OPTIMAL BESOV DIFFERENTIABILITY FOR ENTROPY SOLUTIONS OF THE EIKONAL EQUATION

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Abstract. In this paper we study the Eikonal equation in a bounded planar domain. We prove the equivalence among optimal Besov regularity, the finiteness of every entropy production and the validity of a kinetic formulation.

1. Introduction

In this paper we study the Eikonal equation in a planar domain: given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ we consider solutions $m : \Omega \to \mathbb{R}^2$ to the following constrained equation

$$|m| = 1 \quad \text{ a.e. in } \Omega, \qquad \nabla \cdot m = 0 \quad \text{ in } \mathcal{D}'(\Omega). \tag{M}$$

The eikonal equation (M) is very flexible, and uniqueness or regularity cannot be expected for such weak solutions, even imposing a boundary datum (the equation, on simply connected domains, is equivalent to solving $|\nabla u| = 1$ for a scalar function u, for which for instance the theory of viscosity solutions singles out a distinguished subclass of solutions).

On the other hand, solutions to (M) coming from physical models usually possess extra information that limit this flexibility. This equation emerges in the description of several physical phenomena, collectively called line-energy Ginzburg-Landau models, that describe for instance smectic liquid crystals, soft ferromagnetic films, blister formations, and broadly speaking phase transition phenomena where the order parameter is a gradient [22].

For example one can consider the Aviles-Giga energy

$$AG_\varepsilon(u_\varepsilon, \Omega) := \int_\Omega \varepsilon |\nabla^2 u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - |\nabla u_\varepsilon|^2)^2 dx, \qquad \Omega \subset \mathbb{R}^d \tag{AG}$$

(with appropriate boundary conditions): this energy has been introduced in [5] to study liquid crystal configurations, and in the two dimensional case was considered by Gioia and Ortiz as a model energy for the deformation of thin film blisters undergoing biaxial compression [37]. The functional (AG) can be thought of as a vectorial Modica Mortola energy, where the fields are forced to be gradients; equivalently, $\varepsilon AG_\varepsilon$ can be seen as a singular perturbation (accounting for the bending energy of the film) of the elastic energy. Competition between these two terms favors concentration along the jump discontinuities of the limit gradient ∇u, with a limit energy believed to be asymptotically

$$\frac{1}{3} \int_{Jump(\nabla u)} \left| \nabla^+ u - \nabla^- u \right|^3 d\mathcal{H}^1.$$

Solutions to (M) can be obtained from sequences (u_ε) with equibounded energy $AG_\varepsilon(u_\varepsilon) \leq E$, by setting

$$m_\varepsilon := \nabla^\perp u_\varepsilon, \qquad m_\varepsilon \to m, \tag{1.1}$$

and observing that any pointwise limit m satisfies the unitary constraint $|m| = 1$ as well as the linear constraint $0 = \nabla \cdot \nabla^\perp u_\varepsilon \to \nabla \cdot m$ (see [2, 22] for the precise compactness statements).

In [33], Jin and Kohn studied the energy AG_ε and its variations (under suitably boundary conditions), and discovered that the divergences $\nabla \cdot \Sigma(\nabla u_\varepsilon)$ of suitable vectorial renormalizations of the gradient fields ∇u_ε are measures providing nontrivial asymptotic lower bounds for $AG_\varepsilon(u_\varepsilon)$. The explicit form of $\Gamma - \lim_\varepsilon AG_\varepsilon$ and its domain have been subject to intensive study, see [2, 6, 22, 13], where partial results on the $\Gamma - \liminf$ already conjectured in [5, 33] have been obtained.

Unit vector fields m obtained through the limit procedure (1.1) enjoy further regularity properties: they are entropy solutions. After recognizing that (M) can be interpreted as a perturbation of Burgers' equation, in [22] the parallel between these vectorial renormalizations of the eikonal equation and entropy solutions of scalar conservation laws was pushed forward, and a family ENT of entropies $\Phi\colon S^1 \to \mathbb{R}^2$ such that $\nabla \cdot \Phi(m)$ detect the singularities of m, has been singled out. It became therefore natural to study *entropy solutions* to (M), namely vector fields satisfying the further property that

$$\nabla \cdot \Phi(m) \in \mathcal{M}(\Omega) \quad \forall \Phi \in ENT,$$

where $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω. As in the case of hyperbolic conservation laws, such additional informations imply further regularity and compactness of the set of solutions, in the spirit of Tartar's compensated compactness [44].

A quantitative statement of compactness, in the form of fractional differentiability was afterwards obtained, only for solutions of (M) specifically arising as limits of $\nabla^\perp u_\varepsilon$ (1.1), by Jabin and Perthame in [31]. They prove that such solutions satisfy a kinetic formulation: the equilibrium function (Maxwellian)

$$\chi(x,\xi) = \mathbb{1}_{m(x)\cdot\xi>0},$$

defined for $\xi \in \mathbb{R}^2$, solves a transport equation of the form

$$\xi \cdot \nabla_x \chi(x,\xi) = \partial_{\xi_1}\sigma_1 + \partial_{\xi_2}\sigma_2 + \sigma_3, \tag{1.2}$$

for some locally finite measures σ_ℓ, see [31, Theorem 1.1]. Recall that in the realm of scalar conservation law, the validity of a kinetic formulation is equivalent to the finiteness of all entropy productions [35, 39]. With the help of methods coming from velocity averaging, the authors of [31] are able to prove that such solutions possess some fractional differentiability: $m \in W^{\frac{1}{5}-,\frac{5}{3}-}_{\mathrm{loc}}$; better Sobolev regularity $W^{\frac{1}{3}-,\frac{3}{2}-}_{\mathrm{loc}}$ was established by the same authors in a subsequent work [32]. Examples by De Lellis and Westdickenberg [21] show that this regularity is optimal in the number of derivatives (1/3) but leave room for improving the integrability.

Similar results hold for weak solutions of Burgers' equation $\partial_t u + \partial_x \frac{1}{2}u^2 = 0$ whose entropy productions are finite measures (but may change sign). This should come as no surprise since Burgers' equation formally arises when considering solutions of (M) which are small perturbations of the constant solution $m_0 = (1,0)$ (see e.g. the discussion in [38, p.143]). In the case of Burgers' equation, solutions with finite entropy production are shown by Golse and Perthame [26] to lie in $B^{1/3}_{3,\infty}$, which is the optimal regularity according to [21].

We wish to mention also a similar model arising in the theory of micromagnetism and studied by Rivière and Serfaty in [42, 43]. There, solutions of (M) also appear as limits of sequences with bounded energy depending on a parameter ε, and they enjoy a kinetic formulation. In that model the unit constraint is imposed already at the ε level, thus enforcing a topological restriction, while the divergence free condition is only reached in the limit, via the penalization of a nonlocal term. This feature makes the limit problem quite different from ours (motivated by the Aviles-Giga functional): there, the field m_ε possesses an H^1 lifting $e^{i\varphi}$ (excluding vortices at the ε level - Bloch lines), that enables the use of a convenient family of entropies which control jumps of the angle φ. For this model, a quite thorough study of the rectifiability properties of entropy solutions has been carried out in [4]. Similar rectifiability properties were then obtained for the present model [18] and for higher dimensional scalar conservation laws [19, 15]. An interesting and more sophisticated model describing (almost horizontal) micromagnetism in three dimensions, exhibiting different types of transition layers (one and two dimensional Néel walls and Bloch lines) has been considered in [1].

Another distinguished subset of solutions to (M) are the so-called zero energy states, for which the field m is again as in (1.1) with the additional property that $\lim_\varepsilon AG_\varepsilon(u_\varepsilon) = 0$. Such solutions have no entropy production: $\nabla \cdot \Phi(m) = 0$ for all $\Phi \in ENT$. This yields stronger regularity and rigidity properties, as shown by Jabin, Otto and Perthame [30]: m is locally Lipschitz outside a locally finite set of points (the *vortices*, that asymptotically carry no energy), and in any convex neighborhood of one of them (say p), it holds $m(x) = \pm\frac{(x-p)^\perp}{|x-p|}$ (see also [9] for similar results in higher dimensions). Recently Lorent and Peng [36] showed that the vanishing of only two particular entropy productions (instead of all $\Phi \in ENT$) is needed to obtain this conclusion. An indication on the minimal regularity of m needed to trigger such an improvement was further studied by De Lellis and Ignat in [17], where it is proved that if $m \in W^{\frac{1}{3},3}_{\mathrm{loc}}$ then there is no entropy production. The $\frac{1}{3}$ differentiability exponent seems to be somehow critical in several problems, notably the problem of energy conservation for the Euler equations (Onsager conjecture) [12, 11, 20, 29, 10, 23].

In this article we prove the following (see Theorem 2.6 and Section 3 for the precise definitions):

Theorem. *Let m satisfy* (M). *The following three conditions are equivalent:*

(i) *m has locally finite entropy production;*
(ii) *m satisfies a kinetic formulation;*
(iii) $m \in B^{1/3}_{3,\infty,\mathrm{loc}}(\Omega)$.

This Theorem improves the previous literature in several aspects: the kinetic formulation is deduced from the mere knowledge that all entropy productions are finite, instead of the stronger requirement that m be the limit of an Aviles-Giga sequence (1.1). Whether or not the latter is *strictly* stronger is a nontrivial and, to the authors' knowledge, open question (related to the upper Γ-limit of AG_ε – what can be checked by estimating the energy of a convolution is that maps $m \in B^{1/2}_{2,\infty} \cap B^{1/4}_{4,\infty} \subsetneq B^{1/3}_{3,\infty}$ are limits of Aviles-Giga sequences, see [41]). Moreover our kinetic formulation (see (KIN) below) takes a simpler form than (1.2).

The fractional differentiability $B^{1/3}_{3,\infty}$ that we deduce from the kinetic formulation entails improved integrability compared to the previous known one [30]. As already mentioned, the corresponding result for Burgers' equation is due to Golse and Perthame [26]. Their proof relies on a kinetic formulation in which the equilibrium function χ satisfies some monotonicity assumption. This monotonicity is not present in our case, which requires substantial modification of their method.

This fractional differentiability is necessary *and sufficient* (hence optimal). Moreover our calculations also show that slightly better summability (e.g. $m \in B^{1/3}_{3,q}$ for some $q < \infty$, see § 4) already triggers the aforementioned enhanced regularity (m locally Lipschitz outside a discrete set). This criticality of $B^{1/3}_{3,\infty}$ is due to the commutator estimates employed in the argument: similarly to the case of Euler equations, “energy conservation” for functions with slightly better differentiability properties can be proved [17, 12].

The proof of the Besov regularity from the kinetic formulation (implication (ii)⇒(iii) in the Theorem) employs an *interaction estimate* due to Varadhan [45], that was used in [26] and in [24]: as in those works we build a quantity $\Delta(x,z)$ which depends on the equilibrium function $\chi(x,\xi)$ and which controls the cubic increment $|m(x+z)-m(x)|^3$. The above-mentioned interaction estimate, together with the kinetic formulation, provides an upper bound on $\int_\Omega \Delta(x,z)dx$ in terms of $|z|$, hence the Besov regularity. To prove (i)⇒(ii), i.e. the validity of a kinetic formulation from the knowledge of having finite entropy production, we employ a Banach-Steinhaus argument as in [19, 8]. The other implication (iii)⇒(i) follows from a careful integration by compensation inspired by [17].

After proving the above Theorem, we explore several questions that come up naturally. As already mentioned, in the model studied by Rivière and Serfaty in [42, 43], the solutions of (M) that arise can be written as $m = e^{i\varphi}$ with some control on the lifting φ. Analogues of our entropy productions and kinetic formulation play a crucial role, and the kinetic defect measure (which is linked to the kinetic formulation) provides a *sharp* lower bound for the energy [43]. We show that the corresponding property is *not* present in our case.

A second natural question regards the set of entropies necessary to obtain the Besov regularity. Lorent and Peng prove in [36] that the vanishing of only two particular entropy productions is enough to force all the entropy productions to vanish: is the mere finiteness of these two particular entropy productions also enough to ensure the optimal regularity? We are unable to fully answer this question, but adapting some arguments in [25] we do obtain some (lower) regularity.

