On local and nonlocal Moser-Trudinger inequalities

Inauguraldissertation

zur

Erlangung der Würde eines Doktors der Philosophie

vorgelegt der

Philosophisch-Naturwissenschaftlichen Fakultät

der Universität Basel

von

Stefano Iula

aus

Italien

Basel, 2017

Genehmigt von der Philosophisch-Naturwissenschaftlichen Fakultät
auf Antrag von

Prof. Dr. Luca Martinazzi
Prof. Dr. Bernhard Ruf

Basel, den 18.04.2017

Prof. Dr. Martin Spiess
Dekan

Contents

Chapter 1

Introduction

Moser-Trudinger inequalities arise naturally in the study of the critical case of the well known Sobolev embeddings, which are one of the most useful tools in analysis as they play a crucial role in the study of existence, regularity and uniqueness of solutions to partial differential equations of different nature. In this Chapter we will introduce the reader to the topic and we will discuss the main results contained in this thesis.

1.1 The Moser-Trudinger inequality

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. If $p < n$ then

$$\sup_{u \in W_0^{1,p}(\Omega), \|\nabla u\|_{L^p(\Omega)}^p \leq 1} \int_\Omega |u|^q \, dx < +\infty \tag{1.1.1}$$

if and only if $1 \leq q \leq p^*$, where $p^* := \frac{np}{n-p}$. Here $\|\nabla u\|_{L^p(\Omega)}^p = \int_\Omega |\nabla u|^p \, dx$ is the Dirichlet norm of u. Shortly, we write

$$W_0^{1,p}(\Omega) \subset L^q(\Omega) \qquad 1 \leq q \leq p^*.$$

If we now consider the limiting case $p = n$, we have that every polynomial growth is allowed, in the sense that (1.1.1) holds for any $q \geq 1$. Namely, for any $q \geq 1$ we have

$$W_0^{1,n}(\Omega) \subset L^q(\Omega).$$

As $p \to n$, formally, $p^* \sim \infty$ and one would expect functions in $W_0^{1,n}$ to be bounded. It is a well known fact, though, that this is not the case. Indeed denote by $|\cdot|$ the standard Euclidean norm in $\mathbb{R}^n$ and define $u \colon \mathbb{R}^n \to \mathbb{R}$

$$u(x) := \begin{cases} \log|\log|x|| & \text{for } 0 < |x| < \frac{1}{e} \\ 0 & \text{elsewhere.} \end{cases}$$

Let now $\Omega \subset \mathbb{R}^n$ be a domain that contains the unit ball centered at the origin. It is easy to check that $u \in W_0^{1,n}(\Omega)$. Clearly though, $u \notin L^\infty(\Omega)$. It is then natural to look for the maximal growth function $g\colon \mathbb{R} \to \mathbb{R}^+$ such that

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|^n_{L^n(\Omega)} \le 1} \int_\Omega g(u)\, dx < +\infty.$$

The first result in this direction is due to Yudovich [53], Pohozaev [84], and Trudinger [98], who proved independently that functions in $W_0^{1,n}(\Omega)$ enjoy a uniform exponential-type integrability property. They showed that there exist constants $\beta > 0$ and $C > 0$, depending only on the dimension n, such that

$$\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega e^{\beta |u|^{\frac{n}{n-1}}}\, dx \le C|\Omega|. \tag{1.1.2}$$

Their proofs rely on the same idea of developing the exponential function in power series. However, this does not produce the optimal exponent β. Few years later J. Moser [74] solved this problem using a symmetrization argument and proved a sharp version of (1.1.2), which is now called Moser-Trudinger inequality.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with finite measure, $n \ge 2$ and ω_{n-1} the volume of the unit sphere in $\mathbb{R}^n$. Then there exist constants $C = C(n) > 0$ and $\beta_n := n\omega_{n-1}^{\frac{1}{n-1}}$ such that*

$$\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega e^{\beta_n |u|^{\frac{n}{n-1}}}\, dx \le C|\Omega|. \tag{1.1.3}$$

Moreover, the constant β_n is sharp in the sense that

$$\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega e^{\beta |u|^{\frac{n}{n-1}}}\, dx = +\infty \tag{1.1.4}$$

for $\beta > \beta_n$.

We remark that the supremum in (1.1.3) becomes infinite as soon as we slightly modify the integrand, namely

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \le 1} \int_\Omega f(|u|) e^{\beta_n |u|^{\frac{n}{n-1}}}\, dx = +\infty \tag{1.1.5}$$

for any measurable function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t \to +\infty} f(t) = \infty$. This can be proved, for instance, using the same test functions defined in [74]. In [2] Adams, exploiting Riesz potentials, extended Moser's result to higher order Sobolev spaces $W_0^{k,p}(\Omega)$,

$k > 1$, $p = \frac{n}{k}$.

The same result holds if we consider a smooth closed surface. Namely, if (Σ, g) is a smooth, closed Riemannian surface and

$$\mathcal{H} := \left\{ u \in H^1(\Sigma) \ : \ \int_\Sigma |\nabla u|^2 dv_g \le 1, \ \int_\Sigma u \, dv_g = 0 \right\},$$

Fontana [41] proved that

$$\sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi u^2} dv_g < +\infty \tag{1.1.6}$$

and

$$\sup_{u \in \mathcal{H}} \int_\Sigma e^{\beta u^2} dv_g = +\infty \tag{1.1.7}$$

for any $\beta > 4\pi$. Sharp Moser-Trudinger inequalities appear naturally when studying the classical problem of prescribing the Gaussian curvature of a compact Riemannian surface. Given a smooth closed surface (Σ, g) and a function $K \in C^\infty(\Sigma)$ one would like to investigate whether there exists a metric $\tilde{g}$, conformal to g, that has K as Gaussian curvature. We recall that a metric $\tilde{g}$ is conformal to g if there exists a smooth function u so that $\tilde{g} = e^u g$, that is if and only if u solves

$$-\frac{1}{2}\Delta_g u = K e^u - K_g, \tag{1.1.8}$$

where K_g and Δ_g are the Gaussian curvature and the Laplace-Beltrami operator of (Σ, g) respectively.

Let us denote the Euler characteristic of Σ by $\chi(\Sigma)$ and recall the Gauss-Bonnet theorem

$$\int_\Sigma K_g \, dv_g = 2\pi\chi(\Sigma).$$

It is not difficult to see that, if we suppose $\chi(\Sigma) \neq 0$ and K_g constant, then (1.1.8) is equivalent to

$$-\Delta_g u = \rho \left(\frac{K e^u}{\int_\Sigma K e^u \, dv_g} - \frac{1}{|\Sigma|} \right), \tag{1.1.9}$$

where $\rho = 4\pi\chi(\Sigma)$ and $|\Sigma|$ is the measure of Σ. Equation (1.1.9) is known as Liouville equation. One can exploit the variational structure of the problem and look for solutions to equation (1.1.9) as critical points of the associated energy functional

$$J_\rho(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 \, dv_g + \frac{\rho}{|\Sigma|}\int_\Sigma u \, dv_g - \rho \log\left(\frac{1}{|\Sigma|}\int_\Sigma K e^u \, dv_g \right). \tag{1.1.10}$$

Looking at the form of J_ρ, it becomes clear how results like Moser-Trudinger inequalities turn out to be game changers when one tries to apply direct minimization methods to solve problems of this type. For a general compact surface Σ, Kazdan and Warner ([54]) gave necessary and sufficient conditions on the sign of K when $\chi(\Sigma) = 0$, and some necessary condition in the case $\chi(\Sigma) < 0$. In [75] Moser improved these results and considered the case $\chi(\Sigma) > 0$, that is $(\Sigma, g) = (S^2, g_c)$, where g_c is the standard metric on S^2. He proved that, for an even function f, the only necessary condition for (1.1.8) to be solvable with $K = f$, is for f to be positive somewhere. For functions with antipodal symmetry, the critical exponent in Theorem 1.1 can be improved, namely inequality (1.1.3) holds up to $\beta = 8\pi$. In particular, Theorem 1.1 implies that $J_{8\pi}$ is bounded from below and that J_ρ is coercive on the space

$$H_0 := \left\{ u \in H^1(\Sigma) \colon \int_\Sigma u \, dv_g = 0 \right\}$$

for $\rho < 8\pi$. Hence, using direct minimization, Moser proved existence of solutions of (1.1.8). If this simmetry assumption is dropped, minimization techniques are not strong enough and one needs to assume some nondegeneracy of the critical points of K and use, for instance, a min-max scheme or a curvature flow approach, see [21], [20], [93]. To prove existence results in the case $\rho \geq 8\pi$, improved Moser-Trudinger inequalities and non-trivial variational and topological methods are required, see [35], [36], [66], [94].

A more general problem concerns the study of compact surfaces with conical singularities. We recall that, given a finite number of points $p_1, \dots, p_m \in \Sigma$, a smooth metric $\overline{g}$ on $\Sigma \backslash \{p_1, \dots, p_m\}$ is said to have conical singularities of order $\alpha_1, \dots, \alpha_m$ in $p_1, \dots, p_m$ if $\overline{g} = hg$ with g smooth metric on Σ and $h \in C^1(\Sigma \backslash \{p_1, \dots, p_m\})$ is a positive function satisfying

$$h(x) \approx d(x, p_i)^{2\alpha_i} \qquad \text{with} \quad \alpha_i > -1 \;\text{ near }\; p_i \quad i = 1, \dots, m, \tag{1.1.11}$$

where d represents the Riemann distance on Σ. In other words, $\overline{g}$ is a metric of the form $e^u g$ where g is a smooth metric on Σ, and $u \in C^\infty(\Sigma \setminus \{p_1, \dots, p_m\})$ satisfies

$$|u(x) + 2\alpha_i \log d(x, p_i)| \leq C \qquad \text{near } p_i, \quad i = 1, \dots, m.$$

A metric of this form has Gaussian curvature K if and only if the function u is a distributional solution to the singular Gaussian curvature equation

$$-\Delta_g u = 2K e^u - 2K_g - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i}, \tag{1.1.12}$$

see for instance [10]. Define

$$\rho := 4\pi \left(\chi(\Sigma) + \sum_{i=1}^m \alpha_i \right).$$

Similarly to the case without singularities, if $\rho \neq 0$ and K_g is constant, equation (1.1.12) is equivalent to the singular Liouville equation

$$-\Delta_g u = \rho\left(\frac{Ke^u}{\int_\Sigma Ke^u d\,dv_g} - \frac{1}{|\Sigma|}\right) - 4\pi\sum_{i=1}^m \alpha_i\left(\delta_{p_i} - \frac{1}{|\Sigma|}\right). \tag{1.1.13}$$

Finding solutions to (1.1.13) is equivalent to proving existence of critical points of the singular Moser-Trudinger functional

$$J_\rho^{sing} := \frac{1}{2}\int_\Sigma |\nabla u|^2\,dv_g + \frac{\rho}{|\Sigma|}\int_\Sigma u\,dv_g - \rho\log\left(\frac{1}{|\Sigma|}\int_\Sigma he^u\,dv_g\right),$$

where $h \in C^1(\Sigma\backslash\{p_1,\dots,p_m\})$ is as in (1.1.11). Inspired by what Moser did for J_ρ, Troyanov tried to minimize J_ρ^{sing} (see [97], [27]) by finding a sharp version of the Moser-Trudinger inequality for metrics with conical singularities. In particular, he proved that J_ρ^{sing} is bounded from below on $H^1(\Sigma)$, coercive on H_0 if $\rho < 8\pi(1+\bar{\alpha})$ and it is bounded from below if $\rho = 8\pi(1+\bar{\alpha})$, where $\overline{\alpha} = \min\left\{0, \min_{1\le i\le m}\alpha_i\right\}$. In the first case the coercivity of J_ρ^{sing} yields existence of minimum points. As for the regular case, to treat the case $\rho > 8\pi(1+\bar{\alpha})$ different approaches are needed (see e.g. [36], [66], [23], [24], [25], [26]).

It is worth to mention that, even though usually we look at (1.1.9) and (1.1.13) in the context of Riemannian Geometry, they also have been widely studied in mathematical physics. Indeed they appear in the description of Abelian vortices in Chern-Simmons-Higgs theory and have applications in fluid dynamics, as well as in Superconductivity and Electroweak theory (see [73], [99], [95], [45]). If we denote by G the Green's function of (Σ, g), i.e. the solution of

$$\begin{cases} -\Delta_g G(x,\cdot) = \delta_x & \text{on } \Sigma \\ \int_\Sigma G(x,y)\,dv_g(y) = 0, \end{cases}$$

the change of variable $u \to u + 4\pi\sum_{i=1}^m \alpha_i G(x,p_i)$ reduces equation (1.1.13) to

$$-\Delta_g u = \rho\left(\frac{he^u}{\int_\Sigma he^u\,dv_g} - \frac{1}{|\Sigma|}\right), \tag{1.1.14}$$

which is nothing but equation (1.1.9) with K replaced by the singular weight

$$h(x) = Ke^{-4\pi\sum_{i=1}^m \alpha_i G_{p_i}}.$$

Several generalizations and applications of the Moser-Trudinger inequality have appeared in the course of the last decades. This thesis covers two problems related to Theorem 1.1.

1.2 Existence of extremal functions

In the first part of this work, we will focus our attention to the case $n = 2$ and set $H^1_0(\Omega) := W^{1,2}_0(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is open and bounded. In this setting the sharp exponent for the Moser-Trudinger inequality is $\beta = 4\pi$ and, according to Theorem 1.1, we have

$$\sup_{u\in H^1_0(\Omega),\ \int_\Omega |\nabla u|^2dx\le 1} \int_\Omega e^{4\pi u^2}dx < +\infty, \tag{1.2.1}$$

and

$$\sup_{u\in H^1_0(\Omega),\ \int_\Omega |\nabla u|^2dx\le 1} \int_\Omega e^{\beta u^2}dx = +\infty \tag{1.2.2}$$

for $\beta > 4\pi$.

The first issue that we will address is the existence of extremal functions for (1.2.1). While there is no function realizing equality for the critical Sobolev embedding, one can prove that the supremum in (1.2.1) is always attained. This was proved in [19] by Carleson and Chang for the unit disk $D \subseteq \mathbb{R}^2$, and by Flucher ([40]) for arbitrary bounded domains (see also [91] and [62], [67]).

The proof of these results is based on a concentration-compactness alternative stated by P. L. Lions ([63]): for a sequence $u_k \in H^1_0(\Omega)$ such that $\|\nabla u_k\|_{L^2(\Omega)} = 1$ one has, up to subsequences, either

$$\int_\Omega e^{4\pi u_k^2}dx \to \int_\Omega e^{4\pi u^2}dx,$$

where u is the weak limit of u_k, or u_k concentrates in a point $x \in \overline{\Omega}$, that is

$$|\nabla u|^2dx \rightharpoonup \delta_x \qquad \text{and} \qquad u_k \rightharpoonup 0. \tag{1.2.3}$$

The key step in [19] consists in proving that if a sequence of radially symmetric functions $u_k \in H^1_0(D)$ concentrates at 0, then

$$\limsup_{k\to\infty} \int_D e^{4\pi u_k^2}dx \le \pi(1+e). \tag{1.2.4}$$

Since for the unit disk the supremum in (1.2.1) is strictly greater than $\pi(1+e)$, one can exclude concentration for maximizing sequences by means of (1.2.4) and therefore prove existence of extremal functions for (1.2.1). In [40] Flucher observed that concentration at arbitrary points of a general domain Ω can always be reduced, through properly defined rearrangements, to concentration of radially symmetric functions on the unit disk. In particular, he proved that if $u_k \in H^1_0(\Omega)$ satisfies $\|\nabla u_k\|_2 = 1$ and (1.2.3), then

$$\limsup_{k\to\infty}\int_\Omega e^{4\pi u_k^2}dx \le \pi e^{1+4\pi A_\Omega(x)} + |\Omega|. \tag{1.2.5}$$

where $A_\Omega(x)$ is the Robin function of Ω, that is the trace of the regular part of the Green function of Ω. He also proved

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2dx\le 1}\int_\Omega e^{4\pi u^2}dx > \pi e^{1+4\pi \max_{\overline{\Omega}} A_\Omega} + |\Omega|,$$

which implies the existence of extremals for (1.2.1) on Ω. This method turns out to work also when considering the problem on a closed smooth Riemannian manifold (Σ, g). In this case, again by excluding concentration for maximizing sequences, Li [57] (see also [59], [58]) was able to prove existence of extremal functions for (1.1.6).

Here we are interested in Moser-Trudinger type inequalities in the presence of singular potentials. The model for this problem is given by the singular metric $|x|^{2\alpha}|dx|^2$ on a bounded domain $\Omega\subset\mathbb{R}^2$ containing the origin. In [5] Adimurthi and Sandeep observed that for any $\alpha\in(-1,0]$,

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2dx\le 1}\int_\Omega |x|^{2\alpha}e^{4\pi(1+\alpha)u^2}dx < +\infty, \tag{1.2.6}$$

and

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2dx\le 1}\int_\Omega |x|^{2\alpha}e^{\beta u^2}dx = +\infty \tag{1.2.7}$$

for any $\beta > 4\pi(1+\alpha)$. Exploiting the ideas of Flucher, existence of extremals for (1.2.6) has recently been proved in [32] and [31].

In the case $\alpha\neq 0$, applying the strategy in [19], one can again exclude concentration for maximizing sequences using the following estimate, which can be obtained from (1.2.4) using a simple change of variables (see [5], [31]).

Theorem 1.2. *Let $u_k\in H_0^1(D)$ be such that $\int_D|\nabla u_k|^2dx\le 1$ and $u_k\rightharpoonup 0$ in $H_0^1(D)$, then for any $\alpha\in(-1,0]$ we have*

$$\limsup_{k\to\infty}\int_D |x|^{2\alpha}e^{4\pi(1+\alpha)u_k^2}dx \le \frac{\pi(1+e)}{1+\alpha}. \tag{1.2.8}$$

In the first part of this thesis we will give a simplified version of the argument in [19] and show that (1.2.4) (and therefore (1.2.8)) can be deduced from Onofri's inequality for the unit disk:

Proposition 1.3 (See [80], [12])**.** *For any $u\in H_0^1(D)$ we have*

$$\log\left(\frac{1}{\pi}\int_D e^u dx\right) \le \frac{1}{16\pi}\int_D |\nabla u|^2dx + 1. \tag{1.2.9}$$

The analysis can be pushed further and Theorem 1.2 can be used to prove existence of extremals for several generalized versions of (1.2.1). Let $\Omega \subset \mathbb{R}^2$ be open and bounded. In [4] Adimurthi and Druet proved that

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2 dx\le 1} \int_\Omega e^{4\pi u^2(1+\lambda\|u\|^2_{L^2(\Omega)})} dx < +\infty, \tag{1.2.10}$$

for any $\lambda < \lambda(\Omega)$, where $\lambda(\Omega)$ is the first eigenvalue of $-\Delta$ with respect to Dirichlet boundary conditions. This bound on λ is sharp, that is

$$\sup_{u\in H_0^1(\Omega),\int_\Omega |\nabla u|^2 dx\le 1} \int_\Omega e^{4\pi u^2(1+\lambda(\Omega)\|u\|^2_{L^2(\Omega)})} dx = \infty. \tag{1.2.11}$$

Existence of extremal functions for sufficiently small λ for this improved inequality has been proved in [64] and [101]. Similar results hold for compact surfaces on the space $\mathcal{H}$. We refer to [96], [102] and references therein for further improved inequalities.

We will focus on Adimurthi-Druet type inequalities on compact surfaces with conical singularities. Given a smooth closed Riemannian surface (Σ, g), and a finite number of points $p_1,\dots,p_m \in \Sigma$, we will consider functionals of the form

$$E^{\beta,\lambda,q}_{\Sigma,h}(u) := \int_\Sigma h e^{\beta u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g, \tag{1.2.12}$$

where $\lambda, \beta \ge 0$, $q > 1$, and $h \in C^1(\Sigma\backslash\{p_1,\dots,p_m\})$ is a positive function satisfying (1.1.11). The functional (1.2.12) naturally appears in the analysis of Moser-Trudinger embeddings for the singular surface $(\Sigma, \overline{g})$ (see [97]). If $m = 0$ and $h \equiv 1$, the family $E^{\beta,\lambda,q}_{\Sigma,1}$ corresponds to the one studied in [64]. In particular, one has

$$\sup_{u\in\mathcal{H}} E^{4\pi,\lambda,q}_{\Sigma,1} < +\infty \quad \Longleftrightarrow \quad \lambda < \lambda_q(\Sigma, g), \tag{1.2.13}$$

where

$$\lambda_q(\Sigma, g) := \inf_{u\in\mathcal{H}} \frac{\int_\Sigma |\nabla u|^2 dv_g}{\|u\|^2_{L^q(\Sigma,g)}}.$$

As it happens for (1.2.6), if h has singularities (i.e. $\alpha \in (-1, 0]$), the critical exponent becomes smaller. More precisely, in [97] Troyanov (see also [27]) proved that if h is a positive function satisfying (1.1.11), then

$$\sup_{u\in\mathcal{H}} E^{\beta,0,q}_{\Sigma,h} < +\infty \qquad \Longleftrightarrow \qquad \beta \le 4\pi(1+\overline{\alpha}), \tag{1.2.14}$$

where $\overline{\alpha} = \min\left\{0, \min_{1\le i\le m} \alpha_i\right\}$. Here we combine (1.2.13) and (1.2.14) to obtain the following singular version of (1.2.13).

Theorem 1.4. *Let (Σ, g) be a smooth, closed, surface. If $h \in C^1(\Sigma\backslash\{p_1,\dots,p_m\})$ is a positive function satisfying* (1.1.11)*, then for any $\beta \in [0, 4\pi(1+\overline{\alpha})]$ and $\lambda \in [0, \lambda_q(\Sigma, g))$*

we have

$$\sup_{u\in\mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h}(u) < +\infty. \tag{1.2.15}$$

The supremum is attained if $\beta < 4\pi(1+\overline{\alpha})$, or if $\beta = 4\pi(1+\overline{\alpha})$ and λ is sufficiently small. Moreover

$$\sup_{u\in\mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h}(u) = +\infty$$

for $\beta > 4\pi(1+\overline{\alpha})$, or $\beta = 4\pi(1+\overline{\alpha})$ and $\lambda > \lambda_q(\Sigma, g)$.

Note that we do not treat the case $\beta = 4\pi(1+\overline{\alpha})$ and $\lambda = \lambda_q(\Sigma, g)$ (see Remark 2.5). It is worth to remark that in Theorem 1.4 it is possible to replace $\|\cdot\|_{L^q(\Sigma,g)}$, $\lambda_q(\Sigma, g)$, and $\mathcal{H}$ with $\|\cdot\|_{L^q(\Sigma,g_h)}$, $\lambda_q(\Sigma, g_h)$, and

$$\mathcal{H}_h := \left\{ u \in H^1_0(\Sigma) \ : \ \int_\Sigma |\nabla_{g_h} u|^2 dv_{g_h} \le 1, \ \int_\Sigma u \, dv_{g_h} = 0 \right\},$$

where $g_h := hg$. In particular, we can extend the Adimurthi-Druet inequality to compact surfaces with conical singularities.

Theorem 1.5. *Let (Σ, g) be a closed surface with conical singularities of order $\alpha_1, \ldots, \alpha_m > -1$ in $p_1, \ldots, p_m \in \Sigma$. Then for any $0 \le \lambda < \lambda_q(\Sigma, g)$ we have*

$$\sup_{u\in\mathcal{H}} \int_\Sigma e^{4\pi(1+\overline{\alpha})u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g < +\infty.$$

The supremum is attained for $\beta < 4\pi(1+\overline{\alpha})$, or for $\beta = 4\pi(1+\overline{\alpha})$ and sufficiently small λ. Moreover

$$\sup_{u\in\mathcal{H}} \int_\Sigma e^{\beta u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g = +\infty,$$

if $\beta > 4\pi(1+\overline{\alpha})$, or $\beta = 4\pi(1+\overline{\alpha})$ and $\lambda > \lambda_q(\Sigma, g)$.

The proof of Theorem 1.4 follows the ideas in [19] and [40] and makes use of Lion's concentration-compactness alternative discussed above. To exclude concentration of maximizing sequences a careful blow-up analysis is required. Indeed we shall see how, after a suitable scaling, our sequence converges to a solution of a (possibly singular) Liouville-type equation on $\mathbb{R}^2$ (see Proposition 2.14). The behaviour of this sequence depends on the nature of the blow-up point $p \in \Sigma$. A key step in this analysis is a classification result for solutions to the singular Liouville equation on $\mathbb{R}^2$ (see Section 2.2).

1.3 Moser-Trudinger inequalities in dimension one

In the second part of this thesis we tackle a different problem related to Moser-Trudinger inequalities. We will investigate fractional analogues of (1.1.3) and their sharpness, restricting ourselves to the one dimensional case. In particular, using variational techniques in the setting of Bessel-potential spaces, we will discuss the existence of critical points of a functional associated to (1.1.3). We will also present some results on a recent generalization of (1.1.3) on Sobolev-Slobodeckij spaces (see [81]).

Let us recall some basic notions on fractional Sobolev spaces. Consider the space of functions $L_s(\mathbb{R})$ defined by

$$L_s(\mathbb{R}) = \left\{ u \in L^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+|x|^{1+2s}} dx < \infty \right\}, \tag{1.3.1}$$

for $s \in (0,1)$. For a function $u \in L_s(\mathbb{R})$ we define $(-\Delta)^s u$ as a tempered distribution as follows:

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}} u (-\Delta)^s \varphi dx, \quad \varphi \in \mathcal{S}, \tag{1.3.2}$$

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing smooth functions and for $\varphi \in \mathcal{S}$ we set

$$(-\Delta)^s \varphi := \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi}).$$

Here the Fourier transform is defined by

$$\hat{\varphi}(\xi) \equiv \mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x)\, dx.$$

Notice that the convergence of the integral in (1.3.2) follows from the fact that for $\varphi \in \mathcal{S}$ one has

$$|(-\Delta)^s \varphi(x)| \le C(1+|x|^{1+2s})^{-1}.$$

For $s \in (0,1)$ and $p \in [1,\infty]$ we define the Bessel-potential space

$$H^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) \colon (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}) \right\}, \tag{1.3.3}$$

and its subspace

$$\tilde{H}^{s,p}(I) := \{ u \in L^p(\mathbb{R}) : u \equiv 0 \text{ in } \mathbb{R} \setminus I, \ (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}) \}, \tag{1.3.4}$$

where $I \Subset \mathbb{R}$ is a bounded interval. Both spaces are endowed with the norm

$$\|u\|^p_{H^{s,p}(\mathbb{R})} := \|u\|^p_{L^p(\mathbb{R})} + \|(-\Delta)^{\frac{s}{2}} u\|^p_{L^p(\mathbb{R})}. \tag{1.3.5}$$

The first result that we shall present is a fractional version of Theorem 1.1.

Theorem 1.6. *For any $p \in (1, +\infty)$ set $p' = \frac{p}{p-1}$ and*

$$\alpha_p := \frac{1}{2}\left[2\cos\left(\frac{\pi}{2p}\right)\Gamma\left(\frac{1}{p}\right)\right]^{p'}, \quad \Gamma(z) := \int_0^{+\infty} t^{z-1}e^{-t}\,dt. \tag{1.3.6}$$

Then for any interval $I \Subset \mathbb{R}$ and $\alpha \le \alpha_p$ we have

$$\sup_{u \in \tilde{H}^{\frac{1}{p},p}(I),\, \|(-\Delta)^{\frac{1}{2p}}u\|_{L^p(I)} \le 1} \int_I \left(e^{\alpha|u|^{p'}} - 1\right) dx = C_p|I|, \tag{1.3.7}$$

and $\alpha = \alpha_p$ is the largest constant for which (1.3.7) *holds. In fact for any function $h : [0,\infty) \to [0,\infty)$ with*

$$\lim_{t\to\infty} h(t) = \infty \tag{1.3.8}$$

we have

$$\sup_{u \in \tilde{H}^{\frac{1}{p},p}(I),\, \|(-\Delta)^{\frac{1}{2p}}u\|_{L^p(I)} \le 1} \int_I h(u)\left(e^{\alpha_p|u|^{p'}} - 1\right) dx = \infty. \tag{1.3.9}$$

To understand the main issues in the proof of Theorem 1.6 we recall the following analogue of (1.3.7)

$$\sup_{u = c_p I_{\frac{1}{p}} * f \,:\, \mathrm{supp}(f) \subset \bar{I},\, \|f\|_{L^p(I)} \le 1} \int_I e^{\alpha_p|u|^{p'}} dx = C_p|I|, \quad I_{\frac{1}{p}}(x) := |x|^{\frac{1}{p}-1}. \tag{1.3.10}$$

Inequality (1.3.10) is well-known (also in higher dimension, see e.g. [100, Theorem 3.1]), since it follows easily from Theorem 2 in [2], up to choosing c_p so that

$$c_p(-\Delta)^{\frac{1}{2p}} I_{\frac{1}{p}} = \delta_0, \tag{1.3.11}$$

as we shall see in Section 3.2 (compare to Lemma 3.4).

In (1.3.10) one requires that the support of $f = (-\Delta)^{\frac{1}{2p}}u$ is bounded; following Adams [2] one would be tempted to write $u = I_{\frac{1}{p}} * (-\Delta)^{\frac{1}{2p}}u$ and apply (1.3.10), but the support of $(-\Delta)^{\frac{1}{2p}}u$ is in general not bounded, when u is compactly supported.

In order to circumvent this issue, we rely on a Green representation formula of the form

$$u(x) = \int_I G_{\frac{1}{2p}}(x,y)(-\Delta)^{\frac{1}{2p}}u(y)dy,$$

and show that $|G_{\frac{1}{2p}}(x,y)| \le I_{\frac{1}{p}}(x-y)$ for $x \ne y$. This might follow from the explicit formula of $G_s(x,y)$, which is known on an interval, see e.g. [14] and [18], but we prefer to follow a more self-contained path, only using the maximum principle.

More delicate is the proof of (1.3.9). We will construct functions u supported in $\bar{I}$ with $(-\Delta)^{\frac{1}{2p}}u = f$ for some prescribed function $f \in L^p(I)$ suitably concentrated. Then with a barrier argument we will show that $u \in \tilde{H}^{\frac{1}{p},p}(I)$, i.e. $(-\Delta)^{\frac{1}{2p}}u \in L^p(\mathbb{R})$. This is not obvious because $(-\Delta)^{\frac{1}{2p}}$ is a non-local operator and even if $u \equiv 0$ in I^c, $(-\Delta)^{\frac{1}{2p}}u$ does not vanish outside I, and a priori it could even concentrate on ∂I.

Remark 1.1. *An alternative approach to* (1.3.9) *uses the Riesz potential and a cut-off function ψ, as done in [71] following a suggestion of A. Schikorra. This works in every dimension and for arbitrary powers of $-\Delta$, but it is less efficient in the sense that the $\|(-\Delta)^s\psi\|_{L^p}$ is not sufficiently small, and* (1.3.9) *(or its higher-order analog) can be proven only for functions h such that $\lim_{t\to\infty}(t^{-p'}h(t)) = \infty$. On the other hand, the approach used here to prove* (1.3.9) *for every h satisfying* (1.3.8) *does not work for higher-order operators, since for instance if for $\Omega \Subset \mathbb{R}^4$ we take $u \in W_0^{1,2}(\Omega)$ solving $\Delta u = f \in L^2(\Omega)$, then we do not have in general $u \in W^{2,2}(\mathbb{R}^4)$.*

Remark 1.2. *Notice that in* (1.3.7)*, instead of the standard $H^{\frac{1}{p},p}$-norm defined in* (1.3.5)*, we are using the smaller norm $\|u\|^* := \|(-\Delta)^{\frac{1}{2p}}u\|_{L^p(I)}$, which turns out to be equivalent to the full norm $\|u\|_{H^{\frac{1}{p},p}(\mathbb{R})}$ on $\tilde{H}^{\frac{1}{p},p}(I)$ (see [44]).*

A subcritical version of (1.3.7) in Theorem 1.6 has been recently proved by A. Iannizzotto and M. Squassina [48, Cor. 2.4] in the case $p = 2$. Namely they were able to show that

$$\sup_{u\in\tilde{H}^{\frac{1}{2},2}(I)\,:\,\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R})}\le 1}\int_I e^{\alpha u^2}dx \le C_\alpha|I|, \quad \text{for } \alpha < \pi.$$

For further generalization of Theorem 1.6, we refer for instance to [71], [46].

When replacing a bounded interval I by $\mathbb{R}$, an estimate of the form (1.3.7) cannot hold, for instance because of the scaling of (1.3.7), or simply because the quantity $\|(-\Delta)^{\frac{1}{2p}}u\|_{L^p(\mathbb{R})}$ vanishes on constants. This suggests that, in order to have an inequality on $\mathbb{R}$, one should use the full Sobolev norm including the L^p-norm of u (see Remark 1.2). This was done by Bernhard Ruf [88] in the case of $H^{1,2}(\mathbb{R}^2)$. We shall adapt his technique to the case $H^{\frac{1}{2},2}(\mathbb{R})$.

Theorem 1.7. *We have*

$$\sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}),\ \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}\le 1}\int_{\mathbb{R}}\left(e^{\pi u^2}-1\right)dx < \infty, \tag{1.3.12}$$

where $\|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}$ is defined in (1.3.5)*. Moreover, for any function $h : [0,\infty) \to [0,\infty)$ satisfying*

$$\lim_{t\to\infty}(t^{-2}h(t)) = \infty \tag{1.3.13}$$

we have

$$\sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}),\ \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}\le 1}\int_{\mathbb{R}}h(u)\left(e^{\pi u^2}-1\right)dx = \infty. \tag{1.3.14}$$

In particular the constant π in (1.3.12) *is sharp.*

The issue of dealing with a nonlocal operator naturally leads to some open questions. A main ingredient in the proof of (1.3.12) is a fractional Pólya-Szegő inequality which seems to be known only in the L^2 setting, being based mainly on Fourier transform techniques.

Open question 1. *Does an L^p-version of Theorem 1.7 hold, i.e. can we replace $H^{\frac{1}{2},2}$ with $H^{\frac{1}{p},p}$ in* (1.3.12)*?*

The reason for requiring (1.3.13) in Theorem 1.7 (contrary to Theorem 1.6, where (1.3.8) suffices) is that the test functions for (1.3.14) will be constructed using a cut-off procedure, and due to the nonlocal nature of the $H^{\frac{1}{2},2}$-norm, giving a precise estimate for the norm of such test functions is difficult.

Open question 2. *In analogy with Theorem 1.6, does* (1.3.14) *hold for every h satisfying* (1.3.8)*?*

A positive answer to this question has been recently provided by Hyder ([47][Theorem 1.2]).

The usual approach to fractional Moser-Trudinger inequalities is via Bessel potential spaces $H^{s,p}$ (see Section 3.2). Here, we focus our attention on the case (in general different from the one of Bessel potential spaces) of Sobolev Slobodeckij spaces (see definitions below), which has been recently proposed, together with some open questions, by Parini and Ruf. In [81] they considered $\Omega \subset \mathbb{R}^n$ to be a bounded and open domain, $n \geq 2$ and $sp = n$. They were able to prove the existence of $\beta_* > 0$ such that the corresponding version of inequality (1.1.3) is satisfied for any $\beta \in (0, \beta_*)$ (see also [83]). Even though the result is not sharp, in the sense that the value of the optimal exponent is not yet known, an explicit upper bound for the optimal exponent β^* is given.

As a first step, we extend the results in [81] to the case $n = 1$. For any $s \in (0,1)$ and $p > 1$, the Sobolev-Slobodeckij space $W^{s,p}(\mathbb{R})$ is defined as

$$W^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) \colon [u]_{W^{s,p}(\mathbb{R})} < +\infty \right\}$$

where $[u]_{W^{s,p}(\mathbb{R})}$ is the Gagliardo seminorm defined by

$$[u]_{W^{s,p}(\mathbb{R})} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x-y|^2} \, dx \, dy \right)^{\frac{1}{p}} . \tag{1.3.15}$$

We will often write $[\cdot] := [\cdot]_{W^{s,p}(\mathbb{R})}$. The space $W^{s,p}(\mathbb{R})$ is a Banach space with respect to the norm

$$||u||_{W^{s,p}(\mathbb{R})} := \left(||u||^p_{L^p(\mathbb{R})} + [u]^p_{W^{s,p}(\mathbb{R})} \right)^{\frac{1}{p}} . \tag{1.3.16}$$

Let I be an open interval in $\mathbb{R}$. We define the space $\tilde{W}_0^{s,p}(I)$ as the closure of $C_0^\infty(I)$ with respect to the norm $\|u\|_{W^{s,p}(\mathbb{R})}$. An equivalent definition for $\tilde{W}_0^{s,p}(I)$ can be obtained taking the completion of $C_0^\infty(I)$ with respect to the seminorm $[u]_{W^{s,p}(\mathbb{R})}$ (see [17, Remark 2.5]).

With a mild adaptation of the techniques used in [81], we are able to prove that their result holds also in dimension one.

Theorem 1.8. *Let $s \in (0,1)$ and $p > 1$ be such that $sp = 1$. There exists $\beta_* = \beta_*(s) > 0$ such that for all $\beta \in [0, \beta_*)$ it holds*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \leq 1} \int_I e^{\beta |u|^{\frac{1}{1-s}}} \, dx < \infty. \tag{1.3.17}$$

Moreover, there exists $\beta^* = \beta^*(s) := \gamma_s^{\frac{s}{1-s}}$ such that the supremum in (1.3.17) *is infinite for any $\beta \in (\beta^*, +\infty)$.*

It is worth to remark that, as already pointed out in [81], the exponent $\beta^*(\frac{1}{2})$ is equal to $2\pi^2$ and it coincides, up to a normalization constant, with the optimal exponent π determined in [50] in the setting of Bessel potential spaces (cfr. Theorem 1.6).

We move now to the case $I = \mathbb{R}$, pushing further the analysis of [81]. As we already commented above for Theorem 1.7, an inequality of the form (1.3.17) cannot hold if we don't consider the full $W^{s,p}(\mathbb{R})$-norm, i.e. we take into account also the term $\|u\|_{L^p(\mathbb{R})}$, (see also [50], [46] for the case of Bessel potential spaces). We define

$$\Phi(t) := e^t - \sum_{k=0}^{\lceil p-2 \rceil} \frac{t^k}{k!}, \tag{1.3.18}$$

where $\lceil p-2 \rceil$ is the smallest integer greater than, or equal to $p-2$.

