=0}^L \mathbf{Q}_{L-\ell} \left( u_\ell(\cdot) - u_{\ell-1}(\cdot) \right) \right\|_{L^2(B)} = \mathcal{O}(h_L^2).$$

## 6. NUMERICAL RESULTS

In our numerical example, we consider the reference domain  $D$  to be the ellipse with semi-axis 0.6 and 0.4. We represent its boundary by  $\gamma_{\text{ref}} : [0, 2\pi) \rightarrow \partial D$  in polar coordinates and perturb this parametrization in accordance with

$$\gamma(\omega, \varphi) = \gamma_{\text{ref}}(\varphi) + \varepsilon \sum_{i=0}^5 \{y_{-k}[\omega] \sin(k\varphi) + y_k[\omega] \cos(k\varphi)\}$$

where  $y_k \in (-0.5, 0.5)$  for all  $k = -5, -4, \dots, 5$  and  $\varepsilon = 0.05$ . The random parametrization  $\gamma[\omega]$  induces the random domain  $\mathfrak{D}[\omega]$ . The fixed subset  $B \subset D$  is given as the ball of radius 0.2, centered in the origin. For an illustration of four random draws, see Figure 2.

level	finite elements	boundary elements
1	37	32
2	129	64
3	481	128
4	1857	256
5	7297	512
6	28929	1024
7	115201	2048
8	459777	4096

TABLE 1. Number of the degrees of freedom of the finite element method and the boundary element method.

On the random domain  $\mathfrak{D}[\omega]$ , let the Poisson equation

$$-\Delta u[\omega] = 1 \text{ in } \mathfrak{D}[\omega], \quad u[\omega] = 0 \text{ on } \partial\mathfrak{D}[\omega]$$

be given. A suitable Newton potential is analytically given by  $\mathcal{N}_f = -(x_1^2 + x_2^2)/4$ . We consider the  $L^2$ -tracking type functional

$$\text{QoI}(u) = \mathbb{E} \left[ \frac{1}{2} \int_B |u[\omega] - \bar{u}|^2 \, d\mathbf{x} \right]$$

as quantity of interest, where  $\bar{u}$  is a given function. The coarse triangulation of  $B$ , based on Zlámal's curved finite elements [29], consists of 14 curved triangles on the coarse grid, which are then uniformly refined to get the triangulation on the finer grids. The 14 triangles correspond to eight piecewise linear and constant boundary elements each on the boundary  $\partial B$ . At the boundary  $\partial D$ , we likewise consider eight piecewise linear and constant boundary elements each on level 0. When applying uniform refinement, we arrive at the numbers of degrees of freedom in the finite and boundary element spaces found in Table 1.

In order to compute the quantity of interest, we will employ the quasi-Monte Carlo method based on the Halton sequence, see [15] for example. Since the exact solution is unknown, we compute first the quantity of interest on the spatial discretization level 8 by using 10 000 Halton points. Next, we compute the solution by the multilevel quasi-Monte Carlo method. Namely, for the multilevel quasi-Monte Carlo method on level  $L$ , we apply  $N_\ell = 2^{L-\ell}N_L$  and  $N_\ell = 4^{L-\ell}N_L$  Halton points, respectively, on the coarser levels  $0 \leq \ell \leq L$ , where we choose  $N_L = 10, 25, 50$  fine grid samples.

As it is seen in Figure 3, we always observe the same linear and quadratic convergence rate, respectively, but with different constants involved. Notice that linear

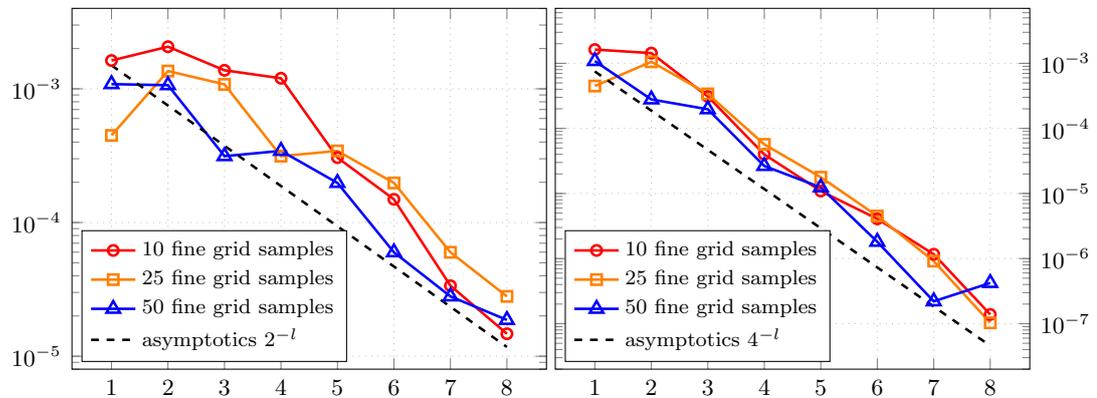


FIGURE 3. Absolute error of the output functional for different numbers of fine grid samples when increasing the number of quadrature point linearly (left) and quadratically (right) per level.

convergence is in accordance with (5.3) while quadratic convergence is in accordance with (5.4).

