Universität Basel
Wirtschaftswissenschaftliche Fakultät

WWZ

June 2015

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WWZ Working Paper 2015/03

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A publication of the Center of Business and Economics (WWZ), University of Basel.

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June 11, 2015

Abstract

This paper presents a model in which sampling biases are evolutionary stable. We consider the sampling best response dynamics for a two-strategy population game having a unique equilibrium that is in mixed strategies. Allowing players to use differing sampling procedures, we model evolutionary competition between such procedures with a variant of the replicator dynamics that discriminates on the basis of average fitness among players with the same procedure. Using results on slow-fast systems, we find that the sampling bias in stable procedures is generically non-zero, that the size of the bias is the more extreme the closer the mixed equilibrium is to the boundary of $(0, 1)$, and that, if sample size increases, then the bias eventually decreases. Based on these observations, we argue that the presence of biases can be explained by an evolutionary second-best effect correcting for suboptimal choices induced by playing best response to small samples.

Keywords: Sampling Best Response Dynamics, Sampling Bias, Evolutionary Second-Best, Two-Speed Dynamics

JEL-Classifications: C73, D83

1 Introduction

Consider a player in a large population who has to take a binary decision, where the optimal choice depends on the share of the other players in the population choosing the first option over the second option. Assume that all other players have made their choices, but that it is technically unfeasible to monitor all the choices, and hence that our player has to rely on a limited sample of the population in order to get a picture of the probabilities at which the respective options are chosen by the other players. Experimental evidence about choice under uncertainty suggests that humans making probability judgments based on the observed frequency in finite samples are prone to errors that cause systematic biases in their probability estimates (Hertwig et al., 2004, 2006). The source of such errors, however, is disputed, and might either lie in biases in the inference from the sample on the true

*This paper has evolved from the second chapter of my PhD-thesis at the University of Basel. I am indebted to Yvan Lengwiler for his ongoing support and to Georg Nöldeke for numerous insightful suggestions. Further thanks go to Jean-Michel Benkert for helpful comments, Nick Netzer for asking just the right questions, Larry Samuelson for a very fruitful discussion, and William Sandholm for invaluable feedback in an early stage of the project. All remaining errors are my own.

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probability (Kahneman et al., 1982), or in biases in the sampling procedure generating the sample from the population (Fiedler and Juslin, 2006; Freytag and Fiedler, 2006). Answers to the question about the true error source are necessarily empirical; however, even if the question could be resolved conclusively, it is not clear at all from an evolutionary perspective why such biases in the estimates would persist in the first place, because they seem to lead humans to systematically take suboptimal decisions.

This paper tries to shed some light on this question by assuming the latter of aforementioned error sources, i.e. by assuming that humans are *naive intuitive statisticians* (Juslin et al., 2007). Naive intuitive statisticians infer the fraction of players choosing either option from the frequencies at which these options occur in the sample, but ignore the fact that the sampling process leading to their sample might be biased. We assume the presence of such a sampling bias. We assume that the source of the bias is internal to the judging player in the sense that players make for example observation errors (i.e. sometimes mistake a choice of the first option in their sample for a choice of the second option, and vice versa) or encoding errors (i.e. sometimes incorrectly store the sample before evaluating it) that both drive a wedge between the probabilities at which the choices appear in the samples for the players and the true probabilities of these choices in the population. The premise in the following analysis is that evolution determines the magnitude of these sampling biases, and that it does so in a way that the resulting sampling process yields fitness-maximizing choices. We derive a simple model in which sample biases indeed induce fitness-maximizing behavior.

In the framework that we use the players' decisions are interdependent and decisions are made repeatedly. Players assume the choices of the others to be constant over time, but now and then engage in probability judgements by sampling and thereupon adapt their strategy choice. More specifically, the fraction of players choosing the first option, in the following called the population state, follows a sampling best response dynamics (Sandholm, 2001a; Oyama et al., 2015). The sampling best response dynamics describes the evolution of the population state when players of a large population repeatedly face a game in which the payoffs associated with the choices available depend on the current population state. Players hold subjective beliefs about the population state, take the population state for stationary, myopically choose a best response to their belief whenever playing the game, and with constant arrival rate receive independent opportunities for belief revision. If such a revision opportunity arrives for a player, the player obtains a finite population sample and takes the frequency of the observed choices in the sample as the new belief of the current population state.

In contrast to Sandholm (2001a) and Oyama et al. (2015) who analyze unbiased sampling best response dynamics in games with pure strategy equilibria, this paper analyzes sampling best response dynamics in two-strategy games with a unique equilibrium that is in mixed strategies, and explicitly allows for biased sampling. The starting point of our analysis is the observation that – as laid out in the two motivating examples to follow – for any unbiased sampling procedure with a given sample size, we can find payoffs such that, if the sampling best response dynamics is at rest, there exist differing sampling procedures which yield higher expected utility at that rest point. Such sampling procedures either use a different sample size or involve biased sampling probabilities. We will take these observations to build a model in which different biases, and possibly sample sizes, stand in evolutionary competition.

1.1 Two Motivating Examples

In both examples, we consider a situation in which players from a unit mass population are repeatedly pairwise matched to play a two-strategy game. The strategies are labeled by 1 and 2. If the fraction of players choosing strategy 1 in the population is given by $z \in [0,1]$, then the expected payoffs $U_i(z)$ to the players when choosing strategy $i = 1,2$ are given by $U_1(z) = -(1+\epsilon)z$, $\epsilon > 0$, and $U_2(z) = -(1-z)$. Because payoffs depend on the choices of the other players only in terms of z, the game considered is an instance of a population game (Sandholm, 2010) with z being called the population state. The game has a Hawk-Dove structure with unique Nash equilibrium $z^* = 1/(2+\epsilon) < 1/2$.

The derivation of the sampling best response dynamics in the following examples is informal, and for a more formal treatment we refer the reader to Section 2.2.1. The first example compares sampling procedures that differ in the sampling probabilities, the second example compares sampling procedures that differ in the sample sizes.

Example 1. *Suppose that revising agents employ sampling with a sample size of one, and then play best response to the sample. That is, a revising agent chooses strategy* 1 *whenever he observes strategy* 2 *in his sample, which happens with probability* $P_1(z) = 1 - z$. *Assuming that individual revision opportunities occur according to Poisson processes that are independent across players and arrive with rate one, the best response dynamics are given by*

$$\dot{z} = P_1(z) - z, \tag{1}$$

where $\dot{z}$ *indicates the derivative with respect to time. System* (1) *has a unique rest point at* $z = 1/2$. *The average utility in the population at* $z = 1/2$ *is given by*

$$V(1/2) = U_1(1/2)/2 + U_2(1/2)/2. \tag{2}$$

Now, consider an alternative sampling procedure that also has a sample size of one but samples strategy 1 *agents with probability* $p(z) \in [0,1]$ *at population state* $z \in [0,1]$, *satisfying* $p(1/2) > 1/2$. *Players using this sampling procedure will play strategy* 1 *with probability* $P_2(z) = 1 - p(z)$, *and their average utility at* $z = 1/2$ *is given by*

$$W(1/2) = (1 - p(1/2))U_1(1/2) + p(1/2)U_2(1/2).$$

Because $z^* < 1/2$ *holds, it follows that* $U_1(1/2) - U_2(1/2) < 0$, *and hence that*

$$V(1/2) - W(1/2) = (1/2 + p(1/2) - 1)[U_1(1/2) - U_2(1/2)] < 0.$$

That is, a biased sampling probability $p(z)$ *satisfying* $p(1/2) > 1/2$ *yields higher utility when the current fraction of strategy* 1 *agents in the population is* $1/2$. *The left panel in Figure 1 depicts this situation graphically for some continuous* $p(z)$ *that satisfies* $p(1/2) > 1/2$: *at the rest point* $z = 1/2$ *induced by players choosing strategy* 1 *with probability* $P_1(z)$, *the biased sampling procedure with agents choosing strategy* 1 *with probability* $P_2(z)$ *is closer to the best reply correspondence (red), and thus generates higher utility.*

Example 2. *Assume that the equilibrium* z^* *lies in* $(\underline{z}, 1/2)$ *where* $\underline{z} < 1/2$ *is equal to the unique* $z \in (0,1)$ *solving* $(1-z)^2 - z = 0$. *Suppose there are two sampling procedures, both sampling strategy* 1 *players with a probability equal to the current population state* z, *but with the first using a sample size of* $k = 1$, *and the second using a sample size of* $k = 2$.

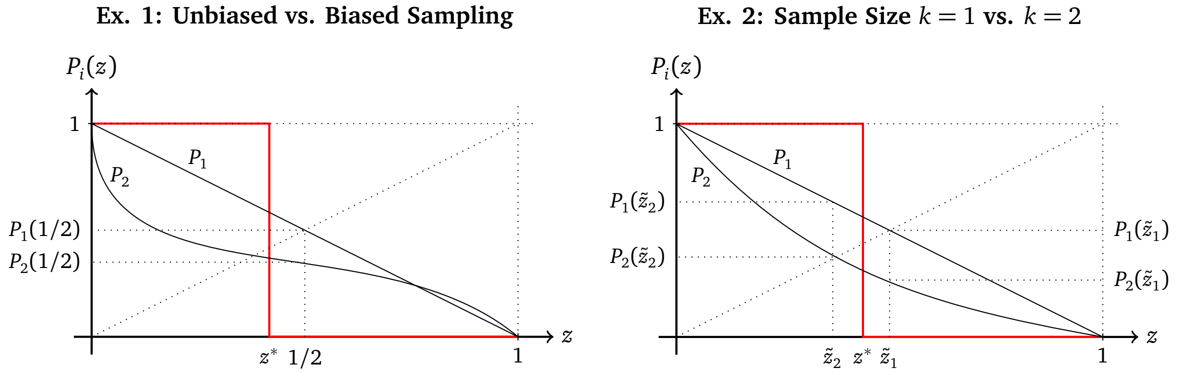

Figure 1: The solid black graphs depict the probability $P_i(z)$ of choosing strategy 1 at population state $z \in [0,1]$. Rest points of the best response dynamics thus induced – denoted in the left panel by 1/2 and in the right panel by $\tilde{z}_i$ – can be read off by projecting the intersection of $P_i(z)$ and the 45-degree line onto the x-axis. The best response correspondence when the equilibrium is z^* is depicted in red.

First, suppose that all players follow the sampling procedure with sample size $k = 1$. Then, the best response dynamics is again as in (1) with a rest point $\tilde{z}_1 = 1/2$, and average utility in the population $V(\tilde{z}_1)$ given by (2). At $\tilde{z}_1 = 1/2$, the average utility for players employing the procedure with sample size $k = 2$, who – playing best response to their sample – choose strategy 1 with probability $P_2(z) = (1-z)^2$ (i.e. if and only if they see two strategy 2 player in the sample), is given by

$$W(\tilde{z}_1) = (1-\tilde{z}_1)^2 U_1(\tilde{z}_1) + (1-(1-\tilde{z}_1)^2) U_2(\tilde{z}_1).$$

Because at $\tilde{z}_1 = 1/2$ strategy 2 yields a higher payoff than strategy 1, and because $1-\tilde{z}_1 < 1-(1-\tilde{z}_1)^2$ holds, we have $W(\tilde{z}_1) > V(\tilde{z}_1)$. That is, using sampling procedure with sample size $k = 2$ instead of sampling procedure with sample size $k = 1$ yields a higher utility.

Conversely, suppose all players follow the sampling procedure with sample size $k = 2$. The best response dynamics is given by

$$\dot{z} = P_2(z) - z,$$

with the unique rest point $\tilde{z}_2$ being equal to the unique $z \in (0,1)$ solving $(1-z)^2 - z = 0$. The average utility in the population at $\tilde{z}_2$ is given by

$$W(\tilde{z}_2) = \tilde{z}_2 U_1(\tilde{z}_2) + (1-\tilde{z}_2) U_2(\tilde{z}_2).$$

At $\tilde{z}_2$, the average utility for players employing the procedure with sample size $k = 1$, who play strategy 1 if and only if they see a strategy 2 player in the sample, is given by

$$V(\tilde{z}_2) = (1-\tilde{z}_2) U_1(\tilde{z}_2) + \tilde{z}_2 U_2(\tilde{z}_2).$$

Because at $\tilde{z}_2 < z^*$ strategy 1 yields a higher payoff than strategy 2, and because $1-\tilde{z}_2 > \tilde{z}_2$ holds, we have $V(\tilde{z}_2) > W(\tilde{z}_2)$. That is, using sampling procedure with sample size $k = 1$ instead of sampling procedure with sample size $k = 2$ yields a higher utility.