Theorem 1.9. *Let $s \in (0,1)$ and $p > 1$ be such that $sp = 1$. There exists $\beta_* = \beta_*(s) > 0$ such that for all $\beta \in [0, \beta_*)$ it holds*

$$\sup_{u \in W^{s,p}(\mathbb{R}), \|u\|_{W^{s,p}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} \Phi(\beta |u|^{\frac{1}{1-s}}) \, dx < \infty. \tag{1.3.19}$$

Moreover the supremum in (1.3.17) *is infinite for any $\beta \in (\beta^*, +\infty)$, where β^* is as in Theorem 1.8*

As we shall see, Theorem 1.8 and 1.9 are sharp in the sense of (1.1.5). Indeed one of the open questions in [81] was whether an inequality of the type

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{\tilde{W}_0^{s,p}(I)} \leq 1} \int_I f(|u|) e^{\beta |u|^{\frac{1}{1-s}}} \, dx < +\infty,$$

where $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is such that $f(t) \to \infty$ as $t \to \infty$ holds true for the same exponents of the standard Moser-Trudinger inequality (see [46],[50]). For $n = 1$ we prove the following

Theorem 1.10. *Let $I \subset \mathbb{R}$ be a bounded interval, $s \in (0,1)$ and $p > 1$ such that $sp = 1$. We have*

$$\sup_{u \in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})} \le 1} \int_I f(|u|) e^{\beta^* |u|^{\frac{1}{1-s}}} \, dx = \infty, \tag{1.3.20}$$

$$\sup_{u \in W^{s,p}(\mathbb{R}), \|u\|_{W^{s,p}(\mathbb{R})} \le 1} \int_{\mathbb{R}} f(|u|) \Phi(\beta^* |u|^{\frac{1}{1-s}}) \, dx = \infty, \tag{1.3.21}$$

where $f\colon [0,\infty) \to [0,\infty)$ is any Borel measurable function such that $\lim_{t \to +\infty} f(t) = \infty$.

1.4 Critical points for the fractional Moser-Trudinger inequality

As an application of Theorem 1.6, we investigate the existence of critical points of functionals associated to inequality (1.3.7) in the case $p = 2$. The results that we are going to present were first proven by Adimurthi [3] in dimension $n \ge 2$ with $(-\Delta)^{\frac{1}{2}}$ replaced by the n-Laplacian.

Denote

$$H := \tilde{H}^{\frac{1}{2},2}(I), \quad \|u\|_H := \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}. \tag{1.4.1}$$

By Remark 1.2 this norm is equivalent to the full $H^{\frac{1}{2},2}$-norm on $\tilde{H}^{\frac{1}{2},2}(I)$.

This also follows from the following Poincaré-type inequality (see e.g. [89, Lemma 6]):

$$\|u\|_{L^2(I)}^2 \le \frac{1}{\lambda_1(I)} \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}^2 \quad \text{for } u \in \tilde{H}^{\frac{1}{2},2}(I), \tag{1.4.2}$$

where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on $\tilde{H}^{\frac{1}{2},2}(I)$ (see Lemma 3.2, Section 3.3).

Since we often integrate by parts and $(-\Delta)^s u$ is not in general supported in I even if $u \in C_c^\infty(I)$, it is more natural to consider the slightly weaker inequality

$$\sup_{u \in H,\, \|u\|_H^2 \le 2\pi} \int_I \left(e^{\frac{1}{2}u^2} - 1 \right) dx = C|I|, \tag{1.4.3}$$

where we use the slightly different norm given in (1.4.1). The reason for using the constant $\frac{1}{2}$ instead of $\beta_2 = \pi$ in the exponential and having $\|u\|_H^2 \le 2\pi$ instead of $\|u\|_H^2 \le 1$ is mostly cosmetic, and becomes more apparent when studying the blow-up behaviour of critical points of functionals associated to (1.4.3) (see (1.4.5) below, and compare to [65] and [70]).

We want to investigate the existence of solutions of the non-local equation

$$(-\Delta)^{\frac{1}{2}} u = \lambda u e^{\frac{1}{2}u^2} \quad \text{in } I, \quad u \equiv 0 \text{ in } \mathbb{R} \setminus I. \tag{1.4.4}$$

Theorem 1.11. *Let $I \subset \mathbb{R}$ be a bounded interval and $\lambda_1(I)$ denote the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on $H = \tilde{H}^{\frac{1}{2},2}(I)$. Then for every $\lambda \in (0, \lambda_1(I))$ Problem* (1.4.4) *has at least one* positive *solution $u \in H$ in the sense of* (1.4.6)*. When $\lambda \geq \lambda_1(I)$ or $\lambda \leq 0$ Problem* (1.4.4) *has no non-trivial non-negative solutions.*

Equation (1.4.4) is the equation satisfied by critical points of the functional $E : M_\Lambda \to \mathbb{R}$, where

$$E(u) = \int_I \left(e^{\frac{1}{2}u^2} - 1 \right) dx, \quad M_\Lambda := \{u \in H : \|u\|_H^2 = \Lambda\},$$

$\Lambda > 0$ is given, λ is a Lagrange multiplier.

Since with the variational interpretation of (1.4.4) that we discussed it is not possible to prescribe λ, we will follow the approach of Adimurthi and see solutions of (1.4.4) as critical points of the functional

$$J : H \to \mathbb{R}, \quad J(u) = \frac{1}{2}\|u\|_H^2 - \lambda \int_I \left(e^{\frac{1}{2}u^2} - 1 \right) dx. \tag{1.4.5}$$

We can compute the derivative of J

$$\langle J'(u), v \rangle := \frac{d}{dt} J(u + tv) \Big|_{t=0} = (u, v)_H - \lambda \int_I u v e^{\frac{1}{2}u^2} dx,$$

for any $u, v \in H$, where

$$(u, v)_H := \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} v \, dx.$$

In particular we have that if $u \in H$ and $J'(u) = 0$, then u is a weak solution of Problem (1.4.4) in the sense that

$$(u, v)_H = \lambda \int_I u v e^{\frac{1}{2}u^2} dx, \quad \text{for all } v \in H. \tag{1.4.6}$$

That this Hilbert-space definition of (1.4.4) is equivalent to the definition in sense of tempered distributions given by (1.3.2) is discussed in the introduction of [65].

To find critical points of J we will follow a method of Nehari, as done by Adimurthi [3].

In the two papers, [76], [77] Nehari introduced a method which turned out to be very useful in critical point theory. Consider X a real Banach space and $F \in C^1(X, \mathbb{R})$ a functional. The Frechet derivative of F at u is an element of the dual space X^*. Suppose that $u \neq 0$ is a critical point of F, i.e. $F'(u) = 0$ and define

$$N := \left\{ u \in X \setminus \{0\} : \langle F'(u), u \rangle = 0 \right\}.$$

Then naturally $u \in N$ and we see how N is as a natural constraint for the problem of finding nontrivial critical points of F. Set now

$$c := \inf_{u \in N} F(u).$$

Under appropriate conditions on F one hopes that c is attained at some $u_0 \in N$ and that u_0 is a critical point of F. More generally, $u \in X$ is a nontrivial critical point of F if and only if $u \in N$ and u is critical for the restriction of F to N. In view of this one can apply critical point theory on N to find critical points of F.

It becomes now clear that an important point is to understand whether J satisfies the Palais-Smale condition or not. We will prove the following:

Proposition 1.12. *The functional J satisfies the Palais-Smale condition at any level $c \in (-\infty, \pi)$, i.e. any sequence (u_k) with*

$$J(u_k) \to c \in (-\infty, \pi), \quad \|J'(u_k)\|_{H'} \to 0 \quad \text{as } k \to \infty \tag{1.4.7}$$

admits a subsequence strongly converging in H.

To prove Theorem 1.11 one constructs a sequence (u_k) which is almost of Palais-Smale type for J, in the sense that $J(u_k) \to \bar{c}$ for some $\bar{c} \in \mathbb{R}$ and $\langle J'(u_k), u_k \rangle = 0$. It is crucial to show that $\bar{c} < \pi$ and this will follow from (1.3.9) with $p = 2$ and $h(t) = |t|^2$. Interestingly, in the general case $s > 1$, $n \geq 2$, $p = \frac{n}{s}$, the analog of (1.3.9) is known only when s is integer or when h satisfies $\lim_{t \to \infty}(t^{-p'}h(t)) = \infty$ (see [71] and Remark 1.1 above).

Let us briefly discuss the blow-up behaviour of solutions to (1.4.4). Extending previous works in even dimension (see e.g. [6], [37], [70], [86]) A. Maalaoui, L. Martinazzi and A. Schikorra [65] studied the blow-up of sequences of solutions to the equation

$$(-\Delta)^{\frac{n}{2}} u = \lambda u e^{\frac{n}{2}u^2} \quad \text{in } \Omega \Subset \mathbb{R}^n$$

with suitable Dirichlet-type boundary conditions when n is odd. The moving plane technique for the fractional Laplacian (see [13]) implies that a non-negative solution to (1.4.4) is symmetric and monotone decreasing from the center of I. Then it is not difficult to check that in dimension one Theorem 1.5 and Proposition 2.8 of [65] yield:

Theorem 1.13. *Fix $I = (-R, R) \Subset \mathbb{R}$ and let $(u_k) \subset H = \tilde{H}^{\frac{1}{2},2}(I)$ be a sequence of non-negative solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\frac{1}{2}u_k^2} \quad \text{in } I, \tag{1.4.8}$$

in the sense of (1.4.6)*. Let* $m_k := \sup_I u_k$ *and assume that*

$$\Lambda := \limsup_{k\to\infty} \|u_k\|_H^2 < \infty.$$

Then up to extracting a subsequence we have that either

(i) $u_k \to u_\infty$ *in* $C^\ell_{loc}(I)\cap C^0(\bar{I})$ *for every* $\ell \geq 0$*, where* $u_\infty \in C^\ell_{loc}(I)\cap C^0(\bar{I})\cap H$ *solves*

$$(-\Delta)^{\frac{1}{2}} u_\infty = \lambda_\infty u_\infty e^{\frac{1}{2}u_\infty^2} \quad \text{in } I, \tag{1.4.9}$$

for some $\lambda_\infty \in (0, \lambda_1(I))$*, or*

(ii) $u_k \to u_\infty$ *weakly in* H *and strongly in* $C^0_{loc}(\bar{I}\setminus\{0\})$ *where* u_∞ *is a solution to* (1.4.9)*. Moreover, setting* r_k *such that* $\lambda_k r_k m_k^2 e^{\frac{1}{2}m_k^2}$ *and*

$$\eta_k(x) := m_k(u_k(r_k x) - m_k) + \log 2, \quad \eta_\infty(x) := \log\left(\frac{2}{1+|x|^2}\right), \tag{1.4.10}$$

one has $\eta_k \to \eta_\infty$ *in* $C^\ell_{loc}(\mathbb{R})$ *for every* $\ell \geq 0$ *and* $\Lambda \geq \|u_\infty\|_H^2 + 2\pi$*.*

The function η_∞ appearing in (1.4.10) solves the equation

$$(-\Delta)^{\frac{1}{2}} \eta_\infty = e^{\eta_\infty} \quad \text{in } \mathbb{R},$$

which has been recently interpreted in terms of holomorphic immersions of a disk (or the half-plane) by F. Da Lio, L. Martinazzi and T. Rivière [33].

Theorem 1.13 should be compared with the two dimensional case, where the analogous equation $-\Delta u = \lambda u e^{u^2}$ on the unit disk has a more precise blow-up behaviour, see e.g. [8], [6], [37], [67].

The content of this thesis is part of various research papers. Chapter 2 refers to the topics in the joint work with Gabriele Mancini [51]. Chapter 3 describes the results obtained in [49] and, jointly with Ali Maalaoui and Luca Martinazzi, in [50].

Acknowledgments

I would like to express my gratitude to my advisor, Prof. Luca Martinazzi, who introduced me to the interesting topics of this thesis. He taught me how to enjoy research and how to overcome difficulties, never giving up on me. I thank him for his precious guidance, collaboration, and advice, which made this work possible.

I am indebted to Prof. Bernhard Ruf for accepting to referee this thesis.

A special thanks goes to Ali Hyder, who has sat next to me for the past four years and has always been there, answering to all my questions with priceless patience.

I am deeply grateful to Gabriele Mancini for giving me the opportunity to work by his side and for having shared with me his passion and dedication.

Chapter 2

Extremal functions for singular Moser-Trudinger embeddings

In this Chapter we will discuss the existence of extremal functions for singular Moser-Trudinger embeddings. In Section 2.1 we propose a simple proof of Theorem 1.2 and discuss some Onofri-type inequalities. In particular, we will show how to deduce (1.2.9) from the standard Onofri inquality on S^2 and discuss its extensions to singular disks. In Section 2.2 we provide a complete and self-contained proof of a useful classification result for solutions to the singular Liouville equation, which will be crucial in our analysis. The rest of the Chapter is devoted to the proof of Theorem 1.4. In section 2.3 we will state some useful lemmas and prove existence of extremals for $E_{\Sigma,h}^{\beta,\lambda,q}$ in the subcritical case, that is when $\beta < 4\pi(1+\overline{\alpha})$. In Section 2.4 we will deal with the blow-up analysis for maximizing sequences for the critical case $\beta = 4\pi(1+\overline{\alpha})$ and we will prove an estimate similar to (1.2.5), which implies the finiteness of the supremum in (1.2.15). Finally, in Section 2.5 we will exploit a properly defined family of test functions and complete the proof of Theorem 1.4.

2.1 A Carleson-Chang type estimate via Onofri's inequality

We show how Theorem 1.2 can be proved directly by means of (1.2.9), which we shall prove at the end of this section.

Throughout this chapter we will consider the space

$$H := \left\{ u \in H_0^1(D) \ : \ \int_D |\nabla u|^2 dx \le 1 \right\}$$

and, for any $\alpha \in (-1, 0]$, the functional

$$E_\alpha(u) := \int_D |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx.$$

By (1.2.6) we have $\sup_H E_\alpha < +\infty$. For any $\delta > 0$, we will denote with D_δ the disk with radius δ centered at 0.

Remark 2.1. *With a trivial change of variables, one immediately gets that if $\delta > 0$ and $u \in H_0^1(D_\delta)$ are such that $\int_{D_\delta} |\nabla u|^2 dx \le 1$, then*

$$\int_{D_\delta} |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx \le \delta^{2(1+\alpha)} \sup_H E_\alpha.$$

In order to control the values of the Moser-Trudinger functional on a small scale, we will need the following scaled version of (1.2.9) (cfr. Lemma 1 in [19]).

Corollary 2.1. *For any $\delta, \tau > 0$ and $c \in \mathbb{R}$ we have*

$$\int_{D_\delta} e^{cu}\, dx \le \pi e^{1+\frac{c^2\tau}{16\pi}} \delta^2$$

for any $u \in H_0^1(D_\delta)$ such that $\int_{D_\delta} |\nabla u|^2\, dx \le \tau$.

As in the original proof in [19], we will first assume $\alpha = 0$ and work with radially symmetric functions. For this reason we introduce the spaces

$$H^1_{0,rad}(D) := \left\{ u \in H_0^1(D) \;:\; u \text{ is radially symmetric and decreasing} \right\}.$$

and

$$H_{rad} := H \cap H^1_{0,rad}(D).$$

Functions in H_{rad} satisfy the following useful decay estimate.

Lemma 2.1. *For any $u \in H_{rad}$, we have*

$$u(x)^2 \le -\frac{1}{2\pi}\left(1 - \int_{D_{|x|}} |\nabla u|^2 dy\right) \log|x|, \qquad \forall\, x \in D \setminus \{0\}.$$

Proof. We bound

$$
\begin{aligned}
|u(x)| &\le \int_{|x|}^{1} |u'(t)|dt \le \left(\int_{|x|}^{1} t u'(t)^2 dt\right)^{\frac{1}{2}} (-\log|x|)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{2\pi}} \left(\int_{D\setminus D_{|x|}} |\nabla u|^2 dy\right)^{\frac{1}{2}} (-\log|x|)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{2\pi}} \left(1 - \int_{D_{|x|}} |\nabla u|^2 dy\right)^{\frac{1}{2}} (-\log|x|)^{\frac{1}{2}} .
\end{aligned}
$$

□

On a sufficiently small scale, it is possible to control E_0 using only Corollary 2.1, Lemma 2.1, and Remark 2.1.

Lemma 2.2. *Take $u_k \in H_{rad}$ and $\delta_k \in (0,1)$. If $\delta_k \to 0$ and*

$$\int_{D_{\delta_k}} |\nabla u_k|^2 dx \to 0, \tag{2.1.1}$$

then

$$\limsup_{k\to\infty} \int_{D_{\delta_k}} e^{4\pi u_k^2} dx \le \pi e.$$

Proof. Take $v_k := u_k - u_k(\delta_k) \in H_0^1(D_{\delta_k})$ and set $\tau_k := \int_{D_{\delta_k}} |\nabla v_k|^2 dx = \int_{D_{\delta_k}} |\nabla u_k|^2 dx$. If $\tau_k = 0$, then $u_k \equiv u_k(\delta_k)$ in D_{δ_k} and, using Lemma 2.1, we find

$$\int_{D_{\delta_k}} e^{4\pi u_k^2} dx = \pi \delta_k^2 e^{4\pi u_k(\delta_k)^2} \le \pi < \pi e.$$

Thus, w.l.o.g. we can assume $\tau_k > 0$ for every $k \in \mathbb{N}$. By Holder's inequality and Remark 2.1 we have

$$
\begin{aligned}
\int_{D_{\delta_k}} e^{4\pi u_k^2} dx &= e^{4\pi u_k(\delta_k)^2} \int_{D_{\delta_k}} e^{4\pi v_k^2 + 8\pi u_k(\delta_k) v_k} dx \\
&\le e^{4\pi u_k(\delta_k)^2} \left(\int_{D_{\delta_k}} e^{4\pi \frac{v_k^2}{\tau_k}} dx\right)^{\tau_k} \left(\int_{D_{\delta_k}} e^{\frac{8\pi u_k(\delta_k) v_k}{1-\tau_k}} dx\right)^{1-\tau_k} \qquad (2.1.2) \\
&\le e^{4\pi u_k(\delta_k)^2} \left(\delta_k^2 \sup_H E_0\right)^{\tau_k} \left(\int_{D_{\delta_k}} e^{\frac{8\pi u_k(\delta_k) v_k}{1-\tau_k}} dx\right)^{1-\tau_k} .
\end{aligned}
$$

Applying Corollary 2.1 with $\tau = \tau_k, \delta = \delta_k$, and $c = \frac{8\pi u_k(\delta_k)}{1-\tau_k}$, we find

$$\int_{D_{\delta_k}} e^{\frac{8\pi u_k(\delta_k) v_k}{1-\tau_k}} dx \le \delta_k^2 \pi e^{1 + \frac{4\pi u_k(\delta_k)^2}{(1-\tau_k)^2} \tau_k}.$$

Thus from (2.1.2) it follows

$$\begin{aligned}\int_{D_{\delta_k}} e^{4\pi u_k^2} dx &\le \delta_k^2 \left(\sup_H E_0\right)^{\tau_k} (\pi e)^{1-\tau_k}\, e^{4\pi u_k^2(\delta_k)+\frac{4\pi u_k(\delta_k)^2\tau_k}{(1-\tau_k)}} \\ &= \delta_k^2 \left(\sup_H E_0\right)^{\tau_k} (\pi e)^{1-\tau_k}\, e^{\frac{4\pi u_k(\delta)^2}{1-\tau_k}}.\end{aligned}$$

Lemma 2.1 yields

$$\delta_k^2 e^{4\pi \frac{u_k(\delta_k)^2}{1-\tau_k}} \le 1,$$

therefore

$$\int_{D_{\delta_k}} e^{4\pi u_k^2} dx \le \left(\sup_H E_0\right)^{\tau_k} (\pi e)^{1-\tau_k}.$$

Since $\tau_k \to 0$, we obtain the conclusion by taking the lim sup as $k \to \infty$ on both sides. □

In order to prove Theorem 1.2 on H_{rad} for $\alpha = 0$, it is sufficient to show that, if $u_k \rightharpoonup 0$, there exists a sequence δ_k satisfying the hypotheses of Lemma 2.2 and such that

$$\int_{D\setminus D_{\delta_k}} \left(e^{4\pi u_k^2} - 1\right) dx \to 0. \tag{2.1.3}$$

Note that, by dominated convergence theorem, (2.1.3) holds if there exists $f \in L^1(D)$ such that

$$e^{4\pi u_k^2} \le f \tag{2.1.4}$$

in $D\setminus D_{\delta_k}$. In the next lemma we will chose a function $f \in L^1(D)$ with critical growth near 0 (i.e. $f(x) \approx \frac{1}{|x|^2 \log^2 |x|}$) and define δ_k so that (2.1.4) is satisfied.

Lemma 2.3. *Take $u_k \in H_{rad}$ such that*

$$\sup_{D\setminus D_r} u_k \to 0 \qquad \forall\, r \in (0,1). \tag{2.1.5}$$

Then there exists a sequence $\delta_k \in (0,1)$ such that

1. $\delta_k \to 0$.
2. $\tau_k := \int_{D_{\delta_k}} |\nabla u_k|^2 dx \to 0$.
3. $\int_{D\setminus D_{\delta_k}} e^{4\pi u_k^2} dx \to \pi$.

Proof. We consider the function

$$f(x) := \begin{cases} \frac{1}{|x|^2 \log^2 |x|} & |x| \le e^{-1} \\ e^2 & |x| \in (e^{-1}, 1]. \end{cases} \tag{2.1.6}$$

Note that $f \in L^1(D)$ and

$$\inf_{(0,1)} f = e^2. \tag{2.1.7}$$

Let us fix $\gamma_k \in (0, \frac{1}{k})$ such that $\int_{D_{\gamma_k}} |\nabla u_k|^2 dx \le \frac{1}{k}$. We define

$$\widetilde{\delta}_k := \inf \left\{ r \in (0,1) \;:\; e^{4\pi u_k^2(x)} \le f(x) \;\text{ for } r \le |x| \le 1 \right\} \in [0,1).$$

and

$$\delta_k := \begin{cases} \widetilde{\delta}_k & \text{if } \widetilde{\delta}_k > 0 \\ \gamma_k & \text{if } \widetilde{\delta}_k = 0. \end{cases}$$

By definition we have

$$e^{4\pi u_k^2} \le f \qquad \text{in} \quad D \backslash D_{\delta_k},$$

thus *3* follows by dominated convergence Theorem. To conclude the proof it suffices to prove that, if $k_\ell \nearrow +\infty$ is chosen so that $\delta_{k_\ell} = \widetilde{\delta}_{k_\ell}$ for any ℓ, then

$$\lim_{\ell\to\infty} \delta_{k_\ell} = \lim_{\ell\to\infty} \tau_{k_\ell} = 0. \tag{2.1.8}$$

For such k_ℓ one has

$$e^{4\pi u_{k_\ell}(\delta_{k_\ell})^2} = f(\delta_{k_\ell}). \tag{2.1.9}$$

In particular using (2.1.7) we obtain

$$e^{4\pi u_{k_\ell}(\delta_{k_\ell})^2} = f(\delta_{k_\ell}) \ge e^2 > 1,$$

which, together with (2.1.5), yields $\delta_{k_\ell} \overset{\ell\to\infty}{\to} 0$. Finally, Lemma 2.1 and (2.1.9) imply

$$1 \ge \delta_{k_\ell}^{2(1-\tau_{k_\ell})} e^{4\pi u_{k_\ell}(\delta_{k_\ell})^2} = \frac{\delta_{k_\ell}^{-2\tau_{k_\ell}}}{\log^2 \delta_{k_\ell}},$$

so that $\tau_{k_\ell} \overset{\ell\to\infty}{\to} 0$ (otherwise the limit of the RHS would be $+\infty$). □

Combining Lemma 2.2 and Lemma 2.3 we immediately get (1.2.4) for radially symmetric functions:

Proposition 2.2. *Let $u_k \in H_{rad}$ and $\alpha \in (-1, +\infty]$. If*

$$\sup_{D\backslash D_r} u_k \to 0,$$

for any $r \in (0,1)$, then

$$\limsup_{k\to\infty} E_\alpha(u_k) \le \frac{\pi(1+e)}{(1+\alpha)}.$$

Proof. If $\alpha = 0$, the proof follows directly applying Lemma 2.3 and Lemma 2.2.

If $\alpha \neq 0$, consider

$$v_k(x) = (1+\alpha)^{\frac{1}{2}} u_k(|x|^{\frac{1}{1+\alpha}}).$$

We have

$$\int_D |\nabla v_k|^2\, dx = \int_D |\nabla u_k|^2\, dx$$

and hence $v_k \in H_{rad}$. Moreover we compute

$$\int_D |x|^{2\alpha} e^{(1+\alpha)u_k^2}\, dx = \frac{1}{1+\alpha} \int_D e^{4\pi v_k^2}\, dx,$$

and the claim follows at once from the case $\alpha = 0$.

□

To pass from Proposition 2.2 to Theorem 1.2 we will use symmetric rearrangements. We recall that given a measurable function $u : \mathbb{R}^2 \to [0, +\infty)$, the symmetric decreasing rearrangement of u is the unique right-continuous radially symmetric and decreasing function $u^* : \mathbb{R}^2 \to [0, +\infty)$ such that

$$|\{u > t\}| = |\{u^* > t\}| \qquad \forall\, t > 0.$$

Among the properties of u^* we recall:

1. If $u \in L^p(\mathbb{R}^2)$, then $u^* \in L^p(\mathbb{R}^2)$ and $\|u_*\|_p = \|u\|_p$.
2. If $u \in H_0^1(D)$, then $u^* \in H^1_{0,rad}(D)$ and
$$\int_D |\nabla u^*|^2 dx \le \int_D |\nabla u|^2 dx. \tag{2.1.10}$$
3. If $u, v : \mathbb{R}^2 \to [0, +\infty)$, then
$$\int_{\mathbb{R}^2} u^*(x) v^*(x) dx \ge \int_{\mathbb{R}^2} u(x) v(x) dx. \tag{2.1.11}$$
In particular, if $u \in H_0^1(D)$ and $\alpha \le 0$,
$$\int_D |x|^{2\alpha} e^{u^*} dx \ge \int_D |x|^{2\alpha} e^{u} dx. \tag{2.1.12}$$

Note that (2.1.12) does not hold if $\alpha > 0$. We refer to [56] for a more detailed introduction to symmetric rearrangements

Proof of Theorem 1.2. Take $u_k \in H$ such that $u_k \rightharpoonup 0$ and let u_k^* be its symmetric decreasing rearrangement. Then $u_k^* \in H_{rad}$ and, since $\|u_k^*\|_2 = \|u_k\|_2 \to 0$, we have

$\sup_{D\setminus D_r} u_k^* \to 0$ for any $r > 0$. Thus, from (2.1.12) and Proposition 2.2 we get

$$\limsup_{k\to\infty} E_\alpha(u_k) \le \limsup_{k\to\infty} E_\alpha(u_k^*) \le \frac{\pi(1+e)}{1+\alpha}.$$

□

Later on we will need the following local version of Theorem 1.2.

Corollary 2.3. *Fix $\delta > 0$, $\alpha \in (-1, 0]$ and take $u_k \in H_0^1(D_\delta)$ such that $\int_{D_\delta} |\nabla u_k|^2 dx \le 1$ and $u_k \rightharpoonup 0$ in $H_0^1(D_\delta)$. For any choice of sequences $\delta_k \to 0$, $x_k \in \Omega$ such that $D_{\delta_k}(x_k) \subset D_\delta$ we have*

$$\limsup_{k\to\infty} \int_{D_{\delta_k}(x_k)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_k^2} dx \le \frac{\pi e}{1+\alpha} \delta^{2(1+\alpha)}.$$

Proof. Let us define $\widetilde{u}_k(x) := u_k(\delta x)$. Note that $\widetilde{u}_k \in H$ and it satisfies the hypotheses of Theorem 1.2. Hence

$$\begin{aligned}\limsup_{k\to\infty} \int_{D_\delta} |x|^{2\alpha}(e^{4\pi u_k^2} - 1)\, dx &= \delta^{2(1+\alpha)} \limsup_{k\to\infty} \int_D |x|^{2\alpha}(e^{4\pi \widetilde{u}_k^2} - 1)\, dx \\ &\le \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}.\end{aligned}$$

Thus we get

$$\begin{aligned}\limsup_{k\to\infty} \int_{D_{\delta_k}(x_k)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_k^2} dx &= \limsup_{k\to\infty} \int_{D_{\delta_k}(x_k)} |x|^{2\alpha} \left(e^{4\pi(1+\alpha)u_k^2} - 1\right) dx \\ &\le \int_{D_\delta} |x|^{2\alpha}(e^{4\pi u_k^2} - 1) dx \\ &\le \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}.\end{aligned}$$

□

We remark that, thanks to Theorem 1.2, in order to prove existence of extremal functions for E_α with $\alpha \in (-1, 0]$, it is enough to prove that

$$\sup_H E_\alpha > \frac{\pi(1+e)}{1+\alpha},$$

as we shall now show (see [19], [32]).

Proposition 2.4. *For any $\alpha \in (-1, 0]$ there exists a function $u_\alpha \in H$ such that*

$$E_\alpha(u_\alpha) = \sup_H E_\alpha. \tag{2.1.13}$$

Proof. We start showing that

$$\sup_H E_\alpha > \frac{\pi(1+e)}{1+\alpha}. \tag{2.1.14}$$

Let us consider the family of functions

$$u_\varepsilon(x) = \begin{cases} c_\varepsilon - \frac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right)+L_\varepsilon}{4\pi(1+\alpha)c_\varepsilon} & |x| \le \gamma_\varepsilon \varepsilon \\ -\frac{1}{2\pi c_\varepsilon}\log|x| & \gamma_\varepsilon\varepsilon \le |x| \le 1. \end{cases}$$

where $\gamma_\varepsilon = |\log\varepsilon|^{\frac{1}{1+\alpha}}$ and c_ε, L_ε will be chosen later. Io order to have $u_\varepsilon \in H_0^1(D)$ we require

$$4\pi(1+\alpha)c_\varepsilon^2 - L_\varepsilon = \log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - 2(1+\alpha)\log\varepsilon \tag{2.1.15}$$

By direct computations

$$\int_{D_{\gamma_\varepsilon\varepsilon}} |\nabla u_\varepsilon|^2 dx = \frac{1}{4\pi(1+\alpha)c_\varepsilon^2}\left(\log(1+\gamma_\varepsilon^{2(1+\alpha)}) - \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{21+\alpha}}\right)$$

and

$$\int_{D\setminus D_{\gamma_\varepsilon\varepsilon}} |\nabla u_\varepsilon|^2 dx = -\frac{1}{2\pi c_\varepsilon^2}\log(\varepsilon\gamma_\varepsilon),$$

so that

$$\int_D |\nabla u_\varepsilon|^2 dx = \frac{1}{4\pi(1+\alpha)c_\varepsilon^2}\left(\log\frac{1+\gamma^{2(1+\alpha)}}{\gamma^{2(1+\alpha)}} - \frac{\gamma^{2(1+\alpha)}}{1+\gamma^{2(1+\alpha)}} - 2(1+\alpha)\log\varepsilon\right).$$

In particular $u_\varepsilon \in H$ if we choose

$$4\pi(1+\alpha)c_\varepsilon^2 = \log\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}} - \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} - 2(1+\alpha)\log\varepsilon. \tag{2.1.16}$$

From (2.1.15) and (2.1.16) we have

$$L_\varepsilon = -\frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} = -1 + O(\gamma_\varepsilon^{-2(1+\alpha)}). \tag{2.1.17}$$

and

$$2\pi c_\varepsilon^2 = |\log\varepsilon|(1+o_\varepsilon(1)). \tag{2.1.18}$$

To compute $E_\alpha(u_\varepsilon)$ we observe first that in $D_{\gamma_\varepsilon \varepsilon}$

$$u_\varepsilon^2 = c_\varepsilon^2 \left(1 - \frac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon}{4\pi(1+\alpha)c_\varepsilon^2}\right)^2 \geq c_\varepsilon^2 \left(1 - \frac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon}{2\pi(1+\alpha)c_\varepsilon^2}\right)$$
$$= c_\varepsilon^2 - \frac{1}{2\pi(1+\alpha)}\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) - \frac{L_\varepsilon}{2\pi(1+\alpha)}.$$

Thus, using also (2.1.15) and (2.1.17),

$$\int_{D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_\varepsilon^2} dx \geq \frac{\pi\varepsilon^{2(1+\alpha)}}{1+\alpha}\frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} e^{4\pi(1+\alpha)c_\varepsilon^2 - 2L_\varepsilon} = \frac{\pi e^{-L_\varepsilon}}{1+\alpha} = \frac{\pi e}{1+\alpha} + O(\gamma_\varepsilon^{-2(1+\alpha)})$$

Finally, since $e^{4\pi(1+\alpha)u_\varepsilon^2} \geq 1 + 4\pi(1+\alpha)u_\varepsilon^2$ and

$$(1+\alpha)\int_{D\setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha}\log^2|x| dx \geq \delta > 0,$$

using (2.1.18) we get

$$\int_{D\setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_\varepsilon^2} dx \geq \int_{D\setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} dx + \frac{(1+\alpha)}{\pi c_\varepsilon^2}\int_{D\setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha}\log^2|x| dx$$
$$\geq \frac{\pi}{1+\alpha} + O((\gamma_\varepsilon\varepsilon)^{2(1+\alpha)}) + \frac{\delta}{\pi c_\varepsilon^2}$$
$$= \frac{\pi}{1+\alpha} + \frac{2\delta}{|\log\varepsilon|}(1+o_\varepsilon(1)) + O((\gamma_\varepsilon\varepsilon)^{2(1+\alpha)}).$$

Therefore

$$E(u_\varepsilon) \geq \frac{\pi(1+e)}{1+\alpha} + \frac{2\delta}{|\log\varepsilon|}(1+o_\varepsilon(1)) + O((\gamma_\varepsilon\varepsilon)^{2(1+\alpha)}) + O(\gamma_\varepsilon^{-2(1+\alpha)}).$$

Since $\gamma_\varepsilon = |\log\varepsilon|^{\frac{1}{1+\alpha}}$ one has

$$|\log\varepsilon|(\gamma_\varepsilon\varepsilon)^{2(1+\alpha)} = |\log\varepsilon|^3\varepsilon^{2(1+\alpha)} = o_\varepsilon(1)$$

and

$$|\log\varepsilon|\gamma_\varepsilon^{-2(1+\alpha)} = |\log\varepsilon|^{-1} = o_\varepsilon(1)$$

so that, for sufficiently small ε,

$$E(u_\varepsilon) \geq \frac{\pi(1+e)}{1+\alpha} + \frac{2\delta}{|\log\varepsilon|}(1+o_\varepsilon(1)) > \frac{\pi(1+e)}{1+\alpha}.$$

Now we conclude the proof showing that for any $\alpha \in (-1, 0]$ there exists a function $u_\alpha \in H$ satisfying (2.1.13). Let $u_k \in H$ be a maximizing sequence for E_α. Up to subsequences, we may assume $u_k \rightharpoonup u$. If $u = 0$, then by Theorem 1.2 we would have

$$\sup_H E_\alpha = \lim_{k\to\infty} E_\alpha(u_k) \le \frac{\pi(1+e)}{1+\alpha},$$

which contradicts (2.1.14). Thus $u \neq 0$. Since

$$\limsup_{k\to\infty} \|\nabla(u_k - u)\|_2^2 = 1 - \|\nabla u\|_2 < \gamma < 1,$$

by (1.2.6) we find

$$\int_D |x|^{2\alpha} e^{\frac{4\pi s(1+\alpha)}{\gamma}(u_k-u)^2} dx \le C$$

for some $s > 1$. If we take $1 < p < \frac{1}{\gamma}$, then

$$pu_k^2 = p(u_k - u)^2 + pu^2 + 2pu(u_k - u) \le \frac{1}{\gamma}(u_k - u)^2 + C_{\gamma,p}u^2$$

so that

$$\int_D |x|^{2\alpha} e^{4\pi p(1+\alpha)u_k^2} dx \le \int_D |x|^{2\alpha} e^{\frac{4\pi(1+\alpha)}{\gamma}(u_k-u)^2} e^{C_{\gamma,p}u^2} dx$$
$$\le \left(\int_D |x|^{2\alpha} e^{\frac{4\pi s(1+\alpha)}{\gamma}(u_k-u)^2} dx\right)^{\frac{1}{s}} \left(\int_D |x|^{2\alpha} e^{s' C_{\gamma,\varepsilon} u^2} dx\right)^{\frac{1}{s'}} \le C.$$

Applying Vitali's convergence Theorem to the measure $|x|^{2\alpha}dx$ we find

$$E_\alpha(u_k) \to E_\alpha(u),$$

which concludes the proof. □

Onofri-type inequalities for disks

We shall now prove Proposition 1.3 and discuss how to get singular Onofri-type inequalities for the unit disk.

Let (Σ, g) be a smooth closed Riemannian surface. As a consequence of (1.7) one gets

$$\log\left(\frac{1}{|\Sigma|}\int_\Sigma e^{u-\overline{u}} dv_g\right) \le \frac{1}{16\pi}\int_\Sigma |\nabla_g u|^2 dv_g + C(\Sigma, g). \tag{2.1.19}$$

While it is well known that the coefficient $\frac{1}{16\pi}$ is sharp, the optimal value of $C(\Sigma, g)$ is harder to determine. For the special case of the standard Euclidean sphere (S^2, g_0), Onofri ([80]) proved that $C(S^2, g_0) = 0$ and gave a complete characterization of the extremal functions for (2.1.19).

Proposition 2.5 ([80])**.** *For any $u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi}\int_{S_2} |\nabla_{g_0} u|^2 dv_{g_0},$$

with equality holding if and only if $e^u g_0$ is a metric on S^2 with positive constant Gaussian curvature, or, equivalently, $u = \log|\det d\varphi| + c$ with $c \in \mathbb{R}$ and $\varphi : S^2 \to S^2$ a conformal diffeomorphism of S^2.

As we shall see, Proposition 1.3 is easily proved by means of the stereographic projection.