A last natural question concerns the validity of global regularity estimates: when the entropy productions are finite in the whole Ω, can we obtain Besov estimates up to the boundary? We present some results in this direction.

The article is structured as follows: after some preliminary notations in the next Section 2, in Section 3 we state and prove the main Theorem, and in Section 4 we gather some results related to the zero energy states and to the above mentioned further natural questions.

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2. Notations and statement of the problem

Given $\xi, \eta \in \mathbb{R}^2$ we identify them with complex number in order to define their scalar and vector products:

$$\bar{\xi}\eta = \xi \cdot \eta + i\xi \wedge \eta.$$

Equivalently: $\xi \cdot \eta = \xi_1\eta_1 + \xi_2\eta_2$, $\xi \wedge \eta = \xi_1\eta_2 - \xi_2\eta_1$. For the sake of clarity we will not always identify unit complex numbers with rotations of the plane: in such occasions, rotations by an angle θ will be denoted by $R_\theta \in SO(2)$.

We will measure the smoothness of our unit fields m in the scale of Besov spaces on a domain: in order to keep the notation light, we give the definition only for the exponents we need; we refer the reader to [46] for an overview of the definitions; see also [7]. If $f : \Omega \to \mathbb{R}$ and $z \in \mathbb{R}^n$, we define $D^z f(x)$ to be the increment $f(x+z) - f(x)$ if both $x, x+z \in \Omega$, and zero otherwise.

Definition 2.1 (Besov spaces on domains, [46, Theorem 1.118]). Let Ω be a bounded Lipschitz domain and $1 \le q \le \infty$. A function $f : \Omega \to \mathbb{R}$ belongs to $B^{\frac{1}{3}}_{3,q}(\Omega)$ if

$$\|f\|_{L^3(\Omega)} + \left(\int_0^1 \left(t^{-\frac{1}{3}} \sup_{|z|\le t} \|D^z f\|_{L^3(\Omega)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty.$$

The expression on the left hand side provides a quasi-norm. Moreover we denote with $B^{\frac{1}{3}}_{3,q,\mathrm{loc}}(\Omega) = \bigcap_{U\subset\subset\Omega} B^{\frac{1}{3}}_{3,q}(U)$.

It will be convenient to denote, for $U \subset \Omega$,

$$N_t(f, U) := \sup_{|z|\le t} t^{-\frac{1}{3}} \|D^z f\|_{L^3(U)} :$$

with this quantity we can also define the distinguished subspace, lying between $B^{\frac{1}{3}}_{3,q}$ with $q < \infty$ and $B^{\frac{1}{3}}_{3,\infty}$:

$$B^{\frac{1}{3}}_{3,c_0}(\Omega) = B^{\frac{1}{3}}_{3,\infty}(\Omega) \cap \{f : \lim_{t\to 0} N_t(f, \Omega) = 0\}.$$

In order to detect and describe the singularities of solutions of equation (M), it is customary to test the equation on suitable renormalization of the solution (see [22, 28, 17]):

Definition 2.2. A function $\Phi \in C^2(S^1, \mathbb{R}^2)$ is an entropy for the equation (M) if

$$e^{it} \cdot \frac{d}{dt}\left[\Phi(e^{it})\right] = 0 \qquad \forall t \in \mathbb{R}.$$

The set of all entropies is denoted by ENT.

This definition is designed so that any smooth unit field m solving (M) satisfies $\nabla\cdot\Phi(m) = 0$ for $\Phi \in ENT$. In contrast, if m has only bounded variation, $\nabla \cdot \Phi(m)$ will be a measure concentrated on the jump set of m, called the entropy production associated to Φ. In other words, BV-type jump discontinuities of m are detected by such divergences: already in [33], in the context of the Aviles-Giga functional (AG), a special family of "cubic" entropies $\Sigma_{\alpha_1,\alpha_2}$ were introduced, depending on a chosen orthonormal frame of coordinates (α_1, α_2):

$$\Sigma_{\alpha_1,\alpha_2}(z) = \frac{4}{3}\left((z\cdot\alpha_2)^3\alpha_1 + (z\cdot\alpha_1)^3\alpha_2\right). \tag{2.1}$$

The maps $\Sigma_{\alpha_1,\alpha_2}$ are easily seen to belong to ENT. The divergences $\nabla\cdot\Sigma_{\alpha_1,\alpha_2}(m) = \nabla\cdot\Sigma_{\alpha_1,\alpha_2}(\nabla^\perp u)$ of these entropies detect the jump discontinuities of ∇u, according to the relative orientation of the discontinuity set $J_{\nabla u}$ with respect to the chosen frame (α_1, α_2). An optimization procedure over the frame bundle provides the lower bound

$$\Gamma - \liminf_{\varepsilon} AG_\varepsilon(u) \ge \frac{1}{3}\int_{J_{\nabla u}} \left|\nabla^+ u - \nabla^- u\right|^3 d\mathcal{H}^1, \tag{2.2}$$

(in the $W^{1,3}$ topology) at functions u such that $\nabla u \in BV(\Omega)$ and $|\nabla u| = 1$ almost everywhere. Here $J_{\nabla u}$ is the jump set of the gradient and $\nabla^\pm u$ are its traces on $J_{\nabla u}$, see [2]. The cubic power of the jump appearing in (2.2) hints at the Besov scale $B^{\frac{1}{3}}_{3,q}$ we are considering here. For functions u with $\nabla u \in BV$, the right-hand side of (2.2) can be conveniently expressed in terms of the entropy productions, since it holds

$$\frac{1}{3}\left|\nabla^+ u - \nabla^- u\right|^3 \mathcal{H}^1 \llcorner J_{\nabla u} = \bigvee_{(\alpha_1,\alpha_2)} \|\nabla\cdot\Sigma_{\alpha_1,\alpha_2}(u)\|.$$

This is proved in [2, Theorem 3.8] (see also [28]). Here $\|\mu\|$ denotes the total variation measure of a complex-valued measure μ, and the symbol $\bigvee$ denotes the least upper bound of a family of measures [3, Definition 1.68]:

$$\bigvee_{\alpha\in A}\mu_\alpha(E) := \sup\left\{\sum_{\{\alpha'\}\subset A}\mu_{\alpha'}(E_{\alpha'}) : \{E_{\alpha'}\}\text{ pairwise disjoint, } E=\bigcup_{\alpha'}E_{\alpha'}\right\}.$$

Hence the estimate (2.2) provides a control of the entropy production associated to the cubic entropies (2.1) by the Aviles-Giga energy. In fact for any entropy $\Phi\in ENT$ it is shown in [22] that limits m of sequences $m_\varepsilon=\nabla^\perp u_\varepsilon$ with $AG_\varepsilon(u_\varepsilon)\leq M$ satisfy

$$\|\nabla\cdot\Phi(m)\|(\Omega)\leq C\|D^2\Phi\|_\infty \liminf_{\varepsilon\to 0} AG_\varepsilon(u_\varepsilon).$$

In particular all the entropy productions are finite measures. This motivates the following

Definition 2.3. We say that a vector field m solving (M) has *locally finite weak entropy production* in Ω if for every $\Phi\in ENT$ we have

$$\nabla\cdot\Phi(m)\in\mathcal{M}_{\mathrm{loc}}(\Omega). \tag{wFEP}$$

If furthermore

$$\bigvee_{\Phi\in ENT,\,\|D^2\Phi\|_\infty\leq 1}\|\nabla\cdot\Phi(m)\|\in\mathcal{M}_{\mathrm{loc}}(\Omega), \tag{sFEP}$$

we say that m has *locally finite strong entropy production* in Ω.

Remark 2.4. Limits of sequences $m_\varepsilon=\nabla^\perp u_\varepsilon$ with $AG_\varepsilon(u_\varepsilon)\leq M$ satisfy (sFEP).

Definition 2.5. We say that a vector field m solving (M) satisfies the kinetic formulation if there exists a Radon measure $\sigma\in\mathcal{M}_{\mathrm{loc}}(\Omega\times\mathbb{R}/2\pi\mathbb{Z})$ such that

$$e^{is}\cdot\nabla_x\mathbb{1}_{e^{is}\cdot m(x)>0}=\partial_s\sigma \qquad \text{in } \mathcal{D}'(\Omega\times\mathbb{R}/2\pi\mathbb{Z}). \tag{KIN}$$

The main Theorem of this paper is the following:

Theorem 2.6. *Let m satisfy* (M)*. The following four conditions are equivalent:*

(i) *m has locally finite weak entropy production,* (wFEP)*;*
(ii) *m satisfies the kinetic equation* (KIN)*;*
(iii) *$m\in B^{1/3}_{3,\infty,\mathrm{loc}}(\Omega)$;*
(iv) *m has locally finite strong entropy production,* (sFEP).

Remark 2.7. It is interesting to recall the following boundary behaviour of solutions of conservation laws with finite entropy production [47, Theorem 1.1] (see also [14, Theorem 2.5]): the field m admit a strong L^1 trace on $\partial\Omega$, in the sense that there exists a function $v\in L^\infty(\partial\Omega,S^1)$ such that

$$\operatorname*{ess\,lim}_{s\to 0}\int_{\partial\Omega}|u(\psi(s,x))-v(x)|d\mathcal{H}^1(x)=0,$$

where ψ is a suitable parametrization of a neighborhood of $\partial\Omega$.

3. Proof of the Main Theorem

The proof of Theorem 2.6 is divided into three propositions. The implication (iv)⇒(i) is trivial.

3.1. **Finite entropy implies kinetic formulation.**

Proposition 3.1. *If m has weak finite entropy production* (wFEP)*, then it satisfies the kinetic formulation* (KIN)*.*

We will need to construct a suitable family of entropies Φ_f parametrized (linearly) by continuous functions on S^1:

$$C^0(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R}) \ni f \mapsto \Phi_f \in ENT.$$

The construction is done in several steps. First define $\tilde{f} \in C^0(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$ by removing the null and the first Fourier modes:

$$\tilde{f}(t) = f(t) - \left(\frac{1}{2\pi}\int_0^{2\pi} f(s)\,ds\right) - \left(\frac{1}{\pi}\int_0^{2\pi} f(s)\cos s\,ds\right)\cos t - \left(\frac{1}{\pi}\int_0^{2\pi} f(s)\sin s\,ds\right)\sin t.$$

Then define $\psi_f \in C^1(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$ by

$$\psi_f(t) = \int_0^t \tilde{f}(s)\,ds.$$

Note that ψ_f is 2π-periodic since $\int_0^{2\pi}\tilde{f} = 0$. Moreover it holds

$$\int_0^{2\pi} \psi_f(s)e^{is}\,ds = 0,$$

since $\int_0^{2\pi}\tilde{f}(s)e^{is}\,ds = 0$. This allows us to define $\varphi_f \in C^2(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R}^2)$ by

$$\varphi_f(t) = \int_0^t \psi_f(s)ie^{is}\,ds.$$

Finally define $\Phi_f \in C^2(\mathbb{S}^1,\mathbb{R}^2)$ by

$$\Phi_f(e^{it}) = -i\varphi_f(t-\pi/2) + i\varphi_f(t+\pi/2).$$

Then it holds

$$\begin{aligned} e^{it}\cdot\frac{d}{dt}\left[\Phi_f(e^{it})\right] &= e^{it}\cdot\left(-i\psi_f(t-\pi/2)ie^{i(t-\pi/2)} + i\psi_f(t+\pi/2)ie^{i(t+\pi/2)}\right) \\ &= -\left(\psi_f(t-\pi/2)+\psi_f(t+\pi/2)\right)\,e^{it}\cdot(ie^{it}) = 0, \end{aligned}$$

so that $\Phi_f \in ENT$. Note that the map $f \mapsto \Phi_f$ is linear, and that $\|\Phi_f\|_{C^2} \le C\|f\|_{C^0}$ for some constant $C > 0$.