The right panel in Figure 1 depicts this situation graphically: At the rest point $\tilde{z}_1$ induced by players choosing strategy 1 with probability $P_1(z)$, the sampling procedure with agents choosing strategy 1 with $P_2(z)$ is closer to the best reply correspondence (red), and thus generates higher utility. Conversely, at the rest point $\tilde{z}_2$ induced by players choosing strategy 1 with probability $P_2(z)$, the sampling procedure with agents choosing strategy 1 with $P_1(z)$ is closer to the best reply correspondence, and thus generates higher utility.

Both examples describe a situation in which unbiased sampling yields a rest point at which a suitably chosen different sampling procedure yields a higher utility to the players using that alternative procedure. These findings suggest that, if we take utility as a measure for evolutionary fitness, then there is scope for an analysis of evolutionary competition between sampling procedures that differ in sample size and sampling bias.

1.2 Preview of the Model

We propose to model evolutionary competition between sampling procedures by adapting the classical replicator dynamics (Taylor and Jonker, 1978). To this end, we define a family of sampling procedures that revising players might use, and divide the population in two subpopulations with all players in a subpopulation using an identical procedure, but with the procedures differing between subpopulations. As a consequence, the distributions of strategies chosen by revising players differ across subpopulations, and because utility depends on the strategy distribution in the whole population, a wedge is driven between average utilities in the subpopulations. Equating utility with fitness, we then assume that the share of the subpopulation with higher than population average fitness level grows, whereas the share of the subpopulation with lower than population average fitness level shrinks.

Combining the sampling best response dynamics with this adapted replicator dynamics yields a dynamical system that consists of the sampling best response dynamics determining the strategy distributions within subpopulations, and of the replicator dynamics determining the share of the respective subpopulations. The key to our analysis lies in understanding this dynamical system as a slow-fast system. Slow-fast systems are characterized by two different time-scales. The variables subject to the faster time-scale are taken together in what is called the fast node, and the others in the slow node, respectively (cf. e.g. Section 39 in Wasow, 1965). In the context of this paper, we understand the best response dynamics as the fast, and replicator dynamics as the slow node.

The speed of the sampling best response dynamics is determined by the arrival rate of revision opportunities: a higher arrival rate corresponds to a faster sampling best response dynamics. For our analysis, we resort to a well-known result on slow-fast systems known as Tikhonov's theorem and an extension thereof (Theorems 1 and 2 in Lobry et al., 1998). These results characterize the convergence, when the arrival rate grows large, of the solution of the slow-fast system to the solution of a tractable reduced system that describes the dynamics of the subpopulation shares when an infinitely fast sampling best response dynamics is assumed.

At the center of our stability analysis are sampling-monomorphic rest points, by which we mean rest points in which all revising players in the population use the same sampling procedure. At such rest points, we consider intrusion of a small share of mutants with a procedure that is different – in a sense to be made precise – from the incumbent procedure. We call a procedure stable within a family of procedures, if the sampling-monomorphic rest point is practically asymptotically stable (Byrnes and Isidori, 2002; Boudjellaba and Sari, 2009) for any mutant procedure in that family. Practical asymptotic stability of a rest point requires that for all sufficiently high arrival rates of revision opportunities, the solution trajectory remains close to the rest point, and converges in the long run back to the rest point when the arrival rate approaches infinity. Practical asymptotic stability of a rest point is weaker than asymptotic stability for any positive arrival rate of revision opportunities, but stronger than merely requiring asymptotic stability in the limit of infinite arrival rates.

1.3 Preview of the Results

We consider families of sampling procedures with a finite sample size and with sampling probabilities for strategy 1 players given by increasing differentiable functions of the current population state. We say that a sampling procedure supports a population state if it yields that population state in the sampling-monomorphic rest point, and show that a sampling procedure is stable within the family considered if and only if it supports the unique mixed Nash-equilibrium of the game (Proposition 1). This implies that the only population state that can obtain under stable sampling procedures is the mixed Nash equilibrium of the game.

We then analyze the nature of stable sampling procedures, with a particular focus on the bias in such procedures. A sampling procedure is called locally unbiased at some population state if the probability of sampling strategy 1 players at that population state corresponds to the actual share of strategy 1 players, a sampling procedure is called locally biased if this does not hold, and the difference between the probability of sampling strategy 1 players and the actual share of strategy 1 players is called the sampling bias at that population state. Our first result on stable sampling (Proposition 2) establishes that for all but a countable set of population states, there is no unbiased sampling procedure supporting that population state, if we assume that the population state is also the equilibrium of the underlying game. Because for a sampling procedure to be stable it must support the equilibrium of the underlying game, we thus say that stable sampling is generically biased.

This result suggests that, in combination with the limited information that sampling yields for the players, simply playing best response to the strategy frequencies observed in the sample induces evolutionary suboptimal decisions. Thus, the sampling bias can be interpreted as an evolutionary second-best solution (Waldman, 1994) for players acting on their sample in such a simplistic way. Evolutionary second-best solutions describe behavioral distortions that are not evolutionary optimal by themselves but that re-align an agent's behavior with fitness maximization in the presence of some other distortion that drives a wedge between the optimal behavior from the player's point of view and the fitness maximizing behavior from nature's point of view, and that is – for whatever reasons – too costly for nature to amend directly. See Section 1.4 for a discussion of the related literature. We support this interpretation with a second result which shows that the stable sampling bias vanishes as the sample size grows large (Proposition 3).

We further look at how the size of the stable bias relates to the asymmetry of the Nash equilibrium. We call a mixed Nash equilibrium symmetric if the two strategies are played with equal probability, and call it the more asymmetric the higher the absolute difference between the two probabilities. For any fixed sample size, the stable bias is not monotone in the asymmetry of the equilibrium. Rather, with the equilibrium population state approaching either zero or one, the absolute difference between the actual share of a strategy in the population and its sampling probability approaches one in any stable sampling procedure with a finite sample size (Proposition 4). Hence, for sufficiently asymmetric equilibria the evolutionary need for correction through biased sampling only vanishes slowly in growing sample sizes k. We thus conclude the stable sampling bias does not converge uniformly to zero in the asymmetry of the equilibrium when the sample size k approaches infinity (Corollary 1).

Last, we extend the sampling procedure as in Oyama et al. (2015) and consider a model of sampling with agents using random sample sizes. We show that allowing for random sample sizes has the effect that the set of population states that can be supported by unbiased sampling procedures, given they are the equilibrium of the underlying game, is not of measure zero anymore: Any sufficiently symmetric equilibrium can be supported by an unbiased sampling procedure that uses random sample sizes. Nevertheless, for sufficiently asymmetric equilibria correction through biased sampling is still required for stability (Proposition 5), so biased sampling can still occur as an evolutionary second best solution.

1.4 Related Literature

This paper relates to three different strands of literature: First, and most obviously, it relates to models of best response and other learning dynamics both with and without random sampling. Second, it relates to evolutionary models of preferences that also, but only implicitly, build on two-speed dynamics. Third, there is a literature on evolutionary second-best approaches to which this paper contributes. We discuss the strands in turn.

Learning dynamics

Besides in the models of Sandholm (2001a) and Oyama et al. (2015), random strategy sampling with finite samples – albeit always in unbiased form, too – appears in several different learning dynamics, such as in models of herding (Ellison and Fudenberg, 1993, 1995; Banerjee and Fudenberg, 2004) or in the adaptive play model of Young (1993). Neither of these papers asks about the evolutionary stability of random sampling as we do in this paper, but rather they focus on the stability properties of certain rest points of the dynamics induced.

Sandholm (2001a) finds that, in population games with a finite strategy set, any $1/k$-dominant equilibrium is globally stable under the unbiased sampling best response dynamics, given players draw a sample of size $k \geq 2$. Oyama et al. (2015) extend the analysis and the stability result to unbiased sampling procedures that involve randomizing over different sample sizes, and to a set-notion of $1/k$-dominance. These global stability results are the first obtained under deterministic population dynamics, as opposed to selection results between pure strategy equilibria under probabilistic population dynamics in which players are prone to making small random mistakes as e.g. in Young (1993). In contrast to these papers, we consider games with a unique equilibrium that is in mixed strategies.

The sampling best response dynamics is a perturbed version of the standard best response dynamics (Gilboa and Matsui, 1991) in which the players learn the true population state when receiving the opportunity for belief revision. The perturbation of the best response dynamics induced by sampling differs from the kind of perturbation analyzed in Hopkins (1999), Hofbauer and Hopkins (2005) or Hofbauer and Sandholm (2007) who assume idiosyncratic shocks to players' preferences at the time of revision.

Models of indirect preference evolution

Two-speed dynamics have been employed in models of preference evolution, albeit only implicitly. For example, Sandholm (2001b), Dekel et al. (2007), or Alger and Weibull (2013) deal with the limit case of play adapting infinitely fast to changes in the distribution of preferences. They do so by assuming that players are aware of the changing nature of preferences, and always play equilibrium – if preferences are observed, given the commonly known preferences in a match, or, if preferences are not observed, given the current distribution of preferences. As an exception, Sandholm (2001b, Appendix A) discusses how to relax in his setup the assumption of infinitely fast adaptation of play at states where equilibrium play is discontinuous in the distribution of the preferences. In contrast to Sandholm (2001b) who remains unspecific about the adaptive process leading back to equilibrium play, we are very specific about the dynamics leading to equilibrium play.

Evolutionary second-best approaches

Examples of evolutionary second-best explanations, that is, for constraints that drive a wedge between the optimal behavior from the player's point of view and the fitness maximizing behavior from nature's point of view and corresponding second-best solutions that re-align an agent's behavior with fitness maximization (cf. Waldman, 1994), include incomplete information about the environment and relative consumption effects in preferences (Samuelson, 2004; Samuelson and Nöldeke, 2005), incomplete information about the environment and menu dependent preferences (Samuelson and Swinkels, 2006), incomplete information about the environment and an S-shaped value function (Netzer, 2009), a failure to ascribe private information to other agents and an endowment effect in preferences (Frenkel et al., 2015), or S-shaped value functions and non-linear probability weighting (Herold and Netzer, 2013). This paper contributes to this literature in examining the interplay between sampling biases and the simple decision rule to play best response to the frequency of strategy occurrences in a finite sample.

2 The Model

The model consists of an underlying population game, and of the dynamics that we obtain by infinitely repeating the game in continuous time. We first present the underlying game, and then describe the dynamics.

2.1 The Underlying Game

A population game with two strategies labeled 1 and 2 is played by players of a unit mass population. The population state, i.e. the share of players choosing strategy 1, is denoted by $z \in [0,1]$, and the payoff to a player choosing strategy $i \in 1,2$ is given by $U_i(z)$. We make three assumptions about the payoff difference $h(z) = U_1(z) - U_2(z)$ that are essential for our results.

(U1) $h(z)$ is Lipschitz-continuous on $[0,1]$.

(U2) $h(z)$ is non-increasing.

(U3) There is unique $z^* \in (0,1)$ such that $h(z^*) = 0$.

(U1) is a technical assumption required for uniqueness of solution to the dynamical system set up in the following. For all our results, we additionally rely on (U2) and (U3). (U2) and (U3) together imply that $h(z)$ is strictly decreasing at z^* and that the set of states z to which strategy 1 is a (weak) best response is a proper, closed interval given by $[0,z^*]$. So, neither strategy is dominated, and we have $h(0), h(1) \neq 0$.

The above formulation of a population game encompasses for example random pairwise matching in a Hawk-Dove game as described in Examples 1 and 2, but also more general playing-the-field games in which choosing strategy 1 has a negative externality on the payoff of other players' payoff when choosing strategy 1. In any population game considered, the unique Nash equilibrium is in mixed strategies and given by z^*.

