Two Contributions to the Representation Theory of Algebraic Groups

Inauguraldissertation

zur
Erlangung der Würde eines Doktors der Philosophie

vorgelegt der
Philosophisch-Naturwissenschaftlichen Fakultät
der Universität Basel

von

Karin Baur

aus Zürich (ZH) und Gaiserwald (SG)

Basel, 2002

Genehmigt von der Philosophisch-Naturwissenschaftlichen Fakultät auf Antrag von

Prof. Dr. H. Kraft
Prof. Dr. N. R. Wallach

Basel, den 30. April 2002 — Prof. Dr. A. Zuberbühler, Dekan

Zusammenfassung

Sei V ein endlich-dimensionaler, komplexer Vektorraum. Eine Teilmenge X in V hat die *Trennungseigenschaft*, falls das Folgende gilt: Für je zwei linear unabhängige lineare Funktionen l, m auf V existiert ein Punkt x in X mit $l(x) = 0$ und $m(x) \neq 0$. Wir interessieren uns für den Fall $V = \mathbb{C}[x, y]_n$, d.h. V ist eine irreduzible Darstellung von SL_2. Die Teilmengen, die wir untersuchen, sind Bahnabschlüsse von Elementen aus $\mathbb{C}[x, y]_n$. Wir beschreiben die Bahnen, die die Trennungseigenschaft erfüllen:

Der Abschluss von O_f hat die Trennungseigenschaft genau dann, wenn f einen linearen Faktor der Vielfachheit eins enthält.

Im zweiten Teil der Dissertation untersuchen wir Tensorprodukte $V_\lambda \otimes V_\mu$ von irreduziblen Darstellungen von G (dabei ist G eine reduktive, komplexe algebraische Gruppe). Im Allgemeinen ist ein solches Tensorprodukt nicht mehr irreduzibel. Es ist eine grundlegende Frage, wie die irreduziblen Komponenten in das Tensorprodukt eingebettet sind. Eine besondere Komponente ist die so genannte Cartankomponente $V_{\lambda+\mu}$, die Komponente mit dem grössten Höchstgewicht. Die Cartankomponente taucht genau einmal auf in der Zerlegung.

Eine weitere interessante Teilmenge von $V_\lambda \otimes V_\mu$ ist die Menge der zerlegbaren Tensoren. Insbesondere stellt sich die folgende Frage:

Ist die Menge der zerlegbaren Tensoren in der Cartankomponente des Tensorprodukts gerade der Abschluss der G–Bahn des Tensors der Höchstgewichtsvektoren?

Falls dies der Fall ist, so sagen wir, dass die Cartankomponente des Tensorproduktes *klein* ist.

Wir zeigen, dass die Cartankomponente im Allgemeinen klein ist. Wir stellen vor, was bei $G = \mathrm{SL}_2$ und $G = \mathrm{SL}_3$ passiert und diskutieren die Darstellungen der speziellen linearen Gruppe.

Abstract

Let V be a finite dimensional complex vector space. A subset X in V has the *separation property* if the following holds: For any pair l, m of linearly independent linear functions on V there is a point x in X such that $l(x) = 0$ and $m(x) \neq 0$. We study the the case where $V = \mathbb{C}[x, y]_n$ is an irreducible representation of SL_2. The subsets we are interested in are the closures of SL_2–orbits O_f of forms in $\mathbb{C}[x, y]_n$. We give an explicit description of those orbits that have the separation property:

The closure of O_f has the separation property if and only if the form f contains a linear factor of multiplicity one.

In the second part of this thesis we study tensor products $V_\lambda \otimes V_\mu$ of irreducible G–representations (where G is a reductive complex algebraic group). In general, such a tensor product is not irreducible anymore. It is a fundamental question how the irreducible components are embedded in the tensor product. A special component of the tensor product is the so-called Cartan component $V_{\lambda+\mu}$ which is the component with the maximal highest weight. It appears exactly once in the decomposition.

Another interesting subset of $V_\lambda \otimes V_\mu$ is the set of decomposable tensors. The following question arises in this context:

Is the set of decomposable tensors in the Cartan component of such a tensor product given as the closure of the G–orbit of a highest weight vector?

If this is the case we say that the Cartan component is *small*. We show that in general, Cartan components are small. We present what happens for $G = \mathrm{SL}_2$ and $G = \mathrm{SL}_3$ and discuss the representations of the special linear group in detail.

Acknowledgments

Many thanks go to Hanspeter Kraft, who introduced me to this interesting subject, for his constant support, for uncountable suggestions and ideas discussed during the work on this thesis.

Also I am thankful to Bert Kostant, Joseph Landsberg, Dmitri Shmel'kin and Evgeni Tevelev for making helpful comments and sharing their ideas. Especially, I thank Nolan Wallach for his suggestions and for the support during my stay at UCSD. In addition, I thank my colleagues from the Department of Mathematics in Basel.

This thesis was supported by the research grant 2134 - 057486 from the Swiss National Science Foundation (Marie Heim–Vögtlin Beiträge).

Contents

Part I
On the Separation Property of Orbits in Representation Spaces

1 Introduction

Let X be a subset of a vector space V. We say that X has the *separation property (SP)* if for any pair α, β of linearly independent linear functions on V there exists a point $x \in X$ such that $\alpha(x) = 0$ and $\beta(x) \neq 0$. Equivalently, X has the SP if for any hyperplane H in V the intersection $X \cap H$ linearly spans H.

Note that if X has the separation property, then every subset Y containing X inherits the SP from X. Thus our goal is to find minimal subsets in V that have the separation property.

In the first section we discuss the situation where the vector space is a representation of SL_2 given as the binary forms of degree n, $V_n := \mathbb{C}[x, y]_n$. The subsets we are studying are the closures $\overline{O_f}$ of SL_2–orbits where f is a form in V_n.

For a general discussion of the separation property in representation spaces see the recent article [KW02] by Kraft and Wallach.

Of special interest is the minimal orbit O_{x^n} in V_n. Unfortunately, $\overline{O_{x^n}}$ does not have the separation property for $n \geq 2$. Naturally the next candidate to study is the orbit $O_{x^{n-1}y}$. We show that its closure has the SP. The main result of section 3 characterises the orbits in V_n:

Theorem. *The closure $\overline{O_f}$ has the separation property if and only if f contains a linear factor of multiplicity one.*

2 Clebsch–Gordan Decomposition

Let $\mathbb{C}[x_1,x_2]_n \otimes \mathbb{C}[x_1,x_2]_m$ be a tensor product of irreducible representations of SL_2. Its decomposition into irreducible components is the so-called Clebsch–Gordan decomposition:

Embed the tensor product in the vector space $\mathbb{C}[x_1,x_2,y_1,y_2]_{(n,m)}$ as follows. Denote the vector space $\mathbb{C}[x_1,x_2,y_1,y_2]_{(n,m)}$ by $V_{(n,m)}$ and let $\mathbb{C}[x_1,x_2]_n$ be the subspace $V_{(n,0)}$ and $\mathbb{C}[x_1,x_2]_m$ the subspace $V_{(0,m)}$ of $V_{(n,m)}$. W.l.o.g. we assume that $n \geq m$. Consider the following differential operators:

$$\begin{aligned} \triangle_{xy} : V_{(n,m)} &\rightarrow V_{(n+1,m-1)} \\ fh &\mapsto (x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2})(fh) \\ \Omega^1 : V_{(n,m)} &\rightarrow V_{(n-1,m-1)} \\ fh &\mapsto (\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1})(fh) \end{aligned}$$

Then one can prove the following result (see e.g. the lecture notes of Kraft and Procesi, [KP00], §9.1 and 9.2.).

Proposition 2.1. Clebsch–Gordan Decomposition
For every $0 \leq i \leq m = min(n,m)$ *there is an* SL_2*–equivariant isomorphism*

$$\mathbb{C}[x_1,x_2,y_1,y_2]_{(n,m)} \xrightarrow{\sim} \bigoplus_{i=0}^{m} \mathbb{C}[x_1,x_2]_{(n+m-2i,0)}$$

given by

$$fh \mapsto (\ldots, \triangle_{xy}^{m-i}\Omega^i(fh), \ldots).$$

We usually write τ_i for the projection operator $\tau_i := \triangle_{xy}^{m-i}\Omega^i$. In particular, the Cartan component V_{n+m} of the tensor product $V_n \otimes V_m$ corresponds to the zero set of $\tau_1, \tau_2, \ldots, \tau_m$ in $V_{(n,m)}$.

Example 2.2. Non-zero decomposable tensors of the Cartan component $\mathbb{C}[x_1,x_2]_{n+1}$ of the tensor product $\mathbb{C}[x_1,x_2]_n \otimes \mathbb{C}[x_1,x_2]_1$ are of the form $l^n \otimes l$.

Proof. Let $l_1 \cdots l_n \otimes m$ be a tensor of the Cartan component, let $m = cx_1 + dx_2$. Then by Proposition 2.1 the projection operator $\Omega : V_{(n,1)} \rightarrow V_{(n-1,0)}$ sends $l_1 \cdots l_n \cdot (cy_1 + dy_2)$ to zero. There are two possibilities:

(i) The factors l_i are all linear dependent, w.l.o.g. let $l_1 \cdots l_n = x_1^n$. Then $\Omega(x_1^n(cy_1 + dy_2)) = dnx_1^{n-1}$. This is zero if and only if d is zero. Hence the tensor is of the form $cx_1^n \otimes x_1$ with $c \in \mathbb{C}$.

(ii) The factors l_i span $\mathbb{C}[x_1, x_2]_1$. W.l.o.g. let $l_1 \cdots l_n$ contain $x_1 \cdot x_2$. Applying Ω to $x_1 \cdot x_2 l_3 \cdots l_n(cy_1 + dy_2)$ implies $m = 0$ hence the tensor is zero. □

3 Separation Property for Binary Forms

Definition 3.1. Let V be a complex vector space. A subset $X \subset V$ is said to have the *separation property* (SP) if for every pair α, β of linearly independent linear functions on V there exists $x \in X$ such that $\alpha(x) = 0$ and $\beta(x) \neq 0$.

The separation property for X means that for any pair $H \neq H' \subset V$ of hyperplanes the intersection $H \cap X$ is not included in H'. Or, equivalently, for any hyperplane $H \subset V$ the linear span of $H \cap X$ equals H.

Remark 3.2. Let $X \subset V$ have the separation property. Let $Y \subset V$ be a subset containing X. Then Y also has the separation property: For each pair α, β of linearly independent linear functions on Y there exists $x \in X \subset Y$ which separates α from β.

This observation explains that our goal is to find minimal subsets that have the separation property: every subset containing such a minimal subset inherits the (SP) from it.

An interesting example are orbits in a representation space. Let $O_{\min}$ be the orbit of the highest weight vector of an irreducible representation V. Assume that $\overline{O_{\min}}$ has the (SP). Hanspeter Kraft and Nolan Wallach have shown that in this case, every non-zero G–stable subvariety has the separation property (see [KW02], §5, Proposition 5).

A first example to look at are irreducible representations of SL_2, i.e. the vector spaces $V_n := \mathbb{C}[x, y]_n$ of binary forms of degree n (where SL_2 acts by substitution of the variables). The question is whether for given $f \in V_n$ the closure $\overline{O_f} \subset V_n$ has the (SP) or not.

Example 3.3. Let $O_{\min}$ be the orbit of the highest weight vector x^n in V_n. Then $\overline{O_{\min}}$ has the separation property if and only if $n = 1$.

Proof. Let $n \geq 2$. For $f = \sum_{i=0}^{n} f_i x^{n-i} y^i \in V_n$ define $\alpha(f) := f_0$ and $\beta(f) := f_1$. Then α and β are linearly independent.

Note that $\overline{O_{\min}} = O_{\min} \cup \{0\}$ (cf. Proposition 3.4 in part II of this thesis) hence every non-zero element g of $\overline{O_{\min}}$ is of the form l^n for some $l = ax + by \in V_1$. Write $l^n = a^n x^n + na^{n-1} b x^{n-1} y + \dots$. Then $\alpha(g) = 0$ implies $a = 0$ hence $\beta(g) = 0$. In other words $\overline{O_{\min}}$ does not have the separation property for $n \geq 2$.

For $n = 1$ let α and β be two linearly independent linear functions on V_1. W.l.o.g. let $\alpha(f_0 x + f_1 y) = f_0$ and $\beta(f_0 x + f_1 y) = f_1$ (note that α and β form a basis for V_1^*).

Then for every $f_1 \neq 0$ we have $\alpha(f_1 y) = 0$ and $\beta(f_1 y) = f_1 \neq 0$ (and similarly, for every $f_0 \neq 0$ we have $\beta(f_0 x) = 0$ and $\alpha(f_0 x) = f_0 \neq 0$). In other words: for every pair of linearly independent linear functions on V_1 we can find $f \in V_1$ such that $\alpha(f) = 0$ and $\beta(f) \neq 0$. □

Since for $n > 1$ the orbit $O_{\min} = O_{x^n}$ does not have the separation property we cannot apply the result of KRAFT and WALLACH mentioned above. In particular, we cannot expect that every non-zero orbit in V_n has the separation property. Nevertheless we are able to characterise the orbits in V_n having the separation property:

Theorem 3.4. *Let $f \in \mathbb{C}[x, y]_n$ be a binary form of degree $n \geq 1$. Then the following two properties are equivalent:*
(i) The closure $\overline{O_f}$ has the separation property.
(ii) The form f contains a linear factor of multiplicity one.

Proof. We first assume that the closure $\overline{O_f}$ has the (SP) and show that f contains a linear factor of multiplicity one.

Suppose that every factor of f has multiplicity at least two. Note that every non-zero element in $\overline{O_f}$ contains every factor with multiplicity greater or equal to two.

We show that there exist two linearly independent linear functions α, β on V_n such that $\alpha(f) = 0$ implies $\beta(f) = 0$ for every $f \in \overline{O_f}$. We use the idea of the proof of Example 3.3 above.

For $g(x, y) = g_0 x^n + g_1 x^{n-1} y + \cdots + g_{n-1} x y^{n-1} + g_n y^n \in V_n$ let α and β be the linearly independent linear form given as $\alpha(g) := g_0$ and $\beta(g) := g_1$. Now take any element of $\overline{O_f}$, i.e. any form $g(x, y) = \prod_{i=1}^{s} (a_i x + b_i y)^{r_i}$ with $r_i \geq 2$ for each i and $\sum r_i = n$. We write

$$g(x,y) = \underbrace{a_1^{r_1} \cdots a_s^{r_s}}_{g_0} x^n + \underbrace{\sum_{j=1}^{s} a_1^{r_1} \cdots a_j^{r_j - 1} \cdots a_s^{r_s} b_j \, x^{n-1} y}_{g_1} + \ldots$$

Note that since $r_i \geq 2$ the coefficient g_1 contains the factor $a_1 \cdots a_s$. If $\alpha(g)$ is zero one of the coefficients a_i has to be zero and so $\beta(g)$ is also zero. Hence $\alpha(g) = 0$ implies $\beta(g) = 0$ for any $g \in \overline{O_f}$.

It remains to show that if f contains a linear factor of multiplicity one, then $\overline{O_f}$ has the separation property. We proceed with two steps:

(A) Consider $xy^{n-1} \in V_n := \mathbb{C}[x, y]_n$. We prove that for every pair $H_1 \neq H_2$ of hyperplanes in V_n the intersection $H_1 \cap \overline{O_{xy^{n-1}}}$ is not included in $H_2 \cap \overline{O_{xy^{n-1}}}$. Hence the closure of the orbit $O_{xy^{n-1}}$ has the SP.

(B) Let f contain a linear factor of multiplicity one. We show that $\overline{O_{xy^{n-1}}}$ is a subset of $\overline{O_f}$. Hence $\overline{O_f}$ has the separation property (see Remark 3.2).
Proof of (A). Suppose that there exist hyperplanes $H_1 \neq H_2$ such that the intersection $H_1 \cap \overline{O_{xy^{n-1}}}$ is contained in $H_2 \cap \overline{O_{xy^{n-1}}}$. Let $H_i = \mathcal{V}(l_i)$ be the zero set of the form $l_i \in \mathcal{O}(V_n)_1 = V_n^*$. Let $V_1 = \mathbb{C}[x,y]_1$ and consider

$$\begin{array}{rcl} \varphi\colon V_1 \times V_1 & \to & V_n \\ (a,b) & \mapsto & ab^{n-1}. \end{array}$$

Step (1): The morphism φ is SL_2–equivariant, bihomogeneous of degree $(1, n-1)$ and its image is the closure of $O_{xy^{n-1}}$. In particular, the comorphism φ^* maps $\mathcal{O}(V_n)_1 \cong V_n$ into the tensor product $\mathcal{O}(V_1)_1 \otimes \mathcal{O}(V_1)_{n-1} \cong V_1 \otimes V_{n-1}$. Since this map is non-zero, it identifies V_n with the component V_n of $V_1 \otimes V_{n-1}$. Let $\tilde{l}_i := \varphi^*(l_i) \in \mathcal{O}(V_1)_1 \otimes \mathcal{O}(V_1)_{n-1}$ be the pull-back of l_i. Then $\tilde{l}_i$ belongs to the component V_n of $V_1 \otimes V_{n-1}$.
Step (2): We proceed by showing that if $\tilde{l}_1$ and $\tilde{l}_2$ are linearly independent, then they belong to the component V_{n-2} of $V_1 \otimes V_{n-1}$, contradicting $\tilde{l}_i \in V_n$ from step (1):

Consider

$$\begin{array}{ccl} V_1 \times V_1 & \xrightarrow{\varphi} & V_n \supset H_i = \mathcal{V}(l_i) \\ & \searrow & \\ & & \overline{O_{xy^{n-1}}} \end{array}$$

Since the image of φ is $\overline{O_{xy^{n-1}}}$ we have $\varphi^{-1}(H_i) = \varphi^{-1}(H_i \cap \overline{O_{xy^{n-1}}})$ which is the zero set $\mathcal{V}_{V_1 \times V_1}(\tilde{l}_i)$. By assumption, $H_1 \cap \overline{O_{xy^{n-1}}}$ is contained in $H_2 \cap \overline{O_{xy^{n-1}}}$, hence $\mathcal{V}_{V_1 \times V_1}(\tilde{l}_1)$ is contained in $\mathcal{V}_{V_1 \times V_1}(\tilde{l}_2)$. Thus every factor of $\tilde{l}_1$ appears as a factor of $\tilde{l}_2$.

Choose coordinates to identify $\mathcal{O}(V_1)_1 \otimes \mathcal{O}(V_1)_{n-1}$ with $\mathbb{C}[\alpha,\beta,\gamma,\delta]_{(1,n-1)}$. We decompose $\tilde{l}_i$ into prime factors. Both $\tilde{l}_1$ and $\tilde{l}_2$ contain a factor of bidegree $(1,r)$ and linear factors in γ, δ:

$$\begin{array}{rcl} \tilde{l}_1 & = & qm^2 M_1 \\ \tilde{l}_2 & = & qmm_1 M_2 \end{array}$$

where q is of bidegree $(1,r)$ in $\mathbb{C}[\alpha,\beta,\gamma,\delta]$ for some $r > 0$, and the forms m, m_1 lie in $\mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,1)}$. Note that $r = 0$ would imply $q \equiv 0$ in equation (2) below. Furthermore, $M_i \in \mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,n-r-3)}$ are such that each factor of M_1 is a factor of M_2. We know by step (1) that $\tilde{l}_1$ and $\tilde{l}_2$ belong to the component V_n of $V_1 \otimes V_{n-1}$. We apply the Clebsch–Gordan decomposition (see Proposition 2.1) to $\tilde{l}_i$: The form $\tilde{l}_i$ belongs to the component $\mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,n)}$ if and only if the differential operator $\Omega = \frac{\partial^2}{\partial\alpha\partial\delta} - \frac{\partial^2}{\partial\beta\partial\gamma}$ of $\tilde{l}_i$ vanishes.

Let $q = q_1\alpha + q_2\beta$ with $q_i \in \mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,r)}$ and

$$\begin{aligned}\tilde{l}_1 &= \alpha q_1 m^2 M_1 + \beta q_2 m^2 M_1 \\ \tilde{l}_2 &= \alpha q_1 m m_1 M_2 + \beta q_2 m m_1 M_2.\end{aligned}$$

The condition $\Omega\tilde{l}_1 = 0$ yields

$$\frac{\partial}{\partial\delta}\underbrace{q_1 m^2 M_1}_{=:S_1} = \frac{\partial}{\partial\gamma}\underbrace{q_2 m^2 M_1}_{=:S_2} \tag{1}$$

and $\Omega\tilde{l}_2 = 0$ yields

$$\frac{\partial}{\partial\delta}(\underbrace{q_1 m m_1 M_2}_{S_1\frac{m_1M_2}{mM_1}}) = \frac{\partial}{\partial\gamma}(\underbrace{q_2 m m_1 M_2}_{S_2\frac{m_1M_2}{mM_1}})$$

If we replace $q_i m m_1 M_2$ by $S_i\frac{m_1M_2}{mM_1}$ we get

$$(\frac{\partial}{\partial\delta}S_1)\frac{m_1M_2}{mM_1} + S_1\frac{\partial}{\partial\delta}\frac{m_1M_2}{mM_1} = (\frac{\partial}{\partial\gamma}S_2)\frac{m_1M_2}{mM_1} + S_2\frac{\partial}{\partial\gamma}\frac{m_1M_2}{mM_1}.$$

Since $\frac{\partial}{\partial\delta}S_1 = \frac{\partial}{\partial\gamma}S_2$ by equation (1), the last equation implies

$$S_1\frac{\partial}{\partial\delta}\frac{m_1M_2}{mM_1} = S_2\frac{\partial}{\partial\gamma}\frac{m_1M_2}{mM_1}.$$

Consider $\frac{m_1M_2}{mM_1}$. It is a rational function in γ and δ of degree zero. Since m_1M_2 and mM_1 are linearly independent, $\frac{m_1M_2}{mM_1}$ is not a constant and so its derivates with respect to γ and to δ do not vanish identically.

Recall the Euler Identity: for any homogeneous $f \in \mathbb{C}(\gamma,\delta)$ we have $\gamma\frac{\partial}{\partial\gamma}f + \delta\frac{\partial}{\partial\delta}f = \deg f \cdot f$. Applying Euler Identity to $\frac{m_1M_2}{mM_1}$ we get $\gamma\frac{\partial}{\partial\gamma}\frac{m_1M_2}{mM_1} + \delta\frac{\partial}{\partial\delta}\frac{m_1M_2}{mM_1} = 0$. Thus

$$\frac{S_1}{S_2} = \frac{\frac{\partial}{\partial\gamma}\frac{m_1M_2}{mM_1}}{\frac{\partial}{\partial\delta}\frac{m_1M_2}{mM_1}} = -\frac{\delta}{\gamma}.$$

In other words, $S_1\gamma + S_2\delta = 0$. If we replace S_i by $q_i m^2 M_1$, we get

$$q_1\gamma + q_2\delta = 0. \tag{2}$$

This implies $q_2 = \gamma \cdot R$, $q_1 = -\delta \cdot R$ for some $R \in \mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,r-1)}$, hence

$$q = -R(\alpha\delta - \beta\gamma).$$

Therefore the forms $\tilde{l}_i$ contain the factor $\alpha\delta - \beta\gamma$,

$$\begin{aligned}\tilde{l}_1 &= -(\alpha\delta - \beta\gamma)Rm^2M_1 \\ \tilde{l}_2 &= -(\alpha\delta - \beta\gamma)Rmm_1M_2,\end{aligned}$$

where Rm^2M_1 and Rmm_1M_2 belong to $\mathbb{C}[\alpha,\beta,\gamma,\delta]_{(0,n-2)}$. Hence $\tilde{l}_i$ are multiples of the generating invariant of $\mathbb{C}[\alpha,\beta,\gamma,\delta]^{\mathrm{SL}_2}$. In other words, if $\triangle$ denotes the generating invariant of $\mathcal{O}(V_1 \times V_1)^{\mathrm{SL}_2}$, we have $\tilde{l}_i \in \triangle \cdot \mathcal{O}(V_1 \times V_1)_{(0,n-2)}$ which is isomorphic to V_{n-2}.

Thus there exists no pair $H_1 \neq H_2$ of hyperplanes in V_n such that the intersection $H_1 \cap \overline{O_{xy^{n-1}}}$ is included in $H_2 \cap \overline{O_{xy^{n-1}}}$. Therefore $\overline{O_{xy^{n-1}}}$ has the separation property.

The proof of step (B) follows by Lemma 3.5 below. $\square$

Lemma 3.5. *Let $f \in \mathbb{C}[x,y]_n$ be a form which contains a linear factor of multiplicity one. Then the form xy^{n-1} is a degeneration of f.*

Proof. We have to show that $\overline{O_{xy^{n-1}}} \subset \overline{O_f}$.

