

# THE DECOMPOSITION GROUP OF A LINE IN THE PLANE

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ABSTRACT. We show that the decomposition group of a line  $L$  in the plane, i.e. the subgroup of plane birational transformations that send  $L$  to itself birationally, is generated by its elements of degree 1 and one element of degree 2, and that it does not decompose as a non-trivial amalgamated product.

## 1. INTRODUCTION

We denote by  $\text{Bir}(\mathbb{P}^2)$  the group of birational transformations of the projective plane  $\mathbb{P}^2 = \text{Proj}(k[x, y, z])$ , where  $k$  is an algebraically closed field. Let  $C \subset \mathbb{P}^2$  be a curve, and let

$$\text{Dec}(C) = \{\varphi \in \text{Bir}(\mathbb{P}^2), \varphi(C) \subset C \text{ and } \varphi|_C : C \dashrightarrow C \text{ is birational}\}.$$

This group has been studied for curves of genus  $\geq 1$  in [BPV2009], where it is linked to the classification of finite subgroups of  $\text{Bir}(\mathbb{P}^2)$ . It has a natural subgroup  $\text{Ine}(C)$ , the *inertia group* of  $C$ , consisting of elements that fix  $C$ , and Blanc, Pan and Vust give the following result: for any line  $L \subset \mathbb{P}^2$ , the action of  $\text{Dec}(L)$  on  $L$  induces a split exact sequence

$$0 \longrightarrow \text{Ine}(L) \longrightarrow \text{Dec}(L) \longrightarrow \text{PGL}_2 = \text{Aut}(L) \longrightarrow 0$$

and  $\text{Ine}(L)$  is neither finite nor abelian and also it doesn't leave any pencil of rational curves invariant [BPV2009, Proposition 4.1]. Further they ask the question whether  $\text{Dec}(L)$  is generated by its elements of degree 1 and 2 [BPV2009, Question 4.1.2].

We give an affirmative answer to their question in the form of the following result, similar to the Noether-Castelnuovo theorem [Cas1901] which states that  $\text{Bir}(\mathbb{P}^2)$  is generated by  $\sigma : [x : y : z] \mapsto [yz : xz : xy]$  and  $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3$ .

**Theorem 1.** *For any line  $L \subset \mathbb{P}^2$ , the group  $\text{Dec}(L)$  is generated by  $\text{Dec}(L) \cap \text{PGL}_3$  and any of its quadratic elements having three proper base points in  $\mathbb{P}^2$ .*

The similarities between  $\text{Dec}(L)$  and  $\text{Bir}(\mathbb{P}^2)$  go further than this. Cornulier shows in [Cor2013] that  $\text{Bir}(\mathbb{P}^2)$  cannot be written as an amalgamated product in any nontrivial way, and we modify his proof to obtain an analogous result for  $\text{Dec}(L)$ .

**Theorem 2.** *The decomposition group  $\text{Dec}(L)$  of a line  $L \subset \mathbb{P}^2$  does not decompose as a non-trivial amalgam.*

The article is organised as follows: in Section 2 we show that for any element of  $\text{Dec}(L)$  we can find a decomposition in  $\text{Bir}(\mathbb{P}^2)$  into quadratic maps such that the successive images of  $L$  are curves (Proposition 2.6), i.e. the line is not contracted to a point at any

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time. We then show in Section 3 that we can modify this decomposition, still in  $\text{Bir}(\mathbb{P}^2)$ , into de Jonquières maps where all of the successive images of  $L$  have degree 1, i.e. they are lines. Finally we prove Theorem 1. Our main sources of inspiration for techniques and ideas in Section 3 have been [AC2002, §8.4, §8.5] and [Bla2012]. In Section 4 we prove Theorem 2 using ideas that are strongly inspired by [Cor2013].

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## 2. AVOIDING TO CONTRACT $L$

Given a birational map  $\rho: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ , the Noether-Castelnuovo theorem states that there is a decomposition  $\rho = \rho_m \rho_{m-1} \dots \rho_1$  of  $\rho$  where each  $\rho_i$  is a quadratic map with three proper base points. This decomposition is far from unique, and the aim of this section is to show that if  $\rho \in \text{Dec}(L)$ , we can choose the  $\rho_i$  so that none of the successive birational maps  $(\rho_i \dots \rho_1: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2)_{i=1}^m$  contracts  $L$  to a point. This is Proposition 2.6.

Given a birational map  $\varphi: X \dashrightarrow Y$  between smooth projective surfaces, and a curve  $C \subset X$  which is contracted by  $\varphi$ , we denote by  $\pi_1: Z_1 \rightarrow Y$  the blowup of the point  $\varphi(C) \in Y$ . If  $C$  is contracted also by the birational map  $\pi_1^{-1}\varphi: X \dashrightarrow Z_1$ , we denote by  $\pi_2: Z_2 \rightarrow Z_1$  the blowup of  $(\pi_1^{-1}\varphi)(C) \in Z_1$  and consider the birational map  $(\pi_1\pi_2)^{-1}\varphi: X \dashrightarrow Z_2$ . If this map too contracts  $C$ , we denote by  $\pi_3: Z_3 \rightarrow Z_2$  the blowup of the point onto which  $C$  is contracted. Repeating this procedure a finite number of times  $D \in \mathbb{N}$ , we finally arrive at a variety  $Z := Z_D$  and a birational morphism  $\pi := \pi_1\pi_2 \dots \pi_D: Z \rightarrow Y$  such that  $(\pi^{-1}\varphi)$  does not contract  $C$ . Then  $(\pi^{-1}\varphi)|_C: C \dashrightarrow (\pi^{-1}\varphi)(C)$  is a birational map.

**Definition 2.1.** In the above situation, we denote by  $D(C, \varphi) \in \mathbb{N}$  the minimal number of blowups which are needed in order to not contract the curve  $C$  and we say that  $C$  is contracted  $D(C, \varphi)$  times by  $\varphi$ . In particular, a curve  $C$  is sent to a curve by  $\varphi$  if and only if  $D(C, \varphi) = 0$ .

**Remark 2.2.** The integer  $D(C, \varphi)$  can equivalently be defined as the order of vanishing of  $K_Z - \pi^*(K_Y)$  along  $(\pi^{-1}\varphi)(C)$ .

We recall the following well known fact, which will be used a number of times in the sequel.

**Lemma 2.3.** *Let  $\varphi_1, \varphi_2 \in \text{Bir}(\mathbb{P}^2)$  be birational maps of degree 2 with proper base points  $p_1, p_2, p_3$  and  $q_1, q_2, q_3$  respectively. If  $\varphi_1$  and  $\varphi_2$  have (exactly) two common base points, say  $p_1 = q_1$  and  $p_2 = q_2$ , then the composition  $\tau = \varphi_2\varphi_1^{-1}$  is quadratic. Furthermore the three base points of  $\tau$  are proper points of  $\mathbb{P}^2$  if and only if  $q_3$  is not on any of the lines joining two of the  $p_i$ .*

*Proof.* The lemma is proved by Figure 1, where squares and circles in  $\mathbb{P}_2^2$  denote the base points of  $\varphi_1$  and  $\varphi_2$  respectively. The crosses in  $\mathbb{P}_1^2$  denote the base points of  $\varphi_1^{-1}$  (corresponding to the lines in  $\mathbb{P}_2^2$ ), and the conics in  $\mathbb{P}_1^2$  and  $\mathbb{P}_2^2$  denote the pullback of a general line  $\ell \in \mathbb{P}_3^2$ .

If  $q_3$  is not on any of the three lines, the base points of  $\tau$  are  $E_1, E_2, \varphi_1(q_3)$ . If  $q_3$  is on one of the three lines, then the base points of  $\tau$  are  $E_1, E_2$  and a point infinitely close to the  $E_i$  which corresponds to the line that  $q_3$  is on.  $\square$

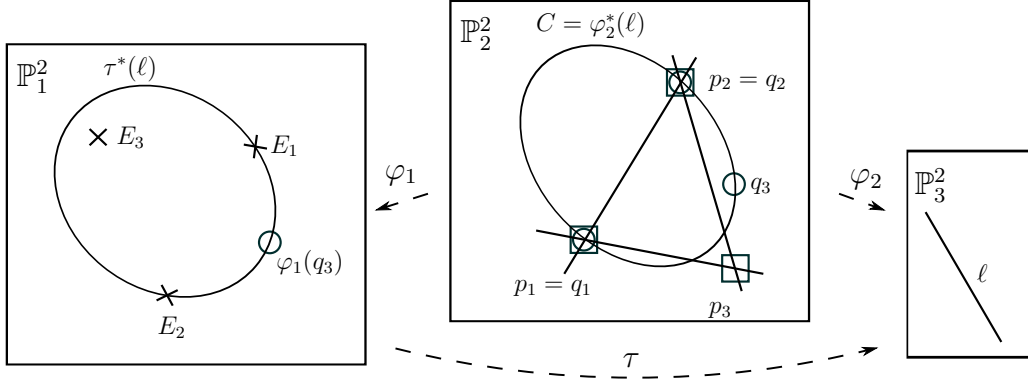


FIGURE 1. The composition of  $\varphi_1$  and  $\varphi_2$  in Lemma 2.3

The following lemma describes how the number of times that a line is contracted changes when composing with a quadratic transformation of  $\mathbb{P}^2$  with three proper base points.