Proof. Let us fix Euclidean coordinates (x_1, x_2, x_3) on $S^2 \subseteq \mathbb{R}^3$ and denote $N := (0,0,1)$ and $S = (0,0,-1)$ the north and the south pole. Let us consider the stereographic projection $\pi : S^2 \backslash \{N\} \to \mathbb{R}^2$,

$$\pi(x) := \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

It is well known that π is a conformal diffeomorphism and

$$\left(\pi^{-1}\right)^* g_0 = e^{u_0}|dx|^2, \tag{2.1.20}$$

where

$$u_0(x) = \log\left(\frac{4}{(1+|x|^2)^2}\right) \tag{2.1.21}$$

satisfies

$$-\Delta u_0 = 2e^{u_0} \qquad \text{on } \mathbb{R}^2. \tag{2.1.22}$$

Given $r > 0$, let $D_r := \{x \in \mathbb{R}^2 \;:\; |x| < r\}$ be the disk of radius r and $S^2_r = \pi^{-1}(D_r)$. We consider the map $T_r : H^1_0(D_r) \to H^1(S^2)$, defined by

$$T_r u(x) := \begin{cases} u(\pi(x)) - u_0(\pi(x)) & \text{on } S^2_r \\ -2\log\left(\frac{2}{1+r^2}\right) & \text{on } S^2\backslash S^2_r. \end{cases}$$

Using (2.1.20) we find

$$\int_{S^2} e^{T_r u} dv_{g_0} \ge \int_{S^2_r} e^{T_r u} dv_{g_0} = \int_{D_r} e^{T_r u(\pi^{-1}(y))} e^{u_0} dy = \int_{D_r} e^{u(y)} dy. \tag{2.1.23}$$

Moreover, by (2.1.22),

$$\begin{aligned}
\int_{S^2_r} |\nabla T_r u|^2 dv_{g_0} &= \int_{D_r} |\nabla u|^2 dx - 2\int_{D_r} \nabla u_0 \cdot \nabla u \; dy + \int_{D_r} |\nabla u_0|^2 dy \\
&= \int_{D_r} |\nabla u|^2 dy - 4\int_{D_r} u e^{u_0} dy + \int_{D_r} |\nabla u_0|^2 dy \\
&= \int_{D_r} |\nabla u|^2 dy - 4\int_{S^2_r} T_r u \; dv_{g_0} + \left(\int_{D_r} |\nabla u_0|^2 dy - 4\int_{D_r} u_0 e^{u_0} dy\right).
\end{aligned}$$

With a direct computation it is easy to check that

$$\int_{D_r} |\nabla u_0|^2 dy = 16\pi \left(\log(1+r^2) - \frac{r^2}{1+r^2} \right)$$

and

$$\int_{D_r} u_0 e^{u_0} dy = 8\pi \log 2 - 8\pi + o(1),$$

where $o(1) \to 0$ as $r \to +\infty$. Thus we get

$$\int_{S^2} |\nabla T_r u|^2 \, dv_{g_0} + 4 \int_{S^2} T_r u \, dv_{g_0} = \int_{D_r} |\nabla u|^2 dy + 16\pi \left(\log(1+r^2) + 1 - 2\log 2 + o(1) \right). \tag{2.1.24}$$

Using (2.1.23), (2.1.24), and Proposition 2.5, we can conclude

$$\begin{aligned} \log\left(\frac{1}{\pi} \int_{D_r} e^u dy \right) &\le \log\left(\frac{1}{\pi} \int_{S^2} e^{T_r u} dv_{g_0} \right) \\ &\le \frac{1}{16\pi} \left(\int_{S^2} |\nabla T u|^2 dv_{g_0} + 2 \int_{S^2} T u \, dv_{g_0} \right) + 2\log 2 \\ &\le \frac{1}{16\pi} \int_{D_r} |\nabla u|^2 dy + \log(1+r^2) + 1 + o(1). \end{aligned} \tag{2.1.25}$$

Now, if $u \in H^1_0(D)$, we can apply (2.1.25) to $u_r(y) = u(\frac{y}{r})$ and, since

$$\int_D e^u dx = \frac{1}{r^2} \int_{D_r} e^{u_r(y)} dy \qquad \text{and} \qquad \int_D |\nabla u|^2 dx = \int_{D_r} |\nabla u_r|^2 dy,$$

we find

$$\log\left(\frac{1}{\pi} \int_D e^u dx \right) \le \frac{1}{16\pi} \int_D |\nabla u|^2 dx + 1 + o(1).$$

As $r \to \infty$ we get the conclusion. □

As in [5], starting from (1.2.9) we can use a simple change of variables to obtain singular Onofri-type inequalities for the unit disk.

Proposition 2.6. *Let $-1 < \alpha \le 0$. Then for any $u \in H^1_0(D)$ we have*

$$\log\left(\frac{1+\alpha}{\pi} \int_D |x|^{2\alpha} e^u dx \right) \le \frac{1}{16\pi(1+\alpha)} \int_D |\nabla u|^2 \, dx + 1. \tag{2.1.26}$$

Moreover, if we restrict ourselves to the space $H^1_{0,rad}(D)$, (2.1.26) *holds true for any $\alpha \in (-1, +\infty]$.*

Proof. As we did in the proof of Proposition 2.2, for $u \in H^1_{0,rad}(D)$ we consider the function $v(x) = u(|x|^{\frac{1}{1+\alpha}})$, which is again in $H^1_{0,rad}(D)$. The second claim follows at once applying (1.2.9) to v. As for the first claim, if $\alpha \le 0$ we can use symmetric rearrangements to remove the symmetry assumption, as we did in the proof of Theorem 1.2.

□

Since

$$\int_D |x|^{2\alpha} dx = \frac{\pi}{1+\alpha},$$

Proposition 2.6 can be written in a simpler form in terms of the singular metric $g_\alpha = |x|^{2\alpha}|dx|^2$.

Corollary 2.7. *If $u \in H_0^1(D)$ and $-1 < \alpha \le 0$ (or $\alpha > 0$ and $u \in H^1_{0,rad}(D)$), we have*

$$\log\left(\frac{1}{|D|_\alpha}\int_D e^u dv_{g_\alpha}\right) \le \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dv_{g_\alpha} + 1,$$

where $|D|_\alpha = \frac{\pi}{(1+\alpha)}$ is the measure of D with respect to g_α.

We stress that the constant 1 appearing in Proposition 2.6 is sharp.

Proposition 2.8. *For any $-1 < \alpha \le 0$ we have*

$$\inf_{u\in H_0^1(D)} \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx - \log\left(\frac{1}{|D|_\alpha}\int_D |x|^{2\alpha} e^u dx\right) = -1.$$

Moreover, if we restrict ourselves to the space $H^1_{0,rad}(D)$, the conclusion above holds true for any $\alpha \in (-1,+\infty)$.

Proof. Let us denote

$$E_\alpha(u) := \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx - \log\left(\frac{1}{|D|_\alpha}\int_D |x|^{2\alpha} e^u dv_g\right).$$

It is sufficient to exhibit a family of functions $u_\varepsilon \in H^1_{0,rad}(D)$ such that $E_\alpha(u_\varepsilon) \overset{\varepsilon\to 0}{\to} -1$. Take $\gamma_\varepsilon \overset{\varepsilon\to 0}{\to} +\infty$ such that $\varepsilon\gamma_\varepsilon \overset{\varepsilon\to 0}{\to} 0$, and define

$$u_\varepsilon(x) = \begin{cases} -2\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon & \text{for } |x| \le \gamma_\varepsilon\varepsilon \\ -4(1+\alpha)\log|x| & \text{for } \gamma_\varepsilon\varepsilon \le |x| \le 1, \end{cases}$$

where the quantity

$$L_\varepsilon := 2\log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - 4(1+\alpha)\log\varepsilon$$

is chosen so that $u_\varepsilon \in H_0^1(D)$. Simple computations show that

$$\frac{1}{16\pi(1+\alpha)}\int_D |\nabla u_\varepsilon|^2 dx = -1 - 2(1+\alpha)\log\varepsilon + o_\varepsilon(1)$$

and

$$\int_D |x|^{2\alpha} e^{u_\varepsilon} dx = \frac{\varepsilon^{2(1+\alpha)} \gamma_\varepsilon^{2(1+\alpha)} e^{L_\varepsilon} \pi}{(1+\alpha)(1+\gamma_\varepsilon^{2(1+\alpha)})} + \frac{\pi}{1+\alpha} \left(\frac{1}{(\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}} - 1 \right)$$
$$= \frac{\pi \varepsilon^{-2(1+\alpha)}}{1+\alpha} (1 + o_\varepsilon(1)).$$

Thus

$$E_\alpha(u_\varepsilon) \to -1.$$

□

To conclude we remark that Propositions 2.6 and 2.8 can also be deduced directly using the singular versions of Proposition 2.5 proved in [68], [69]. We also point out that, as we did for the Carleson-Chang type estimates, one can have a singular version of the Onofri inequality (1.2.9) (see Proposition 2.6). In particular, one can deduce the following generalized version of Corollary 2.1.

Corollary 2.9. *Fix $\delta, \tau > 0$, $c \in \mathbb{R}$, and $\alpha \in (-1, 0]$. We have*

$$\int_{D_\delta} |x|^{2\alpha} e^{cu}\, dx \le \frac{\pi e^{1+\frac{c^2\tau}{16\pi(1+\alpha)}} \delta^{2(1+\alpha)}}{1+\alpha},$$

for any $u \in H_0^1(D_\delta)$ such that $\int_{D_\delta} |\nabla u|^2\, dx \le \tau$.

2.2 Classification of solutions to the singular Liouville equation

In this section we will deal with a singular version of the well known Liouville equation. More precisely, we consider $\alpha > -1$ and study some qualitative properties of solutions of

$$\begin{cases} -\Delta u = |x|^{2\alpha} e^u \text{ on } \mathbb{R}^2, \\ \Theta = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2\alpha} e^u\, dx < \infty. \end{cases} \tag{2.2.1}$$

We would like to thank Prof. Gabriella Tarantello who, after reading the results in this section, pointed us to [42], where Theorem 2.10 is proved in a more general setting.

Problems of this type come from different areas of mathematics and physics ([11], [51], [55], [68], [79]).

The existence and the qualitative properties of solutions to Problem (2.2.1) have been studied in different settings (see for example [9], [28], [29], [30], [72], [78] and the references therein).

If $\alpha = 0$ all the solutions to Problem (2.2.1) have been classified and are known to be radially symmetric (see [28]). The case $\alpha > 0$ has a richer structure. Indeed a symmetry result can be recovered using the method of moving planes [28]. This can be done only if $\Theta > 2$, u behaves logaritmically at infinity and $\alpha < 0$ (this is true for more general potentials, see [29] for a reference). Notice that the assumption on u here it is not restrictive (cfr. [29]). In fact any solution to Problem (2.2.1) has a logarithmic behaviour at infinity, namely we have

$$u(x) = -\Theta \log |x| + O(1) \tag{2.2.2}$$

and in particular it holds

$$\Theta = 4(\alpha + 1). \tag{2.2.3}$$

Condition (2.2.3) can be seen as a Kazdan-Warner type condition and it is crucial, for instance, in classification type results as the one proposed in [85] or to perform a fine blow up analysis when singular potentials are involved [51], [68].

In view of the results stated in [29], Prajapat and Tarantello [85] exploit the necessity of condition (2.2.3) to classify the solutions of

$$\begin{cases} -\Delta u = |x|^{2\alpha} e^u \text{ on } \mathbb{R}^2, \\ \Theta = 4(\alpha + 1), \end{cases} \tag{2.2.4}$$

where $\alpha > -1$. Namely they showed that any solution of (2.2.4) is radially symmetric for $\alpha \notin \mathbb{N}$, while there are no radially symmetric solutions for $\alpha \in \mathbb{N}$, $\alpha \geq 1$ (see [22]).

Remark 2.2. *Notice that for $\alpha \in (-1, 0)$ the condition $\Theta = 4(1+\alpha)$ in Problem* (2.2.4) *is an assumption and does not follow from the result in [29]. The validity of condition* (2.2.3) *for any $\alpha > -1$ will play a crucial role later in Section 2.4.*

Here we consider Problem (2.2.1) for any $\alpha > -1$ and give a unified proof, consistent with the one proposed in [29], of the asymptotic behaviour (2.2.2) and condition (2.2.3).

Theorem 2.10. *Let $\alpha > -1$ and let u be a solution to*

$$-\Delta u = |x|^{2\alpha} e^u \qquad \text{on } \mathbb{R}^2$$

with

$$\Theta = \frac{1}{2\pi}\int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx < \infty.$$

Then we have

$$u(x) = -\Theta \log|x| + O(1),$$

with $\Theta > 2(1+\alpha)$*. Moreover it holds*

$$\Theta = 4(1+\alpha).$$

Proof of Theorem 2.10

We begin proving two premilinar lemmas that will be used later in the proof. In what follows $B_R(x)$ will denote the ball of radius R centered at x (the dependence on x will often be omitted if $x = 0$) and C will denote a generic constant that can change from line to line.

Lemma 2.4. *Let* $\alpha > -1$ *and* u *be a solution to Problem* (2.2.1)*. Then for any* $x \in \mathbb{R}^2$ *we have*

$$\fint_{B_R(x)} u^+ \, dy \to 0,$$

as $R \to \infty$*.*

Proof. Fix $x \in \mathbb{R}^2$ and consider $\alpha \geq 0$. We trivially bound

$$\begin{aligned}\fint_{B_R(x)} u^+ \, dy &\leq \fint_{B_R(x)} e^u \, dy \\ &\leq \frac{C}{R^2}\int_{\mathbb{R}^2} |y|^{2\alpha} e^u \, dy \to 0 \qquad \text{as } R \to +\infty.\end{aligned}$$

Consider now $\alpha \in (-1, 0)$. With $u^+ \leq e^u$, multiplying and dividing by $|y|^{2\alpha}$ we get

$$\begin{aligned}\fint_{B_R(x)} u^+ \, dy &\leq \fint_{B_R(x)} e^u \, dy \\ &\leq \frac{C(R+|x|)^{-2\alpha}}{R^2}\int_{B_R(x)} |y|^{2\alpha} e^u \, dy \\ &\leq \frac{C(R+|x|)^{-2\alpha}}{R^2},\end{aligned}$$

where we used that for $y \in B_R(x)$ we have $|y| \leq R + |x|$ and that $\int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx < \infty$. The claim follows letting $R \to \infty$ since $\alpha \in (-1, 0)$. □

Lemma 2.5. *Let* $f \in L^\infty(B_1)$*. Consider* $\alpha > -1$ *and let and* u *be a solution to*

$$-\Delta u = |x|^{2\alpha} f \quad \text{in } B_1. \tag{2.2.5}$$

There exist $C>0$ such that

(*i*) $|\nabla u(x)| \leq C|x|^{2\alpha+1}$ *if* $\alpha < -\frac{1}{2}$,

(*ii*) $|\nabla u(x)| \leq -C\log|x|$ *if* $\alpha = -\frac{1}{2}$,

(*iii*) $|\nabla u(x)| \leq C$ *if* $\alpha > -\frac{1}{2}$.

Proof. Consider $\widetilde{u}$ to be a solution to

$$\begin{cases} -\Delta\widetilde{u} = |x|^{2\alpha}f & \text{in } B_1,\\ \widetilde{u} = 0 & \text{on } \partial B_1. \end{cases}$$

It is clear that the difference $u-\widetilde{u}$ is harmonic and hence $C^\infty(B_1)$. Therefore it is enough to prove our statement for $\widetilde{u}$. First observe that $h := |x|^{2\alpha}f \in L^p(B_1)$ for some $p>1$ and standard elliptic estimates imply that for $\alpha > -\frac{1}{2}$ we have $\widetilde{u} \in C^1(B_1)$ and (*iii*) follows. To prove (*i*) and (*ii*) we will make use of a Green representation formula. We write

$$\widetilde{u}(x) = \int_{B_1} G(x,y)|y|^{2\alpha}f(y)\,dy.$$

It follows immediately with $|\nabla G(x,y)| \leq C\frac{1}{|x-y|}$ that

$$|\nabla\widetilde{u}|(x) \leq C||f||_{L^\infty(B_1)}\int_{B_1}\frac{|y|^{2\alpha}}{|x-y|}\,dy.$$

With $|x|t=y$ we get

$$\int_{B_1}\frac{|y|^{2\alpha}}{|x-y|}\,dy = |x|^{2\alpha+1}\int_{B_{\frac{1}{|x|}}}\frac{|t|^{2\alpha}}{|\frac{x}{|x|}-t|}\,dt.$$

Let us define the sets $A_1 := \left\{|\frac{x}{|x|}-y| \leq \frac{1}{2}\right\}$, $A_2 := \{|y|\leq 2\}$ and $A_3 := \left\{2 \leq |y| \leq \frac{1}{|x|}\right\}$. We have

$$\begin{aligned}
&\int_{\left\{|y|\leq\frac{1}{|x|}\right\}}\frac{|y|^{2\alpha}}{|\frac{x}{|x|}-y|}\,dy\\
&\leq \frac{1}{2^{2\alpha}}\int_{A_1}\frac{1}{|\frac{x}{|x|}-y|}\,dy + 2\int_{A_2}|y|^{2\alpha}\,dy + 2\int_{A_3}|y|^{2\alpha-1}\,dy\\
&\leq C + 2\int_{A_3}|y|^{2\alpha-1}\,dy.
\end{aligned}$$

If now $\alpha < \frac{1}{2}$ we have

$$\int_{A_3}|y|^{2\alpha-1}\,dy \leq C.$$

On the other hand if $\alpha = -\frac{1}{2}$ we compute

$$\int_{A_3} |y|^{2\alpha-1}\,dy = 2\pi \log \frac{1}{2|x|} \leq C(-\log|x|).$$

□

We will mainly follow the proof in [29]. As a first step we will prove the following proposition.

Proposition 2.11. *Let u be a solution to* (2.2.1) *and consider $\alpha > -1$. We have*

$$\frac{u(x)}{\log|x|} \to -\Theta,$$

uniformly as $|x| \to \infty$.

Define the function v as follows

$$v(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{|y|}{|x-y|}\right) |y|^{2\alpha} e^{u(y)}\,dy. \tag{2.2.6}$$

Lemma 2.6. *Let u be a solution to Problem* (2.2.1) *and v as in* (2.2.6)*. Then for $|x| \geq 4$ we have*

$$v(x) \geq -\Theta \log|x| + C. \tag{2.2.7}$$

Proof. Fix $x \in \mathbb{R}^2$ such that $|x| \geq 4$. Decompose $\mathbb{R}^2 = A_1 \cup A_2 \cup B_2$, where $B_2 = B_2(0)$, $A_1 = B_{|x|/2}(x)$, $A_2 = \mathbb{R}^2 \setminus (A_1 \cup B_2)$. Let $y \in A_1$. Notice that A_1, A_2, B_2 are disjoint sets. An easy application of the triangular inequality leads to

$$\int_{A_1} \log \frac{|y|}{|x-y|} |y|^{2\alpha} e^u\,dy \geq 0. \tag{2.2.8}$$

Let us now consider $y \in A_2$. Since $|y|, |x| \geq 2$ it holds

$$\int_{A_2} \log \frac{|y|}{|x-y|} |y|^{2\alpha} e^u\,dy \geq -\log|x| \int_{A_2} |y|^{2\alpha} e^u\,dy. \tag{2.2.9}$$

As for $y \in B_2$ we have that $\log|x-y| \leq \log|x| + C$. Note that $|y|^{2\alpha} \in L^1(B_2)$ for $\alpha > -1$. With $u \in L^\infty_{loc}(\mathbb{R}^2)$ we can bound

$$\begin{aligned}
&\int_{B_2} \log \frac{|y|}{|x-y|} |y|^{2\alpha} e^u\,dy \\
&\geq \int_{B_2} \log|y| |y|^{2\alpha} e^u\,dy - \log|x| \int_{B_2} |y|^{2\alpha} e^u\,dy \\
&- C \int_{B_2} |y|^{2\alpha} e^u\,dy \geq -\log|x| \int_{B_2} |y|^{2\alpha} e^u\,dy + C.
\end{aligned} \tag{2.2.10}$$

Combining (2.2.8), (2.2.9) and (2.2.10) we get

$$\begin{aligned} v(x) &\geq \frac{1}{2\pi}\int_{A_2\cup B_2} \log\frac{|y|}{|x-y|}|y|^{2\alpha}e^u\,dy \\ &\geq -\frac{1}{2\pi}\log|x|\int_{A_2\cup B_2}|y|^{2\alpha}e^u\,dy + C \\ &\geq -\Theta\log|x| + C, \end{aligned}$$

proving our claim. □

Lemma 2.7. *Let* u *be a solution to* (2.2.1) *and* v *defined as in* (2.2.6)*. Then* $u = v + C$*.*

Proof. Define $w := u - v$. It is straightforward that $\Delta w = 0$. We will prove that w is constant. Consider $x \in \mathbb{R}^2$ and fix some $R > 0$. Since w is harmonic in $\mathbb{R}^2$, thanks to the mean value theorem we have

$$|w(x)| \leq \frac{C}{R}\int_{B_R(x)}|w(y)|\,dy.$$

It follows that (see [38, Theorem 7, pg. 29]) for a reference)

$$|Dw(x)| \leq \frac{C}{R}\fint_{B_R(x)}|w(y)|\,dy \leq -\frac{C}{R}\fint_{B_R(x)}w(y)\,dy + \frac{C}{R}\fint_{B_R(x)}w^+(y)\,dy,$$

where in the last inequality we used that $w = w^+ + w^-$ and $|w| = w^+ - w^-$, hence $|w| = 2w^+ - w$. Again the mean value theorem implies that

$$\frac{C}{R}\fint_{B_R(x)}w(y)\,dy = \frac{C}{R}w(x) \to 0,$$

as $R \to \infty$ for any fixed x. Moreover with our definition of w, Lemma 2.6 and Lemma 2.4, we have

$$\frac{C}{R}\fint_{B_R(x)}w^+(y)\,dy \leq \frac{C}{R}\fint_{B_R(x)}w^+(y)\,dy + \frac{C}{R}\fint_{B_R(x)}\log|y|\,dy + \frac{C}{R} \to 0,$$

as $R \to \infty$, proving that $Dw \to 0$ as $R \to \infty$.

□

Lemma 2.8. *Let* u *be a solution to* (2.2.1) *and consider* Θ *as in* (2.2.1)*. We have*

$$\Theta > 2(1+\alpha).$$

Proof. Fix $R > 0$. Since u solves (2.2.1), using Lemma 2.6 we bound

$$
\begin{aligned}
C &\geq \int_{\mathbb{R}^2} |x|^{2\alpha} e^u \, dx \geq \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{2\alpha} e^u \, dx \\
&= \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{2\alpha} e^{v+C} \, dx \geq C \int_{\mathbb{R}^2 \setminus B_R(0)} |x|^{2\alpha - \Theta} \, dx.
\end{aligned}
$$

Hence $\Theta > 2(\alpha + 1)$. □

Lemma 2.9. *For every $p \in (1, \infty)$ there exists $C = C(p, \alpha) > 0$ such that for $|x|$ large*

$$
|x|^{2p\alpha} \int_{B_1(x)} e^{pu(y)} dy \leq C.
$$

Proof. First we observe that for any $\varepsilon > 0$ there exists $K > 0$ such that for $|x| \geq K$

$$
v(x) \leq \left(-\frac{\Theta}{2\pi} + \varepsilon \right) \log |x| + \frac{1}{2\pi} \int_{B_1(x)} \log \left(\frac{1}{|x-y|} \right) |y|^{2\alpha} e^u dy. \tag{2.2.11}
$$

The proof of (2.2.11) is very similar to the proof of (2.11) in [61, Lemma 2.4], and it will be omitted here. We shall rewrite (2.2.11) as

$$
v(x) \leq (-\Theta + \varepsilon) \log |x| + \frac{1}{2\pi} \int_{B_R^c} \log \left(\frac{1}{|x-y|} \right) \chi_{|x-y|<1} |y|^{2\alpha} e^u dy, \quad |x| \geq K.
$$

We set

$$
\|f\|_R = \|f\|_{L^1(B_R^c)} \quad \text{and } f(y) := |y|^{2\alpha} e^{u(y)}.
$$

Notice that $\|f\|_R \leq \delta$ for large R since $f \in L^1(\mathbb{R}^2)$. From Jensen's inequality follows for $|x| \geq \max\{R+2, K\}$

$$
\begin{aligned}
e^{pv(x)} &\leq |x|^{p(-\Theta+\varepsilon)} exp \left(\int_{B_R^c} \frac{p}{2\pi} \|f\|_R \log \left(\frac{1}{|x-y|} \right) \chi_{|x-y|<1} \frac{f(y)}{\|f\|_R} dy \right) \\
&\leq |x|^{p(-\Theta+\varepsilon)} \int_{B_R^c} exp \left(\frac{p}{2\pi} \|f\|_R \log \left(\frac{1}{|x-y|} \right) \chi_{|x-y|<1} \right) \frac{f(y)}{\|f\|_R} dy \\
&= |x|^{p(-\Theta+\varepsilon)} \left(\int_{B_1(x)} \left(\frac{1}{|x-y|} \right)^{\frac{p}{2\pi}\|f\|_R} \frac{f(y)}{\|f\|_R} dy + \int_{B_R^c \cap B_1(x)^c} \frac{f(y)}{\|f\|_R} dy \right) \\
&\leq |x|^{p(-\Theta+\varepsilon)} \left(\int_{B_1(x)} \left(\frac{1}{|x-y|} \right)^{\frac{p}{2\pi}\|f\|_R} \frac{f(y)}{\|f\|_R} dy + 1 \right).
\end{aligned}
$$

Again for $|x_0| \geq \max\{R+2, K\}$ we get

$$\begin{aligned}
\int_{B_1(x_0)} e^{pv(x)}dx &\leq C|x_0|^{p(-\Theta+\varepsilon)} \int_{B_1(x_0)} \left(\int_{B_1(x)} \left(\frac{1}{|x-y|}\right)^{\frac{p}{2\pi}\|f\|_R} \chi(x,y)_{\{|x-y|<1\}} \frac{f(y)}{\|f\|_R} dy + 1 \right) dx \\
&\leq C|x_0|^{p(-\Theta+\varepsilon)} \int_{B_1(x_0)} \left(\int_{B_2(x_0)} \left(\frac{1}{|x-y|}\right)^{\frac{p}{2\pi}\|f\|_R} \chi(x,y)_{\{|x-y|<1\}} \frac{f(y)}{\|f\|_R} dy + 1 \right) dx \\
&\leq C|x_0|^{p(-\Theta+\varepsilon)} \left(\int_{B_4(x_0)} \frac{f(y)}{\|f\|_R} \int_{B_1(x_0)} \left(\frac{1}{|x-y|}\right)^{\frac{p}{2\pi}\|f\|_R} \chi(x,y)_{\{|x-y|<1\}} \, dx\, dy + C \right) \\
&\leq C|x_0|^{p(-\Theta+\varepsilon)}.
\end{aligned}$$

Since $u = v + C$ thanks to Lemma 2.7, for $|x_0| \geq \max\{R+2, K\}$, one has

$$\begin{aligned}
|x_0|^{2p\alpha} \int_{B_1(x_0)} e^{pu(y)}dy &\leq C|x_0|^{2p\alpha} \int_{B_1(x_0)} e^{pv(y)}dy \\
&\leq C|x_0|^{p(-\Theta+\varepsilon+2\alpha)}.
\end{aligned}$$

From Lemma 2.8 we get that $p(-\Theta+\varepsilon+2\alpha) < 0$ for any $p \in (1, +\infty)$ and the claim follows. □

Lemma 2.10. *We have*

$$v(x) \leq (-\Theta+\varepsilon)\log|x| + C$$

for $|x|$ large.

Proof. The result will follow from (2.2.11) once we get a control on the second term in the RHS of (2.2.11), which could be really big. With Hölder's inequality we have for $p \in (1, +\infty)$

$$\begin{aligned}
\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)|y|^{2\alpha}e^u dy &\leq \left(\int_{B_1(x)} |y|^{2p\alpha}e^{pu}\,dy\right)^{\frac{1}{p}} \left(\int_{B_1(x)} \log\left(\frac{1}{|x-y|}\right)^{p'}\right)^{\frac{1}{p'}} \\
&\leq C\left(\int_{B_1(x)} |y|^{2p\alpha}e^{pu}\,dy\right)^{\frac{1}{p}},
\end{aligned} \tag{2.2.12}$$

where p' satisfies $1/p + 1/p' = 1$. Lemma 2.9 gives the boundedness of the last term in (2.2.12), concluding the proof.

□

Proof of Proposition 2.11. Since $u = C + v$, thanks to Lemma 2.6 and Lemma 2.10, we bound

$$-\Theta\log|x| + C \leq C + v \leq (-\Theta+\varepsilon)\log|x| + C.$$

Therefore we get the thesis as $|x| \to \infty$. □

To prove Theorem 2.10 it remains to compute the exact value of Θ, which is the content of the next proposition.

Proposition 2.12. *Let u be a solution of* (2.2.1) *and Θ defined as in* (2.2.1)*. We have*

$$\Theta = 4(1+\alpha).$$

The proof will follow from the next lemmas. Define

$$\varphi(x) = u(x) + \Theta \log |x| \tag{2.2.13}$$

$$\widetilde{\varphi}(x) = \varphi\left(\frac{x}{|x|^2}\right). \tag{2.2.14}$$

Lemma 2.11. *Let $\widetilde{\varphi}$ be as in* (2.2.14)*. We have*

$$\widetilde{\varphi}(x) = o(|\log |x||)$$

as $|x| \to 0$.

Proof. Using (2.2.14) and $|\frac{x}{|x|^2}| = \frac{1}{|x|}$ we compute

$$\frac{\widetilde{\varphi}(x)}{\log |x|} = \varphi\left(\frac{x}{|x|^2}\right)\frac{1}{\log |x|} = -\frac{u\left(\frac{x}{|x|^2}\right)}{\log\left|\frac{x}{|x|^2}\right|} - \Theta.$$

Thanks to Proposition 2.11, as $|x| \to 0$ we get the thesis. □

Lemma 2.12. *Let $\alpha > -1$ and consider u a solution to* (2.2.1)*. We have*

$$u(x) = -\Theta \log |x| + O(1).$$

Proof. Observe that using (2.2.13), (2.2.14) and (2.2.1) we get that $\widetilde{\varphi}$ satisfies

$$-\Delta\widetilde{\varphi}(x) = |x|^{\Theta-4-2\alpha} e^{\widetilde{\varphi}(x)} \qquad \text{in } \mathbb{R}^2 \setminus \{0\}. \tag{2.2.15}$$

Moreover from Lemma 2.8 we have that $\Theta - 4 - 2\alpha > -2$ and in particular that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\varepsilon \log |x| \le \widetilde{\varphi}(x) \le -\varepsilon \log |x| \qquad \text{in } B_R(0).$$

Therefore $e^{\widetilde{\varphi}} \leq |x|^{-\varepsilon}$ in $B_R(0)$. By choosing ε such that $\Theta - 4 - 2\alpha - \varepsilon > -2$ we get there exists $p > 1$ so that

$$-\Delta\widetilde{\varphi} = |x|^{\Theta-4-2\alpha} e^{\widetilde{\varphi}} \in L^p(B_R(0)). \tag{2.2.16}$$

Let now η be such that

$$\begin{cases} -\Delta \quad \eta = |x|^{\Theta-4-2\alpha} e^{\varphi} & \text{in } B_R(0), \\ \qquad \eta = 0 \quad \text{on } \partial B_R(0). \end{cases}$$

Standard elliptic estimates and (2.2.16) imply that $\eta \in C^0(B_R(0))$.

A direct application of the Removable Singularity Theorem to $\Phi := \eta - \widetilde{\varphi}$ yelds

$$|\widetilde{\varphi}| \leq C \qquad \text{in } B_R(0), \tag{2.2.17}$$
$$-\Delta\widetilde{\varphi} = |x|^{\Theta-4-2\alpha} e^{\widetilde{\varphi}} \qquad \text{in } \mathbb{R}^2. \tag{2.2.18}$$

It follows from (2.2.14) that φ is bounded for $|x| > \frac{1}{R}$ and hence that $u = -\Theta \log|x| + O(1)$ for $|x| > \frac{1}{R}$, concluding the proof. ☐

From (2.2.17) and (2.2.18) we have that $\widetilde{\varphi}$ solves, for some small $R > 0$, an equation of the form

$$-\Delta\widetilde{\varphi} = |x|^{2s} f \qquad \text{in } B_R(0),$$

where $s > -1$ and $f \in L^\infty(B_R(0))$. Thanks to Lemma 2.5 we have that there exists $\gamma \in [0, 1)$ such that in $B_R(0)$ it holds

$$|\nabla\widetilde{\varphi}(x)| \leq C(-\log|x|)|x|^{-\gamma}. \tag{2.2.19}$$

We will see how this implies estimates for $\nabla\varphi$.

Lemma 2.13. *Let $\gamma \in [0, 1)$. We have*

$$|\nabla\varphi| \leq C\,(\log|x|)\,|x|^{\gamma-2} \qquad \textit{in } \mathbb{R}^2 \setminus B_R(0).$$

Proof. It is immediate to check that it holds

$$\varphi(x) = \widetilde{\varphi}\left(\frac{x}{|x|^2}\right).$$

With a direct computation and (2.2.19) applied to $\widetilde{\varphi}\left(\frac{x}{|x|^2}\right)$ we get

$$|\nabla\varphi(x)| = \frac{|\nabla\widetilde{\varphi}\left(\frac{x}{|x|^2}\right)|}{|x|^2} \leq C\left(\log|x|\right)|x|^{\gamma-2}.$$

□

We are in position now to prove Proposition 2.12.

Proof of Proposition 2.12. Multiplying the equation in (2.2.1) by $x\cdot\nabla u$ and integrating by parts on $B_R := B_R(0)$ we get

$$\begin{aligned}
&\frac{1}{2}\int_{\partial B_R}|\nabla u|^2 x\cdot\nu\,d\sigma - \int_{\partial B_R}\frac{\partial u}{\partial\nu}\nabla u\cdot x\,d\sigma \\
&= \int_{\partial B_R}|x|^{2\alpha}e^u x\cdot\nu\,dx - 2(\alpha+1)\int_{B_R}|x|^{2\alpha}e^u\,dx \\
&= I_1 + I_2 = I_3 + I_4.
\end{aligned} \tag{2.2.20}$$

We compute each integral separately. Using (2.2.13) and Lemma 2.13 we get

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{\partial B_R}|\nabla u|^2 x\cdot\nu\,d\sigma = \frac{1}{2}R\int_{\partial B_R}|\nabla u|^2\,d\sigma \\
&= \pi\Theta^2 - 2\frac{\Theta}{R}\int_{\partial B_R}\nabla\varphi\cdot x\,d\sigma + o(1) \\
&= \pi\Theta^2 + o(1),
\end{aligned}$$

where $o(1)\to 0$ as $R\to\infty$. With a similar computation we get also

$$I_2 = 2\pi\Theta^2 + o(1).$$

As for I_3, using Theorem 2.12 and Lemma 2.8 we have

$$\begin{aligned}
I_3 &= \int_{\partial B_R}R^{2\alpha+1}e^u\,d\sigma = R^{2\alpha+1}\int_{\partial B_R}e^{-\Theta\log R+O(1)}\,d\sigma \\
&= O(R^{2\alpha+1-\Theta}) = o(1).
\end{aligned}$$

At last, from (2.2.1) it is immediate that

$$I_4 = -4(1+\alpha)\pi\Theta,$$

and the thesis follows at once.

□

2.3 Extremal functions on compact surfaces: notations and preliminaries

Let (Σ, g) be a smooth closed Riemannian surface. We will fix $p_1, \dots, p_m \in \Sigma$ and consider a positive function $h \in C^1(\Sigma \backslash \{p_1, \dots, p_m\})$ satisfying (1.1.11). More precisely, denoting by d the Riemannian distance on (Σ, g) and by B_r the corresponding metric ball, we will assume that for some $\delta > 0$,

$$\frac{h}{d(\,\cdot\,, p_i)^{2\alpha_i}} \in C^1_+(B_\delta(p_i)) := \big\{ f \in C^1(B_\delta(p_i)) \;:\; f > 0 \big\} \qquad \text{for } i = 1, \dots, m. \tag{2.3.1}$$

In order to distinguish the singular points $p_1, \dots, p_m$ from the regular ones, we introduce a singularity index function

$$\alpha(x) := \begin{cases} \alpha_i & \text{if } x = p_i \\ 0 & x \in \Sigma \backslash \{p_1, \dots, p_m\}. \end{cases} \tag{2.3.2}$$

Clearly condition (2.3.1) implies that the limit

$$K(p) := \lim_{q \to p} \frac{h(q)}{d(q,p)^{2\alpha(p)}} \tag{2.3.3}$$

exists and it is strictly positive for any $p \in \Sigma$. We will study functionals of the form (1.2.12) on the space

$$\mathcal{H} := \left\{ u \in H^1(\Sigma) \;:\; \int_\Sigma |\nabla u|^2 dv_g \le 1, \; \int_\Sigma u \, dv_g = 0 \right\}.$$

To simplify the notation we set

$$\overline{\alpha} := \min \left\{ 0, \min_{1 \le i \le m} \alpha_i \right\}$$

and

$$\overline{\beta} := 4\pi(1 + \overline{\alpha}).$$

Given $s \ge 1$, the symbols $\|\cdot\|_s$, $L^s(\Sigma)$ will denote the standard L^s-norm and L^s-space on Σ with respect to the metric g. Since we will deal with the singular metric $g_h = gh$ we will also consider

$$\|u\|_{s,h} := \int_\Sigma |u|^s dv_{g_h} = \int_\Sigma h\, |u|^s dv_g$$

and

$$L^s(\Sigma, g_h) := \{ u : \Sigma \to \mathbb{R} \;\text{ Borel-measurable}, \; \|u\|_{s,h} < +\infty \}.$$

In this section we will prove the existence of an extremal function for $E^{\beta,\lambda,q}_{\Sigma,h}$ for the subcritical case $\beta < \overline{\beta}$. We begin by stating some well known but useful results.

Lemma 2.1. *If $u \in H^1(\Sigma)$, then $e^{u^2} \in L^s(\Sigma) \cap L^s(\Sigma, g_h)$, for any $s \geq 1$.*

Proof. Thanks to (2.3.1) we have $h \in L^r(\Sigma)$ for some $r > 1$, hence it is sufficient to prove that $e^{u^2} \in L^s(\Sigma)$ for any $s \geq 1$. Moreover, since

$$e^{su^2} = e^{s(u-\overline{u})^2 + 2s(u-\overline{u})\overline{u} + s\overline{u}^2} \leq e^{2s(u-\overline{u})^2} e^{2s\overline{u}^2},$$

without loss of generality we can assume $\overline{u} = 0$. Take $\varepsilon > 0$ such that $2s\varepsilon \leq 4\pi$ and a function $v \in C^1(\Sigma)$ satisfying $\|\nabla_g(v-u)\|_2^2 \leq \varepsilon$ and $\int_\Sigma v \, dv_g = 0$. By (1.7), we have

$$\|e^{2s(u-v)^2}\|_1 + \|e^{2s\varepsilon \frac{u^2}{\|\nabla u\|_2}}\|_1 < +\infty. \tag{2.3.4}$$

Note that

$$e^{su^2} \leq e^{s(u-v)^2} e^{2suv}. \tag{2.3.5}$$

By (2.3.4), we have $e^{s(u-v)^2} \in L^2(\Sigma)$ and, since $v \in L^\infty(\Sigma)$,

$$e^{2suv} \leq e^{s\varepsilon \frac{u^2}{\|\nabla u\|_2^2}} e^{C(\varepsilon, s, \|\nabla u\|_2) v^2} \in L^2(\Sigma).$$

Hence, using (2.3.5) and Holder's inequality, we get $e^{su^2} \in L^1(\Sigma)$. □

Lemma 2.2. *If $u_k \in \mathcal{H}$ and $u_k \rightharpoonup u \neq 0$ weakly in $H^1(\Sigma)$, then*

$$\sup_k \int_\Sigma h e^{p\overline{\beta} u_k^2} dv_g < +\infty$$

for any $1 \leq p < \frac{1}{1-\|\nabla u\|_2^2}$.

