Bell Correlations in a Many-Body System with Finite Statistics

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A recent experiment reported the first violation of a Bell correlation witness in a many-body system [Science **352**, 441 (2016)]. Following discussions in this Letter, we address here the question of the statistics required to witness Bell correlated states, i.e., states violating a Bell inequality, in such experiments. We start by deriving multipartite Bell inequalities involving an arbitrary number of measurement settings, two outcomes per party and one- and two-body correlators only. Based on these inequalities, we then build up improved witnesses able to detect Bell correlated states in many-body systems using two collective measurements only. These witnesses can potentially detect Bell correlations in states with an arbitrarily low amount of spin squeezing. We then establish an upper bound on the statistics needed to convincingly conclude that a measured state is Bell correlated.

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Introduction.—Bell nonlocality, as revealed by the violation of a Bell inequality, constitutes one of the strongest forms of nonclassicality [1,2]. However, its demonstration has long been restricted to systems involving few particles [3–7]. Recently, the discovery of multipartite Bell inequalities that only rely on one- and two-body correlators opened up new possibilities [8]. Although these inequalities have not yet lead to the realization of a multipartite Bell test, they have been used to derive witnesses able to detect Bell correlated states, i.e., states capable of violating a Bell inequality [9,10].

These witnesses have triggered two experiments [9,11] which successfully detect the presence of Bell correlations in a many-body system under the assumption of Gaussian statistics [12,13]. The witness used in Refs. [9,11] involves one- and two-body correlation functions and takes the form $\mathcal{W} \geq 0$, where the inequality is satisfied by measurements on states that are not Bell correlated. Observation of a negative value for $\mathcal{W}$ then leads to the conclusion that the measured system is Bell correlated. However, reaching such a conclusion in the presence of finite statistics requires special care [14,15]. In particular, an assessment of the probability with which a non-Bell-correlated state could be responsible for the observed data is required before concluding about the presence of Bell correlations without further assumptions.

Concretely, the witness of Refs. [9] has the property of admitting a quantum violation lower bounded by a constant $\mathcal{W}_{\text{opt}} < 0$, while the largest possible value $\mathcal{W}_{\max} > 0$ is achievable by a product state and increases linearly with the size of the system N. These properties imply that a small number of measurement rounds on a state of the form

$$\rho = (1-q)|\psi\rangle\langle\psi| + q(|\uparrow\rangle\langle\uparrow|)^{\otimes N}, \tag{1}$$

where $\mathcal{W}(|\psi\rangle) = \mathcal{W}_{\text{opt}}$, $\mathcal{W}(|\uparrow\rangle^{\otimes N}) = \mathcal{W}_{\max}$ and q is small, is likely to produce a negative estimate of $\mathcal{W}$, even though the state is not detected by the witness in the limit of infinitely many measurement rounds [9]. This state thus imposes a lower bound on the number of measurement rounds required to exclude, through such witnesses, all non-Bell-correlated states with high confidence. Contrary to other assessments, this lower bound increases with the number of particles involved in the many-body system. Therefore, it is not captured by the standard deviation of one- and two-body correlation functions (which on the contrary decreases as the number of particles increases).

For small systems, this dependence of the number of measurement rounds on the size of the measured system merely represents a technical overhead: a conclusion may still be obtained at the price of performing few more measurements. For large systems, however, any bound on the number of measurements that can be performed imposes a hard limit on the maximal size of systems on which a reliable conclusion can be drawn. The question of statistical significance thus constitutes a fundamental question for many-body systems.

It is worth noting that states of the form (1) put similar bounds on the number of measurement rounds required to perform any hypothesis tests in a many-body system satisfying the conditions above. This includes in particular tests of entanglement [16–19] based on the entanglement witnesses of Ref. [20–22].