2.2 The Dynamics

Time is continuous and the game described above is repeated infinitely. We do not refer to time explicitly, but all variables of the dynamics set up below are understood to implicitly depend on time. The dynamics consist of the sampling best response dynamics and of the replicator dynamics. We treat the two in turn.

2.2.1 Sampling Best Response Dynamics

At every point in time, each player holds an individual belief $\hat{z} \in [0,1]$ about the current population state z, and myopically plays best response to $\hat{z}$. Players employ a deterministic tie-breaking rule in case of indifference and play strategy 1 whenever $\hat{z} \in [0, z^*]$.[1] Players take the population state for stationary – that is, there is no updating of $\hat{z}$ based on the history of play – but from time to time, players receive individual opportunities for belief revision. Such individual revision opportunities occur according to Poisson processes that are independent across players, and arrive with rate $\lambda > 0$. A revising player forms a new belief $\hat{z}$ about the current population state by randomly sampling players in the population, and then plays best response to $\hat{z}$ until the next opportunity to revise arises.

A sampling procedure is described by the tuple $\{k, p\}$ where $k \in \mathbb{N}_+$ denotes the sample size and $p : [0,1] \to [0,1]$ returns the sampling probability of strategy 1 for every population state $z \in [0,1]$. Throughout the following, we assume that $p(z)$ is increasing and continuously differentiable[2] on $[0,1]$.

Definition 1. A sampling procedure $\{k, p\}$ is called

- locally unbiased at $z \in [0,1]$ if $p(z) = z$,
- locally biased at $z \in [0,1]$ if $p(z) \neq z$.

Definition 2. $d(z) = p(z) - z$ is the sampling bias at z.

We consider sampling procedures with finite sample sizes, and collect all sampling procedures $\{k, p\}$ for which it holds that $k \leq b \in \mathbb{N}_+$ in the family $\mathscr{S}_b$. Let $m \leq k$ be the number of strategy 1 players observed in the sample. As in Sandholm (2001a), we assume that agents take the frequency of strategy 1 observations as the population state, i.e. we have $\hat{z} = m/k$. Under $\{k, p\}$, the distribution of m is binomial with the parameters $(k, p(z))$ depending on the population state z. The probability that a revising agent plays strategy 1 after sampling is given by

$$P(z|z^*, \{k, p\}) = \sum_{i=0}^{\lfloor k \cdot z^* \rfloor} \binom{k}{i} p(z)^i (1 - p(z))^{k-i}. \tag{3}$$

[1] The particular tie-breaking rule in case of indifference is not important for any of our subsequent results. All results go through for any (probabilistic) tie-breaking rule.

[2] We say that a function $g : [0,1]^m \to \mathbb{R}^m$, $m \in \mathbb{N}_+$, is continuously differentiable on $[0,1]^m$ if g is continuously differentiable at every interior point of $[0,1]^m$, and, additionally, the relevant one-sided partial derivatives of g exist and are continuous at all boundary points of $[0,1]^m$.

Observe that although the equilibrium z^* is held fixed at the moment, we refer to it explicitly in $P(z|z^*, \{k,p\})$. This will make the interpretation of the coming results more transparent. For later use, it will prove helpful to write (3) as

$$P(z|z^*, \{k,p\}) = I_{1-p(z)}(k - \lfloor kz^* \rfloor, \lfloor kz^* \rfloor + 1), \tag{4}$$

where $I_x(\alpha, \beta)$ denotes the cumulative distribution function of the beta distribution with coefficients α, β. The right-hand side in (4) follows from the fact that the value $F_X(x)$ of the cumulative distribution function F_X of a binomial variable $X \sim B(n,p)$ evaluated at x is equal to the value $F_Y(1-p)$ of the distribution function F_Y of a beta-distributed variable $Y \sim \text{Beta}(n-x, x+1)$ evaluated at $1-p$ (cf. Olver et al., 2010). From the right-hand side of (4), we see that $P(z|z^*, \{k,p\})$ is decreasing in z for any z^* and sampling procedure $\{k,p\}$.

The population is divided in two subpopulations labeled by 1 and 2. All players in subpopulation $q = 1,2$ use sampling procedure $\{k,p\}_q \in \mathcal{S}_b$. The mass of subpopulation 1 is denoted by $y \in [0,1]$. We call the population sampling-monomorphic if $y \in \{0,1\}$: The value $y = 0$ corresponds to the case in which all players in the population use sampling procedure $\{k,p\}_2 \in \mathcal{S}_b$, while $y = 1$ corresponds to the case in which all players in the population use sampling procedure $\{k,p\}_1 \in \mathcal{S}_b$. Strategy shares within subpopulations are denoted by $x_q \in [0,1]$, such that the current population state z is given by $z = yx_1 + (1-y)x_2$. We sometimes write the population state z as a function $z(x,y)$, with $x = (x_1, x_2) \in [0,1]^2$, in order to emphasize its dependence on (x,y).

We follow the literature (Hopkins, 1999; Hofbauer and Sandholm, 2007) in assuming (i) that the fraction of players revising their belief in a short time period of length $d\tau > 0$ is given by $\lambda d\tau$, and (ii) that the fraction of revising players in subpopulation q choosing strategy 1 at population state z is $P(z(t)|z^*, \{k,p\}_q)$. This allows us to express the share of players using strategy 1 in subpopulation q at time $t + d\tau$, $d\tau > 0$, as $x_q(t + d\tau) = (1 - \lambda d\tau)x_q(t) + \lambda d\tau P(z(t)|z^*, \{k,p\}_q)$. Taking the limit $d\tau \to 0$, the dynamics of strategy share $x_q \in [0,1]$ in subpopulations $q = 1,2$ is given by

$$\dot{x}_q = \lambda \cdot \left(P(z|z^*, \{k,p\}_q) - x_q\right),$$

where $\dot{x}_q$ indicates the derivative with respect to time t. Defining

$$R = (P(.|z^*, \{k,p\}_1), P(.|z^*, \{k,p\}_2)),$$

we rewrite the sampling best response dynamics for $(x,y) \in [0,1]^3$ compactly as

$$\dot{x} = \lambda \cdot [R(z(x,y)) - x]. \tag{5}$$

We refer to system (5) as the sampling best response node.

2.2.2 Evolution

Turning to the evolution of subpopulation share $y \in [0,1]$, we adapt the idea of the replicator dynamics on strategies (Taylor and Jonker, 1978). Let average subpopulation utility in subpopulation $q = 1,2$ be given by

$$\bar{U}_q(x,y) \equiv x_q U_1(z(x,y)) + (1 - x_q)U_2(z(x,y)).$$

For $(x,y) \in [0,1]^3$, the evolution of y is given by

$$\dot{y} = y(1-y)\left(\bar{U}_1(x,y) - \bar{U}_2(x,y)\right),$$

which we rewrite as

$$\dot{y} = y\,(1-y)\,(x_1 - x_2)\,h\,(z(x,y))\,. \tag{6}$$

We call (6) the evolution node. Equation (6) has a natural interpretation: As long as we are not in a sampling-monomorphic state with $y \in \{0,1\}$, the share y of sampling procedure $\{k,p\}_1$ grows if either strategy 1 yields a higher utility than strategy 2 and the subpopulation using $\{k,p\}_1$ has currently a higher fraction of players choosing strategy 1, or if strategy 2 yields a higher utility than strategy 1 and the subpopulation using $\{k,p\}_1$ has currently a higher fraction of players choosing strategy 2.

2.2.3 The final system of interest

Taking together the sampling best response node (5) and the evolution node (6), we arrive at the following final system of interest for $(x,y) \in [0,1]^3$.

$$\begin{aligned} \lambda^{-1} \cdot \dot{x} &= R\,(z(x,y)) - x \\ \dot{y} &= y\,(1-y)\,(x_1 - x_2)\,h\,(z(x,y)) \end{aligned} \tag{7}$$

Observe that the right-hand side of (7) is Lipschitz-continuous on $[0,1]^3$: The sampling best response node can be written as the difference $R\,(z(x,y)) - f\,(x,y)$ where f is clearly Lipschitz-continuous and $R \circ z$ is Lipschitz-continuous by continuous differentiability of R on $[0,1]^3$ and by linearity of z. The evolution node can be written as product $g(x,y)h(z(x,y))$ where g is clearly Lipschitz-continuous, and $h \circ z$ is Lipschitz-continuous because both h and z are Lipschitz-continuous. Because system (7) never leaves $[0,1]^3$ the Picard-Lindelöf theorem applies, and we get that, for any initial condition $(x_0, y_0) \in [0,1]^3$, a unique global solution $(x(t), y(t)) \in [0,1]^3$, $t \geq 0$ to (7) exists.

3 Stability

This section describes our stability criterion for sampling procedure $\{k,p\} \in \mathscr{S}_b$, and gives a necessary and sufficient condition for sampling procedure $\{k,p\} \in \mathscr{S}_b$ to be stable.

3.1 $\mathscr{S}_b$-Stable Sampling

Let $(\tilde{x}, \tilde{y}) \in [0,1]^3$ be a rest point of (7). We start with the definition of practical asymptotic stability (cf. Boudjellaba and Sari, 2009) that we employ in the following. Let $\|.\|$ denote the standard Euclidean norm.

Definition 3. [Practically Asymptotic Stability] A rest point $(\tilde{x}, \tilde{y})$ of (7) is *practically asymptotically stable* if conditions (a) and (b) below hold.

(a) For every $\theta > 0$ there exists $\delta(\theta) > 0$ and $\bar{\lambda} > 0$ such that

$$\|(x_0, y_0) - (\tilde{x}, \tilde{y})\| < \delta(\theta) \;\Rightarrow\; \|(x(t), y(t)) - (\tilde{x}, \tilde{y})\| < \theta, \quad \forall t \geq 0,\; \forall \lambda > \bar{\lambda}.$$

(b) There exists $\delta > 0$ such that if $\|(x_0, y_0) - (\tilde{x}, \tilde{y})\| < \delta$, then

$$\lim_{\lambda \to \infty} \lim_{t \to \infty} \|(x(t), y(t)) - (\tilde{x}, \tilde{y})\| = 0.$$

Practical asymptotic stability[3] of the rest point $(\tilde{x},\tilde{y})$ is weaker than asymptotic stability of rest point $(\tilde{x},\tilde{y})$ for arbitrary $\lambda > 0$ (cf. Definition 8 in Appendix A): Whereas asymptotic stability for arbitrary $\lambda > 0$ requires solution trajectories to stay close to $(\tilde{x},\tilde{y})$ and ultimately converge back to $(\tilde{x},\tilde{y})$ for any $\lambda > 0$, practical asymptotic stability requires trajectories starting close to $(\tilde{x},\tilde{y})$ to stay close for sufficiently high λ only, and merely in the limit of $\lambda \to \infty$ requires trajectories to converge back to $(\tilde{x},\tilde{y})$ in the long run. Asymptotic stability of the rest point $(\tilde{x},\tilde{y})$ for arbitrary $\lambda > 0$ cannot easily be checked without assuming differentiability of the right-hand side in (7), and even if we did assume differentiability, we would have to deal with non-hyperbolic rest points when $\tilde{y} \in \{0,1\}$. Practical stability of the rest point $(\tilde{x},\tilde{y})$, on the other hand, can be checked using the results presented below.

In the following we restrict attention to sampling-monomorphic rest points with $\tilde{y} = 1$. The population state in such a sampling-monomorphic rest point is equal to $\tilde{x}_1$. By construction of the sampling best response node, $\tilde{x}_1$ only depends on $\{k,p\}_1$, and because $P(z|z^*,\{k,p\})$ is decreasing in z for any sampling procedure $\{k,p\}$ as can be inferred from (4), $\tilde{x}_1$ is unique. We say that $\{k,p\}_1 \in \mathscr{S}_b$ supports the population state $z = \tilde{x}_1$ given equilibrium z^*:

Definition 4. Sampling procedure $\{k,p\}$ supports $z \in [0,1]$ given equilibrium $z^* \in (0,1)$ if $z = P(z|z^*,\{k,p\})$ holds.

