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Multilevel quadrature for elliptic stochastic partial differential equations

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(joint work with Michael Peters and Markus Siebenmorgen)

1. Introduction

This talk is concerned with elliptic second order boundary value problems with random diffusion. In parametrized form, such problems are of the form

$$\begin{aligned} &\text{find } u \in L^2_\rho\big(\square; H^1_0(D)\big) \text{ such that} \\ &\qquad - \operatorname{div}\big(\alpha(\mathbf{y})\nabla u(\mathbf{y})\big) = f \text{ in } D \text{ for all } \mathbf{y} \in \square, \end{aligned} \tag{1}$$

where $D \subset \mathbb{R}^d$ is the physical domain, $\rho : \square \to \mathbb{R}_{\geq 0}$ is the joint density function, and $\square = (-1,1)^m$ (in the uniformly elliptic case) or $\square = \mathbb{R}^m$ (in the log-normal case) is the parameter domain of the stochastic variable. The quantities of interest are the solution's expectation

$$\mathbb{E}_u(\mathbf{x}) = \int_\square u(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{2}$$

its variance, or even higher order moments.

A principal approach to compute (2) is the Monte Carlo method. However, it is extremely expensive to generate a large number of suitable samples and to solve the deterministic boundary value problem (1) on each sample. To overcome this obstruction, the multilevel Monte Carlo method (MLMC) has been developed in [1]. From the stochastic point of view, it is a variance reduction technique which considerably decreases the complexity. The idea is to combine the Monte Carlo quadrature of the stochastic variable with a multilevel splitting of the Bochner space which contains the random solution. Then, to compute (2), most samples can be performed on coarse spatial discretizations while only a few samples must be performed on fine spatial discretizations. This proceeding is a sparse grid approximation of the expectation. If we replace the Monte Carlo quadrature by another quadrature rule for high-dimensional integrals, we obtain for example the multilevel quasi Monte Carlo method (MLQMC) or the multilevel Gaussian quadrature method (MLGQ).

2. Quadrature in the stochastic variable

To compute the integral (2), we have to provide a sequence of quadrature formulae $\{Q_\ell\}$ for the Bochner integral

$$I : L^2_\rho(\square; X) \to X, \quad Iv = \int_\square v(\cdot, \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

where $X \subset L^2(D)$ denotes a Banach space. The quadrature formula

$$Q_\ell : L^2_\rho(\square; X) \to X, \quad Q_\ell v = \sum_{i=1}^{N_\ell} \omega_{\ell,i} v(\cdot, \boldsymbol{\xi}_{\ell,i})\rho(\boldsymbol{\xi}_{\ell,i}) \tag{3}$$

is supposed to fulfill the error bound

$$\|(I - Q_\ell)v\|_X \lesssim 2^{-\ell}\|v\|_{\mathcal{H}(\square;X)}$$

uniformly in $\ell \in \mathbb{N}$, where $\mathcal{H}(\square; X) \subset L^2_\rho(\square, X)$ is a suitable Bochner space.

3. Finite element approximation in the spatial variable

In order to apply the quadrature formula (3), we shall calculate the solution $u(\mathbf{y}) \in H^1_0(D)$ of the diffusion problem (1) in certain points $\mathbf{y} \in \square$. To this end, consider a coarse grid triangulation/tetrahedralization $\mathcal{T}_0 = \{\tau_{0,k}\}$ of the domain D. Then, for $\ell \geq 1$, a uniform and shape regular triangulation/tetrahedralization $\mathcal{T}_\ell = \{\tau_{\ell,k}\}$ is recursively obtained by uniformly refining each triangle/tetrahedron $\tau_{\ell-1,k}$ into 2^n triangles/tetrahedrons with diameter $h_\ell \sim 2^{-\ell}$. Then, define the finite element spaces

$$\mathcal{S}_\ell(D) := \{v \in C(D) : v|_{\partial D} = 0 \text{ and } v|_\tau \text{ is linear for all } \tau \in \mathcal{T}_\ell\} \subset H^1_0(D)$$

and let

$$G_\ell(\mathbf{y}) : H^1_0(D) \to \mathcal{S}_\ell(D), \quad v \mapsto v_\ell$$

denote the Galerkin projection related with (1), given by Galerkin orthogonality

$$\int_D \alpha(\mathbf{x}, \mathbf{y})\nabla\big(v(\mathbf{x}) - v_\ell(\mathbf{x})\big)\nabla w_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \text{ for all } w_j \in \mathcal{S}_\ell(D).$$

Then, the approximate solution $G_\ell(\mathbf{y})u(\mathbf{y}) \in \mathcal{S}_\ell(D)$ to (1) of a finite element method in the space $\mathcal{S}_\ell(D)$ satisfies the error estimate

$$\|u(\mathbf{y}) - G_\ell(\mathbf{y})u_\ell(\mathbf{y})\|_{H^1(D)} \lesssim 2^{-\ell}\sqrt{\frac{\alpha_{\min}(\mathbf{y})}{\alpha_{\min}(\mathbf{y})}}\|u(\mathbf{y})\|_{H^2(D)}$$

provided that the domain D is convex and $f \in L^2(D)$.