Remark 3.2. Note that $\Phi_{\cos(2t)} = -\frac{1}{2}\Sigma_{e_1,e_2}$ and that $\Phi_{\sin(2t)} = -\frac{1}{2}\Sigma_{\varepsilon_1,\varepsilon_2}$, where (e_1,e_2) is the standard basis and $(\varepsilon_1,\varepsilon_2)$ is its rotation by $\pi/4$. In particular, the classical entropies for the Aviles-Giga functional discovered by Jin and Kohn are parametrized by the first nontrivial modes of f (those with wavenumber 2).

The reason for defining the family of entropies $\{\Phi_f\}$ as above lies in its connection to the left-hand side of the kinetic formulation:

Lemma 3.3. *Let $\nu := e^{it}\cdot\nabla_x\left(\mathbb{1}_{e^{it}\cdot m(x)>0}\right) \in \mathcal{D}'(\Omega\times\mathbb{R}/2\pi\mathbb{Z})$. For any $f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$ and $\zeta \in C_c^\infty(\Omega)$ it holds*

$$\langle \nu, \zeta\otimes\psi_f\rangle = -\langle \nabla\cdot\Phi_f(m),\zeta\rangle.$$

Proof. We have

$$\langle \nu, \psi_f(t)\zeta(x)\rangle = -\int_\Omega \nabla\zeta(x)\cdot\int_{\mathbb{R}/2\pi\mathbb{Z}} \psi_f(t)\mathbb{1}_{e^{it}\cdot m>0}e^{it}\,dt\,dx,$$

and for all $x \in \Omega$, writing $m(x) = e^{i\alpha}$ we compute

$$\begin{aligned} \int_{\mathbb{R}/2\pi\mathbb{Z}} \psi_f(t) \mathbb{1}_{e^{it}\cdot m>0} e^{it}\, dt &= \int_{\alpha-\pi/2}^{\alpha+\pi/2} \psi_f(t) e^{it}\, dt \\ &= \int_{\alpha-\pi/2}^{\alpha+\pi/2} \frac{1}{i} \varphi_f'(t)\, dt \\ &= -i\varphi_f(\alpha+\pi/2) + i\varphi_f(\alpha-\pi/2) \\ &= \Phi_f(m), \end{aligned}$$

hence $\langle \nu, \zeta \otimes \psi_f \rangle = -\int_\Omega \Phi_f(m) \cdot \nabla\zeta\, dx$. □

The next lemma provides the measure σ appearing in the right-hand side of the kinetic formulation: as in [40, Theorem 3.1.6], the entropy production of the solution u of a conservation law under a certain entropy S can be written as an integral of S'' against the so-called entropy measure. In our case, observe that Φ_f is obtained by integrating f twice.

Lemma 3.4. *If m has locally finite weak entropy production in Ω, then there exists $\sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ satisfying*

$$\langle \nabla \cdot \Phi_f(m), \zeta \rangle = \iint_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} f(t)\zeta(x) d\sigma(x,t), \tag{3.1}$$

for every $\zeta \in C_c^\infty(\Omega)$ and every $f \in C^0(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$.

Proof. We consider, for any fixed $\zeta \in C_c^\infty(\Omega)$, the linear functional $T_\zeta \colon C^0(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}) \to \mathbb{R}$ given by

$$T_\zeta(f) = \langle \nabla \cdot \Phi_f(m), \zeta \rangle.$$

Each functional T_ζ is continuous, since

$$|T_\zeta(f)| \le \|\Phi_f\|_\infty \|\nabla\zeta\|_\infty \le C\|\nabla\zeta\|_\infty \|f\|_\infty.$$

On the other hand, for any $U \subset\subset \Omega$ and $f \in C^0(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$, by (wFEP) it holds

$$|T_\zeta(f)| \le \|\nabla \cdot \Phi_f(m)\|_{\mathcal{M}(U)} \qquad \forall \zeta \in C_c^\infty(U),\ \|\zeta\|_\infty \le 1.$$

Applying Banach-Steinhaus' theorem we deduce the existence of $C(U) > 0$ such that

$$|\langle \nabla \cdot \Phi_f(m), \zeta \rangle| \le C(U) \|f\|_\infty \|\zeta\|_\infty,$$

for all $f \in C^0(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$ and $\zeta \in C_c^\infty(U)$. Since tensor products are dense in $C_c^0(\Omega\times\mathbb{R}/2\pi\mathbb{Z})$, by Riesz' representation theorem this implies the existence of $\sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega\times\mathbb{R}/2\pi\mathbb{Z})$ satisfying (3.1). □

By Lemma 3.3 and 3.4 above, and since by definition $f = \psi_f'$, we have

$$\langle \nu - \partial_t\sigma, \psi_f(t)\zeta(x) \rangle = 0 \qquad \forall f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}),\ \zeta \in C_c^\infty(\Omega).$$

However ψ_f cannot be any arbitrary function $\psi \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$. In fact it holds

$$\{\psi_f \colon f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\} = \left\{ \psi \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}) \colon \psi(0) = 0 \text{ and } \int_0^{2\pi} \psi_f(s) e^{is}\, ds = 0 \right\}.$$

In other words, we have thus far determined ν up to the Fourier modes $\{1, \cos t, \sin t\}$ in the t-variable. The next lemma takes care of those modes.

Lemma 3.5. *For all $\zeta \in C_c^\infty(\Omega)$ it holds*

$$\langle \nu, \zeta(x) \rangle = \langle \nu, \zeta(x)\cos t \rangle = \langle \nu, \zeta(x)\sin t \rangle = 0.$$

Proof. We compute

$$\begin{aligned}
\langle \nu, \zeta(x) \rangle &= -\int_\Omega \nabla\zeta(x) \cdot \int_{\mathbb{R}/2\pi\mathbb{Z}} \mathbb{1}_{e^{it}\cdot m>0} e^{it}\, dt\, dx \\
&= -\int_\Omega \nabla\zeta(x) \cdot (2m(x))\, dx \\
&= 2\langle \nabla\cdot m, \zeta\rangle = 0, \\
\langle \nu, \zeta(x)\cos t\rangle &= -\int_\Omega \nabla\zeta(x) \cdot \int_{\mathbb{R}/2\pi\mathbb{Z}} \cos t \mathbb{1}_{e^{it}\cdot m>0} e^{it}\, dt\, dx \\
&= \int_\Omega \nabla\zeta(x) \cdot \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} dx = 0,
\end{aligned}$$

and similarly $\langle \nu, \zeta(x)\sin t\rangle = 0$. □

Remark 3.6. A similar computation shows that

$$\langle \nu, \zeta(x)\cos((2k+1)t)\rangle = \langle \nu, \zeta(x)\sin((2k+1)t)\rangle = 0 \qquad \forall k \in \mathbb{N}.$$

Therefore the measure σ does not have odd frequency Fourier modes. It can also be checked directly that for $f(t) = \cos((2k+1)t)$ and $f(t) = \sin((2k+1)t)$ it holds $\Phi_f \equiv 0$, which implies the same conclusion.

Proof of Proposition 3.1. For $f(t) = \cos t$ or $f(t) = \sin t$ we have $\tilde{f} = 0$ and therefore $\Phi_f = 0$. By Lemma 3.4 this implies

$$\langle \partial_t\sigma, \psi(t)\zeta(x)\rangle = 0 \quad \text{ for } \psi(t) = 1 \text{ or } \cos t \text{ or } \sin t.$$

We deduce that

$$\langle \nu - \partial_t\sigma, \psi(t)\zeta(x)\rangle = 0,$$

for any $\zeta \in C_c^\infty(\Omega)$ and $\psi \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$, which proves (KIN). □

3.2. Kinetic formulation implies Besov regularity.

Proposition 3.7. *If m satisfies the kinetic equation* (KIN)*, then it belongs to* $B^{1/3}_{3,\infty;\mathrm{loc}}(\Omega)$.

The proof of Theorem 2.6 is inspired from the kinetic averaging lemma in [26] for 1D scalar conservation laws, and the way it is revisited in [24]. We make use of the following quantity to control spatial increments of m at a fixed scale h in the direction e. Let $m : \Omega \to S^1$ be measurable: given $h > 0$, $|e| = 1$ and $x \in \Omega$ we set

$$\Delta(x,h,e) = \iint_{S^1\times S^1} \varphi(\xi,\eta)(\xi\wedge\eta) D_e^h\chi(x,\xi) D_e^h\chi(x,\eta) d\xi d\eta,$$

where $\chi(x,\xi) = \mathbb{1}_{\xi\cdot m(x)>0}$, $D_e^h\chi(x,\cdot) = D^{he}\chi(x,\cdot) = \chi(x+he,\cdot) - \chi(x,\cdot)$ and

$$\varphi(\xi,\eta) = (\mathbb{1}_{\xi\cdot\eta>0} - \mathbb{1}_{\xi\cdot\eta<0})(\mathbb{1}_{\xi\wedge\eta>0} - \mathbb{1}_{\xi\wedge\eta<0}). \tag{3.2}$$

The next Lemma describes the coerciveness properties of the function $\Delta(x,h,e)$, with respect to the averaged quantities

$$\frac{1}{2}\int_{S^1} \xi\chi(x+he,\xi) d\xi = m(x+he) \qquad \text{and} \qquad \frac{1}{2}\int_{S^1} \xi\chi(x,\xi) d\xi = m(x).$$

Lemma 3.8. *Given $m\colon \Omega \to S^1$, $x \in \Omega$ and $0 < h < dist(x,\partial\Omega)$, it holds:*

$$\Delta(x,h,e) \gtrsim |m(x+he) - m(x)|^3 = |D_e^h m(x)|^3.$$

Proof. It holds $\Delta(x,h,e) = \Xi(m(x+he), m(x))$ where

$$\Xi(m_1,m_2) = \iint_{S^1\times S^1} \varphi(\xi,\eta)(\xi\wedge\eta)\left(\mathbb{1}_{\xi\cdot m_1>0} - \mathbb{1}_{\xi\cdot m_2>0}\right)\left(\mathbb{1}_{\eta\cdot m_1>0} - \mathbb{1}_{\eta\cdot m_2>0}\right).$$

Therefore it suffices to prove that

$$\Xi(m_1,m_2) \gtrsim |m_1-m_2|^3 \qquad \forall m_1,m_2\in S^1.$$

It is easily checked that $\Xi(m_1,m_2)=\Xi(m_2,m_1)$ and, since

$$\varphi(R\xi,R\eta) = \varphi(\xi,\eta) \qquad \forall R\in SO(2), \tag{3.3}$$

that $\Xi(Rm_1,Rm_2)=\Xi(m_1,m_2)$ for all $R\in SO(2)$. Therefore it is enough to consider the case $m_1=e^{-i\beta}$, $m_2=e^{i\beta}$ for some $\beta\in[0,\pi/2]$ and to prove

$$\Xi(e^{-i\beta},e^{i\beta}) \gtrsim \beta^3 \qquad \forall \beta\in[0,\pi/2].$$

The function φ defined in (3.2) that appears in the definition of Ξ satisfies

$$\varphi(e^{i\theta},e^{i\psi}) = \widetilde{\varphi}(\psi-\theta), \qquad \widetilde{\varphi}(\omega) = \mathbb{1}_{\omega\in(0,\pi/2)\bmod \pi} - \mathbb{1}_{\omega\in(\pi/2,\pi)\bmod \pi}.$$

We compute

$$\begin{aligned}
\Xi(e^{-i\beta},e^{i\beta}) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \widetilde{\varphi}(\psi-\theta)\sin(\psi-\theta) \\
&\qquad \cdot\left(\mathbb{1}_{e^{i\theta}\cdot e^{-i\beta}>0} - \mathbb{1}_{e^{i\theta}\cdot e^{i\beta}>0}\right)\left(\mathbb{1}_{e^{i\psi}\cdot e^{-i\beta}>0} - \mathbb{1}_{e^{i\psi}\cdot e^{i\beta}>0}\right) d\theta d\psi \\
&= \int_{-\pi}^{\pi} \widetilde{\varphi}(\omega)\sin(\omega)\gamma(\omega)d\omega,
\end{aligned}$$

$$\text{where } \gamma(\omega) = \int_{-\pi}^{\pi} \overline{\chi}(\theta)\overline{\chi}(\theta+\omega)\,d\theta,$$

$$\text{and } \overline{\chi}(\theta) = \mathbb{1}_{e^{i\theta}\cdot e^{-i\beta}>0} - \mathbb{1}_{e^{i\theta}\cdot e^{i\beta}>0}.$$

