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A Tug of War Team Contest

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Abstract

This paper analyzes a tug of war contest between two teams. In each round of the tug of war a pair of agents from the opposing teams competes in a private value all-pay auction with asymmetric type distributions and effort effectiveness. Whichever team arrives first at a given lead in terms of battle victories over the opponent wins the tug of war. There exists a unique Markov-perfect equilibrium in bidding strategies that depend on the player's valuation and on the history through the current state of the tug of war only. We derive rich comparative statics for this equilibrium by using the fact that the states of the tug of war evolve according to a time-homogeneous absorbing Markov chain.

JEL classification: D74, F51, H56

Keywords: Team Contests, Multi-Stage Contests, Tug of War, All-Pay Auction, Absorbing Markov Chain

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1. Introduction

In the basic contest model, a finite number of agents expends efforts in order to win a fixed, indivisible prize. The allocation of the prize is a function of the efforts expended, and effort costs accrue irrespective of the final allocation. Canonical examples for contests include R&D-competition, sports contests, political elections, lobbying games, or wars. Konrad (2009) provides an exhaustive overview of the literature. The basic model assumes that competition is between individual agents and that the contest is static. In many contest situations, however, there is competition between groups of agents rather than between single agents. Further, a contest is often divided into several battles that take place over time. These two observations have led to two separate strands of literature. Nevertheless, the combination of the two, that is, the analysis of competition between groups in dynamic contests, which is the topic of this paper, has received comparably little attention so far.

This paper considers a contest between two groups, here called teams, that consists of a sequence of battles between a pair of agents (or units) from the opposing teams. The battles are linked by a tug of war structure which is represented by a number of states located on a

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horizontal line: Depending on the outcome of the battles, the tug of war state moves either to the left or to the right until one of the final states at the respective ends of the line is reached. The tug of war starts symmetrically, that is, both teams need to arrive at the same lead in battle victories over the opponent in order to win. The crucial feature of the tug of war contest is that the winner is not determined in terms of of absolute gains, but in terms of gain differences over the adversary. The tug of war is a multi-stage contest with a potentially infinite number of rounds, because the state might move back and forth several times during the contest.

Real life tug of war contests, like the sport contests after which they are named, typically exhibit this pendulum-like character. For example, the dispute over the settlement for the victims of the Ponzi scheme by Bernard Madoff that started after his firm filed bankruptcy in 2009 lasted for more than four years, exhibited several agreed, blocked and re-agreed deals, and was called a "legal tug of war" by the Wall Street Journal in 2013.[1] Militarized disputes, too, often exhibit changing advantages of the parties involved. For example, in the Afghan war that started in 2001, the town of Garmsir in the Helmland province marked the border between British controlled territory and Taliban controlled territory between 2006 and 2008 and changed hands in these years at least three times. Accordingly, the Washington Post described this back and forth as a tug of war.[2]

The battles in our tug of war are modeled as asymmetric private value all-pay auctions. We assume that each agent of the matched pair has a private valuation for the victory of his team, called that agents' type, and that there is no intrinsic value to a victory in battle. The all-pay auction is asymmetric with respect the distributions of these types, but also with respect to a commonly known effort effectiveness parameter of the agents. Because we believe that private values are – at least to some degree – a salient feature of many real life contests, we focus on the private value all-pay auction case in our analysis. Nevertheless, we note that the model is flexible enough to also allow the study of public information about valuations, and alternative contest technologies for the battles. The last section contains a discussion of such alternatives.

A crucial assumption of our tug of war model is that each team has a countably infinite number of agents that are called to play sequentially in every round of the tug of war. This implies that each agent is active only once during the course of the contest, and thus has an influence on the contest outcome only through the outcome of his specific battle. This feature guarantees that the kind of discouragement effect familiar from the literature on dynamic contests (cf. Konrad, 2012) is absent by construction: No agent has future costs which might affect his choice of action today. The assumption of large teams will thus sharpen our interpretation of the effort dynamics that we observe across the tug of war states, and over time. Furthermore, the implication that players do not have an influence on the course of the course contest except through the outcome of their battles squares with the

[1] Wall Street Journal, The Law Blog, "Latest Salvo In Legal Tug Of War Over Madoff Settlement", on Aug 7, 2013: *blogs.wsj.com/law.*

[2] Washington Post, "British Troops, Taliban In a Tug of War Over Afghan Province", on March 30, 2008: *www.washingtonpost.com.*

reality of, e.g., complex legal disputes that involve specialists competing for partial victories in their respective fields of expertise only, or with that of long term militarized disputes where the composition of the troops involved changes over time.

When called upon to play, the players fully observe the public history of the game, consisting, for every round of the tug of war, in the chosen efforts of the players and the tug of war state in the following round. We restrict attention to strategies that we call Markov. Markov strategies only condition on the private type and on the current tug of war state. We show that there exists a unique equilibrium in such strategies (Proposition 1) that we call Markov-perfect equilibrium. We show existence by establishing strategic equivalence between the battles of the tug of war, given the other players follow their Markov-perfect equilibrium strategies, and a simple asymmetric one-shot all-pay auction with scaled valuations. For this latter game, we can adapt the standard uniqueness result of Amann and Leininger (1996).

A first observation about the Markov-perfect equilibrium is that battle-winning probabilities are state-independent, i.e. only depend on the type distributions and on the effort effectiveness parameters of the agents. The result is due to the fact that, in equilibrium, the agents in battle fight over symmetric and strictly positive gains in the contest-winning probability. These gains serve as a scaling factor of the valuations to be won for both players, and because the ex ante equilibrium probability to win the asymmetric all-pay auction is independent of such a scaling, state-independent battle-winning probabilities follow. As a consequence, we can use results on simple random walks with two absorbing states to derive results on the expected duration of the tug of war, and on the contest-winning probabilities of the teams. State-independent battle-winning probabilities imply that in contrast to other dynamic contests, where early battle victories increase the likelihood of victory in later rounds (e.g. Klumpp and Polborn, 2006), there is no such momentum effect in our tug of war. The relation between type distributions, effort effectiveness and battle-winning probabilities is exhibited in Proposition 2.

The core contribution of this paper consists in four results about expected effort provision across states and over time. The first result (Proposition 3) shows that expected individual efforts of either team increase in the stronger team's closeness to defeat, where we call a team stronger than the opponent team if it has a battle-winning probability of more than one half. This monotonicity in effort provision across states is in contrast to the non-monotone effort dynamics known from single-agent tug of wars (Konrad and Kovenock, 2005; Agastya and McAfee, 2006, see the next section for more details), and crucially depends on the assumption that we have teams competing rather than single agents. It can be explained by the interplay of two opposing effects: a dynamic free-riding effect and a discouragement effect, with the latter dominating for the weaker team and the former dominating for the stronger team (and both effects exactly balancing when teams have equal strength). To see the free-riding effect, consider first the extreme case of an agent from the stronger team who expects that his team will win all future battles with probability one. Such an agent has no incentive to exert any effort because his team will win the tug of war anyway – the agent can free-ride on the future efforts of his fellow team members, unless, of course, when the agent is called upon to play in the state where battle defeat is equivalent to defeat in the tug of war, in which case the effort of the agent is essential. The same dynamic free-riding affect arises for the stronger team, albeit in diluted form, whenever battle-outcome probabilities are strictly below one: The closer the stronger team is to victory, the higher is the probability that a defeat in a given battle is compensated by the efforts of future team members, thus reducing the incentive to exert effort for the current agent. For an intuition of the discouragement effect, it is instructive to consider the extreme case of an agent who expects his team to loose all future battles with probability one. Such an agent has no incentive to exert any effort because his team will loose the tug of war anyway – the lack of future efforts of his fellow team members discourages the exertion of effort by the agent under consideration, unless, of course, when the agent is called upon to play in the state where battle victory is equivalent to the tug of war victory, in which case the agent has the strongest possible incentive to exert effort. The same dynamic discouragement effect arises for agents of the weaker team in diluted form whenever the battle-winning probability is strictly above zero: The closer the weaker team is to defeat, the more likely it becomes that a victory in a current battle is squandered by the failure of future team members to win their battles, thus reducing the incentives to exert effort for the current agent. Crucially, the dynamic free-riding and the discouragement effect discussed above both pull in the same direction, implying lower incentives to exert effort for the agents of both teams the closer the strong team is to victory. As a consequence, the fact that it is not only the expectation of future battle-winning probabilities but also the effort of ones own opponent which determines equilibrium effort choices does not upset the intuition.

In spite of varying equilibrium efforts across states, it turns out that the ex ante, i.e. prior to the contest, expected efforts of a given round $t \geq 0$, conditional on the contest not having ended, are independent of t. This is shown in our second result (Proposition 4). An intuition for this result is obtained by considering the interplay between state-independent battle-winning probabilities and the effort monotonicity across states: The former implies that in any state, the tug of war is more likely to move towards victory of the stronger team rather than it is to move towards victory of the weaker team, the latter implies that the expected efforts in the state reached in the first case are lower than in the state reached in the second case. In equilibrium, the two effects exactly balance. We can show the result formally by exploiting two facts about equilibrium play: first, the gains in contest winning probability for which the players in the battles compete are a martingale, and second, expected equilibrium efforts in a given state are linear in these gains.

The third and the fourth result (Propositions 5 and 6) relate expected summed efforts to the imbalance of powers and to the lead required to win, respectively. Using the observations that expected summed efforts are given by the product of the expected length and the expected efforts in the symmetric position, and that the expected length decreases in the imbalance of powers (i.e. in the distance of the battle-winning probabilities from one half), we get that expected summed efforts decline in the imbalance of powers, given that total efforts in the corresponding simple contest also decrease. On the other hand, we find that, as we let the lead in battle victories required to win the tug of war go to infinity, the expected summed efforts over time approach infinity when the teams have equal equilibrium battle-winning probabilities, but vanish completely when the teams are asymmetric with respect to their battle-winning probabilities.

Additionally, we look at the special case of asymmetric uniform type distributions with the lower bound of the support being zero for both teams. We derive closed form expressions for the equilibrium strategies, the battle-winning probability, and the contest-winning probability. The ratio of a player's expected efforts in battle to those of the opponent player decreases in the effort effectiveness of that player, and increases in the upper bound of his type distribution. The ratio of a team's contest-winning probability to that of the opponent team increases both in the effort effectiveness of that team's players and in the upper bound of the type distribution. The lead required to win thereby assumes the role of a discriminatory parameter with respect to the differences in the respective team attributes which is reminiscent of the role of the discriminatory parameter r in the Tullock contest success function (Tullock, 1980; Skaperdas, 1996) determining the impact of the difference in the players' efforts on their winning-probabilities.

The sharpness of our results owes a great deal to the assumption that the players do not discount the future. In the discussion, we briefly talk about how the analysis changes when we drop this assumption. Further, we discuss model variants with alternative contest technologies in the battles and with public information about valuations, and show that our results are robust to assuming that the prize has a public good character with commonly known value to the members of the winning team but not necessarily to assuming differing contest technologies.

The paper is organized as follows: The next section discusses the related literature. Section 3 then outlines the general model and defines the equilibrium notions that we use. Section 4 establishes existence of a unique Markov-perfect equilibrium, and discusses the properties of the tug of war state evolution induced by the Markov-perfect equilibrium. Section 5 contains the main results on effort provision. The example of uniform distribution is analyzed in Section 6. The discussion is in Section 7 , and all proofs are in the appendix.

2. Related Literature

The tug of war model in this paper relates to at least three strands of literature: group contests, dynamic contest, and asymmetric all-pay auctions. Furthermore, the model is a contribution to the research program on team contests with pairwise battles initiated by Fu et al. (2015), which we also discuss in some detail.

In group contest models there are, as the name suggests, groups competing over the prize, and the probability to win for a group depends on the efforts of its members. An important class of group contest models assumes that the probability to win the prize depends on the aggregate group efforts. Such contests are analyzed in Katz et al. (1990) and Baik (1993) assuming that the prize has a pure public good character, in Nitzan (1991) assuming that the group must divide the prize according to some exogenous sharing rule, or in Waerneryd (1998) who assumes that the members of the winning group again engage in wasteful competition over the prize (for further references, see e.g. Section 4.2 in Corchón 2007, or Chapter 6 in Konrad 2009). The main difference of our model to these models is that in the tug of war team contest it is not aggregate efforts that are decisive for victory but individual efforts that are compared pairwise over multiple rounds.

