A NOTE ON THE INITIAL-BOUNDARY VALUE PROBLEM FOR CONTINUITY EQUATIONS WITH ROUGH COEFFICIENTS

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ABSTRACT. In this note we establish the existence of solutions of initial-boundary value problems for continuity equations with low regularity coefficients. We also announce a uniqueness result and some related counterexamples.

1. **Introduction.**

We are concerned with the continuity equation

$$\partial_t u + \mathrm{div}\,(bu) = cu + f, \tag{1}$$

where $b :]0, T[\times\Omega \to \mathbb{R}^d$, $c :]0, T[\times\Omega \to \mathbb{R}$ and $f :]0, T[\times\Omega \to \mathbb{R}$ are given functions and the unknown is $u :]0, T[\times\Omega \to \mathbb{R}$. Finally, $\Omega \subseteq \mathbb{R}^d$ is an open set and div denotes the divergence computed with respect to the space variable only. The investigation of (1) in the case when b has low regularity is the object of several recent research papers. Here we only quote the two milestones provided by the works by DiPerna and Lions [11] and by Ambrosio [1], which deal with the case when b enjoys Sobolev and BV regularity, respectively. We refer to the lecture notes by Ambrosio and Crippa [2] for a more extended bibliography. Both [11] and [1] establish existence and uniqueness results for the Cauchy problem obtained by coupling (1) with an initial datum in the case when $\Omega = \mathbb{R}^d$. We also point out that these results are motivated by applications to different classes of nonlinear PDEs, see the lecture notes by De Lellis [10] and the informal overview by Crippa and Spinolo [9] for the applications concerning systems of conservation laws.

This note aims at establishing existence of solutions of the initial-boundary value problem for (1) under weak regularity assumptions on b. More precisely, if all functions were smooth up to the boundary then the formulation of our problem would read as follows:

$$\begin{cases} \partial_t u + \mathrm{div}(bu) = cu + f & \text{in }]0, T[\times\Omega \\ u = \bar{g} & \text{on } \Gamma^- \\ u = \bar{u} & \text{on } \{0\} \times \Omega, \end{cases} \tag{2}$$

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where $\bar{u}$ and $\bar{g}$ are bounded smooth functions and Γ^- is the subset of $]0,T[\times\partial\Omega$ where the characteristics are entering the domain $]0,T[\times\Omega$. Note, however, that if b and u are not sufficiently regular (if, for instance, u is only an L^∞ function), then their values on zero-measure sets are not well defined. However, in [8] (see also § 2 in here) we introduce a distributional formulation of (2) and we consequently provide a definition of distributional solution, see Definition 2.2 below. This is done by relying on the theory of normal traces for low regularity vector fields, see the works by Anzellotti [4], Chen and Frid [6], Chen, Torres and Ziemer [7] and Ambrosio, Crippa and Maniglia [3]. The main result of this note reads as follows.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz continuous boundary. Assume that the following conditions hold:*

- $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$ *and* $\operatorname{div} b \in L^\infty(]0,T[\times\Omega)$;
- $c \in L^\infty(]0,T[\times\Omega)$ *and* $f \in L^\infty(]0,T[\times\Omega)$.

Then for every $\bar{u} \in L^\infty(\Omega)$ and $\bar{g} \in L^\infty(\Gamma^-)$ there is a distributional solution of problem (2).

Three remarks are here in order. First, we refer to the book by Leoni [12, Definition 12.10] for the definition of open set with uniformly Lipschitz continuous boundary.

Second, the proof of Theorem 1.1 closely follows an argument due to Boyer [5]. The main novelties of Theorem 1.1 compared to the analysis in [5] are: (i) we replace the condition $\operatorname{div} b \equiv 0$ with $\operatorname{div} b \in L^\infty$ and (ii) we remove the assumptions that $c \equiv 0$ and that Ω is bounded.