In this Letter, we address this statistical problem in the case of Bell correlation detection by providing a number of measurement rounds sufficient to exclude non-Bell-correlated states from an observed witness violation. Let us mention that in Refs. [9,11], this finite statistics issue is circumvented by the addition of an assumption on the set of local states being tested. This has the effect of reducing the scope of the conclusion: the data reported in Refs. [9,11], are only able to exclude a subset of all non-Bell-correlated states (as pointed out in the references). Here, we show that such additional assumptions are not required in experiments on many-body systems, and thus argue that they should be avoided in the future.

In order to minimize the amount of statistics required to reach our conclusion, we start by investigating improved Bell correlation witnesses. For this, we first derive Bell inequalities with two-body correlators and an arbitrary number of settings. This allows us to obtain Bell correlation witnesses that are more resistant to noise compared to the one known to date [9]. We then analyse the statistical properties of these witnesses and provide an upper bound on the number of measurement rounds needed to rule out all local states in a many-body system. We show that this upper bound is linear in the number of particles, hence demonstrating the possibility of reliable detection of Bell correlations in systems with a large number of particles.

Symmetric two-body correlator Bell inequalities with an arbitrary number of settings.—Multipartite Bell inequalities that are symmetric under exchange of parties and which involve only one- and two-body correlators have been proposed in scenarios where each party uses two measurement settings and receives an outcome among two possible results [8]. Similar inequalities were also obtained for translationally invariant systems [23], or based on Hamiltonians [24]. Here, we derive a similar family of Bell inequalities that is invariant under arbitrary permutations of parties but allows for an arbitrary number of measurement settings per party.

Let us consider a scenario in which N parties can each perform one of m possible measurements $M_k^{(i)}$ ($k = 0, \ldots, m-1$; $i = 1, \ldots, N$) with binary outcomes ± 1. We write the following inequality:

$$I_{N,m} = \sum_{k=0}^{m-1} \alpha_k S_k + \frac{1}{2}\sum_{k,l} S_{kl} \geq -\beta_c, \qquad (2)$$

where $\alpha_k = m - 2k - 1$, β_c is the local bound, and the symmetrized correlators are defined as

$$S_k := \sum_{i=1}^{N} \langle M_k^{(i)} \rangle, \qquad S_{kl} := \sum_{i \neq j} \langle M_k^{(i)} M_l^{(j)} \rangle. \qquad (3)$$

Let us show that Eq. (2) is a valid Bell inequality for $\beta_c = \lfloor (m^2 N/2) \rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller or equal to x. Below, we assume that m is even; see Appendix A in the Supplemental Material [25] for the case of odd m.

Since $I_{N,m}$ is linear in the probabilities and local behaviors can be decomposed as a convex combination of deterministic local strategies, the local bound of Eq. (2) can be reached by a deterministic local strategy [1]. We thus restrict our attention to these strategies and write

$$\langle M_k^{(i)} \rangle = x_k^i = \pm 1 \quad \Rightarrow S_{kl} = S_k S_l - \sum_{i=1}^{N} x_k^i x_l^i, \qquad (4)$$

where x_k^i is the (deterministic) outcome party i produces when asked question k. This directly leads to the following decomposition:

$$I_{N,m} = \sum_{k=0}^{\frac{m}{2}-1} \alpha_k (S_k - S_{m-k-1}) + \frac{1}{2} B^2 - \frac{1}{2} C \geq -\beta_c, \qquad (5)$$

with $B := \sum_{k=0}^{m-1} S_k$ and $C := \sum_{i=1}^{N} (\sum_{k=0}^{m-1} x_k^i)^2$. Because of the symmetry under exchange of parties of this Bell expression, it is convenient to introduce, following Ref. [8], variables counting the number of parties that use a specific deterministic strategy:

