Multistep Predictions for Multivariate GARCH Models: Closed Form Solution and the Value for Portfolio Management*

Jaroslava Hlouskova
Institute for Advanced Studies
Vienna

Kurt Schmidheiny
Universitat Pompeu Fabra

Martin Wagner†
Institute for Advanced Studies
Vienna

Abstract

This paper derives the closed form solution for multistep predictions of the conditional means and covariances for multivariate ARMA-GARCH models. These predictions are useful e.g. in mean-variance portfolio analysis when the rebalancing frequency is lower than the data frequency. In this situation the conditional mean and the conditional covariance matrix of the cumulated higher frequency returns are required as inputs in the mean-variance portfolio problem. The empirical value of the result is evaluated by comparing the performance of quarterly and monthly rebalanced portfolios using monthly MSCI index data across a large set of GARCH models. Using correct multistep predictions generally results in lower risk and higher returns.

JEL Classification: C32; C61; G11
Keywords: Multivariate GARCH models; Volatility forecasts; Portfolio optimization; Minimum variance portfolio

---

*The helpful comments of the editor, Richard T. Baillie, and the anonymous referee are gratefully acknowledged. We furthermore thank Robert Kunst and Beatrix Pawelczak for their comments. The usual disclaimer applies. The support of OLZ and Partners, Asset and Liability Management AG is gratefully acknowledged.

†Corresponding author: Institute for Advanced Studies, Department of Economics and Finance, Stumpergasse 56, A-1060 Vienna, Austria; Tel.: ++43 +1 59991 150, Fax: ++43 +1 59991 163, Email: `Martin.Wagner@ihs.ac.at`

1 Introduction

This paper derives the closed form solution for multistep predictions of the conditional means and covariances from multivariate GARCH models. These predictions are useful in mean-variance portfolio analysis, when the rebalancing frequency is lower than the data frequency. In the application the empirical value of this result is evaluated in the performance of quarterly rebalanced portfolios based on correct three-step predictions. We compare their performance with that of quarterly rebalanced portfolios incorrectly based on one-step predictions and with the performance of monthly rebalanced portfolios. We use monthly Morgan Stanley Capital International (MSCI) index data for six regions.

Multistep prediction in GARCH models has been considered previously in e.g. Baillie and Bollerslev (1992). They derive the minimum mean squared error forecasts for the conditional mean and the conditional variance of univariate GARCH processes. We extend their results to the multivariate case and derive closed form representations for the conditional mean and the conditional covariances h-steps ahead. In addition we derive the explicit formula for the conditional covariance of the sum of the conditional means up to h-steps ahead. This corresponds to the conditional variance of the cumulative returns over an h-period horizon, when modelling asset returns.

In our empirical application portfolios are adjusted quarterly based on GARCH models estimated with monthly data. This implies that the conditional variances of monthly returns cumulated over three months have to be computed. The empirical part of our study is related to Ledoit, Santa-Clara and Wolf (2003), who apply one-step predictions from multivariate GARCH models for portfolio selection using - as we do - MSCI regional indices. However, our study is based on multistep predictions and the results are based on a larger set of GARCH models.

In particular, the *value* of the derived multistep predictions for portfolio management is evaluated on monthly data for six regional MSCI indices during the evaluation period January 1992 to December 2003. The minimum variance portfolios are tracked for 48 different GARCH models, both for monthly and quarterly rebalancing. In the latter case the quarterly rebalanced portfolios *correctly* based on multistep predictions and those *incorrectly* based on one-step predictions are evaluated. We find that using correct multistep predictions generally results in lower risk and higher returns. Furthermore, the correctly computed quarterly rebalanced portfolios exhibit higher returns than monthly rebalanced portfolios.

The paper is organized as follows: In section 2 the multistep prediction problem is discussed. Section 3 contains the empirical application in portfolio management. Section 4 briefly summarizes and provides conclusions.

2 Multistep Prediction in Multivariate GARCH Models

This section derives the closed form solution for the multistep minimum mean squared error (MSE) prediction of the conditional means, conditional variances and conditional covariances for multivariate GARCH models. Based on these results we also present the solution for the conditional variance of the sum of the predictions over h-periods. The results of this section can be used for the prediction of *cumulative* returns and their covariance matrices in mean-variance portfolio analysis as explained in section 3.

Since the original contribution of Engle (1982) a large variety of ARCH and GARCH models has been proposed for volatility modelling, see Bollerslev, Engle and Nelson (1994) or Gourieroux (1997) for early discussions of some of the models developed or Bauwens, Laurent and Rombouts (2006) and Li, Ling and McAleer (2002) for more recent surveys.

