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Games without winners: Catching-up with asymmetric spillovers*

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Abstract

Dynamic game with changing leader is studied on the example of R&D co-opetition structure. The leader benefits from higher followers' innovations rate and followers are enjoying a spillover from the leader. Leadership changes because of asymmetric efficiency of investments of players. It is demonstrated that under sufficiently asymmetric players there is no long-run leader in this game and all players act as followers. Moreover this outcome may be the socially optimal one. In decentralised setting additional complex types of dynamics are observed: permanent fluctuations around symmetric (pseudo)equilibrium and chaotic dynamics. This last is possible only once strategies of players are interdependent. Cooperative solution is qualitatively similar for any number of players while market solution is progressively complex given all players are asymmetric. Results are extended to an arbitrary linear-quadratic multi-modal differential game with spillovers and the structure necessary for the onset of non-deterministic chaos is discussed.

Keywords: technological spillovers; heterogeneous innovations; asymmetric players; social optimality; market inefficiency; multi-modal differential games; piecewise-smooth systems.

JEL classification: C61, C73, L16, O32

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1 Introduction

Many economic phenomena are described by piecewise dynamics. This relates to changing leadership in dynamic competition, intellectual property rights reforms leading to the change in regime of the industry and changes in government economic policy in general.

Many models in the literature describe various economic areas with the help of such systems (such as technology catch-up across countries (Motta, Thisse, and Cabrales 1997), change in industrial leadership (Giachetti 2013), technology leap-frogging within industries (Bondarev 2016), (Dawid, Kopel, and Kort 2013) and etc.). However little attention has been paid to the global dynamics of such systems with asymmetric actors. Yet the dynamic properties of such piecewise systems may have a profound impact on our treatment of the efficiency of economic regulation and types of this regulation necessary to correct potential market inefficiencies. In particular, growing complexity of market interactions may call to new types of interventions which are yet to be developed.

In this paper I make use of some recent findings in piecewise-smooth dynamical systems, (Dixon 1995), (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008), (Jeffrey and Colombo 2009) and (Colombo and Jeffrey 2011) and apply them to the standard framework of R&D differential game like (Bondarev 2016) along with learning-by-doing effects as modelled in (Greiner and Bondarev 2017). I show that the growing complexity of r&d may lead to emergence of some complex dynamics as an emergent technical change property, see (Antonelli 2016) for discussion on this issue.

The baseline framework closely follows that of (Bondarev and Greiner 2017). Some (arbitrary) number of R&D firms is engaged both in quality-improving innovations for existing products and in introduction of new products to the market. While firms jointly develop the range of new products, they independently develop all existing products. In that paper only the decentralised open-loop Nash equilibria are analysed and the multiplicity of these equilibria emerging due to learning-by-doing effects are studied. Present paper rather concentrates on the emergent properties of the game because of the piecewise-smooth nature of the dynamical system and explores differences in cooperative and non-cooperative solutions.

It builds up on the results of (Gromov and Gromova 2017) on multi-modal differential games and contributes further to our understanding of the potential types of dynamics in such games. In particular, the results of the aforementioned paper remain valid only for special configuration of the global dynamics whereas generically the emergence of the so-called Teixeira singularity, (Teixeira 1990) is to be expected.

On the example of the aforementioned baseline model I demonstrate that the presence of asymmetry across firms may lead to profound differences in qualitative behavior of the market. However the result itself is more general and applies to *any* piecewise-defined market dynamics with asymmetric actors.

The paper contributes first to the general field of multi-modal differential games which are gaining popularity in economic literature by considering new emergent types of dynamics. Second, it contributes to the R&D policy debate by demonstrating new source of market inefficiency stemming from the asymmetry of firms. Third, it demonstrates the ambiguous role of the competition, since increasing number of firms may divert market equilibrium from the social optimum.

In the next section the model setup is described, Section 3 describes the cooperative (central planner) solution, Section 4 describes non-cooperative setting, Section 5 generalizes the results obtained to an arbitrary multi-modal differential game and Section 6 concludes.

2 The Model

Consider an industry with N multi-product firms engaged in R&D. These firms invest both in horizontal innovations as well as in the improvement of the qualities of existing products, i.e. in vertical innovations. As regards the structure of the R&D competition, assume that the firms cooperate in the development of new products while the quality improvements of the products are undertaken by each firm separately where the followers can benefit from the technological leader through spillovers. The reason for this model structure is that it may be beneficial for firms to cooperate when the development of new products is expensive even if they are competitors otherwise. Further, there exists empirical evidence for such a behavior and in the literature one sometimes refers to such a situation as co-opetition, i.e. the simultaneous existence of cooperation and competition. (Gnyawali and Park 2009) find evidence that in particular small and medium sized enterprises pursue such strategies because they face very large challenges in their attempt to realize technological innovations and co-opetition is an effective means to achieve innovations. But large firms often cooperate, too, in particular in high technology sectors because of several challenges such as shortened product life cycles, the need for large investments in R&D and the importance of technological standards, for example, see (Gnyawali and Park 2011).

The formal intertemporal optimization problem closely follows of the firm k can be written as

$$\forall k \in N \subset \mathbb{R}_{++} : \; J_k \stackrel{\text{def}}{=} \max_{u(\cdot),g(\cdot)} \int_0^{\infty} \mathrm{e}^{-rt} \left(\int_0^{n(t)} \Big[q_k(i,t) - \frac{1}{2} g_k(i,t)^2 \Big] di - \frac{1}{2} u_k(t)^2 \right) dt. \qquad (1)$$

where

- $q_k(i,t)$ is the quality of the product(technology) i at time t achieved by the firm k;
- $g_k(i,t)$ are investments into quality $q_k(i,t)$ by the firm k at time t;
- $n(t)$ is the currently available variety of technologies at t;
- $u_k(t)$ are investments by firm k at time t into the expansion of the variety of technologies

The evolution of the variety of products (technologies) is governed by joint investments of the firms whereas the development of the quality of each new product is individual and subject to the imitation effect:

$$\begin{aligned} \dot{n}(t) &= \left(\sum_{k=1}^{N} \alpha_k u_k(t) \right) - \delta n(t), \\ \dot{q}_k(i,t) &= \psi_k(i) g_k(i,t) - \beta_k q_k(i,t) + \theta \max\left\{0, q_{-k}(i,t) - q_k(i,t)\right\}, \; \forall i \in \mathbf{I} \subset \mathbb{R}_+. \end{aligned} \qquad (2)$$

where

- α_k is the efficiency of investments of firm k into variety expansion;
- $\psi_k(i)$ is the (heterogeneous) efficiency of investments of firm k into quality of product i;
- β_k is the depreciation rate of technology for the firm k;
- θ is the intensity of imitation effect.

The last term in the second equation of (2) gives the *cross-technologies imitation effect*, where θ determines its magnitude.

The efficiency of investments into fundamental research α_k is changing relative to the position of the firm (whether it is the leader[1] of R&D or not):

$$\alpha_k = \begin{cases} \alpha_k^-, \text{if k is not the leader;} \\ \alpha_k^+, \text{if k is the leader} \end{cases} , \; \alpha_k^- \geq \alpha_k^+ \qquad (3)$$

[1]Observe that this is not a standard Stackelberg leadership, the notion of the leader is to be defined further

with $\alpha_k^- - \alpha_k^+$ being the *efficiency asymmetry*: the higher it is the more efficient is the firm in increasing variety of technologies while being the follower.

It is thus assumed that followers are investing more in the fundamental research. This assumption is in line with findings of (Bondarev 2014): under the close loop regime only imitators are investing in new technologies creation and of (Acemoglu and Cao 2015) where only new entrants are creating drastic innovations. In general we can define any α_k efficiency, it suffices to get different efficiencies of fundamental research in different regimes for every player.

This model is close to the setup in (Bondarev and Greiner 2017), where multiple regimes are also obtained, but all players invest with similar efficiency in new products' creation, i. e. $\forall k: \alpha_k^- = \alpha_k^+ = 1$. As such that model is a degenerate case[2] of the more general setup considered here. That model is referred to as the baseline model throughout the text.

The efficiency of quality investments $\psi_k(i)$ in general may take any form. To simplify the analysis we assume it to be such that the resulting system is linear in the state n, that is

$$\psi_k(i) = \sqrt{\psi_k i + \gamma_k}. \tag{4}$$

Specification (4) makes every regime of the game a linear-quadratic one, which is simple to analyze. So current model focuses on the effect of the asymmetry across players, introduced by α_k term rather then on non-linearity effects. It turns out that these asymmetries play crucial role in the complexity of potential developments.

We next define what is called leadership in this differential game. To this end restrict attention to the case $\beta_k = \beta$ so that the leadership in development of each technology i cannot change in time. Then we can define[3]

Definition 1 (Leader and followers)**.**
The player k is the leader in development of technology i as long as this firm has maximal efficiency of investments into i:

$$\psi_k(i) > \psi_{-k}(i) \tag{5}$$

[2]it will be demonstrated that this minor extension of asymmetry may lead to qualitatively different results and substantially more complicated dynamics

[3]under different β_k leadership may change for each i, as shown in (Bondarev 2016), for constant β across players the leader in each technology remains constant as shown in (Bondarev 2014).