Note that $\overline{\chi}(\theta+\pi)=\overline{\chi}(-\theta)=-\overline{\chi}(\theta)$ for almost every $\theta\in\mathbb{R}$. Therefore $\omega\mapsto\widetilde{\varphi}(\omega)\sin(\omega)\gamma(\omega)$ is π-periodic and even, and

$$\Xi(e^{-i\beta},e^{i\beta}) = 4\int_0^{\pi/2} \widetilde{\varphi}(\omega)\sin(\omega)\gamma(\omega)d\omega. \tag{3.4}$$

Moreover the integrand defining γ is π-periodic in θ, hence for all $\omega\in(0,\pi/2)$ we have

$$\gamma(\omega) = 2\int_0^{\pi} \overline{\chi}(\theta)\overline{\chi}(\theta+\omega)\,d\theta.$$

Assume first $\beta\in[0,\pi/4]$. Then for $\theta\in(0,\pi)$ it holds

$$\overline{\chi}(\theta)\overline{\chi}(\theta+\omega) = \begin{cases} \mathbb{1}_{\theta\in[\pi/2-\beta,\pi/2+\beta-\omega)} & \text{if } \omega\in[0,2\beta], \\ 0 & \text{if } \omega\in[2\beta,\pi/2], \end{cases}$$

and we find

$$\begin{aligned}
\gamma(\omega) &= \begin{cases} 2\cdot(2\beta-\omega) & \text{if } \omega\in[0,2\beta], \\ 0 & \text{if } \omega\in[2\beta,\pi/2]. \end{cases} \\
&= 2\cdot(2\beta-\omega)_+ \qquad \forall\omega\in[0,\pi/2],
\end{aligned}$$

Plugging this into (3.4) we deduce

$$\begin{aligned}
\Xi(e^{-i\beta},e^{i\beta}) &= 8\int_0^{2\beta}(2\beta-\omega)\sin\omega\,d\omega \\
&= 8\cdot(2\beta-\sin(2\beta)) \gtrsim \beta^3,
\end{aligned}$$

for all $\beta\in[0,\pi/4]$.

Consider now $\beta \in [\pi/4, \pi/2]$. For $\theta \in [0, \pi]$ we have

$$\overline{\chi}(\theta)\overline{\chi}(\theta+\omega) = \begin{cases} \mathbb{1}_{\theta \in [\pi/2-\beta, \pi/2+\beta-\omega)} & \text{if } \omega \in [0, \pi-2\beta] \\ \mathbb{1}_{\theta \in [\pi/2-\beta, \pi/2+\beta-\omega)} - \mathbb{1}_{\theta \in (3\pi/2-\beta-\omega, \pi/2+\beta)} & \text{if } \omega \in [\pi-2\beta, \pi/2], \end{cases}$$

and therefore

$$\gamma(\omega) = \begin{cases} 2 \cdot (2\beta - \omega) & \text{if } \omega \in [0, \pi-2\beta], \\ 2 \cdot (\pi - 2\omega) & \text{if } \omega \in [\pi-2\beta, \pi/2], \end{cases}$$

$$\begin{aligned} \Xi(e^{-i\beta}, e^{i\beta}) &= 8\int_0^{\pi-2\beta} (2\beta-\omega)\sin\omega\, d\omega + 8\int_{\pi-2\beta}^{\pi/2} (\pi - 2\omega)\sin\omega\, d\omega \\ &= 8 \cdot (2\beta - (\pi-4\beta)\cos(2\beta) - \sin(2\beta)) \\ &\quad + 8 \cdot (2\sin(2\beta) - 2 + (\pi - 4\beta)\cos(2\beta)) \\ &= 8 \cdot (\sin(2\beta) + 2\beta - 2) \geq 8\left(\frac{\pi}{2} - 1\right) \gtrsim \beta^3, \end{aligned}$$

for all $\beta \in [\pi/4, \pi/2]$. □

To obtain bounds for the integral of $\Delta(x, h, e)$ on Ω, when m satisfies the kinetic formulation (KIN), we use the following lemma, that estimates its derivative with respect to h:

Lemma 3.9. *Suppose that m satisfies the kinetic formulation of the eikonal equation* (KIN)*, and that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. We then have for all unit vectors e and $|h| \lesssim dist(\Omega', \partial\Omega'')$:*

$$\int_{\Omega'} \Delta(x, h, e) dx \lesssim |h|(1 + \|\sigma\|_{\mathcal{M}(\Omega''\times\mathbb{R}/2\pi\mathbb{Z})}), \tag{3.5}$$

where the multiplicative constant depends on the distance between Ω' and $\partial\Omega''$.

Proof. Assume (KIN):

$$e^{is} \cdot \nabla_x \mathbb{1}_{e^{is}\cdot m(x)>0} = \partial_s \sigma \qquad \text{in } \mathcal{D}'(\Omega \times \mathbb{R}/2\pi\mathbb{Z}),$$

and let us assume to have intermediate domains $\widetilde{\Omega}, \Omega''$ with $\Omega' \subset\subset \widetilde{\Omega} \subset\subset \Omega'' \subset\subset \Omega$ and such that the distances among the boundaries of the first three are comparable. We perform the calculation of $\partial_h \int_{\Omega'} \Delta(x, h, e) dx$ for a regularized integrand, namely:

- we regularize the equation (KIN) by convolving with respect to x with a smooth approximation of the identity ρ_ε:
$$e^{is} \cdot \nabla_x \chi_\varepsilon(x, e^{is}) = \partial_s \sigma_\varepsilon, \qquad \chi_\varepsilon = \left(\mathbb{1}_{e^{is}\cdot m(x)>0}\right) \underset{x}{*} \rho_\varepsilon, \quad \sigma_\varepsilon = \sigma \underset{x}{*} \rho_\varepsilon; \tag{3.6}$$
here $\varepsilon < \text{dist}(\Omega, \partial\Omega'')$;
- we approximate φ (3.2) by a smooth φ_δ. The calculations below are valid for a generic φ and only use the skew-symmetry property $\varphi(\xi, \eta) = -\varphi(\eta, \xi)$. Assuming in addition the $SO(2)$ invariance property (3.3), and parametrizing with the angle between ξ and η, these conditions amount to require that $\widetilde{\varphi}$: $s \mapsto \varphi(1, e^{is})$ is odd and 2π periodic. In turn, a convolution on the real line with a smooth even kernel, at scale δ, preserves both these properties. Explicitly, we set
$$\varphi_\delta(e^{i\theta}, e^{i\psi}) = \widetilde{\varphi}_\delta(\psi - \theta), \qquad \widetilde{\varphi}_\delta = \widetilde{\varphi} * \rho_\delta,$$
for some smooth even kernel ρ. This approximation has the following properties:
$$\widetilde{\varphi}_\delta \to \widetilde{\varphi} \text{ a.e.}, \qquad |\widetilde{\varphi}_\delta| \leq 1, \qquad \|\widetilde{\varphi}_\delta'\|_{L^1(\mathbb{R}/2\pi\mathbb{Z})} \leq 8. \tag{3.7}$$

The explicit dependence of the function Δ on the parameters ε, δ is omitted in the first calculations.

We assume without loss of generality that $e = e_1$ and use the notations $\chi^h(x,\xi) = \chi(x + he_1, \xi)$ and $D_1^h\chi = \chi^h - \chi$. Let $x \in \tilde{\Omega}$ and $|h| < \mathrm{dist}(\tilde{\Omega}, \partial\Omega'')$. Using the skew-symmetry of φ, we have

$$\begin{aligned}\frac{\partial}{\partial h}\Delta(x,e,h) &= \frac{\partial}{\partial h}\iint_{S^1\times S^1}\varphi(\xi,\eta)(\xi\wedge\eta)D_1^h\chi(x,\xi)D_1^h\chi(x,\eta)d\xi d\eta \\ &= \iint_{S^1\times S^1}\varphi(\xi,\eta)(\xi\wedge\eta)[\partial_1\chi^h(x,\xi)D_1^h\chi(x,\eta) + \partial_1\chi^h(x,\eta)D_1^h\chi(x,\xi)]d\xi d\eta \\ &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)(\xi_1\eta_2)[\partial_1\chi^h(x,\xi)D_1^h\chi(x,\eta) + \partial_1\chi^h(x,\eta)D_1^h\chi(x,\xi)]d\xi d\eta.\end{aligned}$$

Letting $\nu(x, e^{is}) := \partial_s\sigma(x,s)$, we use the the equation (KIN) in the form

$$\xi_1\partial_1\chi(x,\xi) + \xi_2\partial_2\chi(x,\xi) = \nu(x,\xi), \tag{3.8}$$

to replace $\xi_1\partial_1\chi^h(x,\xi)$ in the above and obtain

$$\begin{aligned}\frac{\partial}{\partial h}\Delta(x,e,h) &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2[(\nu^h(x,\xi) - \xi_2\partial_2\chi(x,\xi))D_1^h\chi(x,\eta) + \xi_1\partial_1\chi^h(x,\eta)D_1^h\chi(x,\xi)]d\xi d\eta \\ &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2[(\nu^h(x,\xi) - \xi_2\partial_2\chi(x,\xi))D_1^h\chi(x,\eta) - \xi_1\chi^h(x,\eta)\partial_1 D_1^h\chi(x,\xi)]d\xi d\eta \\ &\quad + \partial_1\left[2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\xi_1\chi^h(x,\eta)D_1^h\chi(x,\xi)d\xi d\eta\right] =: I_1 + \partial_1 A_1.\end{aligned}$$

The term $\partial_1 A_1$ is a boundary term and will be treated at the end. Focusing on I_1, we can expand $\xi_1\partial_1 D_1^h\chi(x,\xi)$ and use (3.8) to deduce

$$\begin{aligned}I_1 &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\Big[(\nu^h(x,\xi) - \xi_2\partial_2\chi(x,\xi))(\chi^h(x,\eta) - \chi(x,\eta)) \\ &\qquad - (\nu^h(x,\xi) - \nu(x,\xi) - \xi_2\partial_2\chi^h(x,\xi) + \xi_2\partial_2\chi(x,\xi))\chi^h(x,\eta)\Big]d\xi d\eta \\ &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\Big[-\nu^h(x,\xi)\chi(x,\eta) + \nu(x,\xi)\chi^h(x,\eta) \\ &\qquad + \xi_2\partial_2\chi^h(x,\xi)\chi(x,\eta) - \xi_2\partial_2\chi(x,\xi)\chi^h(x,\eta)\Big]d\xi d\eta \\ &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2[-\nu^h(x,\xi)\chi(x,\eta) + \nu(x,\xi)\chi^h(x,\eta)] \\ &\quad + 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\xi_2[\partial_2\chi^h(x,\xi)\chi(x,\eta) - \partial_2\chi(x,\xi)\chi^h(x,\eta)]d\xi d\eta.\end{aligned}$$

Exchanging ξ and η only in the last term of the second integral, we can rewrite

$$\begin{aligned}I_1 &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2[-\nu^h(x,\xi)\chi(x,\eta) + \nu(x,\xi)\chi^h(x,\eta)] \\ &\quad + 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\xi_2[\partial_2\chi^h(x,\xi)\chi(x,\eta) + \partial_2\chi(x,\eta)\chi^h(x,\xi)]d\xi d\eta \\ &= 2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2[-\nu^h(x,\xi)\chi(x,\eta) + \nu(x,\xi)\chi^h(x,\eta)] \\ &\quad + \partial_2\left[2\iint_{S^1\times S^1}\varphi(\xi,\eta)\eta_2\xi_2\chi(x,\eta)\chi^h(x,\xi)d\xi d\eta\right] =: I_2 + \partial_2 A_2.\end{aligned}$$

Therefore we have

$$\frac{\partial}{\partial h}\Delta(x,h,e) = I_2 + \partial_1 A_1 + \partial_2 A_2, \tag{3.9}$$

where

$$|A| = |(A_1, A_2)| \leq 8\pi \qquad \text{pointwise } \forall (x,h,e). \tag{3.10}$$

The identities leading from (3.8) to (3.9) are our counterpart of Varadhan's interaction identities (that we mentioned in the introduction), as they are revisited in [24].