Let $f \in V_n = \mathbb{C}[x,y]_n$ contain a linear factor l of multiplicity one, say $f = l \cdot l_2 \cdots l_n$ where the factors $l_2, \ldots, l_n$ possibly appear several times. Then the form $f_1 := l \cdot l_2^{n-1}$ is contained in the closure $\overline{O_f}$, hence $\overline{O_{f_1}} \subset \overline{O_f}$. Since l and l_2 are linearly independent there exists $g \in \mathrm{SL}_2$ such that $g(xy^{n-1})$ is a non-zero multiple of $l \cdot l_2^{n-1}$. Therefore $O_{xy^{n-1}}$ is contained in $\overline{O_{f_1}}$. $\square$

Part II
Decomposable Tensors and Cartan Components

1 Introduction and Results

The idea to study SL_2–orbits of elements $x^k y^{n-k}$ of V_n comes from an early approach to the separation property for binary forms. In doing so we are led to the second part of my thesis.

Section 2

We introduce the notations and present the tools used in the remaining sections.

Section 3

In order to understand SL_2–orbits $O_{x^k y^{n-k}}$ in V_n, we use a translation into a different setting: Consider the map $\varphi_k : V_1 \times V_1 \to V_n$ given by $(a, b) \mapsto a^k b^{n-k}$. For $k < \frac{n}{2}$ the image of φ_k is the closure of the orbit $O_{x^k y^{n-k}}$. The comorphism φ_k^* maps regular functions on V_n of degree one to regular functions on $V_1 \times V_1$ of bidegree $(k, n-k)$,

$$\varphi_k^* \colon \mathcal{O}(V_n)_1 \to \mathcal{O}(V_1)_k \otimes \mathcal{O}(V_1)_{n-k}.$$

As in section 3 of the first part we consider hyperplanes in V_n. A hyperplane H in V_n is given as the zero set of a linear form $l \in \mathcal{O}(V_n)_1$. Since $\mathcal{O}(V_1)_l$ is isomorphic to V_l, we may likewise study the tensor product $V_k \otimes V_{n-k}$ of irreducible SL_2–representations. The comorphism φ_k^* embeds V_n in the tensor product $V_k \otimes V_{n-k}$.

We generalise the situation: let G be a reductive group and consider the tensor product $V_\lambda \otimes V_\mu$ of irreducible representations of G. Note that the tensor product is not irreducible anymore. The following problems arise.

(i) How many irreducible components of the tensor product meet a given tensor?

(ii) Describe the set of decomposable tensors in the component $V_{\lambda+\mu}$ of the tensor product.

Recall that a tensor is called *decomposable* if it can be written as $v \otimes w$. The *rank of a tensor* is the minimum of decomposable tensors needed to write it as their sum.

It turns out that it is rather difficult to find answers to these straightforward questions. We are able to answer question (i) in a special case:

Remark (A). Decomposable tensors lying in one irreducible component of the tensor product $V_\lambda \otimes V_\mu$ belong to the component $V_{\lambda+\mu}$.

In the remaining sections of this thesis we present different methods to solve problem (ii). We recall the decomposition of a tensor product into its irreducible components.

In the case of SL_2–representations the decomposition of a tensor product $V_n \otimes V_m$ of irreducible representations is known as the Clebsch-Gordan decomposition:

$$\mathbb{C}[x,y]_n \otimes \mathbb{C}[x,y]_m = \bigoplus_{i=0}^{\min(n,m)} \mathbb{C}[x,y]_{n+m-2i}.$$

Note that every irreducible component appearing has multiplicity one in the tensor product. One can describe how the irreducible components lie in the tensor product by means of certain differential operators (see section 2 in the first part). While it is a tedious task to apply these differential operators, the example of SL_2–representations is by far the easiest case.

In the general situation let $V_\lambda \otimes V_\mu$ be a tensor product of irreducible representations of G. It decomposes into irreducible components as follows:

$$V_\lambda \otimes V_\mu = \bigoplus_\nu N_\nu V_\nu,$$

where $N_\nu = N_\nu(\lambda,\mu)$ denotes the multiplicity of V_ν in the decomposition. The coefficients N_ν are called Littlewood–Richardson coefficients. They can be calculated combinatorially by the Littlewood–Richardson rule. A recent proof of this rule is given by Littelmann in [Li90], §2.2 and §4.

A special component is the representation $V_{\lambda+\mu}$, the so-called *Cartan component*. It appears exactly once in the decomposition. It is clear that the orbit $G(v_\lambda \otimes v_\mu)$ of a highest weight vector consists of decomposable tensors of the Cartan component $V_{\lambda+\mu}$.

If the closure of $G(v_\lambda \otimes v_\mu)$ describes the set of decomposable tensors of the Cartan component, we say that the representation $V_\lambda \otimes V_\mu$ has a *small* Cartan component.

Tensor products of irreducible SL_2–representations always have small Cartan components (compare with section 7).

The SL_3–representation $S^2\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ is the first example of a tensor product where the Cartan component is not small. It was found by Hanspeter Kraft and Nolan Wallach ([KW98]). One way to see that the Cartan component of $S^2\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ is not small, is to compare the dimensions of the corresponding algebraic sets.
We follow an idea by Bert Kostant ([Ko98]) and use the Casimir operator to prove that decomposable tensors of the Cartan component have the following property.

Theorem (B). *In the K-orbit of every decomposable tensor of the Cartan component $V_{\lambda+\mu}$ lies a tensor $v \otimes w$ (with v and w normed) for which the following holds:*

$$\left(\sum \|v_i\|_\lambda^2 \lambda_i \mid \sum \|w_j\|_\mu^2 \mu_j\right) = (\lambda \mid \mu).$$

(Where $K \subset G$ is maximal compact, $\|v\|_\lambda = \langle v, v\rangle_\lambda$ is the norm induced by a hermitian form on V_λ. Furthermore, $v = \sum v_i$ and $w = \sum w_j$ are the decompositions of the vectors into weight vectors and the form $(\cdot \mid \cdot)$ is a non-degenerate $\mathcal{W}$–invariant symmetric bilinear form on $X_\mathbb{R}$).

We show that there exist decomposable tensors in $S^2\mathbb{C}^3 \otimes \mathbb{C}^{3*}$, for which property (ii) of Theorem C holds, but which do not belong to the Cartan component of this tensor product. Hence a tensor satisfying property (ii) does not necessarily belong to the Cartan component.

Section 4

By the result of Theorem B we are led to study the weight lattice of G, the vector space $X_\mathbb{R}$ spanned by the root lattice and the action of the Weyl group $\mathcal{W}$ on it. The idea is to understand what happens on $X_\mathbb{R}$ and to translate these results back to the situation of the tensor product $V_\lambda \otimes V_\mu$.

In this section we discuss the convex hull $\mathrm{Con}(\lambda)$ spanned by the weights of the irreducible representation V_λ .

We say that the tensor product $V_\lambda \otimes V_\mu$ *has only Weyl-conjugated maximal pairs*, if for each pair $(a, b) \in \mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ with $(a \mid b) = (\lambda \mid \mu)$ there is an element ω of the Weyl group such that $\omega a = \lambda$ and $\omega b = \mu$.

A first result is the following.

Theorem (C). *If $V_\lambda \otimes V_\mu$ has only $\mathcal{W}$–conjugated maximal pairs, then its Cartan component $V_{\lambda+\mu}$ is small.*

The example of the SL_3–representation $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ shows that the converse does not hold: For each tensor $v \otimes w$ of the Cartan component one can give explicitly an element $A \in \mathrm{SL}_3$ such that $A(v \otimes w)$ is a multiple of the highest weight vector $e_1 \otimes e_3^*$. But it is obvious that there exist maximal pairs in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ not lying in the $\mathcal{W}$–orbit of the pair $(\varepsilon_1, \varepsilon_1 + \varepsilon_2)$ of highest weights.

The main result of this section gives a description of those tensor products whose maximal pairs are all $\mathcal{W}$–conjugated:

Theorem (D). *Let $V_\lambda \otimes V_\mu$ be a tensor product of irreducible representations. Then the following assertions are equivalent:*
(i) $V_\lambda \otimes V_\mu$ has only Weyl-conjugated maximal pairs.
(ii) The weights λ and μ are perpendicular to the same simple roots.

In particular, this is the case if both λ and μ are regular. A consequence of Theorems C and D is that generic tensor products of irreducible G–representations have small Cartan components. The cases not treated in Theorem D are the tensor products where the dominant weights are perpendicular to different simple roots. We call such a representation *critical.*

Section 5

We develop a necessary condition for $V_\lambda \otimes V_\mu$ to have a small Cartan component. Denote by $L_{I(\lambda)} \subset G$ the reductive subgroup generated by T together with the root subgroups U_α of the roots perpendicular to λ.

We show that the submodule $< L_{I(\lambda)}(v_\lambda \otimes v_\mu) >$ of $V_\lambda \otimes V_\mu$ consists of decomposable tensors lying in the Cartan component $V_{\lambda+\mu}$. Using this property we can prove the following necessary condition:

Theorem (E). *If the tensor product $V_\lambda \otimes V_\mu$ has a small Cartan component, then the $L_{I(\lambda)}$–orbit of v_μ is dense in $< L_{I(\lambda)} v_\mu >$ and the $L_{I(\mu)}$–orbit of v_λ is dense in $< L_{I(\mu)} v_\lambda >$.*

It is known that there are only few representations of simple groups which contain a dense orbit (see Parshin, Shafarevich [PS94], p. 260). So Theorem E severely restricts the choice of critical representations that can have a small Cartan component.

Section 6

In this section we discuss irreducible representations $V_{a\omega_1 + b\omega_2}$ of SL_3 (where we denote by ω_i the ith fundamental weight of SL_3). We give a classification of the tensor products of SL_3–representations for which the Cartan component is small:

Theorem (F). *The tensor product $V_\lambda \otimes V_\mu$ has a small Cartan component if and only if it is isomorphic to one of the following products:*
$V_{a\omega_1+b\omega_2} \otimes V_{c\omega_1+d\omega_2}$, $V_{a\omega_1} \otimes V_{c\omega_1}$, $V_{a\omega_1+\omega_2} \otimes V_{c\omega_1}$, $V_{\omega_1} \otimes V_{\omega_2}$ *(with $a, b, c, d > 0$).*

Section 7

We use the method of associated cones to discuss tensor products of irreducible representations of SL_2 and to prove the following.

Theorem (G). *Tensor products of irreducible representations of* SL_2 *have small Cartan components.*

Section 8

We investigate irreducible representations of the special linear group in detail. We apply the criterion from Theorem E, section 5, to critical SL_{n+1}–representations and show that their weights need to be nearly regular.

We say that a critical representation $V_\lambda \otimes V_\mu$ is *semi-critical*, if there is an index i such that $l_i = 0$ and $m_i = 1$ (or $m_i = 0$ and $l_i = 1$), i.e. α_i is perpendicular to λ and the corresponding coefficient of μ is one. We call a critical representation *fully critical*, if there is a pair $i \neq j$ such that $l_i = m_j = 0$ and $l_j = m_i = 1$.

Theorem (H). *Let $V_\lambda \otimes V_\mu$ be semi-critical (fully critical) with $\alpha_j \perp \mu$, $l_j = 1$ (and $\alpha_i \perp \lambda$, $m_i = 1$, $i \neq j$) with a small Cartan component. Then the following holds: If there is a connected string of simple roots perpendicular to μ containing α_j (and a connected string of simple roots perpendicular to λ containing α_i), then α_j (and α_i) has to be a vertex of this string.*

The remaining part consists of a case by case study of semi- and fully critical representations:

The Cartan component of a semi-critical representation $V_\lambda \otimes V_\mu$ with regular weight μ is small. We prove that a tensor products $V_{\omega_i} \otimes V_{\omega_j}$ of fundamental representations has a small Cartan component if and only if (i, j) equals $(1, n)$ or $(i, i+1)$.

It remains an open problem what happens in general with semi- and fully critical representations.

2 Preliminaries

We first introduce the notations we will be using in the sequel and recall some facts that can be found e.g. in HUMPHREYS [Hu94], BOURBAKI [Bou75], or ONISHCHIK, VINBERG [OV90].

2.1 Notation

Unless specified otherwise, let G be a connected semi-simple linear reductive group over $\mathbb{C}$. Choose a Borel-subgroup B, a maximal torus T in B and $K \subset G$ a maximal compact subgroup such that $T_K := T \cap K$ is a maximal torus in K. We denote by $\mathfrak{g} := \operatorname{Lie} G$, $\mathfrak{k} := \operatorname{Lie} K$, $\mathfrak{h} := \operatorname{Lie} T$ and $\mathfrak{t} := \operatorname{Lie} T_K$ the corresponding Lie algebras.

For $\alpha \in \mathfrak{h}^*$ define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\} \subset \mathfrak{g}$. Every non-zero $\alpha \in \mathfrak{h}^*$ for which the subspace $\mathfrak{g}_\alpha$ is not zero is called a *root* of $\mathfrak{g}$ (relative to $\mathfrak{h}$). The set of all roots of $\mathfrak{g}$ (relative to $\mathfrak{h}$) is denoted by $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. The *root space decomposition* of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ is the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

To any Lie algebra $\mathfrak{g}$ one can associate a symmetric bilinear form defining $\kappa(X,Y) := \operatorname{Tr}(\operatorname{ad} X \cdot \operatorname{ad} Y)$, the so-called *Killing form*. It is $\mathfrak{g}$–invariant ($\kappa([ZX],Y) + \kappa(X,[ZY]) = 0$). Since $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate.

Lemma 2.1. *For every root α, $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{h}$ relative to the Killing form.*

Proof. The assertion follows from the fact that for all α, $\beta \in \mathfrak{h}^*$ such that $\alpha + \beta \neq 0$, the root space $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to the Killing form (cf. HUMPHREYS [Hu94], Proposition 8.1). $\square$

Lemma 2.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Since the Killing form is non-degenerate, the map $\iota : X \mapsto \kappa(X, \cdot)$ induces an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.*

Observe that ι maps the root subspace $\mathfrak{g}_{-\alpha}$ to $\mathfrak{g}_\alpha^*$. The isomorphism ι induces a bilinear form on $\mathfrak{g}^*$ which is also symmetric, non-degenerate and $\mathfrak{g}$–invariant. We will denote it by $(\cdot \mid \cdot)$. For $l, m \in \mathfrak{g}^*$ let $X_l := \iota^{-1}(l)$ and $X_m := \iota^{-1}(m)$, i.e. $l = \kappa(X_l, \cdot)$ and $m = \kappa(X_m, \cdot)$. Then we define $(l \mid m) := \kappa(X_l, X_m)$.

Recall that Φ spans $\mathfrak{h}^*$. Denote by $\triangle = \{\alpha_1, \ldots, \alpha_l\}$ a basis of $\mathfrak{h}^*$ consisting of roots. The elements of $\triangle$ are called *simple roots* of $\mathfrak{g}$. For $G = \mathrm{SL}_{n+1}$ we use the Bourbaki numbering of the simple roots (see BOURBAKI [Bou68], planche I).

Let $\mathfrak{h}_{\mathbb{Q}}^*$ be the $\mathbb{Q}$–span of $\triangle$ and let $X_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}}^* := \mathfrak{h}_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} \mathbb{R}$ be the real vector space spanned by the simple roots. The form $(\cdot \mid \cdot)$ is positive definite on $X_{\mathbb{R}}$, i.e. $X_{\mathbb{R}}$ is an Euclidean space. For $\alpha \in \Phi$ denote by σ_α the reflection on the hyperplane $\Omega_\alpha := \{\beta \in X_{\mathbb{R}} \mid (\beta \mid \alpha) = 0\}$. Let $\langle \beta \mid \alpha \rangle := \frac{2(\beta|\alpha)}{(\alpha|\alpha)}$ for $\alpha, \beta \in \Phi$. Then the action of σ_α on $\beta \in X_{\mathbb{R}}$ can be written as $\sigma_\alpha(\beta) = \beta - \langle \beta \mid \alpha \rangle \alpha$. One can show that Φ is a reduced root system in $X_{\mathbb{R}}$ (Bourbaki [Bou75], VIII, §2. 2, Théorème 2).

The reflections σ_α, $\alpha \in \triangle$, generate a finite subgroup $\mathcal{W}$ of $\mathrm{GL}(X_{\mathbb{R}})$, the so-called *Weyl group of* Φ. Note that reflections are orthogonal transformations, i.e. they preserve the inner product:

Lemma 2.3. *The inner product* $(\cdot \mid \cdot)$ *is* $\mathcal{W}$*–invariant.*

Recall the group-theoretic description of the Weyl group: If $\mathrm{N}_G(T)$ denotes the normaliser of T in G, $\mathrm{N}_G(T) = \{g \in G \mid ghg^{-1} = h \text{ for all } h \in T\}$, then there is an isomorphism $\mathrm{N}_G(T)/T \xrightarrow{\sim} \mathcal{W}$ (see Goodman, Wallach [GW98], 2.5.1).

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$. We always assume that V is finite-dimensional. If the subspace $V(\lambda) := \{v \in V \mid h.v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$ is not zero, λ is said to be a *weight* of V (relative to $\mathfrak{h}$), $V(\lambda)$ is a *weight subspace* of V and its non-zero vectors are the *weight vectors* corresponding to λ. The vector space V decomposes as $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V(\lambda)$ (weight space decomposition). The dimension of $V(\lambda)$ is called the *multiplicity of* λ *in* V.

Now let $R : G \to \mathrm{GL}(V)$ be a linear representation of G on V and χ an element of the character group $X(T)$ of T. If the subspace $V(\chi) := \{v \in V \mid R(h)v = \chi(h)v \text{ for all } h \in T\}$ is not zero, χ is said to be a *weight* of the representation (with respect to T). If ρ is the differential of the linear representation R then their sets of weights coincide. It is convenient to speak of representations of G or of representations of $\mathfrak{g}$ depending on the context.

We denote the set of weights of V by $\Pi(V)$. If V is irreducible with highest weight λ we write $V = V_\lambda$ and its set of weights will be denoted by $\Pi(\lambda)$. A *highest weight vector of* V_λ is a non-zero element of $V_\lambda(\lambda)$. It will be denoted by v_λ.

Lemma 2.4. *Let* V *be a finite dimensional representation of* $\mathfrak{g}$*. If* λ *is a weight of* V *and* ω *an element of the Weyl group* $\mathcal{W}$*, then* $\omega\lambda$ *has the same multiplicity in* V *as* λ*. The set* $\Pi(V)$ *is* $\mathcal{W}$*–stable.*

Proof. See Bourbaki [Bou75], VIII, §7.1 Corollaire 2. □

If λ is a weight of some finite dimensional $\mathfrak{g}$–module, then $\langle \lambda \mid \alpha \rangle$ is an integer for each $\alpha \in \triangle$. If, furthermore, $\langle \lambda \mid \alpha \rangle$ is non-negative for each

$\alpha \in \triangle$ we say that λ is *dominant* (relative to $\triangle$). The set of dominant weights is denoted by X^+, $X^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda \mid \alpha \rangle \in \mathbb{N} \text{ for all } \alpha \in \triangle\}$. The hyperplanes Ω_α, $\alpha \in \triangle$, partition $X_\mathbb{R}$ into finitely many regions. The connected components of $X_\mathbb{R} \setminus \cup_{\alpha \in \triangle} \Omega_\alpha$ are called the open *Weyl chambers* and their closures the closed Weyl chambers. An element γ of $X_\mathbb{R} \setminus \cup_{\alpha \in \triangle} \Omega_\alpha$ is called *regular* and belongs to exactly one Weyl chamber. The Weyl chamber that consists of the elements $\gamma \in X_\mathbb{R}$ such that $(\gamma \mid \alpha)$ is strictly positive for every simple root α, is denoted by $\mathcal{C}(\triangle)$. It is called the (open) *dominant* Weyl chamber (relative to $\triangle$). We will denote the closure of the dominant Weyl chamber by $X^+_\mathbb{R}$. Note that it is equal to $X^+ \otimes_\mathbb{Z} \mathbb{R}$.

Let $\triangle = \{\alpha_1, \cdots, \alpha_l\}$ be the set of simple roots of $\mathfrak{g}$. The *fundamental weights* $\{\omega_1, \cdots, \omega_l\}$ of $\mathfrak{g}$ (relative to $\triangle$) are defined by $\langle \omega_i \mid \alpha_j \rangle = \delta_{i,j}$. In terms of fundamental weights, the set of dominant weights is described by $\{\sum k_i \omega_i \mid k_i \in \mathbb{N}\}$.

Lemma 2.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Let l, $m \in \mathfrak{g}^*$, $\{X_i\}$ be a basis of $\mathfrak{g}$ and $\{Y_i\}$ its dual relative to the Killing form, i.e. $\kappa(X_i, Y_j) = \delta_{i,j}$. Then one can show that the following holds:*

$$(l \mid m) = \sum_i l(X_i) m(Y_i).$$

In Lemma 2.6, Lemma 2.7, Lemma 2.8 and Lemma 2.9 we list the properties of $\mathfrak{g}$, its dual $\mathfrak{g}^*$ and of their sub-algebras that will be used in the sequel:

Lemma 2.6. *The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.*

Lemma 2.7. *The Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ of K and $\mathfrak{t} \subset \mathfrak{h}$ of T_K are real subspaces. Let $\mathfrak{k}^* := \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{k}) \subset \mathbb{R}\}$ and $\mathfrak{t}^* := \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{t}) \subset \mathbb{R}\}$. Then $\mathfrak{k}^*$ is a real subspace of $\mathfrak{g}^*$. The map $\lambda \mapsto \lambda|_\mathfrak{k}$ gives a canonical isomorphism of $\mathfrak{k}^*$ with the $\mathbb{R}$–dual of $\mathfrak{k}$. Furthermore, $\mathfrak{g}^*$ decomposes as $\mathfrak{g}^* = \mathfrak{k}^* \oplus i\mathfrak{k}^*$. Similar assertions hold for $\mathfrak{t}^* \subset \mathfrak{g}^*$, hence $\mathfrak{h}^* = \mathfrak{t}^* \oplus i\mathfrak{t}^*$.*

Lemma 2.8. *For every root α in Φ, the subspace $\mathfrak{g}^*_\alpha$ is orthogonal to $\mathfrak{h}^*$ (with respect to the Killing form).*

Proof. Recall that the isomorphism $\iota : \mathfrak{g} \to \mathfrak{g}^*$ given by $X \mapsto \kappa(X, \cdot)$ maps $\mathfrak{g}_{-\alpha}$ to $\mathfrak{g}^*_\alpha$. Let $p : \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ be the projection induced by the restriction $l|_\mathfrak{h} : \mathfrak{h} \to \mathbb{C}$ of elements $l \in \mathfrak{g}^*$ to $\mathfrak{h}$.

Since $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate it follows that p induces an isomorphism $\iota(\mathfrak{h}) \xrightarrow{\sim} \mathfrak{h}^*$. This allows us to identify $\mathfrak{h}^*$ with the subspace $\iota(\mathfrak{h}) \subset \mathfrak{g}^*$. Then, for $X_l := \iota^{-1}(l) \in \mathfrak{h}$ and $X_m := \iota^{-1}(m) \in \mathfrak{g}_{-\alpha}$ we have $(l \mid m) = \kappa(X_l, X_m)$ which is zero by Lemma 2.1. ∎

Lemma 2.9. *For arbitrary $\alpha \in X_{\mathbb{R}}$, $\alpha(X)$ is purely imaginary for every $X \in \mathfrak{t}$. In other words:*
$\alpha \in i\mathfrak{t}^*$ and hence $X_{\mathbb{R}} \subset i\mathfrak{t}^*$.

Proof. Every root in $X_{\mathbb{R}}$ takes purely imaginary values on $\mathfrak{t}$ (see Fulton, Harris [FH96], Proposition 26.4) and thus the claim follows since every element of $X_{\mathbb{R}}$ is a real combination of roots. □

2.2 The Casimir Operator

Let $\{X_i\}$ be a base of $\mathfrak{g}$ and $\{Y_i\}$ its dual relative to the Killing form (i.e. $\kappa(X_i, Y_j) = \delta_{i,j}$). The universal *Casimir element* of $\mathfrak{g}$ is defined as $C_{\mathfrak{g}} := \sum X_i Y_i$. It is an element of the universal envelopping algebra $\mathcal{U}(\mathfrak{g})$. $C_{\mathfrak{g}}$ acts as a linear operator on every $\mathfrak{g}$–module V, $C_{\mathfrak{g}}(v) = \sum X_i(Y_i(v))$. We recall some properties of the Casimir operator (to be found e.g. in Humphreys [Hu94], §22.1 and Goodman, Wallach [GW98], 7.3.1):
The Casimir operator is independent of the choice of the basis of $\mathfrak{g}$. Furthermore, $C_{\mathfrak{g}}$ commutes with the action of $\mathfrak{g}$. Hence $C_{\mathfrak{g}}$ acts as scalar multiplication on irreducible representations. Denote by ρ half the sum of the positive roots of $\mathfrak{g}$. Then the scalar by which the Casimir operator is acting on irreducible representations is the following (see Bourbaki [Bou75], VIII, §6.4 Corollaire).

