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Endogenous growth and structural change through vertical and horizontal innovations

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Abstract

This paper combines horizontal and vertical innovations to generate an endogenous growth model allowing for structural change as an endogenous phenomenon. Every industry is profitable only for a limited period of time, making the effective time of existence of the technology endogenous and finite. We find that in such an economy endogenous structural change is the source of ongoing economic growth. Further, the range of existing sectors stays constant as well as growth rates as long as the technologies are symmetric.

Keywords: Endogenous Growth, Creative Destruction, Arrow Replacement Effect, Endogenous Structural Change, Horizontal Innovation, Endogenous Patents

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1 Introduction

The question of how to foster dynamic structural change (i.e. the replacement of older sectors by newer ones) in an economy compatible with ongoing economic growth is of interest for a great many modern economies. Such a dynamic structural change is, for example, of great importance for developing economies and for emerging countries of Asia. It is also of interest for developed countries when considering the transition to more energy efficient industries. Consistent model of this structural change may re-establish an argument for growth through horizontal innovations (i.e. through the creation of new sectors) paving the way for a sustainable development.

In this paper we present an endogenous growth model that allows for both horizontal and vertical innovations simultaneously. New technologies result from R&D investment of innovating firms and are continuously improved by vertical innovations. However, since the potential for improvement of a given technology is limited and since there is competitive pressure in the R&D sector, investment in higher quality of a given technology ends at a certain point in time and R&D firms rather invest in newly created technologies that have a higher potential for further quality enhancement. This makes old technologies disappear from the economy and, thus, generates endogenous structural change as well as ongoing growth.

The endogenous growth literature has a long tradition of identifying technical change as the primary source of sustained economic growth, dating back to the seminal papers by Romer (1990) and by Aghion and Howitt (1992). In these papers, economic growth results from vertical or from horizontal innovations (creative destruction). While most of the early contributions in the endogenous growth literature focus on either vertical or horizontal innovations, there are some recent approaches to model both types of technical progress simultaneously, as in the paper by Peretto and Connolly (2007) for example. However, in that paper horizontal innovations are limited due to the presence of fixed costs and growth results from the further development of existing technologies (productivity growth). In our current paper, we make a similar attempt and extend the model by integrating patents for new technologies in the same way as in Romer (1990). This allows continuous sustained horizontal innovations as well as productivity growth of the existing sectors. The competitive nature of the R&D sector and the fact that the potential for quality improvements of a given technology is bounded, lead to the gradual disappearance of older technologies from the economy and to the emergence of new technologies. This result is sometimes referred to as the Arrow replacement effect, named after Arrow (1962), and is in fact generating endogenous structural change in our framework.

The majority of the economics literature on structural change discusses the reallocation of productive factors from some sectors of the economy to others, but the number of sectors is assumed to be constant, as in Meckl (2002), Huntington (2010), Laitner (2000). However the rapid technical change leads not only to an overall productivity growth but also to a structural transformation of the economy, destroying older sectors and creating newer ones, as already discussed by Schumpeter (1942) and formally treated in Boucekkine et al. (2005). It is this aspect that is taken into account in our approach, in contrast to the papers by Meckl (2002), Huntington (2010), Laitner (2000). One example for such a model with a dynamic number of sectors is Chu (2011). There, however, the number of sectors cannot decrease, thus, excluding the disappearance of sectors as in our model.

The appearance of fundamentally new technologies is usually accounted for by the concept of General Purpose Technologies (GPTs) that have a broad impact on different sectors in the economy. Examples for such GPTs are steam-power, electricity and so on. A literature review of this type of models can be found in Bresnahan (2010). The difference of our model to this approach is that it is not the significance of an innovation which leads to structural change (destroying some sectors and introducing new ones), but rather the existence of limiting factors that change the structure of the economy: all technologies are symmetric, but, since the potential for improvement of a given technology is limited and due to competitive pressure in the R&D sector, older technologies are driven out of the market. To model such a dynamic transformation of the economy we fully account for the endogenous formation of patent prices and for R&D behaviour in the spirit of Nordhaus (1967).

In our model setup, the monopolistic competition in the manufacturing sector together with free entry in the technology sector affect the patent prices in such a way that excessive monopolistic profits are not used for asset accumulation but for the development of newer technologies by the competitive R&D sector. The overall structure of R&D in the model resembles the one of venture capital firms: a new technology is invented by the R&D firm with the intention of its further development up to the point when it becomes productive and the only stimulus for such a development is the patent payment from the manufacturing sector. The patent itself is of unlimited duration, but the endogenous emergence of new technologies limits the time of its usage. Thus, the infinite duration of patents in this setting does not create obstacles for technological progress because the technology itself becomes out-dated at some point. The overall life-cycle of each technology resembles the cycles already mentioned in Albernathy and Utterback (1985). However, in contrast to Albernathy and Utterback (1985), this cycles occurs at an economy-wide level in our model and we present our arguments in a more formal setting.

In the model we present below, the setup of the R&D sector resembles the structure resorted to in optimal control models on the endogenous domain, developed mainly in Belyakov et al. (2011) but it is closer to the homogeneous version of the multi-product monopolist from Bondarev (2012). Horizontal and vertical innovations are interrelated, with the profitability resulting from vertical innovations being the stimulus for inventing new technologies and the spectrum of horizontal innovations determining the limitation for vertical innovations. Structural change, then, is defined as the appearance of new technologies accompanied by the disappearance of old ones the productivity of which cannot be improved any further.

The main contribution of our paper is to present a theoretical model of an economy that grows through structural change that is endogenously determined. New technologies arrive at some constant speed because of the symmetry of all new technologies, thus ensuring a constant range of existing technologies. The symmetry of the technologies leads to equal profitability of newer technologies and to equal incentives to innovate for all incumbent R&D firms. At the same time, all technologies are developed through optimal investment plans, which are identical across the whole range of technologies. As a result, the productivity of the economy grows proportionally to the accumulated financial capital (assets).

The productivity of newer sectors grows faster than that of older ones since the abundance of accumulated assets is higher at the time when the new technology is invented and its development starts. The higher potential of newer technologies for quality improvements attracts capital into their development that is withdrawn from older technologies (since potential profits from newer technologies are higher), thus, creating structural change. To obtain this effect, we assume that that any technology needs maintenance to be of use, implying that there is some depreciation of productivities. Finally, profits in the manufacturing sectors with older technologies are dwindling because labour is reallocated to newer sectors (where it is used more efficiently) and older sectors disappear from the economy.

The rest of the paper is organized as follows. Section 2 introduces the structure of the model. Section 3 provides the results and the analysis. Section 4 concludes with some discussion and possible future extensions. Some of the more tedious mathematical proofs are given in the Appendices at the end of the paper.

2 The Model

There are three types of economic agents: households, producers and R&D firms.

The household sector is represented by one household that maximizes the stream of discounted utilities over an infinite time horizon subject to its budget constraint. Utility arises from a composite consumption good which consists of the integral over all existing differentiated products. The solution of the intertemporal optimization problem gives rise to the usual Euler equation. The modelling of the household sector is standard in economics and can be found in quite a many contributions (as for example in the book Grossman and Helpman (1993)).