Further, we impose a number of state and control constraints:

$$\begin{aligned}
&\forall k \in N: \\
&\forall i \in \mathbf{I}, \forall t \geq 0: \; g_k(i,t)|_{i>n(t)} = 0; && (6)\\
&\forall i \in \mathbf{I}, \forall t \geq 0: \; q_k(i,t) \geq 0; && (7)\\
&0 \leq g_k(i,t) < \infty, \; 0 \leq u_k(t) < \infty; && (8)\\
&n(t) \in \mathbf{I}. && (9)
\end{aligned}$$

Condition (6) states that each non-existent technology has zero investments while it is out of the market. This makes sense from an economic point of view because it states that there is a difference between the invention of a product and its innovation. Condition (7) states that level of each technology cannot be negative. Condition (8) imposes non-negativity constraints and the boundedness of investments, condition (9) constrains the variety to be positive real range and maximum principle is understood in the sense of (Skritek, Stachev, and Veliov 2014), (Aseev and Veliov 2015)[4].

The functions $\psi_k(i)$ determine the efficiency of R&D in quality innovations so that we refer to them as efficiency functions. These are functions of the variety already invented and we will allow for learning by doing effects implying that $\psi(\cdot)$ is an increasing function. This implies catching-up in is possible the space of technologies, but not within each technology development.

Further, denote the firm k the *leader* across i_k technologies as long as $\psi_k(i_k) > \psi_{-k}(i_k)$ i. e. this firm has the highest efficiency of quality investments across all firms for the range $i_k \subseteq \mathbb{R}_{>0}$.

Define the leader of the game at time t as follows:

Definition 2 (Leader of the game).
At time t the player k is called the leader of the game if this player's efficiency of investments into the next technology to be invented, $n(t)$, is maximal:

$$\psi_k(n(t)) > \psi_{-k}(n(t)) \tag{10}$$

It follows that as long as $\psi_k(i)$ are monotonic functions, the leader of the game is the leader for all technologies $i \leq n(t)$.

Now, observe that even with monotonic efficiency functions as (4) of the same type there may exist multiple regimes of this differential game, with low and high variety levels. In this model

[4]this last does not require compactness of the state space, transversality conditions converge to standard ones with constant discount rate

the multi-modality is the result solely of strategic effects and not of potential non-linearities in learning functions and etc., since every mode of the game is linear-quadratic and is known to have only one equilibrium point (see e. g. (Dockner, Jorgensen, Long, and Sorger 2000)).

In particular, if no intersection of efficiency functions exist, the game has a constant leader (in the sense of (10)) and the dynamical system possesses only one steady state, $\bar{n}_1$. If there exists an intersection, there is always at most one such (because of monotonicity and linearity of efficiency functions), one player is the leader in the development of technologies up to the point at which the efficiency functions intersect where the leadership changes. This piecewise-defined system is more complicated and may be characterized by two steady-states. It is apparent that once we have more than two firms, the number of potential equilibria is bounded from above by this number N, but we limit the majority of exposition to the case of two firms. To this end define the level of $n(t)$ where leadership changes from one firm to the other as n^* and subsequently for the case of more than two firms by n_k^* the level where firm k becomes the leader.

It turns out that within this simple framework with linear-quadratic game around each of the steady states we may observe rather interesting dynamics. Namely, once we have a changing leadership, the game becomes a bi-modal one, (see (Gromov and Gromova 2017) for details) and the governing dynamical system is of piecewise-smooth type. These systems may exhibit so-called discontinuity-induced bifurcations (DIBS, see (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008) for details) which are recently actively studied.

In particular we are interested in the existence of so-called Teixeira singularity named after the pioneering work (Teixeira 1990), since it was shown that it can exhibit non-deterministic chaos, (Colombo and Jeffrey 2011). We thus study what economic implications this may have for the associated R&D dynamics both from the decentralised and socially optimal viewpoints and discuss what kind of additional economic regulation it may require.

In what follows both cooperative and non-cooperative setups are considered. In both cases the overall game has two modes: normal one and the switching one defined as follows (following (Di Bernardo, Budd, Champneys, Kowalczyk, Nordmark, Olivar, and Piiroinen 2008)):

Definition 3. *The 2-firms game is in the normal mode, if its optimal trajectory never reaches n^*. The game is in the switching mode, if its optimal trajectory visits n^* at least once. It is in the sliding mode if it stays at n^* for positive time.*

The N-firms game is in the normal mode only if none of $\forall k \in N:\ n_k^*$ is never reached by the optimal trajectory. Otherwise it is said to be in the switching mode. It is in the k-th sliding mode if it stays at n_k^* for positive time.

So in the sliding mode the dynamical system is characterized by the $N-1$ manifold and associated dynamics within it requires special treatment.

It will be demonstrated that both cooperative and non-cooperative dynamics are capable of exhibiting switching modes and sliding modes as optimal ones. However, in cooperative game the dimensionality of the resulting dynamical system is invariant to the number of firms, while this is not true for the non-cooperative case. Thus the cooperative case is studied first as the simpler one.

3 Cooperative game

Consider first the cooperative solution denoted as a social planner.

The social planner problem has the objective functional:

$$W \stackrel{\text{def}}{=} \max_{\vec{u}(\cdot),\vec{g}(\cdot)} \sum_{k}^{N} \int_0^{\infty} \mathrm{e}^{-rt} \left(\int_0^{n(t)} \left[q_k(i,t) - \frac{1}{2} g_k(i,t)^2 \right] di - \frac{1}{2} u_k(t)^2 \right) dt. \tag{11}$$

with dynamic constraints (2).

Limiting ourselves to $N=2$ and after rearranging terms and combining integrals we get the Hamiltonian function as:

$$\begin{aligned}
\mathcal{H}_W = & \int_0^{n(t)} \left[q_1(i,t) + q_2(i,t) - \frac{1}{2}\left(g_1^2(i,t) + g_2^2(i,t)\right] di - \frac{1}{2}\left(u_1^2(t) + u_2^2(t)\right) + \right. \\
& + \lambda_W(\alpha_1 u_1(t) + \alpha_2 u_2(t) - \delta n(t)) + \\
& + \int_0^{n^*} \left[\phi_1 \left(\psi_1(i) g_1(i,t) - \beta q_1(i,t)\right) + \phi_2 \left(\psi_2(i) g_2(i,t) - \beta q_2(i,t) + \theta(q_1(i,t) - q_2(i,t)\right)\right] di + \\
& + \int_{n^*}^{n(t)} \left[\phi_1 \left(\psi_1(i) g_1(i,t) - \beta q_1(i,t) + \theta(q_2(i,t) - q_1(i,t)\right) + \phi_2 \left(\psi_2(i) g_2(i,t) - \beta q_2(i,t)\right)\right] di
\end{aligned} \tag{12}$$

Proceeding as in the baseline model, we get the canonical system for n dynamics as:

$$\begin{aligned}
\dot{n} &= ((\alpha_1)^2 + (\alpha_2)^2)\lambda_W - \delta n, \\
\dot{\lambda}_W &= (r+\delta)\lambda_W - \frac{1}{2} \begin{cases} \frac{\psi_1 n + \gamma_1}{(r+\beta)^2} + \frac{\psi_2 n + \gamma_2}{(r+\beta+\theta)^2}, \text{if } n < n^*; \\ \frac{\psi_1 n + \gamma_1}{(r+\beta+\theta)^2} + \frac{\psi_2 n + \gamma_2}{(r+\beta)^2}, \text{if } n > n^* \end{cases}
\end{aligned} \tag{13}$$

which turns to be a 2-dimensional one irrespective of the number of firms. Denote by the upper (lower) regime of the game the time period such that $n(t) > n^*(n(t) < n^*)$ and associated dynamical systems by f^+, f^-.

The system (13) has unique steady states for each of the regimes, denoted $\{\hat{n}^{\pm}, \hat{u}_1^{\pm}, \hat{u}_2^{\pm}\}$ with superscripts $+$ for upper and $-$ for lower regimes respectively.

Assume the system is non-singular and associated steady-states are (saddle) stable[5]. Due to the piecewise character of the dynamics apart from standard upper and low regimes dynamics it may possess an additional sliding dynamics, once the sliding region is non-empty.

Definition 4. *The sliding region SL of the state-space of* (13) *is the range λ_W at $n = n^*$ (switching manifold) such that*

$$SL := \{\lambda_W, n^*\} : \ \dot{\lambda}_W^+|_{n=n^*} < 0, \ \dot{\lambda}_W^-|_{n=n^*} > 0 \tag{14}$$

The escaping region ES of the state space of (13) *is the geometric place of points:*

$$ES := \{\lambda_W, n^*\} : \ \dot{\lambda}_W^+|_{n=n^*} > 0, \ \dot{\lambda}_W^-|_{n=n^*} < 0 \tag{15}$$

Otherwise the region is called a crossing region.

So the sliding region is a part of the line $n = n^*$ where both regimes' dynamics point into the switching manifold n^* and escaping region is where both regimes' dynamics point out of the switching manifold. The crossing region forms if the direction of dynamics coincide on both sides of n^*.