**Lemma 2.4.** *Let  $\rho: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  be a birational map and let  $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  be a quadratic birational map with base points  $q_1, q_2, q_3 \in \mathbb{P}^2$ . For  $1 \leq i < j \leq 3$  we denote by  $\ell_{ij} \subset \mathbb{P}^2$  the line which joins the base points  $q_i$  and  $q_j$ . If  $D(L, \rho) = k \geq 1$ , we have*

$$D(L, \varphi\rho) = \begin{cases} k + 1 & \text{if } \rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \text{Bp}(\varphi), \\ k & \text{if } \rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}, \\ k & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } (\rho\varphi)(L) \in \text{Bp}(\varphi^{-1}), \\ k - 1 & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } (\rho\varphi)(L) \notin \text{Bp}(\varphi^{-1}). \end{cases}$$

*Proof.* We consider the minimal resolutions of  $\varphi$ ; in Figures 2-5, the filled black dots denote the successive images of  $L$ , i.e.  $\rho(L)$ ,  $(\pi^{-1}\rho)(L)$  and  $(\eta\pi^{-1}\rho)(L)$  respectively.

We argue by Figure 2 and 3 in the case where  $\rho(L)$  does not coincide with any of the base points of  $\varphi$ . If  $\rho(L) \in \ell_{ij}$  for some  $i, j$ , then  $D(L, \varphi\rho) = D(L, \rho) + 1$ , since  $\ell_{ij}$  is contracted by  $\varphi$ . Otherwise, the number of times  $L$  is contracted does not change. Suppose that  $\rho(L) = q_i$  for some  $i$ . If  $D(L, \rho) = 1$ , we have  $(\pi^{-1}\rho)(L) = E_i$ , and then

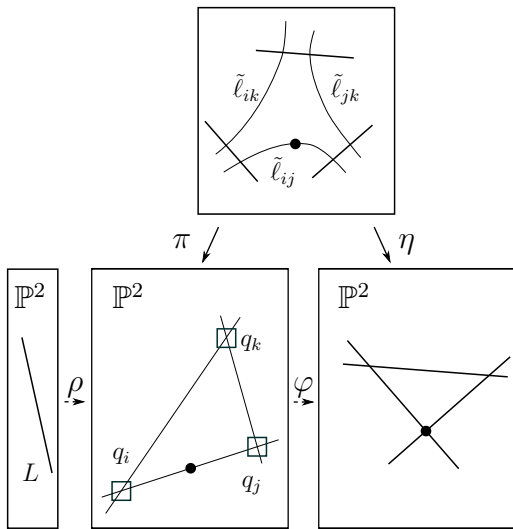


FIGURE 2.  $D(L, \varphi\rho) = k + 1$ ;  
 $\rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \text{Bp}(\varphi)$ .

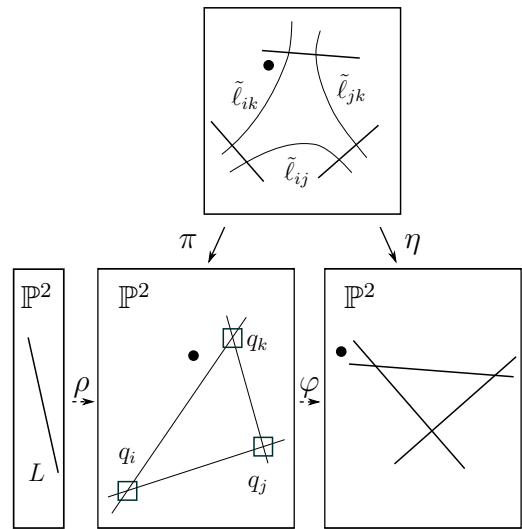


FIGURE 3.  $D(L, \varphi\rho) = k$ ;  
 $\rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}$ .

clearly  $D(L, \varphi\rho) = 0$  since  $E_i$  is not contracted by  $\eta$ . If  $D(L, \rho) \geq 2$  we argue by the Figures 4 and 5.

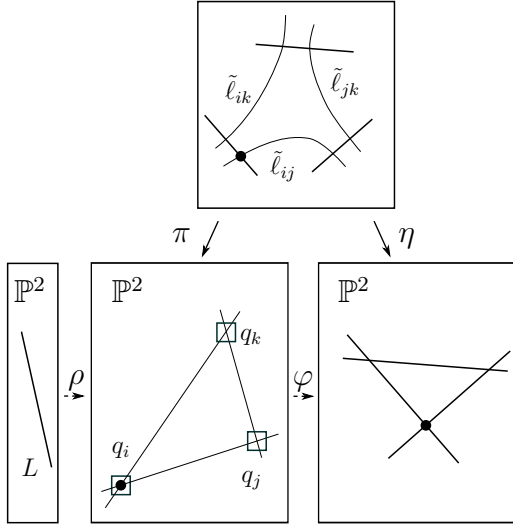


FIGURE 4.  $D(L, \varphi\rho) = k$ ;  
 $\rho(L) = q_i$  and  $(\rho\varphi)(L) \in \text{Bp}(\varphi^{-1})$ .

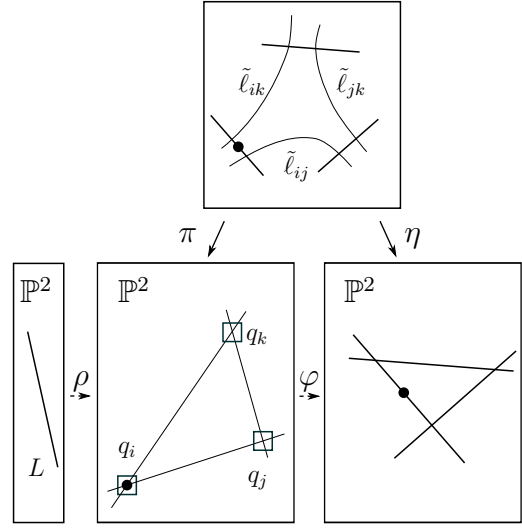


FIGURE 5.  $D(L, \varphi\rho) = k - 1$ ;  
 $\rho(L) = q_i$  and  $(\rho\varphi)(L) \notin \text{Bp}(\varphi^{-1})$ .

□

**Remark 2.5.** If  $D(L, \rho) \geq 2$ , then the point  $(\pi^{-1}\rho)(L)$  in the first neighbourhood of  $\rho(L)$  defines a tangent direction at  $\rho(L) \in \mathbb{P}^2$ . If we take  $\varphi$  as in Lemma 2.4 with  $q_i \in \text{Bp}(\varphi)$  for some  $i$ , then this tangent direction coincides with the direction of one of  $l_{ij}, l_{ik}$  if and only if  $(\rho\varphi)(L) \in \text{Bp}(\varphi^{-1})$ .