We consider a multivariate ARMA process with GARCH errors to model the dynamic behavior of the (conditional) first and second moment of the returns. Let us denote with $r_t \in \mathbb{R}^n$ the vector of returns for n assets. The *mean equation* is of the form

$$r_t = c + A_1 r_{t-1} + \cdots + A_p r_{t-p} + \varepsilon_t + B_1 \varepsilon_{t-1} + \cdots + B_q \varepsilon_{t-q}, \tag{1}$$

with $A_i, B_j \in \mathbb{R}^{n\times n}$. Here ε_t is an n-dimensional random variable such that

$$\varepsilon_t = z_t \Sigma_t^{1/2}, \tag{2}$$

where z_t is i.i.d. with $\mathbb{E}(z_t) = 0$ and $var(z_t) = I$. Throughout the paper I denotes the $n \times n$ identity matrix. $\Sigma_t \in \mathbb{R}^{n\times n}$ is a positive definite, time-dependent covariance matrix measurable with respect to the information set at time $t-1$.

If the investment horizon is larger than one period, predictions for the *cumulative* returns are needed, which in turn require *multistep* predictions. The cumulative returns over an h-period horizon, henceforth denoted as $r_{[t+1:t+h]}$, are straightforwardly calculated from the single period returns, r_{t+i}, as follows[1]

$$r_{[t+1:t+h]} = r_{t+1} + \cdots + r_{t+h}. \tag{3}$$

Thus, the conditional variance matrix of the cumulative returns $r_{[t+1:t+h]}$ is

$$\begin{aligned} var_t(r_{[t+1:t+h]}) &= var_t(r_{t+1} + \cdots + r_{t+h}) \\ &= \sum_{i=1}^{h} var_t(r_{t+i}) + \sum_{i,j=1,i\neq j}^{h} cov_t(r_{t+i}, r_{t+j}), \end{aligned} \tag{4}$$

where throughout the paper the subscript t in $\mathbb{E}_t$, var_t and cov_t indicates that the expected value, variance respectively covariance is conditional upon the information set at time t. From equation (4) the calculation of $var_t(r_{[t+1:t+h]})$ requires the MSE predictors of r_{t+i} for $i = 1, \ldots, h$ and the corresponding conditional variances and covariances. The general formula for computing the required multistep predictions of the conditional variances of r_{t+i} from multivariate ARMA(p, q)-GARCH(k, l) models is presented below. This result is a generalization of the analogue multistep prediction for *univariate* GARCH models discussed in Baillie and Bollerslev (1992).[2] Two remarks on the discussion below are in order: First, the limits for prediction horizon $h \to \infty$ of the results for the minimum MSE predictors of the mean and variance are finite only for stationary processes. Second, the derivations below do not apply to ARCH-in-Mean type models. Detailed derivations are available in our earlier working paper Hlouskova, Schmidheiny and Wagner (2004).

For the derivation of the minimum MSE predictors of r_{t+1}, $r_{[t+1:t+h]}$ and their conditional second moments it is convenient to express the ARMA mean equation (1) in companion form, compare e.g. Baillie (1987, p. 108):

$$\underbrace{\begin{bmatrix} r_t \\ r_{t-1} \\ \vdots \\ r_{t-p+1} \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}}_{R_t} = \underbrace{\begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{K_1 c} + \underbrace{\begin{bmatrix} A_1 & & \ldots & & A_p & B_1 & \ldots & & B_q \\ I & 0 & \ldots & & 0 & 0 & \ldots & & 0 \\ \vdots & & \ldots & & \vdots & \vdots & \ldots & & \vdots \\ 0 & & \ldots & I & 0 & 0 & \ldots & & 0 \\ 0 & & \ldots & & 0 & 0 & \ldots & & 0 \\ 0 & & \ldots & & 0 & I & \ldots & & 0 \\ \vdots & & \ldots & & \vdots & \vdots & \ldots & & \vdots \\ 0 & & \ldots & & 0 & 0 & \ldots & I & 0 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} r_{t-1} \\ r_{t-2} \\ \vdots \\ r_{t-p} \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}}_{R_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{K\varepsilon_t} \tag{5}$$

[1] This follows directly from the definition of the one-period returns, calculated as the logarithmic difference of asset prices.