If the context is clear, we do not explicitly refer to the equilibrium z^*, but merely say that $\{k,p\}$ supports z. Sampling procedure $\{k,p\}_2 \in \mathscr{S}_b$ is called the mutant procedure. Throughout the following, we assume that $\{k,p\}_1$ supporting z is z-different from $\{k,p\}_2$.

Definition 5 (z-difference)**.** $\{k,p\}_1$ is z-different from $\{k,p\}_2$ if

$$P(z|z^*,\{k,p\}_1) \neq P(z|z^*,\{k,p\}_2)$$

holds.

Two sampling procedures are denoted z-different if they do not induce the same distribution over strategies chosen by the revising players at population state $z \in [0,1]$. Because $P(z|.,.)$ is differentiable in z, this is equivalent to requiring that $\exists \omega > 0$ such that $|P(z'|z^*,\{k,p\}_1) - P(z'|z^*,\{k,p\}_2)| \neq 0$, $\forall z' \in (z-\omega, z+\omega) \cap [0,1]$. The reason for the restriction to mutant sampling procedures that are z-different is that evolutionary pressure between sampling procedures at some population state z is only effective when the sampling procedures produce differing distributions over the strategies chosen by revising players. We briefly return to this issue at the end of this section, in subsection 3.3 below.

Observe that for any two sampling procedures $\{k,p\}_1, \{k,p\}_2 \in \mathscr{S}_b$ where $\{k,p\}_1$ supports $z \in [0,1]$, the sampling-monomorphic rest point $(\tilde{x},1)$ of (7) has $\tilde{x} = (z, P(z|z^*,\{k,p\}_2))$. With this, we can state the definition of an $\mathscr{S}_b$-stable sampling rule.

Definition 6 ($\mathscr{S}_b$-stable Sampling)**.** Sampling procedure $\{k,p\}_1 \in \mathscr{S}_b$ supporting $z \in [0,1]$ is $\mathscr{S}_b$-stable, if, keeping $\{k,p\}_1$ fixed, for any $\{k,p\}_2 \in \mathscr{S}_b$ being z-different from $\{k,p\}_1$, the rest point $(\tilde{x},\tilde{y}) = ((z, P(z|z^*,\{k,p\}_2)),1)$ of (7) is practically asymptotically stable.

Informally, this condition states that a sampling procedure $\{k,p\} \in \mathscr{S}_b$ supporting z is $\mathscr{S}_b$-stable if it is resistant against intrusion of any mutant procedure in $\mathscr{S}_b$ that is z-different.

[3]The original formulation is: *There is $\delta > 0$ such that for all $\theta > 0$ there is $\bar{\lambda} > 0$ and $\bar{t} > 0$ such that for all $\lambda > \bar{\lambda}$, any solution $(x(t),y(t))$ to* (7) *with $\|(x_0,y_0)-(\tilde{x},\tilde{y})\| < \delta$ satisfies $\|(x(t),y(t))-(\tilde{x},\tilde{y})\| < \theta$ for all $t > \bar{t}$.* It is standard to verify that this is analogous to Definition 3.

3.2 Analysis

For any two sampling procedures $\{k,p\}_1, \{k,p\}_2 \in \mathscr{S}_b$, let $x^* : [0,1] \to [0,1]^2$ be an implicit function satisfying $x^*(y) = R(z(x^*(y), y))$, $\forall y \in [0,1]$, which by the next lemma always exists, and is both unique and differentiable.

Lemma 1. *Fix $\{k,p\}_1, \{k,p\}_2 \in \mathscr{S}_b$. Then, $x^*(y)$ exists and is both unique and differentiable on $[0,1]$.*

The proof makes use of the Jacobi $C(x,y) = \partial_x[R(z(x,y) - x]$ of the sampling best response node (5) for $y \in [0,1]$ held fixed that is given by

$$C(x,y) = \begin{pmatrix} P'(z(x,y)|z^*, \{k,p\}_1)\, y - 1 & P'(z(x,y)|z^*, \{k,p\}_1)(1-y) \\ P'(z(x,y)|z^*, \{k,p\}_2)\, y & P'(z(x,y)|z^*, \{k,p\}_2)(1-y) - 1 \end{pmatrix}, \tag{8}$$

where $P'(z|z^*, \{k,p\})$ denotes the derivative of P with respect to z. We can establish the following property of the Jacobi $C(x,y)$ that is instrumental in the proof of the subsequent lemma:

Lemma 2. *There is $c > 0$ such that the eigenvalues e_1 and e_2 of $C(x^*(y), y)$ have real parts $Re(e_1)$ and $Re(e_2)$ that satisfy $Re(e_1), Re(e_2) < -c$, $\forall y \in [0,1]$.*

The above result implies that $x^*(y)$ is a uniformly asymptotically stable rest point of the sampling best response node (5) for any $y \in [0,1]$. For the following result, let $z^*(y) = y x_1^*(y) + (1-y) x_2^*(y)$. Using Lemma 1 and Lemma 2 we can apply Theorem 1 and Theorem 2 in Lobry et al. (1998) describing the behavior of the solutions to (7) when the arrival rate λ of revision opportunities becomes high. This, together with Definition 3, yields Lemma 3 below.

Lemma 3. *Sampling procedure $\{k,p\}_1$ supporting $z \in [0,1]$ is $\mathscr{S}_b$-stable iff $\bar{y} = 1$ is an asymptotically stable rest point of reduced system*

$$\dot{\bar{y}} = \bar{y}(1-\bar{y})(x_1^*(\bar{y}) - x_2^*(\bar{y}))h(z^*(\bar{y})), \quad \bar{y} \in [0,1] \tag{9}$$

for any procedure $\{k,p\}_2 \in \mathscr{S}_b$ that is z-different from $\{k,p\}_1$.

The reduced system (9) describes the evolution of the subpopulation share y given that the best response process adapts infinitely fast to changes in y. A rest point $\hat{y} \in [0,1]$ of the reduced system (9) is asymptotically stable iff there exists a neighborhood around $\hat{y}$ such that

$$(\bar{y} - \hat{y})(x_1^*(\bar{y}) - x_2^*(\bar{y}))h(z^*(\bar{y})) < 0 \tag{10}$$

holds for all $\bar{y} \neq \hat{y}$ in that neighborhood. By Lipschitz-continuity of the right-hand side of (9) and because the system remains in $[0,1]$, there exists, for any initial condition $\bar{y}_0 \in [0,1]$, a unique global solution $\bar{y}(t) \in [0,1]$ for $t \geq 0$ satisfying (9). Lemma 1 is necessary for Lemma 3 because it guarantees that the reduced system (9) is well-defined, and Lemma 2 is necessary because it guarantees that $x(t)$ converges, when λ approaches infinity, to $x^*(\bar{y}(t))$ pointwise and that $y(t)$ converges to $\bar{y}(t)$ uniformly on any interval on which $\bar{y}(t)$ is defined. Using Lemma 3 together with assumptions (U1)–(U3), we arrive at the main result of this section, stated in Proposition 1.

Proposition 1. *Sampling procedure $\{k,p\} \in \mathscr{S}_b$ is $\mathscr{S}_b$-stable iff*

$$z^* = P(z^*|z^*, \{k,p\}). \tag{11}$$

holds. Such a sampling procedure always exists.

Stability condition (11) requires that procedure $\{k,p\} \in \mathscr{S}_b$ supports the equilibrium z^*, i.e. the fraction of revising agents choosing strategy 1 at the equilibrium z^* must be equal to z^*. The result is intuitive: If the incumbent sampling procedure does not support the equilibrium z^*, then there will be a mutant procedure yielding higher utility at the supported population state, as suggested in Examples 1–2 in the introduction, thus rendering the incumbent procedure unstable. If, on the other hand, the incumbent procedure supports the equilibrium z^* then any sufficiently small fraction of intruding mutant procedures producing a share of revising players choosing strategy 1 that is strictly higher than z^* instantaneously – considering the limit $\lambda = \infty$ – drives the population state strictly above z^*. If so, by continuity of $P(z|z^*,\{k,p\})$ in z, the players in the incumbent subpopulation enjoy a higher average utility than the mutants, and thus the mutants are driven out again. By an analogous argument, any mutant procedure yielding a share of revising players choosing strategy 1 that is strictly below z^* is crowded out again after intrusion. Crucially, this intuition is not upset by considering any sufficiently high arrival rate λ.

3.3 A Remark on z-difference

The reason why $\mathscr{S}_b$-stability is formulated only in terms of z-different sampling procedures is that evolutionary pressure is absent among sampling procedures that induce identical strategy shares at the population state supported by $\{k,p\}_1 \in \mathscr{S}_b$. The following result makes this claim more precise:

Lemma 4. *Assume (U1). Suppose that $\{k,p\}_1 \in \mathscr{S}_b$ supports $z \in [0,1]$ and that $\{k,p\}_2 \in \mathscr{S}_b$ satisfies*

$$P(z|z^*,\{k,p\}_2) = P(z|z^*,\{k,p\}_1).$$

Then, for any $\eta, T > 0$, there exists $\bar{\lambda}, \omega > 0$ such that, $\forall \lambda > \bar{\lambda}$, we have

$$\|y(t) - y_0\| \leq \eta, \forall t \in [0,T], \tag{12}$$

given that $\|x_0 - x^*(y_0)\| < \omega$ holds.

Lemma 4 implies that if (x_0, y_0) are chosen sufficiently close to $(\tilde{x}, 1)$ then any mutant procedure $\{k,p\}_2$ that is not z-different from incumbent procedure $\{k,p\}_1$ supporting z is not crowded out again in the long run: For any finite time $T > 0$ passed after intrusion, the share of mutants $1 - y(t)$ persisting in the population approaches its initial share $1 - y_0$ uniformly over $[0,T]$ as we let the rate λ of revision opportunities approach infinity. Quite intuitively, if a mutant procedure does not induce differing behavior, then evolutionary pressure is absent.

4 Properties of Stable Random Sampling

4.1 Generic Biasedness of Sampling Rules

In this section we show that $\mathscr{S}_b$-stable sampling procedures are generically biased at the respective sampling-monomorphic population state, by which we mean that, for any $b < \infty$, the set of $z \in [0,1]$ such that there is a sampling procedure $\{k,p\} \in \mathscr{S}_b$ that is both unbiased at z and supports z given z is the equilibrium of the underlying game is of measure zero.

To this end, we define $p_k^*(z)$ as the sampling probability that, for given $z \in (0,1)$ and $k \in \mathbb{N}_+$, uniquely solves

$$z = I_{1-p_k^*(z)}(k - \lfloor kz \rfloor, \lfloor kz \rfloor + 1). \tag{13}$$

A sampling procedure $\{k, p\}$ satisfying $p(z) = p_k^*(z)$ supports z given z is the equilibrium of the underlying game. That is, for a given equilibrium $z^* \in (0,1)$, $p_k^*(z^*)$ is the sampling probability that renders a sampling procedure $\{k, p\}$ satisfying $p(z^*) = p_k^*(z^*)$ $\mathscr{S}_b$-stable. Let

$$K_b(z) \equiv \{k \leq b : p_k^*(z) = z\} \tag{14}$$

be the set of sample sizes $k \in \{1, ..., b\}$ to which there correspond sampling procedures that both are unbiased at z and support z given z is the equilibrium of the underlying game. That is, whenever z is an equilibrium of the underlying game and $K_b(z)$ is non-empty, then there exists a sampling procedure $\{k, p\} \in \mathscr{S}_b$ that is unbiased at z and $\mathscr{S}_b$-stable. The proof of the next proposition shows that the set of such z is countable, and we get:

Proposition 2. *For any $b \in \mathbb{N}_+$, the set*

$$\{z \in (0,1) : K_b(z) \neq \emptyset\}$$

has measure zero.