Proposition 2.10. *The Casimir element $C_{\mathfrak{g}}$ acts on the irreducible representation V_μ of highest weight μ as multiplication by $(\mu \mid \mu) + 2(\mu \mid \rho)$.*

2.3 A Moment Map

Let V be a finite dimensional G–module. On V we can choose a K–invariant hermitian scalar product $\langle . , . \rangle$ (cf. Vinberg [Vi89], I.2 Theorem 2), which is $\mathbb{C}$–linear in the second argument. For $v \in V$ let $\|v\| := \sqrt{\langle v, v\rangle}$. If $V = V_\lambda$ we write $\langle . , . \rangle_\lambda$ for the corresponding scalar product and $\|v\|_\lambda := \sqrt{\langle v, v\rangle_\lambda}$

Lemma 2.11. *Let V be finite dimensional, $\langle \cdot, \cdot \rangle$ a K–invariant hermitian scalar product on V. Then the following holds:*
(i) For arbitrary $X \in \mathfrak{k}$, $v \in V$, $\langle v, Xv\rangle$ is purely imaginary.
(ii) Let V_α and V_β be weight spaces with $\alpha \neq \beta$. Then: $V_\alpha \perp V_\beta$.

Proof. Part (i): The K–invariance of the scalar product yields $\langle v, Xv\rangle + \langle Xv, v\rangle = 0$ for all $X \in \mathfrak{k}$, $v \in V$. Hence $\langle v, Xv\rangle$ equals $-\overline{\langle v, Xv\rangle}$.

Part (ii): Take $v \in V_\alpha$, $w \in V_\beta$ and $X \in \mathfrak{t}$ arbitrary. Then $\overline{\alpha(X)} = -\alpha(X)$. (by Lemma 2.9). We use the K–invariance of the scalar product:

$$\begin{aligned} 0 &= \langle Xv, w\rangle + \langle v, Xw\rangle \\ &= \langle \alpha(X)v, w\rangle + \langle v, \beta(X)w\rangle \\ &= (\overline{\alpha(X)} + \beta(X))\langle v, w\rangle \\ &= (-\alpha(X) + \beta(X))\langle v, w\rangle \end{aligned}$$

Since $\alpha \neq \beta$ there exists $X \in \mathfrak{t}$ such that $\alpha(X) \neq \beta(X)$. Thus $\langle v, w\rangle$ needs to be zero. □

Using the K–invariant scalar product we can define the *moment map* M from V to $\mathfrak{g}^*$:

$$M : V \to \mathfrak{g}^*; \quad M(v)(X) := \langle v, Xv\rangle.$$

Lemma 2.12. *For $v \in V$ the moment map $M(v)$ of v is an element of $i\mathfrak{k}^*$.*

Proof. We have seen in Lemma 2.11 (i) that $M(v)(X)$ is purely imaginary for every $X \in \mathfrak{k}$. Hence $M(v)(\mathfrak{k}) \subset i\mathbb{R}$. □

Remark. The moment map is usually defined as

$$\tilde{M}(v)(X) := \frac{1}{2\pi i}\langle v, Xv\rangle \text{ for } X \in \mathfrak{k},$$

see e.g. Brion [Br87] 2.2.

Moment maps have several interesting properties which we will not use here. Essentially, we only use the definition.

Lemma 2.13. *(i) The moment map is K–equivariant.*

(ii) For every $v \in V$ there exists $g \in K$ such that $M(gv) \in i\mathfrak{t}^* \subset \mathfrak{h}^*$.

(iii) For any weight vector $v \in V$ and $X \in \mathfrak{g}$ arbitrary, we have

$$\langle v, Xv\rangle = \langle v, p(X)v\rangle$$

where $p : \mathfrak{g} \to \mathfrak{h}$ is the projection onto $\mathfrak{h}$.

Proof. (i). The assertion follows from the K–invariance of the scalar product: $M(gv)(X) = M(v)(\operatorname{Ad} g^{-1}(X)) = (g \cdot M(v))(X)$, i.e. $M(gv) = g \cdot M(v)$ for all $g \in K$.

(ii). Every element of $\mathfrak{k}$ is semisimple so each of them is conjugated to some element of $\mathfrak{t}$ (compact groups cannot contain any unipotent elements). Similarly, every element of $i\mathfrak{k}^*$ is conjugated to some element of $i\mathfrak{t}^*$.

(iii). Write $X = X_\mathfrak{h} + X_{\mathfrak{h}^\perp}$ with $X_\mathfrak{h} \in \mathfrak{h}$, $X_{\mathfrak{h}^\perp} \in \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$. Then $Xv = p(X)v + X_{\mathfrak{h}^\perp}v$ where the first term is of the same weight as v. The second term consists of components of weights different from $\operatorname{wt} v$. By Lemma 2.11 (ii), they are all perpendicular to v, so $\langle v, X_{\mathfrak{h}^\perp}v\rangle = 0$. □

Lemma 2.14. *Let* $v = \sum v_i$ *be the decomposition of* $v \in V$ *into weight vectors,* $\mathrm{wt}\, v_i = \lambda_i \in \Pi(V)$*. Then:*

$$M(v)|_{\mathfrak{h}} = \sum \|v_i\|^2 \lambda_i$$

where both sides are considered as elements of $i\mathfrak{t}^*$*.*

Proof. By the definition of the moment map,

$$\begin{aligned} M(v)(X) &= \langle \sum v_i, X \sum v_j \rangle \\ &= \langle \sum v_i, \sum \lambda_j(X) v_j \rangle \\ &= \sum_j \lambda_j(X) \langle \sum_i v_i, v_j \rangle \end{aligned}$$

By Lemma 2.11 (ii), weight spaces are perpendicular, so

$$\begin{aligned} \sum_j \lambda_j(X) \langle \sum_i v_i, v_j \rangle &= \sum_j \lambda_j(X) \langle v_j, v_j \rangle \\ &= \sum_j ||v_j||^2 \lambda_j(X) \end{aligned}$$

□

3 Decomposable Tensors in the Cartan Component

3.1 Motivation and First Examples

In section 3 of the first part of this thesis we were studying orbits in the space $V_n = \mathbb{C}[x,y]_n$ of binary forms of given degree. In order to understand if the closure of the orbit $O_{xy^{n-1}}$ has the separation property we used the morphism $\varphi\colon V_1 \times V_1 \to V_n$ given by $(a,b) \mapsto ab^{n-1}$ (see proof of Theorem 3.4).

In an early approach to the separation property we were studying orbits of elements $x^k y^{n-k}$ in V_n. Let $k < \frac{n}{2}$. Similarly as in section 3 (part I), define the map $\varphi_k\colon V_1 \times V_1 \to V_n$ by $(a,b) \mapsto a^k b^{n-k}$. For $k < \frac{n}{2}$ its image is the closure of the orbit $O_{x^k y^{n-k}}$. The comorphism $\varphi_k^*\colon \mathcal{O}(V_n) \to \mathcal{O}(V_1 \times V_1)$ maps regular functions of degree one to regular functions of bidegree $(k, n-k)$,

$$\varphi_k^*\colon \mathcal{O}(V_n)_1 \to \mathcal{O}(V_n)_k \otimes \mathcal{O}(V_n)_{n-k}.$$

We use the correspondence $\mathcal{O}(V_1)_l \cong V_l$. As we have seen in the proof of Theorem 3.4 in part I, φ^* embeds V_n in the tensor product $V_k \otimes V_{n-k}$ of irreducible SL_2–representations. This explains why we are interested in the component V_n of $V_k \otimes V_{n-k} \cong V_n \oplus V_{n-2} \oplus \cdots \oplus V_{n-2k}$ (with $2k < n$).

This example is a special case of a more general situation: Consider a tensor product of two irreducible representations of a reductive group G and its irreducible components. In general, such a tensor product is not irreducible. It can be decomposed into irreducible components,

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \mathfrak{h}^+} N_\nu V_\nu,$$

where the multiplicities $N_\nu = N_\nu(\lambda,\mu)$ are the Littlewood–Richardson coefficients. They can be calculated combinatorially, see for instance the prove of the Littlewood-Richardson rule given in the paper [Li90],§2.2 and §4, of Peter Littelmann. However, it is a fundamental question how these components are embedded in the tensor product $V_\lambda \otimes V_\mu$.

We recall the notion of decomposable tensors:

Definition 3.1. Let U and V be vector spaces over $\mathbb{C}$. If an element of $U \otimes V$ is of the form $u \otimes v$ we say that it is a *decomposable tensor*. The *rank* of a tensor $\sum_{i,j} a_{i,j} u_i \otimes v_j$ in $U \otimes V$ is defined as the minimal number of decomposable tensors needed to write it as their sum. If $V_\lambda \otimes V_\mu$ is a tensor product of irreducible G–modules we denote the set of decomposable tensors of $V_\lambda \otimes V_\mu$ by $\mathrm{Dec}(\lambda,\mu)$.

In this context the following two problems are arising:

(i) How many irreducible components of the tensor product meet a given tensor?

(ii) Describe the set of decomposable tensors in the component $V_{\lambda+\mu}$ of the tensor product. It turns out that it is rather difficult to find answers to these straightforward questions.

We recall a result about the weights of the tensor product $V_\lambda \otimes V_\mu$. It can be found in [Kr85], III.1.5.

Proposition 3.2. *Let G be a connected linear reductive group. Let U and V be G–modules.*

(i) The weights of $U \otimes V$ are of the form $\nu_1 + \nu_2$, with $\nu_1 \in \Pi(U)$ and $\nu_2 \in \Pi(V)$.

(ii) If $U = V_\lambda$ and $V = V_\mu$ are irreducible then $\lambda + \mu$ is a highest weight of $V_\lambda \otimes V_\mu$ and its multiplicity in the tensor product is one.

An interesting component of the decomposition is the irreducible representation $V_{\lambda+\mu}$. It is the component with the maximal possible weight. By Proposition 3.2 (ii) it appears exactly once. The component $V_{\lambda+\mu}$ is called the *Cartan component* of the tensor product $V_\lambda \otimes V_\mu$.

We can give an answer to a special aspect of problem (i), concerning the set of decomposable tensors. It is an interesting and rather surprising fact that if a decomposable tensor lies in one of the irreducible components this component must be the Cartan component $V_{\lambda+\mu}$.

Theorem 3.3. *Let $v \otimes w \in V_\lambda \otimes V_\mu$ be a decomposable tensor. Then there are two possibilities:*

(i) The tensor $v \otimes w$ is an element of the Cartan component $V_{\lambda+\mu}$

(ii) The tensor $v \otimes w$ belongs to more than one irreducible component of the tensor product.

In other words, there is no irreducible component besides the Cartan component that contains decomposable tensors.

Proof. Suppose that $v \otimes w$ is a non-zero tensor lying in one of the irreducible components of $V_\lambda \otimes V_\mu$, say in V_ν. Let $u \in V_\nu$ be a highest weight vector, hence $V_\nu = \langle Gu \rangle$.

Step (1): We show that the highest weight vector of V_ν is decomposable: Note that the closure $\overline{Gu}$ is contained in the closure of every non-zero orbit in V_ν. Since the cone $\mathbb{C}^*(v \otimes w)$ is not zero, the closure of its G–orbit contains Gu. In particular, the highest weight vector u lies in $\overline{G\mathbb{C}^*(v \otimes w)}$. Therefore u is also decomposable, say $u = v_0 \otimes w_0$.

Step (2): Show that V_ν is the Cartan component of the tensor product: Suppose that $\nu \precneqq \lambda + \mu$. Note that $\nu = \mathrm{wt}\, v_0 + \mathrm{wt}\, w_0$ and that v_0 and w_0 are weight vectors of V_λ respectively of V_μ. W.l.o.g. let $\mathrm{wt}\, v_0 \precneqq \lambda$. Recall that if α is a positive root, then highest weight vectors are killed by the root space $\mathfrak{g}_\alpha$, see Bourbaki [Bou75], VIII, §6.1 Lemme 1. Since the weight of v_0 is smaller than λ the vector v_0 is not a highest weight vector of V_λ. In particular, there exists a positive root α such that $\mathfrak{g}_\alpha$ does not send v_0 to zero. Hence for every $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha(v_0 \otimes w_0) = X_\alpha(v_0) \otimes w_0 + v_0 \otimes X_\alpha(w_0)$ is not zero.

On the other hand the tensor $v_0 \otimes w_0$ is a highest weight vector of V_ν. Therefore it is killed by every X_α. Hence ν must be the weight $\lambda + \mu$ □

In the remaining part of this work we present different methods to solve problem (ii). Note that the orbit $G(v_\lambda \otimes v_\mu)$ consists of decomposable tensors in the Cartan component $V_{\lambda+\mu}$. However, it is not easy to see whether there are any other decomposable tensors in $V_{\lambda+\mu}$.

Recall that that a subset X of a vector space is said to be a *cone* if for every $x \in X$, the line $\mathbb{C}^* x$ is a subset of X. The following result can be found in Kraft [Kr85], III.3.5.

Theorem 3.4. *Let G be a reductive group and M a simple non-trivial G–module, $m \in M$ a highest weight vector. Then, the closure $\overline{Gm}$ is a cone and $\overline{Gm} = Gm \cup \{0\}$.*

Hence the closure of $G(v_\lambda \otimes v_\mu)$ is exactly $G(v_\lambda \otimes v_\mu) \cup \{0\}$.

We modify problem (ii) and ask the following:

Question 3.5. *For which dominant weights λ and μ, the set of decomposable tensors in the Cartan component of $V_\lambda \otimes V_\mu$ equals the closure of $G(v_\lambda \otimes v_\mu)$?*

Definition 3.6. We say that a tensor product $V_\lambda \otimes V_\mu$ of irreducible representations has a *small Cartan component* if the the set $\mathrm{Dec}(\lambda, \mu) \cap V_{\lambda+\mu}$ of decomposable tensors in the Cartan component equals the closure of the orbit $G(v_\lambda \otimes v_\mu)$.

Example 3.7. In general, if one of the highest weights is zero, the Cartan component of the corresponding tensor product is not small

Proof. For $\lambda = 0$ the representation V_λ is the trivial representation, hence $V_\lambda \otimes V_\mu = \mathbb{C} \otimes V_\mu = V_\mu$ which is irreducible. Its Cartan component is V_μ itself and all tensors in $\mathbb{C} \otimes V_\mu$ are decomposable.

Therefore the Cartan component of $\mathbb{C} \otimes V_\mu$ is small if and only if the closure of Gv_μ is all of V_μ. Note that there are only a few representations of the classical groups where the orbit of a highest weight vector is dense (cf. table 5.2). Hence, in general, the Cartan component of $\mathbb{C} \otimes V_\mu$ is not small. □

We start by looking at representations of SL_2 on the binary forms $V_n := \mathbb{C}[x,y]_n$ of degree n. We use the so-called Clebsch-Gordan decomposition (see section 2 in the first part of this thesis) to decompose the tensor product:

$$V_n \otimes V_m = \bigoplus_{i=0}^{\min(n,m)} V_{n+m-2i}.$$

Example 3.8. Representations of SL_2
Every tensor product of irreducible representations of SL_2 has a small Cartan component.

Proof. The assertion follows immediately from the result of Theorem 4.18 below. There is also a different approach which uses the method of associated cones as we will see in section 7. □

The first non-trivial example where the Cartan component of a tensor product of irreducible representations is not small is the SL_3–module $S^2\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ (cf. Kraft, Wallach [KW98]). The tensor $e_1^2 + e_2^2 \otimes e_3^*$, e.g., is an element of the Cartan component $\langle \mathrm{SL}_3(e_1^2 \otimes e_3^*)\rangle$, but it does not belong to the closure of the orbit $\mathrm{SL}_3(e_1^2 \otimes e_3^*)$. Another way to show that the Cartan component cannot be small, is to compare the dimensions of the sets (see Proposition 3.9 and Example 3.10). Furthermore, we will give a necessary criterion for small Cartan components in Corollary 5.9 ("dense orbits criterion"). For a picture of the weights of the representations $S^2\mathbb{C}^3$ and $(\mathbb{C}^3)^*$, see Example 4.15.

Proposition 3.9. *Let $V_\lambda \otimes V_\mu$ be a tensor product for with* $\dim V_\lambda + \dim V_\mu - 1 - \dim V_\lambda \dim V_\mu + \dim V_{\lambda+\mu} > \dim G - \dim B + 1$. *Then its Cartan component is not small.*

In particular, if $G = \mathrm{SL}_3$ and $\dim V_\lambda + \dim V_\mu - 1 - \dim V_\lambda \dim V_\mu + \dim V_{\lambda+\mu}$ is at least five then the Cartan component of the tensor product is not small since $\dim \mathrm{SL}_2 - \dim B = 3$.

Proof. We show that in this case the intersection of the decomposable tensors $\mathrm{Dec}(\lambda, \mu)$ of $V_\lambda \otimes V_\mu$ with the Cartan component $V_{\lambda+\mu}$ cannot be contained in the closure of the G–orbit of $v_\lambda \otimes v_\mu$. Let $l := \dim V_\lambda$ and $m := \dim V_\mu$.

Step (1). Note that the set of decomposable tensors of $V_\lambda \otimes V_\mu$ is an algebraic subset of dimension $l+m-1$. Since the Cartan component has codimension $l \cdot m - \dim V_{\lambda+\mu}$ in the tensor product we get

$$\dim(\mathrm{Dec}(\lambda,\mu) \cap V_{\lambda+\mu}) \geq (l+m-1) - (l \cdot m - \dim V_{\lambda+\mu}).$$

Step (2). The stabiliser $\mathrm{Stab}_G(v_\lambda \otimes v_\mu)$ contains the unipotent part U of B and the torus $T' := \mathrm{Ker}(\lambda+\mu)$ lying in T. Since T' has codimension one in T we have $\dim Stab_{v_\lambda \otimes v_\mu} \geq \dim B - 1$. Therefore $\dim G(v_\lambda \otimes v_\mu) \leq \dim G - \dim B + 1$.

Combining steps (1) and (2) yields the assertion. □

Recall that for $G = \mathrm{SL}_3$ the fundamental weights are $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$ (see section 8 for more details).

Example 3.10. The Cartan component of the tensor product

$$V_{b\omega_1} \otimes V_{\omega_2}$$

of SL_3–representations is not small whenever $b \geq 2$.

In particular, the Cartan component of $S^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ is not small.

Proof. Use the dimension formula for irreducible representations (cf. Bourbaki [Bou75], VIII, §9.2 Théorème 2) to see that the representation $V_{a\omega_1+b\omega_2}$ has dimension $\frac{(a+1)(b+1)(a+b+2)}{2}$. Hence

$$\begin{aligned} \dim V_{\omega_1} + \dim V_{b\omega_2} + \dim V_{\omega_1+b\omega_2} & \\ - \dim\ V_{\omega_1} \cdot \dim V_{b\omega_2} &= 3 + (b+1)(b+3) - (b+1)(b+2) \\ &= b+4 \\ &> 5 \qquad \text{for every } b \geq 2 \end{aligned}$$

□

We will see in section 4, Example 4.15 that as soon as $b \geq 2$ there are too many weights of $V_{b\omega_1}$ on the α_1–string through the highest weight $b\omega_1$, namely there are at least three (i.e. $b+1$) weights of $V_{b\omega_1}$.

3.2 Measuring Decomposable Tensors of Cartan Components

Following an idea communicated by Bert Kostant [Ko98], we will use the Casimir operator to measure decomposable tensors of the Cartan component: Let $v \neq 0 \in V_\lambda$, $\Pi(\lambda) = \{\lambda_i\}_{i\in I}$. To v we associate a point $P_\lambda(v) \in X_\mathbb{R}$ by

$$P_\lambda : v \mapsto \sum_{i\in I} \frac{\|v_i\|_\lambda^2}{\|v\|_\lambda^2}\lambda_i$$

where $v = \sum_{i\in I} v_i$ is the decomposition of v into weight vectors, $\mathrm{wt}\, v_i = \lambda_i$ and $\|v\|_\lambda = \sqrt{\langle v, v\rangle_\lambda}$ as in section 2.3.

Recall the symmetric bilinear form $(\cdot \mid \cdot)$ on $X_\mathbb{R}$ that was introduced in Subsection 2.1. We will see that in the K–orbit of every non-zero decomposable tensor of the Cartan component $V_{\lambda+\mu}$ lies a tensor $v \otimes w$ such that $(P_\lambda(v) \mid P_\mu(w)) = (\lambda, \mu)$.

Let $v \otimes w$ be a decomposable tensor lying in the Cartan component $V_{\lambda+\mu}$ of $V_\lambda \otimes V_\mu$. Recall the Casimir operator $C_\mathfrak{g} = \sum X_i Y_i$ where $\{X_i\}$ is a basis of $\mathfrak{g}$ and $\{Y_i\}$ is a dual basis with respect to the Killing form. By Proposition 2.10 the Casimir operator acts on $v \otimes w$ as multiplication by the scalar $\|\lambda + \mu + \rho\|^2 - \|\rho\|^2$ (where $\|\nu\| := \sqrt{(\nu \mid \nu)}$), so

$$C_\mathfrak{g}(v \otimes w) = (\|\lambda\|^2 + \|\mu\|^2 + 2(\lambda \mid \mu) + 2(\lambda \mid \rho) + 2(\mu \mid \rho))v \otimes w.$$

If $v \otimes w$ is a general tensor of $V_\lambda \otimes V_\mu$, $C_\mathfrak{g}$ acts in the following way:

$$\begin{aligned} C_\mathfrak{g}(v \otimes w) &= \textstyle\sum_i X_i\,.(Y_i v \otimes w + v \otimes Y_i w) \\ &= \textstyle\sum_i (X_i Y_i v \otimes w + Y_i v \otimes X_i w + X_i v \otimes Y_i w + v \otimes X_i Y_i w) \\ &= C_\mathfrak{g} v \otimes w + v \otimes C_\mathfrak{g} w + \textstyle\sum_i (Y_i v \otimes X_i w + X_i v \otimes Y_i w). \end{aligned}$$

The operators $C_\mathfrak{g} \otimes 1$ resp. $1 \otimes C_\mathfrak{g}$ have eigenvalues $\|\lambda\|^2 + 2(\lambda \mid \rho)$ resp. $\|\mu\|^2 + 2(\mu \mid \rho)$ on $V_\lambda \otimes V_\mu$. Thus we obtain the following:

Remark 3.11. Let $v \otimes w$ be a decomposable tensor lying in the Cartan component $V_{\lambda+\mu}$ of $V_\lambda \otimes V_\mu$. Then the following holds:

$$2(\lambda \mid \mu) v \otimes w = \sum_i Y_i v \otimes X_i w + \sum_i X_i v \otimes Y_i w. \tag{3}$$

Denote by $\{\lambda_i\}$ the set $\Pi(\lambda)$ of weights of V_λ and let $\Pi(\mu) = \{\mu_j\}$, take $v \in V_\lambda$ and $w \in V_\mu$ arbitrary. We write $v = \sum v_i$ (resp. $w = \sum w_j$) for the decomposition of v (resp. of w) into weight vectors, $\mathrm{wt}\, v_i = \lambda_i$, resp. $\mathrm{wt}\, w_j = \mu_j$.

Theorem 3.12. *Let $v \otimes w \neq 0$ be a tensor of the Cartan component $V_{\lambda+\mu}$. Then there exists $g_0 \in K$ such that for $\overline{v} := g_0 v$ and $\overline{w} := g_0 w$ the following holds:*

$$(P_\lambda(\overline{v}) \mid P_\mu(\overline{w})) = (\lambda \mid \mu).$$

Proof. We proceed in several steps:
(1) Show that $(\lambda \mid \mu) = (\frac{M(v)}{\|v\|_\lambda^2} \mid \frac{M(w)}{\|w\|_\mu^2})$.
(2) There exists some $g_0 \in K$ with $M(g_0 v) \in \mathfrak{h}^*$.
(3) If $M(v) \in \mathfrak{h}^*$ then

$$(M(v) \mid M(w)) = (\sum \|v_i\|_\lambda^2 \lambda_i \mid \sum \|w_j\|_\mu^2 \mu_j).$$

Part (1): Denote by $\langle \cdot, \cdot \rangle$ the induced scalar product on $V_\lambda \otimes V_\mu$ (if v and v' are vectors in V_λ, w and $w' \in V_\mu$, then $\langle v \otimes w, v' \otimes w' \rangle := \langle v, v' \rangle_\lambda \cdot \langle w, w' \rangle_\mu$). In equation (3), we take the scalar product with $\frac{v \otimes w}{\|v\|_\lambda^2 \|w\|_\mu^2}$ from the left and we get:

$$\begin{aligned}
&2(\lambda \mid \mu) \\
= \;& \langle \frac{v \otimes w}{\|v\|_\lambda^2 \|w\|_\mu^2}, \sum_k Y_k v \otimes X_k w + X_k v \otimes Y_k w \rangle \\
= \;& \sum_k \langle \frac{v \otimes w}{\|v\|_\lambda^2 \|w\|_\mu^2}, Y_k v \otimes X_k w \rangle + \sum_k \langle \frac{v \otimes w}{\|v\|_\lambda^2 \|w\|_\mu^2}, X_k v \otimes Y_k w \rangle \\
= \;& (\sum_k \langle v, Y_k v \rangle_\lambda \langle w, X_k w \rangle_\mu + \sum_k \langle v, X_k v \rangle_\lambda \langle w, Y_k w \rangle_\mu) \frac{1}{\|v\|_\lambda^2 \|w\|_\mu^2} \\
= \;& (\sum_k M(v)(Y_k) M(w)(X_k) + \sum_k M(v)(X_k) M(w)(Y_k)) \frac{1}{\|v\|_\lambda^2 \|w\|_\mu^2} \\
= \;& 2 \frac{(M(v) \mid M(w))}{\|v\|_\lambda^2 \|w\|_\mu^2} \qquad \text{(Lemma 2.5)}.
\end{aligned}$$

Part (2): By Lemma 2.12 we know that $M(v)$ is an element of $i\mathfrak{k}^*$. So by Lemma 2.13 (ii) there exists some $g_0 \in K$ with $M(g_0 v) \in i\mathfrak{t}^*$.
Note that $(M(g_0 v) \mid M(g_0 w)) = (g_0 M(v) \mid g_0 M(w)) = (M(v) \mid M(w))$ (follows from the K–equivariance of M, c.f. Lemma 2.13 (i)).
Part (3): Assume that $M(v) \in \mathfrak{h}^*$. By Lemma 2.8 each root subspace $\mathfrak{g}_\alpha^*$ is orthogonal to $\mathfrak{h}^*$ with respect to $(\cdot \mid \cdot)$. Recall the projection from $\mathfrak{g}^*$ to $\mathfrak{h}^*$ induced by the restriction $l|_\mathfrak{h} \colon \mathfrak{h} \to \mathbb{C}$ of elements $l \in \mathfrak{g}^*$ to $\mathfrak{h}$. If l is an element of $\mathfrak{h}^*$, the orthogonality of $\mathfrak{h}^*$ to each $\mathfrak{g}_\alpha^*$ yields $(l \mid M(w)) = (l \mid M|_\mathfrak{h}(w))$. Since $M(v)$ is an element of $i\mathfrak{t}^* \subset \mathfrak{h}^*$ it follows that

$$(M(v) \mid M(w)) = (M(v) \mid M|_\mathfrak{h}(w)).$$

We now use Lemma 2.14 to complete the proof: Let $v = \sum v_i$ and $w = \sum w_j$ be the decomposition of v resp. of w into weight vectors ($v_i = \lambda_i$ and $\operatorname{wt} w_j = \mu_j$).

$$\begin{aligned}(M(v) \mid M|_{\mathfrak{h}}(w)) &= (\sum \|v_i\|^2 \lambda_i \mid M|_{\mathfrak{h}}(w)) \\ &= (\sum \|v_i\|_\lambda^2 \lambda_i \mid \sum \|w_j\|_\mu^2 \mu_j).\end{aligned}$$

□

4 Convex Hulls of Weights and Small Cartan Components

After the first examples of tensor product appeared for which the Cartan component is not small it was thought that only very special tensor products could have a small Cartan component, possibly just the cases where the highest weights are nonzero multiples of each other (cf. Example 4.14). However, we will show in this section that generic tensor product of irreducible representations have small Cartan components.