Once the sliding region is non-empty it may contain the sliding mode[6], defined as:

$$\dot{\lambda}_W^s = (r+\delta)\lambda_W - \frac{(\gamma_1\psi_2 - \gamma_2\psi_1)}{\psi_2 - \psi_1}\frac{r^2 + (2\beta+\theta)r + \beta^2 + \beta\theta + \frac{1}{2}\theta^2}{(r+\beta+\theta)^2(r+\beta)^2} \tag{16}$$

which is a one-dimensional system with the equilibrium

$$\hat{\lambda}^s = \frac{(\gamma_1\psi_2 - \gamma_2\psi_1)}{\psi_2 - \psi_1}\frac{r^2 + (2\beta+\theta)r + \beta^2 + \beta\theta + \frac{1}{2}\theta^2}{(r+\beta+\theta)^2(r+\beta)^2}\frac{1}{r+\delta} > 0 \tag{17}$$

[5] if this is not the case, the problem either do not have a long-run solution (both steady states are unstable), or this solution is uniquely defined (one stable steady state)

[6] the procedure to obtain the sliding dynamics is described in e. g. (Colombo and Jeffrey 2011) to which the reader is referred for technical details.

which is always an unstable one. The point $\{n^*, \hat{\lambda}^s\}$ is called the *pseudoequilibrium* of the dynamical system (13) and thus of the game.

Still, it can be reached from the outside of the sliding mode by choosing appropriate $u_1(0), u_2(0)$ which are free. We first ask the question, under which conditions the cooperative game is in the normal or switching mode, see Def. 3. We need additional definition:

Definition 5. *The equilibrium of the upper (lower) regime of the game is regular, if* $\hat{n}^+ > n^*(\hat{n}^- < n^*)$.

The equilibrium is virtual if $\hat{n}^+ < n^*(\hat{n}^- > n^*)$.

The equilibrium is a boundary one if $\hat{n}^+ = n^*(\hat{n}^- = n^*)$.

Denote $\hat{n}_r^\pm$ and $\hat{n}_v^\pm$ regular and virtual equilibria of the cooperative game respectively. Denote further $\hat{n}_{ss}$ the *Skiba (DNSS) point* of the game, with the property that $\forall \epsilon > 0: \ W^-(n_{ss}, 0) = W^+(n_{ss}, 0)$, $W^-(n_{ss} - \epsilon, 0) > W^+(n_{ss} - \epsilon, 0)$, $W^-(n_{ss} + \epsilon, 0) < W^+(n_{ss} + \epsilon, 0)$,where $+, -$ superscripts denote the value of the (11) in upper and lower regimes respectively. The game starting at n_{ss} is indifferent in converging to either of the equilibria while uniquely converging to one of them off this threshold. We use this notion following (Wagener 2003) and similar papers.

We note that:

Lemma 1. *The cooperative game given by* (11), (2) *is in the switching mode if one of the following holds:*

1. *At least one of the cooperative steady states is virtual, and* $n_0 < n^* < \hat{n}_r^+$ *or* $n_0 > n^* > \hat{n}_r^-$
2. *Both equilibria are regular, but* $n^* > n_0 > \hat{n}_{ss} > \hat{n}_r^-$ *or* $n^* < n_0 < \hat{n}_{ss} < \hat{n}_r^+$ *.*

Otherwise the cooperative game is in the normal mode.

Proof. If both equilibria are virtual, the only long-run dynamics is converging to the switching manifold. If one is regular, the optimal trajectory would converge to it (as long as it is (saddle) stable). Thus as soon as starting position is on the other side of the switching manifold, this last will be visited at least once by the optimal trajectory.

At last, if both equilibria are regular, but the Skiba-point exists, it is optimal to converge to the equilibrium lying at the other side of the n^* by definition of the Skiba-point. □

Note that this Skiba point is not the property of any of the sub-systems f_+, f_- which are linear, but is an emergent property of the switching manifold. We thus refer to it (to distinguish from normal Skiba point) as *pseudo-Skiba point* in analogue to pseudoequilibrium as an equilibrium of the sliding flow[7].

As soon as Lemma 1 holds, there are three qualitative types of dynamics:

1. The optimal trajectory visits the switching manifold finite number of times

2. The optimal trajectory converges to the switching manifold and stays there

3. There are infinitely many crossings of the switching manifold by the optimal trajectory

In the first case the optimality conditions are obtained in the same manner as in (Gromov and Gromova 2017): we first determine the long-run outcome of the game, then define the starting point at the switching manifold, which leads to the optimal trajectory, and compute the remaining part as a fixed end-point free terminal time optimal control problem. Still, this method is valid only if the switching event is a unique one. As soon as there are multiple switches, more then one iteration may be necessary. Still, as long as this number stays finite, the qualitative procedure is the same (but becomes computationally challenging).

In the second case n^* realizes as a pseudo-equilibrium of the game and in the third case we get the pseudo limit-cycle around the switching manifold (stable or unstable). Observe that so far only the first type of dynamics has been considered in the dynamic games literature and we are thus mainly interested in the two others.

We start with the second case.

Lemma 2. *For the pseudoequilibrium to realize as the long-run outcome of the cooperative game it is necessary that sliding region* $SL \neq \emptyset$ *and* $\hat{\lambda}^s \in SL$.

This is sufficient as long as either both $\hat{n}^{\pm}$ *are virtual or* $W(\hat{n}_r^{\pm}, \hat{\lambda}_r^{\pm}, \infty) < W(\hat{\lambda}^s, n^*, \infty)$.

Proof. The pseudoequilibrium is a feasible outcome of the game only if it exists. For this it is necessary that sliding region is non-empty (sliding flow exists) and its equilibrium is within the boundaries of this region.

If both equilibria of the game are virtual, this is the only possible long-run outcome of the game. If at least one of them is regular, then by Lemma 1, for every n_0 one of the equilibria is selected. Then for convergence to the pseudoequilibrium to be optimal, it must yield higher social welfare in total. □

[7] defined further on

Observe, that the Lemma 2 does not include stability requirements for the pseudoequilibrium. Indeed, since this last is the equilibrium of the sliding flow and not of the original system, the trajectory converging to this point will exist always as soon as sliding region is non-empty (by definition of the sliding region).

For the sliding region of the cooperative game to be non-empty the following has to hold:

$$(\alpha_1^-)^2 + (\alpha_2^+)^2 < (\alpha_1^+)^2 + (\alpha_2^-)^2 \tag{18}$$

i. e. the overall investments efficiency in the upper regime should be *lower* than in the lower regime.

The equilibrium of the sliding flow λ^s is given by (17) and has to lie within λ_+, λ_- tangency points, defined via $\dot{n}^+|_{n=n^*} = 0$ and $\dot{n}^-|_{n=n^*} = 0$ respectively:

$$\lambda_+ = \frac{\delta(\gamma_1 - \gamma_2)}{\psi_2 - \psi_1} \frac{1}{(\alpha_1^-)^2 + (\alpha_2^+)^2}, \ \lambda_- = \frac{\delta(\gamma_1 - \gamma_2)}{\psi_2 - \psi_1} \frac{1}{(\alpha_1^+)^2 + (\alpha_2^-)^2} \tag{19}$$

Thus the first (necessary) condition of Lemma 2 can be expressed as

$$\lambda_+ > \hat{\lambda}^s > \lambda_- \tag{20}$$

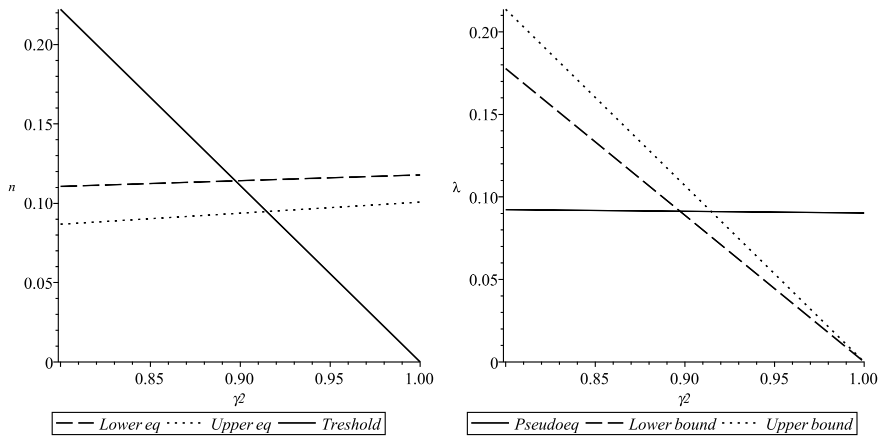
The condition for social planner problem to have virtual steady states is:

$$\hat{n}^+ < n^* < \hat{n}^- \tag{21}$$

with expressions for steady states:

$$\begin{aligned} \hat{n}^- &= \frac{\left((\alpha_1^+)^2 + (\alpha_2^-)^2\right)\left(\frac{1}{2}\frac{\gamma_1}{(r+\beta)^2} + \frac{1}{2}\frac{\gamma_2}{(r+\beta+\theta)^2}\right)}{\left(\delta(r+\delta) - \frac{1}{2}\left((\alpha_1^+)^2 + (\alpha_2^-)^2\right)\left(\frac{\psi_1}{(r+\beta)^2} + \frac{\psi_2}{(r+\beta+\theta)^2}\right)\right)} \\ \hat{n}^+ &= \frac{\left((\alpha_1^-)^2 + (\alpha_2^+)^2\right)\left(\frac{1}{2}\frac{\gamma_1}{(r+\beta+\theta)^2} + \frac{1}{2}\frac{\gamma_2}{(r+\beta)^2}\right)}{\left(\delta(r+\delta) - \frac{1}{2}\left((\alpha_1^-)^2 + (\alpha_2^+)^2\right)\left(\frac{\psi_1}{(r+\beta+\theta)^2} + \frac{\psi_2}{(r+\beta)^2}\right)\right)} \end{aligned} \tag{22}$$

It turns out that both (20) and (21) hold or not simultaneously: that is, either both equilibria are virtual and sliding region exists with the pseudoequilibrium within it, either at least one of the equilibria is regular and the pseudoequilibrium lies in the escaping region (and sliding region is empty). We thus do not have to check optimality via value functions comparison, but rather look at the configuration of steady states of the overall problem. Figures 1, 2 illustrate these two situations with all parameters normalized to one except for $\alpha_1^+ = 0.5, \psi_1 = 0.1$ and α_2^+, γ_2 varying from zero to one.