[2] Alternatively, the temporal aggregation results of Drost and Nijman (1993), derived for a specific class of univariate GARCH models, can be used to obtain multistep predictions.

or more compactly as

$$R_t = K_1 c + \Phi R_{t-1} + K\varepsilon_t. \tag{6}$$

The matrices K_j, $j = 1, \ldots, p+q$ denote $(p+q)n \times n$ matrices of $0_{n\times n}$ sub-matrices except for the j-th sub-matrix which equals I. Furthermore, $K = K_1 + K_{p+1}$, $R_t \in \mathbb{R}^{(p+q)n}$ and $\Phi \in \mathbb{R}^{(p+q)n\times(p+q)n}$.

Recursive substitution in (6), leading to

$$R_t = \sum_{j=0}^{i-1} \Phi^i K_1 c + \Phi^i R_{t-i} + \sum_{j=0}^{i-1} \Phi^i K \varepsilon_{t-j} \tag{7}$$

and straightforward algebra, compare Baillie (1980, p. 366) for the univariate case, show that

$$r_{t+h} = \sum_{i=0}^{h-1} K_1' \Phi^i K_1 c + \sum_{i=0}^{p-1} (K_1' \Phi^h K_{i+1}) r_{t-i} + \sum_{i=0}^{q-1} (K_1' \Phi^h K_{i+p+1}) \varepsilon_{t-i} + \sum_{i=0}^{h-1} (K_1' \Phi^i K) \varepsilon_{t+h-i}. \tag{8}$$

From the above representation (8) the required results can be deduced. The minimum MSE h-step ahead predictor for r_{t+h} is given by

$$\mathbb{E}_t(r_{t+h}) = \sum_{i=0}^{h-1} K_1' \Phi^i K_1 c + \sum_{i=0}^{p-1} (K_1' \Phi^h K_{i+1}) r_{t-i} + \sum_{i=0}^{q-1} (K_1' \Phi^h K_{i+p+1}) \varepsilon_{t-i} \tag{9}$$

and consequently the forecast error $e_{t,h}$ is given by

$$e_{t,h} = \sum_{j=0}^{h-1} (K_1' \Phi^j K) \varepsilon_{t+h-j}. \tag{10}$$

Given the above expression for the forecast error the conditional variance of the minimum MSE predictor is found to be

$$var_t(r_{t+h}) = \mathbb{E}_t(e_{t,h} e_{t,h}') = K_1' \sum_{j=0}^{h-1} \Phi^j K \, \Sigma_{t+h-j,t} (\Phi^j)' K' K_1, \tag{11}$$

with $\Sigma_{t+h-j,t} = var_t(r_{t+h-j})$. The above expression is similar to Yamamoto (1981, p. 487, eq. 3.4), with the difference being that we consider time varying conditional variances $\Sigma_{t+h-j,t}$. We are left to compute the conditional variance of the cumulative returns. This can be done directly by computing, with similar operations as for the variances $var_t(r_{t+j})$, also the covariances $cov_t(r_{t+i}, r_{t+j})$ and by summing all terms or by resorting to aggregation results for ARMA processes as discussed in Lütkepohl (1984), again taking the time varying conditional variances into account. The conditional covariance matrix of the h-period cumulative returns is given by

$$\begin{aligned} var_t(r_{[t+1:t+h]}) &= var_t(r_{t+1} + \cdots + r_{t+h}) \\ &= K_1' \sum_{i=1}^{h} \left[\sum_{k=0}^{i-1} \Phi^k K \, \Sigma_{t+i-k,t} (\Phi^k K)' \right] K_1 \\ &\quad + K_1' \sum_{i,j=1, i \neq j}^{h} \left[\sum_{k=max\{0,i-j\}}^{i-1} \Phi^k K \, \Sigma_{t+i-k,t} (\Phi^{j-i+k} K)' \right] K_1. \end{aligned} \tag{12}$$

For the actual implementation of the above results concerning the predictions of the conditional variances and covariances a recursive formulation is convenient. Denote $\Sigma^R_{t+i,t} = var_t(R_{t+i})$ and $\Sigma^R_{t+i,t+j,t} = cov_t(R_{t+i}, R_{t+j})$. Consider the case $i = j$ first, then (6) implies

$$\Sigma^R_{t+i,t} \quad = \quad var_t(\Phi R_{t+i-1} + K\varepsilon_{t+i}) \quad = \quad \Phi \Sigma^R_{t+i-1,t} \Phi' + K \Sigma_{t+i,t} K' \tag{13}$$