Dynamic contests consist of sequential battles whose outcomes determine the winner. Important dynamic structures for such contests include elimination tournaments with multiple players in which players are pairwise matched and only the winner proceeds to the following round until all but the winner are eliminated (Rosen, 1986; Amegashie, 1999; Gradstein and Konrad, 1999); races between two players in which the players need to arrive at an absolute number $n \geq 2$ of victories over the opponent in order to win (Harris and Vickers, 1985, 1987; Klumpp and Polborn, 2006; Konrad and Kovenock, 2009); and tug of wars between two players in which the players need to arrive at a lead $n \geq 2$ in battle victories over the opponent in order to win. Models analyzing the latter include Harris and Vickers (1987) with battles described by an imperfectly discriminating contest success function, as well as Konrad and Kovenock (2005) and Agastya and McAfee (2006) with a public information all-pay auction at every battle. We extend the dynamic contest literature by letting teams compete in a tug of war.

Also assuming an all-pay auction for the battles, the single-player tug of war models in Konrad and Kovenock (2005) and Agastya and McAfee (2006) are closest to our tug of war. The model in Konrad and Kovenock (2005) also assumes that battle victories have no intrinsic value, but, in contrast to our model, it assumes an endogenous tie-breaking rule assigning victory to the player with a higher continuation value. In equilibrium, the players only spend strictly positive efforts in at most two adjacent tipping states located at the interior of the tug of war, and, apart from the two tipping states, the tug of war moves with certainty towards victory of the player with the higher continuation value. The model in Agastya and McAfee (2006) assumes, in addition to a strictly positive winner prize, a strictly negative loser prize, zero utility when the tug of war proceeds forever, and that in case of a draw the battle has to be re-fought. Because in equilibrium there may be states in which the players prefer a battle draw at no effort costs to winning at a strictly positive cost, the tug of war can get stuck in such states. Given efforts are positive, however, efforts increase towards each of the two final states and the battle-winning probabilities increase for the leading team.

Private value all-pay auctions with asymmetries regarding effort effectiveness and value distributions are analyzed in Amann and Leininger (1996) for the case of differing distributions, by Lien (1990), Clark and Riis (2000) and Feess et al. (2008) for the case of bid handicaps, and by Kirkegaard (2012) for a combination of the two cases. This paper adds to the literature by pointing out the relation between equilibrium behavior and the scaling of valuations, and by placing the auction in the multi-stage framework of a tug of war.

The tug of war model in this paper is an instance of what Fu et al. (2015) call a team contest with pairwise battles. Such a contest is a group contest in which players are pairwise matched across groups, and the outcome of these battles then determine the outcome of the grand contest. In contrast to the tug of war team contest in this paper, Fu et al. (2015) analyze a best-of-n contest about a common prize between two teams with $n \geq 2$ members: Each member of a team is matched with a member of the other team in one of the n battles, and battles are private cost all-pay auctions. Every agent has a distinct cost parameter

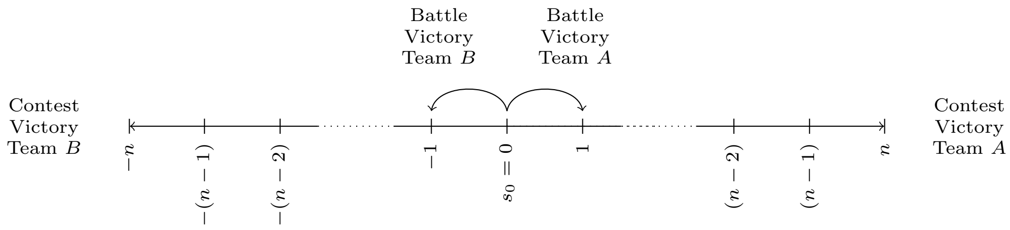

Figure 1: Dynamic structure of the tug of war

distribution, and strictly positive battle prizes are allowed. As in this paper, the sequence of pairwise battles is exogenous. Fu et al. (2015) find that battle-winning probabilities do not depend on the outcomes of the past battles, and that the ex ante (i.e. prior to the contest) expected efforts in a given match do not depend on the particular sequencing of the battles. These findings are particularly striking given that the agents have differing, commonly known cost parameter distributions.

It is an implication of our result establishing the time independence of ex ante expected efforts (Proposition 4) that the sequencing of the pairs is immaterial for the ex ante expected efforts of a given agent in our model, too. In view of our assumption that players in a given team are ex ante symmetric, however, this should not be surprising. Yet, we believe that assuming ex ante symmetric agents is an important strength of our setup: The absence of agent specific factors allows us to characterize the effort dynamics along the course of the contest, and thus provides a rich picture of the forces driving these dynamics in a tug of war over time and across states.

3. The Model

We consider a tug of war contest between two opposing teams, labeled A and B. Both teams have a countably infinite queue of members, labeled $t = 0, 1, \ldots$. The tug of war has a set of *interior states* $S \equiv \{-(n-1), -(n-2), \ldots, (n-1)\}$, and two *final states* $\{-n, n\}$, $n \geq 2$. Provided that the contest has not ended before, in *round* $t = 0, 1, 2\ldots$ of the tug of war, the two agents with the label t compete for the victory in a *battle*. If the state of the contest in round t is $s_t \in S$ and the agent from team A wins the battle, the contest moves to state $s_{t+1} = s_t + 1 \in S$ and continues with round $t+1$ or, in case $s_t + 1 = n$ holds, ends with team A winning the tug of war. Similarly, if agent t from team B wins the battle in round t, the contest moves to state $s_{t+1} = s_t - 1 \in S$ and continues with round $t+1$ or, in case $s_{t+1} = -n$ holds, ends with team B winning the tug of war. The contest starts in round $t = 0$ with initial state $s_0 = 0$, so that whichever team first accumulates n more battle victories than the other team wins the tug of war. We refer to n as the *lead required to win.* Figure 1 illustrates the dynamic structure of a tug of war.

We understand the tug of war as a multi-stage game with observed actions (Fudenberg and Tirole, 1991), where each round $t = 0, 1, ...$ of the tug of war corresponds to a stage, and the tug of war ends in random round $\tau \geq 1$ corresponding to the round in which one of the final states is reached, that is, where $s_\tau \in \{-n, n\}$ holds. At the beginning of each round $t < \tau$, nature assigns the players from either team $i = A, B$ having label t with an individual valuation $v_{i,t}$ for the victory of his team in the tug of war. These individual valuations are independently distributed both within and across teams. Valuations of the players in team i are drawn according to a distribution function F_i that is independent of t and has a strictly positive and continuously differentiable density f_i on its support $[0, \overline{v}_i]$. The distribution functions F_A and F_B are common knowledge. Actual individual valuations are known to the holder alone. In round τ, that is, after the tug of war has ended, the valuations of all the players with labels $t \geq \tau$ realize.

After the valuations have realized in round $t < \tau$, the agents simultaneously choose efforts in battle. If the agent of team A chooses effort $b_{A,t} \in \mathbb{R}_+$ and the competing agent of team B chooses effort $b_{B,t} \in \mathbb{R}_+$, then the agent of team A wins the battle if

$$\alpha_A b_{A,t} > \alpha_B b_{B,t} \tag{1}$$

holds, whereas the agent of team B wins the battle when the reverse strict inequality holds. In case equality holds in (1) then agent A wins the battle with probability 1/2. The strictly positive parameters α_A and α_B appearing in condition (1) are the common *effort effectiveness parameters* of the members of team $i = A, B$. The effort effectiveness parameters are common knowledge and constant over time, and transform the chosen effort $b_{i,t}$ into an *effective effort* $\alpha_i b_{i,t}$ with the battle being won by the agent with the higher effective effort.

The payoff to an agent t of team i, who is called into battle, has valuation $v_{i,t}$, and chooses effort $b_{i,t}$, is $v_{i,t} - b_{i,t}$ if his team wins the tug of war, and $-b_{i,t}$ otherwise. Agents who are not called upon to play receive the payoff $v_{i,t}$ if their team wins the tug of war, and zero otherwise. This formulation of payoffs embodies the assumptions that agents do not discount the future, and that battle victories do not feature an intrinsic value for agents. That is, agents only care about the success of their team on the one hand and their individual effort costs on the other hand.[3]

When entering battlefield in round $t < \tau$, the two players are informed about the efforts expended in the previous rounds and about the states that the tug of war has passed so far. To formalize this idea, denote by h_t, $1 \leq t < \tau$, a history of the tug of war consisting of a sequence $\{a_j\}_{j=0}^{t-1}$ of lists $a_j = (b_{A,j}, b_{B,j}, s_{j+1})$ collecting the efforts $b_{i,j}$ expended by players $i = A, B$ in round j and the tug of war state s_{j+1} at the end of round $j = 0, ..., t-1$.[4] We let $h_0 = \emptyset$, write H_t for the set of all possible such histories h_t, and assume that players observe h_t when called upon to play in round t.

In the case $n = 1$ the game described above reduces to a static private value all-pay auction described by the tuple $\mathcal{T} \equiv (\alpha_A, \alpha_B, F_A, F_B)$, specifying the effort effectiveness parameters and type distributions for the teams. We refer to this game as a *simple contest.* We write $\mathcal{T}_n$ for the tug of war which has the same effort effectiveness parameters and type

[3]The assumption that the discount factor is one is important, as such players do not care about the expected length of the tug of war, given their team wins. We discuss the role of discounting in Section 7.

[4]Whenever $b_{A,j} \neq b_{B,j}$ the information about s_{j+1} is redundant; it needs, however, to be explicitly specified for the case of ties in battle, i.e. $b_{A,j} = b_{B,j}$

distributions in all battles as a simple contest $\mathcal{T}$, but requires a lead of n battles to be won. Much of our analysis will be concerned with understanding the relationship between the equilibria of simple contests and the equilibria of the associated tug of wars.

A strategy of any agent t of team $i = A, B$ is given by a measurable *effort function* $\beta_{i,t} : [0, \overline{v}_i] \times H_t \to \mathbb{R}_+$ relating the valuation $v_{i,t}$ and the history h_t to the effort level $b_{i,t} = \beta_i(v_{i,t}, h_t)$, contingent on being called upon to play in round t. We call $\beta = \{\beta_{A,t}, \beta_{B,t}\}_{t=0}^{\infty}$ a *strategy profile* for the tug of war $\mathcal{T}_n = (\alpha_A, \alpha_B, F_A, F_B)_n$. Given a strategy profile β and a history h_t, we write $U_{i,t}^{\beta}(b_i, v_{i,t}, h_t)$ for the interim utility in the battle of round $t < \tau$ for the agent of team i having valuation $v_{i,t}$ and choosing effort b_i.

Because the valuations are independent both within and across teams, all payoff relevant history is revealed to the players when called upon to play, and hence the move of nature at the beginning of any round $t \geq 1$ after any history $h_t \in H_t$ marks the initial node of a proper subgame (Fudenberg and Tirole, 1991) of the tug of war. Consequently, the strategy profile β is a subgame-perfect equilibrium in the tug of war $\mathcal{T}_n$ if, for every round $t \geq 0$ and for every feasible history $h_t \in H_t$, the strategy pair $(\beta_{A,t}(., h_t), \beta_{B,t}(., h_t))$ forms a Bayes Nash equilibrium in the corresponding battle.

Definition 1 (Subgame-Perfect Equilibrium, SPE)**.** A strategy profile β is SPE for a tug of war $\mathcal{T}_n$ if, for $i = A, B$, we have

$$\beta_{i,t}(v_{i,t}, h_t) \in \arg\max_{b_i \in \mathbb{R}_+} U_{i,t}^{\beta}(b_i, v_{i,t}, h_t) \tag{2}$$

for all $t \geq 0$, $h_t \in H_t$, $v_{i,t} \in [0, \bar{v}_i]$.

Let $H_t^s = \{h_t \in H_t : s_t = s\}$ be the set of histories such that the tug of war state in round $t \geq 0$ is $s \in S$. We will focus in the following analysis on *Markov strategy profiles* β.