Producers of the final output have to buy a blueprint, that is a patent, in order to be able to start the production of the consumption good i. This blueprint also determines the technology in use. The output sector is characterized by monopolistic competition and the present value of future profits of the producer of good i is used to pay for the patent. Thus, the structure of the output sector is similar to the structure of the intermediate goods producers in the Romer (1990) model. However, in contrast to Romer (1990), profits in our approach do not arise over an infinite time horizon, but only over a certain endogenously defined period of time. This results from the fact we allow for the out-dating of goods, giving rise to endogenous structural change.

In particular, there is a continuum of goods indexed by i with an endogenous spectrum. This spectrum can be extended by horizontal innovations. Each good i is provided by a single monopolistic producer which is the holder of the most recent technology in sector i. All products i are fully consumed, as in Ngai and Pissarides (2007), with financial capital (assets) being the separate good, which is used for R&D investments. Since there is a varying continuum of final products, we choose labour as the numeraire, i.e. the wage rate is set equal to one. Thus, all costs, investments and prices come in terms of labour costs.

The modelling of the R&D sector follows the microeconomic model of Bondarev (2012). Firms in the R&D sector are perfectly competitive and invest resources in order to generate horizontal innovations, i.e. blueprints, allowing the production of new products. These are sold to the producers in the final output sector. Each new blueprint, or technology, has zero productivity at the time it is invented but its quality rises over time due to vertical innovations, resulting from the R&D sector investing resources in quality improvements. Thus, we differentiate between the invention and the innovation of a technology. Vertical innovations raise the productivity of the technology sold to the final output sector, thus, guaranteeing this sector positive profits, but only over a certain finite period of time. The latter holds because the potential of a given technology for improvements is limited, implying that it becomes more and more expensive to raise the productivity, the higher is the level already attained. Or, formulated differently, one unit of R&D raises the quality of a technology i the less, the higher is its level. Therefore, at a certain point in time, it is more profitable to invest one unit of R&D into vertical innovations of a younger, i.e. less developed, good rather than to spend that unit into the development of an older technology that has a smaller scope for further improvements. This characteristic of our model makes the profits, resulting from the production of a certain good, finite and it makes old products to disappear from the economy and new ones to come into existence.

Next, we continue with the formal description of the households, followed by manufacturing sectors and R&D activities.

2.1 Households

Households are modelled in a similar way as in Peretto and Connolly (2007). The amount of labour is constant and distributed across the range of final sectors, which are in existence:

$$L = \int\limits_{N_{min}(t)}^{N_{max}(t)} L(i,t)di,$$

$$N_{min}(t) < N_{max}(t) < N(t), \tag{1}$$

where:

- L is the total labour in the economy (equal to population),
- $L(i)$ is the employment in sector i,
- $N(t)$ is the number of products or technologies (range) invented up to time t,
- $N_{max}(t)$ is the range of manufacturing sectors with positive operating profit (any new technology does not immediately yield positive productivity),
- $N_{min}(t)$ is the range of sectors, which have disappeared from the economy up to time t.

Strictly speaking, $N(t)$ gives the number of blueprints developed by the R&D sector that are sold in form of patents to the final goods producers. Since producers of the final good must acquire one blueprint for each good, the number of blueprints equals the number of products in the economy and the blueprint also determines the production technology.

The range of developed sectors is growing over time reflecting the expansion in the variety of products. However, the range of existing sectors, given by $N_{max}(t) - N_{min}(t)$ may grow, decrease or stay constant in time, depending on the characteristics of the process of expansion of variety of technologies, $\dot{N}$. The labour employed by an individual sector is not constant. It is redistributed from older sectors to newer ones.

From now on we omit where possible time arguments keeping in mind dynamic nature of main variables of the model if not stated otherwise.

The objective function of the household is

$$J^H = \int_0^\infty \mathbf{e}^{-\rho t} U(C) dt, \tag{2}$$

with $U(C) = \ln C$ being the utility function from composite consumption C consisting of the continuum of products,

$$C = \left[\int_{N_{min}}^{N_{max}} C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{3}$$

with $1 < \varepsilon < \infty$ being the elasticity of substitution between goods.

The flow budget constraint of the household is

$$\dot{a} = ra + L - \int_{N_{min}}^{N_{max}} P_i C_i di, \tag{4}$$

with L the numeraire so that the wage rate is equal to one and where:

- a is the value of assets being hold by the households, similar to Chu et al. (2012),
- r is the interest rate.

We assume zero depreciation rate of capital for simplicity. Positive depreciation will not essentially change the results of the paper.

We denote consumption expenditures by E:

$$E = \int_{N_{min}}^{N_{max}} P_i C_i di \ , \tag{5}$$

along the same range of existing sectors to condense notation.

The accumulation of assets comes from the difference between consumption expenditures and income of the household, which is the sum of interest earned for existing assets and of labour income.

Consumption of the individual good i is given by (see Appendix A)

$$C_i = E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} \ . \tag{6}$$

The standard Euler equation implies that the optimal growth rate for expenditure is given by

$$\frac{\dot{E}}{E} = r - \rho \ , \tag{7}$$

2.2 Goods Producers

Goods producers employ labour and buy technology (a blueprint) from the R&D sector. With these inputs they produce the goods which they sell to the consumer. Output of good i is given by:

$$Y_i = A_i^{\alpha} L_i \ , \tag{8}$$

where $0 < \alpha < 1$ determines the productivity of the technology in production. Production is linear in labour, since the productivity A_i is the main point of concern here. The productivity A_i is the result of vertical innovations that raise the quality of a given technology and that are generated by the R&D sector (see section 2.3 below). Hence, quality improvement means an increase in efficiency in the sense that final goods producer can generate more output with one unit of labour input.

The profit of firm i is

$$\Pi_i = P_i Y_i - L_i - \Psi \ , \tag{9}$$

where Ψ is a fixed operating cost.[1]

The only use for output of all goods i is consumption, so that $C_i = Y_i$. The only product used for investments is financial capital a which is excluded from this spectrum.

[1]This cost may be treated as resulting from the fact that the firm has to buy a blueprint so that it can produce the good. We assume Ψ to be equal across sectors and do not elaborate on its relationship to the patent price to make exposition simpler.

Firm i, therefore, sets its price to (see Appendix A)

$$P_i = \frac{\varepsilon}{\varepsilon - 1} A_i^{-\alpha} \ . \tag{10}$$

This is the price defined only for the products in the range $N_{max} - N_{min}$. However, since we have fixed operating costs the profit is nonnegative not immediately from the time of invention of technology i, but after some time. At the same time after the product of the given sector i becomes out-dated, the demand for it will decrease down to the point, where no positive profits may be made. All products out of the range $N_{max} - N_{min}$ thus have a price of zero:

$$P_i = \begin{cases} 0, \ t < \tau_{max}(i), \tau_{max}(i) : \Pi_i = 0, \dot{\Pi}_i > 0, \\ \frac{\varepsilon}{\varepsilon-1} A_i^{-\alpha}, \ \tau_{max}(i) < t \leq \tau_{min}(i), \tau_{min}(i) : \Pi_i = 0, \dot{\Pi}_i < 0, \\ 0, t > \tau_{min}(i). \end{cases} \tag{11}$$

Here and throughout the paper we use the following notation:

- $\tau_{min} = N_{min}^{-1}(i)$, time when product (technology) i becomes out-dated and profit of manufacturing decreases below zero;
- $\tau_{max} = N_{max}^{-1}(i)$, time when product (technology) i becomes profitable and manufacturing sector starts production of positive amounts;
- $\tau_0 = N^{-1}(i)$, time when technology i is invented through horizontal innovations process.