Finally, in [8] we prove that the solution of (2) is unique provided that b enjoys BV regularity up to the boundary of Ω. We also discuss some examples showing that, if BV regularity is violated, then (2) admits, in general, infinitely many solutions. In particular, this happens even if b enjoys BV regularity in every open set compactly contained in Ω but the BV regularity deteriorates at the boundary $\partial\Omega$.

This note is organized as follows: in § 2 we provide the distributional formulation of problem (2) and in § 3 we give the proof of Theorem 1.1.

1.1. Notation.

- $\mathcal{L}^n$: the n-dimensional Lebesgue measure.
- $\mathcal{H}^m$: the m-dimensional Hausdorff measure.
- $\operatorname{div} b$: the distributional divergence of the vector field $b :]0,T[\times\Omega \to \mathbb{R}^d$, computed with respect to the space variable only.
- $\nabla\varphi$: the gradient of the Sobolev function $\varphi :]0,T[\times\Omega \to \mathbb{R}$, computed with respect to the space variable only.
- $L^p(\partial\Omega) := L^p(\partial\Omega,\, \mathcal{H}^{d-1})$.
- $L^p(]0,T[\times\partial\Omega) := L^p(]0,T[\times\partial\Omega,\, \mathcal{L}^1 \otimes \mathcal{H}^{d-1})$.
- $\bar{\Omega}$: the closure of the set $\Omega \subseteq \mathbb{R}^d$.

2. Distributional formulation of problem (2).

We first observe that by interpreting the first and the last line of (2) in the sense of distributions we obtain

$$\int_0^T\int_\Omega u\big(\partial_t\eta + b\cdot\nabla\eta\big)dxdt + \int_\Omega \bar{u}\,\eta(0,\cdot)dx + \int_0^T\int_\Omega (c\,u\,\eta + f\,\eta)\,dxdt = 0 \qquad \forall \eta \in \mathcal{C}_c^\infty([0,T[\times\Omega). \tag{3}$$

The following results is proven in [8] and provides a distributional interpretation of (2) under the solely assumptions that b, $\operatorname{div} b$, c and f are all bounded.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz continuous boundary. Assume that $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$ satisfies $\operatorname{div} b \in L^\infty(]0,T[\times\Omega)$. Then the following implications hold:*

i) there is a unique function, which we denote by $\operatorname{Tr} b$, that belongs to the space $L^\infty(]0,T[\times\partial\Omega)$ and satisfies, for every $\varphi \in \mathcal{C}_c^\infty([0,T[\times\mathbb{R}^d)$,

$$\int_0^T\!\!\int_\Omega \varphi \operatorname{div} b \, dxdt + \int_0^T\!\!\int_\Omega \partial_t\varphi + b\cdot\nabla\varphi \, dxdt = \int_0^T\!\!\int_{\partial\Omega} \varphi \operatorname{Tr} b \, d\mathcal{H}^{d-1}dt - \int_\Omega \varphi(0,\cdot)\, dx. \tag{4}$$

Also, the function $\operatorname{Tr} b$ satisfies the inequality $\|\operatorname{Tr} b\|_{L^\infty} \le \|b\|_{L^\infty}$.

ii) Assume moreover that $c, f \in L^\infty([0,T[\times\Omega)$ and that $u \in L^\infty(]0,T[\times\Omega)$ satisfies (3). Then there is a unique function, which we denote by $\operatorname{Tr}(bu)$, that belongs to $L^\infty(]0,T[\times\partial\Omega)$ and satisfies, for every $\varphi \in \mathcal{C}_c^\infty([0,T[\times\mathbb{R}^d)$,

$$\begin{aligned}\int_0^T\int_\Omega u\big(\partial_t\varphi + b\cdot\nabla\varphi\big)dxdt &+ \int_0^T\int_\Omega (c\,u+f)\,\varphi\,dxdt \\ &= \int_0^T\int_{\partial\Omega}\varphi \operatorname{Tr}(bu)\, d\mathcal{H}^{d-1}dt - \int_\Omega \bar u\,\varphi(0,\cdot)\,dx.\end{aligned} \tag{5}$$

Also, the function $\operatorname{Tr}(bu)$ satisfies the inequality $\|\operatorname{Tr}(bu)\|_{L^\infty} \le \|b\|_{L^\infty}\|u\|_{L^\infty}$.