**Proposition 2.6.** *For any given element  $\rho \in \text{Dec}(L)$ , there is a decomposition of  $\rho$  into quadratic maps  $\rho = \rho_m \dots \rho_1$  with three proper base points such that none of the successive compositions  $(\rho_i \dots \rho_1)_{i=1}^m$  contract  $L$  to a point.*

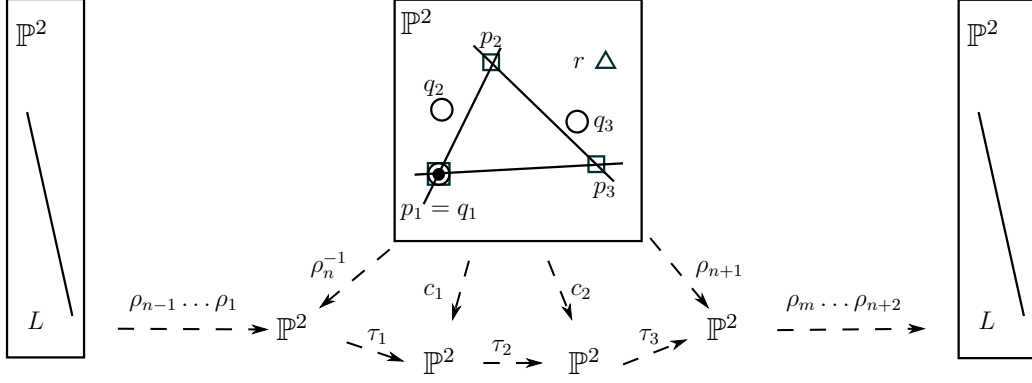
*Proof.* Let  $\rho = \rho_m \dots \rho_1$  be a decomposition of  $\rho$  into quadratic maps with only proper base points. We can assume that  $d := \max\{D(L, \rho_j \dots \rho_1) \mid 1 \leq j \leq m\} > 0$ , otherwise we are done. Let  $n := \max\{j \mid D(L, \rho_j \dots \rho_1) = d\}$ . We denote the base points of  $\rho_n^{-1}$  and  $\rho_{n+1}$  by  $p_1, p_2, p_3$  and  $q_1, q_2, q_3$  respectively.

We first look at the case where  $D(L, \rho_{n-1} \dots \rho_1) = D(L, \rho_{n+1} \dots \rho_1) = d - 1$ . Then composition with  $\rho_n$  and  $\rho_{n+1}$  fall under Cases 1 and 4 of Lemma 2.4, so both  $\rho_n^{-1}$  and  $\rho_{n+1}$  have a base point at  $(\rho_n \dots \rho_1)(L) \in \mathbb{P}^2$ . We may assume that this point is  $p_1 = q_1$ , as in Figure 6. Interchanging the roles of  $q_2$  and  $q_3$  if necessary, we may assume that  $p_1, p_2, q_2$  are not collinear. Let  $r \in \mathbb{P}^2$  be a general point, and let  $c_1$  and  $c_2$  denote quadratic maps with base points  $[p_1, p_2, r]$  and  $[p_1, q_2, r]$  respectively; then the maps  $\tau_1, \tau_2, \tau_3$  (defined by the commutative diagram in Figure 6) are quadratic with three proper base points in  $\mathbb{P}^2$ . Note that  $D(L, \tau_i \dots \tau_1 \rho_{n-1} \dots \rho_1) = d - 1$  for  $i = 1, 2, 3$ . Thus we obtained a new decomposition of  $\rho$  into quadratic maps with three proper base points

$$\rho = \rho_m \dots \rho_{n+2} \tau_3 \tau_2 \tau_1 \rho_{n-1} \dots \rho_1,$$

where the number of instances where  $L$  is contracted  $d$  times has decreased by 1.

Now assume instead that  $D(L, \rho_{n-1} \dots \rho_1) = d$  and  $D(L, \rho_{n+1} \dots \rho_1) = d - 1$ . Then composition with  $\rho_{n+1}$  falls under Case 4 of Lemma 2.4, so  $(\rho_n \dots \rho_1)(L)$  is a base point of  $\rho_{n+1}$ , which we may assume to be  $q_1$ . Furthermore composition with  $\rho_n$  falls under


 FIGURE 6. The decomposition of  $\rho_{n+1}\rho_n$  into quadratic maps  $\tau_1, \tau_2, \tau_3$ 

Cases 2 or 3 of Lemma 2.4, so  $(\rho_n \dots \rho_1)(L)$  either does not lie on a line joining two base points of  $\rho_n^{-1}$ , or  $D(L, \rho_n \dots \rho_1) \geq 2$  and  $(\rho_n \dots \rho_1)(L)$  is a base point of  $\rho_n^{-1}$  (which we may assume to be  $p_1$ , and equal to  $q_1$ ), at the same time as  $(\rho_{n-1} \dots \rho_1)(L)$  is a base point of  $\rho_n$ .

We consider the first case. If  $D(L, \rho_n \dots \rho_1) \geq 2$  so that  $L$  defines a tangent direction at  $(\rho_n \dots \rho_1)(L)$ , then this tangent direction has to be different from at least two of the three directions at  $q_1$  that are defined by the lines through  $q_1$  and the  $p_i$ ,  $i = 1, 2, 3$ . By renumbering the  $p_i$ , we may assume that  $p_2, p_3$  define these two directions (no renumbering is needed if  $D(L, \rho_n \dots \rho_1) = 1$ ). Then with a quadratic map  $c_1 := [q_1, p_2, p_3]$  with base points  $q_1, p_2, p_3$ , we are in Case 4 of Lemma 2.4 and obtain  $D(L, c_1 \rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$ . Let  $r, s \in \mathbb{P}^2$  be two general points and define  $c_2, c_3, c_4$  with three proper base points respectively as  $[q_1, r, p_3]$ ,  $[q_1, r, s]$ ,  $[q_1, q_2, s]$ . Note that the corresponding maps  $\tau_1, \dots, \tau_5$ , defined in an analogous way as in Figure 6, are quadratic with three proper base points. Note also that  $D(L, c_i \rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$  for  $i = 2, 3, 4$ . Only for  $i = 4$  this is not immediately clear, so suppose that this is not the case, i.e.  $D(L, c_4 \rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1)$ . It follows that  $D(L, \rho_n \dots \rho_1) \geq 2$  and that the tangent direction corresponding to  $(\rho_n \dots \rho_1)(L)$  is given by the line through  $q_1$  and  $q_2$ , but this is not possible by the assumption that  $D(L, \rho_{n+1} \dots \rho_1) = d - 1$ .

In the second case we have  $p_1 = q_1$  and the tangent direction at  $p_1 = q_1$  corresponding to  $(\rho_n \dots \rho_1)(L)$  is the direction either of the line through  $p_1$  and  $p_2$  or the line through  $p_1$  and  $p_3$  (see Figure 4). By interchanging the roles of  $p_2$  and  $p_3$  if necessary, we may assume that it corresponds to the direction of the line through  $p_1$  and  $p_3$ . Interchanging the roles of  $q_2$  and  $q_3$  if necessary, we may assume that  $p_1, q_2, p_3$  are not collinear. Let  $r, s \in \mathbb{P}^2$  be general points and define quadratic maps  $c_1, c_2, c_3$  with three proper base points respectively by  $[p_1, p_2, s]$ ,  $[p_1, r, s]$ ,  $[p_1, r, q_2]$ . Then the corresponding maps  $\tau_1, \tau_2, \tau_3, \tau_4$  are quadratic with three proper base points and  $D(L, c_i \rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$  for  $i = 1, 2, 3$ . The latter holds for  $c_1$  since the direction given by  $p_1$  and  $p_2$  is different from the tangent direction corresponding to  $(\rho_n \dots \rho_1)(L)$ , and for  $c_3$  it follows from the assumption that the image of  $L$  is contracted  $d - 1$  times by  $(\rho_{n+1} \dots \rho_1)$  and that  $p_1, q_2, p_3$  are not collinear.

Both in the first and second case, we again arrive at a new decomposition into quadratic maps with three proper base points

$$\rho = \rho_m \dots \rho_{n+2} \tau_j \dots \tau_1 \rho_{n-1} \dots \rho_1 \quad (j \in \{4, 5\}),$$

where the number of instances where  $L$  is contracted  $d$  times has decreased by 1, and we conclude by induction.  $\square$

3. AVOIDING TO SEND  $L$  TO A CURVE OF DEGREE HIGHER THAN 1.

By Proposition 2.6, any element  $\rho \in \text{Dec}(L)$  can be decomposed as

$$\rho = \rho_m \cdots \rho_1$$

where each  $\rho_j$  is quadratic with three proper base points, and all of the successive images  $((\rho_i \cdots \rho_1)(L))_{i=1}^m$  of  $L$  are curves. The aim of this section is to show that the  $\rho_j$  even can be chosen so that all of these curves have degree 1. That is, we find a decomposition of  $\rho$  into quadratic maps such that all the successive images of  $L$  are lines. This means in particular that  $\text{Dec}(L)$  is generated by its elements of degree 1 and 2.

**Definition 3.1.** A birational transformation of  $\mathbb{P}^2$  is called de Jonquières if it preserves the pencil of lines passing through  $[1 : 0 : 0] \in \mathbb{P}^2$ . These transformations form a subgroup of  $\text{Bir}(\mathbb{P}^2)$  which we denote by  $\mathcal{J}$ .

**Remark 3.2.** In [AC2002], a de Jonquières map is defined by the slightly less restrictive property that it sends a pencil of lines to a pencil of lines. Given a map with this property, we can always obtain an element in  $\mathcal{J}$  by composing from left and right with elements of  $\text{PGL}_3$ .