## 7. CONCLUSION

We provided regularity estimates of the solution to elliptic problems on random domains which allow for the application of multilevel quadrature methods. In order to avoid mesh degeneration on coarse grids, we couple finite elements with boundary elements. It has been shown by numerical experiments that this approach is indeed able to exploit the additional regularity we have in the underlying problem without causing numerical problems on too coarse grids.

## REFERENCES

- [1] A. Barth, C. Schwab, and N. Zollinger. Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numer. Math.*, 119(1):123–161, 2011.
- [2] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Models Methods Appl. Sci.*, 22(9):1250023, 2012.
- [3] J. Bramble and J. E. Pasciak. Preconditioner technique for indefinite systems resulting from mixed approximation of elliptic problems. *Math. Comput.*, 50:1–17, 1988.
- [4] J. Bramble, J. E. Pasciak, and J. Xu. Parallel multilevel preconditioners. *Math. Comput.*, 55:1–22, 1990.
- [5] C. Băcuță, H. Li, and V. Nistor. Differential operators on domains with conical points: precise uniform regularity estimates. *Rev. Roumaine de Math. Pures Appl.*, 62(3):383–411, 2017.

- [6] C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. *Numer. Math.*, 107(2):257–293, 2007.
- [7] J. E. Castrillon-Candas, F. Nobile, and R. Tempone. Analytic regularity and collocation approximation for PDEs with random domain deformations. *Comput. Math. Appl.*, 71(6):1173–1197, 2016.
- [8] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup. Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients. *Comput. Vis. Sci.*, 14(1):3–15, 2011.
- [9] M. Costabel and E. P. Stephan. Coupling of finite element and boundary element methods for an elasto-plastic interface problem. *SIAM J. Numer. Anal.*, 27:1212–1226, 1988.
- [10] W. Dahmen and A. Kunoth. Multilevel preconditioning. *Numer. Math.*, 63(3):315–344, 1992.
- [11] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov–Galerkin discretization for parametric operator equations. *SIAM J. Numer. Anal.*, 52(6):2676–2702, 2014.
- [12] K. Eppler and H. Harbrecht. Coupling of FEM and BEM in shape optimization. *Numer. Math.*, 104(1):47–68, 2006.
- [13] M. Griebel, H. Harbrecht, and M. Peters. Multilevel quadrature for elliptic parametric partial differential equations on non-nested meshes. *ArXiv e-prints arXiv:1509.09058*, 2015.
- [14] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 2011.
- [15] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numer. Math.*, 2(1):84–90, 1960.
- [16] H. Han. A new class of variational formulation for the coupling of finite and boundary element methods. *J. Comput. Math.*, 8(3):223–232, 1990.
- [17] H. Harbrecht, F. Paiva, C. Pérez, and R. Schneider. Biorthogonal wavelet approximation for the coupling of FEM-BEM. *Numer. Math.*, 92:325–356, 2002.
- [18] H. Harbrecht, F. Paiva, C. Pérez, and R. Schneider. Wavelet preconditioning for the coupling of FEM-BEM. *Numer. Linear Algebra Appl.*, 3:197–222, 2003.
- [19] H. Harbrecht, M. Peters, and M. Siebenmorgen. On multilevel quadrature for elliptic stochastic partial differential equations. *Sparse Grids and Applications*, 88:161–179, 2013.
- [20] H. Harbrecht, M. Peters, and M. Siebenmorgen. Analysis of the domain mapping method for elliptic diffusion problems on random domains. *Numer. Math.*, 134(4):823–856, 2016.
- [21] H. Harbrecht and M. Schmidlin. Multilevel methods for uncertainty quantification of elliptic PDEs with random anisotropic diffusion. *ArXiv e-prints arXiv:1706.05976*, 2018.

- [22] H. Harbrecht, R. Schneider, and C. Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numer. Math.*, 109(3):385–414, 2008.
- [23] B. Heise and M. Kuhn. Parallel solvers for linear and nonlinear exterior magnetic field problems based upon coupled FE/BE formulations. *Computing*, 56:237–258, 1996.
- [24] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups*, volume 31 of *Amer. Math. Soc. Collog. Publ.* American Mathematical Society, Providence, 1957.
- [25] P. S. Mohan, P. B. Nair, and A. J. Keane. Stochastic projection schemes for deterministic linear elliptic partial differential equations on random domains. *Int. J. Numer. Meth. Eng.*, 85(7):874–895, 2011.
- [26] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung großer vollbesetzter Gleichungssysteme*. B. G. Teubner, Stuttgart, 1998.
- [27] D.M. Tartakovsky and D. Xiu. Stochastic analysis of transport in tubes with rough walls. *J. Comput. Phys.*, 217(1):248–259, 2006.
- [28] D. Xiu and D. M. Tartakovsky. Numerical methods for differential equations in random domains. *SIAM J. Sci. Comput.*, 28(3):1167–1185, 2006.
- [29] A. Zenisek. *Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximation*. Academic Press, London, 1990.