Proof. Observe that

$$e^{p\overline{\beta} u_k^2} \leq e^{p\overline{\beta}(u_k-u)^2} e^{2p\overline{\beta} u_k u}. \tag{2.3.6}$$

Since

$$\frac{1}{p} > 1 - \|\nabla u\|_2^2 \geq \|\nabla u_k\|_2^2 - \|\nabla u\|_2^2 = \|\nabla(u_k-u)\|_2^2 + o(1) \quad \Longrightarrow \quad \limsup_{k\to\infty} \|\nabla(u_k-u)\|_2^2 < \frac{1}{p},$$

by (1.2.14) we get $\|e^{p\overline{\beta}(u_k-u)^2}\|_{s,h} \leq C$ for some $s > 1$. Taking $\frac{1}{s} + \frac{1}{s'} = 1$ and using Lemma 2.1, we have

$$e^{2ps'\overline{\beta} u_k u} \leq e^{\frac{\overline{\beta}}{2} u_k^2} e^{C_{s,\alpha,p} u^2} \in L^1(\Sigma, g_h) \qquad \Longrightarrow \qquad \|e^{2p\overline{\beta} u_k u}\|_{s',h} \leq C.$$

Thus from (2.3.6) we get $\|e^{p\overline{\beta} u_k^2}\|_{1,h} \leq C$. □

Existence of extremals for $\beta < \overline{\beta}$ is a simple consequence of Lemma 2.2 and Vitali's convergence Theorem.

Lemma 2.3. *For any $\beta \in (0, \overline{\beta})$, $\lambda \in [0, \lambda_q(\Sigma, g))$, $q > 1$, we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} < +\infty,$$

and the supremum is attained.

Proof. Let $u_k \in \mathcal{H}$ be a maximizing sequence for $E_{\Sigma,h}^{\beta,\lambda,q}$, and assume $u_k \rightharpoonup u$ weakly in $H^1(\Sigma)$. We claim that $e^{\beta u_k^2(1+\lambda\|u_k\|_q^2)}$ is uniformly bounded in $L^p(\Sigma, g_h)$ for some $p > 1$. In particular, by Vitali's convergence theorem we get $E_{\Sigma,h}^{\beta,\lambda,q}(u_k) \to E_{\Sigma,h}^{\beta,\lambda,q}(u)$ with $E_{\Sigma,h}^{\beta,\lambda,q}(u) < +\infty$. Hence $E_{\Sigma,h}^{\beta,\lambda,q}(u) = \sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q}(u)$, proving the conclusion.

If $u = 0$, then

$$\beta(1 + \lambda\|u_k\|_q^2) \to \beta < \overline{\beta},$$

and the claim is proved taking $1 < p < \frac{\overline{\beta}}{\beta}$ and using (1.2.14). If $u \neq 0$, since

$$(1 - \|\nabla u\|_2^2)(1 + \lambda\|u_k\|_q^2) \leq 1 - \|\nabla u\|_2^2 + \lambda\|u\|_q^2 + o(1) \leq 1 - (\lambda_q(\Sigma) - \lambda)\|u\|_q^2 + o(1) < 1,$$

we can find $p > 1$ such that $\limsup_{k\to\infty} p(1 + \lambda\|u_k\|_q^2) < \dfrac{1}{1 - \|\nabla u\|_2^2}$, and the claim follows from Lemma 2.2. □

The behaviour of extremal functions as $\beta \to \overline{\beta}$ will be studied in Section 2.4. As for now we can study the convergence of the suprema.

Lemma 2.4. *As $\beta \nearrow \overline{\beta}$ we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} \to \sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q}.$$

Proof. Clearly, since $\beta < \overline{\beta}$, we have

$$\limsup_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} \leq \sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q}.$$

On the other hand, by monotone convergence theorem we have

$$\liminf_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} \geq \liminf_{\beta\nearrow\overline{\beta}} E_{\Sigma,h}^{\beta,\lambda,q}(v) = E_{\Sigma,h}^{\overline{\beta},\lambda,q}(v) \qquad \forall\, v \in \mathcal{H},$$

which gives

$$\liminf_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} \geq \sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q}.$$

□

We conclude this section with some Remarks concerning isothermal coordinates and Green's functions. We recall that, given any point $p \in \Sigma$, we can always find a small neighborhood Ω of p and a local chart

$$\psi : \Omega \to D_{\delta_0} \subset \mathbb{R}^2, \tag{2.3.7}$$

such that

$$\psi(p) = 0 \tag{2.3.8}$$

and

$$(\psi^{-1})^* g = e^{\varphi}|dx|^2, \tag{2.3.9}$$

where

$$\varphi \in C^\infty(\overline{D_{\delta_0}}) \qquad \text{and} \qquad \varphi(0) = 0. \tag{2.3.10}$$

For any $\delta < \delta_0$ we will denote $\Omega_\delta := \psi^{-1}(D_\delta)$. More generally, if $D_r(x) \subseteq D_{\delta_0}$, we define $\Omega_r(\psi^{-1}(x)) := \psi^{-1}(D_r(x))$. We stress that (2.3.3) and (2.3.9) also imply

$$(\psi^{-1})^* g_h = |x|^{2\alpha(p)} V(x) e^{\varphi}|dx|^2, \tag{2.3.11}$$

with

$$0 < V \in C^0(\overline{D_{\delta_0}}) \qquad \text{and} \qquad V(0) = K(p). \tag{2.3.12}$$

For any $p \in \Sigma$, we denote G_p^λ the solution of

$$\begin{cases} -\Delta_g G_p^\lambda = \delta_p + \lambda \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda - \dfrac{1}{|\Sigma|}\left(1 + \lambda \|G_p^\lambda\|_q^{2-q} \displaystyle\int_\Sigma |G_p^\lambda|^{q-2} G_p^\lambda dv_g\right) \\ \displaystyle\int_\Sigma G_p^\lambda dv_g = 0. \end{cases} \tag{2.3.13}$$

In local coordinates satisfying (2.3.7)-(2.3.12), we have

$$G_p^\lambda(\psi^{-1}(x)) = -\frac{1}{2\pi}\log|x| + A_p^\lambda + \xi(x), \tag{2.3.14}$$

with $\xi \in C^1(\overline{D_{\delta_0}})$ and $\xi(x) = O(|x|)$. Observe that G_p^0 is the standard Green's function for $-\Delta_g$.

Lemma 2.5. *Fix $p \in \Sigma$. As $\lambda \to 0$, we have $G_p^\lambda \to G_p^0$ in $L^s(\Sigma)$ for any $0 < s < +\infty$, and $A_p^\lambda \to A_p^0$.*

Proof. Let us denote $c_\lambda := \dfrac{\lambda}{|\Sigma|}\|G_p^\lambda\|_q^{2-q}\displaystyle\int_\Sigma |G_p^\lambda|^{q-2} G_p^\lambda dv_g$. Observe that

$$-\Delta_g(G_p^\lambda - G_p^0) = \lambda \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda - c_\lambda.$$

Since

$$\Big\| \|G_p^\lambda\|_q^{2-q} |G_p^\lambda|^{q-2} G_p^\lambda \Big\|_{\frac{q}{q-1}} = \|G_p^\lambda\|_q,$$

by elliptic estimates we find

$$\|G_p^\lambda - G_p^0\|_\infty \le \|G_p^\lambda - G_p^0\|_{W^{2,\frac{q}{q-1}}(\Sigma)} \le C\lambda \|G_p^\lambda\|_q. \tag{2.3.15}$$

In particular

$$\|G_p^\lambda\|_q \le \|G_p^0\|_q + \|G_p^\lambda - G_p^0\|_q \le \|G_p^0\|_q + C\|G_p^\lambda - G_p^0\|_\infty \le \|G_p^0\|_q + C\lambda\|G_p^\lambda\|_q,$$

hence for sufficiently small λ we have

$$\|G_p^\lambda\|_q \le C\|G_p^0\|_q.$$

Thus by (2.3.15), as $\lambda \to 0$ we find

$$\|G_p^\lambda - G_p^0\|_\infty \to 0.$$

In particular, $G_p^\lambda \to G_p^0$ in L^s for any $s > 1$. Since $A_p^\lambda - A_p^0 = (G_p^\lambda - G_p^0)(p)$, we also get $A_p^\lambda \to A_p^0$. □

Lemma 2.6. *Fix $p \in \Sigma$ and let (Ω, ψ) be a local chart satisfying* (2.3.7)-(2.3.12). *As $\delta \to 0$ we have*

$$\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\frac{1}{2\pi}\log\delta + A_p^\lambda + \lambda\|G_p^\lambda\|_q^2 + O(\delta|\log\delta|).$$

Proof. Integrating by parts we have

$$\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\int_{\Sigma\setminus\Omega_\delta} \Delta_g G_p^\lambda \, G_p^\lambda dv_g - \int_{\partial\Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial\nu} d\sigma_g. \tag{2.3.16}$$

For the first term, using the definition of G_p^λ, we get

$$\begin{aligned} -\int_{\Sigma\setminus\Omega_\delta} \Delta_g G_p^\lambda \, G_p^\lambda dv_g &= \lambda\|G_p^\lambda\|_q^{2-q} \int_{\Sigma\setminus\Omega_\delta} |G_p^\lambda|^q dv_g - \left(\frac{1}{|\Sigma|} + c_\lambda\right) \int_{\Sigma\setminus\Omega_\delta} G_p^\lambda \, dv_g \\ &= \lambda\|G_p^\lambda\|_q^2 + o(1). \end{aligned} \tag{2.3.17}$$

For the second term we use (2.3.14) to find

$$-\int_{\partial\Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial\nu} d\sigma_g = -\frac{1}{2\pi}\log\delta + A_p^\lambda + O(\delta|\log\delta|). \tag{2.3.18}$$

□

2.4 Blow-up analysis for the critical exponent

In this section we will study the critical case $\beta = \overline{\beta}$.

Let us fix $q > 1, \lambda \in [0, \lambda_q(\Sigma, g))$ and take a sequence $\beta_k \nearrow \overline{\beta}$ ($\beta_k < \overline{\beta}$ for any $k \in \mathbb{N}$). To simplify the notation we will set $E_k := E^{\beta_k,\lambda,q}_{\Sigma,h}$. By Lemma 2.3, for any k we can take a function $u_k \in \mathcal{H}$ such that

$$E_k(u_k) = \sup_{\mathcal{H}} E_k. \tag{2.4.1}$$

Up to subsequences, we can always assume that

$$u_k \rightharpoonup u_0 \qquad \text{in } H^1(\Sigma) \tag{2.4.2}$$

and

$$u_k \to u_0 \quad \text{in } L^s(\Sigma) \quad \forall\, s \geq 1. \tag{2.4.3}$$

Lemma 2.7. *If $u_0 \neq 0$, then*

$$E_k(u_k) \to E^{\overline{\beta},\lambda,q}_{\Sigma,h}(u_0) < +\infty. \tag{2.4.4}$$

In particular

$$\sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h} < +\infty,$$

and u_0 is an extremal function.

Proof. If $u_0 \neq 0$, we can argue as in Lemma 2.3 to find $p > 1$ such that $e^{\beta_k u_k^2(1+\lambda\|u_k\|_q^2)}$ is uniformly bounded in $L^p(\Sigma, g_h)$. Vitali's convergence Theorem yields (2.4.4). Lemma 2.4 implies

$$\sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h} = E^{\overline{\beta},\lambda,q}_{\Sigma,h}(u_0) < +\infty.$$

□

Thus it is sufficient to study the case $u_0 = 0$, which we will assume for the rest of this section. In the same spirit of Theorem 1.2 and (1.2.5), we will prove the following sharp upper bound for $E_k(u_k)$.

Proposition 2.13. *If $u_0 = 0$, we have*

$$\limsup_{k\to\infty} E_k(u_k) \leq \frac{\pi e}{1+\overline{\alpha}} \max_{p\in\Sigma,\, \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta} A_p^\lambda} + |\Sigma|_{g_h},$$

where A_p^λ is defined as in (2.3.14) *and* $|\Sigma|_{g_h} := \int_\Sigma h \, dv_g$.

Remark 2.3. *We remark that the quantity*

$$\max_{p\in\Sigma,\, \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta} A_p^\lambda}$$

is well defined. Indeed, if $\overline{\alpha} < 0$ *the set of points such that* $\alpha(p) = \overline{\alpha}$ *is finite. On the other hand, if* $\overline{\alpha} = 0$, *we have that* $K \equiv h$ *on* $\Sigma \setminus \{p_1, \dots, p_m\} = \{p \in \Sigma \colon \alpha(p) = \overline{\alpha}\}$, *and* $he^{\overline{\beta} A_p^\lambda}$ *is a continuous function on* Σ *with zeros at the points* $p_1, \dots, p_m$.

In particular, Lemma 2.7 and Proposition 2.13 give a proof of an Adimurthi-Druet type inequality, namely

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} < +\infty.$$

The rest of this section is devoted to the proof of Proposition 2.13.

Lemma 2.8. *Let* $u_k \in \mathcal{H}$ *be a sequence such that* (2.4.1)-(2.4.3) *hold. Then* $\|\nabla u_k\|_2 = 1$ *and there exists* $s > 1$ *such that* $u_k \in \mathcal{H} \cap W^{2,s}(\Sigma)$ *for any* k. *Moreover, there exist* $\gamma_k > 0$, $\lambda_k \geq 0$, *and* $c_k \in \mathbb{R}$ *such that*

$$-\Delta_g u_k = \gamma_k h(x) u_k e^{b_k u_k^2} + s_k(x), \tag{2.4.5}$$

where

$$b_k := \beta_k (1 + \lambda \|u_k\|_q^2), \tag{2.4.6}$$

$$s_k := \lambda_k \|u_k\|_q^{2-q} |u_k|^{q-2} u_k - c_k, \tag{2.4.7}$$

with

$$c_k := \frac{1}{|\Sigma|} \left(\gamma_k \int_\Sigma u_k e^{b_k u_k^2} dv_{g_h} + \lambda_k \|u_k\|_q^{2-q} \int_\Sigma |u_k|^{q-2} u_k dv_g \right). \tag{2.4.8}$$

In particular, since we are assuming $u_0 = 0$, *we have*

$$\limsup_n \gamma_k < +\infty, \quad \gamma_k \int_\Sigma h\, u_k^2 e^{u_k^2} dv_g \to 1, \tag{2.4.9}$$

$$b_k \to \overline{\beta}, \tag{2.4.10}$$

$$\lambda_k \to \lambda, \tag{2.4.11}$$

$$c_k \to 0, \qquad \|s_k\|_{\frac{q}{q-1}} \to 0, \tag{2.4.12}$$

as $k \to +\infty$.

Proof. The maximality of u_k clearly implies $\|\nabla u_k\|_2 = 1$. One can apply Langrange multipliers theorem to verify that u_k satisfies

$$-\Delta_g u_k = \nu_k b_k h(x) u_k e^{b_k u_k^2} + \lambda \nu_k \beta_k \mu_k \|u_k\|_q^{2-q} |u_k|^{q-2} u_k - c_k, \tag{2.4.13}$$

where b_k is defined as in (2.4.6), $\mu_k := \int_\Sigma h\, u_k^2 e^{b_k u_k^2}\, dv_g$,

$$c_k := \frac{1}{|\Sigma|} \left(\gamma_k \int_\Sigma h u_k e^{b_k u_k^2} dv_g + \lambda \nu_k \beta_k \mu_k \|u_k\|_q^{2-q} \int_\Sigma |u_k|^{q-2} u_k \, dv_g \right), \tag{2.4.14}$$

and $\nu_k \in \mathbb{R}$. We define $\gamma_k := \nu_k b_k$, $\lambda_k := \lambda \nu_k \beta_k \mu_k$, and $s_k(x) := \lambda_k \|u_k\|_q^{2-q} |u_k|^{q-2} u_k - c_k$ so that (2.4.5)-(2.4.8) are satisfied. Observe also that

$$\left\| \|u_k\|_q^{2-q} |u_k|^{q-2} u_k \right\|_{\frac{q}{q-1}} = \|u_k\|_q, \tag{2.4.15}$$

hence $s_k \in L^{\frac{q}{q-1}}(\Sigma)$. Choosing $s_0 > 1$ such that $h \in L^{s_0}(\Sigma)$, we can employ Lemma 2.1 and standard elliptic regularity arguments to obtain $u_k \in W^{2,s}(\Sigma)$ for any $1 < s < \min\{s_0, \frac{q}{q-1}\}$.

We shall now prove (2.4.9)-(2.4.12). Since $u_0 = 0$, (2.4.10) follows from (2.4.3). Multiplying (2.4.13) by u_k and integrating on Σ, we get

$$1 = \nu_k b_k \mu_k + \lambda \nu_k \beta_k \mu_k \|u_k\|_q^2 = \nu_k b_k \mu_k \Big(1 + \frac{\lambda \beta_k \|u_k\|_q^2}{b_k}\Big) = \gamma_k \mu_k (1 + o(1)),$$

from which we get the second part of (2.4.9). As a consequence we also have

$$\lambda_k = \lambda \nu_k \beta_k \mu_k = \lambda \gamma_k \mu_k \frac{\beta_k}{b_k} \to \lambda. \tag{2.4.16}$$

Now we prove $\limsup\limits_{k\to\infty} \gamma_k < +\infty$ or, equivalently, $\liminf\limits_{k\to\infty} \mu_k > 0$. For any $t > 0$, we have

$$E_k(u_k) \le \frac{1}{t^2} \int_{\{|u_k|>t\}} h\, u_k^2 e^{b_k u_k^2} dv_g + \int_{\{|u_k|\le t\}} h e^{b_k u_k^2} dv_g \le \frac{1}{t^2} \int_\Sigma h u_k^2 e^{b_k u_k^2} dv_g + |\Sigma|_{g_h} + o(1),$$

from which

$$\liminf_{k\to\infty} \mu_k = \liminf_{k\to\infty} \int_\Sigma h\, u_k^2 e^{b_k u_k^2} dv_g \ge t^2 \left(\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} - |\Sigma|_{g_h} \right) > 0.$$

It remains to prove that $c_k \to 0$ which, with (2.4.15), completes the proof of (2.4.12). For any $t > 0$

$$\gamma_k \int_\Sigma h|u_k| e^{b_k u_k^2} dv_g \le \frac{\gamma_k}{t} \int_{\{|u_k|>t\}} h u_k^2 e^{b_k u_k^2} dv_g + \gamma_k \int_{\{|u_k|\le t\}} h|u_k| e^{b_k u_k^2} dv_g = \frac{1+o(1)}{t} + o(1).$$

Since t can be taken arbitrarily large we find

$$\gamma_k \int_\Sigma h u_k e^{b_k u_k^2} dv_g \to 0. \tag{2.4.17}$$

Finally,

$$\|u_k\|_q^{2-q} \left| \int_\Sigma |u_k|^{q-2} u_k dv_g \right| \le \|u_k\|_q |\Sigma|^{\frac{1}{q}} \to 0, \tag{2.4.18}$$

which, combined with (2.4.8), (2.4.16), and (2.4.17), yields $c_k \to 0$. □

By Lemma 2.8 we know that $u_k \in C^0(\Sigma)$, thus we can take a sequence p_k such that

$$m_k := \max_{\Sigma} |u_k| = u_k(p_k), \tag{2.4.19}$$

where the last equality holds up to changing the sign of u_k. Clearly, if $\sup_k m_k < +\infty$, we would have $E_k(u_k) \to |\Sigma|_{g_h}$, which contradicts Lemma 2.4. Thus, up to subsequences, we will assume

$$m_k \to +\infty \qquad \text{and} \qquad p_k \to p. \tag{2.4.20}$$

Lemma 2.9. *Let $\Omega \subset \Sigma$ be an open subset such that*

$$\limsup_{k\to+\infty} \|\nabla u_k\|_{L^2(\Omega)} < 1.$$

Then

$$\|u_k\|_{L^\infty_{loc}(\Omega)} \leq C.$$

Proof. Fix $\tilde{\Omega} \Subset \Omega$. Take a cut-off function $\xi \in C_0^\infty(\Omega)$ such that $0 \le \xi \le 1$ and $\xi \equiv 1$ in Ω' where $\tilde{\Omega} \Subset \Omega' \Subset \Omega$. Since

$$\begin{aligned}\int_\Sigma |\nabla u_k \xi|^2\, dv_g &= \int_\Omega |\nabla u_k|^2 \xi^2\, dv_g + 2\int_\Omega u_k \xi \nabla u_k \cdot \nabla \xi\, dv_g + \int_\Omega |\nabla \xi|^2 u_k^2\, dv_g \\ &\le (1+\varepsilon) \int_\Omega |\nabla u_k|^2 \xi^2\, dv_g + C_\varepsilon \int_\Omega |\nabla \xi|^2 u_k^2\, dv_g,\end{aligned}$$

and ε can be taken arbitrarily small, we find

$$\limsup_{k\to\infty} \|\nabla(u_k\xi)\|^2_{L^2(\Sigma)} < 1.$$

Thus, applying (1.2.14) to $v_k := \frac{\xi u_k}{\|\nabla(\xi u_k)\|_{L^2(\Sigma)}}$ we find

$$\left\| e^{\overline{\beta} u_k^2 (1+\lambda \|u_k\|_q^2)} \right\|_{L^{s_0}(\Omega', g_h)} \le C \tag{2.4.21}$$

for some $s_0 > 1$. From (2.4.12) and (2.4.21), $-\Delta_g u_k$ is uniformly bounded in $L^s(\Omega')$ for any $s < \min\{s_0, \frac{q}{q-1}\}$. If we take another cut-off function $\widetilde{\xi} \in C_0^\infty(\Omega')$ such that $\widetilde{\xi} \equiv 1$ in $\tilde{\Omega}$, applying elliptic estimates to $\widetilde{\xi} u_k$ in Ω' we find $\sup_{\Omega'} \widetilde{\xi} u_k \le C$, and hence $\sup_{\tilde{\Omega}} u_k \le C$. □

From Lemma 2.9 one can deduce that $|\nabla u_k|^2 \rightharpoonup \delta_p$, that is u_k concentrates at p. Intuitively, it is natural to expect that concentration for maximizing sequences happens in the regions in which h is larger. We will show that p must be a minimum point of the singularity index α defined in (2.3.2). This will clarify the difference between the cases $\overline{\alpha} < 0$ and $\overline{\alpha} = 0$: in the former, the blow-up point p will be one of the singular points $p_1, \dots, p_m$, while in the latter $p \in \Sigma \setminus \{p_1, \dots, p_m\}$ (cfr. Remark 2.4 and Proposition 2.15). The next step consists in studying the behaviour of u_k around p. Arguing as in

[57], we will prove that a suitable scaling of u_k converges to a solution of a (possibly singular) Liouville-type equation on $\mathbb{R}^2$ (see Proposition 2.14).

Again, we consider a local chart (Ω, ψ) satisfying (2.3.7)-(2.3.12). From now on we will denote $x_k := \psi(p_k)$ and

$$v_k = u_k \circ \psi^{-1}. \tag{2.4.22}$$

Define t_k and $\tilde{t}_k$ so that

$$t_k^{2(1+\alpha(p))} \gamma_k m_k^2 e^{b_k m_k^2} = 1, \tag{2.4.23}$$

$$\tilde{t}_k^2 |x_k|^{2\alpha(p)} \gamma_k m_k^2 e^{b_k m_k^2} = 1. \tag{2.4.24}$$

Lemma 2.10. *For any $\beta < \overline{\beta}$ we have*

$$t_k^{2(1+\alpha(p))} m_k^2 e^{\beta m_k^2} \to 0, \qquad \tilde{t}_k^2 |x_k|^{2\alpha(p)} m_k^2 e^{\beta m_k^2} \to 0$$

as $k \to +\infty$. In particular, for any $s \geq 0$ we have

$$\lim_{k \to +\infty} t_k m_k^s = 0, \qquad \lim_{k \to +\infty} \tilde{t}_k m_k^s = 0.$$

Moreover, as $k \to +\infty$, we have

$$\frac{|x_k|}{t_k} \to +\infty \Longleftrightarrow \frac{|x_k|}{\tilde{t}_k} \to +\infty. \tag{2.4.25}$$

Proof. Since the result can be proven both for t_k and $\tilde{t}_k$ with the same argument, we will prove it here only for t_k. By (2.4.9), (2.4.10), and (2.4.23)

$$\begin{aligned} t_k^{2(1+\alpha(p))} m_k^2 e^{\beta m_k^2} &= \frac{e^{(\beta - b_k) m_k^2}}{\gamma_k} = e^{(\beta - b_k) m_k^2} \int_\Sigma h u_k^2 e^{b_k u_k^2} dv_g (1 + o(1)) \\ &\leq \int_\Sigma h u_k^2 e^{\beta u_k^2} dv_g (1 + o(1)). \end{aligned}$$

Take $s = \frac{\overline{\beta}}{\beta}'$ (i.e. $1/s + \beta/\overline{\beta} = 1$) and $s_0 > 1$ such that $h \in L^{s_0}(\Sigma)$. Then

$$\int_\Sigma h u_k^2 e^{\beta u_k^2} dv_g \leq \|u_k^2\|_{s,h} \|e^{\overline{\beta} u_k^2}\|_{1,h}^{\frac{\beta}{\overline{\beta}}} \leq C \|h\|_{s_0}^{\frac{1}{s}} \|u_k^2\|_{ss_0'} \to 0.$$

Finally, to prove (75), it is enough to observe that from (2.4.23) and (2.4.24) one computes

$$\frac{|x_k|}{\tilde{t}_k} = \left(\frac{|x_k|}{t_k} \right)^{1+\alpha(p)}.$$

□

We define now

$$r_k := \begin{cases} \tilde{t}_k & \text{if } \frac{|x_k|}{t_k} \to +\infty \text{ as } k \to +\infty, \\ t_k & \text{otherwise}, \end{cases} \tag{2.4.26}$$

and the function

$$\eta_k(x) := m_k \left(v_k(x_k + r_k x) - m_k \right), \tag{2.4.27}$$

which is defined in $D_{\frac{\delta_0}{r_k}}$.

Proposition 2.14. *Up to subsequences, $\eta_k \to \eta_0$ in $C^0_{loc}(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$. Moreover,*

(i) if $\frac{|x_k|}{r_k} \to +\infty$ as $k \to +\infty$ the function η_0 solves

$$-\Delta\eta_0 = V(0)e^{2\overline{\beta}\eta_0}, \tag{2.4.28}$$

$$\int_{\mathbb{R}^2} V(0)e^{2\overline{\beta}\eta_0}\,dx = 1; \tag{2.4.29}$$

(ii) if $\frac{|x_k|}{r_k} \to \overline{x}$ the function η_0 solves

$$-\Delta\eta_0 = |x+\overline{x}|^{2\alpha(p)}V(0)e^{2\overline{\beta}\eta_0}, \tag{2.4.30}$$

$$\int_{\mathbb{R}^2} |x+\overline{x}|^{2\alpha(p)}V(0)e^{2\overline{\beta}\eta_0}\,dx = 1. \tag{2.4.31}$$

Proof. If $\frac{|x_k|}{t_k} \to +\infty$ as $k \to +\infty$, then $r_k = \tilde{t}_k$ and it follows that η_k as in (2.4.27) satisfies

$$\begin{aligned}
-\Delta\eta_k &= m_k r_k^2 e^{\varphi(x_k+r_kx)}\left(\gamma_k|x_k+r_kx|^{2\alpha(p)}V(x_k+r_kx)e^{b_kv_k^2}v_k(x_k+r_kx) + s_k(x_k+r_kx)\right) \\
&= e^{\varphi(x_k+r_kx)}\left(\left|\frac{x_k}{|x_k|}+\frac{r_k}{|x_k|}x\right|^{2\alpha(p)} V(x_k+r_kx)\left(1+\frac{\eta_k}{m_k^2}\right)e^{b_k\left(2\eta_k+\frac{\eta_k^2}{m_k^2}\right)} + m_kr_k^2s_k(x_k+r_kx)\right).
\end{aligned}$$

Otherwise we have that $r_k = t_k$ and, up to subsequences, $\frac{|x_k|}{t_k} \to \overline{x}$ as $k \to +\infty$. In this case, η_k satisfies

$$\begin{aligned}
-\Delta\eta_k &= m_k r_k^2 e^{\varphi(x_k+r_kx)}\left(\gamma_k\,|x_k+r_kx|^{2\alpha(p)}\,V(x_k+r_kx)e^{b_ku_k^2}v_k(x_k+r_kx) + s_k(x_k+r_kx)\right) \\
&= e^{\varphi(x_k+r_ky)}\left(\left|\frac{x_k}{r_k}+x\right|^{2\alpha(p)} V(x_k+r_kx)\left(1+\frac{\eta_k}{m_k^2}\right)e^{b_k\left(2\eta_k+\frac{\eta_k^2}{m_k^2}\right)} + m_kr_k^2s_k(x_k+r_kx)\right).
\end{aligned}$$

Fix $L > 0$. Observe that from Lemma 2.10 and (2.4.12), we have

$$\begin{aligned} \int_{D_L} |m_k r_k^2 s_k(x_k + r_k x)|^{\frac{q}{q-1}}\, dx &= m_k^{\frac{q}{q-1}} r_k^{\frac{2}{q-1}} \int_{D_{Lr_k}(x_k)} |s_k(x)|^{\frac{q}{q-1}}\, dx \\ &\le m_k^{\frac{q}{q-1}} r_k^{\frac{2}{q-1}} \|s_k\|_{\frac{q}{q-1}}^{\frac{q}{q-1}} \to 0, \end{aligned} \tag{2.4.32}$$

as $k \to +\infty$. Since $2\eta_k + \frac{\eta_k^2}{m_k^2} \le 0$ and $|\eta_k| \le 2m_k^2$, in both case (i) and (ii) we can find $s > 1$ such that

$$\| - \Delta\eta_k\|_{L^s(D_L)} \le C.$$

Moreover $\eta_k(0) = 0$, thus we can exploit Sobolev's embeddings Theorems and Harnack's inequality to find a uniform bound for η_k in $C^{0,\alpha}(D_{\frac{L}{2}})$. Hence, with a diagonal argument, we find a subsequence of η_k such that $\eta_k \to \eta_0$ in $H^1_{loc}(\mathbb{R}^2) \cap C^0_{loc}(\mathbb{R}^2)$. Moreover η_0 solves (2.4.28) or (2.4.30), depending on our choice of r_k. It remains to prove (2.4.29) and (2.4.31) respectively. In order to do this, we observe that in case (i)

$$\begin{aligned} 1 = -\int_\Sigma \Delta_g u_k u_k\, dv_g &= \gamma_k \int_\Sigma h\, u_k^2 e^{b_k u_k^2}\, dv_g + \lambda_k \|u_k\|_q^2 \\ &\ge \gamma_k \int_{\Omega_{Lr_k}(p_k)} h\, u_k^2 e^{b_k u_k^2}\, dv_g + o(1) \\ &= V(0) \int_{D_L} e^{2\overline{\beta}\eta_0}\, dx + o(1). \end{aligned} \tag{2.4.33}$$

In particular it holds (see for instance [28])

$$\lim_{L\to+\infty} V(0) \int_{D_L} e^{2\overline{\beta}\eta_0}\, dx = \frac{1}{1+\overline{\alpha}} \ge 1, \tag{2.4.34}$$

where the last inequality follows from the fact that $\overline{\alpha} \le 0$. Hence with (2.4.33) we obtain (2.4.29). Similarly, in case (ii) we have

$$1 = -\int_\Sigma \Delta_g u_k u_k\, dv_g \ge V(0) \int_{D_L} |x + \overline{x}|^{2\alpha(p)} e^{2\overline{\beta}\eta_0}\, dx + o(1). \tag{2.4.35}$$

On the other hand (cfr. [85])

$$\lim_{L\to+\infty} V(0) \int_{D_L} |x + \overline{x}|^{2\alpha(p)} e^{2\overline{\beta}\eta_0} dx = \frac{1+\alpha(p)}{1+\overline{\alpha}} \ge 1, \tag{2.4.36}$$

where now the last inequality follows from the minimality of $\overline{\alpha}$. Therefore (2.4.31) is proven. □

Remark 2.4. *From the proof of Proposition 2.14 it follows that if* $\overline{\alpha} < 0$ *then by* (2.4.33) *and* (2.4.34) *we have that only case* (ii) *is possible. Moreover from* (2.4.35) *and* (2.4.36) *we get* $\alpha(p) = \overline{\alpha}$, *that is* p *must be one of the singular points* $p_1, \ldots, p_m$.

We stress that Proposition 2.14 gives us information on the nature of the point p only in the case $\overline{\alpha} < 0$. To have a deeper understanding of the case $\overline{\alpha} = 0$ and a more complete analysis of the blow-up behaviour of u_k near the point p we will need few more steps (see Proposition 2.15).

Lemma 2.11. *We have*

(*i*) $\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\Omega_{Lr_k}(p_k)} \gamma_k m_k h u_k e^{b_k u_k^2}\, dv_g = 1$;

(*ii*) $\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\Omega_{Lr_k}(p_k)} \gamma_k h u_k^2 e^{b_k u_k^2}\, dv_g = 1$;

(*iii*) $\lim_{L\to+\infty}\lim_{k\to+\infty}\int_{\Omega_{Lr_k}(p_k)} h e^{b_k u_k^2}\, dv_g = \limsup_{k\to+\infty}\frac{1}{\gamma_k m_k^2}$.

Proof. Both (*i*) and (*ii*) follow easily from Proposition 2.14. We are left with the proof of (*iii*).

By Proposition 2.14, for any $L > 0$ we have

$$\lim_{k\to+\infty}\gamma_k m_k^2\int_{\Omega_{Lr_k}(p_k)} h e^{b_k u_k^2} dv_g = 1 + o_L(1),$$

where $o_L(1)\to 0$ as $L\to\infty$. Hence

$$\limsup_{k\to\infty}\frac{1}{\gamma_k m_k^2} = (1+o_L(1))\limsup_{k\to\infty}\int_{\Omega_{Lr_k(p_k)}} h e^{b_k u_k^2} dv_g,$$

and we can conclude the proof letting $L\to+\infty$. □

Following [57], for any $A > 1$ we define

$$u_k^A := \min\{u_k, \frac{m_k}{A}\}.$$

Lemma 2.12. *For any $A > 1$ we have*

$$\limsup_{k\to\infty}\int_\Sigma |\nabla u_k^A|^2 dv_g = \frac{1}{A}.$$

Proof. Integrating by parts, we have

$$\liminf_{k\to\infty}\int_\Sigma |\nabla u_k^A|^2 dv_g = \liminf_{k\to\infty}\int_\Sigma \nabla u_k^A\cdot\nabla u_k dv_g = \liminf_{k\to+\infty} -\int_\Sigma \Delta_g u_k u_k^A\, dv_g.$$

Fix now $L > 0$. By Proposition 2.14, for sufficiently large k, we get $\Omega_{Lr_k}(p_k)\subseteq\{u_k > \frac{m_k}{A}\}$. Hence, using (2.4.5) and (2.4.7), we find

$$-\int_\Sigma \Delta_g u_k\, u_k^A\, dv_g = \gamma_k\int_\Sigma h u_k e^{b_k u_k^2} u_k^A\, dv_g + o(1) \ge \frac{\gamma_k m_k}{A}\int_{\Omega_{Lr_k}(p_k)} h\, u_k e^{b_k u_k^2}\, dv_g + o(1).$$

Passing to the limit as $k, L \to +\infty$ we obtain

$$\liminf_{k\to\infty} \int_\Sigma |\nabla u_k^A|^2 \, dv_g = \liminf_{k\to+\infty} - \int_\Sigma \Delta_g u_k u_k^A \, dv_g \geq \frac{1}{A}, \tag{2.4.37}$$

where the last inequality follows from Lemma 2.11. Similarly

$$-\int_\Sigma \Delta_g u_k \left(u_k - \frac{m_k}{A}\right)^+ dv_g \geq \gamma_k \int_{\Omega_{Lr_k}(p_k)} h \, u_k e^{b_k u_k^2} \left(u_k - \frac{m_k}{A}\right) dv_g + o(1),$$

and, again from Lemma 2.11, we get

$$\liminf_{k\to\infty} \int_\Sigma |\nabla (u_k - \frac{m_k}{A})^+|^2 \, dv_g \geq \frac{A-1}{A}. \tag{2.4.38}$$

Clearly $u_k = u_k^A + (u_k - \frac{m_k}{A})^+$ and $\int_\Sigma \nabla u_k^A \cdot \nabla (u_k - \frac{m_k}{A})^+ \, dv_g = 0$, thus

$$1 = \int_\Sigma |\nabla u_k|^2 \, dv_g = \int_\Sigma |\nabla u_k^A|^2 \, dv_g + \int_\Sigma |\nabla \left(u_k - \frac{m_k}{A}\right)^+ |^2 \, dv_g,$$

and from (2.4.37) and (2.4.38) we find

$$\lim_{k\to\infty} \int_\Sigma |\nabla u_k^A|^2 \, dv_g = \frac{1}{A} \quad \text{and} \quad \lim_{k\to\infty} \int_\Sigma |\nabla \left(u_k - \frac{m_k}{A}\right)^+ |^2 \, dv_g = \frac{A-1}{A}.$$

□

With Lemma 2.12 we have a first rough version of Proposition 2.13.