Definition 2 (Markov strategy profile)**.** A strategy profile β is Markov if, for both $i = A, B$ and for all $s \in S$, there is a function $\beta_{i,s} : [0, \overline{v}_i] \to \mathbb{R}_+$ such that

$$\beta_{i,t}(., h_t) = \beta_{i,s}(.), \ \forall t \geq 0, h_t \in H_t^s. \tag{3}$$

The strategies $\beta_{i,t}$ in Markov profile β depend on the history h_t only through the current state s_t. We focus on a special class of subgame-perfect equilibrium that we call Markov-Perfect equilibrium (cf. Maskin and Tirole, 2001):

Definition 3 (Markov-Perfect Equilibrium, MPE)**.** A strategy profile β is MPE for a tug of war $\mathcal{T}_n$ if it is both SPE for $\mathcal{T}_n$ and Markov.

In our analysis of the Markov-perfect equilibrium β starting in Section 4.3, we directly deal with the functions $\beta_{i,s}(.)$ that we henceforth call *Markov effort functions*, rather than always referring to the effort functions $\beta_{i,t}(., .)$ actually constituting the profile β. Furthermore, because the strategies in the Markov profile β do not depend on the round index t, we will omit the reference to t in the types $v_{i,t}$ of the corresponding agents, and simply write v_i instead.

4. The Markov-Perfect Equilibrium

This section establishes uniqueness of the Markov-perfect equilibrium, characterizes the equilibrium, and derives first comparative static results for the winning probabilities of the teams and the expected duration of the tug of war.

4.1. Utilities in the MPE

Assuming that the strategy profile β is Markov, we can define the associated ex ante probability that an agent from team A wins a battle in state $s \in S$. These *battle-winning probabilities* are given by

$$p_s^\beta \equiv \int_0^{\bar{v}_A} \int_0^{\bar{v}_B} \Big[\mathbb{1}\{\alpha_A \beta_{A,s}(v_A) > \alpha_B \beta_{B,s}(v_B)\} + \frac{1}{2} \mathbb{1}\{\alpha_A \beta_{A,s}(v_A) = \alpha_B \beta_{B,s}(v_B)\} \Big] dF_B(v_B) dF_A(v_A) \quad (4)$$

where $\mathbb{1}$ is the indicator function. The battle-winning probabilities (4) define a Markov chain of the state space $S \cup \{-n, n\}$, where the states $-n$ and n are absorbing and for $s \in S$ the probability of transiting from state s to state $s+1$ is p_s^β and the probability of transiting to state $s-1$ is $1 - p_s^\beta$. The ex ante probability that team A wins the tug of war given it is currently in state s is then given by the probability $P_{A,s}^\beta$ that the Markov chain defined by the battle-winning probabilities p_s^β is absorbed in state n when starting from state s. Similarly, the probability that team B wins the tug of war given it is currently in state s is given by the probability $P_{B,s}^\beta$ that the Markov chain defined by the battle-winning probabilities p_s^β is absorbed in state $-n$ when starting from state s. We refer to the probability $P_{i,s}^\beta$ as the *contest-winning probability* of the corresponding team given that the current state is s, and let $P_{A,n}^\beta = P_{B,-n}^\beta \equiv 1$ and $P_{A,-n}^\beta = P_{B,n}^\beta \equiv 0$.

Given that all other players stick to their Markov strategy in the Markov strategy profile β, the interim utility $U_{i,t}^\beta$ of player t in team $i = A, B$ depends on h_t only through s_t. Consequently, there are, for all $s \in S$, functions $U_{i,s}^\beta : \mathbb{R}_+ \times [0, \overline{v}_i] \to \mathbb{R}$ such that the interim utilities satisfy $U_{i,t}^\beta(b_i, v_i, h_t) = U_{i,s}^\beta(b_i, v_i)$ for all $t \geq 0$ and $h_t \in H_t^s$. Taken our informational assumptions and, crucially, the fact that every agent is called upon to play at most once into account, the functions $U_{i,s}^\beta(b_i, v_i)$ are given by

$$U_{A,s}^\beta(v_A, b_A) \equiv \Big[P_{A,s+1}^\beta v_A - b_A \Big] \int_0^{\bar{v}_B} \Big[\mathbb{1}\{\alpha_A b_A > \alpha_B \beta_{B,s}(v_B)\} + \frac{1}{2} \mathbb{1}\{\alpha_A b_A = \alpha_B \beta_{B,s}(v_B)\} \Big] dF_B(v_B)$$

$$+ \left[P^{\beta}_{A,s-1} v_A - b_A\right] \int_0^{\bar{v}_B} \Bigg[\mathbb{1}\{\alpha_A b_A < \alpha_B \beta_{B,s}(v_B)\} + \frac{1}{2} \mathbb{1}\{\alpha_A b_A = \alpha_B \beta_{B,s}(v_B)\} \Bigg] dF_B(v_B) \quad (5)$$

$$U^{\beta}_{B,s}(v_B, b_B) \equiv \left[P^{\beta}_{B,s-1} v_B - b_B\right] \int_0^{\bar{v}_A} \Bigg[\mathbb{1}\{\alpha_B b_B > \alpha_A \beta_{A,s}(v_A)\} + \frac{1}{2} \mathbb{1}\{\alpha_B b_B = \alpha_A \beta_{A,s}(v_A)\} \Bigg] dF_A(v_A)$$
$$+ \left[P^{\beta}_{B,s+1} v_B - b_B\right] \int_0^{\bar{v}_A} \Bigg[\mathbb{1}\{\alpha_B b_B < \alpha_A \beta_{A,s}(v_A)\} + \frac{1}{2} \mathbb{1}\{\alpha_B b_B = \alpha_A \beta_{A,s}(v_A)\} \Bigg] dF_A(v_A) \quad (6)$$

The first term on the right side of (5) is the expected payoff $P^{\beta}_{A,s+1} v_A - b_A$ from battle to the agent of team A given that the tug of war continues in $s+1$ multiplied with the probability of winning the battle. The second term is the expected payoff $P^{\beta}_{A,s-1} v_A - b_A$ from battle to the same agent given that the tug of war continues in $s-1$ multiplied with the probability of loosing the battle. The interpretation of (6) is analogous, albeit with inverse signs of the increments to the state variable s in the contest-winning probability $P^{\beta}_{B,s}$.

4.2. Battle Gains and Scaled Simple Contests

Given a Markov strategy profile β for a tug of war $\mathcal{T}_n$, define the *battle gain* in state $s \in S$ for team A by

$$\phi^{\beta}_{A,s} \equiv P^{\beta}_{A,s+1} - P^{\beta}_{A,s-1}$$

and for team B by

$$\phi^{\beta}_{B,s} \equiv P^{\beta}_{B,s-1} - P^{\beta}_{B,s+1}$$

The interpretation of these expressions is straightforward: The battle gain for team $i = A, B$ is the gain in contest-winning probability that accrues to team i from winning rather than loosing a battle in state s. For arbitrary strategy profiles β the battle gain for player A in a given state s can be different from the battle gain for player B in the same state. This will occur only if the battle-winning probabilities associated with the strategy profile β yield strictly positive probability that the tug of war goes on forever without either team winning the contest. The following lemma shows that this cannot happen in equilibrium.

Lemma 1. *If β is an MPE for a tug of war $\mathcal{T}_n$, then the battle-winning probabilities satisfy*

$$0 < p_s^\beta < 1 \text{ for all } s \in S, \tag{7}$$

and the battle gains satisfy

$$\phi_s^\beta \equiv \phi_{A,s}^\beta = \phi_{B,s}^\beta > 0 \text{ for all } s \in S. \tag{8}$$

Condition (8) indicates that whatever team A gains in contest-winning probability by winning a battle is lost by team B and vice versa. Moreover, whether a battle is won or lost always has a non-zero impact on the contest-winning probability of a team. These properties imply that the battle between the agents of teams A and B in state s of the tug of war $\mathcal{T}_n$ is a scaled version of the simple contest $\mathcal{T}$ in which the payoffs of the players are given by

$$u_i(b_i, b_j) \equiv \begin{cases} \phi v_i - b_i & \text{if } \alpha_i b_i > \alpha_j b_j \\ \frac{1}{2}\phi v_i - b_i & \text{if } \alpha_i b_i = \alpha_j b_j \\ -b_i & \text{if } \alpha_i b_i < \alpha_j b_j \end{cases},$$

where $i \neq j \in \{A, B\}$ and $\phi > 0$. We refer to this game as a *scaled simple contest* with (scaling) parameter ϕ. Using Lemma 1, the definitions of the payoff functions (5)–(6) and the fact that the best responses for both players do not depend on the prize received in case of losing the battle, it is straightforward to verify that for a given $s \in S$, the equilibrium condition (2) is equivalent to the requirement that the Markov effort functions $\beta_{A,s}(.)$ and $\beta_{B,s}(.)$ are a Bayes Nash equilibrium in the associated simple contest with scaling parameter ϕ_s^β. We may thus state without further proof:

Lemma 2. *A strategy profile β is an MPE in a tug of war $\mathcal{T}_n$ if and only if for all $s \in S$ the Markov effort functions $\beta_{A,s}(.)$ and $\beta_{B,s}(.)$ are a Bayes Nash equilibrium in the scaled simple contest $\mathcal{T}$ with parameter ϕ_s^β.*

Lemma 2 directs our attention to the equilibria of scaled simple contests. For the case $\alpha_A = \alpha_B$ and distributions F_A and F_B with supports $[0, 1]$ uniqueness of equilibrium for the associated unscaled contest is immediate from Amann and Leininger (1996). Their arguments extend in a straightforward fashion to the case of differing effort effectiveness, distributions with unequal support, and a strictly positive scaling parameter ϕ different from 1, yielding the following result.

Lemma 3. *Given a simple contest $\mathcal{T} = (\alpha_A, \alpha_B, F_A, F_B)$, let $k_i : [0, \bar{v}_i] \to [0, \bar{v}_j]$ be the unique positive solution to the differential equation*

$$k_i'(v_i) = \frac{\alpha_j}{\alpha_i} \frac{k_i(v_i)}{v_i} \frac{f_i(v_i)}{f_j(k_i(v_i))} \tag{9}$$

satisfying the boundary condition $k_i(\bar{v}_i) = \bar{v}_j$ for $i \neq j \in \{A, B\}$. Further, let

$$\beta_i^*(v_i) = \frac{\alpha_j}{\alpha_i} \int_0^{v_i} k_i(v) f_i(v) dv.$$

Then the simple contest $\mathcal{T}$ with scaling parameter $\phi > 0$ has a unique Bayes Nash equilibrium with equilibrium strategy

$$\beta_i^{\phi}(v_i) = \phi \beta_i^*(v_i) \tag{10}$$

for $i = A, B$ and $v_i \in [0, \overline{v}_i]$.

Setting $\phi = 1$ in Lemma 3 provides us with a useful interpretation of the strategies β_i^*: These are nothing but the unique equilibrium strategies of the (unscaled) simple contest $\mathcal{T}$. Throughout the following we let $0 < p < 1$ denote the equilibrium contest-winning probability in a simple contest $\mathcal{T}$. This is given by

$$p \equiv \int_0^{\overline{v}_A} F_B(k_A(v)) dF_A(v) \tag{11}$$

because (cf. the proof of Lemma 3 in the appendix) player A wins the simple contest outright whenever $v_B < k_A(v_A)$ holds and ties occur with probability zero. Throughout the following we refer to p as the *strength* (of team A) in the simple contest $\mathcal{T}$. (We sometimes refer to $(1-p)$ as the strength of team B.) Further, we say that both teams have *equal strength* when they are equally likely to win the simple contest, that is, when $p = 1/2$ holds. When $p > 1/2$ team A is *stronger* than team B. When $p < 1/2$ team B is stronger than team A. Note that it is an immediate implication of (10) that the ex ante probability that player A wins a scaled simple contest with scaling factor $\phi > 0$ is independent of ϕ and given by p.

4.3. Characterization Result

Lemmas 1–3 allow us to demonstrate that there is a unique MPE to any tug of war $\mathcal{T}_n = (\alpha_A, \alpha_B, F_A, F_B)_n$. Further, they provide a full characterization of this equilibrium in terms of the equilibrium strategies, the strength of team A in the underlying simple contest $\mathcal{T}$, and in terms of the parameter n of the tug of war.