(a) Location of steady states

(b) Location of pseudoequilibrium

Figure 1: $\alpha_2^+ = 0.2,\ \gamma_2 \in [0.897, 0.9147]$: Pseudoequilibrium is feasible and optimal

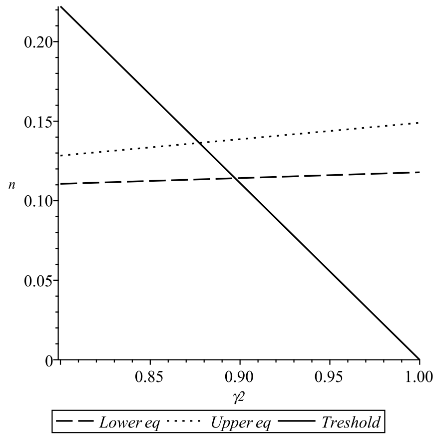
In the Figure 1 there exists a sliding region for any γ_2. For $\gamma_2 \in [0.897, 0.9147]$ it contains the pseudoequilibrium and both original system's steady states are virtual. Thus the only long-run outcome is convergence to the pseudoequilibrium both from the upper and lower regimes. Out of this range one equilibrium is always virtual and the other one is real while pseudoequilibrium is not contained in the sliding region, so convergence is uniquely given by n_0 for any γ_2.

In the Figure 2 for any γ_2 only the escaping region exists, and it contains the pseudoequilibrium for $\gamma_2 \in [0.88, 0.9]$. It may be reached only by setting $n_0 = n^*$. At the same time, for γ_2 in this range both equilibria are regular ones and outside this range one is always virtual. Thus the unique convergence is granted for $\gamma_2 \notin [0.88, 0.9]$: the regular equilibrium is selected (perhaps with finitely many switches). At the same time once $\gamma_2 \in [0.88, 0.9]$, one or the other equilibrium is reached provided $n_0 \neq n^*$ and the location of the Skiba point (if it exists).

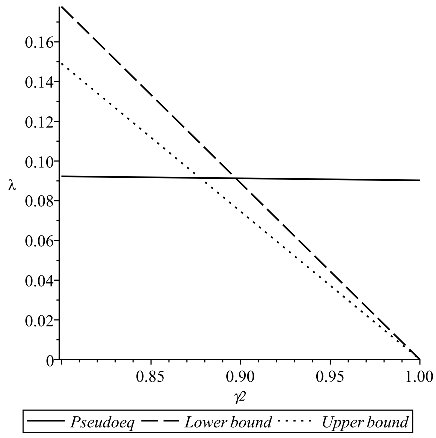
We also note here:

Lemma 3. *As long as the escaping region is non-empty and steady states are regular and saddle-stable, switching dynamics cannot be optimal.*

Proof. As we discussed, the existence of the escaping region implies both equilibria are regular. Once $n_0 < n^*$ for certainty, it might be the case, that convergence to the upper steady state implies higher social welfare W. Still to do so, the optimal trajectory in the upper regime should follow the stable manifold of the upper steady state. If this is a saddle, there exists a single such trajectory, converging from given n_0 to $\hat{n}^+$. If the initial value is below n^*, the upper part of



(a) Location of steady states



(b) Location of pseudoequilibrium

Figure 2: $\alpha_2^+ = 0.7, \gamma_2 \in [0.88, 0.9]$: Pseudoequilibrium is never feasible except for $n_0 = n^*$

solution implies $n(t^*) = n^*$, i. e. final part of the optimal trajectory should start at the switching manifold. But as long as escaping region exists, the stable manifolds of both equilibria start at the escaping region, and thus the optimal trajectory crossing from lower to the upper cannot be found. Thus the only optimal outcome is convergence to the equilibrium of the regime, where n_0 is located. $\square$

Using this observation we claim that the third type of dynamics cannot occur in the cooperative game (11), (2):

Lemma 4. *In cooperative game either one of regular equilibria realizes, or pseudoequilibrium realizes if both equilibria are virtual ones.*

Proof. By Lemma 2 we know that pseudoequilibrium realizes only if the sliding region is nonempty and both equilibria are virtual. If one of those is real and the other virtual, no pseudoequilibrium exists and the only possible outcome is the regular steady state. If both are regular, there exists an escaping region and by Lemma 3 the game is in the normal mode. To rule out the third (quasi-periodic) type of dynamics, observe that for it to be the case, folds have to be formed from both sides of the switching manifold, and they have to coincide, i. e. $\lambda_+ = \lambda_-$. But this can be the case only if $(\alpha_1^-)^2 + (\alpha_2^+)^2 = (\alpha_1^+)^2 + (\alpha_2^-)^2$ implying escaping/sliding region consists of this fold point only. But then the optimal trajectory reaching to the desirable steady state of the original system can always be found and even if the quasi periodic motion exists, it cannot be an optimal outcome. □

Then we have:

Proposition 1. *If both steady states of the social problem are virtual ones, the only socially optimal outcome is the unstable pseudoequilibrium* $\{\hat{\lambda}^s_W, n^*\}$.

It is feasible if and only if

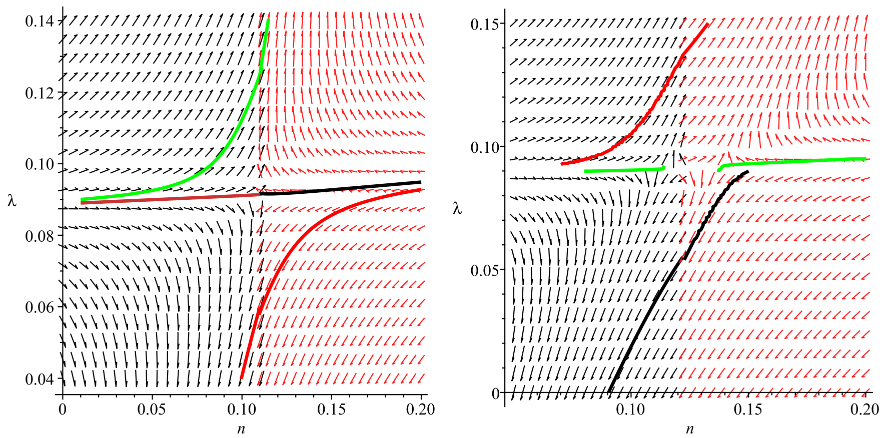
$$\hat{\lambda}^s_W \in [\lambda_-, \lambda_+] \tag{23}$$

otherwise one of two regular equilibria realizes and the game is in the normal mode.

Proof. First claim follows from the fact that the pseudoequilibrium is the only possible long-run steady state (as system is linear) and it is optimal to converge to the steady state by definition of it. The second claim follows from lemma 1 and the fact that the pseudoequilibrium is feasible from the outside of the sliding region only if it lies within the sliding region.

The rest follows from the preceeding analysis. □

Both switching-type and normal dynamics for the cooperative game are illustrated by direction fields at the Figure 3



(a) Two virtual equilibria (b) Two regular equilibria

Figure 3: Two possible outcomes of the cooperative 2-firms game

Figure 3a shows the situation when starting from any side of n^* optimal dynamics inevitably reaches the pseudoequilibrium, since booth regimes cannot reach their respective equilibria.

Figure 3b illustrates the normal situation, when no switching may occur and one of the regular equilibria is selected.

Our analysis has so far demonstrated that in the studied cooperative 2-players R&D game there is always a unique equilibrium (solution). It can be either the permanent technological leadership of one of the firms or the pseudo equilibrium situation, whereas no firm is the leader and thus no imitation may happen. This last outcome is generic and happens for a positive range of parameters. In particular, if initial follower has sufficiently low investment efficiency as a leader, it might be optimal to 'freeze' the dynamics at the boundary between the regimes, since this yields the maximal sustainable level of variety n^*.

At last we observe that the outcome of the cooperative game is invariant to the number of participating firms. Indeed, the aggregate dynamics is characterized as a 2-dimensional system for any N and there are no qualitatively new types of dynamics for such a game. This is known as a topological equivalence property:

Corollary 1. *Any $N \in \mathbb{N}$-player cooperative game type* (13) *is topologically equivalent to a two-persons game, i. e. either regular or pseudoequilbrium type k realises as the long-run outcome without any quasi-periodic solutions.*

The likelihood of the pseudoequilirbium as an outcome comes down with N increase.

If $N \to \infty$ only pseudoequilibrium may realize as an outcome.

Proof. For any natural N the dynamical system is 2-dimensional but with N separate regimes of different firms' leadership (in the sense of Def. 2). Each regime is thus fully equivalent to upper-lower regime of the original (13) 2-persons game. The only difference comes in the multiplicity of switching manifolds. As such for any pairwise neighboring regimes the dynamics is the same as above with the only difference that pseudoequilibrium may realize only if all regimes' equilibria are virtual, which becomes less likely the more regimes are there.