Inserting (6) and (10) into (8) yields labour demand as,

$$L_i^D = \frac{\epsilon - 1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} \ . \tag{12}$$

Labour employed in sector i is thus a function of the relative productivity of labour in sector i. Repeating the arguments made with respect to the price formation, we get a piecewise-defined labour demand:

$$L^D(i) = \begin{cases} 0,\ t < \tau_{max}(i), \tau_{max}(i) : \Pi_i = 0, \dot{\Pi}_i > 0, \\ \frac{\epsilon-1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj},\ \tau_{max}(i) < t \leq \tau_{min}(i), \tau_{min}(i) : \Pi_i = 0, \dot{\Pi}_i < 0, \\ 0, t > \tau_{min}(i). \end{cases} \tag{13}$$

The technology is acquired by the goods producers in the form of a patent and the pricing for this patent follows Nordhaus (1967), Romer (1990) and Grimaud and Rouge (2004). The price of the patent (blueprint) equals the total value of profits which can be derived from it. The manufacturing firm can extract positive profits only for a limited period of time. Thus the patent price is defined as:

$$p_A(i) \stackrel{def}{=} \int\limits_{\tau_{max}}^{\tau_{min}} \mathrm{e}^{-r(t-\tau_0)} \Pi_i dt. \tag{14}$$

The date at which patent i starts, τ_{max}, is endogenously determined by the productivity threshold necessary to gain positive profits, while the effective duration of the patent is endogenously determined from the demand for the manufactured product i, by the point in time, τ_{min}, when the final producer can no longer earn positive profits. Thus, the duration of the patent is determined by two zero-profits conditions.

Further, the patent price is independent of time. It only depends on the ratio of the level of productivity in sector i in time points τ_{max}, τ_{min}. We state this result in Proposition 1.

Proposition 1 *The price of the patent $p_A(i)$ is not a function of time.*

The proof of this proposition is given in Appendix B.

2.3 R&D Sector

The general structure of the R&D sector follows the lines of the paper Bondarev (2012) with homogeneous technologies. In this paper we adopt independent R&D structure, following major endogenous growth literature. Results are the same for in-house R&D management by large multiproduct firms[2] in the spirit of Lambertini (2003), which in fact dominates the TFP growth (see Acemoglu and Cao (2015)).

There are two types of R&D: Productivity-improving (vertical) innovations and variety-expanding (horizontal) innovations. Both types of R&D use financial capital as the only input. Thus, the total sum of both kinds of R&D investments at any time forms the demand for assets in the economy:

$$u(t) + \int_{N_{min}(t)}^{N(t)} g(i,t)di = a^D(t) \ , \tag{15}$$

where

- $u(t)$ are horizontal innovations investments at time t;
- $g(i,t)$ are vertical innovations investments at time t for technology i within the range of invented and not out-dated technologies, $[N_{min}(t), N(t)]$;
- $a^D(t)$ is the total demand for assets.

Both types of investments are optimally set as strategies of associated firms in their optimal control problems. This makes our model different from the classical Romer's case: R&D firms' decisions take into account future potential profits, rather than only the current revenues from selling patents to final producers.

We first describe the problem of R&D investments in horizontal innovations and then proceed to vertical innovations.

[2] this holds true as long as technologies are symmetric

2.3.1 Horizontal innovations

The creation of new technologies (horizontal innovations) follows the setup of Peretto and Connolly (2007). We assume that new technologies appear due to knowledge creation mechanisms that are governed by private initiatives of competitive R&D firms. New technologies are created through R&D investments, $u(t)$, chosen optimally by the firms:

$$\dot{N} = \delta u(t) \ , \tag{16}$$

where the dot denotes the time derivative. These are financed from the assets of the households $a(t)$ and represent a part of the total assets demand a^D in (15). The equation above may be interpreted as a transformation rule of financial assets (being used for investments) into the extension of the existing range of technologies $N(t)$.

The incentive for horizontal innovations is the potential profit from selling the technology to manufacturing firms. We assume that the horizontal R&D firm which invents technology i later develops it through vertical innovations. The two-step sequential optimization is equivalent to the joint optimization in this setup, see Bondarev (2014) for example. Thus, the value of horizontal R&D consists solely in expected future profits from vertical innovations:[3]

$$V_N = \max_{u(\bullet)} \int_0^\infty \mathrm{e}^{-rt} \left(\pi^R(i)|_{i=N} \delta u(t) - \frac{1}{2} u^2(t) \right) dt. \tag{17}$$

Here, the profit of developing the next technology $i = N$, $\pi^R(i)|_{i=N}$, equals the value of vertical innovations into technology i, which is given by:

$$\pi^R(i)|_{i=N} = p_A(N) - \frac{1}{2} \int_{\tau_0(N)}^{\tau_{min}(N)} \mathrm{e}^{-r(t-\tau_0)} g^2(N,t) dt \ , \tag{18}$$

[3] In this sense, we differ between the invention of a new technology and its economic use.

with $g(N,t)$ investments into the development of technology N during the phase when technology i has non-zero productivity. The fact that the value of a horizontal innovation depends only on the next technology is equivalent to the result of Chu (2011) on the presence of an *Arrow replacement effect*: each new technology is owned by a separate R&D firm.

Since the patent price is time-independent from Proposition 1, the expected total profit from developing the next technology is also not a function of time, but of the technology position in the technologies' space i and, thus, the value function of horizontal innovations is state-dependent for $i = N$.

The form of optimal investments into horizontal innovations is given in Proposition 2.

Proposition 2 *With the value of the horizontal innovations given by* (17)*, the optimal investments are proportional to the expected profit from the development of the next invented product*

$$u^* = \delta\pi^R(i)|_{i=N}, \tag{19}$$

and are constant for symmetric technologies.

The proof amounts to constructing the standard Hamilton-Jacobi-Bellman (HJB) equation for this problem. This can be found in the Appendix C.

Horizontal expansion is obtained as a function of the profits resulting from the development of the next-to-be-invented product:

$$N(t) = \delta^2\pi^R(i)|_{i=N}t + N_0. \tag{20}$$

At the same time, both horizontal and vertical R&D are using the assets accumulated by households. Thus, the financial market clearing condition must hold:

$$a^S(t) = a^D(t)\,\forall t, \tag{21}$$

which gives together with (15) and (19):

$$\int\limits_{N_{min}(t)}^{N(t)} g(i,t)di + \delta\pi^R = a^S(t)\ , \tag{22}$$

where we make use of the constancy of horizontal investments from Proposition 2 and the $a^S(t)$ term is the total supply of financial assets coming from households. It contains the cost of capital r, see (4).