For a curve  $C \subset \mathbb{P}^2$  and a point  $p$  in  $\mathbb{P}^2$  or infinitely near, we denote by  $m_C(p)$  the multiplicity of  $C$  in  $p$ . If it is clear from context which curve we are referring to, we will use the notation  $m(p)$ .

**Lemma 3.3.** *Let  $\varphi \in \mathcal{J}$  be of degree  $e \geq 2$ , and  $C \subset \mathbb{P}^2$  a curve of degree  $d$ . Suppose that*

$$\deg(\varphi(C)) \leq d.$$

*Then there exist two base points  $q_1, q_2$  of  $\varphi$  different from  $[1 : 0 : 0]$  such that*

$$m_C([1 : 0 : 0]) + m_C(q_1) + m_C(q_2) \geq d.$$

This inequality can be made strict in case  $\deg(\varphi(C)) < d$ , with a completely analogous proof.

*Proof.* Since  $\varphi \in \mathcal{J}$  is of degree  $e$ , it has exactly  $2e - 1$  base points  $r_0 := [1 : 0 : 0], r_1, \dots, r_{2e-2}$  of multiplicity  $e - 1, 1, \dots, 1$  respectively. Then

$$\begin{aligned} d &\geq \deg(\varphi(C)) = ed - (e - 1)m_C(r_0) - \sum_{i=1}^{e-1} (m_C(r_{2i-1}) + m_C(r_{2i})) \\ &= d + \sum_{i=1}^{e-1} (d - m_C(r_0) - m_C(r_{2i-1}) - m_C(r_{2i})) \end{aligned}$$

Hence there exist  $i_0$  such that  $d \leq m_C(r_0) + m_C(r_{2i_0-1}) + m_C(r_{2i_0})$ .  $\square$

**Remark 3.4.** Note also that we can choose the points  $q_1, q_2$  such that  $q_1$  either is a proper point in  $\mathbb{P}^2$  or in the first neighbourhood of  $[1 : 0 : 0]$ , and that  $q_2$  either is proper point of  $\mathbb{P}^2$  or is in the first neighbourhood of  $[1 : 0 : 0]$  or  $q_1$ .

**Remark 3.5.** A quadratic map sends a pencil of lines through one of its base points to a pencil of lines, and we conclude from Proposition 2.6 and Remark 3.2 that there exists maps  $\alpha_1, \dots, \alpha_{m+1} \in \text{PGL}_3$  and  $\rho_i \in \mathcal{J} \setminus \text{PGL}_3$  such that

$$\rho = \alpha_{m+1} \rho_m \alpha_m \rho_{m-1} \alpha_{m-1} \cdots \alpha_2 \rho_1 \alpha_1$$

and such that all of the successive images of  $L$  with respect to this decomposition are curves.

The following proposition is an analogue of the classical Castelnuovo's Theorem stating that any map in  $\text{Bir}(\mathbb{P}^2)$  is a product of de Jonquières maps.

**Proposition 3.6.** *Let  $\rho \in \text{Dec}(L)$ . Then there exists  $\rho_i \in \mathcal{J} \setminus \text{PGL}_3$  and  $\alpha_i \in \text{PGL}_3$  such that  $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$  and all of the successive images of  $L$  are lines.*

*Proof.* Start with a decomposition  $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$  as in Remark 3.5.

Denote  $C_i := (\rho_i\alpha_i\dots\rho_1\alpha_1)(L) \subset \mathbb{P}^2$ ,  $d_i := \deg(C_i)$  and let

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad n := \max\{i \mid D = d_i\}, \quad k := \sum_{i=1}^n (\deg \rho_i - 1).$$

We use induction on the lexicographically ordered pair  $(D, k)$ .

We may assume that  $D > 1$ , otherwise our goal is already achieved. We may also assume that  $\alpha_{n+1} \notin \mathcal{J}$ , otherwise the pair  $(D, k)$  decreases as we replace the three maps  $\rho_{n+1}, \alpha_{n+1}, \rho_n$  by their composition  $\rho_{n+1}\alpha_{n+1}\rho_n \in \mathcal{J}$ . Indeed, either  $D$  decreases, or  $D$  stays the same while  $k$  decreases at least by  $\deg \rho_n - 1$ . Using Lemma 3.3, we find simple base points  $p_1, p_2$  of  $\rho_n^{-1}$  and simple base points  $\tilde{q}_1, \tilde{q}_2$  of  $\rho_{n+1}$ , all different from  $p_0 := [1 : 0 : 0]$ , such that

$$m_{C_n}(p_0) + m_{C_n}(p_1) + m_{C_n}(p_2) \geq D$$

and

$$m_{\alpha_{n+1}(C_n)}(p_0) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_1) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_2) > D.$$

We choose  $p_1, p_2, \tilde{q}_1, \tilde{q}_2$  as in Remark 3.4. By slight abuse of notation, we denote by  $q_0 = \alpha_{n+1}^{-1}(p_0)$ ,  $q_1 = \alpha_{n+1}^{-1}(\tilde{q}_1)$  and  $q_2 = \alpha_{n+1}^{-1}(\tilde{q}_2)$  respectively the (proper or infinitely near) points in  $\mathbb{P}^2$  that correspond to  $p_0, \tilde{q}_1$ , and  $\tilde{q}_2$  under the isomorphism  $\alpha_{n+1}^{-1}$ . Note that  $p_0$  and  $q_0$  are two distinct points of  $\mathbb{P}^2$  since  $\alpha_{n+1} \notin \mathcal{J}$ . We number the points so that  $m(p_1) \geq m(p_2)$ ,  $m(\tilde{q}_1) \geq m(\tilde{q}_2)$  and so that if  $p_i$  (resp.  $\tilde{q}_i$ ) is infinitely near  $p_j$  (resp.  $\tilde{q}_j$ ), then  $j < i$ .

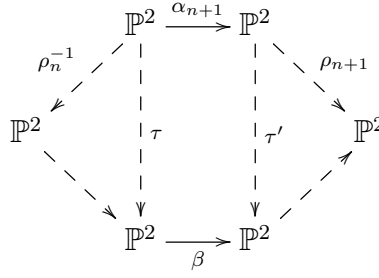
We study two cases separately depending on the multiplicities of the base points.

Case (a):  $m(q_0) \geq m(q_1)$  and  $m(p_0) \geq m(p_1)$ . Then we find two quadratic maps  $\tau', \tau \in \mathcal{J}$  and  $\beta \in \text{PGL}_3$  so that  $\rho_{n+1}\alpha_{n+1}\rho_n = (\rho_{n+1}\tau^{-1})\beta(\tau\rho_n)$  and so that the pair  $(D, k)$  is reduced as we replace the sequence  $(\rho_{n+1}, \alpha_{n+1}, \rho_n)$  by  $(\rho_{n+1}\tau^{-1}, \beta, \tau\rho_n)$ . The procedure goes as follows.

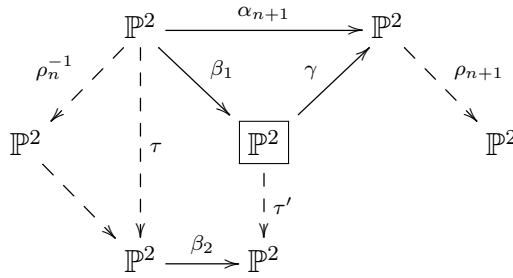
If possible we choose a point  $r \in \{p_1, q_1\} \setminus \{p_0, q_0\}$ . Should this set be empty, i.e.  $p_0 = q_1$  and  $p_1 = q_0$ , we choose  $r = q_2$  instead. The ordering of the points implies that the point  $r$  is either a proper point in  $\mathbb{P}^2$  or in the first neighbourhood of  $p_0$  or  $q_0$ . Furthermore, the assumption implies that  $m(p_0) + m(q_0) + m(r) > D$ , so  $r$  is not on the line passing through  $p_0$  and  $q_0$ . In particular, there exists a quadratic map  $\tau \in \mathcal{J}$  with base points  $p_0, q_0, r$ ; then

$$\deg(\tau(C_n)) = 2D - m(p_0) - m(q_0) - m(r) < D.$$

Choose  $\beta \in \text{PGL}_3$  so that the quadratic map  $\tau' := \beta\tau(\alpha_{n+1})^{-1}$  in the below commutative diagram is de Jonquières – this is possible since  $\tau$  has  $q_0$  as a base point. This decreases the pair  $(D, k)$ .



