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Payoff Shares in Two-Player Contests

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Abstract

In contest models with symmetric valuations, equilibrium payoffs are positive shares of the value of the prize. In contrast to a bargaining situation, these shares sum to less than one because a share of the value is lost due to rent-dissipation. We ask: can every such division into payoff shares arise as the outcome of the unique pure-strategy Nash equilibrium of a simple asymmetric contest in which contestants differ in the effectiveness of their efforts? For two-player contests the answer is shown to be positive.

JEL classification: C72, D72, D74.

Keywords: Contests, Pure-Strategy Equilibrium, Rent-Dissipation

1. Introduction

We study pure-strategy Nash equilibria of imperfectly discriminating contests in which risk-neutral players i simultaneously expend efforts $x_i \geq 0$ in order to increase their probability p_i of winning a prize with value $v > 0$.

There is an extensive literature discussing the existence, uniqueness, and comparative statics of pure-strategy Nash equilibria of such contests under a variety of assumptions on the contest success function, i.e., the map from the efforts of the players into their winning probabilities.[1] Here we focus on contests with two players and consider the class of asymmetric contest

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[1]See Konrad (2009) for a survey of the contest literature. Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) contain existence and uniqueness results of relevance to our analysis; Baik (1994) and Nti (1997) discuss the comparative statics of pure-strategy Nash equilibria in related models.

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success functions $p_i = \alpha_i x_i^r/(\alpha_1 x_1^r + \alpha_2 x_2^r)$, axiomatized in Clark and Riis (1998). We show that, for given v, this class is rich enough to support any positive payoffs for the players, summing to less than the value of the prize, as the outcome of a unique pure-strategy Nash equilibrium for appropriately chosen values of the effectiveness parameters $\alpha_i > 0$ and the decisiveness parameter $r > 0$. Given the parameters of the contest success function, the corresponding equilibrium payoff shares are independent of v. The proof is straightforward and constructive in the sense that for every division of the value v into payoff shares and a dissipated share we provide the parameter values which implement the division.

The motivation for our work is familiar from the economic theory of bargaining (cf. Muthoo, 1999, Chapter 1.2): When a contest is one of many ingredients of an economic model it is convenient to describe a contest in terms of its induced equilibrium payoffs rather than dwelling on the intricacies of the underlying non-cooperative model. Our analysis provides foundations for such an approach by (i) showing that any linear, but otherwise arbitrary, sharing rule can be used to describe the outcome of a contest, and (ii) by exhibiting the simple one-to-one relationship between the parameters of this sharing rule and the parameters of the contest success function.

For the symmetric version ($\alpha_1 = \alpha_2$) of the contest success function the symmetric version of our result (i.e., any division in which both players obtain identical, positive equilibrium payoffs summing to less than the prize can be supported) is immediate from the results in Pérez-Castrillo and Verdier (1992). The point of our analysis is that introducing a simple asymmetry in the contest success function suffices to produce a result which provides a generalization of the asymmetric Nash-bargaining solution to contests.

2. Model

Risk neutral players $i = 1, 2$ simultaneously choose efforts $x_i \geq 0$ at cost x_i. Both players assign value $v > 0$ to winning the prize (and value 0 to not winning the prize). Player i's payoff function is

$$U_i(x_1, x_2) = p_i(x_1, x_2) \cdot v - x_i, \tag{1}$$

where the probability $p_i(x_1, x_2)$ that player i wins the prize is given by the contest success function

$$p_i(x_1, x_2) = \begin{cases} \dfrac{\alpha_i x_i^r}{\alpha_1 x_1^r + \alpha_2 x_2^r} & \text{if } x_1 + x_2 > 0 \\ \dfrac{\alpha_i}{\alpha_1 + \alpha_2} & \text{if } x_1 = x_2 = 0 \end{cases}, \tag{2}$$

with $\alpha_i > 0$ and $r > 0$.[2] As payoff functions are homogenous of degree zero in α_1 and α_2, it is without loss of generality to assume $\alpha_1 + \alpha_2 = 1$ and we will do so throughout the following. Let

$$\mathcal{P} = \{(\alpha_1, \alpha_2, r) \in \mathbb{R}^3_{++} : \alpha_1 + \alpha_2 = 1\} \tag{3}$$

denote the corresponding set of feasible parameters for the contest success function. The parameters of a contest are $(\alpha_1, \alpha_2, r, v) \in \mathcal{P} \times \mathbb{R}_{++}$. A pure-strategy Nash equilibrium, or simply equilibrium, of a contest is a strategy profile $(x_1^*, x_2^*) \in \mathbb{R}^2_+$ satisfying

$$x_1^* \in \operatorname*{argmax}_{x_1 \geq 0} U_1(x_1, x_2^*) \text{ and } x_2^* \in \operatorname*{argmax}_{x_2 \geq 0} U_2(x_1^*, x_2). \tag{4}$$