Lemma 2.13.

$$\limsup_{k\to\infty} E_k(u_k) \leq \lim_{L\to+\infty} \lim_{k\to+\infty} \int_{\Omega_{Lr_k}(p_k)} h e^{b_k u_k^2} \, dv_g + |\Sigma|_{g_h}.$$

Proof. For any $A > 1$ we have

$$E_k(u_k) = \int_{\left\{u_k \geq \frac{m_k}{A}\right\}} h e^{b_k u_k^2} \, dv_g + \int_{\left\{u_k \leq \frac{m_k}{A}\right\}} h e^{b_k (u_k^A)^2} \, dv_g.$$

By (2.4.9),

$$\int_{\left\{u_k \geq \frac{m_k}{A}\right\}} h e^{b_k u_k^2} \, dv_g \leq \frac{A^2}{m_k^2} \int_\Sigma h u_k^2 e^{b_k u_k^2} \, dv_g = \frac{A^2}{\gamma_k m_k^2}(1 + o(1)).$$

For the last integral we apply Lemma 2.12. Since $\limsup_{k\to\infty} \|\nabla u_k^A\|_2^2 \leq \frac{1}{A} < 1$, (1.2.14) implies that $e^{b_k (u_k^A)^2}$ is uniformly bounded in $L^s(\Sigma, g_h)$ for some $s > 1$. Thus, by Vitali's Theorem

$$\int_{\left\{u_k \leq \frac{m_k}{A}\right\}} h e^{b_k (u_k^A)^2} dv_g \leq \int_\Sigma h e^{b_k (u_k^A)^2} dv_g \to |\Sigma|_{g_h}.$$

Therefore we proved

$$\limsup_{k\to\infty} E_k(u_k) \le \limsup_{k\to\infty} \frac{A^2}{\gamma_k m_k^2} + |\Sigma|_{g_h}.$$

As $A \to 1$ we get the conclusion, thanks to Lemma 2.11. □

Lemma 2.14. *We have*

$$\gamma_k m_k h\, u_k e^{b_k u_k^2} \rightharpoonup \delta_p$$

weakly as measures as $k \to +\infty$*.*

Proof. Take $\xi \in C^0(\Sigma)$. For $L > 0$, $A > 1$, we have

$$\begin{aligned}\gamma_k m_k \int_\Sigma h\, u_k e^{b_k u_k^2} \xi dv_g &= \gamma_k m_k \int_{\Omega_{Lr_k}(p_k)} h u_k e^{b_k u_k^2} \xi dv_g \\ &+ \gamma_k m_k \int_{\{u_k > \frac{m_k}{A}\}\setminus\Omega_{Lr_k}(p_k)} h u_k e^{b_k u_k^2} \xi dv_g \\ &+ \gamma_k m_k \int_{\{u_k \le \frac{m_k}{A}\}} h u_k e^{b_k u_k^2} \xi dv_g \\ &=: I_k^1 + I_k^2 + I_k^3.\end{aligned}$$

We have

$$I_k^1 = \gamma_k m_k \int_{\Omega_{Lr_k}(p_k)} h u_k e^{b_k u_k^2} (\xi - \xi(p))\, dv_g + \gamma_k m_k \int_{\Omega_{Lr_k}(p_k)} h u_k e^{b_k u_k^2} \xi(p)\, dv_g.$$

Since $\|\xi - \xi(p)\|_{L_\infty(\Omega_{Lr_k}(p_k))} \to 0$ as $k \to +\infty$, thanks to Lemma 2.11, we have

$$\lim_{L\to\infty} \lim_{k\to\infty} I_k^1 = \xi(p).$$

Similarly, using (2.4.9),

$$\begin{aligned}|I_k^2| &\le m_k \int_{\{u_k > \frac{m_k}{A}\}\setminus\Omega_{Lr_k}(p_k)} \gamma_k h u_k e^{b_k u_k^2} |\xi| dv_g \\ &\le A \int_{\{u_k > \frac{m_k}{A}\}\setminus\Omega_{Lr_k}(p_k)} \gamma_k h u_k^2 e^{b_k u_k^2} |\xi| dv_g \\ &\le A \|\xi\|_{L^\infty(\Sigma)} \left(1 - \int_{\Omega_{Lr_k}(p_k)} \gamma_k h u_k^2 e^{b_k u_k^2} dv_g + o(1)\right).\end{aligned}$$

Therefore, from Lemma 2.11,

$$\lim_{L\to\infty} \lim_{k\to\infty} I_k^2 = 0.$$

For the last integral, by Lemma 2.12 and (1.2.14), there exist $s > 1, C > 0$ such that

$$\int_\Sigma h e^{s\overline{\beta}(u_k^A)^2} dv_g \le C.$$

Thus

$$|I_k^3| \leq \gamma_k m_k \|\xi\|_\infty \int_\Sigma h|u_k| e^{b_k(u_k)^2} dv_g \leq \gamma_k m_k \|\xi\|_\infty \|u_k\|_{s',h} \|e^{\overline{\beta}(u_k)^2}\|_{s,h} = \gamma_k m_k o(1).$$

By (iii) in Lemma 2.11 and Lemma 2.13 we get that $\gamma_k m_k \to 0$ and hence we find $|I_k^3| \to 0$, which gives the conclusion. □

Let now G_p^λ be the Green's function defined in (2.3.13). Using Lemma 2.14 we obtain:

Lemma 2.15. *For any $s > 1$, we have $m_k u_k \to G_p^\lambda$ in $C^0_{loc}(\Sigma \backslash \{p\}) \cap H^1_{loc}(\Sigma \backslash \{p\}) \cap L^s(\Sigma)$.*

Proof. First we observe that $\|m_k u_k\|_q$ is uniformly bounded. If not we could consider the sequence $w_k := \frac{u_k}{\|u_k\|_q}$, which satisfies

$$-\Delta_g w_k = \gamma_k h \frac{u_k}{\|u_k\|_q} e^{b_k u_k^2} + \frac{s_k}{\|u_k\|_q}.$$

Arguing as in Lemma 2.14, one can prove that $\|\gamma_k h m_k u_k e^{b_k u_k^2}\|_1 \leq C$ and hence it follows

$$\frac{\|\gamma_k h u_k e^{b_k u_k^2}\|_1}{\|u_k\|_q} = \frac{\|\gamma_k h m_k u_k e^{b_k u_k^2}\|_1}{\|m_k u_k\|_q} \to 0,$$

as $k \to +\infty$. Moreover it is easy to check, with (2.4.7) and (2.4.8), that

$$\|s_k\|_1 \leq C \|u_k\|_q,$$

and we have a uniform bound for $-\Delta_g w_k$ in $L^1(\Sigma)$. Therefore w_k is uniformly bounded in $W^{1,s}(\Sigma)$, for any $1 < s < 2$ (see [92] for a reference on open sets in $\mathbb{R}^2$). The weak limit w of w_k will satisfy

$$\int_\Sigma \nabla w \cdot \nabla \varphi \, dv_g = \lambda \int_\Sigma |w|^{q-2} w \varphi \, dv_g,$$

for any $\varphi \in C^1(\Sigma)$ such that $\int_\Sigma \varphi \, dv_g = 0$. But, since $\lambda < \lambda_q(\Sigma, g)$, this implies $w = 0$, which contradicts $\|w_k\|_q = 1$. Hence $\|m_k u_k\|_q \leq C$.

This implies that $-\Delta_g(m_k u_k)$ is uniformly bounded in $L^1(\Sigma)$ and, as before, $m_k u_k$ is uniformly bounded in $W^{1,s}(\Sigma)$ for any $s \in (1, 2)$. By Lemma 2.14 we have $m_k u_k \rightharpoonup G_p^\lambda$ weakly in $W^{1,s}(\Sigma)$ for any $s \in (1, 2)$, and strongly in L^r for any $r \geq 1$.

From Lemma 2.9 we get $|\nabla u_k|^2 \rightharpoonup \delta_p$ and u_k is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash \{p\})$. This implies the boundedness of $-\Delta_g(m_k u_k)$ in $L^s_{loc}(\Sigma \backslash \{p\})$ for some $s > 1$, which gives a uniform bound for $m_k u_k$ in $W^{2,s}_{loc}(\Sigma \backslash \{p\})$. Then, by elliptic estimates, we get $m_k u_k \to G_p^\lambda$ in $H^1_{loc}(\Sigma \backslash \{p\}) \cap C^0_{loc}(\Sigma \backslash \{p\})$. □

As we did in the proof of Theorem 1.2, in the next Proposition we will use an Onofri-type inequality (Corollary 2.9) to control the energy on a small scale.

Proposition 2.15. *We have $\alpha(p) = \overline{\alpha}$ and for any $L > 0$*

$$\limsup_{k\to\infty} \int_{\Omega_{Lr_k}(p_k)} h e^{b_k u_k^2} dv_g \leq \frac{\pi K(p) e^{1+\overline{\beta} A_p^\lambda}}{1+\overline{\alpha}}.$$

Proof. Let us observe that

$$\begin{aligned} \int_{D_{Lr_k}(x_k)} |x|^{2\alpha(p)} e^{b_k v_k^2}\, dx &= \int_{D_{Lr_k}(x_k)} |x|^{2(\alpha(p)-\overline{\alpha})+2\overline{\alpha}} e^{b_k v_k^2}\, dx \\ &\leq (Lr_k)^{2(\alpha(p)-\overline{\alpha})} \int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{b_k v_k^2}\, dx. \end{aligned} \tag{2.4.39}$$

Fix $\delta > 0$ and set $\tau_k = \int_{\Omega_\delta} |\nabla u_k|^2 dv_g = \int_{D_\delta} |\nabla v_k|^2 dy$. Observe that, by Lemma 2.15,

$$m_k^2 (1-\tau_k) = \int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g + o(1), \tag{2.4.40}$$

and

$$m_k^2 \|u_k\|_q^2 = \|G_p^\lambda\|_q^2 + o(1). \tag{2.4.41}$$

Since by Lemma 2.6 we have

$$\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\frac{1}{2\pi}\log\delta + O(1) \overset{\delta\to 0}{\to} +\infty, \tag{2.4.42}$$

for δ sufficiently small, we obtain

$$\begin{aligned} \tau_k(1+\lambda\|u_k\|_2^2) &= \left(1 - \frac{1}{m_k^2}\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g + o\left(\frac{1}{m_k^2}\right)\right)\left(1 + \frac{\lambda}{m_k^2}\|G_p^\lambda\|_q^2 + o\left(\frac{1}{m_k^2}\right)\right) \\ &= 1 - \frac{1}{m_k^2}\left(\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g - \lambda\|G_p^\lambda\|_q^2\right) + o\left(\frac{1}{m_k^2}\right) < 1. \end{aligned} \tag{2.4.43}$$

We denote $d_k := \sup_{\partial D_\delta} v_k$ and $w_k := (v_k - d_k)^+ \in H_0^1(D_\delta)$. Applying Holder's inequality we have

$$\begin{aligned} &\int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{b_k v_k^2} dx = e^{b_k d_k^2} \int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{b_k w_k^2 + 2b_k d_k w_k} dx \\ &\leq e^{b_k d_k^2} \left(\int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{\beta_k \frac{w_k^2}{\tau_k}} dx\right)^{\tau_k(1+\lambda\|u_k\|_q^2)} \left(\int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{\frac{2b_k w_k d_k}{1-\tau_k(1+\lambda\|u_k\|_q^2)}}\right)^{1-\tau_k(1+\lambda\|u_k\|_q^2)}. \end{aligned} \tag{2.4.44}$$

Observe that, for $k \to +\infty$, we have that $\frac{w_k}{\sqrt{\tau_k}} \to 0$ uniformly on $D_\delta \setminus D_{\delta'}$, for any $0 < \delta' < \delta$. Thus, applying Corollary 2.3 to the function $\frac{w_k}{\sqrt{\tau_k}}$ with $\delta_k = Lr_k$, we find

$$\limsup_{k\to\infty} \int_{D_{Lr_k}(x_k)} |x|^{2\overline{\alpha}} e^{\beta_k \frac{w_k^2}{\tau_k}}\, dx \leq \frac{\pi e}{1+\overline{\alpha}} \delta^{2(1+\overline{\alpha})}. \tag{2.4.45}$$

Using Corollary 2.9 we find

$$\begin{aligned}\int_{D_{Lr_k(x_k)}} |x|^{2\overline{\alpha}} e^{\frac{2b_k w_k d_k}{1-\tau_k(1+\lambda\|u_k\|_q^2)}} &\leq \int_{D_\delta} |x|^{2\overline{\alpha}} e^{\frac{2b_k w_k d_k}{1-\tau_k(1+\lambda\|u_k\|_q^2)}}\, dx \\ &\leq \frac{\pi e^{1+\frac{4b_k^2 d_k^2 \tau_k}{16\pi(1+\overline{\alpha})(1-\tau_k(1+\lambda\|u_k\|_q^2)^2}}}{1+\overline{\alpha}} \delta^{2(1+\overline{\alpha})} \\ &\leq \frac{\pi e^{1+\frac{b_k d_k^2 \tau_k(1+\lambda\|u_k\|_q^2)}{(1-\tau_k(1+\lambda\|u_k\|_q^2)^2}}}{1+\overline{\alpha}} \delta^{2(1+\overline{\alpha})}.\end{aligned}$$

Combining this with (2.4.39), (2.4.44), and (2.4.45), we find

$$\limsup_{k\to\infty} \int_{D_{Lr_k}(x_k)} |x|^{2\alpha(p)} e^{b_k v_k^2}\, dx \leq \frac{\pi e \delta^{2(1+\overline{\alpha})}}{1+\overline{\alpha}} \limsup_{k\to\infty} (Lr_k)^{2(\alpha(p)-\overline{\alpha})}\, e^{\frac{b_k d_k^2}{1-\tau_k(1+\lambda\|u_k\|_q^2)}}. \tag{2.4.46}$$

Using (2.4.43) and Lemma 2.15,

$$\lim_{k\to\infty} \frac{b_k d_k^2}{1-\tau_k(1+\lambda\|u_k\|_q^2)} = \frac{\overline{\beta}(\sup_{\partial B_\delta} G_p^\lambda)^2}{\left(\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g - \lambda\|G_p^\lambda\|_q^2\right)} =: H(\delta). \tag{2.4.47}$$

Notice that by Lemma 2.6 and (2.3.14)

$$H(\delta) = -2(1+\overline{\alpha})\log\delta + \overline{\beta} A_p^\lambda + o_\delta(1). \tag{2.4.48}$$

With (2.4.46) and (2.4.47) we obtain

$$\begin{aligned}\limsup_{k\to+\infty} \int_{\Omega_{Lr_k}(p_k)} h e^{b_k u_k^2}\, dv_g &= \limsup_{k\to\infty} \int_{D_{Lr_k}(x_k)} V(x)|x|^{2\alpha(p)} e^{b_k v_k^2} dx \\ &\leq \frac{K(p)\pi e \delta^{2(1+\overline{\alpha})}}{1+\overline{\alpha}} e^{H(\delta)} \limsup_{k\to+\infty} (Lr_k)^{2(\alpha(p)-\overline{\alpha})}.\end{aligned} \tag{2.4.49}$$

If $\alpha(p) > \overline{\alpha}$ we would have $(Lr_k)^{2(\alpha(p)-\overline{\alpha})} \to 0$ as $k \to +\infty$. This would imply, using Lemma 2.13, that

$$\limsup_{k\to+\infty} E_k(u_k) \leq |\Sigma_{g_h}|,$$

which is a contradiction since u_k is a maximizing sequence. Hence, it must be $\alpha(p) = \overline{\alpha}$. Therefore, combining (2.4.47), (2.4.48), and (2.4.49), we get

$$\limsup_{k\to+\infty}\int_{\Omega_{Lr_k}(p_k)} he^{b_k u_k^2}\,dv_g \leq \frac{K(p)\pi e\delta^{2(1+\overline{\alpha})}}{1+\overline{\alpha}}e^{H(\delta)} = \frac{K(p)\pi e^{1+\overline{\beta}A_p^\lambda+o_\delta(1)}}{1+\overline{\alpha}}.$$

□

Proof of Proposition 2.13. The proof follows at once from Lemma 2.13 and Proposition 2.15. □

2.5 Test functions and existence of extremals

By Proposition 2.13, in order to prove existence of extremals for $E_{\Sigma,h}^{\overline{\beta},\lambda,q}$, it suffices to show that the value

$$\frac{\pi e}{1+\overline{\alpha}}\max_{p\in\Sigma,\ \alpha(p)=\overline{\alpha}} K(p)e^{\overline{\beta}A_p^\lambda} + |\Sigma|_{g_h}$$

is exceeded. In this section we will show that this is indeed the case if λ is small enough.

Proposition 2.16. *There exists $\lambda_0>0$ such that*

$$\sup_{u\in\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} > \frac{\pi e}{1+\overline{\alpha}}\max_{p\in\Sigma,\ \alpha(p)=\overline{\alpha}} K(p)e^{\overline{\beta}A_p^\lambda} + |\Sigma|_{g_h},$$

for any $0\leq\lambda<\lambda_0$.

Proof. Let $p\in\Sigma$ be such that $\alpha(p)=\overline{\alpha}$ and

$$K(p)e^{\overline{\beta}A_p^\lambda} = \max_{q\in\Sigma,\ \alpha(q)=\overline{\alpha}} K(q)e^{\overline{\beta}A_q^\lambda}.$$

In local coordinates (Ω,ψ) satisfying (2.3.7)-(2.3.12), we define

$$w_\varepsilon(x) := \begin{cases} c_\varepsilon - \dfrac{\log\left(1+\left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right)+L_\varepsilon}{\overline{\beta}c_\varepsilon} & x\in\Omega_{\gamma_\varepsilon\varepsilon}\\ \frac{G_p^\lambda-\eta_\varepsilon\xi}{c_\varepsilon} & x\in\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}\\ \frac{G_p^\lambda}{c_\varepsilon} & x\in\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}\end{cases} \tag{2.5.1}$$

and

$$u_\varepsilon := \frac{w_\varepsilon}{\sqrt{1+\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}},$$

where $c_\varepsilon, L_\varepsilon$ will be chosen later, $\gamma_\varepsilon = |\log\varepsilon|^{\frac{1}{1+\overline{\alpha}}}$, ξ is defined as in (2.3.14), and $\eta_\varepsilon\in C_0^\infty(\Omega_{2\gamma_\varepsilon\varepsilon})$ is a cut-off function such that $\eta_\varepsilon\equiv 1$ in $\Omega_{\gamma_\varepsilon\varepsilon}$ and $\|\nabla\eta_\varepsilon\|_{L^\infty(\Sigma)} = O(\frac{1}{\gamma_\varepsilon\varepsilon})$. In order to have $u_\varepsilon\in H^1(\Sigma)$ we choose L_ε so that

$$\overline{\beta}c_\varepsilon^2 - L_\varepsilon = \log\left(\frac{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}{\gamma_\varepsilon^{2(1+\overline{\alpha})}}\right) + \overline{\beta}A_p^\lambda - 2(1+\overline{\alpha})\log\varepsilon. \tag{2.5.2}$$

Observe that

$$\int_{\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\overline{\beta} c_\varepsilon^2}\left(\log(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}) - 1 + O(|\log\varepsilon|^{-2})\right). \tag{2.5.3}$$

Since $\xi \in C^1(\overline{D_{\delta_0}})$ and $\xi(x) = O(|x|)$, we have

$$\begin{aligned}\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla(\eta_\varepsilon\xi)|^2\,dv_g &= \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla\eta_\varepsilon|^2\xi^2\,dv_g \\ &\quad + \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla\xi|^2\eta_\varepsilon^2\,dv_g + 2\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \eta_\varepsilon\xi\nabla\eta_\varepsilon\cdot\nabla\xi\,dv_g \\ &= O((\gamma_\varepsilon\varepsilon)^2).\end{aligned}$$

Similarly

$$\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \nabla G_p^\lambda\cdot\nabla(\eta_\varepsilon\xi)dv_g = O(\gamma_\varepsilon\varepsilon).$$

By Lemma 2.6 we have

$$\begin{aligned}c_\varepsilon^2\int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla w_\varepsilon|^2 dv_g &= \int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla G_p^\lambda|^2 + O(\gamma_\varepsilon\varepsilon) \\ &= -\frac{1}{2\pi}\log\gamma_\varepsilon\varepsilon + A_p^\lambda + \lambda\|G_p^\lambda\|_q^2 + O(\gamma_\varepsilon\varepsilon|\log(\gamma_\varepsilon\varepsilon)|).\end{aligned}$$

Observe that $\gamma_\varepsilon\varepsilon\log(\gamma_\varepsilon\varepsilon) = o(|\log\varepsilon|^{-2})$. Therefore we get

$$\int_\Sigma |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\overline{\beta}c_\varepsilon^2}\left(-1 - 2(1+\overline{\alpha})\log\varepsilon + \overline{\beta}A_p^\lambda + \overline{\beta}\lambda\|G_p^\lambda\|_q^2 + O(|\log\varepsilon|^{-2})\right).$$

If we chose c_ε so that

$$\overline{\beta}c_\varepsilon^2 = -1 - 2(1+\overline{\alpha})\log\varepsilon + \overline{\beta}A_p^\lambda + O(|\log\varepsilon|^{-2}), \tag{2.5.4}$$

then $u_\varepsilon - \overline{u}_\varepsilon \in \mathcal{H}$. Note also that (2.5.2) and (2.5.4) yield

$$L_\varepsilon = -1 + O(|\log\varepsilon|^{-2}), \tag{2.5.5}$$

and

$$2\pi c_\varepsilon^2 = |\log\varepsilon| + O(1). \tag{2.5.6}$$

Since $0 \le w_\varepsilon \le O(c_\varepsilon)$ in $\Omega_{\gamma_\varepsilon\varepsilon}$, we get

$$\int_{\Omega_{\gamma_\varepsilon\varepsilon}} w_\varepsilon dv_g = O(c_\varepsilon(\gamma_\varepsilon\varepsilon)^2) = o(|\log\varepsilon|^{-2}).$$

Moreover

$$\begin{aligned}\int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} w_\varepsilon\,dv_g &= \int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \frac{G_p^\lambda}{c_\varepsilon}\,dv_g - \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \frac{\eta_\varepsilon\xi}{c_\varepsilon}\,dv_g\\ &= O\left(\frac{(\gamma_\varepsilon\varepsilon)^2|\log(\gamma_\varepsilon\varepsilon)|}{c_\varepsilon}\right) + O\left(\frac{(\gamma_\varepsilon\varepsilon)^3}{c_\varepsilon}\right)\\ &= o(|\log\varepsilon|^{-2}),\end{aligned}$$

therefore

$$\overline{w}_\varepsilon = o(|\log\varepsilon|^{-2}) = o(c_\varepsilon^{-4}). \tag{2.5.7}$$

From (2.5.4), (2.5.5), and (2.5.7), it follows that in $\Omega_{\gamma_\varepsilon\varepsilon}$

$$\overline{\beta}(w_\varepsilon-\overline{w}_\varepsilon)^2 \ge \overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - 2\log\left(1+\left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right) + o(c_\varepsilon^{-2}).$$

We have

$$c_\varepsilon^2\|w_\varepsilon-\overline{w}_\varepsilon\|_q^2 \ge \left(\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} |G_p^\lambda - c_\varepsilon\overline{w}_\varepsilon|^q\,dv_g\right)^{\frac{2}{q}} \ge \|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-2}),$$

where the last inequality follows from (2.5.1) and Bernoulli's inequality, after splitting the integral on regions where $|G_p^\lambda| \ge |c_\varepsilon\overline{w}_\varepsilon|$ and $|G_p^\lambda| \le |c_\varepsilon\overline{w}_\varepsilon|$. Therefore we find

$$\begin{aligned}\frac{1}{1+\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\left(1+\frac{\lambda\|w_\varepsilon-\overline{w}_\varepsilon\|_q^2}{1+\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\right) &\ge \frac{1+2\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-4})}{\left(1+\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2\right)^2}\\ &= 1-\frac{\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^4} + o(c_\varepsilon^{-4}).\end{aligned} \tag{2.5.8}$$

Hence

$$\overline{\beta}(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u}_\varepsilon\|_q^2) \ge \overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - 2\log\left(1+\left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right) - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2}).$$

It follows that

$$\begin{aligned}\int_{\Omega_{\gamma_\varepsilon\varepsilon}} he^{\overline{\beta}(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u}_\varepsilon\|_q^2)}dv_g &\ge \int_{D_{\gamma_\varepsilon\varepsilon}} |x|^{2\overline{\alpha}}(V(0)+O(\gamma_\varepsilon\varepsilon))\frac{e^{\overline{\beta}c_\varepsilon^2-2L_\varepsilon-\frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2}+o(c_\varepsilon^{-2})}}{\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right)^2}dx\\ &= \frac{\pi V(0)\varepsilon^{2(1+\overline{\alpha})}\gamma_\varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})(1+\gamma_\varepsilon^{2(1+\overline{\alpha})})}e^{\overline{\beta}c_\varepsilon^2-2L_\varepsilon-\frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2}+o(c_\varepsilon^{-2})}(1+O(\gamma_\varepsilon\varepsilon))\\ &= \frac{\pi V(0)\varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})}e^{\overline{\beta}c_\varepsilon^2-2L_\varepsilon-\frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2}+o(c_\varepsilon^{-2})}(1+O(c_\varepsilon^{-4})).\end{aligned}$$

Using (2.5.4) and (2.5.5) we find

$$\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon = -2(1+\overline{\alpha})\log\varepsilon + 1 + \overline{\beta}A_p^\lambda + O(c_\varepsilon^{-4}),$$

so that

$$\int_{\Omega_{\gamma_\varepsilon\varepsilon}} he^{\overline{\beta}(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u}_\varepsilon\|_q^2)} = \frac{\pi V(0)e^{1+\overline{\beta}A_p^\lambda}}{(1+\overline{\alpha})}\left(1-\frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2}+o(c_\varepsilon^{-2})\right). \quad (2.5.9)$$

Finally, with (2.5.7) and (2.5.8), we observe that

$$\begin{aligned}
&\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} he^{\overline{\beta}(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u}_\varepsilon\|_q^2)}dv_g \\
&\geq \int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} h\,dv_g + \overline{\beta}(1+\lambda\|u_\varepsilon-\overline{u}_\varepsilon\|_q^2)\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} h(u_\varepsilon-\overline{u}_\varepsilon)^2\,dv_g \\
&\geq |\Sigma|_{g_h} + O((\gamma_\varepsilon\varepsilon)^{2(1+\overline{\alpha})}) + \overline{\beta}\left(1-\frac{\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^4}+o(c_\varepsilon^{-4})\right)\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} h(w_\varepsilon-\overline{w}_\varepsilon)^2 dv_g \\
&= |\Sigma|_{g_h} + \frac{\overline{\beta}\|G_p^\lambda\|_{L^2(\Sigma,g_h)}}{c_\varepsilon^2} + o(c_\varepsilon^{-2}).
\end{aligned} \quad (2.5.10)$$

Hence, from (2.3.12), (2.5.9), and (2.5.10), it follows that

$$E_{\Sigma,h}^{\overline{\beta},\lambda,q}(u_\varepsilon-\overline{u}_\varepsilon) \geq \frac{\pi K(p)}{1+\overline{\alpha}}e^{1+\overline{\beta}A_p^\lambda}+|\Sigma|_{g_h}+\frac{\overline{\beta}}{c_\varepsilon^2}\left(\|G_p^\lambda\|_{L^2(\Sigma,g_h)}-\frac{\pi K(p)e^{1+\overline{\beta}A_p^\lambda}\lambda^2\|G_p^\lambda\|_q^4}{1+\overline{\alpha}}\right)+o(c_\varepsilon^{-2}).$$

By Lemma 2.5, we know that

$$\left(\|G_p^\lambda\|_{L^2(\Sigma,g_h)}-\frac{\pi K(p)e^{1+\overline{\beta}A_p^\lambda}\lambda^2\|G_p^\lambda\|_q^4}{1+\overline{\alpha}}\right)\to \|G_p^0\|_{L^2(\Sigma,g_h)}>0,$$

as $\lambda\to 0$. Thus, for sufficiently small λ, we get the conclusion. □

To finish the proof of Theorem 1.4 we have to treat the case $\lambda > \lambda_q(\Sigma,g)$. We will use a family of test functions similar to the one used in [64].

Lemma 2.16. *If $\beta > \overline{\beta}$, or $\beta = \overline{\beta}$ and $\lambda > \lambda_q(\Sigma,g)$, we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} = +\infty.$$

Proof. Take $p\in\Sigma$ such that $\alpha(p)=\overline{\alpha}$ and consider a local chart (Ω,ψ) satisfying (2.3.7)-(2.3.12). Let us define $v_\varepsilon : D_{\delta_0}\to[0,+\infty)$,

$$v_\varepsilon(x) := \frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log\frac{\delta_0}{\varepsilon}} & |x|\leq\varepsilon \\ \dfrac{\log\frac{\delta_0}{|x|}}{\sqrt{\log\frac{\delta_0}{\varepsilon}}} & \varepsilon\leq|x|\leq\delta_0,\end{cases}$$

and $u_\varepsilon\colon \Sigma \to [0,+\infty)$,

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(\psi(x)) & x \in \Omega \\ 0 & x \in \Sigma \backslash \Omega. \end{cases}$$

It is simple to verify that

$$\int_\Sigma |\nabla u_\varepsilon|^2 dv_g = \int_{D_{\delta_0}} |\nabla v_\varepsilon|^2 dx = 1,$$

which implies $u_\varepsilon - \overline{u}_\varepsilon \in \mathcal{H}$. By direct computation one has

$$\overline{u}_\varepsilon = O\left(\left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{2}}\right). \tag{2.5.11}$$

Hence in Ω_ε

$$(u_\varepsilon - \overline{u}_\varepsilon)^2 = \frac{1}{2\pi} \log\left(\frac{\delta_0}{\varepsilon}\right) + O(1).$$

Thus, if $\beta > \overline{\beta}$, we have

$$\begin{aligned} E^{\beta,\lambda,q}_{\Sigma,h}(u_\varepsilon - \overline{u}_\varepsilon) \geq E^{\beta,0,q}_{\Sigma,h}(u_\varepsilon - \overline{u}_\varepsilon) \geq \int_{\Omega_\varepsilon} h e^{\beta(u_\varepsilon - \overline{u}_\varepsilon)^2} dv_g \geq \frac{C}{\varepsilon^{\frac{\beta}{2\pi}}} \int_{D_\varepsilon} |x|^{2\overline{\alpha}} dx \\ = \frac{C\pi}{1+\overline{\alpha}} \varepsilon^{2(1+\overline{\alpha}) - \frac{\beta}{2\pi}} = \tilde{C} \varepsilon^{\frac{\overline{\beta}-\beta}{2\pi}} \to +\infty, \end{aligned}$$

as $\varepsilon \to 0$. For the case $\beta = \overline{\beta}$ and $\lambda > \lambda_q(\Sigma, g)$, we take a function $u_0 \in H^1(\Sigma)$ such that

$$\begin{cases} \|\nabla u_0\|_2^2 = \lambda_q(\Sigma, g) \|u_0\|_q^2 \\ \int_\Sigma u_0 \, dv_g = 0 \\ \|u_0\|_q^2 = 1. \end{cases} \tag{2.5.12}$$

This function u_0 will also satisfy

$$-\Delta_g u_0 = \lambda_q \|u_0\|_q^{2-q} |u_0|^{q-2} u_0 - c,$$

where

$$c = \frac{\lambda_q}{|\Sigma|} \|u_0\|_q^{2-q} \int_\Sigma |u_0|^{q-2} u_0 \, dv_g.$$

Let us take $t_\varepsilon, r_\varepsilon \to 0$ such that

$$t_\varepsilon^2 |\log \varepsilon| \to +\infty, \qquad \frac{r_\varepsilon}{\varepsilon} \to +\infty, \qquad \text{and} \qquad \frac{\log^2 r_\varepsilon}{t_\varepsilon^2 |\log \varepsilon|} \to 0. \tag{2.5.13}$$

We define

$$w_\varepsilon := u_\varepsilon \eta_\varepsilon + t_\varepsilon u_0, \tag{2.5.14}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega_{2r_\varepsilon})$ is a cut-off function such that $\eta_\varepsilon \equiv 1$ in Ω_{r_ε}, $0 \le \eta_\varepsilon \le 1$, and $|\nabla \eta_\varepsilon| = O(r_\varepsilon^{-1})$. It is straightforward that

$$\overline{w}_\varepsilon = O(|\log \varepsilon|^{-\frac{1}{2}}). \tag{2.5.15}$$

Observe that

$$\|\nabla w_\varepsilon\|_2^2 = \int_\Sigma |\nabla(u_\varepsilon \eta_\varepsilon)|^2 dv_g + t_\varepsilon^2 \|\nabla u_0\|_2^2 + 2t_\varepsilon \int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g.$$

Using the definition of $u_\varepsilon, \eta_\varepsilon$, and (2.5.13), we find

$$\int_\Sigma |\nabla \eta_\varepsilon|^2 u_\varepsilon^2 dv_g = O(r_\varepsilon^{-2}) \int_{\Omega_{2r_\varepsilon} \setminus \Omega_{r_\varepsilon}} u_\varepsilon^2 dv_g = O\left(|\log \varepsilon|^{-1} \log^2 r_\varepsilon\right) = o(t_\varepsilon^2),$$

and

$$\left| \int_\Sigma u_\varepsilon \eta_\varepsilon \nabla u_\varepsilon \cdot \nabla \eta_\varepsilon dv_g \right| \le O(r_\varepsilon^{-1}) \int_{\Omega_{2r_\varepsilon} \setminus \Omega r_\varepsilon} |\nabla u_\varepsilon| u_\varepsilon dv_g = O(|\log r_\varepsilon| |\log \varepsilon|^{-1}) = o(t_\varepsilon^2).$$

Thus

$$\|\nabla(u_\varepsilon \eta_\varepsilon)\|_2^2 = \int_\Sigma |\nabla u_\varepsilon|^2 \eta_\varepsilon^2 dv_g + o(t_\varepsilon^2) \le 1 + o(t_\varepsilon^2).$$

Moreover (2.5.12) gives $\|\nabla u_0\|_2^2 = \lambda_q$ and

$$\left| \int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g \right| = \left| \lambda_q \int_\Sigma (|u_0|^{q-2} u_0 - c) \eta_\varepsilon u_\varepsilon dv_g \right| = O(1) \int_\Sigma u_\varepsilon dv_g = O(|\log \varepsilon|^{-\frac{1}{2}}) = o(t_\varepsilon).$$

Hence we have

$$\|\nabla w_\varepsilon\|_2^2 \le 1 + \lambda_q t_\varepsilon^2 + o(t_\varepsilon^2).$$

Furthermore, by dominated convergence we have

$$\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2 \ge t_\varepsilon^2 \left(\int_{\Sigma \setminus \Omega_{2r_\varepsilon}} |u_0 - \frac{\overline{w}_\varepsilon}{t_\varepsilon}|^q dv_g \right)^{\frac{2}{q}} = t_\varepsilon^2 \|u_0\|_q^2 + o(t_\varepsilon^2) = t_\varepsilon^2 + o(t_\varepsilon^2).$$

Thus we get

$$\frac{1}{\|\nabla w_\varepsilon\|_2^2} \left(1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2} \right) \ge 1 + (\lambda - \lambda_q) t_\varepsilon^2 + o(t_\varepsilon^2).$$

Finally, using (2.5.15), in Ω_ε we find

$$\begin{aligned} \frac{4\pi(1+\overline{\alpha})(w_\varepsilon - \overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2} \left(1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2} \right) &= (2(1+\overline{\alpha})|\log \varepsilon| + O(1)) \left(1 + (\lambda - \lambda_q) t_\varepsilon^2 + o(t_\varepsilon^2)\right) \\ &= -2(1+\overline{\alpha}) \log \varepsilon + (\lambda - \lambda_q) t_\varepsilon^2 |\log \varepsilon| + O(1), \end{aligned}$$

so that

$$\begin{aligned} E_{\Sigma,h}^{\overline{\beta},\lambda,q}\left(\frac{w_\varepsilon-\overline{w}_\varepsilon}{\|\nabla w_\varepsilon\|_2}\right) &\geq \int_{\Omega_\varepsilon} h e^{\frac{4\pi(1+\overline{\alpha})(w_\varepsilon-\overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2}\left(1+\lambda\frac{\|w_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2}\right)} dv_g \\ &\geq c\varepsilon^{-2(1+\overline{\alpha})} e^{(\lambda-\lambda_q)t_\varepsilon^2|\log\varepsilon|+O(1)}\int_{D_\varepsilon}|y|^{2\overline{\alpha}}dy \\ &= \tilde{c}e^{(\lambda-\lambda_q)t_\varepsilon^2|\log\varepsilon|}\to+\infty, \end{aligned}$$

as $\varepsilon\to 0$. □

Remark 2.5. *If there exists a point $p\in\Sigma$ such that $\alpha(p)=\overline{\alpha}$ and $u_0(p)\neq 0$, then one can argue as in [64] to prove*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q}=+\infty$$

also for $\lambda=\lambda_q(\Sigma,g_0)$. This is always true if $\overline{\alpha}=0$.

The proof of Theorem 1.5 is very similar to the one of Theorem 1.4, hence it will not be discussed here.

We conclude this Chapter by observing that, as in [57], [102] and [64], our techniques can be adapted with minor modifications to treat the case of compact surfaces with boundary, which we state here without proof, as it is very similar to that of Theorem 1.4.

Theorem 2.17. *Let (Σ,g) be a smooth, compact, Riemannian surface with boundary. If $p_1,\dots,p_m\in\Sigma\backslash\partial\Sigma$ and $h\in C^1(\Sigma\backslash\{p_1,\dots,p_m\})$ satisfies (1.1.11), then $\forall\,\beta\in[0,4\pi(1+\overline{\alpha})]$ and $\lambda\in[0,\lambda_q(\Sigma,g))$ we have*

$$\sup_{u\in H_0^1(\Sigma),\int_\Sigma|\nabla u|^2dv_g\leq 1} E_{\Sigma,h}^{\beta,\lambda,q}(u)<+\infty.$$

The supremum is attained if $\beta<4\pi(1+\overline{\alpha})$, or if $\beta=4\pi(1+\overline{\alpha})$ and λ is sufficiently small. Furthermore if $\beta>4\pi(1+\overline{\alpha})$, or $\beta=4\pi(1+\overline{\alpha})$ and $\lambda\geq\lambda_q(\Sigma,g)$, we have

$$\sup_{u\in u\in H_0^1(\Sigma),\int_\Sigma|\nabla u|^2dv_g\leq 1} E_{\Sigma,h}^{\beta,\lambda,q}(u)=+\infty.$$

In particular, if $\Sigma=\overline{\Omega}$ is the closure of a bounded domain in $\mathbb{R}^2$, Theorem 2.17 gives the following generalization of the results in [40], [4], [31].

