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Novel results for the anisotropic sparse grid quadrature

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(joint work with Abdul-Lateef Haji-Ali, Michael Peters, Markus Siebenmorgen)

1. Introduction

The anisotropic sparse grid quadrature can be applied for high-dimensional integrands which are analytically extendable into an anisotropic tensor product domain. Taking into account this anisotropy, we end up with a dimension independent error versus cost estimate of the proposed quadrature. In addition, we provide a novel and sharp estimate for the cardinality of the underlying anisotropic index set. To validate the theoretical findings, we present numerical results which demonstrate the remarkable convergence behaviour of the anisotropic sparse grid quadrature in applications.

2. Anisotropic sparse grid quadrature

We introduce anisotropic sparse grid quadrature formulas which extend the original idea of Smolyak's construction from [4]. Consider an increasing sequence of univariate quadratures

$$Q_j : C([-1,1]) \to \mathbb{R}, \quad f \mapsto Q_j f = \sum_{i=1}^{N_j} w_{i,j} f(\xi_{i,j}),$$

where the number N_j of quadrature points satisfies $N_1 \leq N_2 \leq \cdots$ and $N_j \to \infty$ for $j \to \infty$. For given $j \in \mathbb{N}$, we further introduce the difference quadrature operator

$$\Delta_j := Q_j - Q_{j-1}, \quad \text{where} \quad Q_{-1} := 0.$$

Let $\mathbf{w} = (w_1, w_2, \ldots, w_m) \in \mathbb{R}_+^m$ denote a weight vector for the different parameter dimensions. We assume in the following that the weight vector is sorted in ascending order, i.e. $w_1 \leq w_2 \leq \cdots \leq w_m$. Otherwise, we would rearrange the particular dimensions accordingly.

We define the index set

$$X_{\mathbf{w}}(q,m) := \left\{ \mathbf{0} \le \boldsymbol{\alpha} \in \mathbb{N}^m : \sum_{n=1}^{m} \alpha_n w_n \le q \right\}. \tag{1}$$

The *anisotropic sparse grid quadrature operator* of level $q \in \mathbb{N}$ is thus defined by

$$\mathcal{A}_{\mathbf{w}}(q,m) := \sum_{\boldsymbol{\alpha} \in X_{\mathbf{w}}(q,m)} \Delta_{\alpha_1}^{(1)} \otimes \cdots \otimes \Delta_{\alpha_m}^{(m)}. \tag{2}$$

3. Main results

3.1. **Cardinality of the index set.** For computing the number of quadrature points which the quadrature operator $\mathcal{A}_{\mathbf{w}}(q,m)$ applies, we need a sharp estimate on the index set $X_{\mathbf{w}}(q,m)$ from (1). Existing estimates are the well-known Beged-Dov formula (cf. [1])

$$\# X_{\mathbf{w}}(q,m) \le \prod_{n=1}^{m} \frac{q + \|\mathbf{w}\|_1}{n w_n}$$

or the product estimate

$$\# X_{\mathbf{w}}(q,m) \le \prod_{n=1}^{m} \left(\left\lfloor \frac{q}{w_n} \right\rfloor + 1 \right),$$

which is related to the full tensor product quadrature operator. Nonetheless, both estimates are not very sharp and the following, new estimate is much better.

Lemma 1. *The cardinality of the set $X_{\mathbf{w}}(q,m)$ is bounded by*

$$\# X_{\mathbf{w}}(q,m) \le \prod_{n=1}^{m} \left(\frac{q}{n w_n} + 1 \right).$$

3.2. **Error estimate.** We should next provide an error estimate for the anisotropic sparse grid quadrature operator (2). To that end, we should specify the univariate quadratures Q_j. They are supposed to be the *Gauss-Legendre quadrature rule* on $\Gamma := [-1,1]$ with $N_j = \left\lceil \frac{1}{2}(j+2) \right\rceil$ quadrature points. The class of integrands $f_m : [-1,1]^m \to \mathbb{R}$ we consider are functions which admit an analytic extension into the region

$$\boldsymbol{\Sigma}_m = \bigtimes_{n=1}^{m} \{ z \in \mathbb{C} : \operatorname{dist}(z, \Gamma) \le \tau_n \}$$

with $\tau_n \ge c n^r$ $(r > 1)$.

Lemma 2. *Let the weight vector $\mathbf{w}$ in (1) be given by $w_n = \log(\kappa_n)$, where $\kappa_n := \tau_n + \sqrt{1 + \tau_n^2}$. Then, for each $\delta > 0$, there exists a constant $c(\delta)$, independent of m, such that the error of the anisotropic sparse quadrature is bounded by*

$$\left| \big(\mathbf{I} - \mathcal{A}_{\mathbf{w}}(q,m) \big) f_m \right| \le c(\delta, \boldsymbol{\tau}) \exp\big(-q(1-\delta) \big) \| f_m \|_{C(\boldsymbol{\Sigma}_m)}$$

with $c(\delta, \boldsymbol{\tau}) = 4 c(\delta) \left\| \{ \tau_n^{-1} \}_n \right\|_{\ell^1}$. Note that $c(\delta) \to \infty$ as $\delta \to 0$.