Some remarks are here in order. First, if the functions are smooth up to the boundary, then the Gauss-Green formula implies that $\operatorname{Tr} b = b\cdot\vec{n}$ and $\operatorname{Tr}(bu) = ub\cdot\vec{n}$, where $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$. Second, in [8] we exhibit an example where $\operatorname{Tr} b \equiv 0$ but $\operatorname{Tr}(bu) \equiv 1$. The vector field b in the example enjoys BV regularity in every open set compactly contained in Ω, but the BV regularity deteriorates at the boundary. Finally, based on Lemma 2.1, we can rigorously define the sets Γ^- and Γ^{0+} by setting

$$\Gamma^- := \{(t,x) \in]0,T[\times\partial\Omega : \ \operatorname{Tr} b < 0\}, \quad \Gamma^{0+} := \{(t,x) \in]0,T[\times\partial\Omega : \ \operatorname{Tr} b \ge 0\}. \tag{6}$$

We can now introduce the definition of distributional solution of (2).

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^d$ be an open set with uniformly Lipschitz continuous boundary and assume that $b \in L^\infty(]0,T[\times\Omega;\mathbb{R}^d)$, $\operatorname{div} b \in L^\infty(]0,T[\times\Omega)$, $c \in L^\infty(]0,T[\times\Omega)$ and $f \in L^\infty(]0,T[\times\Omega)$. A distributional solution of problem (2) is a function $u \in L^\infty(]0,T[\times\Omega)$ satisfying (3) such that the equality $\operatorname{Tr}(bu) = \bar g \operatorname{Tr} b$ holds on Γ^-.

3. Proof of Theorem 1.1.

In the following, we establish the existence of functions $u \in L^\infty(]0,T[\times\Omega)$ and $\beta \in L^\infty(\Gamma^{0+})$ such that, for every $\varphi \in \mathcal{C}_c^\infty([0,T[\times\mathbb{R}^d)$,

$$\begin{aligned}\int_0^T\int_\Omega u\big(\partial_t\varphi + b\cdot\nabla\varphi\big)dx &+ \int_0^T\int_\Omega (c\,u\,\varphi + f\,\varphi)\,dxdt \\ &= \int_{\Gamma^-}\varphi\,\bar g \operatorname{Tr} b\, d\mathcal{H}^{d-1}dt + \int_{\Gamma^{0+}}\varphi\beta\, d\mathcal{H}^{d-1}dt - \int_\Omega \bar u\,\varphi(0,\cdot)dx.\end{aligned} \tag{7}$$

By comparing the previous expression with (3) and (5) and by recalling Definition 2.2, we infer that u is a distributional solution of problem (2) and that

$$\operatorname{Tr}(bu)(t,x) = \begin{cases} \bar g \operatorname{Tr} b & (t,x) \in \Gamma^- \\ \beta & (t,x) \in \Gamma^{0+}. \end{cases}$$

The proof of Theorem 1.1 is divided into three steps: in § 3.1 we introduce a second order approximation and we state an existence result for the approximate problem.

In § 3.2 we establish a priori bounds on the family of approximate solutions and in § 3.3 we pass to the limit and obtain a distributional solution of (2).