To get an intuition for how the result comes about, we need to take a closer look at the function

$$P(z|z, \{k, \tilde{p}\}) = I_{1-z}(k - \lfloor kz \rfloor, \lfloor kz \rfloor + 1), \tag{15}$$

returning for a given sampling size k and the everywhere unbiased sampling probability $\tilde{p}(z) = z$ the fraction of revising agents choosing strategy 1 given the population state z is also the equilibrium of the underlying game. For some $z \in (0,1)$ to lie in the set $\{z \in (0,1) : K_b(z) \neq \emptyset\}$, there must exists some sample size k such that z is a fixed point of $P(z|z, \{k, \tilde{p}\})$. That such can hold only for a countable set of population states z is established by showing that for every $k \leq b$, the set of fixed points of $P(z|z, \{k, \tilde{p}\})$ is countable, which is straightforward once we appreciate that, for any $k \leq b$, the map $P(z|z, \{k, \tilde{p}\})$ is strictly decreasing whenever it is continuous and that it has but finitely many points of discontinuity.

4.2 Biased Sampling as Evolutionary Second-Best

Proposition 2 suggests that in the absence of sampling biases, sampling $k \in \mathbb{N}_+$ players and then playing best response to the frequency at which the strategies occur in the sample is hardly ever optimal from an evolutionary perspective. Hence, keeping fixed some generic equilibrium $z^* \in (0,1)$, any sample size $k \in \mathbb{N}_+$, and the simple decision rule for our players to naively play best response to the sample, nature will push sampling towards a bias that compensates for the suboptimality of the choices induced. In this sense, sampling biases are an evolutionary-second best distortion given that agents follow too simple a decision rule.

We can strengthen this evolutionary second-best claim by showing that the distortion in the sampling probability required for stability eventually decreases when we increase the sample size. In other words, if we allow agents to acquire more and more information by using bigger samples, then the suboptimality of playing best response to the average strategy occurrences in the sample becomes less severe. Recall that $p_k^*(z)$ implicitly defined in (13) returns the sampling probability at $z \in (0,1)$ such that the sampling procedure $\{k, p\}$ satisfying $p(z) = p_k^*(z)$ is $\mathscr{S}_b$ stable, given z is the equilibrium of the underlying game.

Proposition 3. *For any $z \in (0,1)$, it holds $\lim_{k\to\infty} p_k^*(z) = z$.*

Proposition 3 implies that, for any equilibrium z^*, the bias at z^* of any $\mathscr{S}_b$-stable procedure vanishes when we let the sample size approach infinity. The intuition for this result is that, when the sample size k grows large and the population state is kept fixed at the equilibrium z^*, then already a small sampling bias has drastic effects on the choice of the revising agents in the sense that it drives the share of agents choosing the first option to either extreme of zero or one. Stability requires that this does not happen, and hence the bias at the equilibrium must go to zero as k approaches infinity.

Although the stable bias at the equilibrium goes to zero in the sample size k for any equilibrium z^*, the speed of this convergence depends on the particular locus of z^* in the interval $(0,1)$. To see this, we proceed by further analyzing the relation between the equilibrium z^* and the sampling bias $d(z^*) = p(z^*) - z^*$ required such that sampling is stable for given k. We start with an example examining the sampling probability that renders a sampling procedure with a sample size of $k = 3$ $\mathscr{S}_b$-stable.

Example 3. *Consider a population game satisfying (U1)–(U3) with equilibrium $z^* \in (0,1)$. Assume sampling with a sample size $k = 3$. By Proposition 1, sampling procedure $\{3, p\}$ is $\mathscr{S}_b$-stable if and only if* (11) *holds, that is, if and only if $p = p_3^*$ solves*

$$z^* = \begin{cases} (1-p_3^*(z^*))^3 & \text{if } z^* \in (0,1/3) \\ (1-p_3^*(z^*))^3 + 3p_3^*(z^*)(1-p_3^*(z^*))^2 & \text{if } z^* \in [1/3,2/3) \\ (1-p_3^*(z^*))^3 + 3p_3^*(z^*)(1-p_3^*(z^*))^2 + 3p_3^*(z^*)^2(1-p_3^*(z^*)) & \text{if } z^* \in [2/3,1) \end{cases}. \quad (16)$$

Figure 2 depicts sampling probability $p_3^*(z^*)$ solving (16)*. The discontinuities arise at the points of discontinuity in $\lfloor kz^* \rfloor$. At these points the set of signals $\hat{z} \in \{0, 1/2, 2/3, 1\}$ to which players play strategy* 1 *changes composition. As a consequence, the absolute value $|d(z^*)|$ of the bias required in order that $\{k, p\}$ is $\mathscr{S}_b$-stable is not monotone in z^*, and the sign of $d(z^*)$ changes several times on* $[0,1]$.

From (16)*, it is immediate that for $z^* = 1/2$ it must hold that $p_3^*(z^*) = z^*$, and further that $\mathrm{d}p_3^*(z^*)/\mathrm{d}z^* < 0$ whenever $p_3^*(z^*)$ is differentiable. In particular, for $z^* \in (0,1/3)$ we see, both in* (16) *and Figure 2, that $\mathscr{S}_b$-stable sampling procedure $\{3, p_3^*\}$ needs to oversample strategy* 1 *agents in an ever more extreme fashion as z^* approaches* 0*, and for $z^* \in [2/3,1)$ we see that $\mathscr{S}_b$-stable sampling procedure $\{3, p\}$ needs to undersample strategy* 1 *agents in an ever more extreme fashion as z^* approaches* 1.

As an aside, we can also confirm the logic of the proof to Proposition 2 as discussed above: Substituting $p_3^*(z^*)$ with z^* in (16)*, we see that the right-hand side of* (16)*, corresponding to $P(z^*|z^*, \{3, \tilde{p}\})$ with $\tilde{p}(z) = z$, is strictly decreasing in z^* whenever it is continuous. Because $P(z^*|z^*, \{3, \tilde{p}\})$ has three points of discontinuity, this implies that for $k = 3$, there can be at most three different z^* such that $z^* = P(z^*|z^*, \{3, \tilde{p}\})$.*

In the following, we call a mixed equilibrium symmetric if $z^* = 1/2$, say that it is asymmetric if $z^* \neq 1/2$, and that it is the more asymmetric the higher the absolute difference between z^* and $1 - z^*$, that is, the higher $|1 - 2z^*|$. Example 3 suggests that the absolute value $|d(z^*)|$ of the bias required in order that some $\{k, p\}$ is $\mathscr{S}_b$-stable is highest when the equilibrium z^* is most asymmetric. That this indeed holds for any finite sample size k is established with Proposition 4 below.

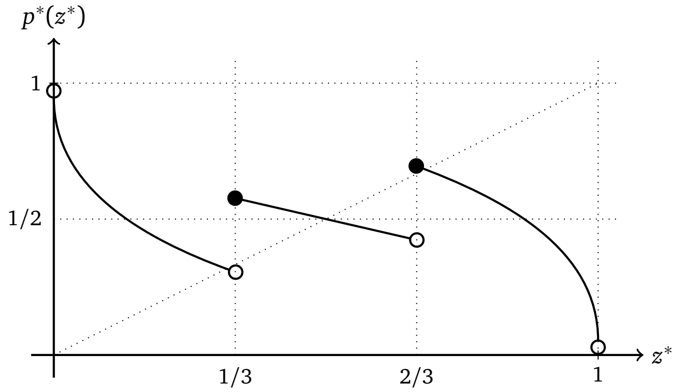

Figure 2: Sampling probability $p_3^*(z^*)$, $z^* \in (0,1)$, solving (16) as discussed in Example 3: The value $p_3^*(z^*)$ corresponds to the sampling probability of the stable sampling rule having a sample size of $k = 3$, given the equilibrium of the underlying game is z^*. The diagonal dotted line corresponds to the unbiased sampling probability.

Proposition 4. *Let z_n be a sequence with $z_n \in (0,1)$, $n = 1,2,\ldots$. If $z_n \to 1$, then for any $k \in \mathbb{N}_+$, it holds $\lim_{n\to\infty} p_k^*(z_n) = 0$. If $z_n \to 0$, then for any $k \in \mathbb{N}_+$, it holds $\lim_{n\to\infty} p_k^*(z_n) = 1$.*

Proposition 4 establishes that for any bounded sample size, there are games with sufficiently asymmetric equilibria such that the absolute value of the bias of the $\mathscr{S}_b$-stable procedure at this equilibrium is close to one. This implies that unbiased sampling is particularly suboptimal when the equilibrium is very asymmetric. With this result we can return to the question about the speed of the convergence of the stable sampling bias when we increase the sample size.

Definition 7. The stable sampling bias vanishes uniformly, if for any $\eta > 0$ there is $K > 0$ such that $\forall k \geq K$ and $\forall z \in (0,1)$ we have $|p_k^*(z) - z| < \eta$.

Combining Proposition 4 and Proposition 3, the next result follows straight away.

Corollary 1. *The stable sampling bias does not vanish uniformly.*

That is, for sufficiently asymmetric equilibria the need for correction through biased sampling only vanishes slowly in the sample size.

4.3 More Sophisticated Sampling

In this last section we explore the question whether there are – still assuming that players choose best response to the frequency of strategy occurrences in the sample – more sophisticated unbiased sampling procedures that yield higher fitness in games that have equilibria z^* which cannot be supported by the unbiased sampling procedures studied so far.

We consider the class of sampling procedures also studied in Oyama et al. (2015) that allow sample sizes to be drawn at random. We assume that the sampling bias is independent of the sample size. Let $\mu = \{\mu_k\}_{k=1}^{\infty}$ be a distribution over sample sizes. A distribution over sample sizes has finite support if there is $K > 0$ such that $\forall k > K$ we have $\mu_k = 0$. We let the tuple $\{\mu, p\}$ stand for the sampling procedure described by μ and p. We collect the sampling procedures using random sample sizes in the family $\mathscr{S}_b^r$, where now b indicates the upper bound on the support of μ.

An agent revising at population state z and using sampling procedure $\{\mu, p\}$ will play strategy 1 with probability

$$P(z|z^*, \{\mu, p\}) = \sum_{k=1}^{\infty} \mu_k P(z|z^*, \{k, p\}). \tag{17}$$

Because all $P(z|z^*, \{\mu, p\}) \in \mathscr{S}_b^r$ are decreasing in z and continuously differentiable, Lemmas 1 – 3 and Proposition 1 can be stated in terms of $\mathscr{S}_b^r$, too. That is, the results for stability of a sampling procedure $\{k, p\} \in \mathscr{S}_b$ carry over to sampling procedures $\{\mu, p\} \in \mathscr{S}_b^r$, and a sampling procedure $\{\mu, p\}$ is $\mathscr{S}_b^r$-stable iff

$$P(z^*|z^*, \{\mu, p\}) = z^*. \tag{18}$$

Condition (18) is the analogue to the stability condition (11) for the family $\mathscr{S}_b$. We next show by Example 4 that this class of sampling procedures extends the set of population states $z \in (0, 1)$ that can be supported by unbiased sampling rules, given z is the equilibrium of the underlying game, to a measureable set.

Example 4. *Consider a population game satisfying (U1)–(U3) with equilibrium $z^* > 1/2$, and two sampling procedures with $p(z) = z$ and sample sizes $k = 1$ and $k = 2$, respectively. The fraction of revising players choosing strategy 1 at z^* under the procedure with $k = 1$ is given by $P_1 = 1 - z^* < z^*$. The fraction of revising players choosing strategy 1 at z^* under the procedure with $k = 2$ is given by $P_2 = (1 - z^*)^2 + 2z^*(1 - z^*)$. Observe that we have*

$$P_2 - z^* = 1 - z^*(1 + z^*).$$

Hence, there is an interval $(1/2, \overline{z})$ with $\overline{z} > 1/2$ such that $\forall z^* \in (1/2, \overline{z})$, we have $P_1 < z^* < P_2$. That is, to any equilibrium $z^* \in (1/2, \overline{z})$ there is an unbiased sampling procedure $\{\mu, p\}$ that appropriately mixes between samples of size $k = 1$ and of size $k = 2$, such that (18) holds.

The example makes clear that for an equilibrium $z^* \in (0, 1)$ there is an unbiased procedure $\{\mu, p\} \in \mathscr{S}_b^r$ that is $\mathscr{S}_b^r$-stable if and only if there exist unbiased sampling procedures $\{k_1, p\}$ and $\{k_2, p\}$, $k_1, k_2 \in \mathbb{N}_+$, such that $P(z^*|z^*, \{k_1, p\}) \leq z^* \leq P(z^*|z^*, \{k_2, p\})$.