Every equilibrium (x_1^*, x_2^*) gives rise to a division of the value of the prize into equilibrium payoffs $u_i^* = U_i(x_1^*, x_2^*)$ for the two players and a rent-dissipation term $d^* = v - u_1^* - u_2^*$. Each player i can assure a positive payoff by choosing the strategy $x_i = 0$ and for any strategy combination (x_1, x_2) the sum of the two players' payoffs is less than v. Hence, for every equilibrium (x_1^*, x_2^*) there exist $(s_1^*, s_2^*, s_3^*) \in \Delta$, where

$$\Delta = \{(s_1, s_2, s_3) \in \mathbb{R}^3_+ : s_1 + s_2 + s_3 = 1\}, \tag{5}$$

such that $u_1^* = s_1^* \cdot v$, $u_2^* = s_2^* \cdot v$, and $d^* = s_3^* \cdot v$. That is, we can view any equilibrium (x_1^*, x_2^*) of the contest as inducing a division of the value of the prize into payoff shares s_1^* and s_2^* for the two contestants and a dissipated share s_3^*. We refer to these shares as equilibrium shares.

[2] Our main result, Proposition 1 remains unchanged for any specification of $p_i(0,0) \geq 0$ satisfying $p_1(0,0) + p_2(0,0) \leq 1$.

3. Result

Proposition 1. *For any* $(s_1^*, s_2^*, s_3^*) \in \Delta$ *satisfying* $s_3^* > 0$ *there exists a unique* $(\alpha_1, \alpha_2, r) \in \mathcal{P}$ *such that any contest with parameters* $(\alpha_1, \alpha_2, r, v)$ *has a unique pure-strategy Nash equilibrium* (x_1^*, x_2^*) *with equilibrium shares* (s_1^*, s_2^*, s_3^*).

To prove this proposition we establish two lemmas. Lemma 1 delineates the set $\mathcal{P}^* \subset \mathcal{P}$ of parameters of the contest success function for which a unique equilibrium exists and determines equilibrium strategies and shares as functions of the parameters. Lemma 2 then completes the proof by exhibiting, for any shares in Δ satisfying $s_3 > 0$, the unique parameters in $\mathcal{P}^*$ yielding these shares as equilibrium shares.

Lemma 1. *A contest with parameters* $(\alpha_1, \alpha_2, r, v) \in \mathcal{P} \times \mathbb{R}_{++}$ *has a pure-strategy Nash equilibrium if and only if* $(\alpha_1, \alpha_2, r) \in \mathcal{P}^*$ *holds, where*

$$\mathcal{P}^* = \left\{ (\alpha_1, \alpha_2, r) \in \mathcal{P} : r \leq \frac{1}{\max\{\alpha_1, \alpha_2\}} \right\}. \tag{6}$$

If a pure-strategy Nash equilibrium exists, it is unique with equilibrium efforts

$$x_1^* = x_2^* = \alpha_1 \alpha_2 r v \tag{7}$$

and equilibrium shares

$$s_1^* = \alpha_1 - \alpha_1 \alpha_2 r \tag{8}$$
$$s_2^* = \alpha_2 - \alpha_1 \alpha_2 r \tag{9}$$
$$s_3^* = 2\alpha_1 \alpha_2 r. \tag{10}$$

Most of the proof of Lemma 1, which we have relegated to the appendix, is straightforward. Taking the existence of equilibrium for granted, uniqueness can be established directly by considering the appropriate first order conditions.[3] These imply (Mills, 1959) that equilibrium efforts are identical

[3]For $0 < r \leq 1$ existence and uniqueness of equilibrium follows from Szidarovszky and Okuguchi (1997) or Cornes and Hartley (2005, Theorem 3). For the case $1 < r \leq 1/\max\{\alpha_1, \alpha_2\}$ existence and uniqueness can be established along the same lines as in Pérez-Castrillo and Verdier (1992) or the proof of Proposition 4 in Cornes and Hartley (2005, Appendix 3). We provide a direct proof to make the paper self-contained.

and given by (7). Equations (8) and (9) for the equilibrium payoff shares s_1^* and s_2^* are then immediate: with equal efforts the probability that contestant i wins the prize is α_i, so that $s_i^* = \alpha_i - x_i^*/v$. Similarly, equal equilibrium efforts imply that the dissipated rent is twice the individual effort, yielding the expression for s_3^* in (10). Further, as equilibrium payoff shares must be positive, it is apparent from (8) and (9) that $r \leq 1/\max\{\alpha_1, \alpha_2\}$ is necessary for the existence of equilibrium. Showing that the same condition suffices for the existence of equilibrium is more subtle as player's payoff functions are not concave in own effort for $r > 1$.