The most important term in the identity (3.9) is I_2, since the extra term is a divergence $\nabla_x \cdot A$ of a bounded vector field, hence it can be treated as a boundary term. In polar coordinates I_2 becomes:

$$\begin{aligned} I_2 &= 2 \iint_{[0,2\pi[\times[0,2\pi[} \widetilde{\varphi}(\psi-\theta)\sin\psi[-\partial_\theta \sigma^h(x,\theta)\chi(x,e^{i\psi}) + \partial_\theta \sigma(x,\theta)\chi^h(x,e^{i\psi})]d\theta d\psi \\ &= 2 \iint_{[0,2\pi[\times[0,2\pi[} \widetilde{\varphi}'(\psi-\theta)\sin\psi[-\sigma^h(x,\theta)\chi(x,e^{i\psi}) + \sigma(x,\theta)\chi^h(x,e^{i\psi})]d\theta d\psi. \end{aligned}$$

Recall now that the above was derived for an approximation φ_δ of φ and for a solution of the regularized kinetic equation (3.6) at scale ε. Writing this dependence explicitly we have:

$$\begin{aligned} I_2 = I_2^{\varepsilon,\delta} &= -2 \int_0^{2\pi} \sigma_\varepsilon^h(x,e^{i\theta}) \int_0^{2\pi} \widetilde{\varphi}_\delta'(\psi-\theta)\chi_\varepsilon(x,e^{i\psi})\sin\psi\, d\psi\, d\theta \\ &\quad + 2 \int_0^{2\pi} \sigma_\varepsilon(x,e^{i\theta}) \int_0^{2\pi} \widetilde{\varphi}_\delta'(\psi-\theta)\chi_\varepsilon^h(x,e^{i\psi})\sin\psi\, d\psi\, d\theta. \end{aligned}$$

Recalling (3.7) and the fact that $|\chi_\varepsilon| \leq 1$ a.e., we deduce

$$\left| I_2^{\varepsilon,\delta} \right| \lesssim \int_0^{2\pi} \left(\left|\sigma_\varepsilon^h(x,\theta)\right| + |\sigma_\varepsilon(x,\theta)| \right)\, d\theta.$$

Plugging this estimate into the identity $\partial_h \Delta^{\varepsilon,\delta} = I_2^{\varepsilon,\delta} + \nabla_x \cdot A^{\varepsilon,\delta}$ tested against any nonnegative $\gamma \in C_c^\infty(\widetilde{\Omega})$, and recalling that $A^{\varepsilon,\delta}$ is a uniformly bounded vector field (3.10), we obtain

$$\begin{aligned} \frac{\partial}{\partial h} \int_\Omega \gamma(x)\Delta^{\varepsilon,\delta}(x,h,e)\, dx &\lesssim \|\gamma\|_{C^0} \left(\|\sigma_\varepsilon^h\|_{L^1(\widetilde{\Omega}\times\mathbb{R}/2\pi\mathbb{Z})} + \|\sigma_\varepsilon\|_{L^1(\widetilde{\Omega}\times\mathbb{R}/2\pi\mathbb{Z})} \right) + \|\nabla\gamma\|_{C^0} \\ &\lesssim \|\gamma\|_{C^0} \|\sigma\|_{\mathcal{M}(\Omega''\times\mathbb{R}/2\pi\mathbb{Z})} + \|\nabla\gamma\|_{C^0}, \end{aligned}$$

for $|h| + \varepsilon \leq \mathrm{dist}(\widetilde{\Omega}, \partial\Omega'')$ and $\delta > 0$. Integrating with respect to h we find that

$$\frac{1}{|h|} \int_\Omega \gamma(x)\Delta^{\varepsilon,\delta}(x,h,e)\, dx \lesssim \|\gamma\|_{C^0} \|\sigma\|_{\mathcal{M}(\Omega''\times\mathbb{R}/2\pi\mathbb{Z})} + \|\nabla\gamma\|_{C^0}.$$

By dominated convergence we may pass to the limit $\varepsilon, \delta \to 0$ in the left-hand side. Then it remains to choose $\gamma \equiv 1$ in Ω' to obtain the claimed estimate (3.5). □

We can now prove Proposition 3.7:

Proof. The proof follows combining the results of Lemmas 3.8 and 3.9. For $t < \mathrm{dist}(\Omega', \partial\Omega'')$,

$$\begin{aligned} [N_t(m,\Omega')]^3 &= \frac{1}{t} \sup_{|e|=1, |h|\leq t} \int_{\Omega'} |D_e^h m(x)|^3 dx \\ &\lesssim \frac{1}{t} \sup_{|e|=1, |h|\leq t} \int_{\Omega'} \Delta(x,e,h) dx \\ &\lesssim 1 + \|\sigma\|_{\mathcal{M}(\Omega''\times\mathbb{R}/2\pi\mathbb{Z})}. \end{aligned}$$

For other values of t up to 1, the triangular inequality and the boundedness of m yield a trivial control on $N_t(f,\Omega)$. Together, these estimates give the desired bound on the local Besov norm $B^{1/3}_{3,\infty}(\Omega')$. □

3.3. **Besov regularity implies finite entropy.** For the proofs of the next propositions and lemmas, we let $m_\varepsilon := m * \rho_\varepsilon$ be a regularization with a standard kernel (with $\mathrm{spt}(\rho) \subset B_1$ and $\nabla\rho = 0$ in $B_{1/2}$).

Proposition 3.10. *If m solves* (M) *and belongs to the space $B^{1/3}_{3,\infty,\mathrm{loc}}(\Omega)$, then m has locally finite strong entropy production* (sFEP)*:*

$$\bigvee_{\Phi\in ENT, \|D^2\Phi\|_\infty \leq 1} \|\nabla\cdot\Phi(m)\|(A) \lesssim [m]^3_{B^{\frac{1}{3}}_{3,\infty}(A)} \qquad \text{for } A \subset\subset \Omega.$$

Proof. For a given $\Phi \in ENT$, we consider its extension $\widetilde{\Phi} \in C^2_c(\mathbb{R}^2, \mathbb{R}^2)$ given in polar coordinates by $\widetilde{\Phi}(re^{i\theta}) = \eta(r)\Phi(e^{i\theta})$, where η is a fixed cut-off function $\eta \in C^\infty_c(0,\infty)$ satisfying $\eta \equiv 0$ outside $(1/2, 2)$ and $\eta(1) = 1$.

Following [22, 17], for the mollified field m_ε we can single out in the entropy production the contribution of the radial oscillation:

$$\nabla\cdot\widetilde{\Phi}(m_\varepsilon) = \Psi(m_\varepsilon)\cdot\nabla(1-|m_\varepsilon|^2),$$

where $\Psi \in C^1_c(\mathbb{R}^2, \mathbb{R}^2)$ is a regular vectorfield. Given a test function $\phi \in C^\infty_c(\Omega)$ we can integrate by parts

$$\langle\nabla\cdot\widetilde{\Phi}(m_\varepsilon), \phi\rangle = -\int_\Omega \phi(x)\Psi(m_\varepsilon(x))\cdot\nabla(1-|m_\varepsilon(x)|^2)dx =: A_\varepsilon[\phi] + B_\varepsilon[\phi],$$

where

$$\begin{aligned} A_\varepsilon[\phi] &= \int_\Omega \nabla\phi(x)\cdot\Psi(m_\varepsilon(x))(1-|m_\varepsilon(x)|^2)dx, \\ B_\varepsilon[\phi] &= \int_\Omega \phi(x)\nabla\cdot[\Psi(m_\varepsilon(x))](1-|m_\varepsilon(x)|^2)dx \\ &= \int_\Omega \phi(x)Tr[D\Psi(m_\varepsilon(x))\nabla m_\varepsilon(x)](1-|m_\varepsilon(x)|^2)dx. \end{aligned}$$

While $A_\varepsilon[\phi] \to 0$, trivially because $|m| = 1$ almost everywhere, the second integral $B_\varepsilon[\phi]$ can be bounded by

$$B_\varepsilon[\phi] \lesssim \|\phi\|_{L^\infty}\|D\Psi\|_{L^\infty}\int_{\mathrm{spt}(\phi)} |\nabla m_\varepsilon(x)||1-|m_\varepsilon(x)|^2|dx.$$

Since 3 and $\frac{3}{2}$ are dual exponents, using Lemmas 3.11 and 3.12 below, the last integral can be bounded by

$$\|1-|m_\varepsilon|^2\|_{L^{\frac{3}{2}}(\mathrm{spt}(\phi))}\|\nabla m_\varepsilon\|_{L^3(\mathrm{spt}(\phi))} \lesssim N_\varepsilon(m, \mathrm{spt}(\phi))^3.$$

Noting that $|D\Psi| \lesssim |D^2\widetilde{\Phi}|$ and letting $\varepsilon \to 0$ we deduce that

$$|\langle\nabla\cdot\Phi(m), \phi\rangle| \lesssim \|\phi\|_{L^\infty}\|D^2\Phi\|_\infty \liminf_{\varepsilon\to 0} N_\varepsilon(m, \mathrm{spt}(\phi))^3,$$

and therefore

$$\|\nabla\cdot\Phi(m)\|(U) \lesssim \|D^2\Phi\|_\infty \liminf_{\varepsilon\to 0} N_\varepsilon(m, \overline{U})^3,$$

for all $U \subset\subset \Omega$. Note that $N_\varepsilon(m, \overline{U})$ involves integrals with respect to x over the sets $\overline{U}$ and $\overline{U} + \varepsilon y$, hence given a finite family of open and distant sets $U_1, \ldots, U_k \subset\subset A \subset\subset \Omega$, and a corresponding family of entropies $\Phi_1, \ldots, \Phi_k$ with $\|D^2\Phi_j\|_\infty \leq 1$, if ε is small enough it holds

$$\sum_j \|\nabla\cdot\Phi_j(m)\|(U_j) \lesssim \liminf_{\varepsilon\to 0}\sum_j N_\varepsilon(m, U_j)^3 = \liminf_{\varepsilon\to 0} N_\varepsilon(m, A)^3.$$

Recalling the definitions of the least upper bound measure and of the Besov seminorm, this implies the conclusion of Proposition 3.10. ☐

In the proof of Proposition 3.10, we used the two following lemmas on the growth of certains norms of the regularized field m_ε. Their proof is an adaptation to the Besov scale of corresponding statements for Sobolev functions, treated in [17].