Even though the numerical calculations in Example 4 suggest that the class of games with payoffs satisfying (U1)–(U3) in which stable sampling rules $\{\mu, p\}$ are unbiased is much richer than if we only consider sampling rules $\{k, p\}$ with a fixed sample size, sampling with random sample sizes can still not accommodate for games with extremely asymmetric equilibria. Proposition 5 below makes this claim more precise, and implies that the set of population states $z \in (0, 1)$ that can be supported by unbiased sampling with random sample size, given z is the equilibrium of the underlying game, is a strict subset of $(0, 1)$.

Proposition 5. *There is, for every $b \in \mathbb{N}_+$, $\bar{z} > 0$ such that $\forall z \in (0, \bar{z}] \cap [\bar{z}, 1)$ there is no $\{\mu, p\} \in \mathscr{S}_b^r$ with p unbiased at z such that $P(z|z, \{\mu, p\}) = z$ holds.*

5 Conclusion

In this paper we have analyzed evolutionary stability of sampling biases. We have argued that sampling biases serve as evolutionary second-best corrections for players that play best response to the frequency of strategy occurrences in finite samples. Such a decision rule is particularly evolutionary suboptimal in games with a highly asymmetric mixed Nash equilibrium. We have modeled the evolution of optimal biases in a framework of sampling best response dynamics determining the population states in two subpopulations that use different sampling procedures. The dynamics of the subpopulation shares is governed by an adapted replicator dynamics that discriminates on the basis of average fitness within subpopulations. We have interpreted the resulting dynamical system as two-speed dynamics, which has allowed us to assess stability of a sampling procedure in terms of a lower-dimensional tractable reduced system.

The two-speed dynamics analyzed here are likely to be important in other contexts as well. For example, models of indirect preference evolution implicitly employ two-speed dynamics with play adapting infinitely fast to changes in the preferences: The results on slow-fast systems employed in this paper suggest a route to analyze the conditions under which such a short-cut is feasible. Further, the application of the results on slow-fast systems that we have resorted to is not restricted to the class of games that we have considered here. Extensions to population games with more than two-strategies and with more than one population should be feasible, in particular because the results allow focusing on a reduced system that has the same number of dimensions as the evolutionary node. We leave these directions for future research.

A Asymptotic Stability

Definition 8 (Asymptotic Stability, see e.g. Weibull, 1997.)**.** Consider system $\dot{u} = f(u)$, $f : [0,1]^n \to \mathbb{R}^n$, with initial condition $u_0 \in [0,1]^n$, solution $u(t)$, $t \geq 0$, and rest point $\tilde{u}$ so that $f(\tilde{u}) = 0$. Rest point $\tilde{u}$ is called asymptotically stable if it holds that

(a) for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $||u_0 - \tilde{u}|| < \delta(\epsilon)$, then $||u(t) - \tilde{u}|| < \epsilon$, $\forall t \geq 0$, and

(b) there exists $\delta > 0$ such that if $||u_0 - \tilde{u}|| < \delta$, then $\lim_{t\to\infty} ||u(t) - \tilde{u}|| = 0$.

If (a) holds then rest point $\tilde{u}$ is called neutrally stable; rest point $\tilde{u}$ is called unstable if it is neither asymptotically nor neutrally stable. If f is differentiable, a sufficient condition for asymptotic stability of $\tilde{u}$ is that all eigenvalues of the Jacobi $\partial_u f(\tilde{u})$ have strictly negative real parts, and a sufficient condition for $\tilde{u}$ to be unstable is that all eigenvalues of the Jacobi $\partial_u f(\tilde{u})$ have strictly positive real parts.

B Proofs

B.1 Proof of Lemma 1

We first show existence. By definition, $z(x, y)$ is linear in the elements of $x \in [0,1]^2$, and hence continuous in x. From the differentiability of p_1 and p_2 it follows that $R(z)$ is differentiable, and hence continuous in z. Hence, for any fixed $y \in [0,1]$, the composite function $R(z(., y))$ continuously maps the compact and convex set $[0,1]^2$ into itself. We can apply Brouwer's fixed point theorem.

Uniqueness can be shown as follows: Recall that $P'(z|z^*, \{k, p\}_q) \leq 0$, for all $z \in [0,1]$ and any $\{k, p\}_q \in \mathscr{S}_b$. Suppose the claim is false, that is, suppose that there are $x \neq x' \in X^*(y)$. It must hold that

$$P(yx_1 + (1-y)x_2|z^*, \{k, p\}_1) = x_1 \tag{a}$$

$$P(yx_1+(1-y)x_2|z^*,\{k,p\}_2)=x_2 \tag{b}$$
$$P(yx_1'+(1-y)x_2'|z^*,\{k,p\}_1)=x_1' \tag{a'}$$
$$P(yx_1'+(1-y)x_2'|z^*,\{k,p\}_2)=x_2' \tag{b'}$$

Assume that $x_1 > x_1'$. As $P(.|z^*,\{k,p\}_q)$, $q=1,2$, are decreasing, it follows by (a) and (a') that $yx_1+(1-y)x_2 < yx_1'+(1-y)x_2'$. From $yx_1+(1-y)x_2 < yx_1'+(1-y)x_2'$ and $x_1 > x_1'$ it follows that $x_2 < x_2'$. On the other hand, from $yx_1+(1-y)x_2 < yx_1'+(1-y)x_2'$, (b), (b') and the fact that $P(.|z^*,\{k,p\}_q)$, $q=1,2$, are decreasing it follows that $x_2 \geq x_2'$. A contradiction. The argument can be repeated for $x_1 < x_1'$, as well as for subpopulation 2. This gives us uniqueness.

To establish differentiability, we observe that the determinant of the Jacobi $C(x,y)$ given in (8) reads as

$$\det(C(x,y)) = -P'(z(x,y)|z^*,\{k,p\}_1)\,y - P'(z(x,y)|z^*,\{k,p\}_2)(1-y)+1, \tag{19}$$

which is strictly positive and bounded away from zero for all $(x,y)\in[0,1]^3$. Hence, $C(x,y)$ is invertible for all $(x,y)\in[0,1]^3$, and because the right-hand side of the sampling best response dynamics (5) is differentiable on $[0,1]^3$, we can compute $\partial_y x^*(y)$ from $x^*(y)=R(z(x^*(y),y))$ as

$$\partial_y x^*(y) = -[C(x^*(y),y)]^{-1}\left[\,\partial_y R(z(x,y))\big|_{x=x^*(y)}\right]. \tag{20}$$

B.2 Proof of Lemma 2

The trace of the Jacobi $C(x,y)$ given in (8) reads as

$$\mathrm{tr}(C(x,y)) = P'(z(x,y)|z^*,\{k,p\}_1)\,y + P'(z(x,y)|z^*,\{k,p\}_2)(1-y)-2, \tag{21}$$

which is negative and strictly bounded away from zero for all $(x,y)\in[0,1]^3$. Together with the fact that the determinant given in (19) is positive and strictly bounded away from zero for all $(x,y)\in[0,1]^3$, we have the claim by observing that the two eigenvalues satisfy $e_1+e_2=\mathrm{tr}(C(x,y))$ and $e_1e_2=\det(C(x,y))$.

B.3 Proof of Lemma 3

We start with Lemma 5 below that describes the convergence of the unique solution $(x(t),y(t))$, $t\geq 0$, of (7) with initial condition $(x_0,y_0)\in[0,1]^3$ when λ approaches infinity. Let $\bar{y}(t)$ be the unique solution of (9) with initial condition $\bar{y}_0=y_0$.

Lemma 5. *Assume (U1), and fix any $\eta>0$.*

(a) For any $T<\infty$, $\exists L,\bar{\lambda},\omega>0$ such that $\forall\lambda>\bar{\lambda}$, we have

$$\|y(t)-\bar{y}(t)\| < \eta, \quad \forall t\in[0,T] \tag{22}$$
$$\|x(t)-x^*(\bar{y}(t))\| < \eta, \quad \forall t\in[\lambda^{-1}L,T], \tag{23}$$

given that $\|x_0-x^*(y_0)\|\leq\omega$ holds.

(b) *Suppose* $\bar{y} = 1$ *is an asymptotically stable rest point of reduced system* (9). *Then,* $\exists L, \bar{\lambda}, \delta, \omega > 0$ *such that* $\forall \lambda > \bar{\lambda}$, *we have*

$$\|y(t) - \bar{y}(t)\| < \eta, \quad \forall t \geq 0 \tag{24}$$

$$\|x(t) - x^*(\bar{y}(t))\| < \eta, \quad \forall t \geq \lambda^{-1} L, \tag{25}$$

given that $\|y_0 - 1\| \leq \delta$ *and* $\|x_0 - x^*(y_0)\| \leq \omega$ *hold jointly.*

Proof. For the following, let $f : [0,1]^3 \to \mathbb{R}^2$ be shorthand for the vector field of the sampling best response node, i.e.

$$f(x,y) = \lambda \cdot [R(z(x,y)) - x], \tag{26}$$

and let $g : [0,1]^3 \to \mathbb{R}$ be shorthand for the vector field of the evolutionary node, i.e.

$$g(x,y) = y(1-y)(x_1 - x_2)h(z(x,y)). \tag{27}$$

We consider a continuously differentiable function $q : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$q(x,y) = \begin{bmatrix} \bar{P}_1(z(x,y)) - x_1 \\ \bar{P}_2(z(x,y)) - x_2 \end{bmatrix}, \tag{28}$$

with continuously differentiable $\bar{P}_q : \mathbb{R} \to \mathbb{R}$, $\bar{P}_q(v) = P(v|z^*, \{k,p\}_q)$ for $v \in [0,1]$, $q = 1,2$ and $z(x,y) = yx_1 + (1-y)x_2$. That is, we have $q(x,y) = f(x,y)$ on $[0,1]^3$. Further, we define $\bar{C}(x,y) = \partial_x q(x,y)$, and assume that all eigenvalues of $\bar{C}(x,y)$ have strictly negative real parts and are strictly bounded away from zero on $\mathbb{R}^3 \setminus [0,1]^3$. Together with Lemma 2, we then have that all eigenvalues of $\bar{C}(x,y)$ have strictly negative real parts that are bounded away from zero on the whole $\mathbb{R}^3$, and hence that $\bar{C}(x,y)$ is invertible for all $(x,y) \in \mathbb{R}^3$. Together with the fact that q is continuously differentiable on $\mathbb{R}^3$, it follows by the implicit function theorem that there exists unique continuously differentiable function $\bar{x}^* : \mathbb{R} \to \mathbb{R}^2$ that satisfies $q(\bar{x}^*(y), y) = 0$, $\forall y \in \mathbb{R}$, and hence satisfies $\bar{x}^*(y) = x^*(y)$ on $[0,1]$.

We now adapt Theorems 1 and 2 in Lobry et al. (1998) to our setting. As discussed in the text, $g(x,y)$ is Lipschitz-continuous on $[0,1]^3$ as a consequence of (U1). We consider a Lipschitz-continuous function $r : \mathbb{R}^3 \to \mathbb{R}$ satisfying $r(x,y) = g(x,y)$ for $(x,y) \in [0,1]^3$, and use function r together with function q described above to define dynamical system (29) for $(x,y) \in \mathbb{R}^3$ and $0 < \epsilon < 1$.

$$\begin{array}{rcl} \epsilon \cdot \dot{x} &=& q(x,y) \\ \dot{y} &=& r(x,y) \end{array}. \tag{29}$$

We are interested in the convergence when $\epsilon \to 0$ of the unique solution $(x(t), y(t))$, $t \geq 0$ of (29) with initial condition $(x_0, y_0) \in [0,1]^3$, whose existence follows from the Picard-Lindelöf theorem because any solution with initial condition in $[0,1]^3$ never leaves $[0,1]^3$. To this end, fix arbitrary $\rho > 0$, and define the following system for $\bar{y} \in (-\rho, 1+\rho)$:

$$\dot{\bar{y}} = r(\bar{x}^*(\bar{y}), \bar{y}). \tag{30}$$

By Lipschitz-continuity of the right-hand side of (30) and because the system remains in $[0,1]$, there exists, for any initial condition $\bar{y}_0 \in [0,1]$, a unique global solution $\bar{y}(t) \in [0,1]$ for $t \geq 0$ satisfying (30). The conditions that Lobry et al. (1998) impose on (29) and (30) are as follows:

(H1) *For all* y, $q(x,y)$ *is locally Lipschitz-continuous in* x.