4.1 Convex Hulls of Weights

For a dominant weight λ let $\Pi(\lambda) = \{\lambda_i\}_{i\in I}$ be the set of weights of the irreducible representation V_λ.

Definition 4.1. Denote by $\mathrm{Con}(\lambda) := \{\sum_{i\in I} a_i\lambda_i \mid a_i \geq 0, \sum a_i = 1\} \subset X_\mathbb{R}$ the convex hull of the weights of V_λ. We will call $\mathrm{Con}(\lambda)$ the *convex hull of* λ.

Note that $\Pi(\lambda)$ is $\mathcal{W}$–stable. It consists of all dominant weights $\prec \lambda$ and of their Weyl-orbits, cf. §13.4 and §21.3 in Humphreys [Hu94]. Thus we have the following:

Lemma 4.2. *The convex hull of λ forms a polyeder which is equal to the convex hull of the Weyl group orbit $\mathcal{W}\lambda$ of λ: Let $\mathcal{W}\lambda = \{\lambda_i\}_{i\in I_0}$. Then* $\mathrm{Con}(\lambda) = \{\sum_{i\in I_0} a_i\lambda_i \mid a_i \geq 0, \sum a_i = 1\}$.

To any non-zero vector $v = \sum_{i\in I} v_i \in V_\lambda$ where $v = \sum_{i\in I}$ is the decomposition of v into weight vectors we have associated a point in the convex hull of λ by the following map (cf. section 3.2):

Definition 4.3. Define $P_\lambda : V_\lambda \setminus \{0\} \to \mathrm{Con}(\lambda)$ by

$$P_\lambda : v \mapsto \sum_{i\in I} \frac{\|v_i\|_\lambda^2}{\|v\|_\lambda^2}\lambda_i.$$

We call $P_\lambda(v) \in \mathrm{Con}(\lambda)$ the point *associated to the vector v of V_λ*.

Note that the map P_λ is not injective: If for instance, v is any non-zero vector of V_λ, then it is clear that $P_\lambda(av) = P_\lambda(v)$ for every $a \in \mathbb{C}^*$. In other words, $P_\lambda^{-1}(P_\lambda(v)) \supset \mathbb{C}^*v$ for every non-zero v.

Lemma 4.4. *Let $a \in \mathrm{Con}(\lambda)$ be a point in the convex hull of λ. Let $\triangle = \{a_l\}_{l\in L}$ be the simple roots. Then there exist nonnegative coefficients r_l such that*

$$a = \lambda - \sum_{l\in L} r_l\alpha_l.$$

Proof. Let $a = \sum_{i\in I} a_i\lambda_i \in \mathrm{Con}(\lambda)$ with $a_i \geq 0$ for every $i \in I$ and $\sum_{i\in I} a_i = 1$. Recall that $\lambda_i = \lambda - \sum_{l\in L} k_{i_l}\alpha_{i_l}$ with $\alpha_{i_l} \in \triangle$ and $k_{i_l} \in \mathbb{N}$. Thus

$$\begin{aligned} a &= \sum_{i\in I} a_i(\lambda - \sum_{l\in L} k_{i_l}\alpha_{i_l}) \\ &= \lambda - \sum_{i\in I, l\in L} a_i k_{i_l}\alpha_{i_l} \end{aligned}$$

and hence the assertion follows (all coefficients are non-negative). □

Recall the $\mathfrak{g}$–invariant scalar product $(\cdot \mid \cdot)$ on $X_{\mathbb{R}}$ introduced in Subsection 2.1.

Lemma 4.5. $\mathrm{Max}\{(a \mid b) \mid a \in \mathrm{Con}(\lambda), b \in \mathrm{Con}(\mu)\} = (\lambda \mid \mu)$.

Proof. Take $a \in \mathrm{Con}(\lambda)$ and $b \in \mathrm{Con}(\mu)$ with $(a \mid b)$ maximal. Recall that $(a \mid b)$ is $\mathcal{W}$–invariant (cf. Lemma 2.3), i.e. $(a \mid b) = (wa \mid wb)$ for every $w \in \mathcal{W}$. So w.l.o.g., we can assume that $a \in \mathcal{C}(\triangle)$. We use a generalization of an argument of Joseph in [Jo95], A.1.17.

Write $a = \lambda - \sum r_i\alpha_i$ and $b = \mu - \sum s_i\alpha_i$ with coefficients r_i, $s_i \in \mathbb{R}_{\geq 0}$ (cf. Lemma 4.4) and $\alpha_i \in \triangle$. Then the assertion follows easily:

$$\begin{aligned} (a \mid b) &= (a \mid \mu) - \sum s_i(a \mid \alpha_i) \\ &\leq (a \mid \mu) \\ &= (\lambda \mid \mu) - \sum r_i(\alpha_i \mid \mu) \\ &\leq (\lambda \mid \mu) \end{aligned}$$

where the last inequality is due to $(\alpha \mid \mu) \geq 0$ for each $\alpha \in \triangle$. □

Corollary 4.6. *Let $v_0 \otimes w_0$ be an element of the Cartan component of $V_\lambda \otimes V_\mu$. Then there exists $g_0 \in K$ such that for the tensor $v \otimes w := g_0(v_0 \otimes w_0)$ the value $(P_\lambda(v) \mid P_\mu(w))$ is maximal, i.e. $(P_\lambda(v) \mid P_\mu(w)) = \max\{(P_\lambda(\overline{v}) \mid P_\mu(\overline{w})) \mid \overline{v} \otimes \overline{w} \in V_\lambda \otimes V_\mu\}$.*

Proof. Follows from Theorem 3.12 and Lemma 4.5. □

Generalising the statements above to points in $X_\mathbb{R}$ we define the *convex hull of* P as the convex hull of the Weyl orbit of P, $\mathrm{Con}(P) := \{\sum_{i\in I_0} a_i P_i \mid a_i \geq 0, \sum a_i = 1\}$ where the Weyl orbit of P is the set $\mathcal{W}P = \{P_i\}_{i\in I_0}$.

Remark 4.7. It is easy to see that the statements of Lemma 4.4 and of Lemma 4.5 still hold if we take points P and Q in the closed dominant Weyl chamber $X_\mathbb{R}^+$ instead of the dominant weights λ and μ.

Lemma 4.8. *Let P, $Q \in X_\mathbb{R}$ be elements of the same Weyl chamber. Then*

$$(P \mid Q) \geq (P \mid \omega Q) \text{ for all } \omega \in \mathcal{W}.$$

If P is a regular point, we have:

$$(P \mid Q) > (P \mid \omega Q) \text{ for all } \omega \text{ such that } \omega Q \neq Q.$$

Proof. The first part follows from Lemma 4.5 applied to P and Q instead of λ and μ (see Remark 4.7).

For the second part we assume w.l.o.g. that P and Q are elements of $X_\mathbb{R}^+$. Since P lies in the (open) dominant Weyl chamber, $(P \mid \alpha)$ is strictly positive for every simple root α. Let $\omega \in \mathcal{W}$ such that $\omega Q \neq Q$. By Lemma 4.4 (and Remark 4.7), we can write $\omega Q = Q - \sum_i s_i \alpha_i$ with non-negative coefficients s_i which do not vanish simultaneously. Then, $(P \mid \omega Q) = (P \mid Q) - \sum s_i (P \mid \alpha_i) < (P \mid Q)$. □

The following result will be useful.

Lemma 4.9. *Let λ_0 be a dominant weight, $\Pi(\lambda_0) = \{\lambda_i\}_{i\in I}$. Let $a = \sum a_i \lambda_i = \omega\lambda_0$ be a vertex of $\mathrm{Con}(\lambda_0)$. Then, all but one coefficient of a are zero, i.e. there exists i_0 such that $a_{i_0} = 1$ and $a_i = 0$ for every $i \neq i_0$.*

Proof. Assume w.l.o.g. that $\sum_{i\in I} a_i \lambda_i = \lambda_0$ is the highest weight. We consider the scalar product of a with λ_0,

$$(\lambda_0 \mid \lambda_0) = \sum_{i\in I} a_i (\lambda_i \mid \lambda_0) = a_0 (\lambda_0 \mid \lambda_0) + \sum_{i\neq 0} a_i (\lambda_i \mid \lambda_0).$$

Part (A): We show that the scalar product $(\lambda_i \mid \lambda_0)$ of λ_0 with any weight $\lambda_i \neq \lambda_0$ is strictly smaller than $\|\lambda_0\| := (\lambda_0 \mid \lambda_0)$.
Let $i \neq 0$, let φ be the angle between λ_0 and λ_i. Note that

$$\begin{aligned}(\lambda_i \mid \lambda_0) &= \cos\varphi \|\lambda_i\| \|\lambda_0\| \\ &\leq \|\lambda_i\| \|\lambda_0\| \\ &\leq \|\lambda_0\| \|\lambda_0\|.\end{aligned}$$

Case (i). If λ_i is not a vertex, i.e. not an element of $\mathcal{W}\lambda_0$ then $\|\lambda_i\|$ is strictly smaller than $\|\lambda_0\|$.
Case (ii). If λ_i belongs to $\mathcal{W}\lambda_0$ then the angle φ is non-zero (in fact, it is not acute). Hence $\cos\varphi < 1$.
In both cases, the value $(\lambda_i \mid \lambda_0)$ is smaller than $(\lambda_0 \mid \lambda_0)$.
Part (B): We show that the only non-zero coefficient of a is a_0.
Suppose that there exists $j \neq 0$ such that $a_j \neq 0$. Then,

$$\begin{aligned}(\lambda_0 \mid \lambda_0) &= a_0(\lambda_0 \mid \lambda_0) + \sum_{i\neq 0} a_i(\lambda_i \mid \lambda_0) \\ &< a_0(\lambda_0 \mid \lambda_0) + \sum_{i\neq 0} a_i(\lambda_0 \mid \lambda_0) \qquad \text{by part (A)} \\ &= \sum_{i\in I} a_i\|\lambda_0\|^2 = \|\lambda_0\|^2\end{aligned}$$

which is a contradiction. Hence $a_0 = 1$. □

We use Lemma 4.9 above to understand the map $P_\lambda : V_\lambda \setminus \{0\}$ better. By Lemma 4.9 no vertex can be written as a linear combination of different weights. Hence the inverse image of each vertex can be described as follows.

Corollary 4.10. *Let ν be a vertex of $\Pi(\lambda)$. Then $P_\lambda^{-1}(\nu) = V_\lambda(\nu) \setminus \{0\}$.*

4.2 Maximal Pairs and Small Cartan Components

Recall that in the K–orbit of every non-zero decomposable tensor of the Cartan component of $V_\lambda \otimes V_\mu$ lies a tensor $v \otimes w$ for which the value $(P_\lambda(v) \mid P_\mu(w))$ is the maximal value obtained among decomposable tensors (cf. Theorem 3.12 and Corollary 4.6). This observation explains why we often use the following composition of maps as a dictionary between the tensor products and the product of their convex hulls of weights.

$$\begin{array}{ccccc}\mathrm{Dec}(\lambda,\mu)\setminus\{0\} & \xrightarrow{P_\lambda\times P_\mu} & \mathrm{Con}(\lambda)\times\mathrm{Con}(\mu) & \xrightarrow{(\cdot|\cdot)} & [0,(\lambda\mid\mu)] \\ v\otimes w & \mapsto & (P_\lambda(v),P_\mu(w)) & \mapsto & (P_\lambda(v)\mid P_\mu(w))\end{array}$$

In particular, we study the inverse image of the value $(\lambda \mid \mu)$. Note that the composition of the maps is not injective. The inverse image of $(\lambda \mid \mu)$ contains at least $\mathbb{C}^*(v_\lambda \otimes v_\mu)$.

On one hand we use this composition of maps to show that a given tensor product of irreducible representations has a small Cartan component: Suppose that we can show that all tensors $v \otimes w$ mapping to $(\lambda \mid \mu)$ *behave well*, i.e. that they belong to the G–orbit of the highest weight vector $v_\lambda \otimes v_\mu$. Then the Cartan component of the tensor product $V_\lambda \otimes V_\mu$ is small.

On the other hand we use the maps to prove that the Cartan component of a given tensor product is not small: The idea is to find tensor $v \otimes w$ among the tensors mapping to $(\lambda \mid \mu)$ such that $v \otimes w$ lies in the Cartan component $V_{\lambda+\mu}$ but *not* in the G–orbit of $v_\lambda \otimes v_\mu$. Then the Cartan component is not small. This explains why pairs (a, b) in the product of the convex hulls for which the value $(a \mid b)$ is maximal are of particular interest.

Definition 4.11. (i) We say that a pair (a, b) in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ is *maximal* if $(a \mid b) = (\lambda \mid \mu)$.
(ii) We say that $V_\lambda \otimes V_\mu$ *has only* $\mathcal{W}$*–conjugated maximal pairs* if for each maximal pair (a, b) in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ there is an element $\omega \in \mathcal{W}$ such that $\omega a = \lambda$ and $\omega b = \mu$.

Theorem 4.12. *If* $V_\lambda \otimes V_\mu$ *has only* $\mathcal{W}$*–conjugated maximal pairs then its Cartan component is small.*

Proof. Let $\Pi(\lambda) = \{\lambda_i\}_{i \in I}$ and $\Pi(\mu) = \{\mu_j\}_{j \in J}$ be the set of weights of V_λ resp. of V_μ.

Recall the map $P_\lambda : V_\lambda \to \mathrm{Con}(\lambda)$ defined by $P_\lambda(v) := \sum_{i \in I} \frac{\|v_i\|_\lambda^2}{\|v\|_\lambda^2} \lambda_i$ where $v = \sum_{i \in I} v_i$ is the decomposition of v into weight vectors. Let $v_0 \otimes w_0$ be a tensor of the Cartan component $V_{\lambda+\mu}$. Then by Theorem 3.12 and Corollary 4.6 there is a tensor $v \otimes w$ in the K–orbit of $v_0 \otimes w_0$ such that $(P_\lambda(v), P_\mu(w))$ is a maximal pair, i.e.

$$(P_\lambda(v) \mid P_\mu(w)) = (\sum_{i \in I} \frac{\|v_i\|_\lambda^2}{\|v\|_\lambda^2} \lambda_i \mid \sum_{j \in J} \frac{\|w_j\|_\mu^2}{\|w\|_\mu^2} \mu_j) = (\lambda \mid \mu).$$

By assumption, there exists ω_0 in $\mathcal{W}$ such that $\omega_0 P_\lambda(v) = \lambda$ and $\omega_0 P_\mu(w) = \mu$. In particular, $P_\lambda(v)$ and $P_\mu(w)$ are vertices of the corresponding convex hulls. Then by Lemma 4.9, all but one of the coefficients $\frac{\|v_i\|_\lambda^2}{\|v\|_\lambda^2}$ and $\frac{\|w_j\|_\mu^2}{\|w\|_\mu^2}$ vanish. In other words, $P_\lambda(v) = \lambda_{i_0}$ and $P_\mu(w) = \mu_{j_0}$ for some indices i_0, j_0 and $v = v_{i_0}$, $w = w_{j_0}$ are weight vectors.

Recall that $\mathcal{W}$ is isomorphic to $\mathrm{N}_G(T)/T$. Let $g_0 \in \mathrm{N}_G(T) \subset G$ be a representative for ω_0. Then $g_0(v \otimes w)$ is a non-zero multiple of $v_\lambda \otimes v_\mu$. $\square$

Remark 4.13. The converse does not hold. We will see in Example 4.16 that there exist tensor products whose Cartan component is small and for which there exist maximal pairs in $\mathrm{Con}(\lambda) \otimes \mathrm{Con}(\mu)$ which do not belong to $\mathcal{W}(\lambda, \mu)$.

Example 4.14. If the highest weights λ, μ are non-zero multiples of each other, then the tensor product $V_\lambda \otimes V_\mu$ has a small Cartan component.

Proof. Let $\mu = r\lambda$ for some $r \in \mathbb{Q}^*$. Let $\|a\| := (a \mid a)$.
For any pair $(a, b) \in \mathrm{Con}(\lambda) \times \mathrm{Con}(r\lambda)$ let φ be the angle between a and b. Note that the value $(a \mid b) = \cos\varphi \|a\|\|b\|$ is smaller than or equal to $\|a\|\|b\|$. Equality holds if and only if $\cos\varphi = 1$, i.e. if and only if $\varphi = 0$. Furthermore, since $\|a\| \le \|\lambda\|$ and $\|b\| \le r\|\lambda\|$, we have $\|a\|\|b\| \le r\|\lambda\|^2$, where equality holds if and only if $\|a\| = \|\lambda\|$ and $\|b\| = r\|\lambda\|$, i.e. if a is a vertex of $\mathrm{Con}(\lambda)$ and if b is a vertex of $\mathrm{Con}(r\lambda)$.

Hence if (a, b) is a maximal pair, the angle φ is zero and a and b are vertices of the corresponding convex hulls. Hence there exists $\omega_0 \in \mathcal{W}$ such that $\omega_0 a = \lambda$. Since b is a positive multiple of a, the vertex $\omega_0 b$ is a positive multiple of $\omega_0 a = \lambda$, and so $\omega_0 b = r\lambda$. Thus the assertion follows with Theorem 4.12. □

The Cartan component of the SL_3–representation $S^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ is not small. In Example 3.10 we showed this by comparing the dimensions of the corresponding algebraic sets. We will now study the geometry of the weights involved. Note that the highest weight of the second symmetric power $S^2(\mathbb{C}^3)$ is $2\omega_1$ and that the representation $(\mathbb{C}^3)^*$ has highest weight ω_2 (see section 8 for a description of the fundamental weights for the general linear group).

Example 4.15. The SL_3–representation $S^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ has maximal pairs which are not Weyl-conjugates of the pair $(2\omega_1, \omega_2)$.

Proof. The weights of $S^2(\mathbb{C}^3) = V_{2\omega_1}$ are $\{2\omega_1, \omega_2, 2\omega_2 - 2\omega_1, -\omega_1, -2\omega_2, \omega_1 - \omega_2\}$ and the representation $(\mathbb{C}^3)^* = V_{\omega_2}$ has the weights $\{\omega_2, -\omega_1, \omega_1 - \omega_2\}$ (see figure 1 below). Note that the α_1–string through the highest weight $2\omega_1$

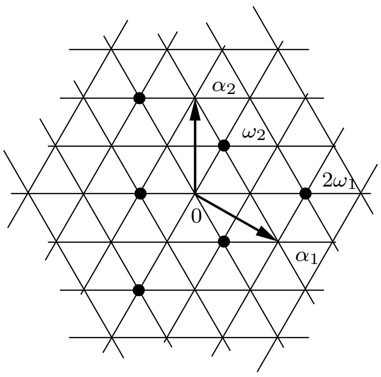

Figure 1: Weights of $S^2\mathbb{C}^3$ and of $(\mathbb{C}^3)^*$

of $S^2(\mathbb{C}^3)$ is perpendicular to the highest weight ω_2 of $(\mathbb{C}^3)^*$. Since every $a := q2\omega_1 + (1-q)2(\omega_2 - \omega_1)$ in $\mathrm{Con}(2\omega_1)$ with $0 \le q \le 1$ lies on the line following the α_1–string through $2\omega_1$, its scalar product $(a \mid \omega_2)$ with ω_2 is maximal, i.e. $(a \mid \omega_2) = (2\omega_1 \mid \omega_2)$. But whenever $q \notin \{0,1\}$ there is no element $\omega \in \mathcal{W}$ such that $\omega a = 2\omega_1$. In particular, there are maximal pairs in $\mathrm{Con}(2\omega_1) \times \mathrm{Con}(\omega_2)$ which do not belong to the Weyl group orbit of $(2\omega_1, \omega_2)$. □

Example 4.16. For the SL_3–representation $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ there exist maximal pairs which are not $\mathcal{W}$–conjugated to (ω_1, ω_2). However, the Cartan component of $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ is small.

Proof. Part (A): We show that there exist maximal pairs in $\mathrm{Con}(\omega_1) \times \mathrm{Con}(\omega_2)$ which are not Weyl-conjugated to (ω_1, ω_2).
The representation $\mathbb{C}^3 = V_{\omega_1}$ has the weights $\{\omega_1, \omega_2 - \omega_1, -\omega_2\}$ and $(\mathbb{C}^3)^*$ has the weights $\{\omega_2, -\omega_1, \omega_1 - \omega_2\}$ (see figure 2 below).

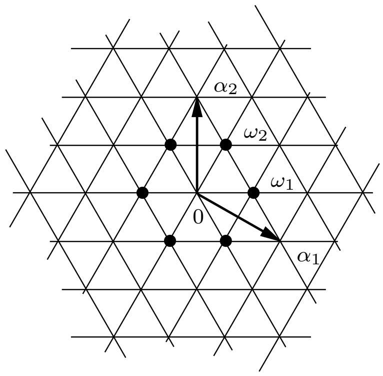

Figure 2: Weights of $\mathbb{C}^3$ and of $(\mathbb{C}^3)^*$

Note that the α_1–string through the highest weight ω_1 is perpendicular to the highest weight ω_2 and that the α_2–string through the highest weight ω_2 is perpendicular to the highest weight ω_1. Let l be the line following the α_1–string through ω_1, i.e. the line through ω_1 and $\omega_2 - \omega_1$. Similarly, let m be the line following the α_2–string through ω_2. Let $L := l \cap \mathrm{Con}(\omega_1)$ and $M := m \cap \mathrm{Con}(\omega_2)$ the intersection of these lines with the convex hulls.

It is easy to see that each pair in $\mathcal{W}(L \times \{\omega_2\}) \cup \mathcal{W}(\{\omega_1\} \times M)$ is maximal. But on the other hand, it is obvious that no pair (a, ω_2) such that a is not a vertex in $\mathrm{Con}(\omega_1)$ (and similarly, no pair (ω_1, b) where b is not a vertex in $\mathrm{Con}(\omega_2)$) lies in the Weyl orbit of (ω_1, ω_2). Hence the tensor product $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ has maximal pairs which are not Weyl-conjugated to (ω_1, ω_2).

Part (B): It remains to show the Cartan component of the representation $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ is small.
This assertion follows from the fact that for every n the Cartan component of the SL_n–representation $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ is small (cf. section 8, Proposition 8.6). □

Note that we can see directly what happens with the tensors that are mapping to the value $(\omega_1 \mid \omega_2)$. Choose any tensor $v \otimes w$ for which $(P_\lambda(v) \mid P_\mu(w))$ is a maximal. By Theorem 3.12, in the SU_3-orbit of every non-zero decomposable tensor of the Cartan component lies a tensor with this property.