By the result in Lemma 1 that guarantees that equilibrium battle gains satisfy $\phi_s^{\beta} > 0$ and the fact that the ex ante probability p that player A wins a scaled simple contest is independent of the parameter $\phi > 0$, it follows from Lemma 2 that the equilibrium battle-winning probabilities satisfy $p_s^{\beta} = p$ for all $s \in S$. Together with the specification of a starting state $s \in S$, the probability $0 < p < 1$ can be taken to define a simple random walk on the integers. The contest-winning probability $P_{A,s}^{\beta}$ then corresponds to the probability of the event that this random walk hits n before it hits $-n$ and $P_{B,s}^{\beta}$ is the probability of the event that the random walk hits $-n$ before it hits n. Interpreting $s + n$ as the initial wealth of an agent and $2n$ as his target level of wealth, this corresponds to the classical gambler's ruin problem as described and analyzed in Feller (1968, Chapter XIV). It follows that equilibrium contest-winning probabilities for the tug of war are uniquely determined and given by

$$P^*_{A,s} \equiv \begin{cases} \dfrac{1-\left(\dfrac{1-p}{p}\right)^{s+n}}{1-\left(\dfrac{1-p}{p}\right)^{2n}} & \text{if } p \neq \dfrac{1}{2} \\ \frac{s+n}{2n} & \text{if } p = \dfrac{1}{2} \end{cases} \tag{12}$$

$$P^*_{B,s} \equiv 1 - P^*_{A,s} \tag{13}$$

for all $s \in S$.[5] The corresponding equilibrium battle gains are then given by

$$\phi^*_s \equiv \begin{cases} \dfrac{\left(\dfrac{1-p}{p}\right)^{s+n-1} - \left(\dfrac{1-p}{p}\right)^{s+n+1}}{1-\left(\dfrac{1-p}{p}\right)^{2n}} & \text{if } p \neq \dfrac{1}{2} \\ \frac{1}{n} & \text{if } p = \dfrac{1}{2} \end{cases} \tag{14}$$

Substituting ϕ^*_s into (10) and applying the sufficiency part of Lemma 2 yields the following result.

Proposition 1. *Let $\mathcal{T}$ be a simple contest with equilibrium strategies β^*_i and strength p, and consider the corresponding tug of war $\mathcal{T}_n$ with $n \geq 2$. Then, there is a unique MPE β. In this MPE, for all $s \in S$, the battle gains are given by ϕ^*_s that depend on p as described in* (14)*, and the equilibrium Markov effort functions are given by*

$$\beta_{i,s}(v_i) = \phi^*_s \beta^*_i(v_i) \tag{15}$$

for all $s \in S$ and $i = A, B$.

4.4. The Determination of Strength

It is clear from Proposition 1 and the description of the equilibrium battle gains ϕ^*_s in (14) that p, the strength of team A, plays an essential role in determining equilibrium behavior and the resulting dynamics in a tug of war. Hence, it is instructive to relate the strength of team A to the parameters describing a simple contest. Because uniqueness of equilibrium in a symmetric game implies symmetry of equilibrium, it immediately follows that the teams have equal strength if the effort effectiveness parameters and the distributions of valuations are identical. For the case of asymmetric simple contests, Lemma 1 in Kirkegaard (2012) is directly applicable and yields the following result:

[5] We arrive at (12) from Equation 2.4 in Feller (1968, p. 345) by dividing both numerator and denominator by $(q/p)^a$, and letting $q = 1-p$, $a = 2n$ and $z = s - n$ with the latter two substitutions being due to the fact that our states are labeled with $\{-n, ..., n\}$ whereas Feller (1968) uses labels $\{0, ..., 2n\}$.

Proposition 2. *Consider a simple contest $\mathcal{T} = (\alpha_A, \alpha_B, F_A, F_B)$:*

(1) Let the simple contest $\mathcal{T}$ satisfy $\alpha_A = \alpha_B$ and $F_A = F_B$. Then teams A and B have equal strength.

(2) Given the other parameters of a simple contest, the strength of team A is strictly increasing in α_A and strictly decreasing in α_B. In particular, in a simple contest with $F_A = F_B$, team A (B) is stronger than team B (A) if and only if $\alpha_A > \alpha_B$ ($\alpha_A < \alpha_B$) holds.

(3) Let the simple contest $\mathcal{T}$ satisfy $\alpha_A = \alpha_B$. If F_A is strictly greater (smaller) than F_B in the usual stochastic order, then team A (B) is stronger than team B (A).

Our approach in the following will be to state results in terms of p, serving as a summary statistic for the equilibrium in the associated simple contest, and the parameter n. This allows us to focus on the aspects of the problem which are intrinsic to the dynamic structure of the team contest, rather than on the equilibrium analysis of the simple contest, which has been thoroughly investigated in the literature (Lien, 1990; Amann and Leininger, 1996; Feess et al., 2008; Kirkegaard, 2012). Section 6 complements the results from Propositions 1 and 2 by providing closed-form solutions for the equilibrium Markov effort functions $\beta_{i,s}$ and for the battle-winning probability p for the case in which the valuation for both teams are uniformly distributed, allowing for asymmetries both in the effort effectiveness parameters α_i and in the support of the uniform distributions.[6]

4.5. Contest-Winning Probabilities and Expected Contest Duration

In Section 5 we explore the consequences of Proposition 1 for expected efforts in the equilibrium of a tug of war. Before turning to this task we note that by using standard results from the analysis of the gambler's ruin problem both the ex ante contest-winning probabilities and the expected duration of the tug of war in the MPE can be determined explicitly as a function of p and n. In addition, we note some comparative static properties.

As the starting state of the tug of war is $s_0 = 0$, the ex ante probability that team A wins the contest is obtained by setting $s = 0$ in the expression for $P^*_{A,s}$ given in (12). It is straightforward that this can be rewritten as in the statement of the following result and satisfies the monotonicity properties asserted:[7]

Corollary 1. *Let p be the strength of team A in the simple contest $\mathcal{T}$. Then the ex ante probability that team A wins the tug of war $\mathcal{T}_n$ is given by*

$$P^*_{A,0} = \frac{p^n}{p^n + (1-p)^n} \tag{16}$$

which is strictly increasing in p, and strictly increasing (decreasing) in n for $p > 1/2$ ($p < 1/2$).

[6] For the case of identical effort effectiveness parameters Amann and Leininger (1996) provide further examples in which the equilibrium strategies for a simple contest can be determined explicitly.

[7] To see the monotonicity properties, observe that from equation (16) we can write $1/P^*_{A,0} = 1 + ((1-p)/p)^n$ which is strictly decreasing in p, and strictly increasing (decreasing) in n for $p > 1/2$ ($p < 1/2$).

It is obvious from (16) that $P^*_{A,0} = 1/2$ holds if $p = 1/2$, so that when both teams have equal strength the contest-winning probability is independent of the lead n needed to win the tug of war. When team A is stronger than team B, the contest-winning probability of team A is strictly increasing in the lead n (exceeding in particular, the battle-winning probability p for $n \geq 2$). Thus, n can be interpreted as a decisiveness parameter, magnifying the differences in team strength that are driven by asymmetries in the simple contest (cf. Proposition 2). Corollary 1 will be of particular interest in Section 6, where we look at the special case of uniform distributions and derive an explicit expression for p.

The *duration* τ of a tug of war is the number of battles taking place until one of the teams wins the contest. In terms of the random walk described by the parameters p and n (and the initial state $s_0 = 0$) τ corresponds to the hitting time for the set $\{-n, n\}$ and we can again apply results from the analysis of the gambler's ruin problem to characterize the expected duration of the tug of war.

Corollary 2. *Let p be the strength of team A in the simple contest $\mathcal{T}$. Then the expected duration of the tug of war $\mathcal{T}_n$, $n \geq 2$, is given by*

$$E[\tau] = \begin{cases} n\left(\dfrac{1}{2p-1}\right)\left(\dfrac{p^n - (1-p)^n}{(1-p)^n + p^n}\right) & \text{if } p \neq \dfrac{1}{2} \\ n^2 & \text{if } p = \dfrac{1}{2} \end{cases}, \tag{17}$$

which is strictly increasing in n, and strictly increasing (decreasing) in p for $p < 1/2$ ($p > 1/2$).

It is immediate from (17) that for a given n the expected duration of the tug of war depends only on the absolute value of the difference between the strength of the two teams $I(p) = |2p - 1|$, but not on which team is the stronger one, with a greater *imbalance of strength* $I(p)$ implying a shorter expected duration.

As an aside, the implication that a higher imbalance of strength leads to shorter expected duration is in line with empirical findings regarding militarized disputes: Slantchev (2004) presents evidence that wars tend to last the longer the lower is the material asymmetry between the warring parties.

5. Equilibrium Efforts

In this section we explore the consequences of Proposition 1 for the expected efforts in the MPE of the tug or war. We begin by investigating how expected efforts depend on the state, then consider expected efforts conditional on the round of the tug of war and, finally, look at total expected efforts over the duration of the war.

5.1. Expected Efforts across States

Let

$$e^*_i = \int_0^{\bar{v}_i} \beta^*_i(v_i) dF_i(v_i)$$

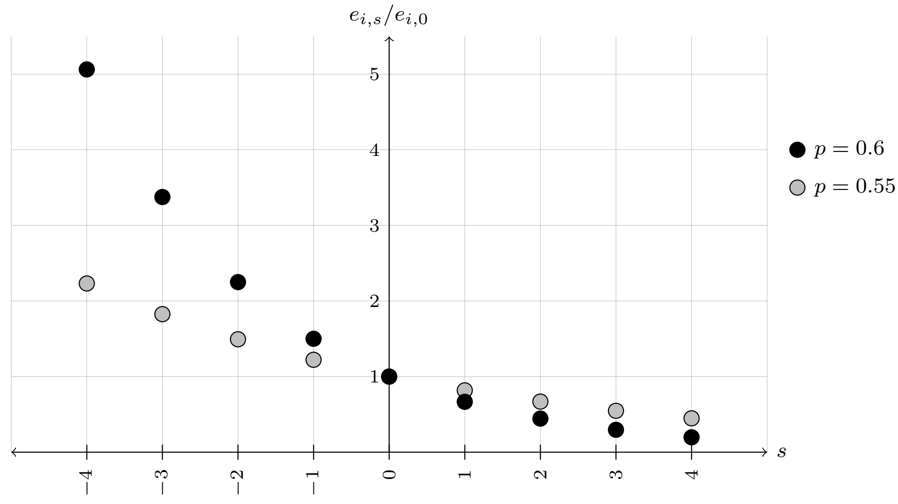

Figure 2: Normalized Expected Efforts $e_{i,s}/e_{i,0}$ across States

denote the expected equilibrium efforts in the simple contest $\mathcal{T}$. From Proposition 1, conditional on the state s, the expected equilibrium efforts of the players in the tug of war $\mathcal{T}_n$ are

$$e_{i,s} \equiv \phi_s^* e_i^*. \tag{18}$$

The expected efforts in state s in the tug of war depend on the state s only through the battle gain given by ϕ_s^* and do so in a linear way. In particular, we can use (14) to obtain

$$e_{i,s} = r^s e_{i,0}, \tag{19}$$

where $r = (1 - p)/p$ is the *relative strength* (of team B). The following result is then immediate.

Proposition 3. *If team A is stronger than team B, then expected efforts $e_{i,s}$ in the tug of war $\mathcal{T}_n$ are strictly decreasing in s for both teams $i = A, B$. If team B is stronger than team A, then expected efforts $e_{i,s}$ in the tug of war $\mathcal{T}_n$ are strictly increasing in s for both teams $i = A, B$. If both teams $i = A, B$ have equal strength, then expected efforts $e_{i,s}$ in the tug of war $\mathcal{T}_n$ are independent of the state.*

Proposition 3 establishes that expected equilibrium efforts decrease monotonically in the stronger team's closeness to victory. In order to get a picture of the comparative statics of these effort dynamics in the strength parameter p, we refer to Figure 2. Figure 2 depicts normalized expected efforts $e_{i,s}/e_{i,0}$ – which are equal for both $i = A, B$, as is evident from (19) – across the states $s \in \{-4, ..., 4\}$ for $p = 0.55$, and $p = 0.6$. Clearly, the expected efforts of both teams decrease in s. Furthermore, Figure 2 suggests that an increase in the imbalance of strength increases the volatility of efforts across states in the sense that the ratio between normalized expected efforts in the states $-s$ and s is higher with $p = 0.6$ than it is with more equal teams under $p = 0.55$ for any $s \in \{1, 2, 3, 4\}$. Considering (19), we see that this is indeed the case and that it holds for any tug of war with $n \geq 2$ because, for $p > 1/2$ ($p < 1/2$), r^s increases (decreases) in p for any negative $s \in S$ and decreases (increases) in p for positive $s \in S$.