as $cov_t(K\varepsilon_{t+i}, R_{t+i-1}) = 0$. For $i > j$ (7) implies that

$$\Sigma^R_{t+i,t+j,t} \quad = \quad cov_t(\Phi^{i-j} R_{t+j} + \textstyle\sum_{k=0}^{i-j-1} \Phi^k K \varepsilon_{t+i-k}, R_{t+j}) \quad = \quad \Phi^{i-j} \Sigma^R_{t+j,t} \tag{14}$$

as $cov_t(K\varepsilon_{t+i-k}, R_{t+j}) = 0$ for $k = 0, \ldots, i-j-1$. Similarly, for $i < j$ it holds that $\Sigma^R_{t+i,t+j,t} = \Sigma^R_{t+i,t}(\Phi^{j-i})'$. Combining the above derivations an alternative, recursive formulation for the conditional variance of the h-period cumulative returns is given by

$$\begin{aligned} var_t(r_{[t+1:t+h]}) &= K_1' \left(\sum_{i=1}^{h} \Sigma^R_{t+i,t} + \sum_{i=2}^{h} \sum_{j=1}^{i-1} \Phi^{i-j} \Sigma^R_{t+j,t} \right) K_1 + \\ &+ K_1' \left(\sum_{i=1}^{h-1} \sum_{j=i+1}^{h} \Sigma^R_{t+i,t} (\Phi^{j-i})' \right) K_1, \end{aligned} \tag{15}$$

where the conditional variance matrices $\Sigma^R_{t+i,t}$ are calculated according to the recursion (13) for $i = 1, \ldots, h$.

3 An Empirical Application in Portfolio Management

In the previous section we have shown how multistep predictions are obtained for ARMA-GARCH models. These become useful for portfolio management when the data frequency is higher than the rebalancing frequency, a situation often faced by portfolio managers. In this section we assess the practical implications of this result for portfolio selection by comparing the portfolio performance with higher rebalancing frequency (one-month) to lower rebalancing frequency (three-month) using higher frequency (one-month) data. Consequently, the former portfolio selection has to be based on one-step predictions and the latter on predictions up to three steps ahead. The quantitative importance of correct three-step predictions is evaluated by computing several performance measures of portfolios rebalanced at a three-month frequency but incorrectly based on one-step predictions. This also allows to identify the sets of models leading to the best portfolio performance, according to optimality criteria such as lowest risk, highest return or highest Sharpe ratio. Note, however, that the interesting exercise of finding an *optimal* rebalancing interval is beyond the scope of this paper.

3.1 Portfolio Optimization

The empirical application is performed within the framework of mean-variance (MV) portfolio analysis (Markowitz, 1952 and 1956). MV analysis assumes that the investor's decisions and hence the optimal portfolio only depend on the expected return and the conditional variance of the portfolio return, the latter measuring risk. Considering n risky assets and an investment horizon of one period, the investor faces the following decision problem at time t:

$$\underset{x_t}{Min} \quad \sigma^2_{pt+1} = \sum_{i,j=1}^{n} x_{it} x_{jt} cov_t(r_{it+1}, r_{jt+1})$$

$$s.t. \quad \mathbb{E}_t(r_{pt+1}) = \sum_{i=1}^{n} x_{it}\mathbb{E}_t(r_{it+1}) = \overline{r}, \quad \sum_{i=1}^{n} x_{it} = 1, \quad x_{it} \geq 0,$$

where r_{pt+1} and σ^2_{pt+1} denote the portfolio return and portfolio variance, respectively.[3] Given a fixed value of the expected return, $\mathbb{E}_t(r_{pt+1}) = \overline{r}$, the fractions, x_{it}, of wealth invested in an individual asset i, are chosen to minimize the risk of the portfolio return. In addition, we assume nonnegative x_{it}, i.e. short sales are prohibited.[4] $\mathbb{E}_t(r_{it+1})$ and $cov_t(r_{it+1}, r_{jt+1})$ are

[3]Mean-variance portfolio optimization is based on *discrete* returns, which implies that the portfolio return is a weighted average of individual asset returns, as seen in the above equation. The predictions from the multivariate GARCH models are, however, based on *continuous* (log) returns for the following reason: the cumulative returns over multiple periods are linear in the individual period returns when using continuous returns but non-linear (and thus not analytically tractable) for discrete returns. We pursue the following *pragmatic* strategy: we predict continuous returns using the multivariate GARCH models. We then - as is common in the literature, compare e.g. Ledoit, Santa-Clara and Wolf (2003) - use these predictions in the mean-variance optimization to get the optimal portfolio weights. In order to provide a realistic assessment of the portfolio performance, in the evaluation we calculate the *discrete* returns of the portfolios.

