An application of Q-curvature to an embedding of critical type

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Let $\Omega \subset \mathbb{R}^{2m}$ be open, bounded and with smooth boundary, and let a sequence $\lambda_k \to 0^+$ be given. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of positive smooth solutions to

$$(1)\qquad \begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Problem (1) arises from the Adams-Moser-Trudinger inequality [1, 10, 13]:

$$(2)\qquad \sup_{u\in H_0^m(\Omega),\, \|u\|^2_{H_0^m}\le \Lambda_1} \frac{1}{|\Omega|}\int_\Omega e^{mu^2}dx = c_0(m) < +\infty,$$

where $c_0(m)$ is a dimensional constant, $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, and $H_0^m(\Omega)$ is the Beppo-Levi space defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$(3)\qquad \|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx\right)^{\frac{1}{2}},$$

where $\Delta^{\frac{m}{2}} u := \nabla \Delta^{\frac{m-1}{2}} u$ for m odd. In fact critical points of (2) under the constraint $\|u\|^2_{H_0^m} = \Lambda_1$ solve (1). Then we have the following concentration-compactness result:

Theorem 1 ([9]). *Let (u_k) be a sequence of solutions to (1) such that*

$$(4)\qquad \limsup_{k\to\infty} \|u_k\|^2_{H_0^m} = \limsup_{k\to\infty}\int_\Omega \lambda_k u_k^2 e^{mu_k^2}dx = \Lambda < \infty.$$

Then up to a subsequence either

(i) $\Lambda = 0$ and $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, or

(ii) There exists a positive integer I such that $\Lambda \ge I\Lambda_1$, and there is a finite set $S = \{x^{(1)}, \ldots, x^{(I)}\}$ such that

$$u_k \to 0 \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S),$$

and

$$\lambda_k u_k^2 e^{mu_k^2} \rightharpoonup \sum_{i=1}^I \alpha_i \delta_{x^{(i)}}, \quad \alpha_i \ge \Lambda_1,$$

weakly in the sense of measures.

Theorem 1 was proven by Adimurthi and M. Struwe [3] and Adimurthi and O. Druet [2] in the case $m = 1$, and by F. Robert and M. Struwe [11] for $m = 2$. Recently O. Druet [6] for the case $m = 1$, and M. Struwe [12] for $m = 2$ improved the previous results by showing that in case (ii) of Theorem 1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m > 2$ is still an open question.

Part (ii) of the theorem shows an interesting threshold phenomenon: below the critical energy level Λ_1 we always have compactness. Moreover Λ_1 is the total Q-curvature of the sphere (see [8] for a short discussion of Q-curvature). We shall briefly explain how this remarkable connection with Riemannian geometry arises. It is easy to see that if we are not in case (i) of the theorem, then $\sup_\Omega u_k \to \infty$ as $k \to \infty$. Then one can blow up, i.e. define the scaled functions

$$\eta_k(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) \quad \text{for } x \in r_k^{-1}\Omega - x_k,$$

where x_k is such that $u_k(x_k) = \max_\Omega u_k$ and $r_k \to 0$ is a suitably chosen scaling factor. Then it turns out that

$$\eta_k(x) \to \eta_0(x) \quad \text{in} \quad C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}), \quad \text{as } k \to \infty, \tag{5}$$

where η_0 is a solution of the Liouville-type equation

$$(-\Delta)^m \eta = (2m-1)! e^{2m\eta} \quad \text{on } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2m\eta} dx < \infty. \tag{6}$$

We recall (see e.g. [8]) that if η solves $(-\Delta)^m \eta = V e^{2m\eta}$ on $\mathbb{R}^{2m}$, then the conformal metric $g_\eta := e^{2\eta}|dx|^2$ has Q-curvature V, where $|dx|^2$ denotes the Euclidean metric. Now the problem is to understand what is the solution η_0 or (equivalently) what is the conformal metric g_{η_0}.

A bunch of solution to (6) is given by the so-called *standard solutions*

$$\eta_{\lambda,x_0}(x) = \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2}, \quad \lambda > 0, x_0 \in \mathbb{R}^{2m}.$$

These are “spherical” solutions, as the metric $e^{2\eta_{\lambda,x_0}}|dx|^2$ can be obtained by pulling-back the metric of the round sphere S^{2m} onto $\mathbb{R}^{2m}$ via the stereographic projection and a Möbius diffeomorphism.