$$a_{j_1<\ldots<j_n} := \#\{i \in \{1, \ldots, N\} | x_k^i = -1 \mathrm{iff} k \in \{j_1, \ldots, j_n\}\},$$
$$\bar{a}_{j_1<\ldots<j_n} := \#\{i \in \{1, \ldots, N\} | x_k^i = +1 \mathrm{iff} k \in \{j_1, \ldots, j_n\}\},$$
$$n \leq \frac{m}{2}, \qquad \bar{a}_{j_1,\ldots,j_{\frac{m}{2}}} \equiv 0, \qquad (6)$$

where # denotes the set cardinality. Since each party has to choose a strategy, the variables sum up to N:

$$\sum_{\text{all variables}} = \sum_{n=0}^{\frac{m}{2}} \sum_{j_1<\ldots<j_n} (a_{j_1\ldots j_n} + \bar{a}_{j_1\ldots j_n}) = N. \qquad (7)$$

The correlators can now be expressed as

$$S_k = \sum_{n=0}^{\frac{m}{2}} \sum_{j_1<\ldots<j_n} (a_{j_1\ldots j_n} - \bar{a}_{j_1\ldots j_n}) y_k^{j_1\ldots j_n}, \qquad (8)$$

with $y_k^{j_1\ldots j_n} = -1$ if $k \in \{j_1, \ldots, j_n\}$, and $+1$ otherwise.

The first term of (5) concerns the difference between two correlators. Let us see how this term decomposes as a function of the number of indices present in its variables. From Eq. (8), it is clear that a variable with n indices only appears in the difference $S_k - S_l$ if $y_k^{j_1\ldots j_n} \neq y_l^{j_1\ldots j_n}$. But the corresponding strategy only has n differing outcomes and each correlator in this term only appears once, so a variable with n indices appears in at most n of these differences. Moreover, if it appears, it does so with a factor ± 2. The coefficient in front of a variable with n indices in the first sum of Eq. (5) thus cannot be smaller than $-2\sum_{k=0}^{n-1} \alpha_k = 2n(n-m)$.

The second term of Eq. (5) can be bounded as $B^2 \geq 0$, while the third one can be expressed as

$$C = \sum_{n=0}^{\frac{m}{2}} \sum_{j_1<\ldots<j_n} (a_{j_1\ldots j_n} + \bar{a}_{j_1\ldots j_n})(m-2n)^2. \qquad (9)$$

Putting everything together and using property (7), we arrive at

$$I_{N,m} \geq \sum_{k=0}^{\frac{m}{2}-1} \alpha_k (S_k - S_{m-k-1}) - \frac{1}{2} C$$
$$\geq -\frac{m^2}{2} \sum_{\text{all variables}} = -\frac{m^2 N}{2} = -\beta_c, \qquad (10)$$

which concludes the proof.

Note that this bound is achieved for $a_{01\ldots(m/2)-1} = N$, i.e., when for each party exactly the first half of the m measurements yields the result -1. Note also that the Bell inequality (2) does not reduce to Eq. (6) of Ref. [8] when $m = 2$. Indeed, while none of these inequalities is a facet of the local polytope, the latter one is a facet of the symmetrized 2-body correlator local polytope [8,30].

From Bell inequalities to Bell-correlation witnesses.— Let us now derive a set of Bell-correlation witnesses assuming a certain form for the measurement operators. Here, no assumptions are made on the measured state.

Following Ref. [9], we start from inequality (2) and introduce spin measurements along the axes $\vec{d}_k$, $k=0,\ldots,m-1$, as well as the collective spin observables $\hat{S}_k$:

$$M_k^{(i)} = \vec{d}_k \cdot \vec{\sigma}^{(i)}, \qquad \hat{S}_k = \frac{1}{2}\sum_{i=1}^{N} M_k^{(i)}, \tag{11}$$

where $\vec{\sigma}$ is the Pauli vector acting on a spin-$\frac{1}{2}$ system. The correlators can be expressed in terms of these total spin observables and the measurement directions [8]:

$$S_k = 2\langle \hat{S}_k \rangle, \quad S_{kl} = 2[\langle \hat{S}_k \hat{S}_l \rangle + \langle \hat{S}_l \hat{S}_k \rangle] - N\vec{d}_k \cdot \vec{d}_l. \tag{12}$$