This may be used to define total investments into vertical innovations (given by the integral term):

$$\int\limits_{N_{min}(t)}^{N(t)} g(i,t)di = G(t),$$

$$G(t) = a^S(t) - u(t), \tag{23}$$

where $g(i,t)$ are investments into the improvement of productivity in sector i and $G(t)$ denotes total assets available for vertical innovations. The $G(t)$ is thus the function of the cost of capital $r(t)$ and profit from inventions $\pi^R(i,t)$.

2.3.2 Vertical innovations

Productivity-improving innovations (vertical innovations) lead to a rise in efficiency of technologies that have zero productivity upon their invention. This productivity can be developed through specific investments for every product.

Profits from R&D result from sales of blueprints to manufacturing firms. These sales come in the form of patents for each new technology i and all of the investments into the development of each new technology (vertical innovations) are financed from this patent payment. Costs of R&D are costs of development of the productivity through technology-specific investments g_i. These investments are financed from financial capital a just as for horizontal innovations.

The profit associated with the development of technology i is given by:

$$\pi^R(i) = p_A(i) - \frac{1}{2} \int_{\tau_0}^{\tau_{min}} \mathrm{e}^{-r(t-\tau_0)} g^2(i,t) dt, \tag{24}$$

with investments going into the increase of productivity:

$$\dot{A}(i,t) = \gamma g(i,t) - \beta A(i,t) \ , \tag{25}$$

where γ is the efficiency of investments into the productivity increase (equal for all sectors) and β is the cost of supporting the productivity at the current level. The presence of the parameter β also reflects the fact that one unit of R&D raises the quality of a given technology more when the level of quality is still low. Hence, the more a given technology has already been impoved, by vertical R&D investment, the more difficult it becomes to generate an additional increase in its quality. This results from the fact that the potential of a given technology for improvement is limited.

At any time, there exists a range of $N(t) - N_{min}(t)$ of new technologies and, thus, exactly this range of vertical R&D investments. It should be noted that the range of the manufacturing sector is different and is given by $N_{max}(t) - N_{min}(t)$.

In order to assure that vertical R&D investments are chosen optimally, the R&D sector solves a dynamical problem of optimal investment plans subject to the availability of resources (the price for assets r is the same and constant in equilibrium, see 2.4 for the formal proof).

The optimization problem for vertical R&D investments, then, reads:

$$V = \max_{g} \int_{0}^{\infty} e^{-rt} dt \int_{N_{min}(t)}^{N(t)} p_A(i) di - \int_{0}^{\infty} e^{-rt} \int_{N_{min}(t)}^{N(t)} \frac{1}{2} g^2(i,t) di dt, \tag{26}$$

$$s.t. \tag{27}$$

$$\forall i \in [N_{min}, N] \subset \mathbb{R}_+ : \dot{A}(i,t) = \gamma g(i,t) - \beta A(t), \tag{28}$$

$$\int_{N_{min}(t)}^{N(t)} g(i,t) di = G(t), \tag{29}$$

with $G(t) = a^S(t) - \delta \pi^R$ determined by the financial market clearing condition (22). For those technologies, which are outside of the operating range, $i < N_{min}$ there is no development, since the price of the patent, p_A, pays only for the development of the technology during the operational time, $t \in [\tau_{max}(i), \tau_{min}(i)]$.

Applying the Maximum Principle, we derive optimal investments as a function of the shadow costs of investments, $\psi(i,t)$, with the latter being a function of the patent price:

$$\dot{\psi}(i,t) = (r + \beta)\psi(i,t) - \frac{\partial p_A(i)}{\partial A(i)},$$

$$g^*(i,t) = \gamma \psi(i,t) - \frac{\int_{N_{min}(t)}^{N(t)} \gamma \psi(i,t) di - G(t)}{N(t) - N_{min}(t)}. \tag{30}$$

We now establish auxiliary Proposition 3 which will help us to obtain the symmetric solution of the model:

Proposition 3 *The effect of a rise in productivity with respect to the price of the patent is the same for all technologies,*

$$\forall i : \frac{\partial p_A(i)}{\partial A_i} = C_A = const. \tag{31}$$

Where the notation $\frac{\partial p_A(i)}{\partial A_i}$ means the differentiation of an integral, see for example Flanders (1973) for treatment. The proof can be found in the Appendix D.

Using Propositions 1 and 3 it can be demonstrated that the shadow costs of investments are the same across all existing technologies:

$$\psi(i,t) = \psi^* = \frac{C_A}{r+\beta}. \tag{32}$$

Then, investments into productivities of all the existing technologies are symmetric:

$$g^*(t) = \frac{G(t)}{N(t) - N_{min}(t)}, \tag{33}$$

but the dynamics of the productivities differ by the depreciation rate:

$$\dot{A}(i,t) = \gamma \frac{G(t)}{N(t) - N_{min}(t)} - \beta A(i,t). \tag{34}$$

To fully define the vertical innovations dynamics, we make use of the same arguments as for final prices and for labour demand to obtain the optimal investments plans for each i in piecewise form:

$$g^*(i,t) = \begin{cases} 0, t < \tau_0(i), \\ \frac{G(t)}{N(t)-N_{min}(t)}, \tau_{min}(i) > t > \tau_0(i), \\ 0, t > \tau_{min}(i). \end{cases} \tag{35}$$

Then, the associated evolution paths of the technologies are given by:

$$\dot{A}(i,t) = \begin{cases} 0, t < \tau_0(i), \\ \gamma \frac{G(t)}{N(t)-N_{min}(t)} - \beta A(i,t), \tau_{min}(i) > t > \tau_0(i), \\ -\beta A(i,t), t > \tau_{min}(i). \end{cases} \tag{36}$$

2.4 Markets clearing

2.4.1 Final goods and capital markets

Now, we are in the position to demonstrate that total expenditures per capita do not grow in time and are constant. For this, consider the final goods market clearing condition:

$$E(t) = \int_{N_{min}}^{N} P(i,t)C(i,t)di = \int_{N_{min}}^{N} P(i,t)Y(i,t)di = \frac{\epsilon}{\epsilon - 1}\int_{N_{min}}^{N} L(i,t)di = \frac{\epsilon}{\epsilon - 1}L. \tag{37}$$

Since technology cancels out from product prices, final goods market clearing reduces to the proportionality of expenditures to a fraction of labour income. Total labour is assumed to be constant so that total expenditures are also constant. This follows from the choice of labour as the numeraire: prices of final goods adjust in a way such that expenditures stay constant, although consumption grows.

Using $\dot{E} = 0$ and the Euler equation, we can derive the interest rate in equilibrium:

$$\frac{\dot{E}}{E} = r - \rho = 0 \rightarrow r = \rho. \tag{38}$$

The real interest rate is constant, since financial assets are the only good unaffected by labour while prices movements cancel out.