(H2) *(i) The function* $\bar{x}^* : [-\rho, 1+\rho] \to \mathbb{R}^2$ *is continuous. (ii) For all* $y \in [-\rho, 1+\rho]$, $\bar{x}^*(y)$ *is an isolated root of* $p(x,y)$, *(iii)* $[-\rho, 1+\rho]$ *is compact.*

(H3) *(i) For each* $y \in [-\rho, 1+\rho]$, *the point* $x = \bar{x}^*(y)$ *is an asymptotically stable rest point of* $\dot{x} = \epsilon^{-1}q(x,y)$. *(ii) The basin of attraction of* $x = \bar{x}^*(y)$ *is uniform over* $[-\rho, 1+\rho]$, *that is, there exists* $a > 0$ *such that for all* $y \in [-\rho, 1+\rho]$, *the ball* $\mathscr{B} = \{x \in \mathbb{R}^2 : \|x - \bar{x}^*(y)\| \leq a\}$ *is a basin of attraction of* $\bar{x}^*(y)$.

(H4) $r(\bar{x}^*(\bar{y}), \bar{y})$ *is locally Lipschitz continuous in* $\bar{y} \in (-\rho, 1+\rho)$.

(H5) x_0 *lies in the basin of attraction of* $\bar{x}^*(y_0)$.

Note that we have replaced the conditions (H1) and (H4) in Lobry et al. (1998) by their sufficient conditions as identified in Lobry et al. (1999). Theorem 1 in Lobry et al. (1998) then implies the following:

Lemma 6 (cf. Theorem 1 in Lobry et al., 1998). *Suppose (H1) – (H5) to be satisfied. Let* $T < \infty$. *For every* $\eta > 0$, *there exists* $\bar{\epsilon}, L > 0$ *such that* $\forall \epsilon < \bar{\epsilon}$ *we have*

$$\|y(t) - \bar{y}(t)\| < \eta, \;\; 0 \leq t \leq T \tag{31}$$

$$\|x(t) - \bar{x}^*(\bar{y}(t))\| < \eta, \;\; \epsilon L \leq t \leq T. \tag{32}$$

If we further assume:

(H6) $\bar{y} = 1$ *is an asymptotically stable rest point of* (30), *and* y_0 *lies in the basin of attraction of* $\bar{y} = 1$,

then the following is a consequence of Theorem 2 in in Lobry et al. (1998).

Lemma 7 (cf. Theorem 2 in Lobry et al., 1998). *Suppose (H1) – (H6) to be satisfied. For every* $\eta > 0$, *there exists* $\bar{\epsilon}, L > 0$ *such that* $\forall \epsilon < \bar{\epsilon}$ *we have*

$$\|y(t) - \bar{y}(t)\| < \eta, \;\; t \geq 0 \tag{33}$$

$$\|x(t) - \bar{x}^*(\bar{y}(t))\| < \eta, \;\; t \geq \epsilon L. \tag{34}$$

We next argue that (H1) to (H4) are satisfied under the assumptions that we make on q and r. (H1) holds by assumption. (H2.i) holds since $\bar{x}^*$ is differentiable and hence continuous on $[-\rho, 1+\rho]$, (H2.ii) holds because it follows from Lemma 2 and our assumption on $\bar{C}(x,y)$ that $\forall y \in [-\rho, 1+\rho]$, the eigenvalues $\bar{C}(\bar{x}^*(y), y)$ have strictly negative real parts, and (H2.iii) holds because $[-\rho, 1+\rho]$ is bounded and closed. (H3) is guaranteed by Lemma 2 and our assumption on $\bar{C}(x,y)$: (i) $\forall y \in [-\rho, 1+\rho]$, the eigenvalues $\bar{C}(\bar{x}^*(y), y)$ have strictly negative real parts, and hence point $\bar{x}^*(y)$ is asymptotically stable $\forall y \in [-\rho, 1+\rho]$. (ii) Uniformity follows because the eigenvalues of $\bar{C}(\bar{x}^*(y), y)$ are uniformly bounded away from zero on $[-\rho, 1+\rho]$. Lastly, from the fact that $\bar{x}^*(y)$ is continuously differentiable, it follows that the right-hand side of (30) is Lipschitz-continuous, and hence (H4) holds.

Finally, we note that there exist $\omega > 0$ such that if we choose $\|x_0 - \bar{x}^*(y_0)\| < \omega$ then (H5) holds. This allows us to prove part (a) of Lemma 5: As observed in the text, the solution $(x(t), y(t))$ of (7) with initial condition $(x_0, y_0) \in [0,1]^3$ stays in $[0,1]^3$ for $t \geq 0$ and is unique. Therefore, the solution $(x(t), y(t))$ of system (29) with initial condition $(x_0, y_0) \in [0,1]^3$ coincides with the solution of (7) with the same initial condition, and hence is independent of the functional form assumed for q and r on $\mathbb{R}^3 \setminus [0,1]^3$. Since $\bar{x}^* = x^*$ on $[0,1]$, we get the claim by applying Lemma 6, and setting $\epsilon = \lambda^{-1}$ and $\bar{\epsilon} = \bar{\lambda}^{-1}$.

Claim (b) follows straight away: If $\bar{y} = 1$ is an asymptotically stable rest point of (30), then there exist $\delta, \omega > 0$ such that if we choose $\|x_0 - \bar{x}^*(y_0)\| < \omega$ and $\|y_0 - 1\| < \delta$, then both (H5) and (H6) hold. Repeating the argument given in the last paragraph, applying Lemma 7, and setting $\epsilon = \lambda^{-1}$ and $\bar{\epsilon} = \bar{\lambda}^{-1}$ then finishes the proof for claim (b). □

The rest of the proof consists of three auxiliary lemmas that together imply Lemma 3. The first is an observation needed below and also later on.

Lemma 8. *Let* $z^*(y) = yx_1^*(y) + (1-y)x_2^*(y)$ *and consider any* $\hat{y} \in [0,1]$.

- *If* $x_1^*(\hat{y}) > x_2^*(\hat{y})$, *then* $\mathrm{d}z^*(y)/\mathrm{d}y|_{y=\hat{y}} > 0$.
- *If* $x_1^*(\hat{y}) < x_2^*(\hat{y})$, *then* $\mathrm{d}z^*(y)/\mathrm{d}y|_{y=\hat{y}} < 0$.
- *If* $x_1^*(\hat{y}) = x_2^*(\hat{y})$, *then* $\mathrm{d}z^*(y)/\mathrm{d}y|_{y=\hat{y}} = 0$.

Proof. By definition, $x^*(y)$ solves $x^*(y) = R(z(x^*(y), y))$ for all $y \in [0,1]$ and is differentiable by Lemma 1. If $x = R(z(x,y))$, then $z = yx_1 + (1-y)x_2$ satisfies

$$yP(z|z^*, \{k,p\}_1) + (1-y)P(z|z^*, \{k,p\}_2) = z. \tag{35}$$

Totally differentiating (35), we get

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{P(z|z^*, \{k,p\}_1) - P(z|z^*, \{k,p\}_2)}{1 - yP'(z|z^*, \{k,p\}_1) - (1-y)P'(z|z^*, \{k,p\}_2)}. \tag{36}$$

The denominator of (36) corresponds to the determinant of $C(x,y)$ which we know to be strictly positive. Because at any rest point (x,y) of the sampling best response node it holds that $x_q^*(y) = P(z|z^*, \{k,p\}_q)$, $q = 1,2$, the claim follows. □

Using Lemma 8, we get the following result that concerns the stability of the rest point of reduced system (9).

Lemma 9. *Assume (U1) – (U3). Fix* $\{k,p\}_1 \in \mathscr{S}_b$ *supporting* $z \in [0,1]$ *and choose any* $\{k,p\}_2 \in \mathscr{S}_b$ *that is z-different from* $\{k,p\}_1$. *Then,* $\bar{y} = 1$ *is either an asymptotically stable rest point of reduced system* (9) *or an unstable rest point of the reduced system* (9).

Proof. Again, let $z^*(y) = yx_1^*(y) + (1-y)x_2^*(y)$. To get the claim, first suppose that $h(z^*(1)) \neq 0$. Because $h(z)$ is continuous by (U1) and $z^*(y)$ is differentiable by Lemma 1, if follows that $h(z^*(y))$ is continuous in y. Hence $\exists \epsilon > 0$ such that the right-hand side of (9) is either strictly positive or strictly negative $\forall \bar{y} \in [1-\epsilon, 1]$ and consequently $\bar{y} = 1$ is either an asymptotically stable or an unstable rest point of reduced system (9).

Second, suppose that $h(z^*(1)) = 0$. Because $x_1^*(1) \neq x_2^*(1)$ by z-difference of $\{k,p\}_1$ and $\{k,p\}_2$ it follows from Lemma 8 that $\mathrm{d}z/\mathrm{d}y|_{y=1} \neq 0$. Together with the fact that $h(z)$ is strictly decreasing at $z^*(1)$, by (U3), it follows that $\exists \epsilon > 0$ such that the right-hand side of (9) is either strictly positive or strictly negative $\forall \bar{y} \in [1-\epsilon, 1)$. Consequently, it follows that for the case $h(z^*(1)) = 0$, too, $\bar{y} = 1$ is either an asymptotically stable or an unstable rest point of reduced system (9). This gives us the claim. □

Using Lemma 5, we get the following third observation:

Lemma 10. *Assume (U1) – (U3), and fix $\{k,p\}_1 \in \mathscr{S}_b$ supporting $z \in [0,1]$.*

(a) If $\bar{y} = 1$ is an asymptotically stable rest point of (9) *for any $\{k,p\}_2 \in \mathscr{S}_b$ that is z-different from $\{k,p\}_1$, then F_1 is $\mathscr{S}_b$-stable.*

(b) If $\exists\{k,p\}_2 \in \mathscr{S}_b$, $\{k,p\}_2$ being z-different from $\{k,p\}_1$, such that $\bar{y} = 1$ is an unstable rest point of (9)*, then $\{k,p\}_1$ is not $\mathscr{S}_b$-stable.*

Proof. We start with claim (a). If $\bar{y} = 1$ is an asymptotically stable rest point of reduced system (9) for any $\{k,p\}_2 \in \mathscr{S}_b$ that is z-different from $\{k,p\}_1$, then part (b) of Lemma 5 applies for any $\{k,p\}_2 \in \mathscr{S}_b$ that is z-different from $\{k,p\}_1$. From this follows (b) in Definition 3 for any $\{k,p\}_2 \in \mathscr{S}_b$ that is z-different from $\{k,p\}_1$ straight away.