This defines the Bell operators

$$\hat{W}_{N,m} \coloneqq 2\sum_{k=0}^{m-1} \alpha_k \hat{S}_k + 2\sum_{k,l} \hat{S}_k \hat{S}_l - \frac{N}{2}\sum_{k,l} \vec{d}_k \cdot \vec{d}_l + \left\lfloor \frac{m^2 N}{2} \right\rfloor, \tag{13}$$

whose expectation values are positive for states that are not Bell correlated. Note that the expectation value of these operators need not be negative for all Bell correlated states and every choice of measurement directions, though. A negative value may only be achieved for specific choices of states and measurement settings.

We now consider measurement directions $\vec{d}_k = \vec{a}\cos(\vartheta_k) + \vec{b}\sin(\vartheta_k)$ lying in a plane spanned by two orthonormal vectors $\vec{a}$ and $\vec{b}$, with the antisymmetric angle distribution $\vartheta_{m-k-1} = -\vartheta_k$. Note that the coefficients α_k share the same antisymmetry. Defining $\mathcal{W}_m \coloneqq \langle \hat{W}_{N,m}/(2\hat{N}) \rangle$ for even m, we arrive at the following family of witnesses:

$$\mathcal{W}_m = \mathcal{C}_b \sum_{k=0}^{\frac{m}{2}-1} \alpha_k \sin(\vartheta_k) - (1-\zeta_a^2)\left(\sum_{k=0}^{\frac{m}{2}-1} \cos(\vartheta_k)\right)^2 + \frac{m^2}{4}, \tag{14}$$

with $\mathcal{W}_m \geq 0$ for states that are not Bell correlated. These Bell correlation witnesses depend on $m/2$ angles ϑ_k and involve just two quantities to be measured: the scaled collective spin $\mathcal{C}_b \coloneqq \langle \hat{S}_{\vec{b}}/(\hat{N}/2) \rangle$ and the scaled second moment $\zeta_a^2 \coloneqq \langle \hat{S}_{\vec{a}}^2/(\hat{N}/4) \rangle$.

The tightest constraints on $\mathcal{C}_b$ and ζ_a^2 that allow for a violation of $\mathcal{W}_m \geq 0$ are obtained by minimizing $\mathcal{W}_m$ over the angles ϑ_k. Solving $\partial \mathcal{W}_m / \partial \vartheta_k = 0$ yields (see Appendix B in the Supplemental Material [25]):

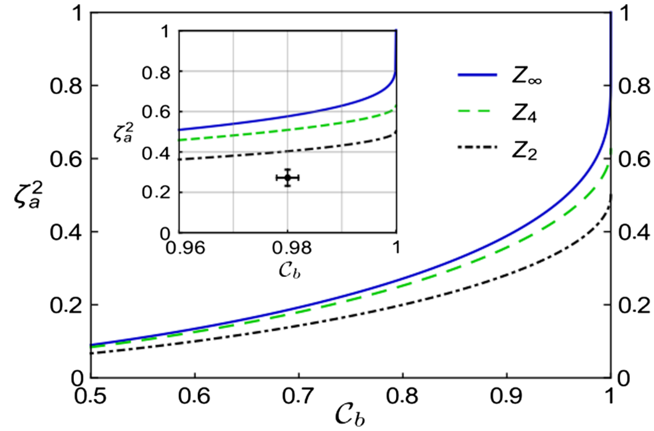

FIG. 1. Plots of the critical lines Z_2, Z_4, and Z_∞. The witness obtained from the Bell inequality with 4 settings already provides a significant improvement over the case of 2 settings. The black point in the inset shows the data point from Ref. [9], with $N = 476 \pm 21$.

$$\vartheta_k = -\arctan[\lambda_m(m-2k-1)], \tag{15}$$

$$\frac{\mathcal{C}_b}{2\lambda_m(1-\zeta_a^2)} = \sum_{k=0}^{\frac{m}{2}-1} \cos(\vartheta_k). \tag{16}$$

Equation (16) is a self-consistency equation for λ_m that has to be satisfied in order to minimize $\mathcal{W}_m$.