The optimal evolution of assets can be found by solving (4) for a with $\dot{E} = 0$. Further, using $E = L\epsilon/(\epsilon - 1)$, we obtain the change in assets as

$$\dot{a} = ra - \frac{1}{\epsilon - 1}L, \tag{39}$$

which can be solved to obtain the assets as a function of time,

$$a(t) = \mathrm{e}^{rt}\left(a_0 - \frac{1}{(\epsilon - 1)r}L\right) + \frac{1}{r(\epsilon - 1)}L. \tag{40}$$

Assets accumulation is positive as long as the initial assets of households are sufficiently large:

$$a_0 > \frac{1}{\epsilon - 1} \frac{1}{r} L. \tag{41}$$

As long as (41) holds, assets increase exponentially. Since horizontal investments are constant(see Proposition 2) we have

$$\forall t : \ G(t) > 0, \tag{42}$$

and since $N(t) - N_{min}(t) \geq 0$ by definition, we also have

$$\forall t \in [\tau_0(i), \tau_{min}(i)] : \ g_i^*(t) > 0. \tag{43}$$

2.4.2 Labour market clearing

Labour market clearing condition is given if the following holds:

$$\int_{N_{min}}^{N} L^D(i,t)di = L = L \int_{N_{min}}^{N} \frac{A_i^{-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)} dj} di,$$

$$\int_{N_{min}}^{N} \frac{A_i^{-\alpha(1-\epsilon)}}{\int_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)} dj} di = \frac{\int_{N_{min}}^{N} A_i^{-\alpha(1-\epsilon)} di}{\int_{N_{min}}^{N} A_j^{-\alpha(1-\epsilon)} dj} = 1. \tag{44}$$

But this last condition is automatically satisfied, hence the labour market is cleared.

We should also like to point out that our focus is on the relation that exists between structural change and endogenous growth with both vertical and horizontal innovations and on the question of whether such a model can produce a balanced growth path and, if so, by which properties it is characterized. Therefore, we neglect adjustment mechanisms that make the economy reallocate labour and investment expenditures from one sector to another one.

3 Analysis and results

To finally solve for vertical innovations as well as for the range of existing sectors, we need the following results, derived above:

- Total expenditures are constant by (37);
- The evolution of assets is given by (40);
- Horizontal innovations are linear functions of time, given by (20).

3.1 Variety expansion

In what follows we demonstrate that structural change in the economy can be represented as a 1-dimensional shift operator over t of size $\delta\pi^R$, that is the range of existing sectors is constant but its composition varies.

First, note that (34) can be explicitly solved only after $N_{min}(t)$ has been determined. This is the range of out-dated sectors at time t. This quantity is determined by the zero profit condition of the manufacturing sector with this index:

$$N_{min}(t) : \frac{1}{\epsilon - 1} L \frac{A_{N_{min}}^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} - \Psi = 0. \tag{45}$$

The definition of $N_{max}(t)$ follows the same form with the only difference that this is the index of a sector which enters the market:

$$N_{max}(t) : \frac{1}{\epsilon - 1} L \frac{A_{N_{max}}^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} - \Psi = 0. \tag{46}$$

Comparing (45) and (46) leads to the following Proposition.

Proposition 4 *The productivity of the oldest operational sector, $A_{N_{min}}$, is equal to the productivity of the newest operational sector, $A_{N_{max}}$, at the time when the first is leaving the economy and the latter is entering its operational phase:*

$$A_{N_{min}} = \left((\Psi/L)(\epsilon - 1) \int_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj \right)^{1/\alpha(\epsilon-1)} = A_{N_{max}}. \tag{47}$$

At the same time, the productivity of each sector grows within its operational phase,

$$A_i(\tau_{min}(i)) > A_i(\tau_{max}(i)). \tag{48}$$

For any sector i these two relations are fulfilled at times $\tau_{min}(i)$ and $\tau_{max}(i)$, respectively, denoting the time of the disappearance of the sector and the time of its appearance in the economy. In both moments profits of the sector are zero, but the overall accumulated productivity differs. At $t = \tau_{max}$ the profit of sector i grows, $\dot{\Pi}(i) > 0$, while at $t = \tau_{min}$ the profit decreases, $\dot{\Pi}(i) < 0$. This makes the difference between N_{max} and N_{min}.

It can be shown that the sign of the derivative of the profit function depends on the relation

$$\dot{\Pi}(i) \lesseqqgtr 0 \Leftrightarrow \frac{\dot{A}(i)}{A(i)} - \left(\int_{N_{min}}^{N_{max}} \frac{\dot{A}(j)}{A(j)} dj \right) \lesseqqgtr 0. \tag{49}$$

Note that this implies $\dot{N}_{max} - \dot{N}_{min} = 0$. At the same time, the profit evolves differently for these two technologies:

$$\dot{\Pi}(N_{min}) < 0, \dot{\Pi}(N_{min}) > 0. \tag{50}$$

We first compute the derivative of the profit for an arbitrary technology:

$$\dot{\Pi}(i) = \frac{A(i,t)^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\alpha(\epsilon-1) \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) \right) + \frac{A(i,t)^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\frac{A(N_{max},t)\dot{N}_{max} - A(N_{min},t)\dot{N}_{min}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \right). \tag{51}$$

Making use of (47), we get

$$\dot{\Pi}(i) = \frac{A(i,t)^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{\alpha(\epsilon-1)} dj} \left(\alpha(\epsilon-1) \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) + (\Psi/L)(\epsilon-1)(\dot{N}_{max} - \dot{N}_{min}) \right). \tag{52}$$

Noting that the maximum profit for any sector i is reached at the point of $\dot{\Pi}(i) = 0$, it follows that the growth of N_{min} and N_{max} is equal and given by:

$$\dot{\Pi}(i) = 0 \Leftrightarrow \left(\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj \right) = \frac{\Psi}{\alpha L}(\dot{N}_{max} - \dot{N}_{min}). \tag{53}$$

However, the bracket in the lefthand side has to be equal to zero since the growth rate of productivity of sector i and the average growth rate of productivity in the economy are identical. Since all the technologies are symmetric except for the time of their invention, it is straightforward to state that the maximum profit for the given industry is reached at the point where its productivity grows at the average rate of the economy. Otherwise, there will be still room for improvements of the technology or the technology is already out-dated. From this it follows that $\dot{N}_{max} - \dot{N}_{min} = 0$. This proves conjectured (49) and is stated in the following Proposition.

Proposition 5 *New sectors emerge to operational phase at the same speed as older* *sectors disappear from the economy,* $\dot{N}_{max} - \dot{N}_{min} = 0$.

Moreover, it shows that the range of existing sectors in the economy is constant if $\dot{N}_{max} = \dot{N}_{min} = \dot{N}$. However, the last identity has yet to be proven.