Lemma 3.11. *If $m \in B^{1/3}_{3,\infty,\mathrm{loc}}(\Omega)$, and $\Omega' \subset\subset \Omega$, then for every $\varepsilon \lesssim dist(\Omega', \partial\Omega)$*

$$\int_{\Omega'} |\nabla m_\varepsilon|^3 dx \lesssim \varepsilon^{-2} N_\varepsilon(m, \Omega')^3.$$

Proof. As in [17, Proof of Proposition 3, Step 6(ii)], for ε small enough we have the pointwise bound:

$$|\nabla m_\varepsilon(x)| \leq \frac{\|\nabla\rho\|_\infty}{\varepsilon^3} \int_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)} |m(x+z) - m(x)| dz.$$

Applying Jensen inequality and integrating in Ω' one obtains

$$\begin{aligned}
\int_{\Omega'} |\nabla m_\varepsilon(x)|^3 dx &\lesssim \frac{1}{\varepsilon^5} \int_{\Omega'} \int_{B_\varepsilon(0)\setminus B_{\varepsilon/2}(0)} |m(x+z) - m(x)|^3 dz dx \\
&= \frac{1}{\varepsilon^3} \int_{\Omega'} \fint_{B_\varepsilon(0)} |D^z m(x)|^3 dz dx \\
&\leq \frac{1}{\varepsilon^2} N_\varepsilon(m, \Omega')^3.
\end{aligned}$$

□

Lemma 3.12. *With the notations of Lemma 3.11, it holds:*

$$\int_{\Omega'} (1 - |m_\varepsilon|^2)^{3/2} dx \lesssim \varepsilon N_\varepsilon(m, \Omega')^3.$$

Proof. As in [17, Proof of Proposition 3, Step 6(i)], using that $|m| = 1$ almost everywhere we obtain the pointwise bound

$$\begin{aligned}
|1 - |m_\varepsilon|^2|(x) &\lesssim \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)\rho_\varepsilon(x-z)|m(y) - m(z)|^2 dy dz \\
&\lesssim \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)|m(y) - m(x)|^2 dy = \int_{B_\varepsilon(0)} \rho_\varepsilon(z)|m(x+z) - m(x)|^2 dz
\end{aligned}$$

Then by Hölder's inequality

$$\begin{aligned}
\int_{\Omega'} |(1 - |m_\varepsilon(x)|^2)|^{\frac{3}{2}} dx &\lesssim \int_{\Omega'} \int_{B_\varepsilon(0)} \rho_\varepsilon(z)|D^z m(x)|^3 dz dx \\
&\leq \sup_{|z|\leq\varepsilon} \|D^z m\|^3_{L^3(\Omega')} \\
&\leq \varepsilon N_\varepsilon(m, \Omega')^3.
\end{aligned}$$

□

To conclude the proof of Theorem 2.6 we remark that the implication (iv)⇒(i) is trivial.

4. Corollaries and further comments

4.1. **Sharp differentiability for zero energy states.** We observe that if $m \in B^{1/3}_{3,c_0,\mathrm{loc}}(\Omega)$ (in particular when $m \in B^{1/3}_{3,q,\mathrm{loc}}(\Omega)$, $q < \infty$), then thanks to Lemmas 3.11 and 3.12 we have

$$\varepsilon^{\frac{2}{3}} \nabla m_\varepsilon \to 0 \quad \text{in} \quad L^3_{\mathrm{loc}}(\Omega) \text{ and } \quad \varepsilon^{-\frac{2}{3}}(1 - |m_\varepsilon|^2) \to 0 \quad \text{in} \quad L^{\frac{3}{2}}_{\mathrm{loc}}(\Omega).$$

Therefore the conclusion of Proposition 3.10 can be refined to

$$\|\nabla \cdot \Phi(m)\| = 0 \qquad \text{for every } \Phi \in ENT.$$

That is, slightly better regularity rules out entropy production. This in turn implies much stronger regularity properties: m is locally Lipschitz outside a locally finite set of vortices, [30].

4.2. **The mass of the entropy measure $\|\sigma\|(\Omega)$.** In the micromagnetics model studied by Rivière-Serfaty in [43], twice the total variation of the kinetic measure provides a sharp asymptotic lower bound for the energy, [43, Theorem 1]. In this paragraph we investigate whether this property holds for our model (M), at least in the BV case. Recall [13] that for $m = \nabla^\perp u \in BV(\Omega)$ satisfying (M) it holds

$$\big(\Gamma - \lim AG_\varepsilon\big)(u) = \frac{1}{3}\int_{J_m} \big|m^+ - m^-\big|^3 d\mathcal{H}^1.$$

Hence the question we raise is whether this equals $2\|\sigma\|(\Omega)$.

For a given $m \in BV(\Omega)$ satisfying (M) we compute $\|\sigma\|$ as follows. In light of Lemma 3.4 it holds

$$\|\sigma\| = \bigvee_{|f|\leq 1} \|\nabla\cdot\Phi_f(m)\|.$$

On the other hand, Remark 3.2 and the results in [2] ensure that

$$\begin{aligned} 2\|\sigma\| &= 2\bigvee_{|f|\leq 1}\|\nabla\cdot\Phi_f(m)\| \\ &\geq \bigvee_{(\alpha_1,\alpha_2)} \|\nabla\cdot\Sigma_{\alpha_1,\alpha_2}(m)\| = \frac{1}{3}|m^+ - m^-|^3\mathcal{H}^1 \llcorner J_m. \end{aligned} \tag{4.1}$$

Proposition 4.1. *If Dm has a nontrivial jump part, then the inequality in* (4.1) *is strict.*

Proof. According to [28, Theorem 3], since the set $\{\Phi_f : |f|\leq 1\}$ is symmetric (stable under multiplication by -1) and equivariant (stable under conjugation by any rotation), it holds

$$\bigvee_{|f|\leq 1}\|\nabla\cdot\Phi_f(m)\| = c(\big|m^+ - m^-\big|)\mathcal{H}^1\llcorner J_m. \tag{4.2}$$

for a certain cost function c. This cost function is given by

$$c(s) = \sup\left\{\big(\Phi_f(m^+) - \Phi_f(m^-)\big)\cdot\nu\right\},$$

where the supremum is taken among :

- all possible jumps $m^\pm \in S^1$ of size $|m^+ - m^-| = s$,
- all possible normal vectors $\nu \in S^1$ with the admissibility condition $(m^+ - m^-)\cdot\nu = 0$ (due to the divergence constraint $\nabla\cdot m = 0$),
- and all possible f with $|f|\leq 1$.

Using again the symmetry and equivariance of $\{\Phi_f\}$, we can simplify this as

$$c(s) = \sup\left\{\big(\Phi_f(e^{i\beta}) - \Phi_f(e^{-i\beta})\big)\cdot e_1,\ |f|\leq 1\right\} \quad \text{for } s = 2\sin\beta,\ \beta\in[0,\pi/2].$$

For angles $\beta\in[-\pi/2,\pi/2]$ it holds

$$\begin{aligned} e_1\cdot\Phi_f(e^{i\beta}) &= \mathfrak{Re}(-i\varphi_f(\beta-\pi/2) + i\varphi_f(\beta+\pi/2)) \\ &= \mathfrak{Re}\left(\int_0^{\beta-\pi/2}\psi_f(s)e^{is}\,ds - \int_0^{\beta+\pi/2}\psi_f(s)e^{is}\,ds\right) \\ &= -\int_{\beta-\pi/2}^{\beta+\pi/2}\psi_f(s)\cos s\,ds = -\left[\psi_f\sin\right]_{\beta-\pi/2}^{\beta+\pi/2} + \int_{\beta-\pi/2}^{\beta+\pi/2}\tilde f(s)\sin s\,ds \\ &= -\cos\beta\left(\int_0^{\beta+\pi/2}\tilde f + \int_0^{\beta-\pi/2}\tilde f\right) + \int_{\beta-\pi/2}^{\beta+\pi/2}\tilde f\sin \end{aligned}$$

$$
\begin{aligned}
&= -\cos\beta\Bigg[\int_0^{\beta+\pi/2} f - \frac{1}{2\pi}(\beta+\pi/2)\int_0^{2\pi} f - \frac{1}{\pi}\left(\int_0^{\beta+\pi/2}\cos\right)\int_0^{2\pi} f\cos \\
&\qquad - \frac{1}{\pi}\left(\int_0^{\beta+\pi/2}\sin\right)\int_0^{2\pi} f\sin + \int_0^{\beta-\pi/2} f - \frac{1}{2\pi}(\beta-\pi/2)\int_0^{2\pi} f \\
&\qquad - \frac{1}{\pi}\left(\int_0^{\beta-\pi/2}\cos\right)\int_0^{2\pi} f\cos - \frac{1}{\pi}\left(\int_0^{\beta-\pi/2}\sin\right)\int_0^{2\pi} f\sin\Bigg] \\
&\quad + \int_{\beta-\pi/2}^{\beta+\pi/2} f\sin - \frac{1}{2\pi}\left(\int_{\beta-\pi/2}^{\beta+\pi/2}\sin\right)\int_0^{2\pi} f \\
&\quad - \frac{1}{\pi}\left(\int_{\beta-\pi/2}^{\beta+\pi/2}\cos\sin\right)\int_0^{2\pi} f\cos - \frac{1}{\pi}\left(\int_{\beta-\pi/2}^{\beta+\pi/2}\sin^2\right)\int_0^{2\pi} f\sin \\
&= -\cos\beta\left[\int_0^{\beta+\pi/2} f + \int_0^{\beta-\pi/2} f - \frac{\beta}{\pi}\int_0^{2\pi} f - \frac{2}{\pi}\int_0^{2\pi} f\sin\right] \\
&\quad + \int_{\beta-\pi/2}^{\beta+\pi/2} f\sin - \frac{1}{\pi}\sin\beta\int_0^{2\pi} f - \frac{1}{2}\int_0^{2\pi} f\sin\,.
\end{aligned}
$$

Hence for any $\beta \in [0,\pi/2]$,

$$
\begin{aligned}
e_1\cdot\left(\Phi_f(e^{i\beta}) - \Phi_f(e^{-i\beta})\right) &= \cos\beta\left[\int_0^{-\beta+\pi/2} f + \int_0^{-\beta-\pi/2} f - \int_0^{\beta+\pi/2} f - \int_0^{\beta-\pi/2} f + \frac{2\beta}{\pi}\int_0^{2\pi} f\right] \\
&\quad + \int_{\beta-\pi/2}^{\beta+\pi/2} f\sin - \int_{-\beta-\pi/2}^{-\beta+\pi/2} f\sin - \frac{2}{\pi}\sin\beta\int_0^{2\pi} f \\
&= -\cos\beta\int_{-\beta-\pi/2}^{\beta-\pi/2} f - \cos\beta\int_{-\beta+\pi/2}^{\beta+\pi/2} f + \int_{-\beta+\pi/2}^{\beta+\pi/2} f\sin - \int_{-\beta-\pi/2}^{\beta-\pi/2} f\sin \\
&\quad - \frac{2}{\pi}(\sin\beta - \beta\cos\beta)\int_0^{2\pi} f \\
&= \int_0^{2\pi} g_\beta f,
\end{aligned}
$$

where g_β is π-periodic and

$$
g_\beta(t) = (\sin t - \cos\beta)\mathbb{1}_{\pi/2-\beta\le t\le\pi/2+\beta} - \frac{2}{\pi}(\sin\beta - \beta\cos\beta) \qquad \forall t\in[0,\pi].
$$

The above computation with $f(t)=\cos(2t)$ yields an entropy production equal to $(2\sin\beta)^3/6$, as expected. On the other hand the supremum of the above quantity over $|f|\le 1$ is given by $\|g_\beta\|_{L^1(0,2\pi)}$. This supremum is not attained by a continuous function when $\beta>0$. In other words, for any jump of size $s>0$ we have $c(s) > s^3/6$. In view of (4.2) this shows that equality in (4.1) can not happen unless Dm has a trivial jump part. □

To calculate the value of $c(s)$ we observe the following. Since g_β is π-periodic and even it holds

$$
\|g_\beta\|_{L^1(0,2\pi)} = 4\int_0^{\pi/2}|g_\beta|.
$$

The function g_β is negative in $[0,t_\beta)$ and positive in $(t_\beta,\pi/2]$, where $t_\beta\in[\pi/2-\beta,\pi/2]$ is characterized by

$$
\sin t_\beta - \cos\beta = \frac{2}{\pi}(\sin\beta - \beta\cos\beta).
$$

Moreover it holds that $\int_0^{\pi/2} g_\beta = 0$, hence we find

$$\int_0^{\pi/2} |g_\beta| = \int_0^{t_\beta} (-g_\beta) + \int_{t_\beta}^{\pi/2} g_\beta = 2\int_{t_\beta}^{\pi/2} g_\beta$$
$$= 2\cos t_\beta - 2(\pi/2 - t_\beta)\left(\cos\beta + \frac{2}{\pi}(\sin\beta - \beta\cos\beta)\right).$$

With this expression it can be checked that

$$\|g_\beta\|_{L^1(0,2\pi)} \sim \frac{1}{6}(2\beta)^3 \qquad \text{as } \beta \to 0,$$

hence $c(s) \sim s^3/6$ for $s \to 0$, so that the measure $\|\sigma\|$ does behave like the right-hand side of (4.1) for very small jumps.