Corollary 2.18. *Let $\Omega\subseteq\mathbb{R}^2$ be a bounded domain. For any choice of $V\in C^1(\overline{\Omega})$, $V>0$, $\alpha_1,\dots,\alpha_m>-1$, $x_1,\dots,x_m\in\Omega$, $q>1$ and $\lambda\in[0,\lambda_q(\Omega))$, the supremum*

$$\sup_{u\in H_0^1(\Omega),\,\int_\Omega|\nabla u|^2dx\leq 1}\int_\Omega V(x)\prod_{i=1}^m|x-x_i|^{2\alpha_i}e^{4\pi(1+\overline{\alpha})u^2\left(1+\lambda\|u\|_{L^q(\Omega)}^2\right)}dx$$

is finite. Moreover, it is attained if λ is sufficiently small.

Chapter 3

Fractional Moser-Trudinger type inequalities in dimension one

This chapter is organized as follows. In Section 3.1 we recall some definitions and useful results on fractional Sobolev spaces and fractional Laplace operators. In Section 3.2 we investigate fractional analogues of Theorem 1.1. In particular, we shall prove Theorem 1.6, Theorem 1.7, Theorem 1.8, Theorem 1.9 and Theorem 1.10. To conclude, in Section 3.3 we discuss the existence of critical points of the functional associated to (1.2.2), proving Proposition 1.12 and Theorem 1.11.

3.1 Sobolev spaces of fractional order

In this section we introduce some relevant fractional function spaces. We will discuss some results that we be useful in the next sections. We refer to [87], [34], [90], [39] for a more detailed discussion on the topics presented here.

We define

$$W^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : [u]^p_{W^{s,p}(\mathbb{R})} := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dxdy < \infty \right\}, \qquad (3.1.1)$$

and we will denote by I an interval such that $I \Subset \mathbb{R}$. Throughout this Chapter we will also use the following notation:

$$H := \tilde{H}^{\frac{1}{2},2}(I), \quad \|u\|_H := \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})},$$

where $\tilde{H}^{\frac{1}{2},2}(I)$ is defined as in (1.3.4) for a bounded interval $I \Subset \mathbb{R}$.

Proposition 3.1. *For $s \in (0,1)$ we have, $[u]_{W^{s,2}(\mathbb{R})} < \infty$ if and only if $(-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R})$, and in this case*

$$[u]_{W^{s,2}(\mathbb{R})} = C_s \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R})},$$

where $[u]_{W^{s,2}(\mathbb{R})}$ *is as in* (3.1.1) *and* C_s *depends only on* s*. In particular* $H^{s,2}(\mathbb{R}) = W^{s,2}(\mathbb{R})$*.*

Proof. See e.g. Proposition 3.6 in [34]. □

Define the bilinear form

$$\mathcal{B}_s(u,v) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}dxdy, \quad \text{for } u,v \in H^{s,2}(\mathbb{R}),$$

where the double integral is well defined thanks to Hölder's inequality and Proposition 3.1.

The following simple and well-known existence result proves useful. A proof can be found (in a more general setting) in [39].

Theorem 3.2. *Given* $s \in (0,1)$*,* $f \in L^2(I)$ *and* $g : \mathbb{R} \to \mathbb{R}$ *such that*

$$\int_I \int_{\mathbb{R}} \frac{(g(x)-g(y))^2}{|x-y|^{1+2s}}dxdy < \infty, \tag{3.1.2}$$

there exists a unique function $u \in \tilde{H}^{s,2}(I) + g$ *solving the problem*

$$\mathcal{B}_s(u,v) = \int_{\mathbb{R}} fvdx \quad \textit{for every } v \in \tilde{H}^{s,2}(I). \tag{3.1.3}$$

Moreover such u *satisfies* $(-\Delta)^s u = \frac{C_s}{2} f$ *in* I *in the sense of distributions, i.e.*

$$\int_{\mathbb{R}} u(-\Delta)^s \varphi dx = \frac{C_s}{2}\int_{\mathbb{R}} f\varphi dx \quad \textit{for every } \varphi \in C_c^\infty(I), \tag{3.1.4}$$

where C_s *is the constant in Proposition 3.6.*

The following version of the maximum principle is a special case of Theorem 4.1 in [39].

Proposition 3.3. *Let* $u \in \tilde{H}^{s,2}(I) + g$ *solve* (3.1.3) *for some* $f \in L^2(I)$ *with* $f \geq 0$ *and* g *satisfying* (3.1.2) *and* $g \geq 0$ *in* I^c*. Then* $u \geq 0$*.*

Proof. From Proposition 3.1 it easily follows $u^- := \min\{u, 0\} \in \tilde{H}^{s,2}(I)$. Then according to (3.1.3) we have

$$\begin{aligned} 0 \geq \mathcal{B}_s(u,u^-) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u^+(x)+u^-(x)-u^+(y)-u^-(y))(u^-(x)-u^-(y))}{|x-y|^{1+2s}}dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{1+2s}}dxdy - 2\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{u^+(x)u^-(y)}{|x-y|^{1+2s}}dxdy \end{aligned}$$

where we used that $u^+u^- = 0$. Since the second term in the last equality is non-negative, it follows at once that $u^- \equiv 0$, hence $u \geq 0$. □

Proposition 3.4. *Let $u \in \tilde{H}^{s,2}(I)$ be as in Theorem 3.2 (with $g = 0$), where we further assume $f \in L^\infty(I)$. Then*

$$|u(x)| \le C\|f\|_{L^\infty(I)}(\mathrm{dist}(x, \partial I))^s$$

for every $x \in I$. In particular u is bounded in I and continuous at ∂I.

Proof. This proof is inspired from [87], where a much stronger result is proven, i.e. $u/(\mathrm{dist}(\cdot, \partial I))^s \in C^\alpha(\bar{I})$ for some $\alpha > 0$.

To prove the proposition we assume that $I = (-1, 1)$ and recall that

$$w(x) := \begin{cases} (1 - |x|^2)^s & \text{for } x \in (-1, 1) \\ 0 & \text{for } |x| \ge 1 \end{cases}$$

belongs to $\tilde{H}^{s,2}(I)$ and solves $(-\Delta)^s w = \gamma_s$ for a positive constant γ_s, in the sense of Theorem 3.2, i.e. (3.1.3) holds with $u = w$ and $f \equiv \gamma_s$ (see e.g. [43]). Then

$$-\frac{(-\Delta)^s w}{\gamma_s} \le \frac{(-\Delta)^s u}{\|f\|_{L^\infty(I)}} \le \frac{(-\Delta)^s w}{\gamma_s}$$

and Proposition 3.3 gives at once

$$-\frac{\|f\|_{L^\infty(I)}}{\gamma_s} w \le u \le \frac{\|f\|_{L^\infty(I)}}{\gamma_s} w \quad \text{in } I.$$

We conclude noticing that $0 \le w(x) \le 2^s(\mathrm{dist}(x, \partial I))^s$. □

The following density result is known for an arbitrary domain in $\mathbb{R}^n$. On the other hand, its proof is quite complex in such a generality, hence we provide a short elementary proof which fits the case of an interval.

Lemma 3.1. *For $s \in (0, 1)$ and $p \in [1, \infty)$ the sets $C_c^\infty(I)$ ($I \Subset \mathbb{R}$ is a bounded interval) is dense in $\tilde{H}^{s,p}(I)$.*

Proof. Without loss of generality we consider $I = (-1, 1)$. Given $u \in \tilde{H}^{s,p}(I)$ and $\lambda > 1$, set $u_\lambda(x) := u(\lambda x)$. We claim that $u_\lambda \to u$ in $\tilde{H}^{s,p}(I)$ as $\lambda \to 1$. Indeed

$$\|u_\lambda - u\|^p_{H^{s,p}(\mathbb{R})} = \|u - u_\lambda\|^p_{L^p(\mathbb{R})} + \|\lambda^s f_\lambda - f\|^p_{L^p(\mathbb{R})},$$

where $f = (-\Delta)^{\frac{s}{2}} u$ and $f_\lambda(x) := f(\lambda x)$. Since $f \in L^p(\mathbb{R})$ it follows that $\|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})} \to 0$ as $\lambda \to 1$, since this is obviously true for $f \in C^0(\mathbb{R})$ with compact support, and for a general $f \in L^p(\mathbb{R})$ it can be proven by approximation in the following standard way. Given $\varepsilon > 0$ choose $f_\varepsilon \in C^0(\mathbb{R})$ with compact support and $\|f_\varepsilon - f\|_{L^p(\mathbb{R})} \le \varepsilon$. Then by the Minkowski inequality

$$\begin{aligned}\|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})} &\le \|\lambda^s f_\lambda - \lambda^s f_{\varepsilon,\lambda}\|_{L^p(\mathbb{R})} + \|\lambda^s f_{\varepsilon,\lambda} - f_\varepsilon\|_{L^p(\mathbb{R})} + \|f_\varepsilon - f\|_{L^p(\mathbb{R})}\\ &\le \varepsilon\lambda^{s-\frac{1}{p}} + \|\lambda^s f_{\varepsilon,\lambda} - f_\varepsilon\|_{L^p(\mathbb{R})} + \varepsilon,\end{aligned}$$

and it suffices to let $\lambda \to 1$ and $\varepsilon \to 0$. Similarly $\|u - u_\lambda\|^p_{L^p(\mathbb{R})} \to 0$ as $\lambda \to 1$.

Now given $\delta > 0$ fix $\lambda > 1$ such that $\|u_\lambda - u\|_{H^{s,p}(\mathbb{R})} < \delta$ and let ρ be a mollifying kernel, i.e. a smooth non-negative function supported in I with $\int_I \rho dx = 1$. Also set $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. Then noticing that u_λ is supported in $[-\lambda^{-1}, \lambda^{-1}] \Subset I$, for $\varepsilon > 0$ sufficiently small we have that $\rho_\varepsilon * u_\lambda \in C^\infty_c(I)$. To conclude the proof notice that

$$\rho_\varepsilon * u_\lambda \to u_\lambda \text{ in } \tilde{H}^{s,p}(I) \text{ as } \varepsilon \to 0,$$

since

$$(-\Delta)^{\frac{s}{2}}(\rho_\varepsilon * u_\lambda) = \rho_\varepsilon * (-\Delta)^{\frac{s}{2}} u_\lambda \to (-\Delta)^{\frac{s}{2}} u_\lambda \quad \text{in } L^p(\mathbb{R}) \text{ as } \varepsilon \to 0,$$

and use the Minkowski inequality to conclude that $\rho_\varepsilon * u_\lambda \to u$ in $\tilde{H}^{s,p}(I)$ as $\varepsilon \to 0$ and $\lambda \downarrow 1$. □

Proposition 3.5. *Let $I \Subset \mathbb{R}$ be a bounded interval and $s \in (0,1)$. Let $u \in L_s(\mathbb{R})$ satisfy $(-\Delta)^s u \ge 0$ in I (i.e. $\langle u, (-\Delta)^s \varphi\rangle \ge 0$ for every $\varphi \in C^\infty_c(I)$ with $\varphi \ge 0$), $u \ge 0$ in I^c and*

$$\liminf_{x\to\partial I} u(x) \ge 0. \tag{3.1.5}$$

Then $u \ge 0$ in I. More precisely, either $u > 0$ in I, or $u \equiv 0$ in $\mathbb{R}$.

Proof. This is a special case of Proposition 2.17 in [90]. □

Remark 3.1. *The statement of Proposition 2.17 in [90] is slightly different, since it assumes u to be lower-semicontinuous in $\bar{I}$. On the other hand, lower semicontinuity inside I already follows from [90, Prop. 2.15]. What really matters is condition* (3.1.5)*. That an assumption of this kind (possibly weaker) is needed follows for instance from the example of Lemma 3.2.4 in [1].*

The following way of computing the fractional Laplacian of a sufficiently regular function is often used.

Proposition 3.6. *For an open interval $J \subset \mathbb{R}$, let $s \in (0, \frac{1}{2})$ and $u \in L_s(\mathbb{R}) \cap C^{0,\alpha}(J)$ for some $\alpha \in (2s, 1]$, or $s \in [\frac{1}{2}, 1)$ and $u \in L_s(\mathbb{R}) \cap C^{1,\alpha}(J)$ for some $\alpha \in (2s-1, 1]$. Then $((-\Delta)^s u)|_J \in C^0(J)$ and*

$$(-\Delta)^s u(x) = C_s P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy := C_s \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus[x-\varepsilon, x+\varepsilon]} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy$$

for every $x \in J$. This means that

$$\langle(-\Delta)^s u, \varphi\rangle = C_s \int_{\mathbb{R}} \varphi(x)\, P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy\, dx, \quad \textit{for every } \varphi \in C_c^\infty(J).$$

Proof. See e.g. [90, Prop. 2.4] □

Lemma 3.2. *Let $\varphi_1 \in H = \tilde{H}^{\frac{1}{2},2}(I)$ be an eigenfunction corresponding to the first eigenvalue $\lambda_1(I)$ of $(-\Delta)^{\frac{1}{2}}$ on I. Then $\varphi_1 > 0$ a.e. on I or $\varphi_1 < 0$ a.e. on I and the corresponding eigenspace has dimension 1.*

Proof. Recall that the first eigenvalue $\lambda_1(I)$ can be characterised by minimizing the following functional

$$F(u) = \frac{\|u\|_H^2}{\int_I u^2 dx},$$

that is,

$$\lambda_1(I) = \min_{u \in H\setminus\{0\}} F(u).$$

On the other hand using Proposition 3.1 we get that for any $u \in H$

$$\|u\|_H^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dxdy \geq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(|u(x)|-|u(y)|)^2}{(x-y)^2} dxdy = \||u|\|_H^2, \quad (3.1.6)$$

hence, $F(|u|) \leq F(u)$, and $F(u) = F(|u|)$ if and only if u is non-negative or non-positive. Therefore if $F(\varphi_1) = \lambda_1$, then φ_1 does not change sign. Moreover Theorem $A.1$ in [16] gives us $\varphi_1 > 0$ or $\varphi_1 < 0$ almost everywhere in I. Any other eigenfunction corresponding to λ_1 must also have fixed sign, hence it cannot be orthogonal to φ_1, therefore it is a multiple of φ_1. □

Lemma 3.3. *Consider a sequence $(f_k) \subset L^1(I)$ with $f_k \to f$ a.e. and with*

$$\int_{\{f_k > L\}} f_k dx = o(1), \quad (3.1.7)$$

with $o(1) \to 0$ as $L \to \infty$ uniformly with respect to k. Then $f_k \to f$ in $L^1(I)$.

Proof. From the dominated convergence theorem

$$\min\{f_k, L\} \to \min\{f, L\} \quad \text{in } L^1(I),$$

and the convergence of f_k to f in L^1 follows at once from (3.1.7) and the triangle inequality. □

3.2 Fractional Moser-Trudinger type inequalities

We begin this section by recalling Remark 1.2. As it points out, the $\|u\|^* := \|(-\Delta)^{\frac{1}{2p}}u\|_{L^p(I)}$ norm is equivalent to the full norm $\|u\|_{H^{\frac{1}{p},p}(\mathbb{R})}$ on $\tilde{H}^{\frac{1}{p},p}(I)$. This fact does not appear to be obvious, but one can prove it as follows. By Theorem 7.1 in [44] the operator $T: u \mapsto ((-\Delta)^{\frac{1}{2p}}u)|_I$ is Fredholm from $\tilde{H}^{\frac{1}{p},p}(I)$ $(= H_p^{\frac{1}{2p}(\frac{1}{p})}(\bar{I})$ in the notation of [44]) into $L^p(I)$. Moreover T is injective by Lemma 3.5 below. This implies that

$$\|u\|_{H^{\frac{1}{p},p}(\mathbb{R})} \leq C\|Tu\|_{L^p(I)} = C\|u\|^*, \quad \text{for every } u \in \tilde{H}^{\frac{1}{p},p}(I).$$

Proof of Theorem 1.2.2 By a simple scaling argument it suffices to prove (1.3.7) for a given interval, say $I = (-1, 1)$.

Lemma 3.4. *For $s \in \left(0, \frac{1}{2}\right)$ the fundamental solution of $(-\Delta)^s$ on $\mathbb{R}$ is*

$$F_s(x) = \frac{1}{2\cos(s\pi)\Gamma(2s)|x|^{1-2s}},$$

i.e. $(-\Delta)^s F_s = \delta_0$ in the sense of tempered distributions.

Proof. This follows easily e.g. from Theorem 5.9 in [60]. □

Lemma 3.5. *Fix $s \in \left(0, \frac{1}{2}\right)$. For any $x \in I = (-1, 1)$ let $g_x \in C^\infty(\mathbb{R})$ be any function with $g_x(y) = F_s(x - y)$ for $y \in I^c$. Then there exists $H_s(x, \cdot) \in \tilde{H}^{s,2}(I) + g_x$ unique solution to*

$$\begin{cases} (-\Delta)^s H_s(x, \cdot) = 0 & \text{in } I \\ H_s(x, \cdot) = g_x & \text{in } \mathbb{R} \setminus I \end{cases} \tag{3.2.1}$$

and the function

$$G_s(x, y) := F_s(x - y) - H_s(x, y), \qquad (x, y) \in I \times \mathbb{R}$$

is the Green function of $(-\Delta)^s$ on I, i.e. for $x \in I$ it satisfies

$$\begin{cases} (-\Delta)^s G_s(x, \cdot) = \delta_x & \text{in } I \\ G(x, y) = 0 & \text{for } y \in \mathbb{R} \setminus I. \end{cases} \tag{3.2.2}$$

Moreover

$$0 < G_s(x, y) \leq F_s(x - y) \quad \text{for } y \neq x \in I. \tag{3.2.3}$$

Finally, for any function $u \in \tilde{H}^{2s,p}(I)$ $(p \in [1, \infty))$ we have

$$u(x) = \int_I G_s(x, y)(-\Delta)^s u(y)dy, \quad \text{for a.e. } x \in I, \tag{3.2.4}$$

where the right-hand side is well defined for a.e. $x \in I$ thanks to (3.2.3) and Fubini's theorem.

Remark 3.2. *The first equations in* (3.2.1) *above and in* (3.2.2) *below are intended in the sense of distribution, compare to* (1.3.2).

Proof. The existence and non-negativity of $H_s(x,\cdot)$ for every $x \in I$ follow from Theorem 3.2 and Proposition 3.3. The next claim, namely (3.2.2), follows at once from Lemma 3.4 and (3.2.1).

We show now that $G(x,y) \geq 0$ for every $(x,y) \in I \times I$. We claim that

$$\lim_{y \to \pm 1} H_s(x,y) = H_s(x,\pm 1) = F_s(x \mp 1), \tag{3.2.5}$$

hence $G_s(x,y) \to 0$ as $y \to \partial I$, and by Silvestre's maximum principle, Proposition 3.5 below, we also have $G_s(x,\cdot) \geq 0$ for every $x \in I$, hence also (3.2.3) follows. For the proof of (3.2.5) notice that

$$\tilde{H}_s(x,\cdot) := H_s(x,\cdot) - g_x \in \tilde{H}^{s,2}(I)$$

satifies

$$\begin{cases} (-\Delta)^s \tilde{H}_s(x,\cdot) = -(-\Delta)^s g_x & \text{in } I \\ \tilde{H}_s(x,\cdot) = 0 & \text{in } \mathbb{R} \setminus I \end{cases}$$

and $((-\Delta)^s g_x)|_I \in L^\infty(I)$ by Proposition 3.6 (we are using that $g_x \in C^\infty(\mathbb{R})$), hence Proposition 3.4 gives $\tilde{H}_s(x,y) \to 0$ as $y \to \partial I$, and (3.2.5) follows at once.

To prove (3.2.4), let us start considering $u \in C_c^\infty(I)$. Then, according to (3.2.2), we have

$$u(x) = \langle \delta_x, u \rangle = \langle (-\Delta)^s G_s(x,\cdot), u \rangle = \int_I G_s(x,y)(-\Delta)^s u(y) dy.$$

Given now $u \in \tilde{H}^{2s,p}(I)$, let $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(I)$ converge to u in $\tilde{H}^{2s,p}(I)$, i.e.

$$u_k \to u, \quad (-\Delta)^s u_k \to (-\Delta)^s u \quad \text{in } L^p(\mathbb{R}), \text{ hence in } L^1(I),$$

see Lemma 3.1. Then

$$u \overset{L^1(I)}{\longleftarrow} u_k = \int_I G_s(\cdot,y)(-\Delta)^s u_k(y) dy \overset{L^1(I)}{\longrightarrow} \int_I G_s(\cdot,y)(-\Delta)^s u(y) dy,$$

the convergence on the right following from (3.2.3) and Fubini's theorem:

$$\begin{aligned} &\int_I \left| \int_I G_s(x,y) \left[(-\Delta)^s u_k(y) - (-\Delta)^s u(y) \right] dy \right| dx \\ &\quad \leq \int_I \int_I F_s(x-y) \left| (-\Delta)^s u_k(y) - (-\Delta)^s u(y) \right| dx dy \\ &\quad \leq \sup_{y \in I} \|F_s\|_{L^1(I-y)} \|(-\Delta)^s u_k - (-\Delta)^s u\|_{L^1(I)} \to 0 \end{aligned}$$

as $k \to \infty$. Since the convergence in L^1 implies the a.e. convergence (up to a subsequence), (3.2.4) follows. □

Proof of Theorem 1.2.2. Set $s = \frac{1}{2p}$. From Lemma 3.5 we get

$$0 \le (2\alpha_p)^{\frac{p-1}{p}} G_s(x,y) \le I_{\frac{1}{p}}(x-y) = |x-y|^{\frac{1}{p}-1},$$

where G_s is the Green's function of the interval I defined in Lemma 3.5. Choosing $f := |(-\Delta)^{\frac{1}{2p}} u|\big|_I$ and using (3.2.3) and (3.2.4), we bound

$$(2\alpha_p)^{\frac{p-1}{p}} |u(x)| \le (2\alpha_p)^{\frac{p-1}{p}} \int_I G_s(x,y) f(y) dy \le I_{\frac{1}{p}} * f(x)$$

and (1.3.7) follows at once from (1.3.10).

It remains to show (1.3.9). The proof is based on the construction of suitable test functions and it is split into steps.

Step 1. Definition of the test functions. We fix $\tau \ge 1$ and set

$$f(y) = f_\tau(y) := \frac{1}{2\tau} |y|^{-\frac{1}{p}} \chi_{[-\frac{1}{2},-r]\cup[r,\frac{1}{2}]}, \quad r := \frac{e^{-\tau}}{2}. \tag{3.2.6}$$

Notice that

$$\|f\|_{L^p}^p = \frac{2}{(2\tau)^p} \int_r^{\frac{1}{2}} \frac{dy}{y} = \frac{1}{(2\tau)^{p-1}}.$$

Now let $u = u_\tau \in \tilde{H}^{s,2}(I)$ solve

$$\begin{cases} (-\Delta)^s u = f & \text{in } I \\ u \equiv 0 & \text{in } I^c. \end{cases} \tag{3.2.7}$$

in the sense of Theorem 3.2.

Step 2. Proving that $u \in \tilde{H}^{2s,p}(I)$. According to Proposition 3.4 u satisfies

$$|u(x)| \le C \|f\|_{L^\infty} (1-|x|)^s \quad \text{for } x \in I. \tag{3.2.8}$$

We want to prove that $(-\Delta)^s u \in L^p(\mathbb{R})$. Since by Proposition 3.6

$$(-\Delta)^s u(x) = C_s \int_I \frac{-u(y)}{|x-y|^{1+2s}} dy, \quad \text{for } |x| > 1$$

and u is bounded, we see immediately that

$$|(-\Delta)^s u(x)| \le \frac{C}{|x|^{1+2s}}, \quad \text{for } |x| \ge 2,$$

hence

$$\|(-\Delta)^s u\|_{L^q(\mathbb{R}\setminus[-2,2])} < \infty \quad \text{for every } q \in [1,\infty). \tag{3.2.9}$$

Now we claim that

$$(I) := \|(-\Delta)^s u\|_{L^q([-2,2]\setminus[-1,1])} < \infty, \quad q = \max\{p, 2\}. \tag{3.2.10}$$

Again using Proposition 3.6, (3.2.8) and translating, we have

$$(I) = \left(\int_{[-2,2]\setminus[-1,1]} \left| C \int_{-1}^{1} \frac{-u(y)dy}{|y-x|^{1+2s}} \right|^q dx\right)^{\frac{1}{q}} \le C \left(\int_{-1}^{0} \left| \int_0^2 \frac{y^s dy}{(y-x)^{1+2s}} \right|^q dx\right)^{\frac{1}{q}},$$

and using the Minkowski inequality

$$\left(\int_{A_1} \left| \int_{A_2} F(x,y)dy \right|^q dx\right)^{\frac{1}{q}} \le \int_{A_2} \left(\int_{A_1} |F(x,y)|^q dx\right)^{\frac{1}{q}} dy,$$

we get

$$(I) \le C \int_0^2 y^s \left(\int_{-1}^{0} \frac{dx}{(y-x)^{(1+2s)q}}\right)^{\frac{1}{q}} dy \le C \int_0^2 \frac{dy}{y^{1+s-\frac{1}{q}}} < \infty,$$

since $1 + s - \frac{1}{q} < 1$. This proves (3.2.10).

To conclude that $(-\Delta)^s u \in L^p(\mathbb{R})$ it remains to show that $(-\Delta)^s u$ does not concentrate on $\partial I = \{-1, 1\}$, in the sense that the distribution defined by

$$\begin{aligned} \langle T, \varphi \rangle &:= \int_{\mathbb{R}} u(-\Delta)^s \varphi dx - \int_I f \varphi dx - C_s \int_{I^c} \int_{\mathbb{R}} \frac{-u(y)}{|x-y|^{1+2s}} dy\, \varphi(x) dx \\ &=: \langle T_1, \varphi \rangle - \langle T_2, \varphi \rangle - \langle T_3, \varphi \rangle \quad \text{for } \varphi \in C_c^\infty(\mathbb{R}) \end{aligned}$$

vanishes. Notice that $\langle T, \varphi \rangle = 0$ for $\varphi \in C_c^\infty(\mathbb{R} \setminus \partial I)$, since $T_1 = (-\Delta)^s u$, while

$$\langle T_2, \varphi \rangle = \langle (-\Delta)^s u, \varphi \rangle, \quad \langle T_3, \varphi \rangle = 0 \quad \text{for } \varphi \in C_c^\infty(I)$$

by (3.2.7), and

$$\langle T_2, \varphi \rangle = 0, \quad \langle T_3, \varphi \rangle = \langle (-\Delta)^s u, \varphi \rangle \quad \text{for } \varphi \in C_c^\infty(I^c)$$

by Proposition 3.6, and for $\varphi \in C_c^\infty(\mathbb{R} \setminus \partial I)$ we can split $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in C_c^\infty(I)$ and $\varphi_2 \in C_c^\infty(I^c)$. In particular $\mathrm{supp}(T) \subset \partial I$.

It is easy to see that T_1 is a distribution of order at most 1, i.e.

$$\left| \int_{\mathbb{R}} u(-\Delta)^s \varphi dx \right| \le C \|\varphi\|_{C^1(\mathbb{R})}, \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R})$$

(use for instance Proposition 3.6), and that T_2 and T_3 are distributions of order zero, i.e.

$$|\langle T_i, \varphi \rangle| \le C \|\varphi\|_{L^\infty(\mathbb{R})} \quad \text{for } i = 2, 3.$$

Since $\mathrm{supp}(T) \subset \partial I$ it follows from Schwartz's theorem (see e.g. [15, Sec. 6.1.5]) that

$$T = \alpha\delta_{-1} + \beta\delta_1 + \tilde{\alpha}D\delta_{-1} + \tilde{\beta}D\delta_1, \quad \text{for some } \alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \mathbb{R},$$

where $\langle D\delta_{x_0}, \varphi\rangle := -\langle \delta_{x_0}, \varphi'\rangle = -\varphi'(x_0)$ for $\varphi \in C_c^\infty(\mathbb{R})$.

In order to show that $\tilde{\alpha} = 0$, take $\varphi \in C_c^\infty(\mathbb{R})$ with

$$\mathrm{supp}(\varphi) \subset (-1,1), \quad \varphi'(0) = 1, \quad \varphi(0) = 0,$$

and rescale it by setting for $\varphi_\lambda(-1+x) = \lambda\varphi(\lambda^{-1}x)$ for $\lambda > 0$. Since T_2 and T_3 have order 0 it follows

$$|\langle T_i, \varphi_\lambda\rangle| \le C\lambda \to 0 \text{ as } \lambda \to 0, \quad \text{for } i = 2, 3.$$

As for T_1, using Proposition 3.6 we get

$$\begin{aligned}
\frac{\langle T_1, \varphi_\lambda\rangle}{C_s} &= \int_{(B_{2\lambda}(-1))^c} u(x) \int_{B_\lambda(-1)} \frac{-\varphi_\lambda(y)}{|x-y|^{1+2s}} dy dx \\
&\quad + \int_{B_{2\lambda}(-1)} u(x) \int_{(B_{4\lambda}(-1))^c} \frac{\varphi_\lambda(x)}{|x-y|^{1+2s}} dy dx \\
&\quad + \int_{B_{2\lambda}(-1)} u(x) \int_{B_{4\lambda}(-1)} \frac{\varphi_\lambda(x) - \varphi_\lambda(y)}{|x-y|^{1+2s}} dy dx \\
&=: (I) + (II) + (III).
\end{aligned}$$

Since $\|\varphi_\lambda\|_{L^\infty(\mathbb{R})} = C_\varphi\lambda$ and $u \in L^\infty(\mathbb{R})$, one easily bounds $|(I)| + |(II)| \to 0$ as $\lambda \to 0$, and using that $\sup_{\mathbb{R}} |\varphi_\lambda'| = \sup_{\mathbb{R}} |\varphi'|$ we get

$$|(III)| \le \int_{B_{2\lambda}(-1)} |u(x)| \int_{B_{4\lambda}(-1)} \frac{\sup_{\mathbb{R}} |\varphi'|}{|x-y|^{2s}} dy dx \le C\lambda^{1-2s} \int_{B_{2\lambda}(-1)} |u(x)| dx \to 0 \quad \text{as } \lambda \to 0.$$

Since for $\lambda \in (0,1)$ we have $\langle T, \varphi\rangle = -\tilde{\alpha}$, by letting $\lambda \to 0$ it follows that $\tilde{\alpha} = 0$. Similarly one can prove that $\tilde{\beta} = 0$.

We now claim that $\alpha, \beta = 0$. Considering

$$\tilde{u}(x) := u(x) - \alpha F_s(x+1) - \beta F_s(x-1),$$

and recalling that $(-\Delta)^s F_s = \delta_0$, one obtains that

$$(-\Delta)^s \tilde{u} = T_1 - \alpha\delta_{-1} - \beta\delta_1 = T_2 + T_3 \in L^2(\mathbb{R}),$$

hence with Proposition 3.1

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{1+2s}} dx dy = [\tilde{u}]^2_{W^{2s,2}(\mathbb{R})} = C\|(-\Delta)^s \tilde{u}\|^2_{L^2(\mathbb{R})} < \infty,$$

and this gives a contradiction if $\alpha \neq 0$ or $\beta \neq 0$ since the integral on the left-hand side does not converge in these cases.

Then $T = 0$, i.e. $(-\Delta)^s u =: T_1 = T_2 + T_3$ and from (3.2.7), (3.2.9) and (3.2.10) we conclude that $(-\Delta)^s u \in L^p(\mathbb{R})$, hence $u \in \tilde{H}^{2s,p}(I)$, as wished.

Step 3: Conclusion. Recalling that $(-\Delta)^s u = f$ in I, from (3.2.4) we have for $x \in I$

$$\begin{aligned} u(x) &= \int_I G_s(x,y) f(y) dy \\ &= \frac{1}{2\tau(2\alpha_p)^{\frac{p-1}{p}}} \int_{r<|y|<\frac{1}{2}} \frac{1}{|x-y|^{1-\frac{1}{p}}|y|^{\frac{1}{p}}} dy - \int_{r<|y|<\frac{1}{2}} H_s(x,y) f(y) dy \qquad (3.2.11) \\ &=: u_1(x) + u_2(x), \end{aligned}$$

where $H_s(x,y)$ is as in Lemma 3.5.

We now want a lower bound for u in the interval $[-r,r]$. We fix $0 < x \leq r$ and estimate

$$\begin{aligned} u_1(x) &= \frac{1}{2\tau(2\alpha_p)^{\frac{p-1}{p}}} \left(\int_r^{\frac{1}{2}} \frac{dy}{(y-x)^{1-\frac{1}{p}} y^{\frac{1}{p}}} + \int_{-\frac{1}{2}}^{-r} \frac{dy}{|y-x|^{1-\frac{1}{p}}|y|^{\frac{1}{p}}} \right) \\ &\geq \frac{1}{2\tau(2\alpha_p)^{\frac{p-1}{p}}} \left(\int_r^{\frac{1}{2}} \frac{dy}{y} + \int_r^{\frac{1}{2}} \frac{dy}{y+x} \right) \\ &= \frac{1}{2\tau(2\alpha_p)^{\frac{p-1}{p}}} \left(2\tau + \log\left(\frac{1+2x}{1+\frac{x}{r}} \right) \right) \\ &= \frac{1}{(2\alpha_p)^{\frac{p-1}{p}}} + O(\tau^{-1}). \end{aligned}$$

Since H_s is bounded on $[-r,r] \times [-\frac{1}{2}, \frac{1}{2}]$, we have

$$|u_2(x)| \leq C \int_r^{\frac{1}{2}} f(y) dy \leq C\tau^{-1} \int_0^{\frac{1}{2}} |y|^{-\frac{1}{p}} dy = O(\tau^{-1}), \quad x \in [-r,r].$$

Then

$$u = u_\tau \geq \frac{1}{(2\alpha_p)^{\frac{p-1}{p}}} + O(\tau^{-1}) \quad \text{on } [-r,r],$$

as $\tau \to \infty$. We now set

$$w_\tau := (2\tau)^{\frac{p-1}{p}} u_\tau \in \tilde{H}^{\frac{1}{p},p}(I),$$

so that $\|(-\Delta)^s w_\tau\|_{L^p(I)} = 1$, we compute

$$\int_I e^{\alpha_p |w_\tau|^{p'}} dx \geq \int_{-r}^{r} e^{\tau + O(1)} dx \geq \frac{2re^\tau}{C} = \frac{1}{C},$$

and using that $\inf_{[-r,r]} w_\tau \to \infty$ as $\tau \to \infty$, we conclude

$$\lim_{\tau \to \infty} \int_I h(w_\tau) e^{\alpha_p |w_\tau|^{p'}} dx = \infty,$$

whenever h satisfies $\lim_{t\to\infty} h(t) = \infty$. □

A few consequences of Theorem 1.2.2

Lemma 3.6. *Let $u \in H$. Then $u^q e^{pu^2} \in L^1(I)$ for every $p, q > 0$.*

Proof. Since $|u|^q \le C(q)e^{|u|^2}$, it is enough to prove the case $q = 0$. Given $\varepsilon > 0$ (to be fixed later), by Lemma 3.1 there exists $v \in C_c^\infty(I)$ such that

$$\|v - u\|_H^2 < \varepsilon.$$

Using

$$u^2 \le (v-u)^2 + 2vu$$

we bound

$$e^{pu^2} \le e^{p(v-u)^2} e^{2pvu}. \tag{3.2.12}$$

Using the inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ we have

$$e^{2puv} \le e^{\frac{1}{\varepsilon}p^2\|u\|_H^2 v^2} e^{\varepsilon\left(\frac{u}{\|u\|_H}\right)^2},$$

and for ε small enough the right-hand side is bounded in $L^2(I)$ thanks to Theorem 1.2.2. Still by Theorem 1.2.2 we have $e^{p(v-u)^2} \in L^2(I)$ if $\varepsilon > 0$ is small enough, hence going back to (3.2.12) and using that $v \in L^\infty(I)$ is now fixed, we conclude with Hölder's inequality that $e^{pu^2} \in L^1(I)$. □

Lemma 3.7. *For any $q, p \in (1, +\infty)$ the functional*

$$E_{q,p} : H \to \mathbb{R}, \quad E_{q,p}(u) := \int_I |u|^q e^{pu^2} dx$$

is continuous.

Proof. Consider a sequence $u_k \to u$ in H. By Lemma 3.6 (up to changing the exponents) we have that the sequence $f_k := |u_k|^q e^{pu_k^2}$ is bounded in $L^2(I)$. Indeed, it is enough to write $u_k = (u_k - u) + u$ and use the same estimates as in (3.2.12) with u instead of v and u_k instead of u. We now claim that $f_k \to f$ in $L^1(I)$. Indeed up to a subsequence $u_k \to u$ a.e., hence $f_k \to f := |u|^q e^{pu^2}$ a.e.

Then considering that since f_k is bounded in $L^2(I)$ we have

$$\int_{\{f_k > L\}} f_k \, dx \le \frac{1}{L} \int_{\{f_k > L\}} f_k^2 \, dx \le \frac{C}{L} \to 0 \quad \text{as } L \to +\infty,$$

the claim follows at once from Lemma 3.3. □

Lemma 3.8. *The functional $J : H \to \mathbb{R}$ defined in (1.4.5) is of class C^∞.*

Proof. This follows easily from Lemma 3.7, since the first term on the right-hand side of (1.4.5) is simply $\frac{1}{2}\|u\|_H^2$, and the derivatives of the second term are continuous thanks to Lemma 3.7. The details, at least to prove that $J \in C^1(H)$, are essentially as in the proof of Lemma 2.1 of [91]. The higher-order differentials are handled in the same way since they have a similar form, with the non-linear term $e^{\frac{1}{2}u^2}$ just multiplied by polynomial terms. □

The following lemma is a fractional analog of a well-known result of P-L. Lions [63].