The last claim follows from the fact that with $N \to \infty$ the phase space consists solely of switching manifolds and thus the only long-run outcome is the realisation of (the nearest to n_0) pseudo equilibrium. □

We thus observe that for a cooperative solution the number of participating firms may play stabilizing role as long as this number is finite, since this helps to select the regular (thus optimal) equilibrium as an outcome, while if there are infinitely many firms, the regular outcome cannot be reached. This finding agrees with (Bondarev and Greiner 2017), whereas the same claim has been made for decentralized system with symmetric efficiencies but non-linear dynamics. It turns out, that it is the piecewise nature (multiple modes of the game present) of the system which is crucial for the optimality and not the precise type of the underlying dynamical system regimes themselves.

4 Non-cooperative game

We now consider the decentralised system. In this case the problem is given as a differential game with players' objectives (1) and dynamics (2). We start with the minimal example of two firms, $k \in \{1, 2\}$. For simplicity we normalize $\gamma_1 > \gamma_2 > 0$ and $0 < \psi_1 < \psi_2$ such that firm 1 has initial advantage in efficiency but as range of technologies progresses, the other firm catches up. There is only one such regime shift, and the range where firm 1 is the leader, i_1 or firm 2, i_2 are defined as:

$$i_1 : [0, n^*),\ i_2 : (n^*, \infty),\ n^* = \frac{\gamma_1 - \gamma_2}{\psi_2 - \psi_1} \tag{24}$$

and we again term the regime whereas $n(t) \in i_1$ the *lower* regime and $n(t) \in i_2$ the *upper* regime of the R&D system.

Proceeding as in the baseline model we next define dynamics of the R&D system in both regimes:

$$\forall t : n(t) < n^* :\ f_- := \begin{cases} \dot{n}^-(t) = (\alpha_1^+ u_1(t) + \alpha_2^- u_2(t)) - \delta n(t) \\ \dot{u}_1^-(t) = \alpha_1^+ (r + \delta) u_1(t) - \alpha_1^+ \frac{\psi_1 n(t) + \gamma_1}{(r+\beta)^2} \\ \dot{u}_2^-(t) = \alpha_2^- (r + \delta) u_2(t) - \alpha_2^- \frac{\psi_2 n(t) + \gamma_2}{(r+\beta+\theta)^2} \end{cases} \tag{25}$$

$$\forall t : n(t) > n^* :\ f_+ := \begin{cases} \dot{n}^+(t) = (\alpha_1^- u_1(t) + \alpha_2^+ u_2(t)) - \delta n(t) \\ \dot{u}_1^+(t) = \alpha_1^- (r + \delta) u_1(t) - \alpha_1^- \frac{\psi_1 n(t) + \gamma_1}{(r+\beta+\theta)^2} \\ \dot{u}_2^+(t) = \alpha_2^+ (r + \delta) u_2(t) - \alpha_2^+ \frac{\psi_2 n(t) + \gamma_2}{(r+\beta)^2} \end{cases} \tag{26}$$

Denote the steady states of both systems as $\{\bar{n}^\pm, \bar{u}_1^\pm, \bar{u}_2^\pm\}$ points where the system (25) or (26) has zero change. By construction both regimes have unique steady-states as long as both systems are non-singular.

We thus assume:

Assumption 1. *Jacobians of f_-, f_+ are of full rank*

We also require

Assumption 2. *Both* $\{\bar{n}^{\pm}, \bar{u}_1^{\pm}, \bar{u}_2^{\pm}\}$ *are stable or saddle-type, i. e. the Jacobians of both systems* f_-, f_+ *at their respective equilibria have at least one negative eigenvalue.*

These assumptions are not restrictive, since otherwise there are no long-run equilibria in the game.

Linearity of both f_+, f_- implies no Skiba points exist (no multi-stability) for both subsystems but the overall system may possess the (pseudo)Skiba-point as discussed in the previous section. Since investments are optimally chosen, only the stable sub-manifold of the associated equilibrium will be selected as an optimal trajectory of the system. Such stable sub-manifolds exist only if equilibria are stable or of the saddle-type. From now on we refer to the set of players, objectives (1) and dynamics (25)–(26) as the non-cooperative game.

Since investments are optimal controls of the firms, we also need transversality conditions. We define them as follows:

$$\lim_{t\to\infty} \mathrm{e}^{-rt} u_{1,2}(t) = 0 \tag{27}$$

Under given assumptions it follows that we can characterize these modes by the configuration of steady states and initial position of the game:

Lemma 5. *Under Assumptions 1, 2:*

1. *Either* $\bar{n}^- < n^* < \bar{n}^+$ *or* $\max\{\bar{n}^-, \bar{n}^+\} < n^*, n(0) < n^*$ *or* $\min\{\bar{n}^-, \bar{n}^+\} > n^*, n(0) > n^*$, *and the game is in the normal mode*
2. *Either* $\bar{n}^- > n^* > \bar{n}^+$, *or* $\max\{\bar{n}^-, \bar{n}^+\} < n^*, n(0) > n^*$ *or* $\max\{\bar{n}^-, \bar{n}^+\} > n^*, n(0) < n^*$, *and the game is in the switching mode.*

Proof. This is equivalent of Lemma 1 but for the 3-dimensional decentralised system. □

We are mostly interested in the case when $\bar{n}^- > n^* > \bar{n}^+$, since in all other cases the switch occurs at most once[8]. For such configuration of the system to hold the condition is

$$\begin{aligned}
&\frac{1}{2}\,\frac{\left(\frac{\gamma_1\,(\alpha_1^+)^2}{(r+\beta)^2}+\frac{\gamma_2\,(\alpha_2^-)^2}{(r+\beta+\theta)^2}\right)}{\left(\delta(r+\delta)-\frac{1}{2}\left(\frac{\psi_1\,(\alpha_1^+)^2}{(r+\beta)^2}+\frac{\psi_2\,(\alpha_2^-)^2}{(r+\beta+\theta)^2}\right)\right)}>\\
&>\frac{\gamma_1-\gamma_2}{\psi_2-\psi_1}>\\
&\frac{1}{2}\,\frac{\left(\frac{\gamma_1\,(\alpha_1^-)^2}{(r+\beta+\theta)^2}+\frac{\gamma_2\,(\alpha_2^+)^2}{(r+\beta)^2}\right)}{\left(\delta(r+\delta)-\frac{1}{2}\left(\frac{\psi_1\,(\alpha_1^-)^2}{(r+\beta+\theta)^2}+\frac{\psi_2\,(\alpha_2^+)^2}{(r+\beta)^2}\right)\right)}
\end{aligned}\tag{28}$$

i. e. both regimes have only *virtual* (in the sense of Def. 5) equilibria. This is just the condition on investment efficiencies. We observe that

Lemma 6. *As long as* (28) *holds and both equilibria are (saddle-type) stable, system converges to the switching manifold in finite time from any* $n(0)$ *and stays locally in the neighbourhood of* n^* *for infinite time.*

Proof. It follows from the linearity of the system: once (28) holds, no long-run equilibrium may be reached (both are virtual) and no heteroclinic connections across steady states may exist. At the same time optimality implies that $V_1(\bar{n}^+) > V_1(\bar{n}^-)$ and $V_2(\bar{n}^+) < V_2(\bar{n}^-)$ where $V_k(\bar{n}^i)$ denotes the value (maximized discounted profit stream) of firm k in steady state i respectively. Thus from the lower regime system will always converge towards the upper and vice versa. The only long-run trajectories which are compatible are either cycling around the switching manifold, or end up at the pseudoequilibrium within it. □

Lemma 6 implies that it is important to study the dynamics only near the switching manifold. However general theory tells us that behaviour around the switching manifold may be more or less complicated. In particular, we are interested in the *generic* behaviour qualitatively different from the cooperative case. It turns out, that if only one sliding/escaping region exists for the non-cooperative game, dynamics is qualitatively similar to the cooperative case: either pseudoequilibrium realizes or one of the regular steady states. Still in 3 dimensions there exists another (and more generic) possibility, called the *two-fold.*

[8] provided assumption on both steady states are (saddle-type) stable, arguments are the same as for the cooperative game case

Lemma 7. *The dynamics near the switching region is characterized by the two-fold once it holds*

$$\exists!(u_1^*, u_2^*) > 0 : \begin{cases} \alpha_1^+ u_1 + \alpha_2^- u_2 - \delta n^* = 0, \\ \alpha_1^- u_1 + \alpha_2^+ u_2 - \delta n^* = 0 \end{cases} \tag{29}$$

In particular, both sliding and escaping regions are 2-dimensional and there exists two 2-dimensional crossing regions.