4.3. **Partial regularity obtained by using only the Jin-Kohn entropies.** In this paragraph, we show how to obtain fractional differentiability of a solution m of (M) having finite entropy production for every entropy (2.1) in the class of Jin-Kohn:

$$\Sigma_{\alpha_1,\alpha_2}(z) = \frac{4}{3}\left((z\cdot\alpha_2)^3\alpha_1 + (z\cdot\alpha_1)^3\alpha_2\right).$$

Recall that (α_1,α_2) is a positive orthonormal frame $(R_\theta e_1, R_\theta e_2)$, and notice moreover that every entropy is a linear combination of two basic entropies Σ_{e_1,e_2} and $\Sigma_{\varepsilon_1,\varepsilon_2}$:

$$\Sigma_{R_\theta e_1, R_\theta e_2}(z) = \cos(2\theta)\Sigma_{e_1,e_2}(z) + \sin(2\theta)\Sigma_{\varepsilon_1,\varepsilon_2}(z),$$

where $\varepsilon_1 = \frac{e_1+e_2}{\sqrt{2}}$, $\varepsilon_1 = \frac{-e_1+e_2}{\sqrt{2}}$.

In [36] the authors show that whenever the entropy production associated to Σ_{e_1,e_2} and $\Sigma_{\varepsilon_1,\varepsilon_2}$ vanish (which is equivalent to all the Jin-Kohn entropy productions vanishing), then in fact all entropy productions vanish and the rigidity result of [30] applies. Hence it is natural to wonder whether in general, controlling the total variation of these two basic entropy productions is enough to obtain the $B^{1/3}_{3,\infty}$ estimate (which we obtained here using all entropy productions). We do not provide an answer to this question, but show how a method described in [25] can be combined with estimates derived in [36] to obtain a $B^s_{4,\infty}$ estimate for all $s < 1/4$.

To this end we set

$$\Delta_{JK}(x,h,e) = D^h_e \Sigma_{e_1,e_2}(m(x)) \wedge D^h_e \Sigma_{\varepsilon_1,\varepsilon_2}(m(x))$$
$$= \det D^h_e \left(\Sigma_{e_1,e_2}(m), \Sigma_{\varepsilon_1,\varepsilon_2}(m)\right)(x).$$

Here we recall that D^h_e denotes the spatial increment of size h in direction e, that is $D^h_e f(x) = f(x+he) - f(x)$. In [36] the authors study some properties of the set K of 2×2 matrices given by

$$K = \left\{(\Sigma_{e_1,e_2}(m), \Sigma_{\varepsilon_1,\varepsilon_2}(m)) : m \in S^1\right\} \subset \mathbb{R}^{2\times 2}.$$

One of its key properties, obtained in [36, Lemma 7] and inspired from the work of Šverak on the Tartar conjecture [48], is the following inequality:

$$\det(X - Y) \gtrsim |X - Y|^4 \qquad \forall (X,Y) \in K \times K.$$

Therefore the quantity Δ_{JK} can be estimated from below by

$$\Delta_{JK}(x,h,e) \gtrsim \left|D^h_e \left(\Sigma_{e_1,e_2}(m), \Sigma_{\varepsilon_1,\varepsilon_2}(m)\right)(x)\right|^4 \gtrsim |D^h_e m(x)|^4,$$

where the last inequality follows from the (easily checkable) fact that $m \mapsto (\Sigma_{e_1,e_2}, \Sigma_{\varepsilon_1,\varepsilon_2})$ is an immersion.

Following [25], we aim to apply the div-curl Lemma, taking advantage of the fact that

$$\nabla \cdot \Sigma_{e_1,e_2}(m) = \mu_{e_1,e_2}, \qquad \nabla \cdot \Sigma_{\varepsilon_1,\varepsilon_2}(m) = \mu_{\varepsilon_1,\varepsilon_2},$$

are locally finite measures. To this end let us fix χ a smooth cutoff function and set

$$E := \chi D_e^h \Sigma_{e_1,e_2}(m), \qquad B := \chi D_e^h \Sigma_{\varepsilon_1,\varepsilon_2}(m).$$

Lemma 4.2. *For every $p \in]1,\infty[$ the following estimate holds true:*

$$\int_{\mathbb{R}^2} E \wedge B\, dx \lesssim pp' \big(\|E\|_{L^p} \|\nabla \cdot B\|_{W^{-1,p'}} + \|B\|_{L^p} \|\nabla \cdot E\|_{W^{-1,p'}} \big). \tag{4.3}$$

Proof. The proof is nowadays standard, and we report it for the reader's convenience: for $1 < p < \infty$, using the potential theoretic solution ϕ to $\Delta\phi = \nabla \cdot E$, we find that E can be Hodge-decomposed as

$$E = \nabla\phi + \nabla^\perp \psi, \qquad \|\nabla\phi\|_{L^{p'}(\mathbb{R}^2)} \lesssim pp' \|\nabla \cdot E\|_{W^{-1,p'}(\mathbb{R}^2)},$$

([27, Theorem 4.4.1], [49]), which yields

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} E \wedge B\, dx \right| &\le \left| \int \nabla\phi \wedge B \right| + \left| \int \nabla^\perp \psi \wedge B \right| \\
&\lesssim \|\nabla\phi\|_{L^{p'}} \|B\|_{L^p} + \left| \int \left(\psi - \fint_{\mathrm{spt}(B)} \psi \right) \nabla \cdot B \right| \\
&\lesssim pp' \|\nabla \cdot E\|_{W^{-1,p'}(\mathbb{R}^2)} \|B\|_{L^p} + \|\nabla\psi\|_{L^p} \|\nabla \cdot B\|_{W^{-1,p'}(\mathbb{R}^2)} \\
&\lesssim pp' \|\nabla \cdot E\|_{W^{-1,p'}(\mathbb{R}^2)} \|B\|_{L^p} + (1 + pp') \|E\|_{L^p} \|\nabla \cdot B\|_{W^{-1,p'}(\mathbb{R}^2)}.
\end{aligned}$$

The conclusion follows from $pp' \ge 4$. □

Proposition 4.3. *Any solution m to* (M) *such that $\nabla \cdot \Sigma_{\alpha_1,\alpha_2}(m) \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ for $(\alpha_1,\alpha_2) = (e_1,e_2)$ and $(\varepsilon_1,\varepsilon_2)$, belongs to $B^s_{4,\infty;\mathrm{loc}}(\Omega)$ for every $s < 4$.*

Proof. The div-curl estimate of Lemma 4.2 reduces the control of $\int \chi^2 |D_e^h m|^4$ to the estimate of the product

$$\|\chi D_e^h \Sigma_{e_1,e_2}(m)\|_{L^p} \|\nabla \cdot (\chi D_e^h \Sigma_{\varepsilon_1,\varepsilon_2}(m))\|_{W^{-1,p'}}$$

and its companion obtained by exchanging E and B. Let us for simplicity drop the frame index and write Σ instead of $\Sigma_{\alpha_1,\alpha_1}$, and also write D^h instead of D_e^h. We start by estimating the $W^{-1,p'}$ norm of the second factor

$$\nabla \cdot (\chi D_e^h \Sigma(m)) = \chi D^h \mu + D_e^h \Sigma(m) \cdot \nabla\chi.$$

By Sobolev embedding $W^{1,p} \subset C^{1-\frac{2}{p}}$ for $p > 2$, it holds

$$\begin{aligned}
\|\chi D^h \mu\|_{W^{-1,p'}} &= \sup \left\{ \int D^h(\chi\psi) d\mu, \|\psi\|_{W^{1,p}} \le 1 \right\} \\
&\le \sup_{\|\psi\|_{W^{1,p}} \le 1} \|\mu\|_{\mathcal{M}} \|D^h(\chi\psi)\|_\infty \\
&\lesssim \|\mu\|_{\mathcal{M}} |h|^{1-\frac{2}{p}},
\end{aligned}$$

and therefore

$$\|\nabla \cdot (\chi D_e^h \Sigma(m))\|_{W^{-1,p'}} \lesssim \|\mu\|_{\mathcal{M}} |h|^{1-\frac{2}{p}} + |h|.$$

The L^p norms $\|\chi D_e^h \Sigma\|_{L^p}$ are uniformly bounded. Inserting this estimate in (4.3), and choosing $p = -\log(|h|)$ [49], one easily obtains the modulus of continuity

$$\int \chi^2 |D_e^h m|^4 dx \lesssim \int E \wedge B\, dx \lesssim (1 + \|\mu\|_{\mathcal{M}}) |h| \log(\frac{1}{|h|})$$

for $|h| < \exp(-2)$. This implies $m \in B^s_{4,\infty}(\mathrm{spt}\chi)$ for every $s < \frac{1}{4}$. □

Remark 4.4. It is unclear to the authors whether the $\frac{1}{4}$ exponent is optimal or not.

4.4. **Regularity up to the boundary.** It is tempting to conjecture that, under suitable regularity assumptions on $\partial\Omega$, a global estimate

$$\|m\|_{B^{1/3}_{3,\infty}(\Omega)} \lesssim 1 + \|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) \tag{4.4}$$

holds true whenever m solves (M) and satisfies the kinetic formulation (KIN) with a globally finite measure $\sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$.

To obtain (4.4), a natural strategy would be to extend m outside Ω and then apply our local estimates. However, without any further assumption on the trace of m on $\partial\Omega$, it is not straightforward how to extend m so that it still solves (M) and satisfies a kinetic formulation (KIN).

When the boundary of $\partial\Omega$ is flat, say $\Omega = \{x \in \mathbb{R}^2 \colon x_2 < 0\}$, then a simple extension by reflection across the boundary is possible (see the proof of Lemma 4.7 below). But reflecting across a curved boundary does not ensure in general that both the divergence-free and the unit field constraints are retained. Note that, relaxing in a first step the unit field constraint to $|m| \leq 1$, one could then apply Baire category arguments [16] to obtain a solution of (M), but the control on the entropy production would be lost.

Yet another attempt at obtaining a good extension would be to write locally $m = \nabla^\perp u$ and rely on the theory of viscosity solutions for the eikonal equation $|\nabla u|^2 - 1 = 0$, providing via the Hopf-Lax formula a canonical extension outside the domain [34]. But without any strenghtened assumptions on the trace (for instance $u_{\lfloor\partial\Omega} \in C^2$ and $\|\partial_\tau u\|_{C^0(\partial\Omega)} < 1$ ensure that this extension is BV outside Ω, see [34, Theorem 8.2]) it is unclear how to estimate the entropy production of this extension.

In spite of these obstacles, we manage to obtain some positive results regarding regularity estimates up to the boundary: if $\partial\Omega$ is Lipschitz, a simple interpolation argument yields global estimates of m in $B^{1/6}_{3,\infty}(\Omega)$ (see Proposition 4.6). If $\partial\Omega$ is $C^{1,1}$, approximating Ω from the inside with domains having locally flat boundaries, we are able to improve the interpolation argument and obtain a global $B^{2/9}_{3,\infty}(\Omega)$ estimate (see Proposition 4.8).

An important ingredient we begin with, is the following rescaled local estimate.

Lemma 4.5. *If $\delta > 0$, $m \colon B_\delta \to \mathbb{R}^2$ solves* (M) *and*

$$e^{is} \cdot \nabla_x (\mathbb{1}_{e^{is}\cdot m > 0}) = \partial_s \sigma, \qquad \sigma \in \mathcal{M}(B_\delta \times \mathbb{R}/2\pi\mathbb{Z}),$$

then

$$\frac{1}{h} \int_{B_{\delta/2}} |m(x+he) - m(x)|^3 dx \lesssim \delta + \|\sigma\|(B_\delta \times \mathbb{R}/2\pi\mathbb{Z}) \qquad \forall h \in (0, \delta/4),\ e \in S^1,$$

where the inequality is up to a universal constant.