Lemma 3.9. *Consider a sequence $(u_k) \subset H$ with $\|u_k\|_H = 1$ and $u_k \rightharpoonup u$ weakly in H, but not strongly (so that $\|u\|_H < 1$). Then if $u \not\equiv 0$, $e^{\pi u_k^2}$ is bounded in L^p for $1 \le p < \tilde{p} := (1 - \|u\|_H^2)^{-1}$.*

Proof. We split

$$u_k^2 = u^2 - 2u(u - u_k) + (u - u_k)^2.$$

Then $v_k := e^{\pi u_k^2} = v v_{k,1} v_{k,2}$, where $v = e^{\pi |u|^2} \in L^p(I)$ for all $p \ge 1$ by Lemma 3.6, $v_{k,1} = e^{-2\pi u(u-u_k)}$ and $v_{k,2} = e^{\pi (u-u_k)^2}$.
Notice now that from

$$-2p\pi u(u - u_k) \le \pi \left(\frac{p^2}{\varepsilon^2} u^2 + \varepsilon^2 (u - u_k)^2 \right),$$

we get from Lemma 3.6 and Theorem 1.2.2 that $v_{k,1} \in L^q(I)$ for all $q \ge 1$ if $\varepsilon > 0$ is small enough (depending on q). But again from Theorem 1.2.2 $v_{2,k}$ is bounded in $L^p(I)$ for all $p < \tilde{p}$ since

$$\|u_k - u\|_H^2 = 1 - 2\langle u_k, u \rangle + \|u\|_H^2 \to 1 - \|u\|_H^2.$$

Therefore by Hölder's inequality we have that v_k is bounded in $L^p(I)$ for all $p < \tilde{p}$. □

Proof of Theorem 1.7

For a measurable function u we set $|u|^* : \mathbb{R} \to \mathbb{R}_+$ to be its non-increasing symmetric rearrangement, whose definition we shall now recall. For a measurable set $A \subset \mathbb{R}$, we define

$$A^* = (-|A|/2, |A|/2).$$

The set A^* is symmetric (with respect to 0) and $|A^*| = |A|$. For a non-negative measurable function f, such that

$$|\{x \in \mathbb{R} : f(x) > t\}| < \infty \quad \text{for every } t > 0,$$

we define the symmetric non-increasing rearrangement of f by

$$f^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R} : f(y) > t\}^*}(x) dt.$$

Notice that f^* is even, i.e. $f^*(x) = f^*(-x)$ and non-increasing (on $[0,\infty)$).

We will state here the two properties that we shall use in the proof of Theorem 1.7. The following one is proven e.g. in [60, Section 3.3].

Proposition 3.7. *Given a measurable function $F : \mathbb{R} \to \mathbb{R}$ and a non-negative non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ it holds*

$$\int_{\mathbb{R}} F(f)dx = \int_{\mathbb{R}} F(f^*)dx.$$

The following Pólya-Szegő type inequality can be found e.g. in [52] (Inequality (3.6)) or [82].

Theorem 3.8. *Let $u \in H^{s,2}(\mathbb{R})$ for $0 < s < 1$. Then*

$$\int_{\mathbb{R}} |(-\Delta)^s |u|^*|^2 dx \le \int_{\mathbb{R}} |(-\Delta)^s u|^2 dx.$$

Now given $u \in H^{\frac{1}{2},2}(\mathbb{R})$, from Proposition 3.7 we get

$$\int_{\mathbb{R}} \left(e^{\pi u^2} - 1\right) dx = \int_{\mathbb{R}} \left(e^{\pi(|u|^*)^2} - 1\right) dx, \quad \||u|^*\|_{L^2} = \|u\|_{L^2},$$

and according to Theorem 3.13

$$\||u|^*\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} = \||u|^*\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} |u|^*|^2 dx \le \|u\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx = \|u\|^2_{H^{\frac{1}{2},2}(\mathbb{R})}.$$

Therefore in the rest of the proof of (1.3.12) we may assume that $u \in H^{\frac{1}{2},2}(\mathbb{R})$ is even, non-increasing on $[0,\infty)$, and $\|u\|_{H^{\frac{1}{2},2}(\mathbb{R})} \le 1$.

We write

$$\int_{\mathbb{R}} \left(e^{\pi u^2} - 1\right) dx = \int_{\mathbb{R}\setminus I} \left(e^{\pi u^2} - 1\right) dx + \int_I \left(e^{\pi u^2} - 1\right) dx =: (I) + (II),$$

where $I = (-1/2, 1/2)$. We start by bounding (I). By monotone convergence

$$(I) = \sum_{k=1}^{\infty} \int_{I^c} \pi^k \frac{u^{2k}}{k!} dx.$$

Since u is even and non-increasing, for $x \neq 0$ we have

$$u^2(x) \le \frac{1}{2|x|} \int_{-|x|}^{|x|} u^2(y)dy \le \frac{\|u\|^2_{L^2}}{2|x|}, \tag{3.2.13}$$

hence for $k \geq 2$ we bound

$$\int_{I^c} u^{2k} dx \leq 2^{1-k} \|u\|_{L^2(\mathbb{R})}^{2k} \int_{\frac{1}{2}}^{\infty} \frac{1}{x^k} dx = \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{(k-1)}.$$

It follows that

$$\sum_{k=2}^{\infty} \int_{I^c} \pi^k \frac{u^{2k}}{k!} dx \leq \sum_{k=2}^{\infty} \frac{(\pi \|u\|_{L^2}^2)^k}{k!(k-1)}.$$

Thus, since $\|u\|_{L^2(\mathbb{R})} \leq 1$ we estimate

$$(I) \leq \pi \|u\|_{L^2(\mathbb{R})}^2 \left(1 + \sum_{k=1}^{\infty} \frac{\left(\pi \|u\|_{L^2(\mathbb{R})}^2\right)^k}{(k+1)!k} \right) \leq C.$$

We shall now bound (II). We define the function $v : \mathbb{R} \to \mathbb{R}$ as follows

$$v(x) = \begin{cases} u(x) - u(\frac{1}{2}) & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Then with (3.2.13) and the estimate $2a \leq a^2 + 1$, we find

$$\begin{aligned} u^2 &\leq v^2 + 2vu(\tfrac{1}{2}) + u(\tfrac{1}{2})^2 \\ &\leq v^2 + 2v\|u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq v^2 + v^2\|u\|_{L^2(\mathbb{R})}^2 + 1 + \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq v^2 \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) + 2. \end{aligned} \tag{3.2.14}$$

Now, recalling that u is decreasing we have for $x \in I = [-\frac{1}{2}, \frac{1}{2}]$

$$\begin{aligned} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy &= \int_I \frac{(u(x) - u(y))^2}{(x-y)^2} dy + \int_{I^c} \frac{(u(x) - u(\frac{1}{2}))^2}{(x-y)^2} dy \\ &\leq \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy. \end{aligned}$$

Notice that the last integral converges for a.e. $x \in I$ thanks to Proposition 3.1 and Fubini's theorem. Similarly for $x \in I^c$

$$\begin{aligned} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy &= \int_I \frac{(u(\frac{1}{2}) - u(y))^2}{(x-y)^2} dy \\ &\leq \int_I \frac{(u(x) - u(y))^2}{(x-y)^2} dy \\ &\leq \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy. \end{aligned}$$

Integrating with respect to x we obtain

$$\begin{aligned}\|(-\Delta)^{\frac{1}{4}}v\|^2_{L^2(\mathbb{R})} &= \frac{1}{C_s^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v(x)-v(y))^2}{(x-y)^2}dydx\\ &\le \frac{1}{C_s^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx\\ &= \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R})},\end{aligned}$$

where C_s is as in Proposition 3.1 below. Thus, since $\|u\|_H \le 1$,

$$\|(-\Delta)^{\frac{1}{4}}v\|^2_{L^2(\mathbb{R})} \le \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R})} \le 1 - \|u\|^2_{L^2(\mathbb{R})}.$$

Therefore, if we set $w = v\sqrt{1+\|u\|^2_{L^2(\mathbb{R})}}$, we have

$$\|(-\Delta)^{\frac{1}{4}}w\|^2_{L^2(\mathbb{R})} \le \left(1+\|u\|^2_{L^2(\mathbb{R})}\right)\left(1-\|u\|^2_{L^2(\mathbb{R})}\right) \le 1,$$

hence, using the Moser-Trudinger inequality on the interval $I = (-1/2, 1/2)$ (Theorem 1.2.2), one has

$$\int_I e^{\pi w^2}dx < C,$$

and using (3.2.14)

$$\int_I e^{\pi u^2}dx \le e^{2\pi}\int_I e^{\pi w^2}dx \le C,$$

which completes the proof of (1.3.12).

It remains to prove (1.3.14). Given $\tau > 2$ consider the function

$$f = f_\tau := \frac{1}{2\tau\sqrt{|x|}}\chi_{\{x\in\mathbb{R}: r<|x|<\delta\}}, \quad \delta := \frac{1}{\tau}, \quad r := \frac{1}{\tau e^\tau}.$$

Notice that $\|f\|^2_{L^2(\mathbb{R})} = (2\tau)^{-1}$. Fix a smooth even function $\psi : \mathbb{R} \to [0,1]$ with $\psi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathrm{supp}(\psi) \subset (-1,1)$. For $x \in \mathbb{R}$ we set

$$u(x) = \psi(x)(F_{\frac{1}{4}} * f)(x),$$

where $F_{\frac{1}{4}}(x) = (2\pi|x|)^{-\frac{1}{2}}$ is as in Lemma 3.4. Clearly $u \equiv 0$ in $\mathbb{R}\setminus I$, and u is non-negative and even everywhere.

In the rest of the proof $s = \frac{1}{4}$. Notice that $(-\Delta)^s(F_s * f) = f$. This follows easily from Lemma 3.4 and the properties of the Fourier transform, see e.g. [60, Corollary 5.10]. Then we compute

$$(-\Delta)^s u = f + (-\Delta)^s[(\psi-1)(F_s * f)] =: f + v, \tag{3.2.15}$$

and set $g(x,y) = (\psi - 1)(x)F_s(x-y)$. Notice that g is smooth in $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$. We write

$$\begin{aligned} v(x) &= (-\Delta)^s \int_{\mathbb{R}} g(x,y)f(y)dy \\ &= \int_{\{r<|y|<\delta\}} (-\Delta_x)^s g(x,y) f(y)dy, \end{aligned}$$

where we used Proposition 3.6 and Fubini's theorem. With Jensen's inequality

$$\begin{aligned} \|v\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_{\{r<|y|<\delta\}} (-\Delta_x)^s g(x,y)f(y)dy \right|^2 dx \\ &\le 2(\delta - r) \int_{\{r<|y|<\delta\}} f(y)^2 \int_{\mathbb{R}} \left|(-\Delta_x)^s g(x,y)\right|^2 dxdy \\ &\le 2\delta \|f\|^2_{L^2(\mathbb{R})} \sup_{|y|\in[r,\delta]} \int_{\mathbb{R}} \left|(-\Delta_x)^s g(x,y)\right|^2 dx \\ &\le C(\delta\tau^{-1}) = O(\tau^{-2}), \end{aligned} \tag{3.2.16}$$

where we used that

$$\sup_{|y|\in[r,\delta]} \int_{\mathbb{R}} \left|(-\Delta_x)^s g(x,y)\right|^2 dx < \infty.$$

This in turn can be seen noticing that $(-\Delta_x)^s g(x,y)$ is smooth, hence bounded on $[-R,R] \times [r,\delta]$ for every R, and for $|x|$ large and $r \le |y| \le \delta$, using Proposition 3.6

$$\begin{aligned} (-\Delta_x)^s g(x,y) &= C_s \int_{\mathbb{R}} \frac{-F_s(x-y) - (\psi(z)-1)F_s(z-y)}{|z-x|^{1+2s}} dz \\ &= C_s \int_{-1}^{1} \frac{-\psi(z)F_s(z-y)}{|z-x|^{1+2s}} dz - (-\Delta)^s F_s(x-y) \\ &= O(|x|^{-1-2s}) \quad \text{uniformly for } |y| \le \frac{1}{2}, \end{aligned}$$

where we also used that $(-\Delta)^s F_s = 0$ away from the origin, see Lemma 3.4. Actually, with the same estimates we get

$$\begin{aligned} \int_{-\delta}^{\delta} |v|^2 dx &\le 2(\delta - r)\|f\|^2_{L^2(\mathbb{R})} \int_{-\delta}^{\delta} \sup_{(x,y)\in[-\delta,\delta]^2} \left|(-\Delta_x)^s g(x,y)\right|^2 dx \\ &\le C\delta^2 \|f\|^2_{L^2(\mathbb{R})} = O(\tau^{-3}). \end{aligned}$$

Therefore, using Hölder's inequality and that $\mathrm{supp}(f) \subset [-\delta,\delta]$ we get

$$\|(-\Delta)^s u\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2} + \|v\|^2_{L^2} + 2\int_{-\delta}^{\delta} fv dx = \frac{1}{2\tau} + O(\tau^{-2}), \quad \text{as } \tau \to \infty. \tag{3.2.17}$$

We now estimate u. For $0 < x < r$, with the change of variable $\tilde{y} = \sqrt{\frac{y}{x}}$ we have

$$
\begin{aligned}
u(x) &= \frac{1}{2\tau\sqrt{2\pi}} \int_r^\delta \left(\frac{1}{\sqrt{(y-x)y}} + \frac{1}{\sqrt{(y+x)y}} \right) dy \\
&= \frac{1}{\tau\sqrt{2\pi}} \int_{\sqrt{\frac{\tau}{x}}}^{\sqrt{\frac{\delta}{x}}} \left(\frac{1}{\sqrt{\tilde{y}^2-1}} + \frac{1}{\sqrt{\tilde{y}^2+1}} \right) d\tilde{y} \\
&= \frac{1}{\tau\sqrt{2\pi}} \left(\log(\sqrt{\tilde{y}^2-1}+\tilde{y}) \Big|_{\sqrt{\frac{\tau}{x}}}^{\sqrt{\frac{\delta}{x}}} + \log(\sqrt{\tilde{y}^2+1}+\tilde{y}) \Big|_{\sqrt{\frac{\tau}{x}}}^{\sqrt{\frac{\delta}{x}}} \right) \\
&= \frac{1}{\sqrt{2\pi}} + O(\tau^{-1}),
\end{aligned}
$$

with $|\tau O(\tau^{-1})| \leq C$ as $\tau \to \infty$ with C independent of $x \in [0, r]$.

Similarly for $r < x < \delta$ we write

$$
\begin{aligned}
u(x) &\leq \frac{1}{\tau\sqrt{2\pi}} \left[\int_r^x \frac{dy}{\sqrt{(x-y)y}} + \int_x^\delta \frac{dy}{\sqrt{(x-y)y}} \right] \\
&= \frac{2}{\tau\sqrt{2\pi}} \left[\int_{\sqrt{\frac{\tau}{x}}}^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} + \log(\sqrt{\tilde{y}^2-1}+\tilde{y}) \Big|_1^{\sqrt{\frac{\delta}{x}}} \right] \\
&= \frac{1}{\tau\sqrt{2\pi}} \left[\log\left(\frac{\delta}{x}\right) + O(1) \right],
\end{aligned}
$$

since $\int_0^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} < \infty$. Here $|O(1)| \leq C$ as $\tau \to \infty$ with C independent of $x \in (r, \delta)$.

When $\delta < x < 1$ similar to the previous computation, and recalling that $0 \leq \psi \leq 1$,

$$
u(x) \leq \frac{1}{\tau\sqrt{2\pi}} \int_r^\delta \frac{dy}{\sqrt{(x-y)y}} = \frac{2}{\tau\sqrt{2\pi}} \int_{\sqrt{\frac{\tau}{x}}}^{\sqrt{\frac{\delta}{x}}} \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} \leq \frac{2}{\tau\sqrt{2\pi}} \int_0^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} = O(\tau^{-1}),
$$

with $|\tau O(\tau^{-1})| \leq C$ as $\tau \to \infty$ with C independent of $x \in (0, 1)$. Thus

$$
\begin{cases}
u(x) = \frac{1}{\sqrt{2\pi}} + O(\tau^{-1}) & \text{for } 0 < x < r \\
u(x) \leq \frac{2}{\tau\sqrt{2\pi}} \log\left(\frac{\delta}{x}\right) + O(\tau^{-1}) & \text{for } r < x < \delta \\
u(x) = O(\tau^{-1}) & \text{for } \delta < x < 1.
\end{cases}
\tag{3.2.18}
$$

Of course the same bounds hold for $x < 0$ since u is even.

We now want to estimate $\|u\|_{L^2(\mathbb{R})}^2$. We have

$$
\int_0^r u^2 dx = r\left(\frac{1}{2\pi} + O(\tau^{-1})\right) = O(\tau^{-2}).
$$

For $x \in [r, \delta]$ we have from (3.2.18)

$$
u(x)^2 \leq \frac{C}{\tau^2}\left(\log^2\left(\frac{\delta}{x}\right) + \log\left(\frac{\delta}{x}\right) + 1\right) \leq \frac{2C}{\tau^2}\left(\log^2\left(\frac{\delta}{x}\right) + 1\right).
$$

Then, since

$$\int_r^\delta \log^2\left(\frac{\delta}{x}\right) dx = x\left(\log^2\left(\frac{\delta}{x}\right) + 2\log\left(\frac{\delta}{x}\right) + 2\right)\Big|_r^\delta \le 2\delta = O(\tau^{-1}),$$

we bound

$$\int_r^\delta u^2 dx = O(\tau^{-3}).$$

Finally, still using (3.2.18),

$$\int_\delta^1 u^2 dx = O(\tau^{-2}).$$

Also considering (3.2.17), we conclude

$$\|u\|^2_{L^2(\mathbb{R})} = 2\|u\|^2_{L^2([0,1])} = O(\tau^{-2}), \quad \|u\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} = \frac{1}{2\tau} + O(\tau^{-2}). \tag{3.2.19}$$

Setting $w_\tau := u\|u\|^{-1}_{H^{\frac{1}{2},2}(\mathbb{R})}$, and using (3.2.18) and (3.2.19), we conclude

$$\int_{-r}^r |w_\tau|^2 \left(e^{\pi w_\tau^2} - 1\right) dx \ge \int_{-r}^r \left(\frac{\tau + O(1)}{\pi}\right)\left(e^{\tau + O(1)} - 1\right) dx \ge \frac{r\tau e^\tau}{C} = \frac{1}{C},$$

therefore

$$\lim_{\tau\to\infty} \int_{\mathbb{R}} h(w_\tau)\left(e^{\pi w_\tau^2} - 1\right) dx \ge \int_{-r}^r h(w_\tau)\left(e^{\pi w_\tau^2} - 1\right) dx \to \infty$$

as $\tau \to \infty$, for any h satisfying (1.3.13). □

A fractional Moser-Trudinger inequality in Sobolev-Slobodeckij spaces

We start by proving the validity of the Moser-Trudinger inequality (1.3.17). The result for $n \ge 2$ is proved in [81] and the proof in the one dimensional case, which we report here for the sake of completeness, follows by a mild adaptation of the techniques in [81].

Proof of Theorem 1.8

Thanks to [83, Theorem 9.1], using Sobolev embeddings and Hölder's inequality we have that there exists a constant $C > 0$ independent of u such that for any $u \in \tilde{W}_0^{s,p}(I)$

$$||u||_{L^q(\mathbb{R})} \le C[u]_{W^{s,p}(\mathbb{R})} q^{1-s} \tag{3.2.20}$$

for any $q > 1$. For $[u]_{W^{s,p}(\mathbb{R})} \le 1$ we write

$$\int_I e^{\beta|u|^{\frac{1}{1-s}}} dx = \sum_{k=0}^\infty \int_I \frac{\beta^k}{k!}|u|^{\frac{k}{1-s}} dx \le \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{C}{1-s}\beta k\right)^k, \tag{3.2.21}$$

where in the last inequality we used (3.2.20). Thanks to Stirling's formula

$$k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k\left(1+O(\frac{1}{k})\right) \tag{3.2.22}$$

the series in (3.2.21) converges for small β and we recover a bound (uniform w.r.t. u) for

$$\int_I e^{\beta|u|^{\frac{1}{1-s}}}\,dx,$$

yielding (1.3.17).

As a direct consequence of (1.3.17), using the density of $C_c^\infty(I)$ in $\tilde{W}_0^{s,p}(I)$, we have the following corollary (see [81, Proposition 3.2]).

Corollary 3.9. *If $u \in \tilde{W}_0^{s,p}(I)$, for every $\beta > 0$ it holds*

$$\int_I e^{\beta|u|^{\frac{1}{1-s}}}\,dx < \infty.$$

We now give a useful result on the Gagliardo seminorm of radially symmetric functions (see [81, Proposition 4.3]), which will turn out to be useful later on.

Proposition 3.10. *Let $u \in W^{s,p}(\mathbb{R})$ be radially symmetric and let $sp = 1$. Then*

$$[u]_{W^{s,p}(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy = 4\int_0^{+\infty}\int_0^{+\infty} |u(x)-u(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy \tag{3.2.23}$$

Proof. The proof will follow from a direct computation. We split

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy + \int_{-\infty}^0\int_{-\infty}^0 \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy \\
&+ \int_0^{+\infty}\int_{-\infty}^0 \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy + \int_{-\infty}^0\int_0^{+\infty} \frac{|u(x)-u(y)|^p}{|x-y|^2}\,dx\,dy.
\end{aligned}$$

Using a straightforward change of variable and the symmetry of u, we obtain the claim. □

To give an upper bound for the optimal exponent $\bar{\beta}$ such that the supremum in (1.3.17) is finite for $\beta \in [0, \bar{\beta})$, we define the family of functions

$$u_\varepsilon(x) := \begin{cases} |\log\varepsilon|^{1-s} & \text{if } |x| \le \varepsilon \\ \frac{|\log|x||}{|\log\varepsilon|^s} & \text{if } \varepsilon < |x| < 1 \\ 0 & \text{if } |x| \ge 1. \end{cases} \tag{3.2.24}$$

Notice that the restrictions of u_ε to I belong to $\tilde{W}_0^{s,p}(I)$.

Proposition 3.11. *Let $sp = 1$ and $(u_\varepsilon) \subset \tilde{W}_0^{s,p}(I)$ be the family of functions defined in* (3.2.24)*. Then*

$$\lim_{\varepsilon\to 0}[u_\varepsilon]^p_{W^{s,p}(\mathbb{R})} = \gamma_s := 8\,\Gamma(p+1)\sum_{k=0}^{\infty}\frac{1}{(1+2k)^p}. \tag{3.2.25}$$

Proof. We will follow the proof in [81]. Define

$$I(\varepsilon) := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^2}\,dx\,dy. \tag{3.2.26}$$

Using Proposition 3.10 and (3.2.24) we see that $I(\varepsilon)$ can be decomposed as

$$I(\varepsilon) = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon),$$

where

$$I_1(\varepsilon) = \frac{8}{|\log\varepsilon|}\int_\varepsilon^1\int_0^\varepsilon |\log x - \log\varepsilon|^p \frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy,$$

$$I_2(\varepsilon) = \frac{4}{|\log\varepsilon|}\int_\varepsilon^1\int_\varepsilon^1 |\log x - \log y|^p \frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy,$$

$$I_3(\varepsilon) = 8|\log\varepsilon|^{p-1}\int_1^{+\infty}\int_0^\varepsilon \frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy,$$

$$I_4(\varepsilon) = \frac{8}{|\log\varepsilon|}\int_\varepsilon^1\int_1^{+\infty} |\log x|^p \frac{x^2+y^2}{(x^2-y^2)^2}\,dx\,dy.$$

With an integration by parts, it is easy to check that $\lim_{\varepsilon\to 0} I_i(\varepsilon) = 0$ for $i = 1, 3, 4$. As for $I_2(\varepsilon)$, integrating by parts after a change of variables we have

$$
\begin{aligned}
I_2(\varepsilon) &= \frac{4}{|\log\varepsilon|}\left\{\log y\left(\int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx\right)\right\}\Bigg|_{y=\varepsilon}^{y=1} \\
&+ \frac{4}{|\log\varepsilon|}\int_\varepsilon^1 \frac{\log y}{y^2}|\log\frac{1}{y}|^p \frac{\frac{1}{y^2}+1}{\left(\frac{1}{y^2}-1\right)^2}\,dy \\
&- \frac{4\varepsilon}{|\log\varepsilon|}\int_\varepsilon^1 \frac{\log y}{y^2}|\log\frac{\varepsilon}{y}|^p \frac{\left(\frac{\varepsilon}{y}\right)^2+1}{\left(\left(\frac{\varepsilon}{y}\right)^2-1\right)^2}\,dy.
\end{aligned}
$$

A direct computation for the first term gives

$$
\begin{aligned}
&\frac{4}{|\log\varepsilon|}\left\{\log y\left(\int_{\frac{\varepsilon}{y}}^{\frac{1}{y}} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx\right)\right\}\Bigg|_{y=\varepsilon}^{y=1} \\
&= 4\int_1^{\frac{1}{\varepsilon}} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx,
\end{aligned}
$$

which converges to

$$
4\int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx,
$$

as $\varepsilon \to 0$. Moreover, since

$$
\int_0^1 \frac{\log y}{y^2}|\log\frac{1}{y}|^p \frac{\frac{1}{y^2}+1}{\left(\frac{1}{y^2}-1\right)^2}\,dy < +\infty
$$

the second term in the sum converges to 0 as $\varepsilon \to 0$.

After setting $\frac{\varepsilon}{y} = x$, for the last term in the sum we have

$$
\begin{aligned}
&- \frac{4\varepsilon}{|\log\varepsilon|}\int_\varepsilon^1 \frac{\log y}{y^2}|\log\frac{\varepsilon}{y}|^p \frac{\left(\frac{\varepsilon}{y}\right)^2+1}{\left(\left(\frac{\varepsilon}{y}\right)^2-1\right)^2}\,dy \\
&= -\frac{4}{|\log\varepsilon|}\int_\varepsilon^1 \log\left(\frac{\varepsilon}{x}\right)|\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx \\
&= 4\int_\varepsilon^1 |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx - \frac{4}{|\log\varepsilon|}\int_\varepsilon^1 |\log x|^{p+1} \frac{x^2+1}{(x^2-1)^2}\,dx
\end{aligned}
$$

which converges to

$$
4\int_0^1 |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx = 4\int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx
$$

as $\varepsilon \to 0$. Summing up, we have

$$\lim_{\varepsilon\to 0}[u_\varepsilon]^p_{W^{s,p}(\mathbb{R})} = \lim_{\varepsilon\to 0} I_2(\varepsilon) = 8\int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx. \tag{3.2.27}$$

Integrating by parts we obtain

$$\int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx = p\int_1^{+\infty} \frac{|\log x|^{p-1}}{x^2-1}\,dx$$
$$= p\int_0^1 \frac{|\log t|^{p-1}}{1-t^2}\,dt,$$

where we set $t = \frac{1}{x}$. Recall now

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}, \qquad \int_0^1 |\log x|^{p-1} x^{2k}\,dx = \frac{\Gamma(p)}{(1+2k)^p}, \tag{3.2.28}$$

where $\Gamma(\cdot)$ is the Euler Gamma function. Thanks to (3.2.28) we write

$$\int_0^1 \frac{|\log t|^{p-1}}{1-t^2}\,dt = \sum_{k=0}^{\infty}\int_0^1 |\log t|^{p-1} t^{2k}\,dt = \Gamma(p)\sum_{k=0}^{\infty}\frac{1}{(1+2k)^p}, \tag{3.2.29}$$

proving (3.2.25). □

The upper bound for the optimal exponent follows directly from Proposition 3.11.

Proposition 3.12. *Let $sp = 1$. There exists $\beta^* := \gamma_s^{\frac{s}{1-s}}$ such that*

$$\sup_{u\in \tilde{W}_0^{s,p}(I), [u]_{W^{s,p}(\mathbb{R})}\le 1} \int_I e^{\beta |u|^{\frac{1}{1-s}}}\,dx = +\infty \qquad \text{for } \beta \in (\beta^*, +\infty).$$

Proof. Let u_ε be the family of functions in $\tilde{W}_0^{s,p}(I)$ defined in (3.2.24). Thanks to Proposition 3.11 we have that $[u_\varepsilon]_{W^{s,p}(\mathbb{R})} \to (\gamma_s)^{\frac{1}{p}}$ as $\varepsilon \to 0$. Fix $\beta > \gamma_s^{\frac{s}{1-s}}$. For ε small enough, there exists $b > 0$ such that $\beta [u_\varepsilon]^{-\frac{1}{1-s}} \ge b > 1$. If we set $v_\varepsilon := \frac{u_\varepsilon}{[u_\varepsilon]}$ we have

$$\int_I e^{\beta |v_\varepsilon|^{\frac{1}{1-s}}}\,dx \ge \int_{-\varepsilon}^{\varepsilon} e^{\beta |v_\varepsilon|^{\frac{1}{1-s}}}\,dx \ge \int_{-\varepsilon}^{\varepsilon} e^{-b\log\varepsilon}\,dx = 2\varepsilon^{1-b} \to +\infty$$

as $\varepsilon \to 0$, since $b > 1$. □

Proof of Theorem 1.9

We shall adapt a technique by Ruf [88] to our setting.

For a measurable function u we set $|u|^* : \mathbb{R} \to \mathbb{R}_+$ to be its non-increasing symmetric rearrangement, as it is defined in Section 3.2.

The following Pólya-Szegő type inequality can be found e.g. in [7, Theorem 9.2].

Theorem 3.13. *Let* $0 < s < 1$ *and* $u \in W^{s,p}(\mathbb{R})$*. Then*

$$[|u|^*]_W^{s,p}(\mathbb{R}) \leq [u]_W^{s,p}(\mathbb{R}).$$

Now given $u \in W^{s,p}(\mathbb{R})$, from Proposition 3.7 we get

$$\int_{\mathbb{R}} \Phi(\beta(|u|)^{\frac{1}{1-s}})\, dx = \int_{\mathbb{R}} \Phi(\beta(|u|^*)^{\frac{1}{1-s}})\, dx, \quad \||u|^*\|_{L^p} = \|u\|_{L^p},$$

and according to Theorem 3.13

$$\||u|^*\|^p_{W^{s,p}(\mathbb{R})} = \||u|^*\|^p_{L^p(\mathbb{R})} + [|u|^*]^p_{W^{s,p}(\mathbb{R})} \leq \|u\|^p_{L^p(\mathbb{R})} + [u]^p_{W^{s,p}(\mathbb{R})} = \|u\|^p_{W^{s,p}(\mathbb{R})}.$$

Therefore in the rest of the proof of (1.3.19) we may assume that $u \in W^{s,p}(\mathbb{R})$ is even, non-increasing on $[0,\infty)$, and $\|u\|_{W^{s,p}(\mathbb{R})} \leq 1$. We will use a technique by Ruf [88] (see also [50]) and write

$$\begin{aligned}
&\int_{\mathbb{R}} \Phi(\beta(|u|)^{\frac{1}{1-s}})\, dx \\
&= \int_{I^c} \Phi(\beta(|u|)^{\frac{1}{1-s}})\, dx + \int_I \Phi(\beta(|u|)^{\frac{1}{1-s}})\, dx \\
&=: (I) + (II),
\end{aligned}$$

where $I = (-r_0, r_0)$, with $r_0 > 0$ to be chosen. Notice that since u is even and non-increasing, for $x \neq 0$ and $p > 1$, we have

$$|u(x)|^p \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} |u(y)|^p\, dy \leq \frac{\|u\|^p_{L^p}}{2|x|}. \tag{3.2.30}$$

We start by bounding (I). We observe that for $r_0 >> 1$, we have $|u(x)| \leq 1$ on I^c and hence

$$|u|^{\frac{p\lceil p-1\rceil}{p-1}} \leq |u|^p \quad \text{on } I^c,$$

since $\frac{p\lceil p-1\rceil}{p-1} \geq p$. For $k > p-1$ we bound

$$\int_{I^c} (|u|^p)^{\frac{k}{p-1}}\, dx \leq \int_{I^c} \left(\frac{\|u\|^p_{L^p}}{2|x|}\right)^{\frac{k}{p-1}} = \frac{\|u\|_{L^p}^{\frac{pk}{p-1}} r_0^{1-\frac{k}{p-1}}(p-1)}{2^{\frac{k}{p-1}}(k+1-p)}.$$

Hence

$$\begin{aligned}
(I) &= \sum_{k=\lceil p-1\rceil}^{\infty} \int_{I^c} \frac{\beta^k}{k!} |u|^{\frac{kp}{p-1}}\, dx \\
&= \frac{\beta^{\lceil p-1\rceil}}{\lceil p-1\rceil !} \int_{I^c} |u|^{\frac{p\lceil p-1\rceil}{p-1}}\, dx + \sum_{k=\lceil p\rceil}^{\infty} \int_{I^c} \beta^k \frac{|u|^{\frac{kp}{p-1}}}{k!} dx \\
&\leq C(\beta,p)\|u\|_{L^p}^p + r_0(p-1) \sum_{k=\lceil p\rceil}^{\infty} \frac{\beta^k \left(\|u\|_{L^p}^p\right)^{\frac{k}{p-1}}}{k!(k+1-p)(2r_0)^{\frac{k}{p-1}}} \\
&\leq C(\beta,p)\|u\|_{L^p}^p + C \sum_{k=\lceil p\rceil}^{\infty} \left(\frac{\beta}{(2r_0)^{p-1}}\right)^k \frac{1}{k!(k+1-p)} \leq C.
\end{aligned}$$

As for (II), define $v \in \tilde{W}_0^{s,p}(I)$ as follows

$$v(x) = \begin{cases} u(x) - u(r_0) & |x| \leq r_0 \\ 0 & |x| > r_0. \end{cases}$$

Let $x \in I$. We compute using the monotonicity of u

$$\int_0^{\infty} |v(x)-v(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\, dy \leq \int_0^{\infty} |u(x)-u(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\, dy. \tag{3.2.31}$$

Let $x \in I^c$. We have

$$\begin{aligned}
&\int_0^{\infty} |v(x)-v(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\, dy \\
&= \int_I |u(r_0)-u(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\, dy \\
&\leq \int_I |u(x)-u(y)|^p \frac{x^2+y^2}{(x^2-y^2)^2}\, dy.
\end{aligned} \tag{3.2.32}$$

Combining (3.2.31), (3.2.32) and integrating in x, we get

$$[v]^p \leq [u]^p. \tag{3.2.33}$$

Using the definition of v and the inequality $(a+b)^\sigma \leq a^\sigma + \sigma 2^{\sigma-1}(a^{\sigma-1}b + b^\sigma)$ for $a, b \geq 0$ and $\sigma \geq 1$, we have

$$\begin{aligned} u^{\frac{1}{1-s}} &\leq v^{\frac{1}{1-s}} + \frac{1}{1-s} 2^{\frac{s}{1-s}} (v^{\frac{s}{1-s}} u(r_0) + u(r_0)^{\frac{1}{1-s}}) \\ &\leq v^{\frac{1}{1-s}} \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right) + 2^{\frac{s}{1-s}} + \frac{2^{\frac{s}{1-s}}}{1-s} r_0 \\ &= v^{\frac{1}{1-s}} \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right) + C(r_0). \end{aligned} \tag{3.2.34}$$

This implies

$$\begin{aligned} u(x) &\leq v(x) \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{1-s} + C^{1-s}(r_0) \\ &:= w(x) + C^{1-s}(r_0). \end{aligned}$$

From (3.2.33) and the definition of w, we get

$$\begin{aligned} [w]^p &= [v]^p \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{\frac{1-s}{s}} \\ &\leq \left(1 - ||u||_p^p\right) \left(1 + \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)} ||u||_p^p\right)^{\frac{1-s}{s}} \end{aligned} \tag{3.2.35}$$

Consider now the function $f(t) = (1-t)(1+\tau t)^\sigma$, where $\tau := \frac{2^{\frac{2s-1}{1-s}}}{pr_0(1-s)}$ and $\sigma = \frac{1-s}{s} > 0$. We compute

$$f'(t) = (1+\tau t)^{\sigma-1} \left(\tau t(-\sigma - 1) + \tau\sigma - 1\right) \tag{3.2.36}$$

which vanishes for $t_1 = -\frac{1}{\tau} < 0$ and $t_2 = \frac{\tau\sigma - 1}{\tau(\sigma+1)}$. We choose now $r_0 > 2^{\frac{2s-1}{1-s}}$ so that $t_2 < 0$. This implies that f is decreasing in $(0,1)$ and since $f(0) = 1$ we have that $f(t) < 1$ for $t \in (0,1)$, which implies

$$[w]^p \leq 1. \tag{3.2.37}$$

We can apply now Proposition 1.8 on the interval $I = (-r_0, r_0)$ to get that there exists $\beta_* > 0$ such that

$$\int_I e^{\beta_* w^{p'}}\, dx \leq C \tag{3.2.38}$$

and using (3.2.34) we get

$$\int_I e^{\beta_* u^{\frac{1}{1-s}}}\, dx \leq C \int_I e^{\beta_* w^{\frac{1}{1-s}}}\, dx \leq C, \tag{3.2.39}$$

concluding the proof of (1.3.19).

To prove the second part of the claim one can argue as in the previous section, using the sequence of functions u_ε defined in (3.2.24) and taking into account that now the norm we are working with is the full $W^{s,p}$-norm. Indeed we have

$$\|u_\varepsilon\|_{L^p}^p = \int_{\mathbb{R}} |u_\varepsilon|^p \, dx = \int_{|x|\le\varepsilon} \left(|\log\varepsilon|^{p-sp}\right) dx + \int_{\varepsilon<|x|<1} \frac{|\log x|}{|\log\varepsilon|^{sp}} \, dx = O(|\log\varepsilon|^{-1}). \tag{3.2.40}$$

Hence from (3.2.25), it follows that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}^p = \gamma_s. \tag{3.2.41}$$

Choose $M > 0$ large enough so that

$$\Phi(t) \ge \frac{1}{2} e^t, \quad t \ge M.$$

Then one has

$$\begin{aligned} \int_{\mathbb{R}} \Phi\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}}\right) dx &\ge \int_{u_\varepsilon\ge M} \Phi\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}^{\frac{1}{1-s}}\right) dx \\ &\ge \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} e^{\left(\gamma_s^s \frac{u_\varepsilon}{\|u_\varepsilon\|_{W^{s,p}(\mathbb{R})}}\right)^{\frac{1}{1-s}}} dx. \end{aligned} \tag{3.2.42}$$

for ε small enough. Now, thanks to (3.2.41), one can argue as in the proof of Proposition 3.12 to conclude the proof of Theorem 1.9.

Proof of Theorem 1.10

We will start by proving (1.3.20) since the proof of (1.3.21) will follow adapting the reasoning of the previous section.