Using these parameters, we can rewrite our witness in terms of the physical parameters $\mathcal{C}_b$ and ζ_a^2 only. For two measurement directions ($m = 2$), we find that states which are not Bell correlated satisfy

$$\zeta_a^2 \geq Z_2(\mathcal{C}_b) = \frac{1}{2}\left(1 - \sqrt{1-\mathcal{C}_b^2}\right). \tag{17}$$

This recovers the bound obtained from a different inequality in Ref. [9]. Note that in the present case, the argument is more direct since it does not involve $\mathcal{C}_a$, the first moment of the spin operator in the a direction.

Increasing the number of measurement directions allows for the detection of Bell correlations in additional states. In the limit $m \to \infty$, we find (see Appendix B in the Supplemental Material [25])

$$\zeta_a^2 \geq Z_\infty(\mathcal{C}_b) = 1 - \frac{\mathcal{C}_b}{\text{artanh}(\mathcal{C}_b)}. \tag{18}$$

Figure 1 shows the two witnesses (17) and (18) together with the one obtained similarly for $m = 4$ settings in the $\mathcal{C}_b - \zeta_a^2$ plane. The curve Z_∞ reaches the point $\mathcal{C}_b = \zeta_a^2 = 1$, therefore allowing, in principle, for the detection of Bell correlations in presence of arbitrarily low squeezing. It is known, however, that some values of $\mathcal{C}_b$ and ζ_a^2 can only be reached in the limit of a large number of spins [31]. For any fixed N, a finite amount of squeezing is thus necessary in

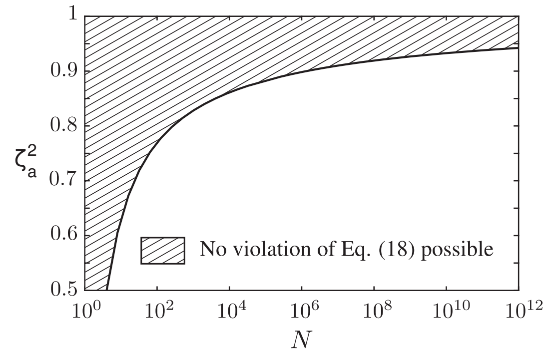

FIG. 2. Upper bound on the value of ζ_a^2 required to see a violation of the Bell correlation witness (18). The bound depends on the number of particles N.

order to allow for the violation of our witness (see Appendix C in the Supplemental Material [25]). The corresponding upper bound on ζ_a^2 is shown in Fig. 2.

Points below the curve Z_m in Fig. 1 indicate a violation of the witness $\mathcal{W}_m \geq 0$ obtained from the corresponding m-settings Bell inequality. Violation of any such bound reveals the presence of a Bell correlated state. However, as discussed in the introduction, conclusions in the presence of finite statistics have to be examined carefully, since in practice, one can never conclude from the violation of a witness that the measured state is Bell correlated with 100% confidence. The point shown in the inset of Fig. 1 corresponds to the data reported in Ref. [9] from measurements on a spin-squeezed Bose-Einstein condensate. This point clearly violates the witnesses for $m = 2, 4, \infty$ by several standard deviations, although the number of measurement rounds is too small to guarantee that the measured state is Bell correlated without further assumptions [9].