W.l.o.g. let $P_\lambda(v) \in L$, say $P_\lambda(v) = q\omega_1 + (1-q)\omega_2$, and $P_\mu(w) = \omega_2$. Hence w is a non-zero multiple of e_3^*. Use Corollary 4.10 to see that v is a non-zero multiple of the vector $\sqrt{q}e_1 + \sqrt{1-q}e_2$.
Let A be the matrix

$$A := \begin{pmatrix} \sqrt{q} & \sqrt{1-q} & * \\ -\sqrt{1-q} & \sqrt{q} & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3 \,.$$

Then $A(v \otimes e_3^*) = e_1 \otimes e_3^*$.

4.3 Dominant Weights and Simple Roots

An interesting property of the representations of Example 4.15 and Example 4.16 is that the two highest weights are perpendicular to different simple roots. Thereby, the α_1–string through the highest weight $2\omega_1$ (resp. through ω_1), which is perpendicular to the other highest weight, contains at least two weights of $V_{2\omega_1}$ (resp. of V_{ω_1}). Because of this orthogonality to ω_2, the value $(a \mid \omega_2)$ is maximal for every a in $\mathrm{Con}(2\omega_1)$ (resp. in $\mathrm{Con}(\omega_1)$) which lies on the line L following this root string. Since there are at least two weights of $V_{2\omega_1}$ (resp. of V_{ω_1}) lying on L, there exist, in particular, points $a \in L$ which are not vertices of the corresponding convex hulls. Hence they do not belong to the Weyl orbit of the highest weights. This observation explains why we will now be studying the relation between highest weights and simple roots. The following result will be useful for the proof of the main theorem of section 4.

Lemma 4.17. *Let $a \in \mathrm{Con}(\lambda)$ and $b \in \mathrm{Con}(\mu)$ such that $(a \mid b) = (\lambda \mid \mu)$.*

(i) Let a be regular. Then b belongs to the Weyl orbit of μ.

(ii) Let a and b be regular. Then there exists some $\omega \in \mathcal{W}$ such that $\omega a = \lambda$ and $\omega b = \mu$.

Note that in both cases, a and b belong to the same Weyl chamber (follows by part two of Lemma 4.8). So, w.l.o.g. let a and b lie in $X_{\mathbb{R}}^+$.

Proof. Part (i): Write $b = \mu - \sum_l r_l \alpha_l$ with non-negative coefficients r_l (cf. Lemma 4.4). Suppose that $b \neq \mu$, i.e. there exists l_0 such that $r_{l_0} > 0$. So, $(a \mid b) = (a \mid \mu) - \sum r_l (a \mid \alpha_l) < (a \mid \mu)$ (since a belongs to the open dominant Weyl chamber, $(a \mid \alpha_l) > 0$ for every simple root α_l), which contradicts the maximality of $(a \mid b)$.
The assertion of part (ii) follows by applying part (i) twice. □

Finally we have all the tools to prove the main result of this section.

Theorem 4.18. *Let λ and μ be dominant weights. The following assertions are equivalent:*
(i) $V_\lambda \otimes V_\mu$ has only $\mathcal{W}$–conjugated maximal pairs
(ii) λ and μ are perpendicular to the same simple roots.

In particular, the Cartan component of $V_\lambda \otimes V_\mu$ is small when both weights are regular.

Proof. We show that the first property implies the second.
Suppose that there exists some simple root α perpendicular to μ and not perpendicular to λ. We claim that in this case, there exist maximal pairs in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ which do not lie in $\mathcal{W}(\lambda, \mu)$: Note that since λ is not perpendicular to α, $\sigma_\alpha(\lambda) \neq \lambda$ and hence the α–string through λ contains at least two weights of V_λ, namely λ and $\sigma_\alpha(\lambda)$. Let l be the line following the α–string through λ and $L := l \cap \mathrm{Con}(\lambda)$. Then it is clear that for every $a \in L$, $(a \mid \mu) = (\lambda \mid \mu)$. But whenever a is not a vertex of $\mathrm{Con}(\lambda)$, i.e. $a \notin \mathcal{W}\lambda$, the pair (a, μ) does not belong to the Weyl orbit of (λ, μ).

It remains to show that property (ii) implies property (i). We proceed in two steps.
Step (1). Assume first that λ and μ are regular dominant weights. Let $(a, b) \in \mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ such that $(a \mid b) = (\lambda \mid \mu)$ and where w.l.o.g. $a \in X_{\mathbb{R}}^+$. Let $\triangle = \{\alpha_l\}_{l \in L}$ be the set of simple weights, $a = \lambda - \sum_{l \in L} r_l \alpha_l$ and $b = \mu - \sum_{l \in L} s_l \alpha_l$ with non-negative coefficients r_l and s_l (cf. Lemma 4.4). Since $a \in X_{\mathbb{R}}^+$, $(a \mid \alpha) \geq 0$ for every simple root α and since μ is regular, $(\alpha \mid \mu)$ is strictly positive for every simple root. Thus

$$
\begin{aligned}
(a \mid b) &= (a \mid \mu) - \sum s_l (a \mid \alpha_l) \\
&\leq (a \mid \mu) \\
&= (\lambda \mid \mu) - \sum r_l (\alpha_l \mid \mu) \\
&\leq (\lambda \mid \mu)
\end{aligned}
$$

which is equal to $(a \mid b)$ by assumption, so equality holds everywhere. Hence every coefficient r_l has to vanish and thus $a = \lambda$. By Lemma 4.17, part (i), the point b belongs to the Weyl orbit of μ and by Lemma 4.8 it is an element of $X_{\mathbb{R}}^+$. Therefore $b = \mu$.
Step (2). Let λ and μ be dominant weights which are perpendicular to the same simple roots, say to $\{\alpha_l\}_{l\in L_0} \subsetneq \triangle$. Consider the root subsystem $\Phi' \subset \Phi$ spanned by the basis $\triangle' := \triangle \setminus \{\alpha_l\}_{l\in L_0}$. Note that $(\lambda \mid \alpha) > 0$ and $(\mu \mid \alpha) > 0$ for every $\alpha \in \triangle'$ where $(\cdot \mid \cdot)$ is the restriction of the inner form to the space $X'_{\mathbb{R}}$ spanned by $\triangle'$. Thus λ and μ are regular dominant weights for the group G' associated to Φ' . Hence $a = \lambda$ and $b = \mu$ by step (1). □

Definition 4.19. Let λ be a dominant weight. Then we define $I(\lambda) \subset \triangle$ as the set of simple roots perpendicular to λ.

Note that the set $I(\lambda) \cap I(\mu)$ of simple roots perpendicular to λ and μ is just the set $I(\lambda + \mu)$ of simple roots perpendicular to their sum.

Corollary 4.20. *Let λ, μ be dominant weights such that $I(\lambda) = I(\mu)$. Then the Cartan component of $V_\lambda \otimes V_\mu$ is small.*

Proof. The assertion follows immediately from Theorems 4.18 and 4.12. □

Definition 4.21. If λ and μ are dominant weights which are not perpendicular to the same simple roots we say that the representation $V_\lambda \otimes V_\mu$ is *critical*. If λ and μ are perpendicular to the same roots we sometimes say that the representation $V_\lambda \otimes V_\mu$ is *not critical*.

The SL_3–representation $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ is an example of a critical representation where the Cartan component is small (cf. Example 4.16).

4.4 Critical Representations

In this subsection we discuss some general properties of critical representations. Let $\triangle = \{\alpha_i\}_{i\in I}$ be the set of simple roots. Recall that for $\lambda = \sum_i l_i\omega_i$ the support of λ, $\operatorname{Supp}\lambda$, is defined as the set of fundamental weights appearing in $\sum l_i\omega_i$, i.e. for which $l_i > 0$. In other words: $\operatorname{Supp}\lambda = \{\omega_i \mid i \in I, (\lambda \mid \alpha_i) > 0\}$.

The next result will give us a useful tool while handling representations that are "nearly" non-critical, i.e. representations $V_\lambda \otimes V_\mu$ where $I(\lambda)$ and $I(\mu)$ are almost the same subsets of $\triangle$ (cf. Corollary 4.23).

Lemma 4.22. *Let $\lambda = \sum l_i\omega_i$ and let $(P, Q) \in \mathrm{Con}(\lambda)\times\mathrm{Con}(\mu)$ be a maximal pair. Assume that Q lies in the closed dominant Weyl chamber $X_{\mathbb{R}}^+$. Then there exist non-negative coefficients s_i such that $Q = \mu - \sum_{i:\omega_i\notin\operatorname{Supp}\lambda} s_i\alpha_i$.*

In other words, if (P,Q) is a maximal pair such that Q belongs to the closed dominant Weyl chamber then $Q = \mu - \sum_{i\in I} s_i\alpha_i$ with non-negative s_i and $s_i = 0$ for every i such that $(\lambda \mid \alpha_i) \neq 0$.

Proof. We write $Q = \mu - \sum_{i\in I} s_i\alpha_i$ and $P = \lambda - \sum_{i\in I} r_i\alpha_i$ with non-negative coefficients s_i, r_i (cf. Lemma 4.2). We have

$$\begin{aligned}(P \mid Q) &= (\lambda - \textstyle\sum r_i\alpha_i \mid Q) \\ &\leq (\lambda \mid Q) \qquad\qquad (\text{since } Q \in X_{\mathbb{R}}^{+}) \\ &= (\lambda \mid \mu - \textstyle\sum s_i\alpha_i) \\ &\leq (\lambda \mid \mu).\end{aligned}$$

Since (P,Q) is a maximal pair, equality holds everywhere. Especially, $(\lambda \mid s_i\alpha_i) = 0$ for each i. But $(\lambda \mid \alpha_i) > 0$ for each i for which ω_i is an element of the support of λ. Hence $s_i = 0$ whenever ω_i belongs to the support of λ. □

Corollary 4.23. *Let λ be regular and μ be perpendicular to one simple root. If (P,Q) is a maximal pair in* $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ *then Q lies in the Weyl-orbit of μ.*

Proof. Follows from Lemma 4.22 since the support of a regular weight consists of all fundamental weights. □

5 Dense Orbits and Small Cartan Components

The goal of this section is to develop a necessary criterion for tensor products $V_\lambda \otimes V_\mu$ to have a small Cartan component.

5.1 A Necessary Condition for Small Cartan Components

We will deal with representations of G and of subgroups of G. The notation V_λ always describes an irreducible representation of the group G. To emphasise on the group that is acting we will denote the representation generated by the G–orbit of a highest weight vector v by $\langle Gv \rangle$.

Recall that there is a $1-1$–correspondence between closed connected subgroups of G and Lie subalgebras of $\mathfrak{g}$. In particular, for each $\alpha \in \Phi$, there is a unique connected T–stable subgroup $U_\alpha \subset G$ such that $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$, cf. Borel [Bo97], IV.13.18. Note that U_α is a unipotent subgroup of G.

Recall that a subgroup of G is called *parabolic* if it contains a Borel subgroup. It is known that every parabolic subgroup P of G can be written as a semi-direct product of its unipotent radical $\operatorname{rad}_U(G)$ and of a reductive group L, $P = L \operatorname{rad}_U(G)$ (see e.g. Humphreys [Hu75], §30.2).

Definition 5.1. Let $I = \{\alpha_1, \ldots, \alpha_r\} \subset \triangle$ be a set of simple roots. Let Φ_I be the subsystem of Φ generated by I.

(i) Denote by P_I the parabolic subgroup of G generated by B together with the root groups U_α, α in Φ_I.

(ii) Let L_I be the Levi factor of $P_{I(\lambda)}$ that contains the torus T.

In particular, L_I is a reductive subgroup of G.

Remark 5.2. Note that the two extreme cases are $I = \emptyset$ and $I = \triangle$. We have $P_\emptyset = B$, $L_\emptyset = T$ and $P_\triangle = L_\triangle = G$.

Recall that for a dominant weight λ the set $I(\lambda) \subset \triangle$ is defined as the set of simple roots perpendicular to λ. Note that $P_{I(\lambda)} = \langle B, U_\alpha \mid \alpha \in \Phi, (\lambda \mid \alpha) = 0 \rangle$.

Lemma 5.3. *Let $P_{I(\lambda)}$ be the parabolic subgroup generated by B and the root groups U_α of the roots perpendicular to α. Then the following holds:*

$P_{I(\lambda)} = \operatorname{Stab}_G \mathbb{C} v_\lambda$.

Proof. The line $\mathbb{C}v_\lambda = V_\lambda(\lambda)$ is B–stable and for each α perpendicular to λ, $V_\lambda(\lambda)$ is fixed by U_α, so $P_{I(\lambda)} \subset \mathrm{Stab}_G \mathbb{C}v_\lambda$.
Suppose that $\mathrm{Stab}_G \mathbb{C}v_\lambda$ is not contained in $P_{I(\lambda)}$. Then there exists some positive root β with $(\beta \mid \lambda) \neq 0$ such that the root group U_β has non-trivial intersection with $\mathrm{Stab}_G \mathbb{C}v_\lambda$. Choose any non-trivial $s \in U_\beta \cap \mathrm{Stab}_G \mathbb{C}v_\lambda$. Let $S \subset U_\beta \cap \mathrm{Stab}_G \mathbb{C}v_\lambda$ be the subgroup generated by s. Hence S stabilises $V_\lambda(\lambda)$ and so its Lie algebra $\mathfrak{s} := \mathrm{Lie}\, S$ stabilises $V_\lambda(\lambda)$ (see Humphreys in [Hu75] §13.2). So, $V_\lambda(\lambda) = \mathfrak{s}V_\lambda(\lambda) \subset \mathfrak{g}_\beta V_\lambda(\lambda) \subset V_\lambda(\lambda+\beta)$. In particular, the weight space $V_\lambda(\lambda+\beta)$ contains the weight space $V_\lambda(\lambda)$ which contradicts the fact, that weight spaces are perpendicular. Therefore $U_\beta \cap \mathrm{Stab}_G \mathbb{C}v_\lambda$ must be trivial for each β with $(\beta \mid \lambda) \neq 0$. □

As before, let V_λ and V_μ be irreducible with highest weight vectors v_λ resp. v_μ. Consider the submodule $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle$ of $V_\lambda \otimes V_\mu$ generated by the $L_{I(\lambda)}$–orbit of the highest weight vector $v_\lambda \otimes v_\mu$.

Lemma 5.4. *The module $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle$ consists of decomposable tensors of the Cartan component $V_{\lambda+\mu}$.*

Proof. It is clear that the module $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle$ lies in the Cartan component $V_{\lambda+\mu} = \langle G(v_\lambda \otimes v_\mu)\rangle$. We show that $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle$ consists of decomposable tensors of the tensor product:

Since $L_{I(\lambda)}$ is a subgroup of the stabiliser of $\mathbb{C}v_\lambda$, the orbit $L_{I(\lambda)}v_\lambda$ is a subset of $\mathbb{C}v_\lambda$. Therefore

$$\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle \;=\; \langle v_\lambda \otimes L_{I(\lambda)}v_\mu\rangle \;=\; v_\lambda \otimes \langle L_{I(\lambda)}v_\mu\rangle.$$

Hence every tensor of $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu)\rangle$ is decomposable. □

The following observation will be useful:

Lemma 5.5. *The $L_{I(\lambda)}$–module $\langle L_{I(\lambda)}v_\mu\rangle$ is irreducible of highest weight μ.*

Proof. Recall that $L_{I(\lambda)}$ is a reductive group. The assertion follows by Kraft [Kr85], III.1.5: The vector v_μ is a non-zero element in $\langle L_{I(\lambda)}v_\mu\rangle_\mu$ of weight μ. It is invariant under the maximal unipotent subgroup U of G contained in B and hence invariant under the unipotent part $U_\lambda := U \cap L_{I(\lambda)}$ of $L_{I(\lambda)}$. Therefore $\langle L_{I(\lambda)}v_\mu\rangle$ is irreducible of highest weight μ. □

Corollary 5.6. *Let λ and μ be dominant weights, let $I(\lambda)$ the set of roots perpendicular to λ. Then the following holds:*
$\overline{L_{I(\lambda)}v_\mu} = L_{I(\lambda)}v_\mu \cup \{0\}$.

Proof. The assertion follows by applying Theorem 3.4 to Lemma 5.5. □

We can now state a necessary condition for representations $V_\lambda \otimes V_\mu$ with small Cartan component.

Theorem 5.7. *If the Cartan component of the G–representation $V_\lambda \otimes V_\mu$ is small, then the following two properties hold:*
1. *The $L_{I(\lambda)}$–orbit of v_μ is dense in $\langle L_{I(\lambda)} v_\mu \rangle$.*
2. *The $L_{I(\mu)}$–orbit of v_λ is dense in $\langle L_{I(\mu)} v_\lambda \rangle$.*

Proof. We prove the first part since the second part follows by the same arguments.
It is clear that $\overline{L_{I(\lambda)} v_\mu}$ is a subset of $\langle L_{I(\lambda)} v_\mu \rangle$. It remains to show that every non-zero element of the module $\langle L_{I(\lambda)} v_\mu \rangle$ lies in the $L_{I(\lambda)}$–orbit or v_μ.

Let $w \neq 0$ in $\langle L_{I(\lambda)} v_\mu \rangle$. Then the tensor $v_\lambda \otimes w$ is a non-zero element of the module $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu) \rangle$. Recall that the module $\langle L_{I(\lambda)}(v_\lambda \otimes v_\mu) \rangle$ is a subset of the set of decomposable tensors in $V_{\lambda+\mu}$ (cf. Lemma 5.4). By assumption, the Cartan component $V_{\lambda+\mu}$ is small, i.e. decomposable tensors in the Cartan component $V_{\lambda+\mu}$ all belong to the closure of the orbit $G(v_\lambda \otimes v_\mu)$. By Theorem 3.4 this closure is just $G(v_\lambda \otimes v_\mu) \cup \{0\}$. Thus the (non-zero) tensor $v_\lambda \otimes w$ lies in the G–orbit of $v_\lambda \otimes v_\mu$. Hence there exists $g \in G$ such that $g(v_\lambda \otimes v_\mu) = v_\lambda \otimes w$. In particular, g belongs to the stabiliser $\mathrm{Stab}_G\, \mathbb{C} v_\lambda$ which is the parabolic subgroup $P_{I(\lambda)}$. Since $P_{I(\lambda)} = L_{I(\lambda)} \,\mathrm{rad}_U(P_{I(\lambda)})$ we can write $g = l \cdot u$ with $l \in L_{I(\lambda)}$, $u \in \mathrm{rad}_U(P_{I(\lambda)})$ which is contained in the unipotent radical U of the Borel subgroup B. Hence u fixes the tensor $v_\lambda \otimes v_\mu$ and so:

$$\begin{aligned} v_\lambda \otimes w &= g(v_\lambda \otimes v_\mu) \\ &= l(v_\lambda \otimes v_\mu) \\ &= c v_\lambda \otimes l v_\mu \end{aligned}$$

for some $c \in \mathbb{C}^*$. Thus $w \in L_{I(\lambda)} v_\mu$. □

The following fact will be useful to reformulate Theorem 5.7.

Lemma 5.8. *Let V be an irreducible representation of a semi-simple group H, let $v \in V$ be a highest weight vector. Then the following properties are equivalent:*
1. $\overline{Hv} = V$.
2. $\overline{\mathbb{C}^* Hv} = V$.
3. $Hv = V \setminus \{0\}$.

Proof. Since H is reductive, $\overline{Hv}$ is a cone and $\overline{Hv} = Hv \cup \{0\}$ (cf. Theorem 3.4). Then the assertion follows easily. □

Our goal is to reformulate the result of Theorem 5.7 in terms of semi-simple subgroups. Denote by $S_{I(\lambda)} := (L_{I(\lambda)}, L_{I(\lambda)}) \subset G$ the semi-simple subgroup of the commutators of $L_{I(\lambda)}$. Hence $L_{I(\lambda)} = \mathrm{Zen}_G(L_{I(\lambda)})S_{I(\lambda)}$ where $\mathrm{Zen}_G(L_{I(\lambda)})$ denotes the center of $L_{I(\lambda)}$ in G. Since $\langle L_{I(\lambda)}v_\mu\rangle$ is an irreducible $L_{I(\lambda)}$–module, Schurs Lemma implies that the center of $L_{I(\lambda)}$ acts by scalar multiplication. Therefore $L_{I(\lambda)}$ operates as $\mathbb{C}^* \times S_{I(\lambda)}$ on $\langle L_{I(\lambda)}v_\mu\rangle$.

Corollary 5.9. Dense Orbits Criterion
If the Cartan component of the G–representation $V_\lambda \otimes V_\mu$ is small then the following holds:

1. *The $S_{I(\lambda)}$–orbit of v_μ is dense in $\langle S_{I(\lambda)}v_\mu\rangle$.*
2. *The $S_{I(\mu)}$–orbit of v_λ is dense in $\langle S_{I(\mu)}v_\lambda\rangle$.*

Proof. The assertion follows by applying Lemma 5.8 to Theorem 5.7. □

Note that the Dense Orbits Criterion (Corollary 5.9) is *not* a sufficient condition. We will see in Proposition 8.9 that Cartan components of tensor products $V_{\omega_k} \otimes V_{\omega_j}$ of fundamental SL_{n+1}–representations are not small whenever $k < j - 1$ and $(k, j) \neq (1, n)$. However, the orbits $S_{I(\omega_j)}v_{\omega_k}$ and $S_{I(\omega_k)}v_{\omega_j}$ are dense in the representations they generate.

Thus there are representations $V_\lambda \otimes V_\mu$ where the $S_{I(\lambda)}$–orbit of v_μ is dense in the representation $\langle S_{I(\lambda)}v_\mu\rangle$ and the $S_{I(\mu)}$–orbit of v_λ is dense in $\langle S_{I(\mu)}v_\lambda\rangle$ but whose Cartan component $V_{\lambda+\mu}$ is not small.

Example 5.10. Suppose both λ and μ are regular. Then the $S_{I(\lambda)}$–orbit of v_μ is dense in $\langle S_{I(\lambda)}v_\mu\rangle$ and the $S_{I(\mu)}$–orbit of v_λ is dense in $\langle S_{I(\mu)}v_\lambda\rangle$.

Proof. Recall that by Proposition 4.18 the Cartan component $V_{\lambda+\mu}$ is small. Then the assertion follows from Corollary 5.9 (Dense Orbits Criterion). □

One can also see directly that these orbits are dense in the corresponding representations: By assumption, the sets $I(\lambda)$ and $I(\mu)$ are empty, hence $L_{I(\lambda)} = L_{I(\mu)} = T$ (cf. Remark 5.2). Since the torus T operates on highest weight vectors as multiplication by scalars, the T–orbits of v_λ resp. of v_μ are dense in the corresponding one-dimensional representations $\langle Tv_\lambda\rangle$ resp. $\langle Tv_\mu\rangle$.

5.2 An Application to Critical Representations

Recall that the Cartan component of the tensor product $V_\lambda \otimes V_\mu$ is small if λ and μ are perpendicular to the same simple roots, i.e. if $I(\lambda) = I(\mu)$ (see Proposition 4.18 and Theorem 4.12). If $I(\lambda)$ is different from $I(\mu)$ the representation $V_\lambda \otimes V_\mu$ is said to be critical. The Dense Orbits Criterion

(Corollary 5.9) is a useful tool to determine whether the Cartan component of a critical representation can be small:

> If the $S_{I(\lambda)}$–orbit of v_μ is not dense in the irreducible representation $\langle S_{I(\lambda)} v_\mu \rangle$ or if the $S_{I(\mu)}$–orbit of v_λ is not dense in $\langle S_{I(\mu)} v_\lambda \rangle$ then the Cartan component of the tensor product $V_\lambda \otimes V_\mu$ is not small.

Recall that the only representations of the classical groups which contain a dense orbit are the following (cf. Parshin, Shafarevich [PS94], p. 260):

G	V	$\dim V$
SL_{n+1} (A_n)	$\mathbb{C}^{n+1}$, $(\mathbb{C}^{n+1})^*$	$n+1$
Sp_{2n}, $n \geq 2$ (C_n)	$\mathbb{C}^{2n}$	$2n$

Table 1: Representations with a Dense Orbit

As an example on how to use the Dense Orbits Criterion we recall the representation $S^2\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$. Note that e_1^2 is highest weight vector of $S^2\mathbb{C}^3$ and that $S_{I(\lambda)}$ is the group SL_2. We check if the SL_2–orbit of e_1^2 is dense in the representation $\langle \mathrm{SL}_2\, e_1^2 \rangle$. Since the orbit $\mathrm{SL}_2\, e_1^2$ is two-dimensional it cannot be a dense subset of the three-dimensional representation $\langle \mathrm{SL}_2\, e_1^2 \rangle$.

6 Example: Representations of SL_3

Recall that the SL_3–representation $S^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ is the first example where the Cartan component is not small. We are thus interested in tensor products of irreducible SL_3–representations. To study them we use results from the previous sections. We need to understand what happens in case the tensor product is a critical representation. Note that critical representations of SL_3 are representations where one of the dominant weights is perpendicular to a simple root, say to α_2 and the other dominant weight is regular or perpendicular to α_1. Corollary 6.3 below helps to understand how the Cartan component lies in such a tensor product.

We start by reminding the notion of a root-string through a weight of a representation. For the moment we do not restrict our attention to the group SL_3. Let Φ be a reduced root system and let $\triangle \subset \Phi$ be a basis.