Case (b):  $m(p_0) < m(p_1)$ . Let  $\tau$  be a quadratic de Jonquières map with base points  $p_0, p_1, p_2$ . This is possible since our assumption implies that  $p_1$  is a proper base point and because  $p_0, p_1, p_2$  are base points of  $\rho_n^{-1}$  of multiplicity  $\deg \rho_n - 1, 1, 1$  respectively and hence not collinear. Choose  $\beta_1 \in \text{PGL}_3$  which exchanges  $p_0$  and  $p_1$ , let  $\gamma = \alpha_{n+1}\beta_1^{-1}$  and choose  $\beta_2 \in \text{PGL}_3$  so that  $\tau' := \beta_2\tau\beta_1^{-1} \in \mathcal{J}$ . The latter is possible since  $\beta_1^{-1}(p_0) = p_1$  is a base point of  $\tau$ , and we have the following diagram.



Since  $\deg(\tau\rho_n) = \deg \rho_n - 1$ , the pair  $(D, k)$  stays unchanged as we replace the sequence  $(\alpha_{n+1}, \rho_n)$  in the decomposition of  $\rho$  by the sequence  $(\gamma, (\tau')^{-1}, \beta_2, \tau\rho_n)$ . In the new decomposition of  $\rho$  the maps  $(\tau')^{-1}$  and  $\gamma$  play the roles that  $\rho_n$  and  $\alpha_{n+1}$  respectively played in the previous decomposition. In the squared  $\mathbb{P}^2$ , we have

$$m(p_0) = m(\beta_1(p_1)) > m(\beta_1(p_0)) = m(p_1).$$

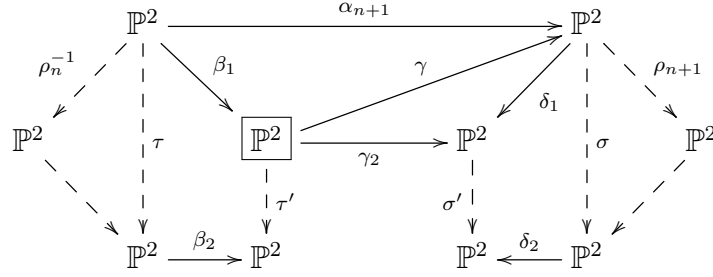
Define  $q'_0 := \gamma^{-1}(p_0)$ ,  $q'_1 := \gamma^{-1}(\tilde{q}_1)$ ,  $q'_2 := \gamma^{-1}(\tilde{q}_2)$ , and note that  $q'_0 = \beta_1(q_0)$ ,  $q'_1 = \beta_1(q_1)$  and  $q'_2 = \beta_1(q_2)$ . In the new decomposition these points play the roles that  $q_0, q_1, q_2$  played in the previous decomposition.

If  $m(q'_0) \geq m(q'_1)$ , we continue as in case (a) with the points  $p_0, p_1, \beta_1(p_2)$  and  $q'_0, q'_1, q'_2$ .

If  $m(q'_0) < m(q'_1)$ , we replace the sequence  $(\rho_{n+1}, \gamma)$  by a new sequence such that, similar to case (a), the roles of  $q'_0$  and  $q'_1$  are exchanged, and we will do this without touching  $p_0, p_1, \beta_1(p_2)$ . The replacement will not change  $(D, k)$  and we can apply case (a) to the new sequence.

As  $m(q'_0) < m(q'_1)$ , the point  $q'_1$  is a proper point of  $\mathbb{P}^2$ . Analogously to the previous case, there exists  $\sigma \in \mathcal{J}$  with base points  $\gamma(q'_0) = p_0, \gamma(q'_1) = \tilde{q}_1, \gamma(q'_2) = \tilde{q}_2$ , and there exists  $\delta_1 \in \text{PGL}_3$  which exchanges  $p_0$  and  $\tilde{q}_1$ . Since  $\delta_1^{-1}(p_0) = \tilde{q}_1$  is a base point of  $\sigma$ , there furthermore exists  $\delta_2 \in \text{PGL}_3$  such that  $\sigma' := \delta_2\sigma\delta_1^{-1} \in \mathcal{J}$ . Let  $\gamma_2 := \delta_1\gamma$ .





Replacing the sequence  $(\rho_{n+1}, \gamma)$  with  $(\rho_{n+1}\sigma^{-1}, \delta_2^{-1}, \sigma', \delta_1\gamma)$  does not change the pair  $(D, k)$ . The latest position with the highest degree is still the squared  $\mathbb{P}^2$  but in the new sequence we have

$$m(\gamma_2^{-1}(p_0)) = m(\beta_1(q_1)) > m(\beta_1(q_0)) = m(\gamma_2^{-1}(\delta_1(\tilde{q}_1)))$$

Since  $p_0, p_1, \beta_1(p_2)$  were undisturbed, the inequality  $m(p_0) > m(p_1)$  still holds, and we proceed as in case (a).

In this proof, we have used several different quadratic maps  $\tau, \tau', \sigma, \sigma'$ . Note that none of these can contract  $C$  (or an image of  $C$ ), since quadratic maps only can contract curves of degree 1.  $\square$

**Remark 3.7.** Suppose that  $\rho \in \mathcal{J}$  preserves a line  $L$ . Then the Noether-equalities imply that  $L$  passes either through  $[1 : 0 : 0]$  and no other base points of  $\rho$ , or that it passes through exactly  $\deg \rho - 1$  simple base points of  $\rho$  and not through  $[1 : 0 : 0]$ .

**Lemma 3.8.** Let  $\rho \in \mathcal{J}$  be of degree  $\geq 2$  and let  $L$  be a line passing through exactly  $\deg \rho - 1$  simple base points of  $\rho$  and not through  $[1 : 0 : 0]$ . Then there exist  $\rho_1, \dots, \rho_i \in \mathcal{J}$  of degree 2 such that  $\rho = \rho_m \cdots \rho_1$  and the successive images of  $L$  are lines.

*Proof.* Note that the curve  $\rho(L)$  is a line not passing through  $\rho(L)$ . Call  $p_0 := [1 : 0 : 0], p_1, \dots, p_{2d-2}$  the base points of  $\rho$ . Without loss of generality, we can assume that  $p_1, \dots, p_{d-1}$  are the simple base points of  $\rho$  that are contained in  $L$  and that  $p_1$  is a proper base point in  $\mathbb{P}^2$ . We do induction on the degree of  $\rho$ .

If there is no simple proper base point  $p_i, i \geq d$ , of  $\rho$  in  $\mathbb{P}^2$  that is not on  $L$ , choose a general point  $r \in \mathbb{P}^2$ . There exists a quadratic transformation  $\tau \in \mathcal{J}$  with base points  $p_0, p_1, r$ . The transformation  $\rho\tau^{-1} \in \mathcal{J}$  is of degree  $\deg \rho$  and sends the line  $\tau(L)$  (which does not contain  $[1 : 0 : 0]$ ) onto the line  $\rho(L)$ . The point  $\rho(r) \in \mathbb{P}^2$  is a base point of  $(\rho\tau^{-1})^{-1}$  not on the line  $\rho(L)$ .

So, we can assume that there exists a proper base point of  $\rho$  in  $\mathbb{P}^2$  that is not on  $L$ , lets call it  $p_d$ . The points  $p_0, p_1, p_d$  are not collinear (because of their multiplicities), hence there exists  $\tau \in \mathcal{J}$  of degree 2 with base points  $p_0, p_1, p_d$ . The map  $\rho\tau^{-1} \in \mathcal{J}$  is of degree  $\deg \rho - 1$  and  $\tau(L)$  is a line passing through exactly  $\deg \rho - 2$  simple base points of  $\rho\tau^{-1}$  and not through  $[1 : 0 : 0]$ .  $\square$

**Lemma 3.9.** Let  $\rho \in \mathcal{J}$  be of degree  $\geq 2$  and let  $L$  be a line passing through  $[1 : 0 : 0]$  and no other base points of  $\rho$ . Then there exist  $\rho_1, \dots, \rho_m \in \mathcal{J}$  of degree 2 such that  $\rho = \rho_m \cdots \rho_1$  and the successive images of  $L$  are lines.

*Proof.* Note that the curve  $\rho(L)$  is a line passing through  $[1 : 0 : 0]$ . We use induction on the degree of  $\rho$ .