Lemma 2. *For any $(s_1^*, s_2^*, s_3^*) \in \Delta$ satisfying $s_3^* > 0$ there exists a unique $(\alpha_1, \alpha_2, r) \in \mathcal{P}^*$, given by*

$$\alpha_1 = \frac{1 + s_2^* - s_1^*}{2} \tag{11}$$

$$\alpha_2 = \frac{1 + s_1^* - s_2^*}{2} \tag{12}$$

$$r = \frac{2s_3^*}{(1 + s_2^* - s_1^*)(1 + s_1^* - s_2^*)}, \tag{13}$$

such that equations (8) – (10) *hold.*

Proof of Lemma 2. First, we show that (α_1, α_2, r) as given by (11) – (13) is in $\mathcal{P}^*$: Adding equations (11) and (12) yields $\alpha_1 + \alpha_2 = 1$. From $(s_1^*, s_2^*, s_3^*) \in \Delta$ and $s_3^* > 0$ we have $|s_2^* - s_1^*| < 1$, so that (11) – (13) imply $\alpha_1 > 0$, $\alpha_2 > 0$, and $r > 0$. Hence, $(\alpha_1, \alpha_2, r) \in \mathcal{P}$. Given (11) and (12), equation (13) can be written as

$$r = \frac{s_3^*}{2\alpha_1\alpha_2} = \frac{s_3^*}{2\min\{\alpha_1, \alpha_2\}\max\{\alpha_1, \alpha_2\}}. \tag{14}$$

Because

$$s_3^* = 1 - s_1^* - s_2^* \leq \min\{1 + s_1^* - s_2^*, 1 + s_2^* - s_1^*\} = 2\min\{\alpha_1, \alpha_2\} \tag{15}$$

equation (14) then implies $r \leq 1/\max\{\alpha_1, \alpha_2\}$, yielding $(\alpha_1, \alpha_2, r) \in \mathcal{P}^*$.

Second, we show that the parameter values given in (11) – (13) are the unique parameter values in $\mathcal{P}^*$ such that (8) – (10) hold. Replacing (α_1, α_2, r) in (8) – (10) by the expressions on the right sides of (11) – (13) and simplifying shows that equations (8) – (10) are satisfied. Vice versa, suppose that for $(\alpha_1, \alpha_2, r) \in \mathcal{P}^*$ equations (8) – (10) hold. Subtracting equation (9) from equation (8), we find $\alpha_1 - \alpha_2 = s_1^* - s_2^*$. Using $\alpha_1 + \alpha_2 = 1$ this yields (11) and (12). From (10) we have $r = s_3^*/(2\alpha_1\alpha_2)$. Replacing α_1 and α_2 by the right sides of (11) and (12) then yields (13). □

4. Discussion

For given parameters $(\alpha_1, \alpha_2, r) \in \mathcal{P}^*$ of the contest success function, the resulting equilibrium shares (s_1^*, s_2^*, s_3^*) are independent of the value of the prize v. Consequently, Proposition 1 may be interpreted as the statement that any linear sharing rule in which players receive a positive share of the prize as a payoff and a strictly positive share of the prize is dissipated can arise as an equilibrium outcome of the asymmetric two-person contest we consider.

The restriction to sharing rules featuring a non-zero amount of rent dissipation, arising from our assumption that the decisiveness parameter r is strictly positive, is natural when considering a contest rather than a bargaining situation. Linearity of the sharing rule holds because the contest success functions we consider is homogeneous of degree zero in effort which corresponds to Axiom A6 both in Skaperdas (1996) and Clark and Riis (1998). More general contest success function, e.g. of the commonly considered ratio-form $p_i = f_i(x_i)/(f_1(x_1) + f_2(x_2))$ (Szidarovszky and Okuguchi, 1997; Cornes and Hartley, 2005), will implement additional, non-linear sharing rules. Given the complexity of characterizing pure-strategy equilibria for such contest success function, it seems unlikely that a sharp characterization analogous to Proposition 1 can be obtained for such an extension. Similarly, while it would be desirable to extend our result to asymmetric contests with more than two players, the straightforward generalization of our result is precluded because multiplicity of equilibria is endemic in such games. The question which linear sharing rules are implementable in n-player contests by homogenous contest success functions when our uniqueness requirement is relaxed appears more tractable. We leave this for further research.