Proof. The condition (29) just says that tangent lines at n^* are in general position and intersect once. This intersection point is

$$u_1^* = \frac{\alpha_2^- - \alpha_2^+}{\alpha_1^- \alpha_2^- - \alpha_1^+ \alpha_2^+} \delta n^*, \; u_2^* = \frac{\alpha_1^- - \alpha_1^+}{\alpha_1^- \alpha_2^- - \alpha_1^+ \alpha_2^+} \delta n^* \tag{30}$$

defined by asymmetry of investment efficiencies across regimes. Since we assume $\alpha_k^- > \alpha_k^+$ this point implies positive u_1^*, u_2^* for positive n^*. The triple u_1^*, u_2^*, n^* defines the location of the *two-fold*, but not its type. Once such an intersection is unique, the tangent lines (29) define a partitioning of the switching manifold into exactly four regions. □

These regions are sets of investments $\{u_1, u_2\} > 0$ defined by:

$$\begin{aligned} CR_1 &:= \{u_1, u_2\} : \; \alpha_1^\pm u_1 + \alpha_2^\pm u_2 - \delta n^* > 0; \\ CR_2 &:= \{u_1, u_2\} : \; \alpha_1^\pm u_1 + \alpha_2^\pm u_2 - \delta n^* < 0, \\ ES &:= \{u_1, u_2\} : \; \alpha_1^+ u_1 + \alpha_2^- u_2 - \delta n^* < 0, \; \alpha_1^- u_1 + \alpha_2^+ u_2 - \delta n^* > 0 \\ SL &:= \{u_1, u_2\} : \alpha_1^+ u_1 + \alpha_2^- u_2 - \delta n^* > 0, \; \alpha_1^- u_1 + \alpha_2^+ u_2 - \delta n^* < 0. \end{aligned} \tag{31}$$

which are crossing regions CR_1, CR_2 and escaping (ES) and sliding (SL) regions defined in the same way as in Def. 4.

The type of the two-fold is given by the sign of the second Lie derivative, at this point defined by (24) and (30). Applied to our example we get

$$\begin{aligned} \mathcal{L}^2_{f_-}(n^*) &:= \frac{1}{2} \frac{\left(2\,(r+\beta)^2 (r+\delta)\, u_1^* - \alpha_1^+ n^* \psi_1\right) \alpha_1^+}{(r+\beta)^2} + \frac{1}{2} \frac{\left(2\,(r+\beta+\theta)^2 (r+\delta)\, u_2^* - \alpha_2^- n^* \psi_2\right) \alpha_2^-}{(r+\beta+\theta)^2} \\ \mathcal{L}^2_{f_+}(n^*) &:= \frac{1}{2} \frac{\left(2\,(r+\beta+\theta)^2 (r+\delta)\, u_1^* - \alpha_1^- n^* \psi_1\right) \alpha_1^-}{(r+\beta+\theta)^2} + \frac{1}{2} \frac{\left(2\,(r+\beta)^2 (r+\delta)\, u_2^* - \alpha_2^+ n^* \psi_2\right) \alpha_2^+}{(r+\beta)^2} \end{aligned} \tag{32}$$

where $\{n^*, u_1^*, u_2^*\}$ is the two-fold point.

For convenience we normalize coefficients such that $\gamma_1 = 1, \psi_2 = 1, \delta = 1, r = 1, \beta = 1$ as in cooperative case and set $\alpha_1^+ = 1, \alpha_2^+ = 1$. The sign of normalized Lie derivatives depend on the quadratic polynomials in investment efficiencies:

$$\begin{aligned} &\psi_1(2+\theta)^2(\alpha_1^+)^2 + 4(\alpha_2^-)^2 - 16(2+\theta)^2 \lesseqgtr 0 \leftrightarrow \mathcal{L}^2_{f_-}(n^*) \gtreqless 0, \\ &\frac{1}{4}(2+\theta)^2(\alpha_2^+)^2 + (\alpha_1^-)^2\psi_1 - 4(2+\theta)^2 \lesseqgtr 0 \leftrightarrow \mathcal{L}^2_{f_+}(n^*) \gtreqless 0 \end{aligned} \tag{33}$$

We get

Lemma 8. *There exist non-empty parameters range such that for $\gamma_2 < \gamma_1, \psi_1 < \psi_2$ it holds $\mathcal{L}^2_{f_-}h(n^*) > 0, \mathcal{L}^2_{f_+}h(n^*) < 0$, i. e. the two-fold exhibits the Teixeira singularity.*

Proof. It suffices to find such a combination of parameters. With normalization as above the Figure 4 provides the range of the follower investment efficiencies such that the non-cooperative game exhibits Teixeira singularity: $\square$

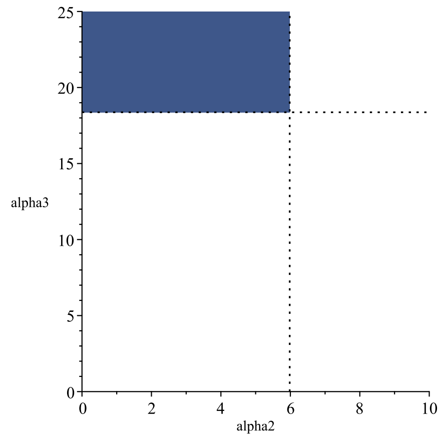


Figure 4: Range of α_1^-, α_2^- for Teixeira singularity

So this singularity appears to be *generic*: once sufficient asymmetry in investment efficiencies is observed, one would expect the appearance of this complex behavior. The importance of Teixeira singularity stems from the fact that this is the only type of the two-fold which is asymptotically stable: trajectories may stay near the switching manifold infinitely long. For other types of the two-fold this is not the case and the trajectory will eventually leave the neighbourhood of a singularity.

For the optimal trajectory to lead into the neighbourhood of this singularity both only virtual steady states should exist and Teixeira singularity. There exists a range of efficiencies for which this is possible, but this range is smaller, as Figure 5 shows.

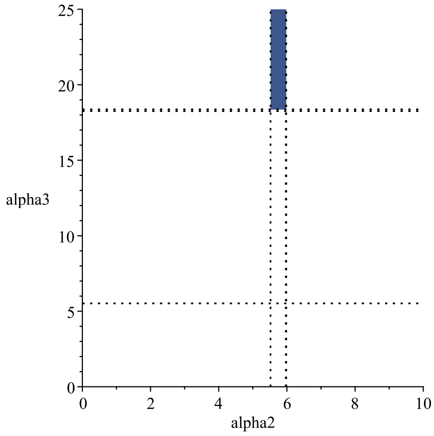


Figure 5: Range of α_1^-, α_2^- for Teixeira singularity and two virtual equilibria

These range of parameters define the partition of the switching manifold as illustrated by Figure 6.

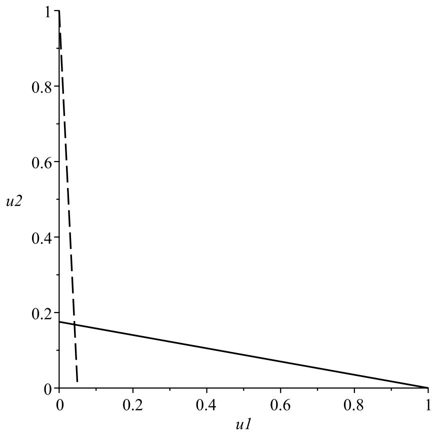


Figure 6: Partition of the switching manifold in regions

We next define the sliding flow following the Filippov convex method:

$$f_s = f_- + \frac{\mathcal{L}_{f_-} n^*}{\mathcal{L}_{f_-} n^* - \mathcal{L}_{f_+} n^*} (f_+ - f_-) \tag{34}$$

where $\mathcal{L}_{f_\pm} n^*$ are given by $n(t)$ dynamics at fixed n^*. This sliding flow is 2-dimensional but highly non-linear:

$$\dot{u}_1^s = \alpha_1^+ (r+\delta) u_1 - \frac{1}{2}\alpha_1^+ C_1 + \frac{\alpha_1^+ u_1 + \alpha_2^- u_2 - \delta n^*}{(\alpha_1^+ - \alpha_1^-) u_1 + (\alpha_2^- - \alpha_2^+) u_2} \left((\alpha_1^- - \alpha_1^+)(r+\delta) u_1 - \frac{1}{2}\alpha_1^- C_2 + \frac{1}{2}\alpha_1^+ C_1 \right)$$
$$\dot{u}_2^s = \alpha_2^- (r+\delta) u_2 - \frac{1}{2}\alpha_2^- C_3 + \frac{\alpha_1^+ u_1 + \alpha_2^- u_2 - \delta n^*}{(\alpha_1^+ - \alpha_1^-) u_1 + (\alpha_2^- - \alpha_2^+) u_2} \left((\alpha_2^+ - \alpha_2^-)(r+\delta) u_2 - \frac{1}{2}\alpha_2^+ C_3 + \frac{1}{2}\alpha_2^- C_4 \right) \tag{35}$$

where $C_1 = \frac{\psi_1 n^* + \gamma_1}{(r+\beta)^2}, C_2 = \frac{\psi_1 n^* + \gamma_1}{(r+\beta+\theta)^2}, C_3 = \frac{\psi_2 n^* + \gamma_2}{(r+\beta)^2}, C_4 = \frac{\psi_2 n^* + \gamma_2}{(r+\beta+\theta)^2}$ are constants depending on investment efficiencies functions in qualities and imitation speed. This sliding flow might have associated equilibrium which is called pseudoequilibrium of the original system:

Definition 6. *The equilibrium* $\{u_1^s, u_2^s\}$ *of the sliding flow* (35) *such that* $n = n^*, \dot{u}_{1,2}^s = 0$ *and* $\{u_1^s, u_2^s\} \in ES \cup SL$ *is called the pseudoequilibrium of the system* (26)-(25).