Indeed, for the economy to be consistent it is necessary that older sectors do not disappear faster than newer sectors emerge. This is given by the condition $\dot{N}_{max} - \dot{N}_{min} = 0$. At the same time, for all the productivities to grow at the same rate it is necessary that the range $N - N_{min}$ stays constant. Otherwise, condition (47) would be violated, since newer technologies would grow faster or slower than older ones if the range was not constant. Thus, we have

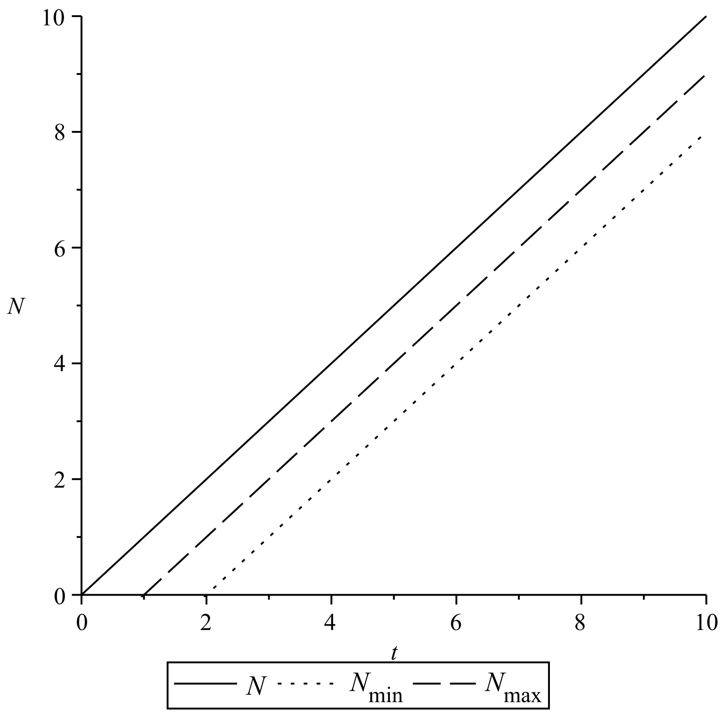
Proposition 6 *Structural change in the economy with homogeneous technologies is represented by the (left) shift operator with shift size* $\delta\pi^R$*. The expansion of variety of technologies is linear and equals the (constant) rate of structural change:*

$$\dot{N}_{max} = \dot{N}_{min} = \dot{N} = \delta\pi^R. \tag{54}$$

For the case of a constant shift length,[4] the structural change is illustrated by Figure 1.

[4] When heterogeneous technologies are assumed, $\gamma(i) \neq const$ the shift length may increase or decrease and variety expansion is non-linear since profits of R&D are not constant across technologies.

Figure 1: Linear variety expansion



3.2 Productivity growth

With the results of the last subsection, we can now determine the time $\tau_{max}(i)$ when sector i enters the market. The latter is obtained from the following two conditions:

$$\tau_{max}(i) : \frac{1}{\epsilon - 1} L \frac{A(i, \tau_{max})^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}(\tau_{max})}^{N_{max}(\tau_{max})} A(j, \tau_{max})^{\alpha(\epsilon-1)} dj} - \Psi = 0, \tag{55}$$

$$\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj > 0, \tag{56}$$

while the time of disappearance of the sector is determined by the pair:

$$\tau_{min}(i) : \frac{1}{\epsilon - 1} L \frac{A(i, \tau_{min})^{\alpha(\epsilon-1)}}{\int\limits_{N_{min}(\tau_{min})}^{N_{max}(\tau_{min})} A(j, \tau_{min})^{\alpha(\epsilon-1)} dj} - \Psi = 0, \tag{57}$$

$$\frac{\dot{A}(i,t)}{A(i,t)} - \int\limits_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)} dj < 0. \tag{58}$$

Comparing these two conditions (for the same technology i) we can see that the growth of each technology *within the time of operation*, $t \in [\tau_{max}(i), \tau_{min}(i)]$ is the same:

$$\forall i \in [N_{min}, N_{max}] : \frac{A(i, \tau_{max}(i))^{\alpha(\epsilon-1)}}{A(i, \tau_{min}(i))^{\alpha(\epsilon-1)}} = \frac{\int\limits_{N_{min}(\tau_{max}(i))}^{N_{max}(\tau_{max}(i))} A(j, \tau_{max}(i))^{\alpha(\epsilon-1)} dj}{\int\limits_{N_{min}(\tau_{min}(i))}^{N_{max}(\tau_{min}(i))} A(j, \tau_{min}(i))^{\alpha(\epsilon-1)} dj}. \tag{59}$$

The productivity growth of each technology is monotonic and proportional to all the others since the time this technology becomes profitable. Combination of (47) and (59) allows to notice that the average productivity grows in time, while the growth rates for all operating technologies are the same. As a result we obtain positive output growth despite a constant range of sectors in operation. This is widely known as the magistrale property of the dynamic system: from the time $\tau_{max}(i)$ onwards each individual technology growth is independent of its time of invention. Proposition 7 states this result.

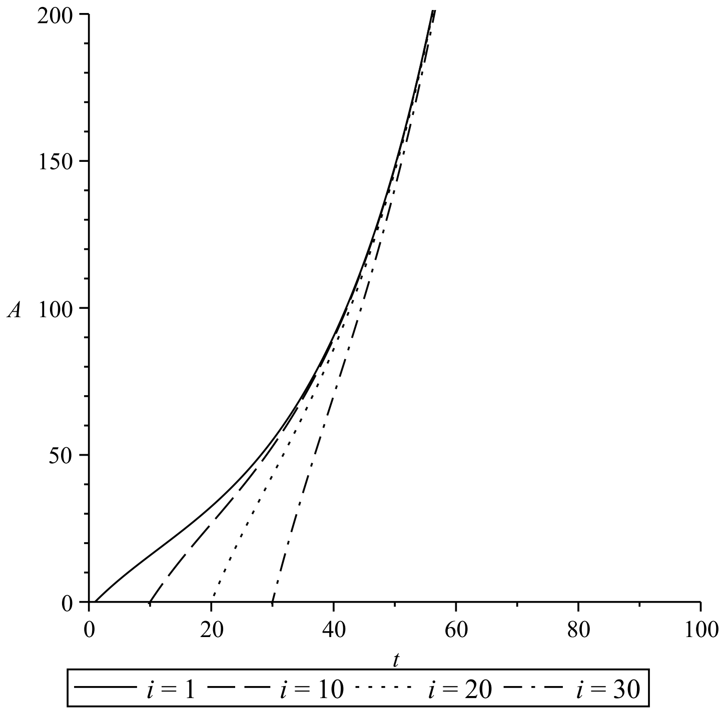
Proposition 7 *The productivities of all technologies grow at the same average speed during the time period of operational activity of the technology,*

$$\dot{A}(i) = \dot{\bar{A}} = \gamma \frac{G}{N - N_{min}} - \beta \bar{A}, \ \forall i \in [N_{min}, N_{max}], \forall t \in [\tau_{max}(i), \tau_{min}(i)], \tag{60}$$

with $\bar{A}$ denoting the average level of productivity.

Since $N - N_{min} = const$ and since $G(t)$ is given by assets minus horizontal investments, the evolution of each technology during its operating time can be recovered. It is illustrated by Figure 2 for some plausible parameters values.

Figure 2: Convergence of productivities to the magistrale



3.3 Output growth

To obtain the output growth rate, recall that aggregate output is given by:

$$Y = \int_{N_{min}}^{N_{max}} \left(\frac{A(i,t)^{\alpha\epsilon}}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}} \right) di = \frac{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj}. \tag{61}$$

The growth of output, then, is:

$$\dot{Y} = \frac{\left(\int\limits_{N_{min}}^{N_{max}} \dot{A}(i,t)^{\alpha\epsilon} di\right) \int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj - \int\limits_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon} di \left(\int\limits_{N_{min}}^{N_{max}} \dot{A}(j,t)^{\alpha(\epsilon-1)} dj\right)}{\left(\int\limits_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)} dj\right)^2} \quad (62)$$

We now can state our last Proposition:

Proposition 8 *The growth rate of the economy is constant and proportional to the growth rate of the productivities of operational technologies times the range of existing sectors (size of the shift operator),*

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{\bar{A}}}{\bar{A}} (N_{max} - N_{min}) > 0. \quad (63)$$

Proof is done by direct computation and can be found in the Appendix E.