For later purposes, it is also instructive to see how the absolute value of $e_{i,s}$ changes when we change n. Using equations (14), (18), and the definition of relative strength $r = (1-p)/p$ we may express the expected equilibrium efforts in state $s \in S$ as

$$e_{i,s} = \begin{cases} \frac{r^{n+s+1} - r^{n+s-1}}{r^{2n} - 1} e_i^* & \text{if } r \neq 1 \\ \frac{1}{n} e_i^* & \text{if } r = 1 \end{cases}. \tag{20}$$

It is straightforward to verify that this expression is strictly decreasing in n with $e_{i,s}$ converging to zero as n goes to infinity. This is intuitive: one would expect the value of winning any fixed battle to be decreasing in the lead that is required to win the contest, and to become insignificant as n goes to infinity.

5.2. Expected Efforts over Time

Conditional on the tug of war not having ended, the expected effort of team i in round t of the contest is given by $E[e_{i,s_t} \mid t < \tau]$, where we extend the definition of $e_{i,s}$ from equation (18) to all integers s, s_t is the state of the random walk defined by p and the initial condition $s_0 = 0$, and τ is the hitting time for the set $\{-n, n\}$. When both teams have equal strength, it is clear that we have

$$E[e_{i,s_t} \mid t < \tau] = e_{i,0} \text{ for all } t = 0, 1, 2, ... \tag{21}$$

because – as noted above – in this case the expected efforts in the tug of war satisfy $e_{i,s} = e_i^*/n$ independent of state. Somewhat surprisingly, equation (21) holds for all tug of wars. The underlying cause for this result is that the equilibrium battle gains are a martingale. This is made clear in the proof of the following proposition, given in the appendix.

Proposition 4. *$E[e_{i,s_t} \mid t < \tau]$ is independent of t, that is,* (21) *holds for all tug of wars $\mathcal{T}_n$.*

The result provides the basis for our analysis of total expected efforts over the duration of the tug of war to which we turn now.

5.3. Expected Total Efforts over the Duration of the Tug of War

Let

$$h_i \equiv E\left[\sum_{t=0}^{\tau-1} e_{i,s_t}\right].$$

denote the expected summed efforts of the members of team i over the duration of a tug of war. We will refer to h_i as the *expected total effort* of team i in the following. Building on the argument proving Proposition 4 we can show that h_i is simply the product of team i's expected equilibrium efforts in the initial state $s_0 = 0$ and the expected duration of the tug of war:

Lemma 4. *Expected total efforts in a tug of war $\mathcal{T}_n$ are given by*

$$h_i = E[\tau]e_{i,0}. \tag{22}$$

The result in Lemma 4 allows us to address the question how p and n affect expected total efforts. Consider the effect of strength first. We have seen in Corollary 2 that expected duration is strictly decreasing in the imbalance of strength. So, in view of Lemma 4 and (18), the proof to the following Proposition establishes that ϕ_0^*, too, strictly decreases in the imbalance of powers $I(p)$. This implies that for two simple contests $\mathcal{T}$ and $\mathcal{T}'$ with associated strength and expected efforts (p, e_A^*, e_B^*) and $(p', e_A^{*'}, e_B^{*'})$ satisfying $I(p) > I(p')$, $e_A^* \leq e_A^{*,'}$, and $e_B^* \leq e_B^{*,'}$, we have $e_{i,0} < e'_{i,0}$ in the corresponding tug of wars $\mathcal{T}_n$ and $\mathcal{T}'_n$. Consequently, in expectation a higher imbalance of strength not only implies a shorter tug of war, but also one in which battles are fought less intensely:

Proposition 5. *Consider two simple contests $\mathcal{T}$ and $\mathcal{T}'$ such that the associated strength and expected efforts (p, e_A^*, e_B^*) and $(p', e_A^{*'}, e_B^{*'})$ satisfy $I(p) > I(p')$, $e_A^* \leq e_A^{*,'}$, and $e_B^* \leq e_B^{*,'}$. Then for any $n \geq 2$ expected total efforts h_i and h'_i in the tug of wars $\mathcal{T}_n$ and $\mathcal{T}'_n$ satisfy $h_i < h'_i$ for $i = A, B$.*

Now consider the effect of n on expected total efforts. This effect is clear when both teams have equal strength: from (17) and (20), we have $E[\tau]e_{i,0} = ne_i^*$, so that for any given underlying simple contest $\mathcal{T}$ with strength $p = 1/2$ expected total efforts are strictly increasing in the parameter n of the associated tug of war $\mathcal{T}_n$. The situation is radically different when one of the teams is stronger than the other one, that is, when $p \neq 1/2$ or, equivalently, when $r \neq 1$ holds in the simple contest $\mathcal{T}$. In this case for large n the expected duration of the tug of war $\mathcal{T}_n$ increases at rate n because the increase in decisiveness of the contest brought about by an increase in n implies that the term multiplying n in (17) converges to a strictly positive constant. On the other hand, as is shown in the proof to the next proposition, the expected effort per round of the contest, $e_{i,0}$, decays at a geometric rate when n is large enough and approaches zero when n goes to infinity. Consequently, for large enough n expected total efforts are strictly decreasing in n and converge to zero as n goes to infinity. Heuristically speaking, for large n the equilibrium is characterized by the steady and almost effortless accumulation of battle wins by the stronger teams:

Proposition 6. *Take a simple contest $\mathcal{T}$. Then it holds for the expected total efforts s_i in the associated tug of war $\mathcal{T}_n$:*

(a) If $I(p) = 0$, then h_i is strictly increasing in n with $\lim_{n\to\infty} h_i = \infty$.

(b) If $I(p) \neq 0$, then $\exists N$ such that for all $n > N$, h_i is strictly decreasing in n, and $\lim_{n\to\infty} h_i = 0$.

6. An Example: Uniform Distributions

This section discusses the special case of uniform type distributions. The uniform distribution allows us to explicitly derive the equilibrium Markov effort functions, which, in turn, yield explicit solutions for the battle-winning probability p and the contest-winning probabilities $P^*_{i,0}$.[8]

6.1. Game and Equilibrium

Individual valuations v_i are uniformly and independently distributed on support $[0, \overline{v}_i]$, with $\overline{v}_i > 0$. If F_i has a larger support than F_j, that is if $\overline{v}_i > \overline{v}_j$, then F_i stochastically dominates F_j in the usual order. We refer to a simple uniform contest by the tuple $\mathcal{T}^u = (\alpha_A, \alpha_B, \overline{v}_A, \overline{v}_B)$. As before, $\mathcal{T}^u_n$ with $n \geq 2$ denotes the corresponding uniform tug of war.

By Proposition 1, we know that a Markov-perfect equilibrium in increasing Markov effort functions $\beta_{i,s}$ with $P^*_{i,s} \in (0,1)$ and $\phi^*_s > 0$ exists for the tug of war $\mathcal{T}^u_n$. Assuming that the agent of team B expends efforts $b_B \geq 0$ in state $s \in S$ according to a Markov effort function $\beta_{B,s}$ that is strictly increasing on $[0, \overline{v}_B]$, agent A expending efforts $b_A \geq 0$ wins the battle, and thus receives $\phi_s v_A$, with probability $F_B(\beta^{-1}_{B,s}(\alpha_A b_A / \alpha_B))$. Consequently, because the distribution of valuations of agent B is uniform on $[0, \overline{v}_B]$, we can write the battle utility of an agent of team A with valuation v_A and effort b_A in state s as

$$U_{A,s}(v_A, b_A) = \beta^{-1}_{B,s}\left(\frac{\alpha_A}{\alpha_B} b_A\right) \frac{\phi_s v_A}{\overline{v}_B} + P_{A,s-1} v_A - b_A.$$

Accordingly, the utility for player B is

$$U_{B,s}(v_B, b_B) = \beta^{-1}_{A,s}\left(\frac{\alpha_B}{\alpha_A} b_B\right) \frac{\phi_s v_B}{\overline{v}_A} + P_{B,s+1} v_B - b_B.$$

The unique and strictly increasing equilibrium Markov effort functions $\beta_{i,s}$ given in the next proposition jointly solve the first-order conditions derived from the battle utilities $U_{i,s}$ for all $v_i \in [0, \overline{v}_i]$, $i = A, B$.

Proposition 7. *In the uniform tug of war $\mathcal{T}^u_n$, $n \geq 2$, the equilibrium Markov effort function for an agent of team i playing against an agent of team j in state $s \in S$ is given by*

$$\beta_{i,s}(v_i) = \phi^*_s \frac{\alpha_j}{\alpha_i \overline{v}_i + \alpha_j \overline{v}_j} \overline{v}_j \overline{v}_i^{\,1 - \frac{\alpha_i \overline{v}_i + \alpha_j \overline{v}_j}{\alpha_i \overline{v}_i}} v_i^{\frac{\alpha_i \overline{v}_i + \alpha_j \overline{v}_j}{\alpha_i \overline{v}_i}}$$

with battle-gains ϕ^*_s as given in (14)*, and the battle-winning probability given by*

$$p = \frac{\alpha_A \overline{v}_A}{\alpha_A \overline{v}_A + \alpha_B \overline{v}_B}. \tag{23}$$

[8] The uniform all-pay auction with asymmetric supports and effort effectiveness parameters is also analyzed in Clark and Riis (2000), albeit without explicit solution for the equilibrium strategies.

When we set $\alpha_A = \alpha_B = 1$ and $\overline{v}_A = \overline{v}_B = 1$, then the equilibrium effort functions $\beta_i^*(v_i) = \beta_{i,s}(v_i)/\phi_s^*$ of the corresponding simple contest $\mathcal{T}^u$ collapse to the well-known equilibrium functions $\beta_i^*(v_i) = v_i^2/2$ of the symmetric all-pay auction with valuations drawn uniformly from $[0, 1]$ for both $i = A, B$ (cf. Krishna, 2002, page 32). The battle-winning probability p in (23) is derived by computing

$$p = \int_0^{\overline{v}_A} \beta_B^{*-1}\left(\frac{\alpha_A}{\alpha_B}\beta_A^*(v)\right)\frac{\mathrm{d}v}{\overline{v}_B\overline{v}_A}$$

and depends on the product of the agents' attributes α_i and $\overline{v}_i$. That is, with respect to battle-winning probabilities, the effort effectiveness parameter α_i and the upper bound $\overline{v}_i$ are substitutes: a high upper bound can compensate for low effort effectiveness, and vice versa.

6.2. Contest-Winning Probabilities and Equilibrium Efforts

By plugging the battle-winning probability p into the expression of $P_{A,0}^*$ which we derived earlier in Corollary 1 we get:

Proposition 8. *The ex ante probability that team A wins the uniform tug of war $\mathcal{T}_n$ is given by*

$$P_{A,0}^* = \frac{(\alpha_A\overline{v}_A)^n}{(\alpha_A\overline{v}_A)^n + (\alpha_B\overline{v}_B)^n}. \tag{24}$$

The ratio of the contest-winning probabilities of the two teams is given by

$$\frac{P_{A,0}^*}{1 - P_{A,0}^*} = \left(\frac{\alpha_A\overline{v}_A}{\alpha_B\overline{v}_B}\right)^n. \tag{25}$$

That is, the lead required to win assumes the role of a discriminatory parameter determining the influence of two teams' strength attributes on the contest-winning probability, where the effort effectiveness parameter and the upper bound on the type distribution enter multiplicatively. As observed in the introduction, the role of n is thus reminiscent of the role of the discriminatory parameter r in the Tullock contest success function that we discuss in some more detail in section 7.1 below.