Observe that once the last condition is violated, the system (35) has an equilibrium located in one of the crossing regions and this cannot be reached by the sliding flow, since this exists only within the $ES \cup SL$. Moreover if $\{u_1^s, u_2^s\} \in ES$ this is a *virtual pseudoequilibrium*: it cannot be reached by the sliding flow, since in the escaping region the flow does not exist in forward time, leaving it immediately upon entrance.

Next, following (Colombo and Jeffrey 2011) we have classification of dynamics around Teixeira singularity[9]:

Proposition 2.
Under conditions of Lemmas 6, 7, 8 the following types of long-run dynamics are possible:

1. *(stable) Limit cycle around the singularity*
2. *If this is unstable, then:*
   (a) *If the pseudoequilibrium is (saddle-type) stable with* $\{u_1^s, u_2^s\} \in SL$*, the system ends up in the pseudoequilibrium of the sliding flow at* $\{\bar{n}^s = n^*, \bar{u}_1^s, \bar{u}_2^s\}$
   (b) *If the pseudoequilibrium is unstable or* $\{u_1^s, u_2^s\} \in ES \cup CR_1 \cup CR_2$*, the system evolves unpredictably around the singularity point with expected average position* $\mathbb{E}_\infty(n, u_1, u_2) = \{n^*, u_1^*, u_2^*\}$

[9] I omit all standard types of dynamics possible once some of Lemmas 6-8 do not hold, since these can be described via the same mechanics as in (Gromov and Gromova 2017).

Proof. In the paper (Colombo and Jeffrey 2011) ten different types of dynamics are reported. Still under condition of (28) the system cannot escape the neighbourhood of the switching manifold, thus eventually it will hit the sliding region which exists due to (29). In the sliding region we either get stable or unstable equilibrium (pseudoequilibrium). If it is (saddle-type) stable, only those trajectories which lie in the stable sub-manifold are optimal and the system ends up in this position. If it is unstable it lies in the escaping region and all orbits from the sliding region are passing through the singularity point, implying non-deterministic chaotic behavior. If this is the case, all orbits recurrently visit the singularity point, but intersect only at that point, hence $\mathbb{E}_\infty(n, u_1, u_2) = \{n^*, u_1^*, u_2^*\}$. $\square$

The Proposition 2 implies that once the system has two virtual equilibria, there are three potential types of associated inefficiency:

1. First, the system ends up not in one of the long-run steady states, but in between those two.
2. Second, it might permanently fluctuate around the singularity point in predictable manner, implying some technologies are constantly re-invented and scrapped and firms fight for the technological leadership
3. Third, these fluctuations may become fully unpredictable exhibiting non-deterministic chaotic behavior[10], i. e. at each point in time none of the players may predict the dynamics for more than some τ time, where τ is the period between recurrent visits of the singularity point by the system. At this point the memory is erased, and the new (unpredictable) cycle starts.

The first inefficiency is stemming from the multi-modality of the game itself: the presence of imitation and its scope define this effect. This inefficiency may be corrected by usual subsidizing schemes, defined via for example, IDP (imputation distribution procedure) as in (Petrosjan and Zaccour 2003), since both decentralized and cooperative dynamics are of the same type (even if cooperative game yields pseudo equilibrium as an outcome). Moreover, this type of dynamics is topologically invariant to the number of firms and has the same properties as the cooperative solution described by Corollary 1.

[10]Roughly speaking, non-deterministic chaos is a set-valued flow, while standard chaotic dynamics is deterministic extremely sensitive to initial conditions. See (Colombo and Jeffrey 2011) for definition.

The second one is a bit more complex, since it requires varying and state-dependent subsidies/transfers for correction. Indeed, since firms constantly change their relative leadership, the subsidy has to be defined relative to this leadership. However theoretically such a corrective policy is feasible. Still, the topological equivalence of the cooperative and non-cooperative system is already lost, since pseudo-limit cycles are possible only for non-cooperative game (this is a 3-dimensional phenomenon).

The third one is of completely new type, since it stems from the endogenous uncertainty of the dynamics of the game. It is much harder to correct by specific state-based subsidies, since the state is in general not known in advance even to firms themselves. Thus we naturally get the second-best case: some constant or approximate subsidizing schemes may implement the pseudoequilibrium (from which the normal first-best subsidy can be used), but there are non-negligible welfare losses on this route. In this paper we do not tackle the question how this third type of inefficiency may be corrected since it turns out to be non-generic for a linear-quadratic setting considered here.

Now observe that the model (26),(25) cannot generate this last type of inefficiency.

Proposition 3. *The decentralised R&D game does not exhibit non-deterministic chaos as long as it is linear-quadratic in each of the modes and there are no cross-players strategic interactions.*

Proof. Indeed for it to occur, the angles between flows and their fold lines, θ^+, θ^- in the upper and lower regimes should be such that $\cot\theta^+ \cot\theta^- > 1$ and $\cot\theta^+, \cot\theta^- < 0$ given by second mixed Lie derivatives of flows around the singularity point $\{n^*, u_1^*, u_2^*\}$:

$$
\begin{aligned}
V^+ &:= \cot\theta^+ = \frac{\mathcal{L}_{f_+}\mathcal{L}_{f_-}(n^*)}{\sqrt{-\left(\mathcal{L}^2_{f_+}n^*\right)\left(\mathcal{L}^2_{f_-}n^*\right)}} \\
V^- &:= \cot\theta^- = \frac{-\mathcal{L}_{f_-}\mathcal{L}_{f_+}(n^*)}{\sqrt{-\left(\mathcal{L}^2_{f_+}n^*\right)\left(\mathcal{L}^2_{f_-}n^*\right)}}
\end{aligned}
\tag{36}
$$

and additional parameter

$$
a^* = \alpha_1^- V^+ + \frac{\alpha_2^+}{V^+} - \alpha_1^+ (V^+)^2 - \alpha_2^- \tag{37}
$$

should be negative (according to (Colombo and Jeffrey 2011)).

In the current setup this cannot be the case, since once cross-investments impacts are zero, $a^* < 0$ as long as $V^+ < 0$, as simple algebra shows. ∎

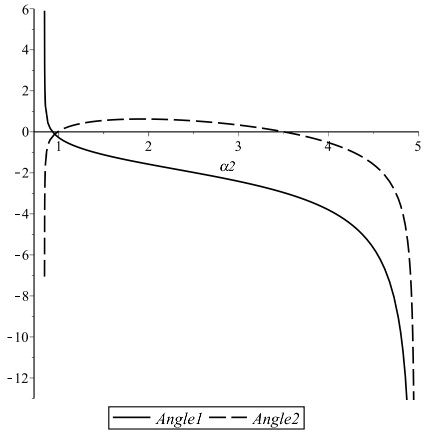


Figure 7: Sign of cotangents of angles between flows

Comparing Eqs. (28) and (22) yields the observation:

Lemma 9. *$\hat{n}^- > \bar{n}^-$ as long as*

$$(\alpha_2^-)^2\gamma_1(r+\beta+\theta)^2 + (\alpha_1^+)^2\gamma_2(r+\beta)^2 > \frac{1}{2}((\alpha_2^-)^4 - (\alpha_1^+)^4)(\psi_2\gamma_1 - \psi_1\gamma_2) > 0 \tag{38}$$

holds. $\hat{n}^+ < \bar{n}^+$ holds as long as

$$(\alpha_2^+)^2\gamma_1(r+\beta)^2 + (\alpha_1^-)^2\gamma_2(r+\beta+\theta)^2 > \frac{1}{2}((\alpha_1^-)^4 - (\alpha_2^+)^4)(\psi_2\gamma_1 - \psi_1\gamma_2) < 0 \tag{39}$$

holds.

As long as both (38), (39) *hold and both non-cooperative equilibria are virtual, both socially optimal steady states are also virtual.*

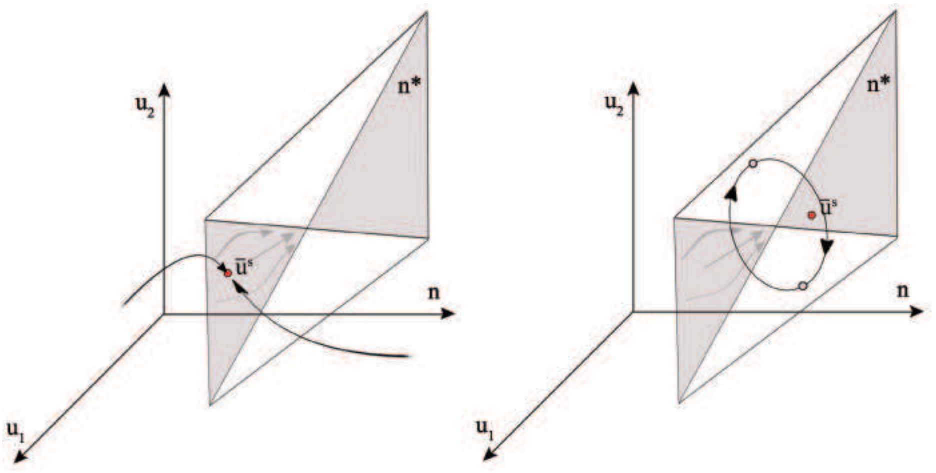
Proof. Both inequalities follow from direct substraction of (28) from (22). If those hold, both social and decentralised steady states are virtual. □

This immediately implies

Corollary 2. *As long as the non-cooperative game admits pseudoequilibrium as the optimal solution, the same holds for the socially optimal solution.*

Proof. As long as the decentralized system exhibits convergence to the pseudoequilibrium it has two virtual equilibria. Then by Lemma 9 the cooperative game also have two virtual equilibria. The only outcome for the cooperative game with two virtual equilibria is the convergence to the pseudoequilibrium (by Lemma 2). □

The analysis here has shown that in the decentralized system of piecewise linear-quadratic type the non-deterministic chaos is not possible. However the Teixeira singularity forms generically and once this is the case, two types of dynamics are possible: convergence to the pseudo equilibrium or stable pseudo-limit cycles around the singularity point. These two types are schematically illustrated by the Figure 8.