Assume that  $\rho$  has no simple proper base points, i.e. all simple base points are infinitely near  $p_0 := [1 : 0 : 0]$ . There exists a base point  $p_1$  of  $\rho$  in the first neighbourhood of  $p_0$ . Choose a general point  $q \in \mathbb{P}^2$ . There exists  $\tau \in \mathcal{J}$  quadratic with base points

$p_0, p_1, q$ . The map  $\rho\tau^{-1} \in \mathcal{J}$  is of degree  $\deg \rho$  and  $\tau(L)$  is a line passing through the base point  $p_0$  of  $\rho\tau^{-1}$  of multiplicity  $\deg \rho - 1$  and through no other base points of  $\rho\tau^{-1}$ . Moreover, the point  $\rho(q)$  is a (simple proper) base point of  $\tau\rho^{-1}$ . Therefore,  $\tau\rho^{-1}$  has a simple proper base point in  $\mathbb{P}^2$  and sends the line  $\rho(L)$  onto the line  $\tau(L)$ , both of which pass through  $p_0$  and no other base points.

So, we can assume that  $\rho$  has at least one simple proper base point  $p_1$ . Let  $p_2$  be a base point of  $\rho$  that is a proper point of  $\mathbb{P}^2$  or in the first neighbourhood of  $p_0$  or  $p_1$ . Because of their multiplicities, the points  $p_0, p_1, p_2$  are not collinear. Hence there exists  $\tau \in \mathcal{J}$  quadratic with base points  $p_0, p_1, p_2$ . The map  $\rho\tau^{-1}$  is a map of degree  $\deg \rho - 1$  and  $\tau(L)$  is a line passing through  $p_0$  and no other base points.  $\square$

**Lemma 3.10.** *Let  $\rho \in \mathcal{J}$  be a map of degree 2 that sends a line  $L$  onto a line. Then there exist quadratic maps  $\rho_1, \dots, \rho_n \in \mathcal{J}$  with only proper base points such that*

$$\rho = \rho_n \cdots \rho_1,$$

and the successive images of  $L$  are lines.

*Proof.* Suppose first that exactly two of the three base points of  $\rho$  are proper. We number

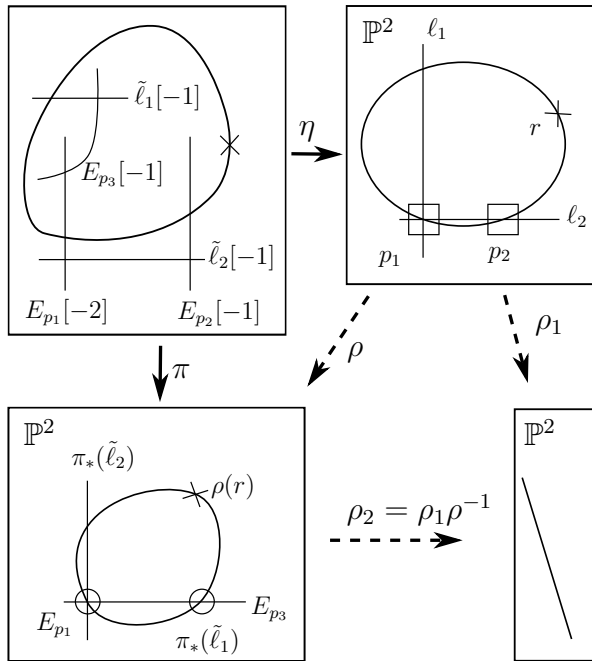


FIGURE 7. Numbers in square brackets denote self-intersection.

the base points so that  $p_1, p_2 \in \mathbb{P}^2$  and so that  $p_3$  is in the first neighbourhood of  $p_1$ , and denote by  $\ell_1 \subset \mathbb{P}^2$  the line through  $p_1$  which has the tangent direction defined by  $p_3$ . Choose a general point  $r \in \mathbb{P}^2$ , and define a quadratic map  $\rho_1$  with three base points  $p_1, p_2, r \in \mathbb{P}^2$ . A minimal resolution of  $\rho$  is given by  $\pi$  and  $\eta$  as in Figure 7; it is obtained by blowing up, in order,  $p_1, p_2, p_3$ , and then contracting in order  $\tilde{\ell}_2 := \eta_*^{-1}(\ell_2)$ ,  $\tilde{\ell}_1 := \eta_*^{-1}(\ell_1)$  and the exceptional divisor corresponding to  $p_1$ . By looking at the pull back of a general line in  $\mathbb{P}^2$  with respect to  $\rho_2 := \rho_1\rho^{-1}$ , we see that this map has three proper base points  $E_{p_1}, \rho(r), \pi_*(\tilde{\ell}_1)$ . This gives us a decomposition of the desired form:  $\rho = \rho_2^{-1}\rho_1$ . Note that since  $\rho$  sends the line  $L$  onto a line,  $L$  has to pass through exactly one of the base points of  $\rho$ , and this base point

has to be proper. Thus  $L$  is sent to a line by  $\rho_1$ . Using the diagram in Figure 7, we can see that this line is further sent by  $\rho_2^{-1}$  to a line through  $E_{p_1}$  if  $L$  passes through  $p_1$  and a line through  $\pi_*(\tilde{\ell}_1)$  if  $L$  passes through  $p_2$ .

If  $[1 : 0 : 0]$  is the only proper base point of  $\rho$ , we reduce to the first case as follows. Denote by  $q$  the base point in the first neighbourhood of  $[1 : 0 : 0]$  and choose a general point  $r \in \mathbb{P}^2$ . Let  $\rho_1$  be a quadratic map with base points  $[1 : 0 : 0], q, r$ , and let  $\rho_2 := \rho_1\rho^{-1}$ . If we denote the base points of  $\rho^{-1}$  by  $q_1, q_2, q_3$  so that  $q_1$  is the proper base point and  $q_2$  the base point in the first neighbourhood of  $q_1$ , then the base points of  $\rho_2$  are  $q_1, q_2, \rho(r)$ , i.e. it has exactly two proper base points.

It is also clear that  $\rho_1$  sends  $L$  to a line, which is further sent by  $\rho_2^{-1}$  to a line through  $q_1$ . Thus we can apply the first part of this proof to each of  $\rho_2^{-1}$  and  $\rho_1$  in  $\rho = \rho_2^{-1}\rho_1$ , and thus get a decomposition of the desired form.  $\square$

**Theorem 1.** *For any line  $L$ , the group  $\text{Dec}(L)$  is generated by  $\text{Dec}(L) \cap \text{PGL}_3$  and any of its quadratic elements having three proper base points in  $\mathbb{P}^2$ .*

*Proof.* By conjugating with an appropriate automorphism of  $\mathbb{P}^2$ , we can assume that  $L$  is given by  $x = y$ . Note that the standard quadratic involution  $\sigma: [x : y : z] \mapsto [yz : xz : xy]$  is contained in  $\text{Dec}(L)$ . It follows from Proposition 3.6, Remark 3.7, and Lemmata 3.8, 3.9 and 3.10 that every element  $\rho \in \text{Dec}(L)$  has a composition  $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$ , where  $\alpha_i \in \text{PGL}_3$  and  $\rho_i \in \mathcal{J}$  are quadratic with only proper base points in  $\mathbb{P}^2$  such that the successive images of  $L$  are lines. By composing the  $\rho_i$  from the left and the right with linear maps, we obtain a decomposition

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$$

where  $\alpha_i \in \text{PGL}_3 \cap \text{Dec}(L)$  and  $\rho_i \in \text{Dec}(L)$  are of degree 2 with only proper base points in  $\mathbb{P}^2$ . It therefore suffices to show that for any quadratic element  $\rho \in \text{Dec}(L)$  having three proper base points in  $\mathbb{P}^2$  there exist  $\alpha, \beta \in \text{Dec}(L) \cap \text{PGL}_3$  such that  $\sigma = \beta\rho\alpha$ .

By Remark 3.7, for any quadratic element of  $\text{Dec}(L)$  the line  $L$  passes through exactly one of its base points in  $\mathbb{P}^2$ .

Let  $q_1 = [0 : 0 : 1]$ ,  $q_2 = [0 : 1 : 0]$ ,  $q_3 = [1 : 0 : 0]$ . They are the base points of  $\sigma$ , and  $\sigma$  sends the pencil of lines through  $q_i$  onto itself. Furthermore,  $q_1 \in L$  but  $q_2, q_3 \notin L$ . Let  $s := [1 : 1 : 1] \in L$ . Remark that  $\sigma(s) = s$  and that no three of  $q_1, q_2, q_3, s$  are collinear.