4. Multilevel quadrature

We now have to combine the quadrature method with the multilevel finite element discretization. To this end, we define the ansatz spaces

$$V_j^{(1)} := \Big\{G_j(\mathbf{y})v(\mathbf{x},\mathbf{y}) : v \in C\big(\Box; H_0^1(D)\big) \text{ and } \mathbf{y} \in \Box\Big\} \subset L_\rho^2\big(\Box; \mathcal{S}_j(D)\big).$$

To compute the expectation (2), we shall apply the quadrature rule Q_j to the finite element solution in $\mathcal{S}_j(D)$ which yields

$$\mathbb{E}_u(\mathbf{x}) \approx Q_j\big(G_j(\mathbf{y})u(\mathbf{x},\mathbf{y})\big) = \sum_{i=0}^{N_j} \omega_{j,i} G_j(\boldsymbol{\xi}_{j,i}) u(\mathbf{x},\boldsymbol{\xi}_{j,i})\rho(\boldsymbol{\xi}_{j,i}). \tag{4}$$

This can be interpreted as the *full* tensor product approximation of the function $\mathbb{E}_u$ in the product space $V_j^{(1)} \otimes V_j^{(2)}$ where the quadrature rule Q_j serves as "space" $V_j^{(2)}$. It produces the error estimate

$$\Big\|\mathbb{E}_u(\mathbf{x}) - Q_j\big(G_j(\mathbf{y})u(\mathbf{x},\mathbf{y})\big)\Big\|_{H^1(D)} \lesssim 2^{-j} j \|u\|_{\mathcal{H}(\Box;H_1(D))\cap L_\rho^2(\Box;H_2(D))}.$$

In contrast to this, setting $G_{-1}(\mathbf{y}) := 0$ for all $\mathbf{y} \in \Box$, the *sparse* tensor product of the spaces $V_j^{(1)}$ and $V_j^{(2)}$ is built with the help of the complement spaces

$$W_\ell^{(1)} := \Big\{\big(G_\ell(\mathbf{y}) - G_{\ell-1}(\mathbf{y})\big)v(\mathbf{x},\mathbf{y}) : v \in C\big(\Box; H_0^1(D)\big) \text{ and } \mathbf{y} \in \Box\Big\} \subset V_\ell^{(1)}$$

in accordance with

$$\widehat{V_j^{(1)} \otimes V_j^{(2)}} = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes \left(\bigoplus_{\ell'=0}^{j-\ell} W_{\ell'}^{(2)}\right) = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes V_{j-\ell}^{(2)}.$$

This means, we consider the sparse tensor product approximation

$$\begin{aligned}\mathbb{E}_u(\mathbf{x}) &\approx \sum_{\ell=0}^{j} Q_{j-\ell}\big(G_\ell(\mathbf{y})u(\mathbf{x},\mathbf{y}) - G_{\ell-1}(\mathbf{y})u(\mathbf{x},\mathbf{y})\big)\\ &= \sum_{\ell=0}^{j}\sum_{i=0}^{N_{j-\ell}} \omega_{j-\ell,i}\big(G_\ell(\boldsymbol{\xi}_{j-\ell,i})u(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i}) - G_{\ell-1}(\boldsymbol{\xi}_{j-\ell,i})u(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i})\big)\rho(\boldsymbol{\xi}_{j-\ell,i}).\end{aligned}$$

Loosely speaking, the function $u \in L_\rho^2\big(\Box; H_0^1(D)\big)$ is divided into j slices which are related to the modulus of its entity. Then, for every slice, the precision of the quadrature is properly chosen. We refer to Figure 1 for a graphical illustration.

Under the assumption that the random solution provides mixed regularity in terms of $u \in \mathcal{H}\big(\Box; H_2(D)\big)$, the multilevel quadrature produces essentially the same accuracy as the standard tensor product quadrature (4):

$$\left\|\mathbb{E}_u(\mathbf{x}) - \sum_{\ell=0}^{j} Q_{j-\ell}\big(G_\ell(\mathbf{y})u(\mathbf{x},\mathbf{y}) - G_{\ell-1}(\mathbf{y})u(\mathbf{x},\mathbf{y})\big)\right\|_{H^1(D)} \lesssim 2^{-j} j \|u\|_{\mathcal{H}(\Box;H_2(D))}.$$

Notice that the multilevel quadrature idea can be generalized also to higher order moments or other output functionals, see [2, 3] for the details.

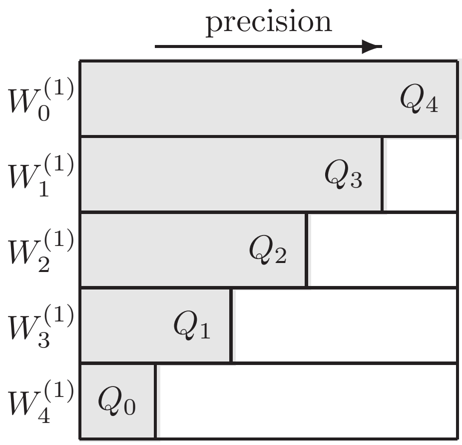

Figure 1. Visualization of the multilevel quadrature.

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Analysis on convex subset of a Riemannian manifold and classical polynomial approximation

Gerard Kerkyacharian

(joint work with P. Petrushev, Y. Xu)

It is a classical topic to look to orthonormal basis of polynomials on a compact set X of R^d, with respect to some Radon measure μ. For exemple : the one dimensional interval (Jacobi), the unit sphere (Spherical harmonics), the ball and the simplex (work of Petrushev, Xu, ...) In this framework, one can be interested in the best approximation of functions by polynomials of fixed degree, in $L_p(\mu)$, and to built a suitable frame for characterization of function spaces related to this approximation. This constructions have been carried using special functions estimates.

We will be interested by spaces where the polynomials give the spectral spaces of some positive selfadjoint operator. Under suitable conditions, a "natural" metric ρ could be defined on X such that (X, ρ, μ) is a homogeneous space, and if the associated semi-group has a good "Gaussian" behavior, then we could apply the procedure developed in recent works by P. Petrushev, T. Coulhon and G.K., to built such frames, and such function spaces.