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Sparse BEM for the heat equation

Helmut Harbrecht
(joint work with Christoph Schwab and Johannes Tausch)

1. Introduction

This talk is concerned with the numerical solution of the heat equation

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } \Omega \times I \\ u &= f \quad \text{on } \Gamma \times I \\ u &= 0 \quad \text{on } \Omega \times \{0\} \end{aligned}$$

via boundary integral equations, where $\Omega \subset \mathbb{R}^3$ is a domain with Lipschitz boundary $\Gamma := \partial\Omega$ and $I = (0, T)$ is a time interval. To this end, we introduce the thermal single layer operator

$$\mathcal{V}g(\mathbf{x}, t) = \int_0^t \int_\Gamma G(\|\mathbf{x} - \mathbf{y}\|, t - \tau) g(\mathbf{y}, \tau) \, \mathrm{d}\sigma_{\mathbf{y}} \, \mathrm{d}\tau$$

where $\mathbf{x} \in \Gamma$ and $G(\cdot, \cdot)$ is the heat kernel, given by

$$G(r, t) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{r^2}{4t}\right), \ t \geq 0 \quad \text{and} \quad G(r, t) = 0, \ t < 0.$$

Thus, the potential ansatz

$$u(\mathbf{x}, t) = \int_0^t \int_\Gamma G(\|\mathbf{x} - \mathbf{y}\|, t - \tau) g(\mathbf{y}, \tau) \, \mathrm{d}\sigma_{\mathbf{y}} \, \mathrm{d}\tau$$

leads to the boundary integral equation

$$\mathcal{V}g = f \quad \text{on } \Gamma \times I. \tag{1}$$

2. Galerkin scheme

The thermal single layer operator is a symmetric, elliptic and continuous operator with respect to the its energy space (see [1] for the details). Hence, we may apply a Galerkin discretization without further restriction.

Consider two sequences of nested ansatz spaces

$$\begin{aligned} V_0^\Gamma \subset V_1^\Gamma \subset \cdots \subset V_j^\Gamma \subset \cdots \subset L^2(\Gamma), \\ V_0^I \subset V_1^I \subset \cdots \subset V_j^I \subset \cdots \subset L^2(I) \end{aligned}$$

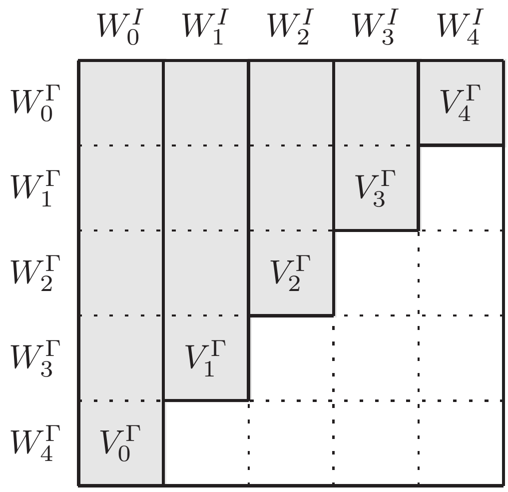

Figure 1. Visualization of the sparse tensor product space.

such that

$$|\Delta_j^\Gamma| = \dim V_j^\Gamma \sim 4^j, \quad |\Delta_j^I| = \dim V_j^I \sim 2^j.$$

Instead of using the full tensor product space $U_j^{\Gamma\times I} := V_j^\Gamma \otimes V_j^I$ for the Galerkin discretization of (1), we shall consider the related sparse tensor product space. The starting point are the multilevel decompositions

$$\begin{aligned} V_j^\Gamma &= W_0^\Gamma \oplus W_1^\Gamma \oplus \cdots \oplus W_j^\Gamma, \\ V_j^I &= W_0^I \oplus W_1^I \oplus \cdots \oplus W_j^I. \end{aligned}$$

Then, the sparse tensor product space is given by

$$\widehat{U}_j^{\Gamma\times I} = \bigoplus_{\ell+\ell'\leq j} W_\ell^\Gamma \otimes W_{\ell'}^I = \bigoplus_{\ell'=0}^{j} \left(\bigoplus_{\ell=0}^{j-\ell'} W_\ell^\Gamma \right) \otimes W_{\ell'}^I = \bigoplus_{\ell'=0}^{j} V_{j-\ell'}^\Gamma \otimes W_{\ell'}^I,$$

as illustrated in Figure 1. Notice that only a wavelet basis in time is necessary to obtain a basis in the sparse tensor product space.

The sparse tensor product space contains much less unknowns compared to the full tensor product space: $\dim \widehat{U}_j^{\Gamma\times I} \sim 4^j$ instead of $\dim U_j^{\Gamma\times I} \sim 8^j$. This means that the time discretization is for free. Nevertheless, the approximation property in the sparse tensor product space is essentially the same as in the full tensor product space, provided that we spent some extra smoothness in terms of the mixed Sobolev spaces. Notice that the construction of the sparse tensor product space can be much improved by using generalized sparse grids and ansatz functions with different polynomial orders in space and in time, see [2].

3. Fast matrix-vector multiplication

For the matrix-vector multiplication, we need to be able to apply the matrix blocks of the form

$$\mathbf{V}_{\boldsymbol{\ell},\boldsymbol{\ell}'} := \langle \mathcal{V}(\Psi^{\Gamma}_{\ell'_1} \otimes \Psi^{I}_{\ell'_2}), \Psi^{\Gamma}_{\ell_1} \otimes \Psi^{I}_{\ell_2} \rangle_{L^2(\Gamma\times I)},$$

where $\|\boldsymbol{\ell}\|_1, \|\boldsymbol{\ell}'\|_1 \leq j$. We aim at approximating such blocks by a low-rank approximation

$$\mathbf{V}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \approx \sum_{i=1}^{M} \mathbf{A}^{(i)}_{\ell_2,\ell'_2} \otimes \mathbf{B}^{(i)}_{\ell_1,\ell'_1} \tag{2}$$

where M is at most a power of j. Then, as proposed in [4], the matrix-vector multiplication can be performed in essentially linear complexity provided that $\mathbf{A}^{(i)}_{\ell_2,\ell'_2}$ and $\mathbf{B}^{(i)}_{\ell_1,\ell'_1}$ can be computed in essentially linear complexity. In particular, by use of prolongations and restriction, it suffices to make available quadratic matrices $\mathbf{A}^{(i)}_{\ell_2,\ell'_2}$ and $\mathbf{B}^{(i)}_{\ell_1,\ell'_1}$ with $\ell_1 = \ell'_1$ and $\ell_2 = \ell'_2$.

A semi-discretization of the heat kernel in time leads to an $\mathcal{H}$-matrix, cf. [3]. Exploiting that this $\mathcal{H}$-matrix is a Toeplitz matrix, we arrive at a low-rank approximation of the form (2). In space, we apply the multipole method as proposed in [5]. Putting these ingredients together, we obtain an algorithm which solves the boundary integral equation (1) in the sparse tensor product space in essentially linear complexity.

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hp–Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes

Paul Houston

(joint work with Paola Antonietti, Andrea Cangiani, Emmanuil Georgoulis, and Stefano Giani)

The numerical approximation of partial differential equations (PDEs) posed on complicated domains which contain ‘small’ geometrical features, or so-called microstructures, is of vital importance in engineering applications. In such situations, an extremely large number of elements may be required for a given mesh generator to