Point (a) in Definition 3 can be shown as follows: Let $\bar{y}(t)$ be a solution to (9) with initial condition $\bar{y}_0$. By asymptotic stability of $\bar{y} = 1$ we have that $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\|\bar{y}_0 - 1\| < \delta \ \ \Rightarrow \ \ \|\bar{y}(t) - 1\| < \epsilon, \ \forall t \geq 0. \tag{37}$$

By continuity of $x^*(y)$ on $[0,1]$ it thus follows that $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\|\bar{y}_0 - 1\| < \delta \ \ \Rightarrow \ \ \|(x^*(\bar{y}(t)), \bar{y}(t)) - (\tilde{x}, 1)\| < \epsilon, \ \forall t \geq 0. \tag{38}$$

Let $(x(t), y(t))$ a solution to (7) with initial condition (x_0, y_0). By Lemma 5 (b) there is to every small enough $\delta > 0$ $\bar{t} > 0$ such that it follows from $\|(x_0, y_0) - (\tilde{x}, 1)\| < \delta$ that $(x(t), y(t)) \to (x^*(\bar{y}(t), \bar{y}(t))$ uniformly on $t \geq \bar{t}$ when $\lambda \to \infty$ and $y_0 = \bar{y}_0$. Together with (38) it thus follows that $\forall \epsilon > 0\ \exists \bar{\lambda}, \bar{t}, \delta > 0$ such that $\forall \lambda \geq \bar{\lambda}$ we have

$$\|(x_0, y_0) - (\tilde{x}, 1)\| < \delta \ \ \Rightarrow \ \ \|(x(t), y(t)) - (\tilde{x}, 1)\| < \epsilon, \ \forall t \geq \bar{t}. \tag{39}$$

Now fix $\epsilon > 0$. By continuity of the solution $(x(t), y(t))$ in the initial conditions (x_0, y_0) (Hale, 2009) it follows for every $\lambda > \bar{\lambda}$ that $\exists \delta' > 0$ such that

$$\|(\bar{x}_0, \bar{y}_0) - (\tilde{x}, 1)\| < \delta' \ \ \Rightarrow \ \ \|(x(t), y(t)) - (\tilde{x}, 1)\| < \epsilon, \ \forall t \geq 0. \tag{40}$$

This gives us point (b) in Definition 3, and hence claim (a).

Turn to claim (b). Consider $\{k,p\}_2 \in \mathscr{S}_b$, $\{k,p\}_2$ being $\tilde{z}$-different from $\{k,p\}_1$, such that $\bar{y} = 1$ is unstable on the reduced system (9). Then, there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon \leq \bar{\epsilon}$, there is no $\delta > 0$ such that $\|\bar{y}_0 - 1\| < \delta$ implies $\|\bar{y}(t) - 1\| < \epsilon$, $\forall t \geq 0$. This, together with Lemma 5 (a), implies that there exist $\eta, \bar{\lambda}, \omega > 0$ with η small enough such that $\forall \lambda > \bar{\lambda}$, condition (a) in Definition 3 fails whenever $\|x_0 - x^*(y_0)\| < \omega$. That is, $\{k,p\}_1$ cannot be $\mathscr{S}_b$-stable. □

The if-part of Lemma 3 follows directly from part (a) of Lemma 10, and the only-if part follows from part (b) of Lemma 10 by observing that if, for some $\{k,p\}_2 \in \mathscr{S}_b$, $\bar{y} = 1$ is not an asymptotically stable rest point of the reduced system (9), then it is unstable by Lemma 9.

B.4 Proof of Proposition 1

We start with the following observation needed later on.

Lemma 11. *For any $u, z \in [0,1]$ and $k \in \mathbb{N}_+$, there exists an increasing and continuously differentiable function $p : [0,1] \to [0,1]$ such that $u = P(z|z^*, \{k,p\})$.*

Proof. Fix $z \in (0,1)$. From (4), we see that for $p(z) = 0$ we have $P(z|z^*, \{k,p\}) = 1$, that for $p(z) = 1$ we have $P(z|z^*, \{k,p\}) = 0$, and that $P(z|z^*, \{k,p\})$ is continuous in $p(z)$. Since $u \in [0,1]$, the claim follows. □

We first show that a sampling procedure $\{k,p\}$ supporting $z \in \{0,1\}$ cannot be $\mathscr{S}_b$-stable. So, consider $z^*(1) \in \{0,1\}$. More specifically, consider $z^*(1) = 1$; the argument for $z^*(1) = 0$ is analogous. By (U2)–(U3), we have $h(z^*(1)) < 0$, and hence, by continuity of the left-hand side of (10) in $\bar{y}$, condition (10) holds for all $\bar{y} \neq 1$ in a neighborhood of 1 iff $(x_1^*(1) - x_2^*(1)) < 0$ holds. But since $x_1^*(1) = 1$ and consequently $x_2^*(1) < 1$, we have a contradiction. Hence, a sampling procedure $\{k,p\}$ supporting $z = 1$ cannot be $\mathscr{S}_b$-stable.

Next, we consider $z^*(1) \in (0,1)$. We want to argue that condition (10) holds for all $\bar{y} \neq 1$ in a neighborhood of 1 iff $z^*(1) = z^*$, where $z^* \in (0,1)$ corresponds to the equilibrium of the underlying game. First, consider the *if*-part. If $z^*(1) = z^*$, then $h(z^*(1)) = 0$. We need to distinguish two cases: (1) $x_1^*(1) - x_2^*(1) < 0$, and (2) $x_1^*(1) - x_2^*(1) > 0$.

(1) If $x_1^*(1) - x_2^*(1) < 0$, then $z^*(y) > z^*(1)$ for all y below, and sufficiently close to 1 by Lemma 8. Because $h(z)$ is strictly decreasing at $z = z^*$ by (U2) and $x^*(y)$ is continuous at $y = 1$, it then follows that (10) holds for all $\bar{y} \neq 1$ in a neighborhood of 1.

(2) If $x_1^*(1) - x_2^*(1) > 0$, then $z^*(y) < z^*(1)$ for all y below, and sufficiently close to 1 by Lemma 8. Because $h(z)$ is strictly decreasing at $z = z^*$ by (U2) and $x^*(y)$ is continuous at $y = 1$, it then follows that (10) holds for all $\bar{y} \neq 1$ in a neighborhood of 1.

For the *only if*-part, we proceed by contradiction: Fix some $\{k,p\}_1 \in \mathscr{S}_b$ supporting $z \neq z^*$, and suppose that (10) holds for all $\bar{y} \neq 1$ in a neighborhood of 1 for any z-different $\{k,p\}_2 \in \mathscr{S}_b$. Consider first the case $h(z^*(1)) > 0$. Because both $z^*(1) \in (0,1)$ and $x_1^*(1) = z^*(1)$ holds it follows that $x_1^*(1) \in (0,1)$. There exists, by Lemma 11 and differentiability of p, a sampling procedure $\{k,p\}_2 \in \mathscr{S}_b$ such that $x_2^*(y) > x_1^*(y)$ holds for all $y \neq 1$ in a neighborhood of 1. Consequently, because $z^*(y)$ is continuous in y, there is a neighborhood around 1 such that the inequality in equation (10) is reversed for all $y \neq 1$ in that neighborhood. We have a contradiction. An analogous argument establishes that $h(z^*(1)) < 0$ cannot hold.

To conclude, we recall that $z^*(1) = z^*$ holds if and only if $z^* = P(z^*|z^*, \{k,p\})$ holds. Finally, existence of a sampling procedure $\{k,p\} \in \mathscr{S}_b$ satisfying $z^* = P(z^*|z^*, \{k,p\})$ follows from Lemma 11.

B.5 Proof of Lemma 4

Because $x_1^*(1) = x_2^*(1)$ holds it follows from Lemma 8 that $\mathrm{d}z^*(y)/\mathrm{d}y|_{y=1} = 0$. Observing that $P(z|z^*, \{k,p\}_1) = P(z|z^*, \{k,p\}_2)$ holds at the population state z being supported by $\{k,p\}_1$, we see that $x_1^*(\bar{y}) = x_2^*(\bar{y})$, $\forall \bar{y} \in [0,1]$ follows. Hence the solution trajectory $\bar{y}(t)$ of reduced system (9) satisfies $\bar{y}(t) = \bar{y}_0$, $\forall t \geq 0$. The statement then follows from equation (22) in Lemma 5.

B.6 Proof of Proposition 2

Let $r(z,k) = I_{1-z}(k - \lfloor kz \rfloor, \lfloor kz \rfloor + 1) - z$, let $f(u;\alpha,\beta)$ be the p.d.f. of $I_u(\alpha,\beta)$, and fix $k \in \{1,\ldots,b\}$. Because at all but at most countable $z \in (0,1)$, $\lfloor kz \rfloor$ is constant in z, it follows that, for all but at most countable $z \in (0,1)$, we have the derivative of r with respect to z given by

$$r_z(z,k) = -f(1-z; k - \lfloor kz \rfloor, \lfloor kz \rfloor + 1) - 1 \in (-\infty, -1). \tag{41}$$

Consequently, the set

$$R_k = \{z \in (0,1) : r(z,k) = 0\} \tag{42}$$

is countable. From this, it follows that R_k is a measure-zero set, and consequently, that the set

$$R = \bigcup_{k \in \{1,\ldots,b\}} R_k \tag{43}$$

is a measure-zero set, too. Because $R = \{z \in (0,1) : K_b(z) \neq \emptyset\}$, we have the claim.

B.7 Proof of Proposition 3

We have $P(z^*|z^*, \{k,p\}) = \text{Prob}\{m < \lfloor kz^* \rfloor\}$ where $m \sim \text{Bin}(k, p(z^*))$, which we can express as

$$\text{Prob}\{m < \lfloor kz^* \rfloor\} = \text{Prob}\left\{ \frac{m - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} < \frac{\lfloor kz^* \rfloor - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} \right\}. \tag{44}$$

From the de Moivre-Laplace Theorem it follows that the distribution of the random variable S_k, defined as

$$S_k = \frac{m - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}}, \tag{45}$$

approaches that of a standard normal variable as we let $k \to \infty$. Hence, we can write

$$\lim_{k\to\infty} \text{Prob}\{m < \lfloor kz^* \rfloor\} = \Phi\left(\lim_{k\to\infty} \frac{\lfloor kz^* \rfloor - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} \right), \tag{46}$$

where $\Phi(.)$ denotes the c.d.f. of a standard normal variable. Observe that we have

$$kz^* - 1 - kp(z^*) \leq \lfloor kz^* \rfloor - kp(z^*) \leq kz^* - kp(z^*). \tag{47}$$

Define

$$f^-(k) = \frac{kz^* - 1 - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} \tag{48}$$

$$f^+(k) = \frac{kz^* - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}}. \tag{49}$$

Because for $p(z^*) > z^*$ we have $\lim_{k\to\infty} f^-(k) = f^+(k) = -\infty$, it follows that

$$\Phi\left(\lim_{k\to\infty} \frac{\lfloor kz^* \rfloor - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} \right) = 0. \tag{50}$$

Conversely, because for $p(z^*) < z^*$ we have $\lim_{k\to\infty} f^-(k) = f^+(k) = \infty$, it follows that

$$\Phi\left(\lim_{k\to\infty} \frac{\lfloor kz^* \rfloor - kp(z^*)}{\sqrt{kp(z^*)(1-p(z^*))}} \right) = 1. \tag{51}$$

Because it must hold that $P(z^*|z^*, \{k,p\}) = z^* \in (0,1)$, for any $k = 1,2,...$, it must follow that $\lim_{k\to\infty} p_k^*(z) = z$.

B.8 Proof of Proposition 4

Let $z_n \in (0,1)$ be a sequence that converges to 1, i.e. $z_n \to 1$, and let p_n be a sequence satisfying

$$z_n = I_{1-p_n}(k - \lfloor kz_n \rfloor, \lfloor kz_n \rfloor + 1). \tag{52}$$

That is, for every z_n we have $p_n = p_k^*(z_n)$. Observe that there is $N > 0$ such that $\forall n > N$, we have $\lfloor kz_n \rfloor = \lfloor kz_{n+1} \rfloor$. Because, for any $\alpha, \beta > 0$, $I_x(\alpha, \beta)$ is continuous in x and because $I_1(\alpha, \beta) = 1$, we must have $p_n \to 0$ as $z_n \to 1$. We have the claim. The case $z_n \to 0$ can be shown using an analogous argument.

B.9 Proof of Proposition 5

Inspection of (3) reveals that if we assume $p(z) = z$, then $\lim_{z\to 0} P(z|z, \{k, p\}) = 1$, and $\lim_{z\to 1} P(z|z, \{k, p\}) = 0$, $\forall k = 1, ..., b$. Because, as noted in the text, in order that

$$P(z|z, \{\mu, p\}) = z$$

holds for $p(z) = z$, there must exist sampling procedures $\{k_1, p\}$ and $\{k_2, p\}$ with $k_1, k_2 \in \mathbb{N}_+$ and $p(z) = z$ satisfying $P(z|z, \{k_1, p\}) \le z \le P(z|z, \{k_2, p\})$, and because μ has finite support, the claim then follows.

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