Let α be a simple root, λ a dominant weight and μ a weight of V_λ. Recall that the weights of the form $\mu - r\alpha$ ($r \in \mathbb{Z}$) in $\Pi(\lambda)$ form a connected string which is called the *α–string through μ*. In particular, the α–string through the highest weight λ consists of the weights $\lambda, \lambda-\alpha, \cdots, \sigma_\alpha(\lambda) = \lambda - \langle \lambda \mid \alpha \rangle \alpha$ where $\langle \lambda \mid \alpha \rangle$ is the number $2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}$.

Let $0 \leq k \leq \langle \lambda \mid \alpha \rangle$. Note that the vectors of weight $\lambda - k\alpha$ are spanned by $X_\alpha^k.v_\lambda$, $X_\alpha \in \mathfrak{g}_\alpha$ (see Humphreys [Hu94] 20.2).

Lemma 6.1. *Let $\alpha \in \triangle$ be a simple root and λ a dominant weight. Then the multiplicity of the weight $\lambda - k\alpha$ in V_λ is one for $k = 0, \ldots, \langle \lambda \mid \alpha \rangle$.*

In other words we have $V_\lambda(\lambda - k\alpha) = \mathbb{C}v_{\lambda - k\alpha}$ for $k = 0, \ldots, \langle \lambda \mid \alpha \rangle$.

Lemma 6.2. *Let λ and μ be dominant weights for SL_{n+1} and α a simple root such that μ is perpendicular to α. Then the weight subspaces $(V_\lambda \otimes V_\mu)(\lambda + \mu - k\alpha)$ are one-dimensional for $k = 0, \ldots, \langle \lambda \mid \alpha \rangle$.*

Proof. We show that the α–string through $\lambda + \mu$ has multiplicity one in $V_\lambda \otimes V_\mu$. Let L be the α–string through λ and M be the set of weights of $V_\lambda \otimes V_\mu$ lying on the α–string through $\lambda + \mu$. Observe that the weights on M possibly have multiplicities bigger than one.

Recall that the weights on the α–string L have multiplicity one in V_λ (see Lemma 6.1). We know that the weights of $V_\lambda \otimes V_\mu$ are pairwise sums of weights of V_λ and of V_μ (cf. Proposition 3.2 (i)).

We show that for every weight $\lambda + \mu - k\alpha$ of M there is only one way to write it as a sum of weights of V_λ and of V_μ. Furthermore, these sums are of the form $\nu_1 + \mu$ where ν_1 is a weight of muliplicity one in V_λ and μ is the highest weight of V_μ:

Since μ is perpendicular to α, the α–strings through λ and through $\lambda+\mu$ are parallels with distance μ. In particular, there is only one way to write an element ν of M as a sum $\nu_1+\nu_2$ with $\nu_1 \in \Pi(\lambda)$ and $\nu_2 \in \Pi(\mu)$ namely $\nu_2 = \mu$ and $\nu_1 = \nu - \mu \in L$. Since the multiplicity of every element of L is one, the multiplicity of ν is one. □

Corollary 6.3. *Let* λ *and* μ *be dominant weights for* SL_{n+1} *and* α *a simple root such that* μ *is perpendicular to* α. *Then* $(V_\lambda \otimes V_\mu)(\lambda+\mu-k\alpha) = V_{\lambda+\mu}(\lambda+\mu-k\alpha) = V_\lambda(\lambda-k\alpha)\otimes v_\mu$ *for* $k=0,\dots,\langle\lambda \mid \alpha\rangle$.

Proof. Note that $\langle\lambda \mid \alpha\rangle = \langle\lambda+\mu \mid \alpha\rangle$ since μ is perpendicular to α. Recall that for every simple root α the weight subspaces $V_\lambda(\lambda-k\alpha)$ and $V_{\lambda+\mu}(\lambda+\mu-k\alpha)$, $k=0,\dots,k_0 := \langle\lambda \mid \alpha\rangle$, of the irreducible representations V_λ resp. $V_{\lambda+\mu}$ are one-dimensional (see Lemma 6.1).

It is clear that the weight subspace $V_{\lambda+\mu}(\lambda+\mu-k\alpha)$ and $V_\lambda(\lambda-k\alpha)\otimes v_\mu$ are subspaces of $(V_\lambda\otimes V_\mu)(\lambda+\mu-k\alpha)$. By Lemma 6.2, $\dim V_\lambda\otimes V_\mu(\lambda+\mu-k\alpha) = 1$ and so equality holds. □

From now on we assume that $G = \mathrm{SL}_3$. Let $\{e_1, e_2, e_3\}$ be a basis of $\mathbb{C}^3$ such that the weight of e_i is ε_i. The simple roots of SL_3 are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$, the fundamental weights are $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$. We mainly use Corollary 6.3 to construct non-zero decomposable tensors of the Cartan component of a representation $V_\lambda \otimes V_\mu$ such that these tensors do not lie in the SL_3–orbit of the highest weight vector $v_\lambda \otimes v_\mu$.

Recall the representation $S^2(\mathbb{C}^3)\otimes(\mathbb{C}^3)^* = V_{2\omega_1}\otimes V_{\omega_2}$ and its weight diagram (see figure 3 below). The crucial point in this example is that the α_1–string through the highest weight $2\omega_1+\omega_2$ contains more than two weights.

Example 6.4. The Cartan component of the representation $S^2(\mathbb{C}^3)\otimes(\mathbb{C}^3)^*$ is not small.

Proof. We have already proved this with different methods (see Example 3.10 or use the Dense Orbits Criterion 5.9). The proof presented here is an illustration of the interplay between properties of the weight lattice and of the tensor product. We construct non-zero decomposable tensors of the Cartan component that do not lie in the SL_3–orbit of $e_1^2 \otimes e_3^*$.

Let $v := ae_1^2 + be_1e_2 + \sqrt{1-a^2-b^2}e_2^2 \neq 0$ with $0 \leq a, b, a^2+b^2 \leq 1$. Note that the point $P_\lambda(v)$ lies on the line following the α_1–string through the highest weight $2\omega_1$. In particular, the value $(P_\lambda(v) \mid \omega_2)$ is maximal.

The tensor $v\otimes e_3^*$ lies in the vector space $(\mathbb{C}e_1^2\oplus\mathbb{C}e_1e_2\oplus\mathbb{C}e_2^2)\otimes e_3^*$ (where $\mathbb{C}e_1^2$ is the weight space $V_{2\omega_1}(2\omega_1)$, the line $\mathbb{C}e_1e_2$ is the weight space $V_{2\omega_1}(2\omega_1-\alpha_1)$

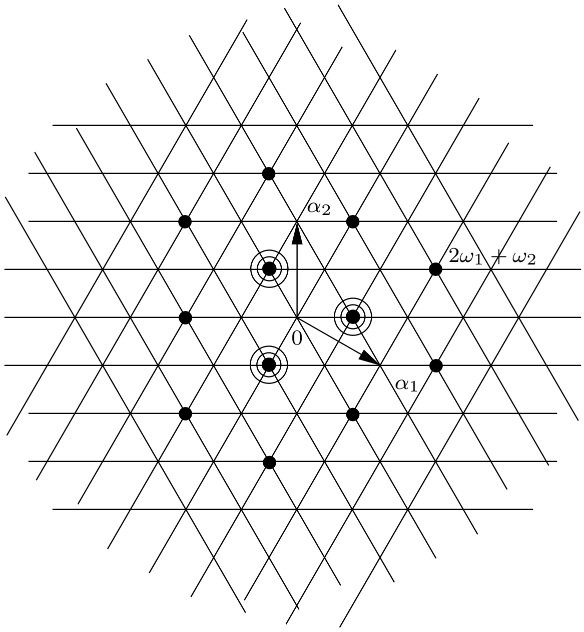

Figure 3: Weight diagram of $S^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$

and $\mathbb{C}e_2^2 = V_{2\omega_1}(2\omega_1 - 2\alpha_1))$. By Corollary 6.3, the tensor $v \otimes e_3^*$ belongs to the Cartan component $V_{2\omega_1+\omega_2}$.

We show that whenever $0 < a < 1$ and $4a\sqrt{1-a^2-b^2} \neq b^2$, the tensor $v \otimes e_3^*$ does not lie in the SL_3–orbit of a highest weight vector.
Let $0 < a < 1$ and $4a\sqrt{1-a^2-b^2} \neq b^2$. Suppose that there exists $A \in \mathrm{SL}_3$ such that $A(v \otimes e_3^*)$ is a multiple of the highest weight vector $e_1^2 \otimes e_3^*$. Hence $A \in \mathrm{Stab}_{\mathrm{SL}_3} \mathbb{C}e_3^*$. Recall that

$$\mathrm{Stab}_{\mathrm{SL}_3}(\mathbb{C}e_3^*) = \{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & a_0 \end{pmatrix} \} \subset \mathrm{SL}_3 \,.$$

W.l.o.g. let $a_0 = 1$ and $A = \begin{pmatrix} d & e & * \\ f & g & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Stab}_{\mathrm{SL}_3} \mathbb{C}e_3^*$. Then $Ae_1 = de_1 + fe_2$, $Ae_2 = ee_1 + ge_2$ and

$$\begin{aligned} Av &= e_1^2(ad^2 + bde + \sqrt{1-a^2-b^2}e^2) \\ &\quad + e_1e_2(2adf + b(dg+ef) + 2\sqrt{1-a^2-b^2}ge) \\ &\quad + e_2^2(af^2 + bfg + \sqrt{1-a^2-b^2}g^2). \end{aligned}$$

Since Av has to be a multiple of e_1^2 we get

$$2adf + b(1+2ef) + 2\sqrt{1-a^2-b^2}ge \;=\; 0 \qquad (4)$$
$$af^2 + bfg + \sqrt{1-a^2-b^2}g^2 \;=\; 0 \qquad (5)$$

(using $\det A = dg - ef = 1$). Note that $g \neq 0$ (if g was zero, we would have $af^2 = 0$ and so $\det A = 0$). Thus we can replace $\sqrt{1-a^2-b^2}g$ by $\frac{-af^2-bf}{g}$ in equation 4 and get

$$\frac{2af}{g} + b \;=\; 0.$$

Since a is not zero, we can replace f by $-\frac{bg}{2a}$ in equation 5 and so

$$\sqrt{1-a^2-b^2} \;=\; \frac{b^2}{4a}$$

which contradicts the assumptions. Note that there always exist such tensors: choose $a, b > 0$ such that $a^2 + b^2 = 1$. Then every tensor $(ae_1^2 + be_1e_2) \otimes e_3^*$ is a non-zero decomposable tensor of the Cartan component which does not lie in the SL_3–orbit of $e_1^2 \otimes e_3^*$. □

The following result tells which tensor products of irreducible representations have a small Cartan component and which do not. Recall that for a dominant weight μ the set $I(\mu)$ is defined as the set of simple roots perpendicular to μ. Note that we exclude the case $\lambda = 0$ of the trivial representation.

Proposition 6.5. *Let λ and μ be dominant weights.*

(A) If the highest weights are regular then the Cartan component of $V_\lambda \otimes V_\mu$ is small.

(B) Let μ be perpendicular to a simple root, say to α_1 and $\mu = m\omega_2$. Then the Cartan component of $V_\lambda \otimes V_\mu$ is small if and only if λ and μ are of the following form:

(i) The weight $\lambda = \sum l_i\omega_i$ is regular and $l_1 = 1$.

(ii) The weight λ is perpendicular to α_1.

(iii) The weight λ is the fundamental weight ω_1 and $m = 1$ (i.e. $\mu = \omega_2$).

It is clear that assertion (ii) of (B) also holds if the weights are perpendicular to α_2.

Proof. For the cases (A) and (B)(ii) there is nothing to show: Since $I(\lambda)$ and $I(\mu)$ are identical, the corresponding representation is not critical and the assertion follows with Theorem 4.18.

Part (1): Case (B)(i). Let λ be regular. We first show that in case $l_1 = 1$ the Cartan component is small.

Let $\lambda = \omega_1 + k\omega_2$ with $k > 0$ be regular. By Theorem 3.12 we know that in the SU_3–orbit of every decomposable tensor of the Cartan component lies a tensor $v \otimes w$ for which the value $(P_\lambda(v) \mid P_\mu(w))$ is maximal. Therefore we are looking for maximal pairs in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$.

It is easy to see that for every point P on the line in $\mathrm{Con}(\lambda)$ following the α_1–string through λ, the value $(P \mid \mu)$ is maximal. Furthermore, these pairs (and their Weyl-conjugates) are the only pairs with $(P \mid \mu) = (\lambda \mid \mu)$. Since the value $\langle \lambda \mid \alpha_1 \rangle$ is one, the α_1–string through λ consists only of the vertices λ and $\lambda - \alpha_1$. Hence the point P is a linear combination only of λ and $\lambda - \alpha_1$. Recall that $P_\lambda^{-1}(\lambda) = \mathbb{C}^* v_\lambda$ and $P_\lambda^{-1}(\lambda - \alpha_1) = \mathbb{C}^* v_{\lambda - \alpha_1}$ (see Corollary 4.10). Hence every vector v mapping to P under the map P_λ lies in $\mathbb{C}^* v_\lambda(\lambda) \oplus \mathbb{C}^* v_{\lambda-\alpha_1}$.

Note that $e_1 \otimes (e_3^*)^k$ is a highest weight vector in V_λ and that $e_2 \otimes (e_3^*)^k$ is a vector of weight $\lambda - \alpha_1$ in V_λ. Thus $v = a(e_1 \otimes (e_3^*)^k) + b(e_2 \otimes (e_3^*)^k)$ for some coefficients $(a, b) \neq (0, 0)$. We show that every such $v \otimes v_\mu$ lies in the SL_3–orbit of $v_\lambda \otimes v_\mu$.

For $b = 0$, any upper triangular matrix of SL_3 will do it. Let $b > 0$ and define $A := \begin{pmatrix} 0 & \frac{1}{b} & * \\ -b & a & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3$.

Then $A(ae_1 + be_2) = -abe_2 + e_1 + abe_2 = e_1$ and $Ae_3^* = e_3^*$.

It remains to show that the Cartan component cannot be small if the coefficient l_1 is bigger than one. This follows from the Dense Orbits Criterion (Corollary 5.9).

Part (2): Case (B)(iii). We already know that the representation $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ has a small Cartan component (see Example 4.16). For $m > 1$ the Cartan component cannot be small by the Dense Orbits Criterion. □

Note that for the cases (B) (i) and (B) (iii) we can give non-zero decomposable tensors of the Cartan component that do not lie in the SL_3–orbit of a highest weight vector $v_\lambda \otimes v_\mu$ using the construction given in the proof of Example 6.4.

Let $\lambda = d\omega_1 + k\omega_2$ and $\mu = m\omega_2$ with $d > 1$, $k \geq 0$ and $m \geq 1$. As in Example 6.4 let $v \otimes e_3^*$ be a non-zero decomposable tensor in the subspace $(\mathbb{C}e_1^2 \oplus \mathbb{C}e_1e_2 \oplus \mathbb{C}e_2^2) \otimes e_3^*$ of the Cartan component that does not lie in the SL_3–orbit of $e_1^2 \otimes e_3^*$. Then the tensor $w := (ve_1^{d-2} \otimes (e_3^*)^k) \otimes (e_3^*)^m$ cannot lie in the SL_3–orbit of $(e_1^d \otimes (e_3^*)^k) \otimes (e_3^*)^m$. By construction, w belongs to $(\mathbb{C}(e_1^d) \oplus \mathbb{C}(e_1^{d-1}e_2) \oplus \mathbb{C}(e_1^{d-2}e_2^2)) \otimes e_3^*$. Hence by Corollary 6.3, w belongs to the Cartan component of the tensor product $V_{d\omega+1+k\omega_2} \otimes V_{m\omega_2}$.

7 An Elementary Approach to Representations of SL_2

The goal of this section is to show that tensor products of irreducible representations of SL_2 have small Cartan component. We prove this in the second subsection. In the first subsection we recall the results and notations we will use for this proof.

7.1 Associated Cones

Let V be a finite-dimensional representation of a reductive group G. For $v \in V$ denote the orbit of v by O_v. If the closure $\overline{O_v}$ contains zero, $\overline{O_v}$ it is called *unstable* and the vector v is said to be *unstable* or is called a *nullform*. If $v \in V$ is not unstable we say that v and $\overline{O_v}$ are *semi-stable*. We use the following classical result (which can be found e.g. in Kraft [Kr85], III.2.3).

Lemma 7.1. Hilbert Criterion
The form $f \in V_n$ is a nullform if and only if there exists a one-parameter sub group (1-PSG) $\lambda : \mathbb{C}^* \to \mathrm{SL}_2$ of SL_2 such that $\lambda(t)f$ tends to zero as $t \to 0$.

The *nullcone* in V is defined as the set of unstable vectors in V: $\mathcal{N}_V := \{v \in V \mid \overline{O_v} \ni 0\}$. If $\pi : V \to V/G$ is the algebraic quotient of V relative to G then the nullcone is given as $\mathcal{N}_V = \pi^{-1}(\pi(0))$, cf. Kraft [Kr85], II.3.3.

Let $X \subset V$ be a subset of V and $I(X) \subset \mathcal{O}(V)$ its defining ideal. For $f \in \mathcal{O}(V)$ let $\operatorname{gr} f$ be the leading term of f. Then the graded ideal of X is defined as the ideal generated by the leading terms of the elements of $I(X)$, $\operatorname{gr} I(X) := \langle \operatorname{gr} f \mid f \in I(X) \rangle$. For any ideal $J \subset \mathcal{O}(V)$ let $\mathcal{V}(J) \subset V$ be the zero set of J.

Definition 7.2. The *cone associated to* X is defined as $\mathcal{C}(X) := \mathcal{V}(\operatorname{gr} I(X))$.

We will use the following property of associated cones (to be found e.g. in Kraft [Kr85], II.4.2).

Proposition 7.3. *(i) The cone $\mathcal{C}(X)$ associated to X is a closed cone in V. It has the same dimension as X.*

(ii) Let $v \in V$ be semi-stable and set $X := O_v$. Then the cone associated to X has the following geometric description: $\mathcal{C}(X) = \overline{\mathbb{C}^* X} \setminus \mathbb{C}^* \overline{X}$.

Note that for arbitrary $v \in V \setminus \mathcal{N}_V$ there exist sequences $(c_i)_{i\in\mathbb{N}} \subset \mathbb{C}^*$ and $(g_i)_{i\in\mathbb{N}} \subset G$ such that $g_i(\lambda_i v)$ tends to a nullform, $v_0 := \lim_{i\to\infty} g_i(c_i v) \in \mathcal{N}_V$. We call the vector v_0 of the nullcone $\mathcal{N}_V$ a *limit point of* v.

Let O_v be an orbit in V. We define $\lim(O_v)$ as union of the limit points of all vectors in O_v, $\lim(O_v) := \bigcup_{\tilde{v}}\{w \in \mathcal{N}_V \mid w \text{ is a limit point of } \tilde{v}\}$. It is a subset of $\mathcal{N}_V$. A consequence of part (ii) of Proposition 7.3 is the following.

Corollary 7.4. *If O_v is a semi-stable orbit in V then $\mathcal{C}(\overline{O_v}) = \lim(O_v)$.*

7.2 Tensor Products of Irreducible SL_2–Representations

Let V_n be the vector space $\mathbb{C}[x,y]_n$. Our goal is to show that every decomposable tensor of the Cartan component V_{n+m} of $V_n \otimes V_m$ is of the form $l^n \otimes l^m$ for a linear form $l \in V_1 = \mathbb{C}[x,y]_1$.

We need the following properties to reach this goal:

Lemma 7.5. *(a) Let $f \otimes h \in V_n \otimes V_m$ an element of the Cartan component V_{n+m} of $V_n \otimes V_m$. If $(f_0, h_0) \in \mathcal{N}_{V_n \oplus V_m}$ is a limit point of $(f,h) \in V_n \oplus V_m$ then the tensor $f_0 \otimes h_0$ belongs to the Cartan component V_{n+m}.*

(b) The nullcone of $V_n \oplus V_m$ has the form

$$\mathcal{N}_{V_n \oplus V_m} = \{(l^r f_1, l^s h_1) \in V_n \oplus V_m \mid l \in V_1, r > \frac{n}{2}, s > \frac{m}{2}\}.$$

(c) Let $l^r f_1 \otimes l^s h_1$ be an element of the Cartan component V_{n+m} with $l \in V_1$, $r > \frac{n}{2}$ and $s > \frac{m}{2}$. Then

$$f_1 = l^{n-r} \quad \text{and} \quad h_1 = l^{m-s}.$$

We give an outline of the proof of these properties:

Idea of Proof: ad (a): Let $(f_0, h_0) = \lim_{j\to\infty} g_j c_j (f,h) \in \mathcal{N}_{V_n \oplus V_m}$. Show that $\tau_i(f_0, h_0) = \lim_{j\to\infty} g_j c_j \tau_i(f,h)$ and use the Clebsch-Gordan decomposition (see part I, Proposition 2.1) to prove (a).

ad (b): Apply the Hilbert Criterion (Lemma 7.1) to elements of the nullcone $\mathcal{N}_{V_n \oplus V_m}$.

ad (c): Consider the action of $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ on $\mathbb{C}[x,y]$,

$$\begin{aligned} e:&\quad y \mapsto x \quad x \mapsto 0 \\ f:&\quad x \mapsto y \quad y \mapsto 0. \end{aligned}$$

Note that ef acts on weight vectors of weight k in V_n as multiplication by $\frac{1}{4}(n(n+2) - k(k-2))$. Use this to show that f_1 is a multiple of x^{n-r} and that h_1 is a multiple of x^{m-s}. □

Lemma 7.6. *Let $f \otimes h$ be an element of the Cartan component V_{n+m} of the tensor product $V_n \otimes V_m$. Assume furthermore that (f,h) is semi-stable. Then the orbit $O_{(f,h)}$ in $V_n \oplus V_m$ is two-dimensional.*

Idea of Proof: If $f \otimes h$ is a non-zero tensor of the Cartan component and (f_0, h_0) a limit point of (f, h) then step (a) shows that the tensor $f_0 \otimes h_0$ belongs to the Cartan component.

By step (b), every such $f_0 \otimes h_0$ is of the form $l^r f_1 \otimes l^s h_1$ for some linear form l and $r > \frac{n}{2}$, $s > \frac{m}{2}$. We apply step (c) to $l^r f_1 \otimes l^s h_1$ and get $f_1 = l^{n-r}$, $h_1 = l^{m-s}$. Therefore every limit point of (f, h) is of the form (l^n, l^m) and the tensor $l^n \otimes l^m$ lies in the Cartan component V_{n+m}. In particular, every limit point (f_0, h_0) of (f, h) lies in the orbit $O_{(x^n, x^m)}$. Note that $O_{(x^n,x^m)}$ is two-dimensional.

It remains to show that the orbit $O_{(f,h)}$ has also dimension two: Since (f, h) is semi-stable we can use Corollary 7.4: Limit points of (f, h) lie in the cone associated to $\overline{O_{(f,h)}}$. Since the set $\lim(O_{(f,h)})$ lies in the orbit $O_{(x^n,x^m)}$ we have $\dim \mathcal{C}\overline{O_{(f,h)}} = \dim \overline{\lim(O_{(f,h)})} \leq \dim O_{(x^n,x^m)} = 2$. By part (i) of Proposition 7.3 the dimension of $\overline{O_{(f,h)}}$ is the same as the dimension of its associated cone $\mathcal{C}\overline{O_{(f,h)}}$.

Furthermore, the stabiliser of (f, h) in SL_2 is at most one-dimensional. Hence the dimension of $O_{(f,h)}$ is two. □

Proposition 7.7. *Let $f \otimes h$ be a non-zero tensor of the Cartan component $\mathbb{C}[x, y]_{n+m}$ of $\mathbb{C}[x, y]_n \otimes \mathbb{C}[x, y]_m$. Then there exists $g \in \mathrm{SL}_2$ such that $g(f \otimes h)$ is a non-zero multiple of $x^n \otimes x^m$.*

Idea of Proof: Assume that (f, h) is unstable, i.e. (f, h) lies in the nullcone $\mathcal{N}_{V_n \oplus V_m}$. By part (b) of Lemma 7.5, (f, h) is of the form $(l^r f_1, l^s h_1)$ for some $l \in V_1$. Since the tensor $l^r f_1 \otimes l^s h_1$ belongs to the Cartan component, part (c) of Lemma 7.5 yields that $f \otimes h$ is of the form $l^n \otimes l^m$. Therefore a multiple of $x^n \otimes x^m$ lies in the orbit $O_{f \otimes h} = O_{l^n \otimes l^m}$.