Hence, the economy with a constant range of changing technologies exhibits positive output growth rate that is proportional to the average growth rate of the productivities of operating technologies. The latter is always positive and proportional to the growth of assets $G(t)$. Thus, the growth rate of the economy is constant for a constant range of sectors. The overall evolution of this economy can be grasped from the 3-d reconstruction at Figure 3, where $\mathbf{Q(t)}$ denotes the total productivity of the economy:

$$\mathbf{Q(t)} = \int_{N_{min}}^{N_{max}} A(i,t) di. \quad (64)$$

4 Conclusion

In this paper we have presented an endogenous growth model of a closed decentralized economy allowing for endogenous structural change, where old sectors permanently disappear and new sectors come into existence. The out-dating of sectors happens due

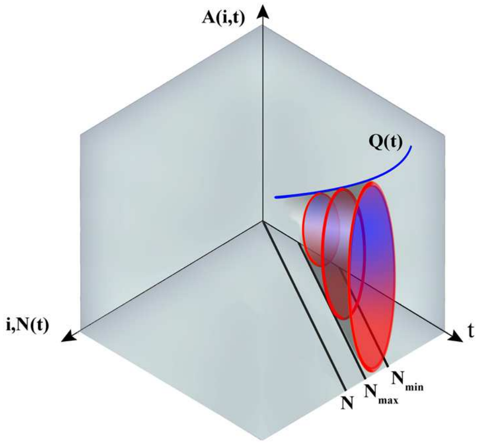


Figure 3: Reconstruction of the economy with endogenous structural change

to the presence of the limited potential of a given technology for quality improvement and due to competitive pressure in the R&D sector. Thus, we are able to present a model that can replicate the real-world phenomena of structural change and ongoing growth by allowing for vertical and horizontal innovations simultaneously.

The overall R&D process is determined by the profits resulting from selling patents to the final goods producers in a given sector. These patents transform monopolistic profits of the manufacturing sector into the resources used for innovative activity in the spirit of Romer (1990). However, the inclusion of productivity growth for all new technologies makes it possible to account for the endogenous process of out-dating of sectors and associated technologies. This result is possible due to the careful consideration of the patent price for a technology as the total additional profit of the manufacturing sector, and not just as the price for the increase in productivity, as in Peretto and Connolly (2007). The evolution of the economy is proportional to the productivity growth in the same way as in the aforementioned paper, but we are able to model structural change as an endogenous phenomenon.

In real economies, patents for technologies do not last forever but are limited to grant incentives for new innovations, following the arguments established already by Nordhaus (1967). Thus, it seems natural that an endogenously determined patents duration should foster structural change through competitive pressure in the R&D sector. The study of Chu et al. (2012) already stressed the importance of the degree of patent protection for the relative speed of vertical and horizontal innovations. However, it was not able to model structural change as an endogenous phenomenon because patents are of an infinite duration in that model.

The key assumption for sustained growth in our framework is the unlimited nature of horizontal innovations and the fact that all of the technologies are symmetric and homogeneous. It would be of interest to extend the model to heterogeneous technologies. This would take into consideration non-constant growth rates that depend on the structure of the space of ideas and on the speed of horizontal innovations.

In the current form, our model does not include any notion of government and taxation and, thus, policy implications cannot be drawn. However, the construction of a model with endogenous structural change is necessary to improve our understanding of the optimal governance of technical change with regard to topical environmental issues, such as climate change for example. Assuming that technologies differ with respect to their pollution intensity, the central authority would possibly want to speed up structural change. However, it is often argued that newer technologies are harder to develop. Taking into account that aspect would lead to heterogeneous innovations in our model. Thus, in order to bring the framework closer to environmental concerns, the heterogeneity of technologies regarding the environment and regarding investment efforts should be modelled. This is considered to be a prospect for future research with the concept of structural change as suggested in the present paper.

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Appendices

A Households and firms optimality conditions

A.1 Derivations for the household

The derivation of equation (6): The Lagrangian of the household is

$$L = \left[\int_{N_{min}}^{N_{max}} C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \left(\int_{N_{min}}^{N_{max}} P_i C_i di - rK + \dot{K} + W \right) . \tag{A.1}$$

The first order condition for consumption good i is

$$C_i^{-\frac{1}{\varepsilon}} C^{\frac{1}{\varepsilon}} = \lambda P_i . \tag{A.2}$$

check power of C Taking the F.O.C. for i and for j and substituting in yields

$$C_i = C_j \left(\frac{P_i}{P_j} \right)^{-\varepsilon} . \tag{A.3}$$

Substituting this back into the equation for expenditure, equation (5) yields

$$C_j \left(\frac{1}{P_j} \right)^{-\varepsilon} \int_{N_{min}}^{N_{max}} P_i^{1-\varepsilon} di = E , \tag{A.4}$$

which can be rearranged to yield expression (6).

A.2 Derivations for the manufacturing sector

The derivation of equation (10): The output by an individual firm Y_i equals to the consumption of that good C_i, so that we can insert equation (6) into the profit function:

$$\begin{aligned} \Pi_i &= P_i Y_i - L_i - \Psi = P_i Y_i - Y_i A_i^{-\alpha} - \Psi \\ &= P_i E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} - E \frac{P_i^{-\varepsilon}}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} A_i^{-\alpha} - \Psi. \end{aligned} \tag{A.5}$$

We use further the assumption of zero mass of each individual product in the price index

$$\frac{\partial \int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj}{\partial P_i} = 0, \tag{A.6}$$

which is usual when the continuum of goods is employed, see for example Peretto and Connolly (2007). Maximizing profit with respect to the price under this non-atomic assumption yields

$$\frac{\partial \Pi_i}{\partial P_i} = \frac{E}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} (1-\varepsilon) P_i^{-\varepsilon} - \frac{E}{\int_{N_{min}}^{N_{max}} P_j^{1-\varepsilon} dj} P_i^{-\varepsilon-1} (-\varepsilon) A_i^{-\alpha} = 0 \, . \tag{A.7}$$

The price is thus

$$P_i = \frac{\varepsilon}{\varepsilon - 1} A_i^{-\alpha} \, . \tag{A.8}$$

B Proof for Proposition 1

1. Using equation (14) we can write the price of a patent i as

$$p_A(i) = \int_{N^{-1}(i)}^{\infty} \mathrm{e}^{-r(t-N^{-1}(i))} \Pi_i dt, \tag{B.1}$$

where $N^{-1}(i)$ is the time when technology i is invented.

2. Denote $\tau_0(i) = N^{-1}(i), \tau_{max}(i) = N^{-1}_{max}(i), \tau_{min}(i) = N^{-1}_{min}(i)$, as the time of the invention of a technology, of it becoming profitable and of it going out of production, respectively.

3. Note that $N \geq N_{max} \geq N_{min}$ implies $\tau_0(i) \leq \tau_{max}(i) \leq \tau_{min}(i)$ as long as $N(t)$ is a non-decreasing function. This last is true as long as $u(t) \geq 0$, which is required by the formulation of the horizontal innovations problem, (16).