Further, the assumption of uniform distribution allows us to state an additional result on the ratio of expected efforts in a battle. Computing $e_{i,s} = \int_0^{\overline{v}_i} \beta_{i,s}(v)\mathrm{d}F_i(v)$ yields

$$\frac{e_{A,s}}{e_{B,s}} = \frac{\overline{v}_A}{\overline{v}_B}\frac{\alpha_A\overline{v}_A + 2\alpha_B\overline{v}_B}{2\alpha_A\overline{v}_A + \alpha_B\overline{v}_B}. \tag{26}$$

Not surprising, the ratio is independent of the state s, as expected efforts are linear in ϕ_s^*. It is easy to see that with symmetric teams (that is, with $\alpha_A = \alpha_B$ and $\overline{v}_A = \overline{v}_B$) we have $e_{A,s} = e_{B,s}$. Furthermore, if we assume equal effort effectiveness $\alpha_A = \alpha_B$, $\overline{v}_A > \overline{v}_B$

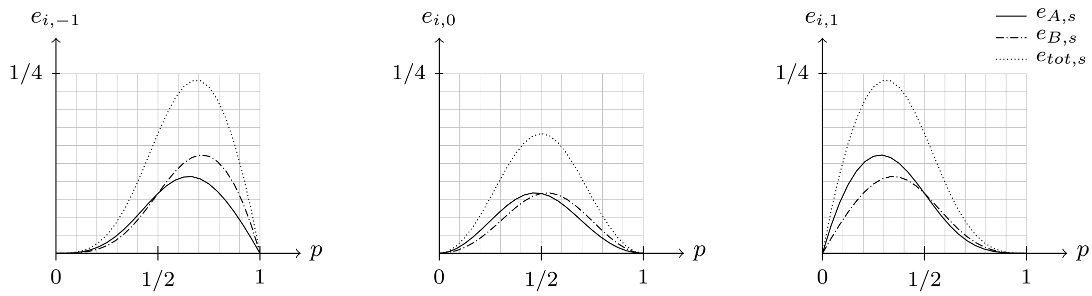

Figure 3: Expected efforts $e_{i,s}$, $s \in \{-1, 0, 1\}$ depending on $p = 1/(1 + \alpha_B/\alpha_A)$

is equivalent to $e_{A,s} > e_{B,s}$ in every state s, and if we assume equal motivation $\overline{v}_A = \overline{v}_B$, $\alpha_A > \alpha_B$ is equivalent to $e_{A,s} < e_{B,s}$ in every state s. Furthermore, by taking the partial derivatives of $e_{A,s}/e_{B,s}$ it is straightforward to verify that the monotonicity properties stated in the next proposition hold.

Proposition 9. *The ratio $e_{A,s}/e_{B,s}$ of expected efforts in battle decreases (increases) in α_A (α_B), and increases (decreases) in $\overline{v}_A$ ($\overline{v}_B$). In particular:*

(a) If $\overline{v}_i = \overline{v}_j$ and $\alpha_i = \alpha_j$ holds, then $e_{i,s} = e_{j,s}, \forall s \in S$.

(b) If $\overline{v}_i = \overline{v}_j$ holds, then: $\alpha_i > \alpha_j \;\Leftrightarrow\; e_{i,s} < e_{j,s}, \forall s \in S$.

(c) If $\alpha_i = \alpha_j$ holds, then: $\overline{v}_i > \overline{v}_j \;\Leftrightarrow\; e_{i,s} > e_{j,s}, \forall s \in S$.

We finish this section by graphically assessing the result on the ratio of expected efforts of Proposition 9, and the result on effort monotonicity across states of Proposition 3. This is done in Figure 3 for a uniform tug of war with $n = 2$, uniform type distributions on $[0, 1]$ for both teams $i = A, B$, and a varying effort effectiveness ratio α_A/α_B. The three sub-figures depict expected individual efforts $e_{i,s}$ for both teams $i = A, B$, as well as total efforts $e_{tot,s} \equiv e_{A,s} + e_{B,s}$ depending on $p = 1/(1 + \alpha_B/\alpha_A) \in (0, 1)$ in the three tug of war states $s = \{-1, 0, 1\}$. When $p > 0.5$ we have $\alpha_A > \alpha_B$, and when $p < 0.5$ we have $\alpha_A < \alpha_B$. In line with Proposition 9, we see that with non-zero imbalance of strength (i.e. with $p \neq 1/2$) the expected individual efforts of the agents of the strong team are below the efforts of the agents of the weak team in every state. The agents of the weak team behave more aggressively. In line with Proposition 3, we see that efforts are monotone over the states, and in particular, that the state with the stronger team closer to victory has the lowest total and individual efforts of all states, whereas the state with the weaker team closer to victory has the highest total and individual efforts.

7. Discussion

7.1. Alternative Contest Technologies and Team prizes

The two crucial features of the equilibrium in the scaled simple contest with scaling factor $\phi > 0$ that allow us to obtain the results in Sections 4.3–4.5 and 5 are that the equilibrium winning probability p is independent of ϕ and that the equilibrium efforts e_i are linear in ϕ. From the former it follows that the tug of war has a Markov-perfect equilibrium with state-independent winning probabilities, and from the second it follows that the efforts in that equilibrium are linear in the battle gains ϕ.

In view of this observation, we now assess the robustness of the results derived in Sections 4.3–4.5 and 5 to alternative contest technologies in the battles and to public information about valuations. We do so by assuming that teams $i = A, B$ compete for a prize that has public good character for the members of the winning team and is of commonly known value $v_i > 0$ to each member of team i.[9] We divide the analysis into perfectly and imperfectly discriminating contest technologies.

Perfectly discriminating contest technologies in battle

A contest technology is called perfectly discriminating if the relation between effort and prize allocation is – apart from knife-edge cases – deterministic. We restrict attention to variants of the all-pay auction, and first consider an all-pay auction with public valuations $v_i > 0$, scaling factor $\phi > 0$, and asymmetric effort effectiveness parameters $\alpha_A, \alpha_B > 0$. Like in the private value case, the probability to win for a player of team A when the players choose effort levels $b_A, b_B \geq 0$ is given by

$$w(b_A, b_B) = \begin{cases} 1 & \text{if } \alpha_A b_A > \alpha_B b_B \\ 1/2 & \text{if } \alpha_A b_A = \alpha_B b_B \\ 0 & \text{if } \alpha_A b_A < \alpha_B b_B \end{cases}, \tag{27}$$

and the probability to win for a player of team B is given by $1 - w(b_A, b_B)$. Existence and uniqueness of the mixed strategy equilibrium in the case $\alpha_A = \alpha_B = 1$ and $\phi = 1$ is well understood, and analyzed in Hillman and Riley (1989) for two players, in Baye et al. (1996) for more than two players, and in Siegel (2009, 2014) for alternative cost functions. The case $\alpha_A \neq \alpha_B$ albeit with $v_A = v_B$ and $\phi = 1$ is treated in in Franke et al. (2014). It is straightforward to extend the analysis of Franke et al. (2014) to the case $v_A \neq v_B$ and $\phi > 0$, and to show that the required equilibrium properties, that is, invariance of p in $\phi > 0$ and linearity of expected efforts in $\phi > 0$, hold for general $\alpha_A, \alpha_B, v_A, v_B, \phi > 0$. We do so in Appendix B.

Introducing other kind of asymmetries into the contest success function might destroy linearity of expected equilibrium efforts in ϕ and invariance of p in ϕ. For example, these features do not hold in the equilibrium of an all-pay auction that gives one player a head start $h > 0$ in terms of efforts, as in the contest success function

$$w(b_A, b_B) = \begin{cases} 1 & \text{if } b_A + h > b_B \\ 1/2 & \text{if } b_A + h = b_B \\ 0 & \text{if } b_A + h < b_B \end{cases}. \tag{28}$$

[9]It is important that team prizes are consistent with the assumption that players in a given team are ex ante symmetric. Dropping this assumption (as would e.g. be the case for individual but commonly known valuations) would of course raise the question of strategic sequencing of the players in the teams – something that we cannot address with the present model.

The unique equilibrium (which is either pure or mixed, depending on the parameter values) for $v_A, v_B, \phi, h > 0$ is derived in Appendix B from which it is easily verified that expected equilibrium efforts are not linear in ϕ, and that winning probabilities are not independent of ϕ.

Imperfectly discriminating contest technologies in battle

A contest technology is called imperfectly discriminating if the relation between effort and prize allocation is not deterministic. In the following, we think of an imperfectly discriminating contest technology in terms of a contest success function $w : \mathbb{R}^2_+ \to [0, 1]$ that is continuously differentiable on $\mathbb{R}^2_{++}$ and that satisfies $w(b_A, b_B) \in (0, 1)$ if $b_A, b_B > 0$, $w(0, b_B) = 0$ for all $b_B > 0$ and $w(b_A, 0) = 1$ for all $b_A > 0$. The function $w(b_A, b_B)$ returns the battle winning probability of agent A when the efforts are $b_A, b_B \geq 0$. As before, we assume that there are no draws, so that the battle-winning probability for agent B is given by $1 - w(b_A, b_B)$. In Appendix B we establish that, for any $v_A, v_B, \phi > 0$ and assuming that a pure-strategy equilibrium exists, homogeneity of degree zero of the contest success function $w(b_A, b_B)$ is necessary and sufficient for equilibrium efforts that are linear in ϕ and equilibrium winning probabilities that are invariant to changes in ϕ. General contest success functions that are homogeneous of degree zero are studied in Baik (2004). A well-studied class of such functions can be parameterized by

$$w(b_A, b_B) = \begin{cases} \dfrac{\alpha_A (b_A)^r}{\alpha_A (b_A)^r + \alpha_B (b_B)^r} & \text{if } b_A + b_B > 0 \\ 1/2 & \text{if } b_A + b_B = 0 \end{cases}, \tag{29}$$

where $\alpha_A, \alpha_B > 0$ are effectiveness parameters and $r > 0$ is called the discriminatory parameter. With $\alpha_A = \alpha_B$, above is the so-called Tullock contest success function that was first analyzed by Tullock (1980) and later axiomatized in Skaperdas (1996). The form with $\alpha_A \neq \alpha_B$ is axiomatized in Clark and Riis (1998). Existence and uniqueness of the pure-strategy equilibrium is well-understood (Pérez-Castrillo and Verdier, 1992; Nti, 1997; Cornes and Hartley, 2005; Häfner and Nöldeke, 2014), and in general requires that the discriminatory parameter r is not too high.

On the other hand, homogeneity of degree zero is not satisfied by the following simple variant $w(b_A, b_B)$ of (29) that gives player A a head start $h > 0$,

$$w(b_A, b_B) = \frac{\alpha_A (b_A)^r + h}{\alpha_A (b_A)^r + \alpha_B (b_B)^r + h}. \tag{30}$$

Other variants $w(b_A, b_B)$ of (29) that are not homogeneous of degree zero include for example those with a player specific discriminatory parameter $r_i > 0$ as analyzed in Cornes and Hartley (2005).

7.2. The Role of Discounting

The crucial feature of the tug of war equilibrium that we have used for our analysis is that the battle gains are equal for both agents at any state $s \in S$. This is established in

Lemma 1. The assumption that agents do not discount the future thereby plays a crucial role, as is shown next.

Suppose β is an MPE of a tug of war with $n \geq 2$, let $\delta \in (0, 1]$ be the common discount factor to all agents, and as before denote the hitting time of the set the final states $\{-n, n\}$ by τ. For a given state $s \in S$, let $E^{\beta}_{A}[\delta^{\tau}|s]$ denote the expected discount factor of the prize given the tug of war is absorbed in state n, i.e. given team A wins. Similarly, let $E^{\beta}_{B}[\delta^{\tau}|s]$ be the expected discount factor for the prize given that team B wins. Then the battle gains for the two player in battle, given state $s \in S$, are given by

$$\phi^{\beta}_{A,s} = P^{\beta}_{A,s+1}E^{\beta}_{A}[\delta^{\tau}|s+1] - P^{\beta}_{A,s-1}E^{\beta}_{A}[\delta^{\tau}|s-1]$$
$$\phi^{\beta}_{B,s} = P^{\beta}_{B,s-1}E^{\beta}_{B}[\delta^{\tau}|s-1] - P^{\beta}_{B,s+1}E^{\beta}_{B}[\delta^{\tau}|s+1].$$

The effect of discounting is best seen by considering the tug of war state $s = n-1$ in which a battle victory for team A implies tug of war victory for team A. The finding in the proof to Lemma 1 that in equilibrium the tug of war ends with a probability of one, and hence that we have $P_{A,s} = 1 - P_{B,s}$, is also valid with discounting, and it follows that in any MPE β the battle gains in state $n-1$ are given by

$$\phi^{\beta}_{A,n-1} = 1 - P^{\beta}_{A,n-2}E^{\beta}_{A}[\delta^{\tau}|n-2]$$
$$\phi^{\beta}_{B,n-1} = (1 - P^{\beta}_{A,n-2})E^{\beta}_{B}[\delta^{\tau}|n-2].$$

From these expressions we see that with $\delta = 1$, we are in the symmetric situation analyzed hitherto, but looking at the other extreme, we see that $\lim_{\delta\to 0}\phi^{\beta}_{A,n-1} = 1$ and $\lim_{\delta\to 0}\phi^{\beta}_{B,n-1} = 0$ holds. Quite intuitively, when the agents discount the future and the expected duration conditional on the own team winning the tug of war differ, then the battle gains cannot be equal. Discounting in the tug of war drives a wedge between the battle gains of the two agents on battlefield. This is in contrast to the best-of-n team contest of Fu et al. (2015), where the duration of the contest is fixed to n rounds and thus a discount factor $\delta < 0$ that is the same for all agents does not affect the fact the battle gains are equal.