Let u_ε be as in (3.2.24). To prove (1.3.20) it is enough to show that there exists a constant $\delta > 0$ such that

$$\int_{-\varepsilon}^{\varepsilon} e^{\beta^*\left(\frac{u_\varepsilon}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx \ge \delta.$$

Indeed, $u_\varepsilon \to +\infty$ uniformly for $|x| < \varepsilon$ as $\varepsilon \to 0$ and we have

$$\sup_{u\in \tilde{W}_0^{s,p}(I),[u]_{W^{s,p}(\mathbb{R})}\le 1} \int_I f(|u|) e^{\beta^*\left(\frac{|u|}{[u]}\right)^{\frac{1}{1-s}}} dx \ge \inf_{|x|<\varepsilon} f(|u_\varepsilon|) \int_{-\varepsilon}^{\varepsilon} e^{\beta^*\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}} dx.$$

From Proposition 3.11, it follows that

$$\lim_{\varepsilon\to 0} \frac{[u_\varepsilon]}{\gamma_s^s} = 1 \tag{3.2.43}$$

and in particular

$$\lim_{\varepsilon\to 0}[u_\varepsilon]^p = 8\int_1^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx = \gamma_s.$$

We compute

$$\lim_{\varepsilon\to 0}\log\frac{1}{\varepsilon}\left([u_\varepsilon]^p - \gamma_s\right) = 8\lim_{\varepsilon\to 0}\log\frac{1}{\varepsilon}\int_{\frac{1}{\varepsilon}}^{+\infty} |\log x|^p \frac{x^2+1}{(x^2-1)^2}\,dx = 0. \tag{3.2.44}$$

Then we can write

$$\frac{[u_\varepsilon]^p}{\gamma_s} \le 1 + (C\log\frac{1}{\varepsilon})^{-1} \tag{3.2.45}$$

and in particular, recalling

$$\lim_{t\to+\infty}\frac{t}{(1+\frac{C}{t})^{\frac{1}{1-s}}} - t = -\frac{1}{1-s},$$

we have

$$\begin{aligned} &\int_{-\varepsilon}^{\varepsilon} e^{\gamma_s^{\frac{s}{1-s}}\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}}\,dx = \int_{-\varepsilon}^{\varepsilon} e^{\left(\frac{\gamma_s^s}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}|u_\varepsilon|^{\frac{1}{1-s}}}\,dx \\ &\ge \int_{-\varepsilon}^{\varepsilon} e^{\frac{\log\frac{1}{\varepsilon}}{(1+C(\log\frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}}\,dx \\ &= 2\varepsilon e^{\frac{\log\frac{1}{\varepsilon}}{(1+C(\log\frac{1}{\varepsilon})^{-1})^{\frac{1}{1-s}}}} \to e^{-\frac{1}{1-s}} \end{aligned} \tag{3.2.46}$$

as $\varepsilon\to 0$. Therefore

$$\int_I e^{\gamma_s^{\frac{s}{1-s}}\left(\frac{|u_\varepsilon|}{[u_\varepsilon]}\right)^{\frac{1}{1-s}}}\,dx \ge \delta \tag{3.2.47}$$

for some $\delta > 0$, proving (1.3.20). We shall now prove (1.3.21). From (3.2.40) and (3.2.44) it follows that

$$\frac{\|u_\varepsilon\|^p_{W^{s,p}(\mathbb{R})}}{\gamma_s} \le 1 + O(|\log\varepsilon|^{-1}). \tag{3.2.48}$$

Now using (3.2.42) and arguing as in (3.2.46) and (3.2.47), we conclude the proof.

3.3 Palais-Smale condition and critical points

In this section we will prove Theorem 1.11. As we already pointed out the main idea of the proof is to construct a sequence (u_k) which is almost of Palais-Smale type for J. Then a modified version of Proposition 1.12 is used, namely Lemma 3.10 below. In order to do so, it is crucial to show that $\bar{c} < \pi$ (Lemma 3.13 below) and this will follow from (1.3.9) with $p = 2$ and $h(t) = |t|^2$.

Proof of Proposition 1.12

For the proof of Proposition 1.12 we will closely follow [3]. Set

$$Q(u) := J(u) - \frac{1}{2}\langle J'(u), u\rangle = \lambda \int_I \left(\left(\frac{u^2}{2} - 1 \right) e^{\frac{1}{2}u^2} + 1 \right) dx. \tag{3.3.1}$$

Remark 3.3. *Notice that the integrand on the right-hand side of* (3.3.1) *is strictly convex and has a minimum at* $u = 0$*; in particular*

$$0 = Q(0) < Q(u) \quad \text{for every } u \in H \setminus \{0\}. \tag{3.3.2}$$

Furthermore by Lemma 1.2.3 the functional Q *is continuous on* H *and by convexity* Q *is also weakly lower semi-continuous.*

Let us also notice that

$$\begin{aligned}\lambda \int_I u^2 e^{\frac{1}{2}u^2}\, dx &= \lambda \int_{\{|u| \le 4\}} u^2 e^{\frac{1}{2}u^2}\, dx + \lambda \int_{\{|u|>4\}} u^2 e^{\frac{1}{2}u^2}\, dx \\ &\le C + \lambda \int_{\{|u>4|\}} u^2 e^{\frac{1}{2}u^2}\, dx \le C + \tilde{C} Q(u)\end{aligned}$$

and hence we have

$$\lambda \int_I u^2 e^{\frac{1}{2}u^2} dx \le C(1 + Q(u)) \quad \text{for every } u \in H. \tag{3.3.3}$$

We consider a Palais-Smale sequence $(u_k)_{k\ge 0}$ with $J(u_k) \to c$. From (1.4.7) we get

$$\langle J'(u_k), u_k\rangle = o(1)\|u_k\|_H \quad \text{as } k \to \infty,$$

and

$$Q(u_k) = J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle = c + o(1) + o(1)\|u_k\|_H. \tag{3.3.4}$$

Then with (3.3.3) we have

$$\lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx \le C\left(1 + \|u_k\|_H\right),$$

hence, using that $Q(u_k) \ge 0$

$$\lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \le C\left(1 + \|u_k\|_H\right),$$

so that

$$J(u_k) \ge \frac{1}{2}\|u_k\|_H^2 - C(1 + \|u_k\|_H).$$

This and the boundedness of $(J(u_k))_{k\ge 0}$ yield that the sequence $(u_k)_{k\ge 0}$ is bounded in H, hence we can extracts a weakly converging subsequence $u_k \rightharpoonup \tilde{u}$ in H. By the compactness of the embedding $H \hookrightarrow L^2$ (see e.g. [34, Theorem 7.1], which we can apply thanks to [34, Proposition 3.6], see Proposition 3.1), up to extracting a further subsequence we can assume that $u_k \to \tilde{u}$ almost everywhere. To complete the proof of the theorem it remains to show that, up to extracting a further subsequence, $u_k \to \tilde{u}$ strongly in H.

By Remark 3.3 we have

$$0 \leq Q(\tilde{u}) \leq \liminf_{k\to\infty} Q(u_k) = \liminf_{k\to\infty} \left(J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle \right) = c \tag{3.3.5}$$

Thus necessarily $c \geq 0$. In other words the Palais-Smale condition is vacantly true when $c < 0$ because no sequence can satisfy (1.4.7).

Clearly (3.3.5) implies $Q(u_k) \to Q(\tilde{u}) = 0$. We now claim that

$$u_k^p e^{\frac{1}{2}u_k^2} \to \tilde{u}^p e^{\frac{1}{2}\tilde{u}^2} \quad \text{in } L^1(I) \quad \text{for } 0 \leq p < 2. \tag{3.3.6}$$

Indeed, up to extracting a further subsequence, from (3.3.3) and (3.3.5) we get

$$\int_{\{|u_k|>L\}} u_k^p e^{\frac{1}{2}u_k^2} dx \leq \frac{1}{L^{2-p}} \int_{\{|u_k|>L\}} u_k^2 e^{\frac{1}{2}u_k^2} dx = O\left(\frac{1}{L^{2-p}}\right),$$

and (3.3.6) follows from Lemma 3.3.

Let us now consider the case $c = 0$. Since $Q(\tilde{u}) = 0$, hence $\tilde{u} \equiv 0$, with (3.3.6) we get

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2\lim_{k\to\infty} \left(J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \right) = 2\lambda \int_I \left(e^{\frac{1}{2}\tilde{u}^2} - 1 \right) dx = 0, \tag{3.3.7}$$

so that $u_k \to 0$ is H and the Palais-Smale condition holds in the case $c = 0$ as well.

The last case is when $c \in (0, \pi)$. We will need the following result which is analogue to Lemma 3.3 in [3].

Lemma 3.10. *Consider a bounded sequence $(u_k) \subset H$ such that u_k converges weakly and almost everywhere to a function $u \in H$. Further assume that:*

1. *there exists $c \in (0, \pi]$ such that $J(u_k) \to c$;*
2. *$\|u\|_H^2 \geq \lambda \int_I u^2 e^{\frac{1}{2}u^2} dx$;*
3. *$\sup_k \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty$;*
4. *either $u \not\equiv 0$ or $c < \pi$.*

Then

$$\lim_{k\to\infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u^2 e^{\frac{1}{2}u^2} dx.$$

Proof. We assume $u \not\equiv 0$ (if $u \equiv 0$ and $c < \pi$ the existence of $\varepsilon > 0$ in (3.3.8) below is obvious). We then have $Q(u) > 0$. On the other hand from assumption 2 we get

$$J(u) = \frac{1}{2}\|u\|_H^2 + Q(u) - \frac{\lambda}{2}\int_I u^2 e^{\frac{1}{2}u^2} dx \geq Q(u) > 0.$$

We also know from the weak convergence of u_k to u in H, the weakly lower semicontinuity of the norm and (3.3.6) that

$$J(u) \leq \lim_{k\to\infty} J(u_k) = c,$$

where the inequality is strict, unless $u_k \to u$ strongly in H (in which case the proof is complete). Then one can choose $\varepsilon > 0$ so that

$$\frac{1+2\varepsilon}{\pi} < \frac{1}{c - J(u)}. \tag{3.3.8}$$

Notice now that if we set $\beta = \lambda \int_I \left(e^{\frac{1}{2}u^2} - 1\right) dx$, then

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2c + 2\beta.$$

Then multiplying (3.3.8) by $\frac{1}{2}\|u_k\|_H^2$ we have for k large enough

$$\frac{1+\varepsilon}{2\pi}\|u_k\|_H^2 \leq \tilde{p} := \frac{1+2\varepsilon}{2\pi}\lim_{k\to\infty}\|u_k\|_H^2 < \frac{c+\beta}{c-J(u)} = \left(1 - \frac{\|u\|_H^2}{2(c+\beta)}\right)^{-1}.$$

By Lemma 3.9 applied to $v_k := \frac{u_k}{\|u_k\|_H}$, we get that the sequence $e^{\tilde{p}\pi v_k^2}$ is bounded in $L^1(I)$, hence $e^{\frac{(1+\varepsilon)}{2}u_k^2}$ is bounded in L^1.
Now we have that

$$\begin{aligned}\int_{\{|u_k|>K\}} u_k^2 e^{\frac{1}{2}u_k^2} dx &= \int_{\{|u_k|>K\}} \left(u_k^2 e^{-\frac{\varepsilon}{2}u_k^2}\right) e^{\frac{1+\varepsilon}{2}u_k^2} dx \\ &\leq o(1)\int_{\{|u_k|>K\}} e^{\frac{1+\varepsilon}{2}u_k^2} dx\end{aligned}$$

with $o(1) \to 0$ as $K \to \infty$, and we conclude with Lemma 3.3. □

We now claim

$$\|\tilde{u}\|_H^2 = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx. \tag{3.3.9}$$

First we show that $\tilde{u} \not\equiv 0$. So for the sake of contradiction, we assume that $\tilde{u} \equiv 0$. By Lemma 3.10

$$\lim_{k\to\infty}\int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = 0.$$

Therefore, also using (3.3.6), we obtain $\lim_{k\to\infty} Q(u_k) = 0$. It follows that

$$0 < c = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \left(Q(u_k) + \frac{1}{2}\langle J'(u_k), u_k\rangle \right) = 0,$$

contradiction, hence $\tilde{u} \not\equiv 0$.

Fix now $\varphi \in C_0^\infty(I) \cap H$. We have $\langle J'(u_k), \varphi\rangle \to 0$ as $k \to \infty$, since (u_k) is a Palais-Smale sequence. But, by weak convergence we have that

$$(u_k, \varphi)_H \to (\tilde{u}, \varphi)_H.$$

Now (3.3.6) implies

$$\int_I \varphi u_k e^{\frac{1}{2}u_k^2} dx \to \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} dx, \quad \text{for every } \varphi \in C_0^\infty(I).$$

Thus we have

$$(\tilde{u}, \varphi)_H = \lambda \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} dx.$$

By density and the fact that $\tilde{u} e^{\frac{1}{2}\tilde{u}^2} \in L^p$ for all $p \geq 1$, we have that

$$(\tilde{u}, \tilde{u})_H = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx,$$

hence (3.3.9) is proven. Therefore, we are under the assumptions of Lemma 3.10, which yields

$$\begin{aligned}
\|\tilde{u}\|_H^2 &\leq \liminf_{k\to\infty} \|u_k\|_H^2 \\
&= 2 \liminf_{k\to\infty} \left[J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \right] \\
&= 2 \liminf_{k\to\infty} \left[\frac{\lambda}{2} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx + \frac{1}{2}\langle J'(u_k), u_k\rangle \right] \\
&= \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx \\
&= \|\tilde{u}\|_H^2.
\end{aligned} \tag{3.3.10}$$

By Hilbert space theory the convergence of the norms implies that $u_k \to \tilde{u}$ strongly in H, and the Palais-Smale condition is proven.

Proof of Theorem 1.11

We start by proving the last claim of Theorem 1.11.

Proposition 3.14. *Let u be a non-negative non-trivial solution to* (1.4.4) *for some $\lambda \in \mathbb{R}$. Then $0 < \lambda < \lambda_1(I)$.*

Proof. Let $\varphi_1 \geq 0$ be as in Lemma 3.2. Then using φ_1 as a test function in (1.4.4) (compare to (1.4.6)) yields

$$\lambda_1(I) \int_I u\varphi_1 dx = \lambda \int_I u\varphi_1 e^{\frac{1}{2}|u|^2} dx > \lambda \int_I u\varphi_1 dx.$$

Hence $\lambda < \lambda_1$. Using u as test function in (1.4.4) gives at once $\lambda > 0$. □

The rest of the section is devoted to the proof of the existence part of Theorem 1.11.

Define the Nehari manifold

$$N(J) := \left\{u \in H \setminus \{0\}; \langle J'(u), u\rangle = 0\right\}.$$

Since, according to (3.3.1)-(3.3.2), $J(u) = Q(u) > 0$ for $u \in N(J)$, we have

$$a(J) := \inf_{u \in N(J)} J(u) \geq 0.$$

Lemma 3.11. *We have $a(J) > 0$.*

Proof. Assume that $a(J) = 0$, then there exists a sequence $(u_k) \subset N(J)$ such that

$$J(u_k) = Q(u_k) \to 0 \quad \text{as } k \to \infty,$$

From (3.3.3) we infer

$$\sup_{k \geq 0} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty, \tag{3.3.11}$$

which, again using the fact that $u_k \in N(J)$, implies that $\|u_k\|_H$ is bounded. Thus, up to extracting a subsequence, we have that u_k weakly converges to a function $u \in H$. From the weak lower semicontinuity of Q we then get

$$0 \leq Q(u) \leq \liminf_{k \to \infty} Q(u_k) = 0,$$

thus $J(u) = Q(u) = 0$ and (3.3.2) implies $u \equiv 0$. On the other hand, we have from (3.3.7) with $\tilde{u}$ replaced by u (which holds with the same proof thanks to (3.3.11))

$$\lim_{k \to \infty} \|u_k\|_H^2 = 2 \lim_{k \to \infty} \left\{ J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1\right) dx \right\} = 0, \tag{3.3.12}$$

therefore we have strong convergence of u_k to 0.

Now, if we let $v_k = \frac{u_k}{\|u_k\|_H}$ and up to a subsequence we assume $v_k \to v$ weakly in H and almost everywhere, we have

$$1 = \|v_k\|_H^2 = \lim_{k \to \infty} \lambda \int_I e^{\frac{1}{2}u_k^2} v_k^2 dx = \lambda \int_I v^2 dx < \lambda_1 \int_I v^2 dx \leq 1, \tag{3.3.13}$$

where the third equality is justified as follows: From the Sobolev imbedding $v_k \to v$ in all $L^p(I)$ for every $p \in [1,\infty)$, while from (3.3.12) and Theorem 1.2.2 we have that for every $q \in [1,\infty)$ the sequence $(e^{\frac{1}{2}u_k^2})$ is bounded in $L^q(I)$, hence from Hölder's inequality we have the desired limit. The last inequality in (3.3.13) follows from the Poincaré inequality, see (1.4.2).

Clearly (3.3.13) is a contradiction, hence $a(J) > 0$. □

Lemma 3.12. *For every $u \in H \setminus \{0\}$ there exists a unique $t = t(u) > 0$ such that $tu \in N(J)$. Moreover, if*

$$\|u\|_H^2 \leq \lambda \int_I u^2 e^{\frac{1}{2}u^2} dx, \tag{3.3.14}$$

then $t(u) \leq 1$ and $t(u) = 1$ if and only if $u \in N(J)$.

Proof. Fix $u \in H \setminus \{0\}$ and for $t \in (0,\infty)$ define the function

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 e^{\frac{1}{2}t^2u^2} dx \right),$$

which can also be written as

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) - t^2 \lambda \int_I u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) dx.$$

Notice that $tu \in N(J)$ if and only if $f(t) = 0$.

From the inequality

$$u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) \geq \frac{1}{2} t^2 u^4$$

we infer

$$f(t) \leq t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) - \frac{1}{2} t^4 \lambda \int_I u^4 dx,$$

hence

$$\lim_{t \to +\infty} f(t) = -\infty.$$

Now notice that the function $t \mapsto \left(e^{\frac{1}{2}t^2u^2} - 1 \right)$ is monotone increasing on $(0,\infty)$, and by Lemma 3.6 we have $\left(e^{\frac{1}{2}u^2} - 1 \right) \in L^p(I)$ for all $p \in [1,\infty)$, so that

$$u^2 \left(e^{\frac{1}{2}u^2} - 1 \right) \in L^1(I).$$

Then by the dominated convergence theorem we get

$$\lim_{t \to 0} \int_I u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) dx = 0.$$

So one has

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) + o(t^2) \quad \text{as } t \to 0.$$

Hence, $f(t) > 0$ for t small, since for $\lambda < \lambda_1(I)$

$$\|u\|_H^2 - \lambda \int_I u^2 dx > 0$$

(compare the proof of Lemma 3.2). Therefore there exists $t = t(u)$ such that $f(t) = 0$, i.e. $tu \in N(J)$. The uniqueness of such t follows noticing that the function

$$t \mapsto \int_I u^2 e^{\frac{1}{2}t^2 u^2} dx$$

is increasing. Keeping this in mind, if we assume (3.3.14), then $f(1) \leq 0$, hence $f(t) \leq 0$ for all $t \geq 1$. This implies at once that $t(u) \leq 1$ and $t(u) = 1$ if and only if $u \in N(J)$. □

Lemma 3.13. *We have $a(J) < \pi$.*

Proof. Take $w \in H$ such that $\|w\|_H = 1$ and let $t = t(w)$ be given as in Lemma 3.12 so that $tw \in N(J)$. Then

$$a(J) \leq J(tw) \leq \frac{t^2}{2}\|w\|_H^2 = \frac{t^2}{2}.$$

Now using the monotonicity of $t \mapsto \int_I w^2 e^{\frac{1}{2}t^2 w^2} dx$ we have

$$\lambda \int_I w^2 e^{a(J) w^2} dx \leq \lambda \int_I w^2 e^{\frac{1}{2}t^2 w^2} dx = \frac{t^2 \|w\|_H^2}{t^2} = 1.$$

Thus

$$\sup_{\|w\|_H = 1} \lambda \int_I w^2 e^{a(J) w^2} dx \leq 1,$$

and Theorem 1.2.2 implies that $a(J) < \pi$. □

Lemma 3.14. *Let $u \in N(J)$ be such that $J'(u) \neq 0$, then $J(u) > a(J)$.*

Proof. We choose $h \in H$ such that $\langle J'(u), h\rangle = 1$, and for $\alpha \in \mathbb{R}$ we consider the path $\sigma_t(\alpha) = \alpha u - th$, $t \in \mathbb{R}$. Remember that by Lemma 3.8 $J \in C^1(H)$. By the chain rule

$$\frac{d}{dt} J(\sigma_t(\alpha)) = -\langle J'(\sigma_t(\alpha)), h\rangle,$$

therefore, if we set $t = 0$, $\alpha = 1$ we find

$$\left.\frac{d}{dt} J(\sigma_t(\alpha))\right|_{t=0,\alpha=1} = -\langle J'(u), h\rangle = -1.$$

Hence there exist $\delta > 0$ and $\varepsilon > 0$ such that for $\alpha \in [1-\varepsilon, 1+\varepsilon]$ and $t \in (0, \delta]$

$$J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u). \tag{3.3.15}$$

Now we consider the function f defined by

$$f_t(\alpha) = \|\sigma_t(\alpha)\|_H^2 - \lambda \int_I \sigma_t(\alpha)^2 e^{\frac{1}{2}\sigma_t(\alpha)^2} dx,$$

which is continuous with respect to t and α by Lemma 1.2.3. Notice that since $u \in N(J)$ we have

$$f_0(\alpha) = \alpha^2 \int_I u^2 \left(e^{\frac{1}{2}u^2} - e^{\frac{1}{2}\alpha^2 u^2}\right) dx$$

and $f_0(1) = 0$. Since the function $\alpha \mapsto u^2(e^{\frac{1}{2}u^2} - e^{\frac{1}{2}\alpha^2 u^2})$ is decreasing, by continuity we can find $\varepsilon_1 \in (0, \varepsilon)$ and $\delta_1 \in (0, \delta)$ such that

$$f_t(1 - \varepsilon_1) > 0, \quad f_t(1 + \varepsilon_1) < 0 \quad \text{for } t \in [0, \delta_1].$$

Then if we fix $t \in (0, \delta_1]$ we can find $\alpha_t \in [1 - \varepsilon_1, 1 + \varepsilon_1]$ such that $f_t(\alpha_t) = 0$, i.e. $\sigma_t(\alpha_t) \in N(J)$, and from (3.3.15) we get

$$a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u).$$

Since

$$\frac{d}{d\alpha} J(\alpha u) = f_0(\alpha),$$

and $f_0(\alpha) > 0$ for $\alpha < 1$ and $f_0(\alpha) < 0$ for $\alpha > 1$, we get

$$J(\alpha u) \leq J(u) \quad \text{for } \alpha \in \mathbb{R},$$

and we conclude that

$$a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u) \leq J(u).$$

□

Proof of Theorem 1.11 (completed). To complete the proof it is enough to show the existence of $u_0 \in N(J)$ such that $J(u_0) = a(J)$. We consider then a minimizing sequence $(u_k) \subset N(J)$.
We assume that u_k changes sign. Then since $u_k \in N(J)$ we have

$$\||u_k|\|_H^2 < \|u_k\|_H^2 = \lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \lambda \int_I |u_k|^2 e^{\frac{1}{2}|u_k|^2} dx,$$

where we used (3.1.6), hence by Lemma 3.12 there exists $t_k = t(|u_k|) < 1$ such that $t_k|u_k| \in N(J)$, whence

$$J(t_k|u_k|) = Q(t_k|u_k|) < Q(|u_k|) = Q(u_k) = J(u_k),$$

where the inequality in the middle depends on the monotonicity of Q. Hence up to replacing u_k with $t_k|u_k|$ we can assume that the minimizing sequence (still denoted by (u_k)) is made of non-negative functions.

Since $J(u_k) = Q(u_k) \leq C$ we infer from (3.3.3)

$$\int_I u_k^2 e^{\frac{1}{2}u_k^2} dx \leq C$$

and for $u_k \in N(J)$ we get

$$\|u_k\|_H \leq C.$$

Thus up to a subsequence u_k weakly converges to a function $u_0 \in H$, and up to a subsequence the convergence is also almost everywhere.

We claim that $u_0 \not\equiv 0$. Indeed if $u_0 \equiv 0$, then from (3.3.6), we have that $\left(e^{\frac{1}{2}u_k^2} - 1\right) \to 0$ in $L^1(I)$. Thus

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2 \lim_{k\to\infty} \left[J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1\right) dx \right] = 2a(J).$$

Then according to Theorem 1.2.2, since $a(J) < \pi$ we have that $e^{\frac{1}{2}u_k^2}$ is bounded in L^p for some $p > 1$, hence weakly converging in $L^p(I)$ to $e^{\frac{1}{2}u_0^2}$. From the compactness of the Sobolev embeddings (see [34, Theorem 7.1], which can be applied thanks to Proposition 3.1), up to a subsequence $u_k^2 \to u_0^2$ strongly in $L^{p'}(I)$, hence

$$\lim_{k\to\infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx = 0,$$

and with Lemma 3.11 and (3.3.1) one gets

$$0 < a(J) = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} Q(u_k) = 0,$$

which is a contradiction.

Next we claim that

$$\|u_0\|_H^2 \leq \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx.$$

So we assume by contradiction that this is not the case, i.e.

$$\|u_0\|_H^2 > \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx.$$

Then from Lemma 3.10, Lemma 3.13 and the weak convergence, we have that

$$\|u_0\|_H^2 \leq \liminf_{k\to\infty} \|u_k\|_H^2 = \liminf_{k\to\infty} \lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx,$$

again leading to a contradiction.

From Lemma 3.12, we have that there exists $0 < t \leq 1$ such that $tu_0 \in N(J)$. Taking Remark 3.3 into account we get

$$a(J) \leq J(tu_0) = Q(tu_0) \leq Q(u_0) \leq \liminf_{k\to\infty} Q(u_k) = a(J).$$

It follows that $t = 1$, since otherwise the second inequality above would be strict. Then $u_0 \in N(J)$ and $J(u_0) = a(J)$. By Lemma 3.14 we have $J'(u_0) = 0$ $\square$

Bibliography

[1] N. Abatangelo. Large S-harmonic functions and boundary blow-up solutions for the fractional Laplacian. *Discrete Contin. Dyn. Syst.*, 35(12):5555–5607, 2015.

[2] D. R. Adams. A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math. (2)*, 128(2):385–398, 1988.

[3] Adimurthi. Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(3):393–413, 1990.

[4] Adimurthi and O. Druet. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. *Comm. Partial Differential Equations*, 29(1-2):295–322, 2004.

[5] Adimurthi and K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl.*, 13(5-6):585–603, 2007.

[6] Adimurthi and M. Struwe. Global compactness properties of semilinear elliptic equations with critical exponential growth. *J. Funct. Anal.*, 175(1):125–167, 2000.

[7] Frederick J. Almgren, Jr. and E. H. Lieb. Symmetric decreasing rearrangement is sometimes continuous. *J. Amer. Math. Soc.*, 2(4):683–773, 1989.

[8] F. V. Atkinson and L. A. Peletier. Ground states and Dirichlet problems for $-\Delta u = f(u)$ in $\mathbf{R}^2$. *Arch. Rational Mech. Anal.*, 96(2):147–165, 1986.

[9] P. Aviles. Conformal complete metrics with prescribed nonnegative Gaussian curvature in $\mathbf{R}^2$. *Invent. Math.*, 83(3):519–544, 1986.

[10] D. Bartolucci, F. De Marchis, and A. Malchiodi. Supercritical conformal metrics on surfaces with conical singularities. *Int. Math. Res. Not. IMRN*, (24):5625–5643, 2011.

[11] J. Bebernes and D. Eberly. *Mathematical problems from combustion theory*, volume 83 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[12] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[13] M. Birkner, J. A. López-Mimbela, and A. Wakolbinger. Comparison results and steady states for the Fujita equation with fractional Laplacian. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(1):83–97, 2005.

[14] R. M. Blumenthal, R. K. Getoor, and D. B. Ray. On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.*, 99:540–554, 1961.

[15] J. M. Bony. *Cours d Analyse Theorie des distrubutions et analyse de Fourier*. Ecole Polytechnique, 2001.

[16] L. Brasco and G. Franzina. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.*, 37(3):769–799, 2014.

[17] L. Brasco, E. Lindgren, and E. Parini. The fractional Cheeger problem. *Interfaces Free Bound.*, 16(3):419–458, 2014.

[18] C. Bucur. Some observations on the Green function for the ball in the fractional Laplace framework. *Commun. Pure Appl. Anal.*, 15(2):657–699, 2016.

[19] L. Carleson and Sun-Yung A. Chang. On the existence of an extremal function for an inequality of J. Moser. *Bull. Sci. Math. (2)*, 110(2):113–127, 1986.

[20] Sun-Yung A. Chang and P. C. Yang. Prescribing Gaussian curvature on S^2. *Acta Math.*, 159(3-4):215–259, 1987.

[21] Sun-Yung A. Chang and P. C. Yang. Conformal deformation of metrics on S^2. *J. Differential Geom.*, 27(2):259–296, 1988.

[22] S. Chanillo and M. Kiessling. Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry. *Comm. Math. Phys.*, 160(2):217–238, 1994.

[23] C.-C. Chen and C.-S. Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. *Comm. Pure Appl. Math.*, 55(6):728–771, 2002.

[24] C.-C. Chen and C.-S. Lin. Topological degree for a mean field equation on Riemann surfaces. *Comm. Pure Appl. Math.*, 56(12):1667–1727, 2003.

[25] C.-C. Chen and C.-S. Lin. Mean field equations of Liouville type with singular data: sharper estimates. *Discrete Contin. Dyn. Syst.*, 28(3):1237–1272, 2010.

[26] C.-C. Chen and C.-S. Lin. Mean field equation of Liouville type with singular data: topological degree. *Comm. Pure Appl. Math.*, 68(6):887–947, 2015.

[27] W. X. Chen. A Trüdinger inequality on surfaces with conical singularities. *Proc. Amer. Math. Soc.*, 108(3):821–832, 1990.

[28] W. X. Chen and C. Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3):615–622, 1991.

[29] W. X. Chen and C. Li. Qualitative properties of solutions to some nonlinear elliptic equations in $\mathbf{R}^2$. *Duke Math. J.*, 71(2):427–439, 1993.

[30] K.-S. Cheng and W.-M. Ni. On the structure of the conformal Gaussian curvature equation on $\mathbf{R}^2$. *Duke Math. J.*, 62(3):721–737, 1991.

[31] G. Csató and P. Roy. Extremal functions for the singular Moser-Trudinger inequality in 2 dimensions. *Calc. Var. Partial Differential Equations*, 54(2):2341–2366, 2015.

[32] G. Csató and P. Roy. The singular moser-trudinger inequality on simply connected domains. *Comm. Partial Differential Equations*, 2015.

[33] F. Da Lio, L. Martinazzi, and T. Rivière. Blow-up analysis of a nonlocal Liouville-type equation. *Anal. PDE*, 8(7):1757–1805, 2015.

[34] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[35] W. Ding, J. Jost, J. Li, and G. Wang. Existence results for mean field equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(5):653–666, 1999.

[36] Z. Djadli. Existence result for the mean field problem on Riemann surfaces of all genuses. *Commun. Contemp. Math.*, 10(2):205–220, 2008.

[37] O. Druet. Multibumps analysis in dimension 2: quantification of blow-up levels. *Duke Math. J.*, 132(2):217–269, 2006.

[38] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[39] M. Felsinger, M. Kassmann, and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.

[40] M. Flucher. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Comment. Math. Helv.*, 67(3):471–497, 1992.

[41] L. Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.*, 68(3):415–454, 1993.

[42] R. Fortini. The role of Liouville type systems in the analysis of selfdual gauge field theories. *Unpublished Ph.D. Thesis*, 2015.

[43] R. K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101:75–90, 1961.

[44] G. Grubb. Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators. *Adv. Math.*, 268:478–528, 2015.

[45] J. Hong, Y. Kim, and P. Y. Pac. Multivortex solutions of the abelian Chern-Simons-Higgs theory. *Phys. Rev. Lett.*, 64(19):2230–2233, 1990.

[46] A. Hyder. Moser functions and fractional moser-trudinger type inequalities. *Nonlinear Anal.*, 146:185–210, 2016.

[47] Ali Hyder. Moser functions and fractional Moser-Trudinger type inequalities. *Nonlinear Anal.*, 146:185–210, 2016.

[48] A. Iannizzotto and M. Squassina. 1/2-Laplacian problems with exponential nonlinearity. *J. Math. Anal. Appl.*, 414(1):372–385, 2014.

[49] S. Iula. A Note on the Moser-Trudinger inequality in Sobolev-Slobodeckij spaces in dimension one. *Preprint, arXiv:1610.00933v1*, 2016.

[50] S. Iula, A. Maalaoui, and L. Martinazzi. A fractional Moser-Trudinger type inequality in one dimension and its critical points. *Differential Integral Equations*, 29(5-6):455–492, 2016.

[51] S. Iula and G. Mancini. Extremal functions for singular moser-trudinger embeddings. *Preprint, arXiv:1601.05666v1*, 2016.

[52] P. Jaming. On the Fourier transform of the symmetric decreasing rearrangements. *Ann. Inst. Fourier (Grenoble)*, 61(1):53–77, 2011.

[53] V. I. Judovič. Some estimates connected with integral operators and with solutions of elliptic equations. *Dokl. Akad. Nauk SSSR*, 138:805–808, 1961.

[54] J. L. Kazdan and F. W. Warner. Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. *Ann. of Math. (2)*, 101:317–331, 1975.

[55] J. L. Kazdan and F. W. Warner. Remarks on some quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 28(5):567–597, 1975.

[56] S. Kesavan. *Symmetrization & applications*, volume 3 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[57] Y. Li. Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. *J. Partial Differential Equations*, 14(2):163–192, 2001.

[58] Y. Li. Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds. *Sci. China Ser. A*, 48(5):618–648, 2005.

[59] Y. Li. Remarks on the extremal functions for the Moser-Trudinger inequality. *Acta Math. Sin. (Engl. Ser.)*, 22(2):545–550, 2006.

[60] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[61] C.-S. Lin. A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$. *Comment. Math. Helv.*, 73(2):206–231, 1998.

[62] K.-C. Lin. Extremal functions for Moser's inequality. *Trans. Amer. Math. Soc.*, 348(7):2663–2671, 1996.

[63] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[64] G. Lu and Y. Yang. Sharp constant and extremal function for the improved Moser-Trudinger inequality involving L^p norm in two dimension. *Discrete Contin. Dyn. Syst.*, 25(3):963–979, 2009.

[65] A. Maalaoui, L. Martinazzi, and A. Schikorra. Blow-up behavior of a fractional Adams-Moser-Trudinger-type inequality in odd dimension. *Comm. Partial Differential Equations*, 41(10):1593–1618, 2016.

[66] A. Malchiodi. Topological methods for an elliptic equation with exponential nonlinearities. *Discrete Contin. Dyn. Syst.*, 21(1):277–294, 2008.

[67] A. Malchiodi and L. Martinazzi. Critical points of the Moser-Trudinger functional on a disk. *J. Eur. Math. Soc. (JEMS)*, 16(5):893–908, 2014.

[68] G. Mancini. Onofri-type inequalities for singular liouville equations. *Journal of Geometric Analysis*, 2015.

[69] G. Mancini. Singular liouville equations on S^2: Sharp inequalities and existence results. *Preprint, arXiv:1508.02090v1*, 2015.

[70] L. Martinazzi. A threshold phenomenon for embeddings of H_0^m into Orlicz spaces. *Calc. Var. Partial Differential Equations*, 36(4):493–506, 2009.

[71] L. Martinazzi. Fractional Adams-Moser-Trudinger type inequalities. *Nonlinear Anal.*, 127:263–278, 2015.

[72] R. C. McOwen. Conformal metrics in $\mathbf{R}^2$ with prescribed Gaussian curvature and positive total curvature. *Indiana Univ. Math. J.*, 34(1):97–104, 1985.

[73] D. Montgomery and G. Joyce. Statistical mechanics of "negative temperature" states. *Phys. Fluids*, 17:1139–1145, 1974.

[74] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.

[75] J. Moser. On a nonlinear problem in differential geometry. pages 273–280, 1973.

[76] Z. Nehari. On a class of nonlinear second-order differential equations. *Trans. Amer. Math. Soc.*, 95:101–123, 1960.

[77] Z. Nehari. Characteristic values associated with a class of non-linear second-order differential equations. *Acta Math.*, 105:141–175, 1961.

[78] W. M. Ni. On the elliptic equation $\Delta u + K(x)e^{2u} = 0$ and conformal metrics with prescribed Gaussian curvatures. *Invent. Math.*, 66(2):343–352, 1982.

[79] M. Nolasco and G. Tarantello. Double vortex condensates in the Chern-Simons-Higgs theory. *Calc. Var. Partial Differential Equations*, 9(1):31–94, 1999.

[80] E. Onofri. On the positivity of the effective action in a theory of random surfaces. *Comm. Math. Phys.*, 86(3):321–326, 1982.

[81] E. Parini and B. Ruf. On the Moser-Trudinger inequality in fractional Sobolev-Slobodeckij spaces. *Preprint, arXiv: 1607.07681v1*, 2015.

[82] Y. J. Park. Fractional Polya-Szego inequality. *J. Chungcheong Math. Soc.*, 24:267–271, 2011.

[83] J. Peetre. Espaces d'interpolation et théorème de Soboleff. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):279–317, 1966.

[84] S. I. Pohozaev. The Sobolev embedding in the case $pl = n$. *Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Energet. Insti. Moscow*, 16:158–170, 1965.

[85] J. Prajapat and G. Tarantello. On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(4):967–985, 2001.

[86] F. Robert and M. Struwe. Asymptotic profile for a fourth order PDE with critical exponential growth in dimension four. *Adv. Nonlinear Stud.*, 4(4):397–415, 2004.

[87] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.

[88] B. Ruf. A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$. *J. Funct. Anal.*, 219(2):340–367, 2005.

[89] R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.*, 33(5):2105–2137, 2013.

[90] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[91] M. Struwe. Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(5):425–464, 1988.

[92] M. Struwe. Positive solutions of critical semilinear elliptic equations on non-contractible planar domains. *J. Eur. Math. Soc. (JEMS)*, 2(4):329–388, 2000.

[93] M. Struwe. A flow approach to Nirenberg's problem. *Duke Math. J.*, 128(1):19–64, 2005.

[94] M. Struwe and G. Tarantello. On multivortex solutions in Chern-Simons gauge theory. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(1):109–121, 1998.

[95] G. Tarantello. *Selfdual gauge field vortices*. Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston, Inc., Boston, MA, 2008. An analytical approach.

[96] C. Tintarev. Trudinger-Moser inequality with remainder terms. *J. Funct. Anal.*, 266(1):55–66, 2014.

[97] M. Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.

[98] N. S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.

[99] A. Tur and V. Yanovsky. Point vortices with a rational necklace: new exact stationary solutions of the two-dimensional Euler equation. *Phys. Fluids*, 16(8):2877–2885, 2004.

[100] J. Xiao and Zh. Zhai. Fractional Sobolev, Moser-Trudinger Morrey-Sobolev inequalities under Lorentz norms. *J. Math. Sci. (N.Y.)*, 166(3):357–376, 2010. Problems in mathematical analysis. No. 45.

[101] Y. Yang. A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface. *Trans. Amer. Math. Soc.*, 359(12):5761–5776 (electronic), 2007.

[102] Y. Yang. Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two. *J. Differential Equations*, 258(9):3161–3193, 2015.