Let  $\rho \in \text{Dec}(L)$  be another quadratic map having three proper base points in  $\mathbb{P}^2$ . Let  $p_1, p_2, p_3$  (resp.  $p'_1, p'_2, p'_3$ ) be its base points (resp. the ones of  $\rho^{-1}$ ). Say  $L$  passes through  $p_1$  and  $\rho$  sends the pencil of lines through  $p_i$  onto the pencil of lines through  $p'_i$ ,  $i = 1, 2, 3$ . Pick a point  $r \in L \setminus \{p_1\}$ , not collinear with  $p_2, p_3$ . Then no three of  $p_1, p_2, p_3, r$  (resp.  $p'_1, p'_2, p'_3, \rho(r)$ ) are collinear. In particular, there exist  $\alpha, \beta \in \text{PGL}_3$  such that

$$\alpha: \begin{cases} q_i \mapsto p_i \\ s \mapsto r \end{cases}, \quad \beta: \begin{cases} p'_i \mapsto q_i \\ \rho(r) \mapsto s \end{cases}$$

Note that  $\alpha, \beta \in \text{Dec}(L) \cap \text{PGL}_3$ . Furthermore, the quadratic maps  $\sigma, \rho' := \beta\rho\alpha \in \text{Dec}(L)$  and their inverse all have the same base points (namely  $q_1, q_2, q_3$ ) and both  $\sigma, \rho'$  send the pencil through  $q_i$  onto itself. Since moreover  $\rho'(s) = \sigma(s) = s$ , we have  $\sigma = \rho'$ .  $\square$

#### 4. $\text{Dec}(L)$ IS NOT AN AMALGAM

Just like  $\text{Bir}(\mathbb{P}^2)$ , its subgroup  $\text{Dec}(L)$  is generated by its linear elements and one quadratic element (Theorem 1). In [Cor2013, Corollary A.2], it is shown that  $\text{Bir}(\mathbb{P}^2)$  is not an amalgamated product. In this section we adjust the proof to our situation and prove that the same statement holds for  $\text{Dec}(L)$ .

The notion of being an amalgamated product is closely related to actions on trees, or, in this case,  $\mathbb{R}$ -trees.

**Definition and Lemma 4.1.** A *real tree*, or  $\mathbb{R}$ -tree, can be defined in the following three equivalent ways [Cis2001]:

- (1) A geodesic space which is 0-hyperbolic in the sense of Gromov.
- (2) A uniquely geodesic metric space for which  $[a, c] \subset [a, b] \cup [b, c]$  for all  $a, b, c$ .

(3) A geodesic metric space with no subspace homeomorphic to the circle.

We say that a real tree is a *complete real tree* if it is complete as a metric space.

Every ordinary tree can be seen as a real tree by endowing it with the usual metric but not every real tree is isometric to an simplicial tree (endowed with the usual metric) [Cis2001, §II.2, Proposition 2.5, Example].

**Definition 4.2.** A group  $G$  has the *property*  $(\text{FR})_\infty$  if for every isometric action of  $G$  on a complete real tree, every element has a fixed point.

We summarize the discussion in [Cor2013, before Remark A.3] in the following result.

**Lemma 4.3.** *If a group  $G$  has property  $(\text{FR})_\infty$ , it does not decompose as non-trivial amalgam.*

We will devote the rest of this section to proving Proposition 4.4 and thereby showing that  $\text{Dec}(L)$  is not an amalgam.

**Proposition 4.4.** *The decomposition group  $\text{Dec}(L)$  has property  $(\text{FR})_\infty$ .*

By convention, from now on,  $\mathcal{T}$  will denote a complete real tree and all actions on  $\mathcal{T}$  are assumed to be isometric.

**Definition 4.5.** Let  $\mathcal{T}$  be a complete real tree.

- (1) A *ray* in  $\mathcal{T}$  is a geodesic embedding  $(x_t)_{t \geq 0}$  of the half-line.
- (2) An *end* in  $\mathcal{T}$  is an equivalence class of rays, where we say that two rays  $x$  and  $y$  are equivalent if there exists  $t, t' \in \mathbb{R}$  such that  $\{x_s; s \geq t\} = \{y'_s; s' \geq t'\}$ .
- (3) Let  $G$  be a group of isometries of  $\mathcal{T}$  and  $\omega$  an end in  $\mathcal{T}$  represented by a ray  $(x_t)_{t \geq 0}$ . The group  $G$  *stably fixes the end*  $\omega$  if for every  $g \in G$  there exists  $t_0 := t_0(g)$  such that  $g$  fixes  $x_t$  for all  $t \geq t_0$ .

**Remark 4.6.** [Cor2013, Lemma A.9] For a group  $G$ , property  $(\text{FR})_\infty$  is equivalent to each of the following statements:

- (1) For every isometric action of  $G$  on a complete real tree, every finitely generated subgroup has a fixed point.
- (2) Every isometric action of  $G$  on a complete real tree has a fixed point or stably fixes an end.

**Definition 4.7.** For a line  $L \subset \mathbb{P}^2$ , define  $\mathcal{A}_L := \text{PGL}_3 \cap \text{Dec}(L)$ . If  $L$  is given by the equation  $f = 0$ , we also use the notation  $\mathcal{A}_{\{f=0\}}$ .

**Lemma 4.8.** *For any line  $L \subset \mathbb{P}^2$  the group  $\mathcal{A}_L$  has property  $(\text{FR})_\infty$ .*

*Proof.* Since for two lines  $L$  and  $L'$  the groups  $\text{Dec}(L)$  and  $\text{Dec}(L')$  are conjugate, it is enough to prove the lemma for one line, say the line given by  $x = 0$ . Note that  $A = (a_{ij})_{1 \leq i, j \leq 3} \in \text{PGL}_3$  is in  $\mathcal{A}_{\{x=0\}}$  if and only if  $a_{12} = a_{13} = 0$ .

Let  $\mathcal{A}_{\{x=0\}}$  act on  $\mathcal{T}$  and let  $F \subset \mathcal{A}_{\{x=0\}}$  be a finite subset. The elements of  $F$  can be written as a product of elementary matrices contained in  $\mathcal{A}_{\{x=0\}}$ ; let  $R$  be the (finitely generated) subring of  $k$  generated by all entries of the elementary matrices contained in  $\mathcal{A}_{\{x=0\}}$  that are needed to obtain the elements in  $F$ . Then  $F$  is contained in  $\text{EL}_3(R)$ , the subgroup of  $\text{SL}_3(R)$  generated by elementary matrices. By the Shalom-Vaserstein theorem (see [EJZ010, Theorem 1.1]),  $\text{EL}_3(R)$  has Kazhdan's property (T) and in particular (as  $\text{EL}_3(R)$  is countable) has a fixed point in  $\mathcal{T}$  [Wat1982, Theorem 2], so  $F$  has a fixed point in  $\mathcal{T}$ . It follows that the subgroup of  $\mathcal{A}_{\{x=0\}}$  generated by  $F$  has a fixed point [Ser1977, §I.6.5, Corollary 3]. In particular, by Remark 4.6 (1),  $\mathcal{A}_{\{x=0\}}$  has property  $(\text{FR})_\infty$ .  $\square$

From now on, we fix  $L$  to be the line given by  $x = y$ . It is enough to prove Proposition 4.4 for this line since  $\text{Dec}(L)$  and  $\text{Dec}(L')$  are conjugate groups (by linear elements) for all lines  $L$  and  $L'$ . As before, we denote the standard quadratic involution by  $\sigma \in \text{Bir}(\mathbb{P}^2)$ ; with our choice of  $L$ , it is contained in  $\text{Dec}(L)$ .

Let  $\mathcal{D}_L \subset \text{PGL}_3$  be the subgroup of diagonal matrices that send  $L$  onto  $L$ , i.e.

$$\mathcal{D}_L := \{\text{diag}(s, s, t) \mid s, t \in \mathbb{C}^*\} \subset \text{PGL}_3.$$

**Lemma 4.9.** *We have  $\langle \mathcal{D}_L, \mu_1, \mu_2, P \rangle = \mathcal{A}_L$ , with the three involutions*

$$\mu_1 := \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L, \mu_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L, \text{ and } P := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L.$$

*Proof.* Given any  $\lambda \in \mathbb{C}^*$ , the matrices

$$A_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix}, B_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix}, \text{ and } C_\lambda := \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$$

belong to  $\langle \mathcal{D}_L, \mu_1, \mu_2, P \rangle$ . Indeed, we have  $A_\lambda = \text{diag}(-\lambda^{-1}, -\lambda^{-1}, 1) \cdot \mu_2 \cdot \text{diag}(\lambda, \lambda, 1)$ ,  $B_\lambda = PA_\lambda P$  and  $C_\lambda = \text{diag}(1, 1, \lambda^{-1}) \cdot \mu_1 \cdot \text{diag}(-1, -1, \lambda)$ .