Appendix

Proof of Lemma 1. There can be no equilibrium with either or both efforts equal to zero: Suppose, without loss of generality, that $x_2 = 0$. Then we have $U_1(x_1, x_2) = 1 - x_1$ for $x_1 > 0$ and $U_1(0, x_2) = \alpha_1 < 1$, so that player 1 has no best response. Hence, every equilibrium satisfies $(x_1, x_2) \in \mathbb{R}^2_{++}$. As the

payoff functions are differentiable on $\mathbb{R}^2_{++}$, the first order conditions

$$\frac{\partial U_1(x_1,x_2)}{\partial x_1} = \frac{\alpha_1\alpha_2 r x_1^{r-1}x_2^r}{(\alpha_1 x_1^r + \alpha_2 x_2^r)^2}v - 1 = 0 \tag{16}$$

$$\frac{\partial U_2(x_1,x_2)}{\partial x_2} = \frac{\alpha_1\alpha_2 r x_2^{r-1}x_1^r}{(\alpha_1 x_1^r + \alpha_2 x_2^r)^2}v - 1 = 0 \tag{17}$$

are then necessary for (x_1, x_2) to be an equilibrium. This yields $x_1 = x_2$. Substituting back into (16) and (17) we obtain (x_1^*, x_2^*) as given in (7) as the unique candidate for an equilibrium with corresponding equilibrium utilities

$$u_1^* = U_1(x_1^*, x_2^*) = [\alpha_1 - \alpha_1\alpha_2 r]\, v \tag{18}$$

$$u_2^* = U_2(x_1^*, x_2^*) = [\alpha_2 - \alpha_1\alpha_2 r]\, v. \tag{19}$$

The expressions for the equilibrium shares in (8) – (10) are then immediate.

Because player i can secure a payoff of zero by choosing $x_i = 0$, any equilibrium must satisfy $\min\{u_1^*, u_2^*\} \geq 0$. From (18) and (19) this condition is equivalent to $r \leq 1/\max\{\alpha_1, \alpha_2\}$. To finish the proof it remains to show that this condition is also sufficient for (x_1^*, x_2^*) to be an equilibrium. Towards this end, consider the second derivatives (well-defined on $\mathbb{R}^2_{++}$)

$$\frac{\partial^2 U_1(x_1,x_2)}{\partial x_1^2} = A_1(x_1,x_2)\left[\alpha_2(r-1)x_2^r - \alpha_1(r+1)x_1^r\right] \tag{20}$$

$$\frac{\partial^2 U_2(x_1,x_2)}{\partial x_2^2} = A_2(x_1,x_2)\left[\alpha_1(r-1)x_1^r - \alpha_2(r+1)x_2^r\right], \tag{21}$$

where $A_1(x_1,x_2) = (\alpha_1\alpha_2 r x_1^{r-2}x_2^r v)/(\alpha_1 x_1^r + \alpha_2 x_2^r)^3 > 0$ and $A_2(x_1,x_2) = (\alpha_1\alpha_2 r x_2^{r-2}x_1^r v)/(\alpha_1 x_1^r + \alpha_2 x_2^r)^3 > 0$. The sign of these derivatives is the sign of the terms in square brackets in (20) and (21). For $0 < r \leq 1$ these terms are strictly negative, so that the payoff of a player is concave in own effort, ensuring that the solution to the first order conditions (16) and (17) satisfies the equilibrium conditions (4). For $r > 1$ the terms in square brackets in (20) and (21) have exactly one sign change in x_1 and x_2, respectively, from positive to negative, so that the same holds for the second derivatives. Consequently, the first derivatives are unimodal (first increasing, then decreasing) in own effort. Hence, if (for given $x_2 > 0$) the first order condition (16) has a solution $\hat{x}_1 > 0$ satisfying $U_1(\hat{x}_1, x_2) \geq U_1(0, x_2) = 0$, then this solution solves $\max_{x_1 \geq 0} U_1(x_1, x_2)$ and, similarly, if (for given $x_1 > 0$) the first order

condition (17) has a solution $\hat{x}_2 > 0$ satisfying $U_2(x_1, \hat{x}_2) \geq U_1(x_1, 0) = 0$, then this solution solves $\max_{x_2 \geq 0} U_2(x_1, x_2)$. From this the desired result is immediate. □

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