Let (f, h) be semi-stable. By Lemma 7.6 the dimension of $O_{f,h}$ is two. Every two-dimensional orbit in $V_n \oplus V_m$ contains (x^n, x^m) or $(x^r y^r, x^s y^s)$ (to see this study one-dimensional stabilisers in SL_2).
Since the tensor $x^r y^r \otimes x^s y^s$ does not belong to the Cartan component, the orbit $O_{(f,h)}$ cannot contain $(x^r y^r, x^s y^s)$. Hence $O_{(f,h)}$ contains (x^n, x^m) and the tensor $x^n \otimes x^m$ lies in the SL_2–orbit of $f \otimes h$. □

8 Further Results and Problems for the Special Linear Group

In this section we present different methods and approaches to representations of the special linear group. As before we denote by $I(\lambda)$ the set of simple roots perpendicular to the dominant weight λ. Recall that a tensor product $V_\lambda \otimes V_\mu$ is called critical if $I(\lambda) \neq I(\mu)$. We have seen that every non-critical tensor product has a small Cartan component (see Corollary 4.20). The main goal is to determine which critical representations have a small Cartan component and which have not. This turns out to be a rather difficult task. There still remain critical representations of SL_{n+1} where it is not clear whether the Cartan component is small.

We recall the construction of the fundamental representations of SL_{n+1}: Let $\{e_k\}_{1\leq k\leq n+1}$ be a basis of $\mathbb{C}^{n+1}$ with $\mathrm{wt}\, e_k = \varepsilon_k$. The fundamental weights are $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $1 \leq k \leq n$. The roots are of the form $\varepsilon_i - \varepsilon_j$ $(i \neq j \leq n+1)$ and the simple roots are $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$ for $k = 1, \ldots, n$.

Proposition 8.1. *Let $\Lambda^k(\mathbb{C}^{n+1})$ be the k–th exterior power of the natural representation of SL_{n+1} on $\mathbb{C}^{n+1}$, $1 \leq k \leq n$. Then $\Lambda^k(\mathbb{C}^{n+1})$ is irreducible of highest weight ω_k and $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is a highest weight vector for $\Lambda^k(\mathbb{C}^{n+1})$.*

Proof. See Theorem 5.1.6 in [GW98] (Goodman and Wallach). □

Definition 8.2. The representations $\Lambda^k(\mathbb{C}^{n+1})$, $k = 1, \ldots, n$, are called the *fundamental representations* of SL_{n+1}.

We usually denote the highest weight vectors $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ by v_k and we often abbreviate the vector space $\mathbb{C}^{n+1}$ by V. The fundamental representations can be used to describe the irreducible representation of a given dominant weight:

Let $\lambda = \sum_{i=1,\ldots,n} l_i\omega_i$ be a dominant weight. Then the tensor $v_1^{l_1} \otimes v_2^{l_2} \otimes \cdots \otimes v_n^{l_n}$ is a highest weight vector for the irreducible representation V_λ and $V_\lambda \subset S^{l_1}V \otimes S^{l_2}(\Lambda^2(V)) \otimes \cdots \otimes S^{l_n}(\Lambda^n(V))$. Note that it is not clear how the irreducible representation V_λ is embedded in this tensor product in general. There are very special cases where V_λ can be described as the kernel of a projection operator. For instance if the dominant weight is a sum of two different fundamental weights $\omega_k + \omega_j$ with $k < j$ we can give a projection operator from $\Lambda^k(V) \otimes \Lambda^j(V)$ to $\Lambda^{k-1}(V) \otimes \Lambda^{j+1}(V)$ whose kernel is $V_{\omega_k+\omega_j}$ (see Subsection 8.3 below).

8.1 Necessary Condition for the Special Linear Group

Suppose that the Cartan component of a given tensor product $V_\lambda \otimes V_\mu$ of irreducible representations is small. We are able to deduce a necessary condition on the highest weights λ and μ, see Proposition 8.3 below. Its result shows the following: if $V_\lambda \otimes V_\mu$ is a critical representation with small Cartan component then the weights λ and μ meet almost the same walls Ω_α of the dominant Weyl chamber.

Recall that $P_{I(\lambda)} \subset \mathrm{SL}_{n+1}$ is the parabolic subgroup spanned by the Borel subgroup B together with the root groups U_α, $\alpha \in \Phi$ such that $(\alpha \mid \lambda) = 0$. The group $L_{I(\lambda)}$ is the Levi subgroup of $P_{I(\lambda)}$ containing the torus T. By Theorem 5.7 the orbit $L_{I(\lambda)} v_\mu$ of a highest weight vector $v_\mu \in V_\mu$ is dense in the irreducible representation $\langle L_{I(\lambda)} v_\mu \rangle$ and similarly, the orbit $L_{I(\mu)} v_\lambda$ is dense in $\langle L_{I(\mu)} v_\lambda \rangle$. Using these facts we can prove the following result:

Proposition 8.3. *Let $\lambda = \sum l_i \omega_i$ and $\mu = \sum m_i \omega_i$ be dominant weights such that the Cartan component of the representation $V_\lambda \otimes V_\mu$ is small. Then the following holds:*

(1) There is at most one index k such that $l_k = 0$ and $m_k > 0$. In this case $m_k = 1$.

(2) There is at most one index j such that $m_j = 0$ and $l_j > 0$. In this case $l_j = 1$.

Observe that the coefficient l_i of λ is zero if and only if α_i belongs to $I(\lambda)$, i.e. if and only if α_i is perpendicular to λ.

Proof. We give the proof of the first assertion. The second follows by the same arguments.

(A) Suppose that there exist two simple roots $\{\alpha_k, \alpha_j\} \subset I(\mu)$ which are not perpendicular to λ. Let $v_\mu := v_1^{m_1} \otimes v_2^{m_2} \otimes \cdots \otimes v_n^{m_n}$ be a highest weight vector of V_μ.

(i) Suppose that the simple roots α_k and α_j are neighbours, w.l.o.g. let $j = k+1$. The root system $\Phi' \subset \Phi$ generated by $I(\mu)$ contains the roots $\pm\alpha_k, \pm\alpha_{k+1}, \pm(\alpha_k + \alpha_{k+1})$. Observe that the subsystem Φ'' that is generated by $\{\pm\alpha_k, \pm\alpha_{k+1}, \pm(\alpha_k + \alpha_{k+1})\} \subset \Phi'$ corresponds to the root system of SL_3. Hence $L_{I(\mu)}$ contains a factor SL_3 on the diagonal and is acting non-trivially as $\mathbb{C}^* \times \mathrm{SL}_3$ on the vectors v_k and v_{k+1}.

$$L_{I(\mu)} = \begin{pmatrix} \square & & & & 0 \\ & \ddots & & & \\ & & \boxed{\mathrm{SL}_3} & & \\ & 0 & & \ddots & \\ & & & & \square \end{pmatrix} \subset \mathrm{SL}_{n+1}$$

By the Dense Orbits Criterion (see Corollary 5.9), the orbit $\mathrm{SL}_3(v_k^{l_k} \otimes v_{k+1}^{l_{k+1}})$ is dense in $\langle \mathrm{SL}_3(v_k^{l_k} \otimes v_{k+1}^{l_{k+1}})\rangle$. By Table 1 in section 5, the only such representations are the natural or the dual (or the trivial) whence $(l_k, l_{k+1}) = (1, 0)$ for the natural representation or $(l_k, l_{k+1}) = (0, 1)$ for the dual representation (or both $l_k = 0$ and $l_{k+1} = 0$). This contradicts the assumption: whenever l_k (or l_{k+1}) equals zero, $\alpha_k \in I(\lambda)$ (resp. $\alpha_{k+1} \in I(\lambda)$).

(ii) In case $|k - j| > 1$, $L_{I(\mu)}$ contains a factor $\mathrm{SL}_2 \times \mathrm{SL}_2$ on the diagonal, and $L_{I(\mu)}$ is acting non-trivially as $\mathrm{GL}_2 \times \mathrm{GL}_2$ on v_k and on v_j. By Corollary 5.9 the $\mathrm{SL}_2 \times \mathrm{SL}_2$–orbit of $v_k^{l_k} \otimes v_j^{l_j}$ has to be dense in the representation $\langle(\mathrm{SL}_2 \times \mathrm{SL}_2)(v_k^{l_k} \otimes v_j^{l_j})\rangle$. This is not possible for strictly positive l_k, l_j: Let $l_k = l_j = 1$. Consider the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 =: M_2$. It has an invariant function, namely the determinant $\det : M_2 \to \mathbb{C}$. Hence the orbit of the highest weight vector $e_1 \otimes e_1$ is not dense.

(B) Suppose that there is one simple root α_j in $I(\mu)$ such that α_j is not perpendicular to λ. Then, $L_{I(\mu)}$ contains a factor SL_2 on the diagonal and is acting non-trivially on v_j as GL_2. We have seen in part one of the proof that since the orbit $\mathrm{SL}_2(v_j^{l_j})$ has to be dense in the representation it generates (see Corollary 5.9), the coefficient l_j equals one. □

Let $V_\lambda \otimes V_\mu$ be a critical representation. Our aim is to decide whether the Cartan component of $V_\lambda \otimes V_\mu$ is small. Proposition 8.3 severely restricts the choice of λ and μ: We can only expect $V_\lambda \otimes V_\mu$ to have a small Cartan component if the highest weights do not differ too much.

Definition 8.4. Let $V_\lambda \otimes V_\mu$ be a critical representation of SL_{n+1}.

(1) Assume that there exists exactly one simple root α_j in $I(\mu)$ such that α_j is not perpendicular to λ and $I(\mu) = I(\lambda) \cup \alpha_j$. Assume further that the coefficient l_j of λ equals one. Then we call $V_\lambda \otimes V_\mu$ *semi-critical.*

(2) Assume that there exists exactly one simple root α_k in $I(\lambda)$ such that α_k is not perpendicular to μ and exactly one simple root α_j in $I(\mu)$ such that α_j is not perpendicular to λ. Assume further that the coefficients l_j and m_k of λ resp. of μ equal one. Then we call $V_\lambda \otimes V_\mu$ *fully critical.*

By Proposition 8.3, every critical representation with small Cartan component is either semi- or fully critical.

8.2 Semi-Critical Representations

We consider very special critical representations. As announced in the title we assume that $V_\lambda \otimes V_\mu$ is semi-critical, i.e. $I(\mu)$ contains a simple root which is not perpendicular to λ. Furthermore, we assume that λ is regular. Hence $I(\mu) = \{\alpha\}$ for some simple root α. The next result shows that the Cartan components of almost all of the SL_{n+1}–representations of this type are not small.

Lemma 8.5. *Let $\lambda = \sum_{i \in I} l_i \omega_i$ be a regular dominant weight and μ be perpendicular to only one simple root, say to α_j. Then $V_\lambda \otimes V_\mu$ has a small Cartan component if and only if $l_j = 1$.*

Proof. (1) If the Cartan component is small then l_j is one by Proposition 8.3.

(2) By Theorem 3.12 we know that every non-zero decomposable tensor of the Cartan component lies in the SL_{n+1}–orbit of a tensor $v \otimes w$ such that $(P_\lambda \mid P_\mu)$ is maximal. We proceed by taking an arbitrary maximal pair and show that the tensors corresponding to such a pair lie in the SL_{n+1}–orbit of a highest weight vector.

Let $(P, Q) \in \mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ be a maximal pair. By Corollary 4.23, Q is an element of the orbit $\mathcal{W}\mu$. W.l.o.g. let $Q = \mu$. Write $P = \lambda - \sum r_i \alpha_i$ with non-negative coefficients r_i.
Then

$$\begin{aligned} (P \mid \mu) &= (\lambda \mid \mu) - \sum r_i (\alpha_i \mid \mu) \\ &\leq (\lambda \mid \mu). \end{aligned}$$

Since (P, μ) is a maximal pair we have equality. Now, $(\alpha_i \mid \mu) > 0$ for each $i \neq j$ and thus r_i has to vanish for each $i \neq j$. Hence

$$P = \lambda - r_j \alpha_j.$$

Note that the only weights of the form $\lambda - s\alpha_j$ in $\Pi(\lambda)$ are the two vertices λ and $\sigma_j(\lambda) = \lambda - \alpha_j$. Therefore P must be a linear combination of only λ and $\lambda - \alpha_j$. The inverse image of λ resp. of $\lambda - \alpha_j$ under P_λ is $P_\lambda^{-1}(\lambda) = \mathbb{C}^* v_\lambda$ resp. $P_\lambda^{-1}(\lambda - \alpha_j) = \mathbb{C}^* v_{\lambda - \alpha_j}$ (see Corollary 4.10). Hence every vector mapping to P under P_λ lies in the vector space $W := V_\lambda(\lambda) \oplus V_\lambda(\lambda - \alpha_j)$. Since Q equals the vertex μ the inverse image $P_\mu^{-1}(Q)$ is $\mathbb{C}^* v_\mu$ and so every non-zero vector of V_μ mapping to Q under P_μ is a highest weight vector.

We proceed by showing that all vectors of $W \otimes v_\mu$ lie in the SL_{n+1}–orbit of $v_\lambda \otimes v_\mu$. Let $A \in \mathrm{SL}_{n+1}$ Since we want Av_μ to be a non-zero multiple of v_μ the matrix A is an element of the stabiliser $\mathrm{Stab}_{\mathrm{SL}_{n+1}} \mathbb{C}v_\mu$. Let A be of the following form: $A = \begin{pmatrix} \ddots & & & * \\ & a & b & \\ & c & d & \\ 0 & & & \ddots \end{pmatrix}$ where the jth diagonal element is a and the $j+1$th element is d and such that the other diagonal elements are ones (hence $ad - bc = \det A = 1$). Let w be a non-zero vector in W, write $w = w_1 v_\lambda + w_2 v_{\lambda-\alpha_j}$ with $(w_1, w_2) \neq (0,0)$.

(i) If $w_2 \neq 0$ consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{w_2} \\ -w_2 & w_1 \end{bmatrix}$.

(ii) If $w_2 = 0$ consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In both cases, $Aw = v_\lambda$ and $Av_\mu = v_\mu$. □

Remark. One can use Corollary 6.3 to see that the vector space $W \otimes v_\mu$ defined in the proof of Lemma 8.5 is in fact a subset of the Cartan component.

8.3 Fundamental Representations

Fundamental representations of SL_{n+1} play an essential role as they serve as models for more complicated irreducible representations. It is clear that tensor products of fundamental representations are fully critical representations. Among these they are very special: The set of simple weights perpendicular to both of the highest weights is as big as possible, namely $I(\lambda+\mu)$ contains $n-2$ simple roots. The other extreme is the case where the set of simple roots perpendicular to λ and μ is empty. We discuss the latter in subsection 8.4 below.

The first case to look at are the tensor products of the natural or the dual representation with another fundamental representation.

Proposition 8.6. *1. The tensor product* $\mathbb{C}^{n+1} \otimes \Lambda^k(\mathbb{C}^{n+1})$ *has a small Cartan component if and only if* $k \in \{2, n\}$.

2. The tensor product $\Lambda^{n-k}(\mathbb{C}^{n+1}) \otimes (\mathbb{C}^{n+1})^*$ *has a small Cartan component if and only if* $k \in \{1, n-1\}$.

Proof. We show the first assertion. The second is essentially the same since the representation $\Lambda^{n-k}(\mathbb{C}^{n+1}) \otimes (\mathbb{C}^{n+1})^*$ is the dual of the representation $\mathbb{C}^{n+1} \otimes \Lambda^{k+1}(\mathbb{C}^{n+1})$.

The tensor product $\mathbb{C}^{n+1} \otimes \Lambda^k(\mathbb{C}^{n+1})$ decomposes as a direct sum of two irreducible representations $V_{\omega_1+\omega_k} \oplus \Lambda^{k+1}(\mathbb{C}^{n+1})$. Consider the map $\varphi : \mathbb{C}^{n+1} \otimes \Lambda^k(\mathbb{C}^{n+1}) \to \Lambda^{k+1}(\mathbb{C}^{n+1})$ given by

$$w \otimes \sum_{i_1<\cdots<i_k} c_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \quad \mapsto \quad \sum_{i_1<\cdots<i_k} c_{i_1\ldots i_k} w \wedge e_{i_1} \wedge \cdots e_{i_k}.$$

Note that the kernel of φ is the Cartan component $V_{\omega_1+\omega_k}$. Let $w \otimes v$ be a nonzero element of the Cartan component. W.l.o.g. let $w = e_1$ (using the action of SL_{n+1}). We write $v = \sum c_{i_1\ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$. We use the fact that $e_1 \otimes v$ is a non-zero element of the kernel of φ: Since $\varphi(e_1 \otimes v) = 0$ every term of v contains e_1 and so

$$v = e_1 \wedge \sum_{1<i_2<\cdots<i_k} c_{1 i_2\ldots i_k} e_{i_2} \wedge \cdots \wedge e_{i_k} =: e_1 \wedge v_0.$$

It remains to show that for every such non-zero $e_1 \otimes e_1 \wedge v_0$ there exists an element $g \in \mathrm{SL}_{n+1}$ sending the tensor to a highest weight vector. In particular, since g maps e_1 to a multiple of itself, g belongs to the stabiliser $\mathrm{Stab}_{\mathrm{SL}_{n+1}}(\mathbb{C}e_1)$ which is the parabolic subgroup

$$P = \left\{ \begin{bmatrix} * & * & \dots & * \\ 0 & * & & \\ \vdots & \vdots & \ddots & \\ 0 & * & & * \end{bmatrix} \right\} \in \mathrm{SL}_{n+1}\,.$$

Let $V' := \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{n+1}$. The group P acts on V' as GL_n and the vector v_0 is an element of the GL_n–representation $\Lambda^{k-1}(V')$. Note that every non-zero $e_1 \otimes e_1 \wedge v_0$ lies in the SL_{n+1}–orbit of the highest weight vector $e_1 \otimes e_1 \wedge \cdots \wedge e_k$ if and only if for each $v_0 \in \Lambda^{k-1}(V')$ there exists $g \in P$ such that gv_0 is a non-zero multiple of $e_2 \wedge \cdots \wedge e_k$. This in turn is the case if and only if the orbit $P(e_2 \wedge \cdots \wedge e_k)$ is dense in $\Lambda^{k-1}(V')$. Hence either $k-1 = 1$ (and $\Lambda^1(V') = V'$ is the natural representation of GL_n) or $k - 1 = \dim V' - 1 = n - 1$ (and $\Lambda^{n-1}(V')$ is the dual representation of GL_n) using Table 5. In other words $\mathbb{C}^{n+1} \otimes \Lambda^k(\mathbb{C}^{n+1})$ has a small Cartan component if and only if $k = 2$ or $k = n$. □

Proposition 8.7. *The tensor products $\Lambda^k(\mathbb{C}^{n+1}) \otimes \Lambda^{k+1}(\mathbb{C}^{n+1})$ have small Cartan components for each $k = 1, \ldots, n-1$.*

We have already shown this for $k = 1$ and $k = n - 1$ in Proposition 8.6. For arbitrary k, a proof was recently found by Christian Ohn [Oh02].

Remark 8.8. Originally, the idea of a proof for $2 \leq k \leq n-2$ was along the following lines:

Consider the vector spaces $U_1 := V_{\omega_k}(\omega_k) \oplus V_{\omega_k}(\omega_k - \alpha_k)$ and $U_2 := V_{\omega_{k+1}}(\omega_{k+1}) \oplus V_{\omega_{k+1}}(\omega_{k+1} - \alpha_{k+1})$.

We claim that the elements of $\mathrm{SL}_{n+1}(U_1 \otimes v_{\omega_{k+1}})$ and of $\mathrm{SL}_{n+1}(v_{\omega_k} \otimes U_2)$ are the only decomposable tensors for which $(P_{\omega_k}(u_1) \mid P_{\omega_{k+1}}(u_2))$ is maximal.

Then every decomposable tensor of the Cartan component $V_{\omega_k+\omega_{k+1}}$ of the representation $\Lambda^k(\mathbb{C}^{n+1}) \otimes \Lambda^{k+1}(\mathbb{C}^{n+1})$ lies in the SL_{n+1}–orbit of $U_1 \otimes v_{\omega_{k+1}}$ or of $v_{\omega_k} \otimes U_2$.

It remains to prove that the two-dimensional vector spaces $U_1 \otimes v_{\omega_{k+1}}$ and $v_{\omega_k} \otimes U_2$ are subsets of the SL_{n+1}–orbit of $v_k \otimes v_{k+1}$: Let $u_1 \otimes u_2 \in U_1 \otimes U_2$ be a non-zero tensor. One can show that there exist $A \in \mathrm{SL}_{n+1}$ such that Au_1 is a non-zero multiple of $v_k = e_1 \wedge \cdots \wedge e_k$. Furthermore, there always exist $A \in \mathrm{SL}_{n+1}$ satisfying this requirement and sending u_2 to the highest weight vector $v_{k+1} = e_1 \wedge \cdots \wedge e_{k+1}$.

Proposition 8.9. *The Cartan component of* $\Lambda^k(\mathbb{C}^{n+1}) \otimes \Lambda^l(\mathbb{C}^{n+1})$ *is not small whenever* $2 \leq l < k-1$, $k \leq n$.

Proof. We write $V := \mathbb{C}^{n+1}$. Recall the projection operator used in the proof of Proposition 8.6: Let $\varphi : \Lambda^k(V) \otimes \Lambda^l(V) \to \Lambda^{k+1}(V) \otimes \Lambda^{l-1}(V)$ be defined by

$$u_1 \wedge \cdots \wedge u_k \otimes w_1 \wedge \cdots \wedge w_l \mapsto \sum_{i=1}^{i} (-1)^l u_1 \wedge \cdots \wedge u_k \wedge w_i \otimes w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_l.$$

Its kernel is the Cartan component of the tensor product, $\ker \varphi = V_{\omega_l+\omega_k}$.

Let $v_k = e_1 \wedge \cdots \wedge e_k$ and define $V' := \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_k$. Then $\varphi(v_k \otimes w) = 0$ for every $w \in \Lambda^l(V')$, hence $v_k \otimes \Lambda^l(V')$ lies in the Cartan component of $\Lambda^k(V) \otimes \Lambda^l(V)$.

It remains to show that the closure of the orbit $\mathrm{SL}_{n+1}(v_k \otimes v_l)$ cannot contain all of $v_k \otimes \Lambda^l(V')$: Suppose that every non-zero $v_k \otimes w$ in $v_k \otimes \Lambda^l(V')$ belongs to $\mathrm{SL}_{n+1}(v_k \otimes v_l)$. This is the case if and only if the $\mathrm{GL}(V')$–orbit of v_l is dense in $\Lambda^l(V')$. The $\mathrm{GL}(V')$–orbit of v_l can only be dense in $\Lambda^l(V')$ if this representation is the natural (and so, $l = 1$) or the dual (hence $l = k-1$). This contradicts the assumptions. Therefore there exist non-zero vectors $v_k \otimes w$ in the Cartan component $V_{\omega_k+\omega_l}$ that do not lie in the SL_{n+1}–orbit of the highest weight vector $v_k \otimes v_l$. □

8.4 Fully Critical Representations

Let λ and μ be dominant weights. Recall that a fully critical representation is a representation $V_\lambda \otimes V_\mu$ for which there exist only two simple roots $\alpha_j \neq \alpha_k$ such that $l_k = m_j = 0$ **and** $l_j = m_k = 1$ (hence λ is perpendicular to α_k but not to α_j and μ is perpendicular to α_j but not to α_k). In contrast to Subsection 8.3 above we do not make any assumptions on the set $I(\lambda + \mu)$ of simple roots perpendicular to λ and μ.

We claim that the Cartan components of most of the fully critical representations are not small. Namely the only fully critical representations with small Cartan components should be those where the two simple roots α_k and α_j are neighbours (i.e. joined in the Dynkin diagram by an edge) or when $k = 1$ and $j = n$.

Lemma 8.10. *Let $\lambda = \sum l_i\omega_i$ and $\mu = \sum m_i\omega_i$ be dominant weights such that $V_\lambda \otimes V_\mu$ is a fully critical representation. Let $(\alpha_k \mid \lambda) = (\alpha_j \mid \mu) = 0$ and $l_j = m_k = 1$.*

(1) Assume furthermore that $(\alpha_k \mid \alpha_j) = 0$. Then:
$V_{\lambda+\mu}(\lambda + \mu - b\alpha_j - c\alpha_k) = (V_\lambda \otimes V_\mu)(\lambda + \mu - b\alpha_j - c\alpha_k) = \mathbb{C}(v_{\lambda - b\alpha_j} \otimes v_{\mu - c\alpha_k})$ *for every $(b, c) \in \{0, 1\}^2$.*

(2) If $j = k + 1$ then:
$V_{\lambda+\mu}(\lambda + \mu - b\alpha_k) = (V_\lambda \otimes V_\mu)(\lambda + \mu - b\alpha_k) = \mathbb{C}(v_\lambda \otimes v_{\mu - b\alpha_k})$ *and*
$V_{\lambda+\mu}(\lambda + \mu - b\alpha_{k+1}) = (V_\lambda \otimes V_\mu)(\lambda + \mu - b\alpha_{k+1}) = \mathbb{C}(v_{\lambda - b\alpha_{k+1}} \otimes v_\mu)$ *for $b = 0, 1$.*

Proof. We prove the first part since the second assertion follows similarly.
(A) It is clear that $V_{\lambda+\mu}(\lambda+\mu) = (V_\lambda \otimes V_\mu)(\lambda+\mu) = V_\lambda(\lambda) \otimes V_\mu(\mu) = \mathbb{C}v_\lambda \otimes v_\mu$. Observe that the weights $\lambda + \mu - b\alpha_k - c\alpha_j$ with $(b, c) \in \{0, 1\}^2$ belong to the Weyl orbit of $\lambda + \mu$:

$$\begin{aligned} s_k(\lambda + \mu) &= \lambda + \mu - \alpha_k \\ s_j(\lambda + \mu) &= \lambda + \mu - \alpha_j \\ s_k s_j(\lambda + \mu) &= s_j s_k(\lambda + \mu) \\ &= \lambda + \mu - \alpha_k - \alpha_j. \end{aligned}$$

Hence they all have multiplicity one in $V_{\lambda+\mu}$ and in $V_\lambda \otimes V_\mu$. Therefore the weight subspaces $V_{\lambda+\mu}(\nu)$ and $(V_\lambda \otimes V_\mu)(\nu)$ coincide for every such $\nu = \lambda + \mu - b\alpha_j - c\alpha_k$.
(B) Recall that the weight subspaces $V_{\lambda-\alpha_j} = \mathbb{C}v_{\lambda-\alpha_j}$ and $V_{\mu-\alpha_k} = \mathbb{C}v_{\mu-\alpha_k}$ are one-dimensional (see Lemma 6.1). Hence the second equality holds. ☐

Lemma 8.11. *Let $V_\lambda \otimes V_\mu$ be a fully critical representation such that $I(\lambda) = \{\alpha_k\}$ and $I(\mu) = \{\alpha_j\}$. Let (P, Q) be a maximal pair in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$. W.l.o.g. let P be an element of the dominant Weyl chamber $X^+_{\mathbb{R}}$.*

(1) Then there exists $\omega \in \mathcal{W}$ and coefficients $0 \leq r, s \leq \frac{1}{2}$ such that ωQ lies in the dominant Weyl chamber $X^+_{\mathbb{R}}$ and such that $P = \lambda - s\alpha_{k+1}$, $\omega Q = \mu - r\alpha_k$.