4. The profit of a manufacturing firm in sector i is given by (9). Substituting for prices, labour and technology in it, one gets:

$$\Pi_i = P_i Y_i - L_i - \Psi = \frac{\epsilon}{\epsilon - 1} A_i^{-\alpha} A_i^{\alpha} L_i - L_i - \Psi = \left(\frac{\epsilon}{\epsilon - 1} - 1\right) L_i - \Psi = \frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi. \tag{B.2}$$

5. The profit is nonnegative only within the interval $t \in [\tau_{max}(i), \tau_{min}(i)]$ such that the patent price is defined also for that interval.

6. Inserting this into the patent price one gets:

$$p_A(i) = \int\limits_{\tau_{max}(i)}^{\tau_{min}(i)} \mathrm{e}^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right) dt. \tag{B.3}$$

7. Formally, taking the definite integral amounts to the difference between two values

of the antiderivative:

$$p_A(i) = \mathbf{F}|_{t=\tau_{min}(i)} \left(\mathrm{e}^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right)\right) -$$
$$- \mathbf{F}|_{t=\tau_{max}(i)} \left(\mathrm{e}^{-r(t-\tau_0(i))} \left(\frac{1}{\epsilon} E \frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)} dj} - \Psi \right)\right). \tag{B.4}$$

8. Without explicit computation of this expressions it is straightforward to see that the resulting patent price is not a function of time, but a difference of two values of such a function at fixed points in time:

$$p_A(i) = \mathbf{F}(i, \tau_{min}(i), \tau_{max}(i)) \neq f(t), \tag{B.5}$$

since $\tau_{min}(i), \tau_{max}(i)$ are functions of the technology index i and are not time-varying. ■.

C Proof of Proposition 2

The HJB equation for the problem given by (17), (16) is:

$$rV = \max_{u(\bullet)} \left\{ \delta\pi^R(i)|_{i=N} u(t) - \frac{1}{2}u^2(t) + \frac{\partial V}{\partial N}\delta u(t) \right\}. \tag{C.1}$$

Taking F.O.C. we have

$$u^* = \delta\pi^R(i)|_{i=N} + \delta\frac{\partial V}{\partial N}. \tag{C.2}$$

Substituting back into the HJB equation, we find that it can be satisfied only for $V = const.$, as long as $\pi^R(i,t)|_{i=N}$ is constant.

This last has to be constant, since there is a free entry condition for vertical innovations: if some of the technologies yielded higher profits, all of the resources would go into the development of only those more profitable technologies. However, the investments are symmetric, thus, requiring constant and equal profits across technologies. Hence, we have

$$u^* = \delta\pi^R(i)|_{i=N} = \delta\pi^R. \tag{C.3}$$

■.

D Proof of Proposition 3

Using Fubini's theorem from (B.3) we can put differentiation sign under the integration term:

$$\frac{\partial p_A(i)}{\partial A_i} = \frac{\partial\left(\int\limits_{\tau_{max}(i)}^{\tau_{min}(i)} \mathrm{e}^{-r(t-\tau_0(i))}\left(\frac{1}{\epsilon}E\frac{A_i^{-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj} - \Psi\right)dt\right)}{\partial A_i} =$$

$$\int\limits_{\tau_{max}(i)}^{\tau_{min}(i)} \mathrm{e}^{-r(t-\tau_0(i))}\left(\frac{E}{\epsilon}\frac{\frac{\partial A_i^{-\alpha(1-\epsilon)}}{\partial A_i}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj}\right)dt = -\int\limits_{\tau_{max}(i)}^{\tau_{min}(i)} \mathrm{e}^{-r(t-\tau_0(i))}\left(\frac{E}{\epsilon}\frac{\alpha(1-\epsilon)A_i^{1-\alpha(1-\epsilon)}}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj}\right)dt. \tag{D.1}$$

Taking the integral in the same way as in Proposition 1, we have

$$\frac{\partial p_A(i)}{\partial A_i} = \mathbf{F}|_{t=\tau_{max}(i)}\left(\mathrm{e}^{-r(t-\tau_0(i))}\frac{1}{\epsilon}E\frac{\alpha(1-\epsilon)}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj}A_i^{1-\alpha(1-\epsilon)}\right) -$$

$$-\mathbf{F}|_{t=\tau_{min}(i)}\left(\mathrm{e}^{-r(t-\tau_0(i))}\frac{1}{\epsilon}E\frac{\alpha(1-\epsilon)}{\int\limits_{N_{min}}^{N_{max}} A_j^{-\alpha(1-\epsilon)}dj}A_i^{1-\alpha(1-\epsilon)}\right). \tag{D.2}$$

This amounts to some function of the increase in productivity A_i from time $\tau_{max}(i)$ until $\tau_{min}(i)$. With symmetric technologies this growth would be the same for all i, although the points in time $\tau_{min}(i), \tau_{max}(i)$ will be different. Note that this expression does not depend on the variable A_i, but only on the level of it at two fixed points in time. This proves that $\partial p_A(i)/\partial A_i = c$ ■.

E Proof of Proposition 8

The direct calculation of output growth rates yields

$$\frac{\dot{Y}}{Y} = \frac{d/dt\left(\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon}di\right)}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon}di} - \frac{d/dt\left(\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}dj\right)}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}dj} \overset{\dot{N}_{min}=\dot{N}_{max}}{=}$$

$$= \frac{\int_{N_{min}}^{N_{max}} \left(\dot{A}(i,t)^{\alpha\epsilon}di\right)}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon}di} - \frac{\int_{N_{min}}^{N_{max}} \left(\dot{A}(j,t)^{\alpha(\epsilon-1)}\right)dj}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}dj} =$$

$$= \alpha\epsilon\frac{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon-1}\dot{A}(i,t)di}{\int_{N_{min}}^{N_{max}} A(i,t)^{\alpha\epsilon}di} - \alpha(\epsilon-1)\frac{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)-1}\dot{A}(j,t)dj}{\int_{N_{min}}^{N_{max}} A(j,t)^{\alpha(\epsilon-1)}dj} =$$

$$= \alpha\epsilon\int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)}di - \alpha(\epsilon-1)\int_{N_{min}}^{N_{max}} \frac{\dot{A}(j,t)}{A(j,t)}dj = \alpha\int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)}di.$$

Using $\dot{A}(i,t) = G/(N - N_{min}) - \beta A(i,t)$ and $A(i,t) = \bar{A}$ we get,

$$\frac{\dot{Y}}{Y} = \alpha\int_{N_{min}}^{N_{max}} \frac{\dot{A}(i,t)}{A(i,t)}di = \alpha\frac{G}{N - N_{min}}\int_{N_{min}}^{N_{max}} A(i,t)^{-1}di - \alpha\int_{N_{min}}^{N_{max}} \beta di =$$

$$= \alpha\frac{G}{N - N_{min}}\frac{N_{max} - N_{min}}{\bar{A}} - \alpha\beta(N_{max} - N_{min}) = \alpha\left(N_{max} - N_{min}\right)\frac{\dot{\bar{A}}}{\bar{A}}. \quad \text{(E.1)}$$

■