Finite statistics.—In this section, we put a bound on the number of experimental runs needed to exclude with a given confidence that a measured state is not Bell correlated. Note that such a conclusion does not follow straightforwardly from the violation of the witness by a fixed number of standard deviations. Indeed, standard deviations inform on the precision of a violation, but fail at excluding arbitrary local models [15], including, e.g., models which may show non-Gaussian statistics with rare events. We thus look here for a number of experimental runs which is sufficient to guarantee a p value lower than a given threshold for the null hypothesis "The measured state is not Bell correlated." Since we are concerned with the characterization of physical systems in the absence of an adversary, we assume that the same state is prepared in each round (i.i.d. assumption).

For this statistical analysis, let us consider a different Bell correlation witness than Eq. (18). Indeed, we derived this inequality in order to maximize the amount of violation for given data, but here we rather wish to maximize the statistical evidence of a violation. For this, we take Eq. (14) and consider the representation of the angles given in Eq. (15), but without taking Eq. (16) into account. In the limit of infinitely many measurement settings, we find (see Appendix B in the Supplemental Material [25])

$$\mathcal{W}_{\text{stat}} = -\mathcal{C}_b\Delta_\nu - (1-\zeta_a^2)\Lambda_\nu^2 + \frac{1}{4} \geq 0, \quad \text{with} \tag{19}$$

$$\Delta_\nu = \frac{\sqrt{1+\nu^2}}{4\nu} - \frac{\operatorname{arsinh}(\nu)}{4\nu^2}, \qquad \Lambda_\nu = \frac{\operatorname{arsinh}(\nu)}{2\nu}, \tag{20}$$

where $\nu = \lim_{m\to\infty}\lambda_m m$ is a free parameter that fully specifies the set of measurement angles.

In order to model the experimental evaluation of $\mathcal{W}_{\text{stat}}$, we introduce the following estimator:

$$\mathcal{T} = \frac{\chi(Z=0)}{q}X + \frac{\chi(Z=1)}{1-q}Y + \left(\frac{1}{4} - \Delta_\nu - \Lambda_\nu^2\right). \tag{21}$$

Here, χ denotes the indicator function and the binary random variable Z accounts for the choice between the measurement of either $\mathcal{C}_b$ or ζ_a. Each measurement round thus allows for the evaluation of the corresponding random variables $X = \Delta_\nu(1-\mathcal{C}_b)$ or $Y = \Lambda_\nu^2\zeta_a^2$. Assuming that Z is independent of X and Y and choosing $q = P[Z=0]$ guarantees that $\mathcal{T}$ is a proper estimator of $\mathcal{W}_{\text{stat}}$, i.e., $\langle\mathcal{T}\rangle = \mathcal{W}$. q then corresponds to the probability of performing a measurement along the b axis. We choose $q = (1 + (\Lambda_\nu^2 N/2\Delta_\nu))^{-1}$ so that the contributions of both measurement choices to $\mathcal{T}$ have the same magnitude; i.e., the maximum values of X/q and $Y/(1-q)$ are equal within the domain $|\mathcal{C}_b| \leq 1$ and $\zeta_a^2 \in [0, N]$. This also guarantees that the spectrum of $\mathcal{T}$ matches that of $\mathcal{W}_{\text{stat}}$.

Suppose the measured state is non-Bell correlated, i.e., that its mean value $\mu = \langle\mathcal{T}\rangle = \mathcal{W}_{\text{stat}} \geq 0$. We are now interested in the probability that after M experimental runs the estimated value $T = (1/M)\sum_{i=1}^M \mathcal{T}_i$ of the witness $\mathcal{W}_{\text{stat}}$ falls below a certain value $t_0 < 0$, with $\mathcal{T}_i$ being the value of the estimator in the ith run.