[4]Jagannathan and Ma (2003) show that imposing short-sale constraints can improve portfolio performance due to avoiding extreme positions resulting from imprecise covariance estimation.

approximated by predictions (e.g. from GARCH models) of asset returns and their covariances over the period from t to $t+1$, given the information available at time t. The above optimization problem leads, by varying $\bar{r}$, to the well-known efficient frontier. Omitting the constraint $\mathbb{E}_t(r_{pt+1}) = \bar{r}$ leads to the *minimum variance portfolio*, which is independent of expected returns. The empirical analysis below considers minimum variance portfolios only. Full details of the implemented models and the empirical analysis are contained in the working paper Hlouskova, Schmidheiny and Wagner (2004).

3.2 Return and Risk Predictions from GARCH Models

The required predictions for both the returns and the conditional covariances of the returns are derived in our study from multivariate GARCH models. The nesting formulation of the mean equations considered in the empirical application is given by the ARMA(1,1) equation

$$r_t = c + Ar_{t-1} + \varepsilon_t + B\varepsilon_{t-1}. \tag{16}$$

Preliminary model selection shows that no higher lags are required for our application. Even in the equations with only one lag many of the coefficients are insignificant. Therefore, we also investigate more parsimonious specifications, where the autoregressive coefficient matrix A, the moving average coefficient matrix B or both are restricted to be diagonal or zero. Note that significant coefficients in A or B in the mean equation are a violation of strong market efficiency. The portfolios based on ARMA models instead of only AR or MA models perform very well, with a majority of the eight ARMA based portfolios being in the top half of the portfolios for all considered performance measures. Two distributions for ε_t are considered: Normally distributed innovations and t-distributed innovations, where in the latter case the degree of freedom of the innovation distribution is estimated itself. The latter possibility is included in order to allow for stronger leptokurtic behavior. Allowing for t-distributions is beneficial and leads to on average quite good performance, especially when considering monthly rebalancing. This is consistent with the stylized fact that leptokurtic behavior is more important at higher data frequencies. See the upper block of Table 1 for a description of the six implemented mean equations. We consider eight different specifications of orders (1,1) for the variance equation, see the lower block of Table 1. The implemented models include the BEKK model of Engle and Kroner (1995) and the vector-diagonal model. The other six multivariate GARCH models are based on univariate GARCH models for appropriately

Table 1: Specifications of implemented GARCH models

Specification of *mean equation*: $r_t = c + Ar_{t-1} + \varepsilon_t + B\varepsilon_{t-1}$			
model	A	B	ε_t
AR(1) diag n	diagonal	0	N(0,Σ_t)
MA(1) diag n	0	diagonal	N(0,Σ_t)
AR(1) full n	unrestr.	0	N(0,Σ_t)
AR(1) diag t	diagonal	0	t-distr.
MA(1) diag t	0	diagonal	t-distr.
ARMA(1,1) full t	unrestr.	unrestr.	t-distr.

Specification of *variance equation*	
model	description
BEKK(1,1)	
Vector Diag(1,1)	vector diagonal model
Diag GARCH(1,1)	pure diagonal GARCH model
Diag EGARCH(1,1)	pure diagonal exponential GARCH model
Diag PGARCH(1,1)	pure diagonal power GARCH model
CCC GARCH(1,1)	constant conditional correlation GARCH model
CCC EGARCH(1,1)	constant conditional correlation exponential GARCH model
CCC PGARCH(1,1)	constant conditional correlation power GARCH model

transformed series (in order to reduce the number of parameters). We have implemented two approaches along this line, the constant conditional correlation (CCC) model of Bollerslev (1990) and the pure diagonal models. For both of these model types we have implemented three underlying univariate GARCH models: the unrestricted GARCH model of Bollerslev (1986), the exponential GARCH (EGARCH) model of Nelson (1991) and the power GARCH (PGARCH) model of Ding, Engle and Granger (1993). We implement all combinations of the six mean equations and eight variance equations, i.e. a total of 48 GARCH models.

As a benchmark portfolio we consider the *naive* portfolio, where both the return and covariance predictions are given by the sample mean and the sample covariance, respectively, over the estimation period. Thus, we need to clarify how we derive multistep predictions for the *naive* portfolio strategy. Since in the quarterly rebalancing the investor is interested in the prediction of the three-month returns and their covariances, we base our naive predictions for the three-month return on the sample mean and covariance matrix of the monthly return series aggregated to three-month returns.[5]

[5] This seems to be more natural than to simply use the empirical mean and covariance matrix of the returns series at the monthly frequency. The latter are used as incorrect forecasts for the quarterly rebalancing of the naive portfolio.