HELMUT HARBRECHT AND MARC SCHMIDLIN, UNIVERSITÄT BASEL, DEPARTEMENT  
MATHEMATIK UND INFORMATIK, SPIEGELGASSE 1, 4051 BASEL, SCHWEIZ

*E-mail address:* {helmut.harbrecht,marc.schmidlin}@unibas.ch

## LATEST PREPRINTS

- | <b>No.</b> | <b>Author:</b> <i>Title</i>  |
|------------|--|
| 2016-31    | <b>M. Dambrine, H. Harbrecht, M. D. Peters, B. Puig</b><br><i>On Bernoulli's free boundary problem with a random boundary</i>  |
| 2016-32    | <b>H. Harbrecht, J. Tausch</b><br><i>A fast sparse grid based space-time boundary element method for the nonstationary heat equation</i>   |
| 2016-33    | <b>S. Iula</b><br><i>A note on the Moser-Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one</i>   |
| 2016-34    | <b>C. Bürli, H. Harbrecht, P. Odermatt, S. Sayasone, N. Chitnis</b><br><i>Mathematical analysis of the transmission dynamics of the liver fluke, <i>Opisthorchis viverrini</i></i> |
| 2017-01    | <b>J. Dölz and T. Gerig, M. Lüthi, H. Harbrecht and T. Vetter</b><br><i>Efficient computation of low-rank Gaussian process models for surface and image registration</i>           |
| 2017-02    | <b>M. J. Grote, M. Mehlin, S. A. Sauter</b><br><i>Convergence analysis of energy conserving explicit local time-stepping methods for the wave equation</i>                         |
| 2017-03    | <b>Y. Bilu, F. Luca, D. Masser</b><br><i>Collinear CM-points</i>   |
| 2017-04    | <b>P. Zaspel</b><br><i>Ensemble Kalman filters for reliability estimation in perfusion inference</i>   |
| 2017-05    | <b>J. Dölz and H. Harbrecht</b><br><i>Hierarchical Matrix Approximation for the Uncertainty Quantification of Potentials on Random Domains</i>                                     |
| 2017-06    | <b>P. Zaspel</b><br><i>Analysis and parallelization strategies for Ruge-Stüben AMG on many-core processor</i>  |
| 2017-07    | <b>H. Harbrecht and M. Schmidlin</b><br><i>Multilevel Methods for Uncertainty Quantification of Elliptic PDEs with Random Anisotropic Diffusion</i>                                |
| 2017-08    | <b>M. Griebel and H. Harbrecht</b><br><i>Singular value decomposition versus sparse grids: Refined complexity Estimates</i>  |

## LATEST PREPRINTS

- | No.     | Author: Title   |
|---------|---|
| 2017-09 | <b>J. Garcke and I. Kalmykov</b><br><i>Efficient Higher Order Time Discretization Schemes for Hamilton-Jacobi-Bellman Equations Based on Diagonally Implicit Symplectic Runge-Kutta Methods</i> |
| 2017-10 | <b>M. J. Grote and U. Nahum</b><br><i>Adaptive Eigenspace Regularization For Inverse Scattering Problems</i>  |
| 2017-11 | <b>J. Dölz, H. Harbrecht, S. Kurz, S. Schöps and F. Wolf</b><br><i>A Fast Isogeometric BEM for the Three Dimensional Laplace- and Helmholtz Problems</i>  |
| 2017-12 | <b>P. Zaspel</b><br><i>Algorithmic patterns for <math>H</math>-matrices on many-core processors</i>   |
| 2017-13 | <b>R. Brügger, R. Croce and H. Harbrecht</b><br><i>Solving a free boundary problem with non-constant coefficients</i>   |
| 2017-14 | <b>M. Dambrine, H. Harbrecht and B. Puig</b><br><i>Incorporating knowledge on the measurement noise in electrical impedance tomography</i>  |
| 2017-15 | <b>C. Bürli, H. Harbrecht, P. Odermatt, S. Sayasone and N. Chitnis</b><br><i>Analysis of Interventions against the Liver Fluke, <i>Opisthorchis viverrini</i></i>                               |
| 2017-16 | <b>D. W. Masser</b><br><i>Abcological anecdotes</i>   |
| 2017-17 | <b>P. Corvaja, D. W. Masser and U. Zannier</b><br><i>Torsion hypersurfaces on abelian schemes and Betti coordinates</i>   |
| 2017-18 | <b>F. Caubet, M. Dambrine and H. Harbrecht</b><br><i>A Newton method for the data completion problem and application to obstacle detection in Electrical Impedance Tomography</i>               |
| 2018-01 | <b>H. Harbrecht and P. Zaspel</b><br><i>On the algebraic construction of sparse multilevel approximations of elliptic tensor product problems</i>   |
| 2018-02 | <b>F. Ghiraldin and X. Lamy</b><br><i>Optimal Besov differentiability for entropy solutions of the eikonal equation</i>   |
| 2018-03 | <b>H. Harbrecht and M. Schmidlin</b><br><i>Multilevel quadrature for elliptic problems on random domains by the coupling of FEM and BEM</i>   |