8. Conclusion

This paper has presented team contest with pairwise battles that are linked by a tug of war structure. The battles are modeled as asymmetric private value all-pay auctions with team specific value distributions and effort effectiveness. There exists a unique Markov-perfect equilibrium with strategies depending on individual types and on the history of play through the current state only. The setup has allowed to analyze the effort dynamics across time and across states, focusing on the incentives arising from such a dynamic structure. For the special case of asymmetric uniform distributions, we have derived closed form solutions for the equilibrium strategies. We have further shown that the results are robust to assuming a winner prize that is a public good of commonly known value for the members of the winning team, but not necessarily to alternative contest-technologies in battles. Further, the critical role of the assumption that players do not discount the future has been emphasized. Both caveats leave ample room for future research.

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Appendix A. Proofs

Appendix A.1. Proofs of Section 4

Proof of Lemma 1. We first establish (7). Suppose there exists an equilibrium β, and a state s such that $p_s^\beta = 0$ holds. (The argument excluding the possibility $p_s^\beta = 1$ is analogous, exchanging the roles of A and B in the following.) By (4) this implies that the inequality $\alpha_A \beta_{A,s}(v_A) < \alpha_B \beta_{B,s}(v_B)$ holds with probability 1. In particular, we have $\beta_{B,s}(v_B) > 0$ for almost all v_B. Further, because team A loses the battle with probability 1, (5) together with the equilibrium condition (2) implies $\beta_{A,s}(v_A) = 0$ for almost all v_A. But then an agent of team B wins the battle with probability 1 with any strictly positive choice of effort. However, for $\beta_B(v_B) > 0$, and any $\epsilon \in (0,1)$, we have $U_{B,s}^\beta(v_B, \epsilon \cdot \beta_{B,s}(v_B)) - U_{B,s}^\beta(v_B, \beta_{B,s}(v_B)) = (1-\epsilon)\beta_{B,s}(v_B) > 0$ for almost all v_B, implying the existence of a profitable deviation. We have a contradiction.

We now turn to (8). It is an immediate consequence of condition (7) that the Markov chain induced by the battle-winning probabilities p_s^β is absorbing (with $-n$ and n being the two absorbing states). This in turn implies (see Chapter 11.2 in Grinstead and Snell, 1997) that the contest-winning probabilities satisfy $P_{A,s}^\beta + P_{B,s}^\beta = 1$ for all $s \in S$, yielding

$$\phi_{A,s}^\beta = \phi_{B,s}^\beta \text{ for all } s \in S.$$

To show that the battle gains are not only identical but also strictly positive, consider the contest-winning probabilities for team A. For $s \in S$ these are linked to the battle-winning probabilities by the recursion

$$P_{A,s}^\beta = p_s^\beta P_{A,s+1} + (1-p_s^\beta) P_{A,s-1}$$

with boundary conditions $P_{A,-n}^\beta = 0$ and $P_{A,n}^\beta = 1$. Observe that we can write

$$\begin{aligned} P_{A,s}^\beta - P_{A,s-1}^\beta &= p_s^\beta P_{A,s+1}^\beta + (1-p_s^\beta) P_{A,s-1}^\beta - P_{A,s-1}^\beta \\ &= p_s^\beta (P_{A,s+1}^\beta - P_{A,s}^\beta) + p_s^\beta (P_{A,s}^\beta - P_{A,s-1}^\beta) \end{aligned}$$

implying that

$$\left[P_{A,s+1}^\beta - P_{A,s}^\beta\right] = \frac{1-p_s^\beta}{p_s^\beta}\left[P_{A,s}^\beta - P_{A,s-1}^\beta\right]$$

holds for all $s \in S$. Since $\left[P^{\beta}_{A,-(n-1)} - P^{\beta}_{A,-n}\right] = P^{\beta}_{A,-(n-1)} > 0$ (as a consequence of (7)), it follows by induction that $\left[P^{\beta}_{A,s+1} - P^{\beta}_{A,s}\right] > 0$, $\forall s \in S$ (again, as a consequence of (7)). That is, $P^{\beta}_{A,s}$ is strictly increasing in s, and hence $\phi^{\beta}_{s} > 0$ holds for all $s \in S$. □

Proof of Lemma 3. The same arguments as in the proofs of Lemmas 1–5 in Amann and Leininger (1996) yield that equilibrium strategies β_A and β_B for the scaled simple contest with parameter $\phi > 0$ are continuous, increasing, and satisfy the boundary conditions $\beta_A(0) = \beta_B(0) = 0$ and

$$\alpha_A \beta_A(\bar{v}_A) = \alpha_B \beta_B(\bar{v}_B) \tag{A.1}$$

with the equality in (A.1) indicating that the maximal effective equilibrium efforts are identical. Further, letting $\underline{v}_i = \max\{v_i : \beta_i(v_i) = 0\}$ for $i = A, B$, we have that β_i is strictly increasing on $[\underline{v}_i, \bar{v}_i]$ and $\min\{\underline{v}_A, \underline{v}_B\} = 0$. This ensures that there exists a function $k_A : [0, \bar{v}_A] \to [0, \bar{v}_B]$ solving the equation

$$\alpha_B \beta_B(k_A(v_A)) = \alpha_A \beta_A(v_A) \tag{A.2}$$

for all $v_A \in [0, \bar{v}_A]$. If $\underline{v}_A = \underline{v}_B = 0$, then k_A is uniquely defined and strictly increasing. If $\underline{v}_B > 0$, we set $k_A(0) = \underline{v}_B$, resulting again in a uniquely defined and strictly increasing k_A. If $\underline{v}_A > 0$, then k_A is uniquely defined from (A.2), satisfies $k_A(v_A) = 0$ for $v_A \le \underline{v}_A$, and is strictly increasing for $v_A \ge \underline{v}_A$.

Consider $\hat{v}_A > \underline{v}_A$. By construction of the function k_A, type $\hat{v}_A$ of player A and type $\hat{v}_B = k_A(\hat{v}_A) > \underline{v}_B$ choose the same effective effort. By the monotonicity properties of the functions β_A, β_B, and k_A noted above, it follows that player B choosing the effort $b_B = \beta_B(k_A(\hat{v}_A))$ will win the auction if and only if the realized type of player A, v_A, satisfies $v_A < \hat{v}_A$ and will loose the auction if and only if $v_A > \hat{v}_A$ holds. Consequently, the payoff to player B with valuation $v_B = k_A(v_A)$ who chooses the effort $b_B = \beta_B(k_A(\hat{v}_A))$ is

$$\phi F_A(\hat{v}_A) k_A(v_A) - \beta_B(k_A(\hat{v}_A)).$$

Equilibrium requires that player B with valuation $v_B = k_A(v_A)$ finds it optimal to choose the effort $\beta_B(k_A(v_A))$. Taking the appropriate first order condition with respect to $\hat{v}_A$ yields

$$\phi f_A(v_A) k_A(v_A) - \beta'_B(k_A(v_A)) k'_A(v_A) = 0. \tag{A.3}$$

Similarly, if player A with valuation v_A chooses the effort $b_A = \beta(\hat{v}_A)$, his probability of winning the auction is $F_B(k_A(\hat{v}_A))$ and the resulting payoff is

$$\phi F_B(k_A(\hat{v}_A)) v_A - \beta_A(\hat{v}_A).$$

The corresponding first order condition with respect to $\hat{v}_A$ yields

$$\phi f_B(k_A(v_A)) k'_A(v_A) v_A - \beta'_A(v_A) = 0. \tag{A.4}$$

Differentiating (A.2) yields

$$\alpha_B \beta'_B(k_A(v_A)) k'_A(v_A) = \alpha_A \beta'_A(v_A).$$

Using this equation to eliminate β'_A from (A.4), solving the resulting equation for β'_B, and substituting into (A.3) then yields that in every equilibrium k_A must satisfy condition (9). That is,

$$k'_A(v_A) = \frac{\alpha_B}{\alpha_A}\frac{k_A(v_A)}{v_A}\frac{f_A(v_A)}{f_B(k_A(v_A))} \tag{A.5}$$

must hold, with (A.1) and (A.2) providing the initial condition $k_A(\bar{v}_A) = \bar{v}_B$. The same argument as given in the proof of Theorem 1 in Amann and Leininger (1996) then imply that k_A is uniquely defined. Substituting from (A.5) into (A.4) and integrating yields the unique candidate for an equilibrium strategy of player A as given in (10). Exchanging the roles of players A and B in this argument provides the corresponding characterization of the equilibrium strategy of player B, yielding a unique equilibrium candidate. Existence of equilibrium is implied by Theorem 7 in Athey (2001). □

Proof of Corollary 2. The expression for $E[\tau]$ in (17) is a straightforward rearrangement of the standard formula for the expected length of play in a gambler's ruin problem (cf. Stern, 1975). For the case $p \neq 1/2$, Corollary 1 implies that the term multiplying n is strictly increasing in n, yielding the claim that the expected duration is strictly increasing in n.

For $p \neq 1/2$ denote the right side of (17) by $L(p, n)$. For $p, q \neq 1/2$ the equality $L(p, n) = L(q, n)$ is equivalent to

$$[q - p]\,[q^n p^n - (1 - p)^n(1 - q)^n] = [1 - q - p]\,[p^n(1 - q)^n - (1 - p)^n q^n]\,.$$

For $n \geq 2$ this equality only holds for $q = p$ or $q = (1-p)$. In particular, for $1/2 < p < q < 1$ we have $L(p, n) \neq L(q, n)$. Because $L(1/2, n) = n^2 > L(1, n) = n$ holds and L is continuous in p, it follows that L is strictly decreasing in p for $p > 1/2$. An analogous argument for the case $p < 1/2$ finishes the proof. □

Appendix A.2. Proofs of Section 5

Proof of Proposition 4. In light of (19) it suffices to show

$$E[r^{s_t} \mid t < \tau] = 1 \text{ for all } t. \tag{A.6}$$

Towards this we first observe that $\lambda_t \equiv r^{s_t}$ is a martingale with respect to s_t as

$$E[\lambda_{t+1} \mid \lambda_0, \dots, \lambda_t] = E[\lambda_{t+1} \mid \lambda_t] = p\lambda_t r + (1 - p)\lambda_t r^{-1} = \lambda_t.$$

Second, applying the optional stopping theorem (cf. Theorem 5.7.4 in Durret, 2010), we obtain

$$E[\lambda_\tau] = E[\lambda_0] = 1, \tag{A.7}$$

and that the corresponding stopped process

$$\bar{\lambda}_t = \begin{cases} \lambda_t & \text{if } t < \tau \\ \lambda_\tau & \text{if } t \geq \tau \end{cases}$$

satisfies

$$E[\bar{\lambda}_t] = 1 \text{ for all } t. \tag{A.8}$$

Third, from Samuels (1975) the random variables τ and λ_τ are independent, implying

$$E[\lambda_\tau \mid \tau \leq t] = E[\lambda_\tau]. \tag{A.9}$$

Observing that

$$E[\bar{\lambda}_t] = P(\tau \leq t)E[\lambda_\tau \mid \tau \leq t] + P(\tau > t)E[\lambda_t \mid \tau > t],$$

we can use equations (A.7)–(A.9) to rewrite this as

$$1 = P(\tau \leq t) + P(\tau > t)E[\lambda_t \mid \tau > t].$$

Because $1 - P(\tau \leq t) = P(\tau > t) > 0$ holds, this implies $E[\lambda_t \mid \tau > t] = 1$, finishing the proof. □