While Chen and Li [5] proved that in the case $m = 1$ the only solutions to (6) are the standard solutions, Chang and Chen [4] showed that for $m > 1$ (6) possesses many other solutions. Therefore the problem of understanding η_0 starts to appear quite subtle, and the following classification result, due to the author [8], turns out to be crucial.

Theorem 2. *Let η be a solution to* (6) *and set*

$$v(x) := \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|} e^{2mu(y)}\right) dy,$$

where γ_m is such that $(-\Delta)^m\left[\frac{1}{\gamma_m}\log\frac{1}{|x|}\right] = \delta_0$. Then $\eta = v + p$, where p is a polynomial of degree at most $2m-2$ and

$$\lim_{|x|\to\infty} \Delta^j v(x) = 0, \quad 1 \le j \le m-1.$$

Moreover the following are equivalent:

(i) *η is a standard solution,*
(ii) *p is constant.*

Finally if η is not a standard solution there exist $1 \le j \le m-1$ and a constant $\alpha \neq 0$ such that

$$\lim_{|x|\to\infty} \Delta^j \eta(x) = \alpha. \tag{7}$$

Now the idea is to use Theorem 2 to prove the following proposition.

Proposition 3. *The function η_0 given by* (5) *is a standard solution to* (6).

Proposition 3 yields

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \lambda_k u_k^2 e^{m u_k^2} dx &\geq (2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx \\ &= (2m-1)! \int_{\mathbb{R}^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = \Lambda_1, \end{aligned}$$

This is the basic reason why $\alpha_i \geq \Lambda_1$ in case (ii) of Theorem 1.

In order to apply Theorem 2, one has to have a better understanding of the asymptotic behavior of the functions η_k and their derivatives. This is achieved in the following proposition, which is central to our argument.

Proposition 4. *For any $R > 0$, $1 \le \ell \le 2m-1$ there exists k_0 such that*

$$u_k(x_k) \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \le C(Rr_k)^{2m-\ell}, \quad \textit{for all } k \ge k_0. \tag{8}$$

Equivalently

$$\int_{B_R(0)} |\nabla^\ell \eta_k| dx \le C R^{2m-\ell}, \quad \textit{for all } k \ge k_0. \tag{9}$$

Observe that taking the limit in (9) one gets

$$\int_{B_R(0)} |\nabla^\ell \eta_0| dx \le C R^{2m-\ell}, \quad k \ge k_0(R), \tag{10}$$

and η_0 has to be a standard solution because (10) is not compatible with (7).

Finally, let us also comment on the proof of Proposition 4. The key idea is to prove that

$$\|\Delta^m (u_k^2)\|_{L^1(\Omega)} \le C. \tag{11}$$

This is an easy consequence of the following Lorentz-space estimate.

Proposition 5. *For every $1 \le \ell \le 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \le C.$$

This can be proven by interpolation observing that (4) implies that $\Delta^m u_k$ is bounded in the Zygmund space $L(\log L)^{\frac{1}{2}}$. Interestingly if we decide to be a bit sloppy and consider that (4) gives bounds for $\Delta^m u_k$ in $L^1(\Omega)$, then we get the bounds $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \le C$ (here $L^{(p,\infty)}$ is the Marcinkievicz space). On the other hand these bounds are too weak to prove (11), hence Proposition 4. This also shows that (8), (9) and (10) are in some sense "sharp".

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Gradient estimates via non-linear potentials

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For the Poisson equation $-\triangle u = \mu$, here considered in the whole $\mathbb{R}^n$ and where μ is in the most general case a Radon measure with finite total mass, it is well-known that it is possible to get pointwise bounds for solutions via the use of Riesz potential

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}\,, \qquad \beta \in (0,n] \tag{1}$$

such as

$$|u(x)| \leq cI_2(|\mu|)(x)\,, \qquad \text{and} \qquad |Du(x)| \leq cI_1(|\mu|)(x)\,. \tag{2}$$

Similar local estimates ca be obtained using the localized version of the Riesz potential $I_\beta(\mu)(x)$ is given by the linear potential

$$\mathbf{I}^\mu_\beta(x_0,R) := \int_0^R \frac{\mu(B(x_0,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad \beta \in (0,n] \tag{3}$$

with $B(x_0,\varrho)$ being the open ball centered at x_0, with radius ϱ. A question is now, *is it possible to give an analogue of estimates* (2) *in the case of general quasilinear*