3.3. **Cost complexity.** The number of quadrature points can be bounded by

$$N(q) := \operatorname{cost}\big(\mathcal{A}_{\mathbf{w}}(q,m)\big) \leq \#X_{\mathbf{w}}(q,m)^2.$$

Further, one has

$$\#X_{\mathbf{w}}(q,m) \leq c(r)\exp\left(\frac{q}{r}\log\big(\log(m)\big)\right) = c(r)\log(m)^{q/r}$$

with a constant $c(r)$ which is independent of m. By combining these estimates with Lemma 2, we derive the convergence rate

$$\big|\big(\mathbf{I} - \mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| \leq c(\delta,\boldsymbol{\tau})N(q)^{-r(1-\delta)/(2\log(\log m))}. \tag{3}$$

However, this estimate is not dimension robust, i.e., it is not independent of the dimension m. By using a result of [2], we arrive at following cost complexity.

Theorem 3. *The error of the anisotropic sparse grid quadrature with* $w_n = \log(\kappa_n)$ *can be bounded in terms of the number* $N(q)$ *of quadrature points according to*

$$\big|\big(\mathbf{I} - \mathcal{A}_{\mathbf{w}}(q,m)\big)f_m\big| \leq c(\boldsymbol{\tau},\beta)N(q)^{-(\beta-1)/2} \tag{4}$$

for all $\beta < r$.

Notice that this rate of convergence is smaller than that in (3) whenever m is fixed.

4. Numerical results

For our numerical tests, we consider a simple quadrature problem, namely, we like to integrate

$$f_m\colon \Gamma^m \to \mathbb{R}, \quad f_m(\mathbf{y}) := 5\left(3 + \sum_{n=1}^{m} n^{-s}y_n\right)^{-1}. \tag{5}$$

The parameter s is chosen according to $s = 2,3,4$. Respective reference solutions are computed for $m = 1000$ dimensions and verified by a quasi-Monte Carlo method.

To validate the rate of convergence (4) with respect to the number of quadrature points N, we approximate the integral (5) for the limit choice $\tau_n = n^{s-1}$ by the m-dimensional anisotropic sparse grid quadrature. In particular, to catch the inherent dimensionality for each choice of the parameter s, we consider $m = 10, 100, 1000$ dimensions. As found in Figure 1 on the next page, we obtain different rates of convergence, dependent on the choice of the parameter s. Especially, the inherent dimension is larger than $m = 100$ only in the case $s = 2$. We observe the rates of convergence $N^{-1.44}$ for $s = 2$, $N^{-2.21}$ for $s = 3$, and N^{-3} for $s = 4$. These are much better than $N^{-1/2}$, N^{-1}, and $N^{-3/2}$, respectively, which are predicted by Theorem 3. Indeed, the rates are at least twice as much as predicted, which issues from the fact that the factor 1/2 in the exponent on the right hand side of (4) issues from the crude estimate $N(q) \leq \#X_{\mathbf{w}}(q,m)^2$, see [3] for details.

More advanced examples from the uncertainty quantification of boundary value problems with random input parameters can be found in [3].

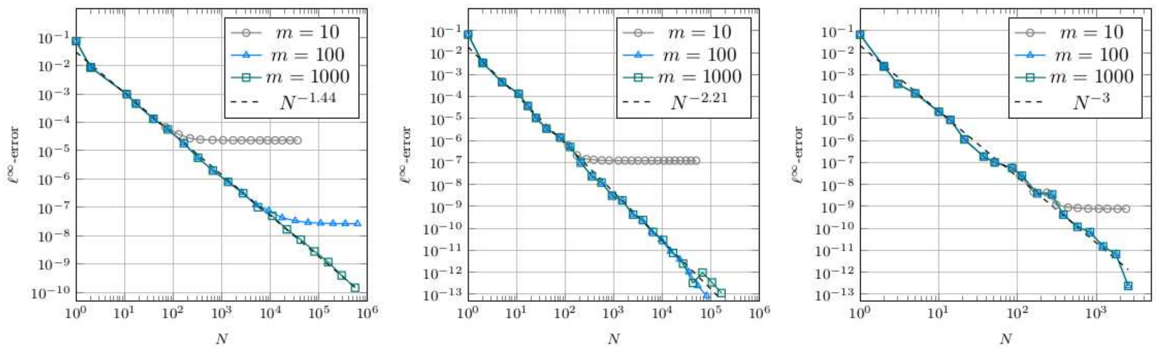

Figure 1. Rates of convergences for the $m = 10, 100, 1000$ dimensions and $s = 2$ (left), $s = 3$ (middle), and $s = 4$ (right).

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Optimal feedback control of semilinear parabolic equations: A "high"-dimensional HJB approach

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(joint work with Dante Kalise)

A procedure for the numerical approximation of high-dimensional Hamilton-Jacobi-Bellman (HJB) equations associated to optimal feedback control problems for semilinear parabolic equations is proposed. Its main ingredients are a pseudospectral collocation approximation of the PDE dynamics, and an iterative method for the nonlinear HJB equation associated to the feedback synthesis. The latter is known as the Successive Galerkin Approximation. It can also be interpreted as Newton iteration for the HJB equation. At every step, the associated linear Generalized HJB equation is approximated via a separable polynomial approximation ansatz. The method requires a stabilizing control as initialisation. Its availability depends on the specific control system. If such a control can not be obtained, then the use of a discount factor and a continuation procedure as the discount factor tends to zero are proposed. Stabilizing feedback controls are obtained from solutions to the HJB equations for systems of dimension up to fourteen, i.e. the dimension of the pseudospectral approximation of the infinite dimensional dynamics has dimension fourteen. Further information can be found in [1].