In statistics, concentration inequalities deal with exactly this issue. In Appendix D in the Supplemental Material [25], we compare four of these inequalities, namely, the Chernoff, Bernstein, Uspensky, and Berry-Esseen ones [25] and show explicitly that in the regime of interest the tightest and, therefore, preferred bound results from the Bernstein inequality:

$$P[T \leq t_0] \leq \exp\left(-\frac{(\mu - t_0)^2 M}{2\sigma_0^2 + \frac{2}{3}(t_u - t_l)(\mu - t_0)}\right) \leq \varepsilon. \tag{22}$$

Here, t_0 is the experimentally observed value of T after M measurement rounds, $t_l = \frac{1}{4} - \Delta_\nu - \Lambda_\nu^2$ and $t_u = \frac{1}{4} + \Delta_\nu + \Lambda_\nu^2(N+1)$ are lower and upper bounds on the random variable $\mathcal{T}$, respectively, and σ_0^2 is its variance for a local state.

We show in Appendix D in the Supplemental Material [25] that the largest p value is obtained by setting $\mu = 0$ and $\sigma_0^2 = -t_l t_u$. A number of measurement rounds

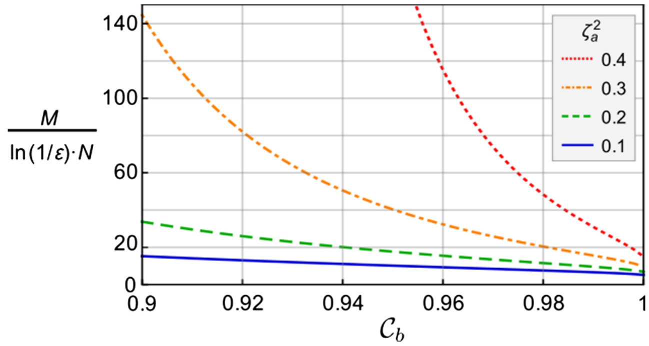

FIG. 3. Number of experimental runs per spins required to rule out non-Bell-correlated states with a confidence of $1-\varepsilon$ as a function of $\mathcal{C}_b$ and ζ_a. For $\mathcal{C}_b = 0.98$ and $\zeta_a^2 = 0.272$ (as reported in Ref. [9]), approximately $17\ln(100) \simeq 80$ runs per spin are sufficient to reach a confidence level of 99%.

sufficient to exclude the null hypothesis with a probability larger than $1-\varepsilon$ is then given by

$$M \geq \frac{-2t_l t_u - \frac{2}{3}(t_u - t_l)t_0}{t_0^2} \ln\left(\frac{1}{\varepsilon}\right). \qquad (23)$$

This quantity can be minimized by choosing the free parameter ν appropriately. As shown in Appendix D in the Supplemental Material [25], optimizing ν at this stage allows us to reduce the number of measurement rounds by ~30%. It is thus clearly advantageous not to consider the witness (18) when evaluating statistical significance.

The number of runs in Eq. (23) depends linearly on t_l and, therefore, also linearly on N. The ratio (M/N) thus tends to a constant for large N (see Appendix D in the Supplemental Material [25] for more details). This implies that a number of measurement rounds growing linearly with the system size is both necessary and sufficient to reliably conclude that the measured state is Bell correlated [9].

Figure 3 depicts the required number of measurement rounds per spin as a function of the scaled collective spin $\mathcal{C}_b$ and the scaled second moment ζ_a^2. For a confidence level of $1-\epsilon = 99\%$, between 20 and 500 measurement rounds per spin are required in the considered parameter region.

Conclusion.—In this Letter, we started by introducing a class of multipartite Bell inequalities involving two-body correlators and an arbitrary number of measurement settings. Assuming collective spin measurements, these inequalities give rise to the witness (18), which can be used to determine whether Bell correlations can be detected in a many-body system. This criterion detects states that were not detected by the previously known witness [9].

We then discussed the role of finite statistics in experiments involving many-body systems. We provided a bound, Eq. (23), on the number of measurement rounds that allows one to detect Bell correlated states without further assumptions. This bound shows that all non-Bell-correlated states can be convincingly ruled out at the cost of performing a number of measurement rounds that grows linearly with the system size. This puts the detection of quantum correlations in many-body systems on firm grounds and opens the way for a possible use of many-body systems in the context of device-independent quantum information processing.

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