3.3 Portfolio Evaluation

We track internationally diversified portfolios denominated in Swiss francs over the period 1992 to 2003. The portfolio wealth is invested in six world regions. The MSCI indices for the United States, Switzerland, Great Britain, Japan, Europe (excluding Great Britain) and Pacific (excluding Japan) are the investment instruments.[6] We use monthly return data from February 1972 to December 2003 for the six indices.

The evaluation with quarterly (respectively monthly) rebalancing proceeds in the following steps:

(1) The monthly return data from February 1972 up to the date of the investment decision are used to predict the covariances of the six regional indices. 49 different predictions are computed: From 48 GARCH models and the naive predictions.

(2) The corresponding minimum variance portfolios are calculated.

(3) The three- and one-month returns are calculated.

(4) The investment decision is repeated every three (one) months from January 1, 1992 to October 1 (December 1), 2003 and the portfolios are rebalanced accordingly.

3.4 Results

Table 2 summarizes the evaluation results for the 48 portfolios based on GARCH models, labelled GARCH portfolios.[7] We report the average risk and return as well as the Sharpe ratio[8] of the 48 GARCH portfolios. For comparison, we also report the results obtained from the *naive* portfolio, based on sample means and covariances. We report risk and Sharpe ratio for annualized quarterly returns.

Let us start by discussing the performance of monthly rebalanced GARCH portfolios. Note first that all GARCH portfolios exhibit lower risk than the naive portfolio, which forcefully demonstrates the substantial value of GARCH modelling. Both, the average across the GARCH portfolios as well as the majority of GARCH portfolios also show higher return (36

[6] Note that we do not include a risk free asset in order to focus on the effect of GARCH predicted correlation structures on portfolio performance.

[7] Detailed tables with risk, return and risk adjusted performance measures such as Sharpe ratio, Jensen's alpha, Treynor's measure as well as shortfall are available from the authors upon request.

[8] The Sharpe ratio is defined as excess return (i.e. return minus riskfree rate) divided by the standard deviation of the excess return. The three-month (respectively one-month) deposit rate is used as riskfree rate. Before December 1996 the deposit rate is approximated by the LIBOR minus five basis points.

Table 2: Main Results of Performance Comparison

	Risk	Return	Sharpe
Average across GARCH portfolios			
- monthly rebalancing	39.00	7.99	0.133
- quarterly rebalancing with 3-step prediction	39.72	8.35	0.135
- quarterly rebalancing with 1-step prediction	39.99	7.89	0.123
Best of GARCH portfolios			
- monthly rebalancing	37.39	9.08	0.167
- quarterly rebalancing with 3-step (correct) prediction	38.47	9.33	0.159
- quarterly rebalancing with 1-step (incorrect) prediction	38.64	9.09	0.156
Naive prediction			
- monthly rebalancing	41.03	7.77	0.120
- quarterly rebalancing with 3-step (correct) prediction	41.38	8.05	0.122
- quarterly rebalancing with 1-step (incorrect) prediction	41.13	7.82	0.117
Number of GARCH models	48	48	48
Comparison of GARCH models			
- quarterly rebal.: correct better than incorrect GARCH prediction	33	48	48
- quarterly (correct prediction) better than monthly rebalancing	1	47	27
Comparison of GARCH models with naive prediction			
- monthly rebal.: GARCH better than 1-step naive prediction	48	36	39
- quarterly rebal.: correct GARCH better than naive prediction	48	40	43
- quarterly rebal.: incorrect GARCH better than naive prediction	48	32	32

Summary of main results. All results apply to quarterly returns. *Return* denotes the mean annualized return of the portfolio. *Risk* denotes the standard deviation of annualized quarterly returns. *Sharpe* ratio is given by excess quarterly return (i.e. return minus riskfree rate) divided by its standard deviation. *correct* means that 3-step predictions for the conditional covariances are used. *incorrect* means that 1-step predictions for the conditional means and covariances are used. *better (best)* means lower risk, higher return and higher Sharpe ratio, respectively.

out of 48) and Sharpe ratio (39 out of 48) than the naive portfolio. The average return (7.99%) across all 48 GARCH portfolios is 22 basis points higher than the return of the naive portfolio. Similar outperformance is also found for daily data by Fleming, Kirby and Ostdiek (2001).