Proof. By scaling one may assume $\delta = 1$ and then this is a consequence of the local estimates obtained in subsection 3.2. □

With Lemma 4.5 at hand we may prove the following:

Proposition 4.6. *Assume Ω is a bounded domain with Lipschitz boundary. If $m \colon \Omega \to \mathbb{R}^2$ solves* (M) *and satisfies the global kinetic formulation*

$$e^{is} \cdot \nabla_x (\mathbb{1}_{e^{is}\cdot m > 0}) = \partial_s \sigma, \qquad \sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z}),$$

then

$$\frac{1}{h^{1/2}} \int_\Omega |m(x+he) - m(x)|^3 dx \lesssim 1 + h^{1/2} \|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) \qquad \forall h > 0,\ e \in S^1,$$

where the inequality is up to a constant depending only on Ω*.*

Proof. Let $h > 0$ and $e \in S^1$. For any $\delta > 0$, by compactness and decomposing the plane in cubes of diameter δ, we may choose a sample $x_1, \ldots, x_N \in \Omega$ with $\mathrm{dist}(x_j, \partial\Omega) \geq 2\delta$ and satisfying the following properties:

$$\{x \in \Omega \colon \mathrm{dist}(x, \partial\Omega) \geq 2\delta\} \subset \bigcup_{j=1}^N B_\delta(x_j),$$

$$\sum_{j=1}^N \|\sigma\|(B_{2\delta}(x_j) \times \mathbb{R}/2\pi\mathbb{Z}) \lesssim \|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}),$$

$$N \lesssim \frac{1}{\delta^2}.$$

Thus, provided $\delta > 4h$ it holds, thanks to Lemma 4.5,

$$\begin{aligned}
\int_\Omega |m(x+he) - m(x)|^3 dx &\leq \sum_{j=1}^N \int_{B_\delta(x_j)} |m(x+he) - m(x)|^3 dx \\
&\quad + \int_{\{\mathrm{dist}(\cdot,\partial\Omega) \leq 2\delta\}} |m(x+he) - m(x)|^3 dx \\
&\lesssim h \sum_{j=1}^N \big(\delta + \|\sigma\|(B_{2\delta}(x_j) \times \mathbb{R}/2\pi\mathbb{Z})\big) + \delta \\
&\lesssim \frac{h}{\delta} + h\|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) + \delta.
\end{aligned}$$

The choice $\delta = h^{1/2}$ allows to conclude whenever $h^{1/2} > 4h$, namely for $h < 1/16$. For $h > 1/16$ the estimate is obvious. □

When $\partial\Omega$ is $C^{1,1}$ (i.e., a C^1 curve with bounded curvature) we can improve this interpolation argument: when the curvature of $\partial\Omega$ is bounded, near every boundary point there exists a half ball of radius $\sim \delta$, whose diameter lies at most $\sim \delta^2$ far from the boundary. On the other hand, entropy solutions defined on a half ball may be easily extended by reflection to the whole ball, keeping the entropy production under control.

Lemma 4.7. *Assume that* $\partial\Omega$ *is* $C^{1,1}$ *and let* $K = \sup_{\partial\Omega} |\mathrm{curvature}|$*. Let* $m \colon \Omega \to \mathbb{R}^2$ *solve* (M) *and satisfy the global kinetic formulation*

$$e^{is} \cdot \nabla_x(\mathbb{1}_{e^{is}\cdot m > 0}) = \partial_s \sigma, \qquad \sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z}).$$

Then for all $x_0 \in \partial\Omega$ *and* $\delta \in (0, \frac{1}{4K})$*, denoting by* $\omega_\delta(x_0)$ *the domain*

$$\omega_\delta(x_0) = B_{\delta/4}(x_0) \cap \left\{x \in \Omega \colon \mathrm{dist}(x, \partial\Omega) \geq 3K\delta^2\right\},$$

it holds

$$\frac{1}{h}\int_{\omega_\delta(x_0)} |m(x+he) - m(x)|^3 dx \lesssim \delta + \|\sigma\|(B_\delta(x_0) \cap \Omega \times \mathbb{R}/2\pi\mathbb{Z}) \qquad \forall h \in (0, K\delta^2),\ e \in S^1,$$

where the inequality is up to a universal constant.

Proof. We denote by ν_0 the outer normal vector to $\partial\Omega$ at x_0. Note that, since the curvature of $\partial\Omega$ is bounded by K, it holds

$$B_{\frac{1}{K}}(x_0 - \frac{1}{K}\nu_0) \subset \Omega \qquad \text{and} \qquad B_{\frac{1}{K}}(x_0 + \frac{1}{K}\nu_0) \subset \mathbb{R}^2 \setminus \Omega.$$

For any $\delta \in (0, \frac{1}{4K})$ we set

$$H_\delta^- = B_\delta(x_0) \cap \left\{x \in \mathbb{R}^2 \colon (x - x_0)\cdot \nu_0 \leq -K\delta^2\right\}.$$

From the inclusions

$$B_\delta(x_0)\cap\left\{x\in\mathbb{R}^2\colon \operatorname{dist}\left(x,B_{\frac{1}{K}}(x_0+\frac{1}{K}\nu_0)\right)\geq 2K\delta^2\right\}\subset H_\delta^-\subset B_{\frac{1}{K}}(x_0-\frac{1}{K}\nu_0),$$

we deduce that

$$B_\delta(x_0)\cap\left\{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)\geq 2K\delta^2\right\}\subset H_\delta^-\subset\Omega. \tag{4.5}$$

We denote by S_δ the orthogonal reflection across the line $\{(x-x_0)\cdot\nu_0=-K\delta^2\}$, and by $S_\delta^\perp$ the orthogonal reflection across the perpendicular line $x_0+\mathbb{R}\nu_0$. Consider the reflected domain $H_\delta^+=S_\delta H_\delta^-$, and define $H_\delta=H_\delta^+\cup H_\delta^-$. Note that since $\delta<\frac{1}{2K}$, it holds $B_{\delta/2}(p)\subset H_\delta$, where $p=x_0-K\delta^2\nu_0$. In H_δ we define a solution $\overline{m}$ of (M) by setting $\overline{m}:=m$ in H_δ^- and $\overline{m}:=S_\delta^\perp\circ m\circ S_\delta$ in H_δ^+. For any $f\in C^0(\mathbb{R}/2\pi\mathbb{Z})$ with $\|f\|_\infty\leq 1$ it can then be easily checked that

$$\|\nabla\cdot\Phi_f(\overline{m})\|(H_\delta)\lesssim\delta+\|\nabla\cdot\Phi_f(m)\|(H_\delta^-)+\|\nabla\cdot\Phi_{\check{f}}(m)\|(H_\delta^-),$$

where $\check{f}(t)=f(-t)$ (see for instance [14, Lemma 3.1]). Since on the other hand Lemma 3.4 ensures that

$$\|\nabla\cdot\Phi_f(m)\|(H_\delta^-)\leq\|\sigma\|(H_\delta^-\times\mathbb{R}/2\pi\mathbb{Z}),$$

we deduce that

$$\|\nabla\cdot\Phi_f(\overline{m})\|(H_\delta)\lesssim\delta+\|\sigma\|(H_\delta^-\times\mathbb{R}/2\pi\mathbb{Z})\qquad\forall f\in C^0(\mathbb{R}/2\pi\mathbb{Z})\text{ with }\|f\|_\infty\leq 1.$$

By the arguments in paragraph 3.1 this implies the existence of $\overline{\sigma}\in\mathcal{M}(H_\delta\times\mathbb{R}/2\pi\mathbb{Z})$ such that

$$\begin{aligned}&e^{is}\cdot\nabla_x(\mathbb{1}_{e^{is}\cdot\overline{m}>0})=\partial_s\overline{\sigma},\\&\|\overline{\sigma}\|(H_\delta\times\mathbb{R}/2\pi\mathbb{Z})\lesssim\delta+\|\sigma\|(H_\delta^-\times\mathbb{R}/2\pi\mathbb{Z}).\end{aligned}$$

Recall that we have the inclusions $B_{\delta/2}(p)\subset H_\delta$. Note moreover that (4.5) ensures that $\omega_\delta\subset B_{\delta/4}(p)$. Hence we may invoke Lemma 4.5 and obtain the following bound, for every $h\in(0,\delta/8)$ and $e\in S^1$:

$$\frac{1}{h}\int_{\omega_\delta}|\overline{m}(x+he)-\overline{m}(x)|^3dx\lesssim\delta+\|\sigma\|(H_\delta^-\times\mathbb{R}/2\pi\mathbb{Z}).$$

The conclusion follows from (4.5) which ensures that $\overline{m}(x)=m(x)$ and $\overline{m}(x+eh)=m(x+eh)$ for any $x\in\omega_\delta$ and $h\in(0,K\delta^2)$, and from the fact that $H_\delta^-\subset B_\delta(x_0)\cap\Omega$. □

Using Lemma 4.7 we obtained the following improved estimate in regular domains.

Proposition 4.8. *Assume Ω is a bounded domain with $C^{1,1}$ boundary. If $m\colon\Omega\to\mathbb{R}^2$ is solves* (M) *and satisfies the global kinetic formulation*

$$e^{is}\cdot\nabla_x(\mathbb{1}_{e^{is}\cdot m>0})=\partial_s\sigma,\qquad\sigma\in\mathcal{M}(\Omega\times\mathbb{R}/2\pi\mathbb{Z}),$$

then

$$\frac{1}{h^{2/3}}\int_\Omega|m(x+he)-m(x)|^3dx\lesssim 1+h^{1/3}\|\sigma\|(\Omega\times\mathbb{R}/2\pi\mathbb{Z})\qquad\forall h>0,\ e\in S^1,$$

where the inequality is up to a constant depending only on Ω.

Proof. Let $h>0$ and $e\in S^1$. For any $\delta>0$ we may find points $x_1,\ldots x_N\in\Omega$ with $\operatorname{dist}(x_j,\partial\Omega)\geq\delta/2$ and points $y_1,\ldots,y_M\in\partial\Omega$ such that

$$\Omega\subset\bigcup_{j=1}^N B_{\delta/4}(x_j)\cup\bigcup_{j=1}^M B_\delta(y_j),$$

$$\sum_{j=1}^{N} \|\sigma\|(B_{\delta/2}(x_j) \times \mathbb{R}/2\pi\mathbb{Z}) + \sum_{j=1}^{M} \|\sigma\|(B_{2\delta}(y_j) \times \mathbb{R}/2\pi\mathbb{Z}) \lesssim \|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}),$$

$$N \lesssim \frac{1}{\delta^2}, \quad M \lesssim \frac{1}{\delta}.$$

Then, provided δ satisfies $1/4K > \delta > \max(4h, \sqrt{h/K})$, thanks to Lemma 4.5 and Lemma 4.7 it holds

$$\begin{aligned}\int_\Omega |m(x+he) - m(x)|^3 dx &\leq \sum_{j=1}^{N} \int_{B_\delta(x_j)} |m(x+he) - m(x)|^3 dx \\ &\quad + \sum_{j=1}^{M} \int_{B_\delta(y_j)\cap\Omega} |m(x+he) - m(x)|^3 dx \\ &\lesssim h\|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) + \frac{h}{\delta} \\ &\quad + \sum_{j=1}^{M} \left|B_\delta(y_j) \cap \{\mathrm{dist}(\cdot, \partial\Omega) \leq 2K\delta^2\}\right| \\ &\lesssim h\|\sigma\|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) + \frac{h}{\delta} + \delta^2.\end{aligned}$$

To conclude, we choose $\delta = h^{1/3}$, which is possible whenever $1/(4K) > h^{1/3}$, $h^{1/3} > 4h$ and $h^{1/3} > \sqrt{h/K}$, i.e. $h < h_0 := \min(K^3, (4K)^{-3}, (1/4)^{3/2})$. For $h \geq h_0$ the estimate is obvious. □

Remark 4.9. The assumptions of Proposition 4.8 can be slighlty relaxed. If $\partial\Omega$ is merely Lipschitz and piecewise $C^{1,1}$, the same proof applies. One just has to pay special attention to the balls covering the boundary: a uniformly bounded number of balls will contain points where the boundary is not $C^{1,1}$ (and therefore where one can not apply Lemma 4.7). This introduces an extra error of order $\sim \delta^2$, and since the other parts of boundary points can be treated exactly as in Proposition 4.8, the conclusion remains.

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