(a) Two virtual equilibria (b) Two regular equilibria

Figure 8: Two possible outcomes of the non-cooperative 2-firms game around Teixeira singularity

The first type corresponds to the pseudoequilibrium of the cooperative solution (Corr. 2) and as such may be implemented through the IDP approach with value functions of players as in (Petrosjan and Zaccour 2003). The second type is new and implies the necessity for time-varying transfers/subsidies, since the decentralized system never comes into any equilibrium (including the pseudoequilibrium).

The last point to consider is the prospect of generalization of the 2-players noncooperative game to the N-players setup. It turns out that the result equivalent to the Corr. 1 is impossible to obtain. The reason for this is that increasing the number of players increases the dimensionality of the system and it cannot be reduced as in the baseline setting (which is a special degenerate case). That is, resulting dynamical systems are not topologically equivalent.

Corollary 3. *As long as the decentralised game is asymmetric, i. e.* $\exists i \neq j : \alpha_i^+ \neq \alpha_i^-$, *any* $N+1$*-players game cannot be described by reducing dimensionality to* N *dimensions.*

Proof. Once $\exists i \neq j : \;\; \alpha_i^+ \neq \alpha_i^-$ holds, shadow costs of investments for each player differ, implying impossibility to aggregate $u_j(t)$ dynamics. Then any additionally player will increase the dimensionality of the dynamical system and topological equivalence is not preserved since the multi-modality of the game. □

This last result has important consequences. In cooperative game (i. e. centralised solution) the number of players does not affect the qualitative type of dynamics, changing only the likelihood of one or another outcome- On the contrary in decentralized equilibrium the higher is the number of players, the more complicated and diverse is the dynamics. Already with 3 players one gets a 4-dimensional non-reducible dynamical system of piecewise type, which does not have full classification of potential dynamic types until now. We thus may conclude that:

Corollary 4. *With increasing number of asymmetric players the relative efficiency of the cooperative solution in comparison with the non-cooperative one goes up and efficiency of regulation goes down.*

It is important to note that this concerns only *asymmetric* players. If these are symmetric we get the familiar structure of monopolistic competition with all regulation tools being fairly well know. Still the structure considered here is not of that type, since heterogeneity of players and may be referred to as the heterogeneous competition.

In particular, this includes a monopoly within a competitive fringe: if we denote monopoly as player 1 and all (symmetric) small firms as an aggregated player 2, we obtain exactly the structure studied above. If all firms are symmetric (perfect competition case) we obtain the degenerate case of the baseline model qualitatively similar to the cooperative solution. However once we have more than two asymmetric players, dynamics cannot be reduced to 3 dimensions and has to be studied further.

5 Generalizations

The results of the above analysis are easily extended to generic piecewise-smooth games linear-quadratic (LQ) in each of the modes.

First we characterize when one might expect the emergence of the non-deterministic chaos and when not. To this end assume the dynamics of the 2-persons game is given by the linear

ODE system type (26)-(25):

$$\dot{x} = a_{11}^{\pm}x + a_{12}^{\pm}u_1 + a_{13}^{\pm}u_2 \tag{40}$$

$$\dot{u}_1 = a_{21}^{\pm}x + a_{22}^{\pm}u_1 + a_{23}^{\pm}u_2 \tag{41}$$

$$\dot{u}_2 = a_{31}^{\pm}x + a_{32}^{\pm}u_1 + a_{33}^{\pm}u_2 \tag{42}$$

where x is the state variable, u_1, u_2 are (optimal) strategies of players, $a_{ij}^{\pm}$ are constant coefficients of impact by variable j on variable i in the upper $(+)$ and lower $(-)$ regimes of the game. We immediately recover our original game by setting a_{23}, a_{32} to zero, i. e. assuming away any dynamic impact of strategies on each other. We then get a remarkably simple result:

Proposition 4. *In the LQ bi-modal 2-players game nondeterministic chaos is possible only if* $a_{23}, a_{32} \neq 0$ *in* (41)-(42). *Otherwise no chaotic dynamics can be observed and all dynamics around the T-singularity (if it exists) is exhausted by the equivalent to Prop. 2.*

Proof. Indeed, once $a_{23}, a_{32} = 0$ there cannot be the situation such that simultaneously $V^+ < 0$ and $a^* < 0$ given by (36) and (37) respectively. At the same time this is necessary to have a nondeterministic chaotic dynamics. Once this is absent, the two other possible types of dynamics around the singularity are given by the equivalent of Prop. 2. $\square$

Observe that this does not extend to N-players LQ games, since those can have up to N modes and N dimensional dynamical system may exhibit completely different types of dynamics (still not fully studied). However this gives a hint that complex dynamics can be achieved already with two players, once cross-strategies feedbacks are allowed for.

The second general result concerns the topological equivalence of cooperative and non-cooperative games.

Proposition 5. *Assume the LQ bi-modal game given by dynamics* (40)-(42). *Then it holds:*

1. *Cooperative game is always described by the two-dimensional system and its equilibria types are qualitatively invariant to the number of players N;*

2. *Non-cooperative game is described by the N-dimensional system and thus its equilibria are qualitatively different in N*

3. *As long as in the N-players system type* (40)-(42) *it holds that* $\forall i \in N: \ a_{1i}^{+} = a_{1i}^{-}$, *the non-cooperative game is topologically equivalent to a 2-dimensional system for any finite N.*

Proof. Point 1 is direct consequence of Corr. 1. Point 2 is consequence of the fact that non-cooperative game exhibit new types of dynamics already for $N = 2$ and this propagates further (e. .g with $N = 3$ there can be so-called cubic tangency points, implying other types of singularities). Point 3 is an observation, that once all investments efficiencies are the same across regimes, we can always aggregate investments into one variable int he same way as for the cooperative solution and get the 2-dimensional system in the same way as in the benchmark model. □

Now the question of interest remains: what are possible classes of (economically meaningful) dynamic games which may exhibit the non-deterministic chaotic behavior? The analysis so far revealed that it does not suffice to have asymmetric players, but rather the optimal strategies of those should be interdependent. That is, strategies have to be not pure state-feedback ones, but interdependent.

For example, if the variety expansion dynamics will include a knowledge spillover,

$$\dot{n} = (\alpha_1^{\pm} u_1(t) + \alpha_2^{\pm} u_2(t))(1 + \epsilon n(t)) - \delta n(t) \tag{43}$$

strategies of firms would become interdependent and non-deterministic chaos is possible. However, this type of dynamics makes the game a non-linear one and explicit solution is difficult to obtain.

It thus remains open for future research, how large is the class of games which may exhibit this additional property. It can be conjectured however, that any game with Stackelberg leadership would belong to this class, since it is known (see e. g. (Dockner, Jorgensen, Long, and Sorger 2000)) that optimal strategies for such games are interdependent.

6 Conclusions

In this paper the consequences of introducing asymmetry in investments into the multi-modal R&D game are studied. The analysis reveals that for centralized solution the dynamics is qualitatively similar to that studied in (Bondarev and Greiner 2017), while for decentralized dynamics this is not the case.

In particular cooperative game may exhibit only two types of dynamics irrespective of the number of firms, while decentralised game has increasingly complex dynamics if number of firms increase. For the simplest 2-players setup we already may observe permanent fluctuations around the switching point as well as convergence to the intermittent pseudo-equilibrium. The last case is equivalent to the cooperative game outcome but the first one cannot be observed in the cooperative game and requires time-varying subsidies to be regulated.

Potentially this setup may exhibit even more complex behavior known as non-deterministic chaos, but once we restrict attention to linear-quadratic games, this cannot happen. The presence of additional spillovers across players' strategies may lead to this type of behavior which would require the development of novel regulation tools.

At last it is demonstrated that the generality of results obtained are far exceeding the particular model considered in the major part of the paper. That is for any linear-quadratic bi-modal game it holds that cooperative and non-cooperative outcomes may be topologically equivalent (have the same number and type of equilibria) only if players are symmetric with respect to their investment efficiencies. Otherwise the non-cooperative dynamics is more complex, the more players are in the game.

At last, necessary condition for the non-deterministic chaos in any game described by a linear differential system has been obtained. Strategies (investments) of players have to be interdependent and not only asymmetric for this type of dynamics to be observed.

This study raises two important questions for future research. First, how much competition in R&D-oriented industry is optimal? According to current paper's results, presence of additional innovators may switch the dynamics to a chaotic one, whereas no one may predict the long-run behavior, so caution may be needed in promoting the R&D competition for asymmetric firms.

Second, what type of regulation may be appropriate for a given industry? It turns out that constant or state-based subsidies are relevant only for the simplest type of dynamics, whereas for sufficiently asymmetric players it becomes time-dependent and even switching. Further research in this direction is apparently necessary to have detailed answers on this policy-oriented questions.

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