Let us now turn to quarterly portfolio rebalancing based on correct multistep predictions. Again, all GARCH portfolios show lower risk and a majority show higher return (40 of 48) and higher Sharpe ratio (43 of 48) than the naive portfolio. The average return across all GARCH portfolios (8.35%) is 30 basis points above the naive portfolio's return. The return of the GARCH portfolio with the highest return and Sharpe ratio based on quarterly rebalancing and multistep predictions is even 128 basis points higher than the naive portfolio's return. That portfolio is based on an unrestricted AR(1) mean equation with normally distributed errors and the BEKK(1,1) GARCH specification. For illustration the asset allocations corresponding to the best and the naive portfolio are displayed in Figure 1. The figure displays a typical

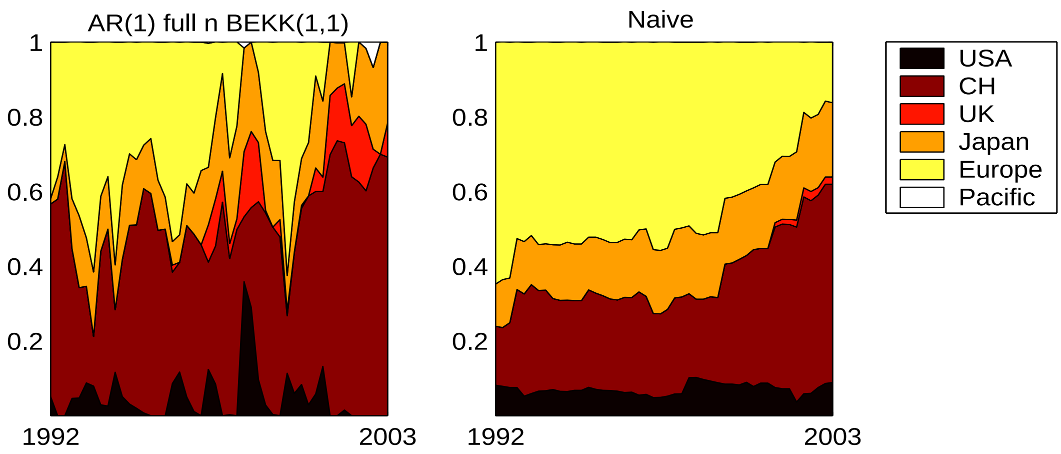

Figure 1: Asset allocations of the best performing GARCH portfolio, AR(1)-full n-BEKK(1,1), and of the naive quarterly rebalanced portfolio.

feature of GARCH based portfolios, namely the larger amount of asset reallocations compared to the naive portfolio.

Thus, the value of GARCH based portfolio selection appears to be substantial at both frequencies. One might expect that the risks of quarterly rebalanced portfolios are higher and their returns lower than for monthly rebalanced portfolios. This, since with monthly rebalancing new information is incorporated faster. Surprisingly, the expected relationship is only observed for risk: all but one quarterly rebalanced portfolio result in higher risk than the corresponding monthly rebalanced portfolio. However, 47 out of 48 quarterly rebalanced portfolios exhibit *higher returns* than the corresponding monthly rebalanced portfolio. On average, the risk of quarterly rebalanced portfolios is 72 basis points higher and the return 36 basis points *higher* than that of monthly rebalanced portfolios. Consequently an ambiguous picture emerges when taking the Sharpe ratio as performance measure: the Sharpe ratio of quarterly and monthly rebalanced portfolios are on average as well as for the individual models almost identical.[9]

Let us finally turn to the assessment of the value of correct multistep predictions by comparing the portfolio performance obtained from quarterly rebalancing based on the incorrect one-step predictions on the one hand and on the correct three-step predictions on the other hand. For the majority (33 of 48) of GARCH portfolios using the correct predictions results

[9] Note that the higher return achieved with lower frequency rebalanced portfolios implies that the results are robust with respect to the consideration of transaction costs.

in lower portfolio risk. The correct predictions reduce the risk by on average 17 basis points. Note also that for all 48 GARCH portfolios the return is higher with the correct multistep method, with the mean difference being 46 basis points. Thus, the correct computation of the predictions is indeed resulting in superior portfolio performance across a large set of GARCH portfolios.

4 Summary and Conclusions

In this paper we have derived the closed form solution for multistep predictions of the conditional means and covariances for multivariate GARCH models and have illustrated their value for portfolio management. Multistep predictions of the conditional means and covariances are needed for mean-variance portfolio analysis when the rebalancing frequency is lower than the data frequency. In order to deal with this problem we have also derived the explicit formula for the conditional covariance matrix of the corresponding cumulative higher frequency returns. The closed form solution for the general ARMA(p, q)-GARCH(k, l) case is provided in section 2 along with a convenient recursive representation.