Proof of Lemma 4. We have

$$E\left[\sum_{t=0}^{\tau-1} e_{i,s_t}\right] = \sum_{T=0}^{\infty} P(\tau = T)\left(\sum_{t=0}^{T-1} E\left[e_{i,s_t} \mid \tau = T\right]\right).$$

Using arguments analogous to the ones appearing in the proof of Proposition 4 we may infer $E\left[e_{i,s_t} \mid \tau = T\right] = e_{i,0}$ for all t and T. Consequently, we have

$$h_i = \sum_{T=0}^{\infty} P(\tau = T)\left(\sum_{t=0}^{T-1} E\left[e_{i,s_t} \mid \tau = T\right]\right) = \sum_{T=0}^{\infty} P(\tau = T)Te_{i,0} = E[\tau]e_{i,0},$$

which is (22). □

Proof of Proposition 5. Fix a tug of war $\mathcal{T}_n$ with $n \geq 2$. We need to show the following: For $r > 1$, ϕ_0^* is strictly decreasing in r, and for $r < 1$, ϕ_0^* is strictly increasing in r. Using the fact that $\sum_{k=0}^{n-1} r^{2k} = (1 - r^{2n})/(1 - r^2)$, we can rewrite ϕ_0^*, defined in (14), as

$$\phi_0^* = \frac{r^{n-1}}{\sum_{k=0}^{n-1} r^{2k}}.$$

From the quotient rule the sign of the partial derivative of this expression with respect to r is identical to the sign of

$$(n-1)r^{n-2}\left[\sum_{k=0}^{n-1} r^{2k}\right] - r^{n-1}\left[\sum_{k=0}^{n-1} 2kr^{2k-1}\right] = \sum_{k=0}^{n-1} (n-1-2k)\, r^{2k+n-2}.$$

The expression on the right side of the equality sign is clearly strictly positive for sufficiently small positive r and strictly negative for sufficiently large r. Further, it is equal to zero at

$r = 1$ (as $\sum_{k=0}^{n-1}(n-1) = n(n-1) = 2\sum_{k=0}^{n-1} k$). Since we have one sign change in the sequence $\{n-1-2k\}_{k=0}^{n-1}$, Descartes' rule of signs implies

$$\sum_{k=0}^{n-1} (n-1-2k)\, r^{2k+n-2} > 0$$

for all $0 < r < 1$ and

$$\sum_{k=0}^{n-1} (n-1-2k)\, r^{2k+n-2} < 0$$

for all $r > 1$, verifying the monotonicity properties. □

Proof of Proposition 6. We need to show that, for $r \neq 1$, $e_{i,s}$ decreases to zero as n approaches infinity, and that it does so at a geometric rate given n is high enough. Writing $e_{i,s}(n)$ to indicate the dependence of the $e_{i,s}$ on n, the fact that $\lim_{n\to\infty} e_{i,s}(n) = 0$ holds for $r \neq 1$ is immediate from (20). Further, (20) implies

$$\frac{e_{i,s}(n+1)}{e_{i,s}(n)} = \frac{r^{2n+1} - r}{r^{2n+2} - 1}$$

which is easily seen to be strictly smaller than 1 for all strictly positive $r \neq 1$, and from which we see that for large n the rate of decay to zero is geometric. □

Appendix B. Alternative Contest Technologies

The following sections provide the formal background of the claims made in the text of Section 7.1.

Perfectly discriminating contest technologies

We look at all-pay auctions, consider mixed strategies over non-negative bids and denote the c.d.f. of player i's bid distribution by $G_i(x)$, $x \geq 0$. For a given contest success function $w(b_A, b_B)$ and a scaling factor $\phi > 0$, the utilities $U_i(b_i)$ for the agents $i = A, B$ having valuation $v_i > 0$ when choosing effort b_i given the opponent plays according to G_j, $j \neq i \in \{A, B\}$ are

$$U_A(b_A) = \int_0^\infty w(b_A, b_B)\mathrm{d}G_B(b_B)\phi v_A - b_A \tag{B.1}$$

$$U_B(b_B) = \int_0^\infty (1 - w(b_A, b_B))\mathrm{d}G_A(b_A)\phi v_B - b_B. \tag{B.2}$$

Let (G_A^*, G_B^*) denote the mixed-strategy Nash equilibrium, and assume the contest success function $w(b_A, b_B)$ is as given in (27). Following the arguments in the proof of Lemma 3.1

in Franke et al. (2014), we get that equilibrium behavior in such an all-pay auction is equal to the equilibrium behavior in an all-pay auction in which expended efforts are equal to effective efforts and the valuations for players $i = A, B$ are given by $\phi\alpha_i v_i$. Let

$$e_i = \int_0^\infty b \mathrm{d}G_i(b)$$

denote the expected efforts of player i. Then, the properties of the equilibrium that are stated in the next proposition follow directly from the equilibrium characterization given in Baye et al. (1996).

Lemma 5. *In the public information all-pay auction with w given in (27) and with v_A, v_B, $\phi > 0$, there exists a unique equilibrium (G_A^*, G_B^*). For $\alpha_i v_i \geq \alpha_j v_j$, $i \neq j \in \{A, B\}$, expected equilibrium efforts are given by $e_i = \phi\alpha_j v_j/2$ and $e_j = \phi(\alpha_j v_j)^2/(2\alpha_i v_i)$, and the winning probability for player A is given by $p = 1 - \alpha_j v_j/(2\alpha_i v_i)$.*

The next result presents the equilibrium of an all-pay auction with public information about valuations where the contest success function is as in (28), giving a head start of $h > 0$ to player A. The characterization of the equilibrium is verified by plugging the respective equilibrium strategies into (B.1) – (B.2) to see that neither agents can profitably deviate. Uniqueness follows immediately from Corollary 1 in Siegel (2014).

Lemma 6. *In the public information all-pay auction with contest success function (28) and with $v_A, v_B, \phi > 0$, there exists a unique equilibrium. If $h \geq \phi v_B$, the equilibrium is in pure strategies in which bidders $i = A, B$ submit bids $b_i = 0$, and equilibrium utilities are given by $U_A^* = v_1$, $U_B^* = 0$. If $h < \phi v_B$, two cases need to be distinguished:*

(a) $\phi(v_B - v_A) < h$. Equilibrium strategies $G_i^*, G_j^* : \mathbb{R}_+ \to [0, 1]$ are given by

$$G_A^*(x) = \begin{cases} \dfrac{x+h}{\phi v_B} & \text{for } x \in [0, \phi v_B - h) \\ 1 & \text{for } x \geq \phi v_B - h \end{cases} \tag{B.3}$$

$$G_B^*(x) = \begin{cases} \dfrac{\phi(v_A - v_B) + h}{\phi v_A} & \text{for } x \in [0, h) \\ \dfrac{\phi(v_A - v_B) + x}{\phi v_A} & \text{for } x \in [h, \phi v_B) \\ 1 & \text{for } x \geq \phi v_B \end{cases} . \tag{B.4}$$

and the equilibrium utilities are given by $U_A^* = \phi(v_A - b_B) + h$ and $U_B = 0$.

(b) $\phi(v_B - v_A) \geq h$. Equilibrium strategies $G_i^*, G_j^* : \mathbb{R}_+ \to [0, 1]$ are given by

$$G_A^*(x) = \begin{cases} \dfrac{\phi(v_B - v_A) + x}{\phi v_B} & \text{for } x \in [0, \phi v_A) \\ 1 & \text{for } x \geq \phi v_A \end{cases} \tag{B.5}$$

$$G^*_B(x) = \begin{cases} 0 & \textit{for } x \in [0, h) \\ \dfrac{x-h}{\phi v_A} & \textit{for } x \in [h, \phi v_A + h) \\ 1 & \textit{for } x \geq \phi v_A + h \end{cases} . \quad \text{(B.6)}$$

and the equilibrium utilities are given by $U^*_A = 0$ *and* $U^*_B = \phi(v_B - v_A) - h$*.*

For example computing e_A using (B.3), it is immediate that expected efforts e_i are not linear in ϕ. Further, it follows from (B.1) that the equilibrium winning probability p of player A satisfies

$$p = \frac{e_A + U^*_A}{\phi v_A}, \quad \text{(B.7)}$$

and is hence clearly not constant in ϕ.

Imperfectly discriminating battle technologies

We consider pure strategies that consist in efforts $b_A, b_B \geq 0$ for the respective agents $i = A, B$. For a given scaling parameter $\phi > 0$, the utilities $U_i(b_A, b_B)$ for the agents $i = A, B$ having valuations $v_A, v_B > 0$ when pure strategies $b_A, b_B \geq 0$ are chosen are given by

$$U_A(b_A, b_B) = w(b_A, b_B)\phi v_A - b_A \quad \text{(B.8)}$$

$$U_B(b_A.b_B) = (1 - w(b_A, b_B))\phi v_B - b_B. \quad \text{(B.9)}$$

Let (b^*_A, b^*_B) denote the pure-strategy Nash equilibrium. The following result states that a contest success function $w(b_A, b_B)$ that is homogeneous of degree zero is necessary and sufficient for equilibrium efforts that are linear in ϕ and equilibrium winning probabilities that are invariant to changes in ϕ.

Lemma 7. *Suppose the imperfectly discriminating scaled contest with contest success function* $w(b_A, b_B)$ *and* $v_A, v_B, \phi > 0$ *has a pure-strategy equilibrium with efforts* (b^*_A, b^*_B)*. Then, the equilibrim efforts are linear in* ϕ *and the battle-winning probability* $p = w(b^*_A, b^*_B)$ *is independent of* ϕ *if and only if* $w(b_A, b_B)$ *is homogeneous of degree zero in* (b_A, b_B)*.*

Proof of Lemma 7. The fact that homogeneity of degree zero in w is necessary is obvious. In order to show sufficiency, we first observe that any equilibrium satisfies $b^*_A, b^*_B > 0$: suppose we have $b^*_A > b^*_B = 0$ (the argument excluding $b^*_B > b^*_A = 0$ is analogous). Then player A wins with probability one and can lower his effort b_A marginally without reducing his winning probability. We have a profitable deviation, and hence contradiction. If, on the other hand, we suppose $b^*_A = b^*_B = 0$, then there is at least one player that can strictly increase his winning probability to one by marginally increasing his efforts. Again, we have profitable deviation, and hence a contradiction. Consequently, the equilibrium efforts $b^*_i > 0$ satisfy the first order conditions

$$w_1(b^*_A, b^*_B)\phi v_A = 1 \quad \text{(B.10)}$$

$$-w_2(b^*_A, b^*_B)\phi v_B = 1, \quad \text{(B.11)}$$

where w_1 and w_2 denote derivatives of w with respect to the first and second argument respectively. These conditions can be rearranged to get

$$w_1(b_A^*, b_B^*)b_A^* + w_2(b_A^*, b_B^*)b_B^* = \frac{b_A^*}{\phi v_A} - \frac{b_B^*}{\phi v_B} \tag{B.12}$$

The proof continues by contradiction: We assume that w is homogeneous of degree zero but that equilibrium efforts b_i^* are not linear in $\phi > 0$ (which is equivalent to assuming that p is not constant in ϕ). By Euler's Theorem it follows that the left-hand side of (B.12) is zero, and hence that we have

$$\frac{b_A^*}{b_B^*} = \frac{v_A}{v_B}. \tag{B.13}$$

Using above relation, we get from (B.10) that

$$w_1(b_A^*, b_A^* \cdot v_B/v_A)\phi v_A = 1 \tag{B.14}$$

must hold. Let (b_A^*, b_B^*) and $(b_A^{*'}, b_B^{*'})$ be the equilibrium efforts for scaling factors ϕ and ϕ', respectively, where the valuations v_A, v_B are kept constant. By non-linearity of the equilibrium efforts in ϕ there must exist values $\phi \neq \phi' > 0$ such that it holds for the values λ, λ' satisfying $\phi = \lambda\phi$ and $b_A^* = \lambda' b_A^{*'}$ that $\lambda \neq \lambda'$. Because w_1 is homogeneous of degree -1, this contradicts the equality in (B.14), thus finishing the proof. □

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