Left multiplication by these corresponds to three types of row operations on matrices in  $\text{PGL}_3$  and right multiplication corresponds in the same way to three types of column operations. We denote them respectively by  $r_1, r_2, r_3, c_1, c_2, c_3$ , and we write  $d$  for multiplication by an element in  $\mathcal{D}_L$ .

Let  $A = (a_{ij})_{1 \leq i, j \leq 3} \in \text{PGL}_3$  be a matrix which is in  $\mathcal{A}_L$ , i.e. such that  $a_{13} = a_{23}$  and  $a_{11} + a_{12} = a_{21} + a_{22}$ . We proceed as follows, using only the above mentioned operations.

$$\begin{aligned} A &= \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow{d} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 1 \end{bmatrix} \xrightarrow{r_3} \begin{bmatrix} * & * & 0 \\ y & z & 0 \\ * & * & 1 \end{bmatrix} \xrightarrow{c_1 \text{ and } c_2} \begin{bmatrix} * & * & 0 \\ y & z & 0 \\ -y & -z & 1 \end{bmatrix} \\ &\xrightarrow{r_3} \begin{bmatrix} * & * & 1 \\ 0 & 0 & 1 \\ -y & -z & 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ * & * & 1 \end{bmatrix} \xrightarrow{r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & * & * \end{bmatrix} \xrightarrow{r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & * & 0 \end{bmatrix} \\ &\xrightarrow{d} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{c_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

In the first step ( $d$ ) we assume that  $a_{33} \neq 0$  – this can always be achieved by performing a row operation of type  $r_1$  on  $A$  if necessary. In the second step ( $r_3$ ), we use that  $a_{13} = a_{23}$ . The entries on place (2, 1) and (2, 2) after the second step are denoted by  $y$  and  $z$  respectively. In the fifth step ( $d$ ), we use that the entry on place (1, 1) is nonzero; this follows from the assumption  $a_{11} + a_{12} = a_{21} + a_{22}$  and that  $A$  is invertible.  $\square$

**Lemma 4.10.** *Suppose that  $\text{Dec}(L)$  acts on  $\mathcal{T}$  so that  $\mathcal{A}_L$  has no fixed points. Then  $\text{Dec}(L)$  stably fixes an end.*

*Proof.* Since  $\mathcal{A}_L$  has property  $(\text{FR})_\infty$  and has no fixed points, it stably fixes an end (Remark 4.6 (2)). Observe that this fixed end is unique: if  $\mathcal{A}_L$  stably fixes two different ends  $\omega_1, \omega_2$ , then  $\mathcal{A}_L$  pointwise fixes the line joining the two ends and has therefore fixed points (this uses that the only isometries on  $\mathbb{R}$  are translations and reflections [Cis2001, §I.2, Lemma 2.1]).

Let  $\omega$ , represented by the ray  $(x_t)_{t \geq 0}$ , be the unique end which is stably fixed by  $\mathcal{A}_L$  and define  $C := \langle \mathcal{D}_L, P \rangle$ . Being a subgroup of  $\mathcal{A}_L$ ,  $C$  obviously also stably fixes  $\omega$ . Note that the end  $\sigma\omega$  is stably fixed by  $\sigma\mathcal{A}_L\sigma^{-1}$ . In particular, since  $\sigma C\sigma^{-1} = C$ , the end  $\sigma\omega$  is also stably fixed by  $C$ . If  $\sigma\omega = \omega$ , then  $\omega$  is stably fixed by  $\sigma$  and by Theorem 1,  $\omega$  is stably fixed by  $\text{Dec}(L)$ . Otherwise, let  $l$  be the line joining  $\omega$  and  $\sigma\omega \neq \omega$ . Since  $C$  stably fixes  $\omega$  and  $\sigma\omega$ , it stably fixes both ends of  $l$ . In particular, the line  $l$  is pointwise fixed by  $C$ . Since  $\mu_1, \mu_2 \in \mathcal{A}_L$ ,  $\mu_1, \mu_2$  stably fix the end  $\omega$  and therefore,  $x_t$  is fixed by  $\mu_1, \mu_2$  for  $t \geq t_0$  for some  $t_0$ , and hence, by Lemma 4.9,  $x_t$  is fixed by all of  $\mathcal{A}_L$  for  $t \geq t_0$ , contradicting the assumption.  $\square$

*Proof of Proposition 4.4.* Recall that  $\mu_1, \mu_2 \in \mathcal{A}_L$  and note that  $\sigma\mu_1$  has order 3 and that  $\sigma\mu_2$  has order 6. It follows that

$$\sigma = (\mu_1\sigma)\mu_1(\mu_1\sigma)^{-1}$$

By Theorem 1,  $\text{Dec}(L)$  is generated by  $\sigma$  and  $\mathcal{A}_L$ . It follows that  $\mathcal{A}_1 := \mathcal{A}_L$  and  $\mathcal{A}_2 := \sigma\mathcal{A}_L\sigma$  generate  $\text{Dec}(L)$ .

Consider an action of  $\text{Dec}(L)$  on  $\mathcal{T}$ . It induces an action of  $\mathcal{A}_L$ , which has property  $(\text{FR})_\infty$  by Lemma 4.8 (i.e.  $\mathcal{A}_L$  has a fixed point or stably fixes an end by Remark 4.6 (2)). If  $\mathcal{A}_L$  has no fixed point, Lemma 4.10 implies that  $\text{Dec}(L)$  stably fixes an end, and then we are done.

Assume that  $\mathcal{A}_L$  has a fixed point. We conclude the proof by showing that in this case, even  $\text{Dec}(L)$  has a fixed point.

For  $i = 1, 2$ , let  $\mathcal{T}_i$  be the set of fixed points of  $\mathcal{A}_i$ . The two trees are exchanged by  $\sigma$ . If  $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$ ,  $\text{Dec}(L)$  has a fixed point since  $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle = \text{Dec}(L)$ . Let us consider the case where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are disjoint.

Let  $\mathcal{S} := [x_1, x_2]$ ,  $x_i \in \mathcal{T}_i$ , be the minimal segment joining the two trees and  $s > 0$  its length. Let  $C := \langle \mathcal{D}_L, P \rangle$ . Then  $\mathcal{S}$  is pointwise fixed by  $C \subset \mathcal{A}_1 \cap \mathcal{A}_2$  and reversed by  $\sigma$ . For  $i = 1, 2$ , the image of  $\mathcal{S}$  by  $\mu_i$  is a segment  $\mu_i(\mathcal{S}) = [x_1, \mu_i x_2]$ . By Lemma 4.9,  $\langle C, \mu_1, \mu_2 \rangle = \mathcal{A}_1$ , so it follows that for  $i = 1$  or  $i = 2$ , we have  $\mu_i(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$ . Otherwise, because  $\mathcal{T}$  is a tree and  $\mathcal{A}_1$  acts by isometries, both  $\mu_1, \mu_2$  fix  $\mathcal{S}$  pointwise and so  $\mathcal{A}_1$  fixes  $\mathcal{S}$  pointwise and in particular it fixes  $x_2$  – this would contradict  $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ . Choose an element  $I \in \{1, 2\}$  such that  $\mu_I(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$ .

Finally we arrive at a contradiction by computing  $d(x_1, (\sigma\mu_I)^k x_1)$  in two different ways. On the one hand we see that this distance is  $sk$ , on the other hand we have  $(\sigma\mu_I)^6 = 1$ . More generally, we show that

$$d((\sigma\mu_I)^k x_1, (\sigma\mu_I)^l x_1) = |k - l|s$$

for all  $k, l$ . Since we are on a real tree, it suffices to show this for  $k, l$  with  $|k - l| \leq 2$  (cf. [Cor2013, Lemma A.4]). By translation, we only have to check it for  $l = 0, k = 1, 2$ . For  $k = 1$ , we have  $d(\sigma\mu_I x_1, x_1) = d(\sigma x_1, x_1) = d(x_2, x_1) = s$ . For  $k = 2$ , the segment  $\mu_I(\mathcal{S}) = [x_1, \mu_I x_2]$  intersects  $\mathcal{S}$  only at  $x_1$ . In particular,  $d(\mu_I x_2, x_2) = 2s$  and hence

$$d(\sigma\mu_I \sigma\mu_I x_1, x_1) = d(\sigma\mu_I \sigma x_1, x_1) = d(\mu_I \sigma x_1, \sigma x_1) = d(\mu_I x_2, x_2) = 2s.$$

It follows that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  cannot be disjoint, and we are done.  $\square$

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