The practical relevance of the theoretical results is assessed empirically with an application to six regional MSCI indices using a large variety of GARCH models. Based on monthly data, the portfolio performance of monthly and quarterly rebalanced portfolios is investigated and compared to the naive portfolio, which is based on the sample mean and covariance. The quarterly rebalancing decision is either correctly based on three-step predictions or incorrectly on one-step predictions. The evaluation period is January 1992 to December 2003. The following main results are obtained: First, portfolios based on GARCH models – labelled GARCH portfolios – have on average higher return, lower risk and higher Sharpe ratio than the naive portfolio. Second, almost all quarterly rebalanced GARCH portfolios based on correct multistep predictions exhibit higher returns and higher risk than the monthly rebalanced GARCH portfolios even in the absence of transaction costs. This is surprising because monthly adjusted portfolios incorporate new information faster and should therefore outperform quarterly adjusted portfolios. Third, portfolios based on correct predictions show on average lower risk than the corresponding portfolios based on incorrect predictions. Furthermore, all GARCH portfolios based on correct predictions result in higher returns and Sharpe ratios than those based on incorrect predictions.

References

Baillie, Richard T., 1980. Predictions from ARMAX models. Journal of Econometrics 12, 365–374.

Baillie, Richard T., 1987. Inference in dynamic models containing surprise variables. Journal of Econometrics 35, 101–117.

Baillie, Richard T., Bollerslev, Tim, 1992. Prediction in dynamic models with time-dependent conditional variances. Journal of Econometrics 52, 91–113.

Bauwens, Luc, Laurent, Sebastien, Rombouts, Jeroen, 2006. Multivariate GARCH models: a survey. Journal of Applied Econometrics 21, 79–109.

Bollerslev, Tim, 1986. Generalized autoregressive conditional heterocedasticity. Journal of Econometrics 31, 307–327.

Bollerslev, Tim, 1990. Modeling the coherence of short-run nominal exchange rates: A multivariate generalized ARCH model. Review of Economics and Statistics 7, 498–505.

Bollerlsev, Tim, Engle, Robert F., Nelson, Daniel B., 1994. ARCH models. In: Engle, Robert F., McFadden, Daniel L. (Eds.), Handbook of Econometrics, Vol. 4, Elsevier, Amsterdam.

Ding, Zhuanxin, Engle, Robert F., Granger, Clive W.J., 1993. A long memory property of stock market returns and a new model. Journal of Empirical Finance 1, 83–106.

Drost, Feike C., Nijman, Theo E., 1993. Temporal aggregation of GARCH processes. Econometrica 61, 909–927.

Engle, Robert F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. Econometrica 50, 987–1008.

Engle, Robert F., Kroner, Kenneth F., 1995. Multivariate simultaneous generalized ARCH. Econometric Theory 11, 122–150.

Fleming, Jeff, Kirby, Chris, Ostdiek, Barbara, 2001. The economic value of volatility timing. The Journal of Finance 56, 329–352.

Gourieroux, Christian, 1997. ARCH Models and Financial Applications. Springer, New York.

Hlouskova, Jaroslava, Schmidheiny, Kurt, Wagner, Martin, 2004. Multistep predictions for multivariate GARCH models: Closed form solution and their value for portfolio management. Working Paper 04.10, Department of Economics, University of Lausanne.

Jagannathan, Ravi, Ma, Tongshu, 2003. Risk reduction in large portfolios: Why imposing the wrong constraints helps. The Journal of Finance 58, 1651–1684.

Ledoit, Olivier, Santa-Clara, Pedro, Wolf, Michael, 2003. Flexible multivariate GARCH modeling with an application to international stock markets. Review of Economics and Statistics 85, 735–747.

Li, W.K., Ling, Shiqing, McAleer, Michael, 2002. Recent theoretical results for time series models with GARCH errors. Journal of Economic Surveys 16, 245–269.

Lütkepohl, Helmut, 1984. Linear aggregation of vector autoregressive moving average processes. Economics Letters 14, 345–350.

Markowitz, Harry M., 1952. Portfolio selection. The Journal of Finance 7, 77–91.

Markowitz, Harry M., 1956. The optimization of a quadratic function subject to linear constraints. Naval Research Logitstics Quarterly 3, 111–133.

Nelson, Daniel B., 1991. Conditional heteroskedasticity in asset returns: a new approach. Econometrica 59, 347–370.

Yamamoto, Taku, 1981. Predictions of multivariate autoregressive moving average models. Biometrika 68, 485–492.