(2) If, furthermore, the two simple roots are neighbours, say $j = k + 1$, then $rs = 0$.

Proof. (1) Use Lemma 4.22 to see that $P = \lambda - s_0\alpha_{k+1}$ and $\omega Q = \mu - r_0\alpha_k$ for some non-negative coefficients s_0, r_0. Note that $\sigma_{k+1}(\lambda) = \lambda - \alpha_{k+1}$ and $\sigma_k(\mu) = \mu - \alpha_k$. Hence the only points in the convex hull $\mathrm{Con}(\lambda)$ of the form $\lambda - s\alpha_{k+1}$ are linear combinations of the vertices λ and $\lambda - \alpha_{k+1}$ and so $s_0 \leq 1$. We assumed that P lies in the dominant Weyl chamber, so $s_0 \leq \frac{1}{2}$. Essentially the same arguments show that $r_0 \leq \frac{1}{2}$.
(2) Since P and ωQ lie in the same chamber, $(P \mid \omega Q) \geq (P \mid \omega'\omega Q)$ for every $\omega' \in \mathcal{W}$ (see Lemma 4.8). In particular, $(P \mid \omega Q) \geq (P \mid Q)$ (choosing ω^{-1} for ω'). By assumption this value is maximal and so

$$\begin{aligned} (\lambda \mid \mu) &= (P \mid \omega Q) \\ &= (\lambda - r\alpha_{k+1} \mid \mu - s\alpha_k) \\ &= (\lambda \mid \mu) + rs \end{aligned}$$

Hence rs equals zero. □

Corollary 8.12. *Let $V_\lambda \otimes V_\mu$ be a fully critical representation such that $I(\lambda) = \{\alpha_k\}$ and $I(\mu) = \{\alpha_j\}$. Then the subspace $(V_\lambda(\lambda) \oplus V_\lambda(\lambda - \alpha_j)) \otimes (V_\mu(\mu) \oplus V_\mu(\mu - \alpha_k))$ of $V_\lambda \otimes V_\mu$ belongs to the Cartan component.*

Proof. Follows immediately from part (1) of Lemma 8.10. □

Proposition 8.13. *Let $V_\lambda \otimes V_\mu$ be a fully critical representation with $I(\lambda) = \{\alpha_k\}$ and $I(\mu) = \{\alpha_{k+1}\}$. Then the Cartan component of $V_\lambda \otimes V_\mu$ is small.*

Proof. Recall that in the SU_{n+1}–orbit of every non-zero decomposable tensor of the Cartan component exists a tensor $v \otimes w$ such that the value $(P_\lambda(v) \mid P_\mu(w))$ is maximal, i.e. equals $(\lambda \mid \mu)$.
(1) By Lemma 8.11 for every maximal pair (P, Q) in $\mathrm{Con}(\lambda) \times \mathrm{Con}(\mu)$ there exists $\omega \in \mathcal{W}$ and $0 \leq r, s \leq \frac{1}{2}$ such that P is λ and $\omega Q = \mu - r\alpha_k$ or $P = \lambda - s\alpha_{k+1}$ and $\omega Q = \mu$.
(2) The vector spaces $W_1 := v_\lambda \otimes (V_\mu(\mu) \oplus V_\mu(\mu - \alpha_k))$ and $W_2 := (V_\lambda(\lambda) \oplus V_\lambda(\lambda - \alpha_{k+1})) \otimes v_\mu$ consist of decomposable tensors of the Cartan component (use part two of Lemma 8.10).

(3) Show that the maximal pairs $(\omega\lambda, \mu - r\alpha_k)$ and $(\omega(\lambda - s\alpha_{k+1}), \mu)$ correspond to tensors in the vector spaces W_1 and W_2:

(i) Let $P = \lambda$ and $\omega Q = \mu - r\alpha_k$. Note that the pair (P, Q) lies in the Weyl-orbit of $(\lambda, \mu - r\alpha_k)$: If $0 < r < \frac{1}{2}$ we use Lemma 4.8 to see that $\omega Q = Q$. Let $r = \frac{1}{2}$. Consider the element $(\sigma_k P, \sigma_k Q)$ in the Weyl-orbit of (P, Q): We have $\sigma_k P = \lambda$ and $\sigma_k(\omega Q) = \mu - \frac{1}{2}\alpha_k$.

(ii) The same arguments show that for $P = \lambda - s\alpha_{k+1}$, $0 \leq s \leq \frac{1}{2}$ and $\omega Q = \mu$ the pair (P, Q) lies in the Weyl-orbit of $(\lambda - s\alpha_{k+1}, \mu)$.

Therefore every maximal pair comes from a tensor in W_1 or in W_2.

(4) Prove that W_1 and W_2 are subsets of the closure of the SL_{n+1}–orbit of $v_\lambda \otimes v_\mu$: Consider a non-zero tensor $w \otimes v_\mu$ where $w = w_1 v_\lambda + w_2 v_{\lambda - \alpha_{k+1}}$. We have seen in the proof of Lemma 8.5 that there exists $A \in \mathrm{SL}_{n+1}$ such that $Aw \otimes v_\mu$ equals $v_\lambda \otimes v_\mu$. □

8.5 Critical Representations

In this subsection we give a description of semi-critical and fully critical representations that can have a small Cartan component. The result gives another useful tool to restrict the search for critical representations with small Cartan component.

Proposition 8.14. *(1) If $V_\lambda \otimes V_\mu$ is a semi-critical representation (i.e. there exists i with $\alpha_i \perp \mu$ and $l_i = 1$) with a small Cartan component, then $I(\mu)$ cannot contain both α_{i-1} and α_{i+1}.*

(2) If $V_\lambda \otimes V_\mu$ is a fully critical representation (i.e. there exists $i \neq j$ with $\alpha_i \perp \mu$, $\alpha_j \perp \lambda$ and $l_i = m_j = 1$) with a small Cartan component, then $I(\lambda + \mu)$ can neither contain both α_{i-1} and α_{i+1} nor both α_{j-1} and α_{j+1}.

In other words: (1) If α_i belongs to a connected string of simple roots perpendicular to μ, then α_i has to be a vertex of this string. (2) The roots α_i resp. α_j have to be vertices of any connected string of simple roots perpendicular to μ resp. to λ.

Proof. Part (1): We use the Dense Orbits Criterion (Corollary 5.9). Assume that $\{\alpha_{i-1}, \alpha_i, \alpha_{i+1}\}$ lie in $I(\mu)$. Then $L_{I(\mu)}$ contains a SL_4-block (corresponding to the simple roots $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$). Note that the coefficients l_{i-1} and l_{i+1} of λ are zero since by assumption, the α_{i-1} and α_{i+1} are perpendicular to λ. Consider the action of $L_{I(\mu)}$ on $v_\lambda = v_1^{l_1} \otimes \cdots \otimes v_n^{l_n}$ (with $v_k = e_1 \wedge \ldots e_k$): The SL_4-block described above is acting on $(v_{i-1})^0 \otimes v_i \otimes v_{i+1}$ in the same way as SL_4 is acting on e_2.

Now since the Cartan component is small, Corollary 5.9 implies that $\langle L_{I(\mu)}v_\lambda\rangle$ is the closure of $L_{I(\mu)}v_\lambda$. This would imply that the representation generated by $\mathrm{SL}_4\, e_2$ is the closure of this orbit which is impossible (cf. table 5.2).
Part (2) follows from applying part (1) twice. □

Note that Proposition 8.9 is an application of this result (with λ and μ fundamental).

8.6 Conclusion

To install an order among irreducible representations of SL_{n+1} we can first divide them into non-critical, semi-critical and fully critical representations. Note that there are restrictions on the coefficients in the latter two cases: Recall that if α_j is only perpendicular to λ then the coefficient m_j of μ equals one and if α_i is only perpendicular to μ then l_i equals one (see Subsection 8.1). Recall that by Proposition 8.14, these roots α_j (and α_i) cannot lie in the interior of a connected string of simple roots in $I(\lambda+\mu)$. The next step is to count the simple roots which are perpendicular to both of the dominant weights.

Definition 8.15. Let λ and μ be dominant weights such that $V_\lambda \otimes V_\mu$ is non-critical, semi-critical or fully critical. Let p be the number of simple roots perpendicular to λ and μ. We define the *type of the representation* $V_\lambda \otimes V_\mu$ as the index $(\#I(\lambda)-p, \#I(\mu)-p, p)$.

It is clear that the first two numbers can only be zero or one if the Cartan component of the corresponding representation is small: The type is $(0,0,p)$ with $0 \le p < n$ corresponds to the family of non-critical representations. The type $(1,0,p)$ (resp. $(0,1,p)$), $0 \le p < n-1$, corresponds to the family of semi-critical representations. Finally, the type $(1,1,p)$, $0 \le p \le n-2$, corresponds to fully critical representations.

Conjecture 8.16. *Let $V_\lambda \otimes V_\mu$ be a tensor product of irreducible representations. Then the following holds:*

(1) Let $V_\lambda \otimes V_\mu$ have type $(0,0,p)$, $p<n$. Then the representation has a small Cartan component.

(2) Let $V_\lambda \otimes V_\mu$ have type $(0,1,p)$ (or $(1,0,p)$), $p<n-1$. Then the representation has a small Cartan component if and only if $I(\mu) \ni \alpha_i \notin I(\lambda)$ is a vertex of the connected string of simple roots in $I(\lambda+\mu)$ it belongs to.

(3) Let $V_\lambda \otimes V_\mu$ have type $(1,1,p)$, $p \le n-2$, let α_j and α_i be the two simple roots that are perpendicular to different highest weights. Then the

representation has a small Cartan component if and only if α_i and α_j are vertices of the connected strings of simple roots in $I(\lambda+\mu)$ they belong to.

Idea of Proof: (1) The representation is not critical, the assertion follows from Corollary 4.20.
(2) The "only if"-part follows from Proposition 8.14 above. The "if"-part for $p > 0$ remains to be proved.
For $p = 0$, the assertion follows from Lemma 8.5.
(3) Again, the "only if"-part follows from Proposition 8.14.
For the "if"-part there are three different situations:
(i) Let $p = n - 2$. Hence we are dealing with the fundamental representation $V_{\omega_k} \otimes V_{\omega_j}$. In this case the assertion follows from Propositions 8.6 and 8.7: The Cartan component is small if and only if (i, j) is $(i, i + 1)$ or $(1, n)$.
(ii) Let $p = 0$. If $j = k + 1$ the assertion follows from Proposition 8.13. In case $i = 1$ and $j = n$ we claim that the Cartan component is small.
(iii) For $0 < p < n - 2$ everything remains to be proved. □

8.7 Process of Reduction

In this subsection we prove a result that can be used to generate a family of critical representations for which the Cartan component is not small. Suppose $V_\lambda \otimes V_\mu$ is a critical representation of SL_{n+1} such that the Cartan component $V_{\lambda+\mu}$ is not small. We can extend the weights λ and μ by zeroes, i.e. if $\lambda = \sum_1^n l_i\omega_i$ let $\lambda' = \sum_1^{n+p} l_i\omega_i$ where the coefficients $l_{n+1}, \ldots, l_{n+p}$ are all zero and define μ' similarly. Then we can use the result of Proposition 8.17 below to show that the Cartan component $V_{\lambda'+\mu'}$ of the SL_{n+p+1} is not small.

The reduction process proceeds in the other direction. The idea is to start with a representation of SL_{n+1} such that the highest weights are both perpendicular to the simple root α_n. We can consider the weights as dominant weights for SL_n and study the tensor product as a representation of SL_n. The idea is to minimise the set $I(\lambda + \mu)$ of simple roots perpendicular to λ and μ.

Unfortunately we cannot eliminate all simple roots in $I(\lambda+\mu)$: Let $V_\lambda \otimes V_\mu$ be a fully critical representation with $(\lambda \mid \alpha_k) = (\mu \mid \alpha_j) = 0$ and such that the coefficients $l_j = m_k$ are one. Suppose that there exists a simple root α_i in $I(\lambda + \mu)$ lying between α_k and α_j (i.e. $k < i < j$). Then Proposition 8.17 cannot be used to eliminate α_i.

Proposition 8.17 (Reduction). *Let $V_\lambda \otimes V_\mu$ be a representation of SL_{n+1} which has a small Cartan component and such that $(\alpha_n \mid \lambda) = (\alpha_n \mid \mu) = 0$. Then $V_\lambda \otimes V_\mu$ considered as a representation of SL_n has a small Cartan component.*

To avoid confusion about the group acting on the modules we denote by $V_\lambda(n)$ resp. $V_\mu(n)$ the irreducible SL_n–modules with highest weight λ resp. μ and maximal vectors $v_{\lambda,n}$ resp. $v_{\mu,n}$. Let $B(n) := \overline{\mathrm{SL}_n(v_{\lambda,n} \otimes v_{\mu,n})}$ be the closure of the SL_n–orbit of the maximal tensor, $(V_{\lambda+\mu})(n)$ the Cartan component and $\mathrm{Dec}(n)$ the set of decomposable tensors in the SL_n– module $(V_\lambda \otimes V_\mu)(n)$.

On the other hand set $B := \overline{\mathrm{SL}_{n+1}(v_\lambda \otimes v_\mu)}$ and let $V_{\lambda+\mu}$ and Dec be the Cartan component resp. the set of decomposable tensors in the SL_{n+1}–representation $V_\lambda \otimes V_\mu$.

Idea of Proof: (1) We first show that $\mathrm{Dec} \cap (V_\lambda \otimes V_\mu)(n)$ equals $\mathrm{Dec}(n)$: Let $V_0 \subset V$ and $W_0 \subset W$ be arbitrary vector spaces. Write $V = V_0 \oplus V'$, $W = W_0 \oplus W'$ and denote by Dec the set of decomposable tensors in $V \otimes W$ and by Dec_0 the set of decomposable tensors in $V_0 \otimes W_0$. Then,

$$V \otimes W = (V_0 \otimes W_0) \oplus (V_0 \otimes W') \oplus (V' \otimes W_0) \oplus (V' \otimes W').$$

It is clear that Dec_0 is a subspace of $\mathrm{Dec} \cap (V_0 \otimes W_0)$. Let $v \otimes w$ be a non-zero tensor in $\mathrm{Dec} \cap V_0 \otimes W_0$. Write $v = v_0 + v'$ and $w = w_0 + w'$. In particular, $v \otimes w = v_0 \otimes w_0 + v_0 \otimes w' + v' \otimes w_0 + v' \otimes w'$ lies in $V_0 \otimes W_0$. Thus only the first term is nonzero and so $v' = w' = 0$. Hence $v \otimes w$ lies in Dec_0 and

$$\mathrm{Dec} \cap V_0 \otimes W_0 = \mathrm{Dec}_0 .$$

(2) We show that $V_{\lambda+\mu} \cap (V_\lambda \otimes V_\mu)(n)$ equals $V_{\lambda+\mu}(n)$: The Cartan component $V_{\lambda+\mu}(n)$ occurs once in this intersection. Suppose that there exists a weight $\nu \neq \lambda + \mu$ such that the irreducible component $V_\nu(n)$ also lies in this intersection.

The weight ν comes from a weight $\nu' := \nu + k\omega_n$ of $V_\lambda \otimes V_\mu$ (with $k \geq 0$). Observe that $V_\nu(n)$ has multiplicity one in the SL_{n+1}–representation $V_{\nu'}$ (branching law). Since the Cartan component $V_{\lambda+\mu}$ has trivial intersection with $V_{\nu'}$, the component $V_\nu(n)$ must be trivial.

(3) It remains to show that $B(n) = B \cap V_{\lambda+\mu}(n)$: The injection $V_{\lambda+\mu}(n) \hookrightarrow V_{\lambda+\mu}$ induces a surjection of the coordinate rings $\mathcal{O}(V_{\lambda+\mu}) \twoheadrightarrow \mathcal{O}(V_{\lambda+\mu}(n)) = \mathcal{O}(V_{\lambda+\mu})/J$ where J denotes the ideal of $V_{\lambda+\mu}(n)$ in $V_{\lambda+\mu}$. Write the coordinate rings as

$$\begin{aligned} \mathcal{O}(V_{\lambda+\mu}) &= \mathbb{C} \oplus V_{(\lambda+\mu)^*} \oplus S^2 V_{(\lambda+\mu)^*} \oplus \cdots \\ \mathcal{O}(V_{\lambda+\mu}(n)) &= \mathbb{C} \oplus V_{(\lambda+\mu)^*}(n) \oplus S^2 V_{(\lambda+\mu)^*}(n) \oplus \cdots \end{aligned}$$

and $S^2 V_{(\lambda+\mu)^*} = V_{(2(\lambda+\mu))^*} \oplus K$ resp. $S^2 V_{2(\lambda+\mu)^*}(n) = V_{2(\lambda+\mu)^*}(n) \oplus K(n)$ where K resp. $K(n)$ is the orthogonal complement. Thus the surjection $\mathcal{O}(V_{\lambda+\mu}) \twoheadrightarrow \mathcal{O}(V_{\lambda+\mu}(n))$ induces a surjection $K \twoheadrightarrow K(n) = K/J$.

By a result of KOSTANT (see BRION [Br85], §4, page 382), the ideal of $B(n)$ (resp. of B) is generated by $K(n)$ (resp. by K):

$$\begin{aligned} I(B(n)) &= \mathcal{O}(V_{\lambda+\mu}(n)) \cdot K(n), \\ I(B) &= \mathcal{O}(V_{\lambda+\mu}) \cdot K. \end{aligned}$$

Hence

$$\begin{aligned} I(B(n)) &= \mathcal{O}(V_{\lambda+\mu}(n)) \cdot K(n) \\ &= (\mathcal{O}(V_{\lambda+\mu})/J) \cdot (K/J) \\ &= (\mathcal{O}(V_{\lambda+\mu}) \cdot K)/J \\ &= I(B)/J \end{aligned}$$

which shows that

$$B \cap V_{\lambda+\mu}(n) = B(n).$$

Combining (1) and (2) yields

$$\mathrm{Dec}(n) \cap \; (V_{\lambda+\mu})(n) \quad = \quad \mathrm{Dec} \cap V_{\lambda+\mu} \cap (V_\lambda \otimes V_\mu)(n).$$

By assumption, this equals

$$B \cap (V_\lambda \otimes V_\mu)(n)$$

which is $B(n)$ by (3). □

Note that this process can be iterated. For instance let λ and μ be highest weights which are perpendicular to the simple roots $\alpha_1, \ldots, \alpha_r$ and $\alpha_s, \ldots, \alpha_n$. Suppose that the corresponding representation has a small Cartan component. Then we can use Proposition 8.17 successively for α_n, $\alpha_{n-1}, \ldots, \alpha_s$ and then for $\alpha_1, \ldots, \alpha_r$.

As announced in the beginning of this subsection the idea is to use Proposition 8.17 in the opposite direction: Let $V_\lambda \otimes V_\mu$ be a fully critical representation of SL_{p+1} (where $p < n$) such that its Cartan component is not small. Extend $\lambda = \sum_i^p l_i \omega_i$ and μ to a weight for SL_{n+1} be zeroes (i.e. set $l_i = m_i = 0$ for every $i = p+1, \ldots, n$). Then the corresponding representation of SL_{n+1} does not have a small Cartan component.

Note that using this process we can prove Proposition 8.9 by applying Proposition 8.17 to Proposition 8.6. If we start with a fully critical representation such that its Cartan component is not small we can produce a whole family of such fully critical representations.

Remark. If Conjecture 8.16 is correct there is no need to use the result from Proposition 8.17.

References

[Bo97] A. Borel, *Linear Algebraic Groups*, Second Enlarged Edition, GTM **126**, Springer-Verlag, New York 1997

[Bou68] N. Bourbaki, *Groupes et algèbres de Lie. Chap. IV - VI*, Hermann, Paris 1968

[Bou75] N. Bourbaki, *Groupes et algèbres de Lie. Chap. 7, 8*, Hermann, Paris 1975

[Br85] M. Brion, *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. École Norm. Sup. **18** (1985), 345–387

[Br87] M. Brion, *Sur l'image de l'application moment*, Seminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, (Paris 1986), Springer Berlin, 1987, 177–192

[FH96] W. Fulton, J. Harris, *Representation Theory. A First Course*, Springer, New York 1996

[GW98] R. Goodman, N. R. Wallach, *Representations and Invariants of the Classical Groups*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge 1998

[Hu75] J. E. Humphreys, *Linear Algebraic Groups*, GTM **21**, Springer, New York 1975

[Hu94] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, GTM **9** (6th corrected printing), Springer, New York 1994

[Jo95] A. Joseph, *Quantum Groups and Their Primitive Ideals*, Springer, Berlin 1995

[Ko98] B. Kostant, *Personal Communication*, June 1998

[Kr85] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Vieweg, Braunschweig 1985

[KP00] H. Kraft, C. Procesi, *Classical Invariant Theory. A Primer*, Lecture Notes, Version 2000

[KW98] H. Kraft, N. R. Wallach, *Personal Communication*, September 1998

[KW02] H. Kraft, N. R. Wallach, *On the Separation Property of Orbits in Representation Spaces*, to appear in J. Algebra

[Li90] P. Littelmann, *A Generalization of the Littlewood-Richardson Rule*, J. Algebra **130** (1990), 328–368

[Oh02] C. Ohn, *Personal Communication*, April 2002

[OV90] A.L. Onishik, E.B. Vinberg, *Lie Groups and Algebraic Groups*, Springer, Berlin 1990

[OV94] A.L. Onishik, E.B. Vinberg, *Linear Groups and Lie Algebras III. Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences Vol. 41, Springer, Berlin 1994

[PS94] A.N. Parshin, I.R. Shafarevich (Eds.), *Algebraic Geometry IV*, Encyclopaedia of Mathematical Sciences Vol. 55, Springer, Berling 1994

[Vi89] E.B. Vinberg, *Linear Representations of Groups*, Birkhäuser, Basel 1989

Curriculum Vitae

Karin Ursula Baur, geboren am 13. Mai 1970 in Zürich.
Heimatorte: Zürich (ZH) und Gaiserwald (SG).

April 77 - Sept. 89	Schulen in Zürich, Matura Typus B
Jan. 90 - Juni 90	Studentin an der Université de Paris III (Sorbonne) Cours de l'Initiation à la Civilisation Française
Okt. 90 - Feb. 97	Studium der Mathematik, der Philosophie und der französischen Literatur an der Universität Zürich
Okt. 93 - April 94	Austauschstudentin an der Université de Paris VI (Pierre et Marie Curie)
Feb. 97	Diplom in Mathematik, Diplomarbeit zum Thema *Beispiel einer Überlagerungskonstruktion im komplexen projektiven Raum* Betreuung durch Prof. Dr. C. Okonek
April - Sept. 97	Stellvertretungen als Mathematiklehrerin
Seit Okt. 97	Doktorandin am mathematischen Institut der Universität Basel Betreuer der Dissertation: Prof. Dr. H. Kraft
Okt. 97 - Sept. 99	Assistentin am mathematischen Institut in Basel und Redaktionssekretärin der *Commentarii Mathematici Helvetici*
Okt. 99 - Feb. 02	Leiterin des Nationalfondsprojekts 2134 - 057486
Seit Feb. 02	Assistentin am mathematischen Institut in Basel
16.5.2002	Doktorkolloquium

Ausbildung bei folgenden Dozenten

Mathematik:
E. Bolthausen, M. Brodmann, P. Gabriel, H. Kraft, N. Manolachae, C. Okonek, A. Telemann, R. Verfürth

Philosophie:
H. Lübbe, G. Kohler, M. Molteni, P. Schulthess

Französische Literatur:
L. Keller