3.1. **Second order approximation.** We introduce a family of approximate problems, whose classical formulation is the following:

$$\begin{cases} \partial_t u_\varepsilon + \operatorname{div}(b u_\varepsilon) = \varepsilon \Delta u_\varepsilon + c\, u_\varepsilon + f & \text{on }]0,T[\times\Omega \\ \varepsilon \dfrac{\partial u_\varepsilon}{\partial \vec{n}} + (u_\varepsilon - \bar{g})\big[\operatorname{Tr} b\big]^- = 0 & \text{on }]0,T[\times\partial\Omega \\ u_\varepsilon = \bar{u} & \text{on } \{0\}\times\Omega. \end{cases} \tag{8}$$

In the previous expression, $\varepsilon > 0$ is a parameter, $\vec{n}$ as usual denotes the outward pointing unit normal vector to $\partial\Omega$ and $[\operatorname{Tr} b]^-$ is the negative part of the function $\operatorname{Tr} b$. In this section we assume that $\bar{u}$, $\bar{g}$ and f, besides being bounded, are also square integrable (see the statement of Lemma 3.2). First, we provide the definition of weak solution of (8). To this end, we introduce the following notation:

- V : the Sobolev space $W^{1,2}(\Omega)$.
- V^* : the dual space of V, endowed with the standard dual norm.
- $\langle F, u\rangle$: the duality between $F \in V^*$ and $u \in V$.
- The bilinear form $B_\varepsilon(t,\cdot) : V \times V \to \mathbb{R}$ is defined for $\mathcal{L}^1$-a.e. $t \in]0,T[$ as

$$\begin{aligned} B_\varepsilon(t,u,v) := & -\int_\Omega u\, b(t,\cdot)\nabla v\, dx + \varepsilon \int_\Omega \nabla u(t,\cdot)\nabla v\, dx \\ & - \int_\Omega c\, u(t,\cdot) v\, dx + \int_{\partial\Omega} u\, v \big[\operatorname{Tr} b\big]^+(t,\cdot)\, d\mathcal{H}^{d-1}, \end{aligned} \tag{9}$$

  where $\big[\operatorname{Tr} b\big]^+$ denotes the positive part of the function $\operatorname{Tr} b$.
- The functional $\mathbf{F}(t) \in V^*$ is defined for $\mathcal{L}^1$-a.e. $t \in]0,T[$ by setting

$$\langle \mathbf{F}(t), v\rangle := \int_{\partial\Omega} v\bar{g}\big[\operatorname{Tr} b\big]^-(t,\cdot)\, d\mathcal{H}^{d-1} + \int_\Omega f(t,\cdot) v\, dx \tag{10}$$

  Note that continuity of $v \mapsto \langle \mathbf{F}(t), v\rangle$ follows from the square integrability of f and from the fact that, under the regularity assumptions we impose on $\partial\Omega$, the trace map is continuous $V \to L^2(\partial\Omega)$, see for example Leoni [12, Theorem 15.23].

The following definition is classical, see for example Salsa [14, §9.3.1].

Definition 3.1. A weak solution of (8) is a function

$$\mathbf{u}_\varepsilon : [0,T] \to V$$

such that

1. $\mathbf{u}_\varepsilon \in L^2\big(]0,T[;V\big)$ and $\dot{\mathbf{u}}_\varepsilon \in (]0,T[;V^*)$, where $\dot{\mathbf{u}}_\varepsilon$ denotes the distributional derivative of $\mathbf{u}_\varepsilon$.
2. For $\mathcal{L}^1$-a.e. $t \in]0,T[$,

$$\langle \dot{\mathbf{u}}_\varepsilon(t), v\rangle + B_\varepsilon(t, \mathbf{u}_\varepsilon(t), v) = \langle \mathbf{F}(t), v\rangle \qquad \forall\, v \in V. \tag{11}$$

3. $\mathbf{u}_\varepsilon(0) = \bar{u}$

Note that requirement 3 above makes sense since by using requirement 1 we infer that $\mathbf{u}_\varepsilon \in \mathcal{C}^0([0,T]; L^2(\Omega))$, see for example Salsa [14, Theorem 7.22]. Also, note that by the bold letter $\mathbf{u}_\varepsilon$ we denote the function taking values in V, while u_ε is the real-valued function $u_\varepsilon(t,\cdot) = \mathbf{u}_\varepsilon(t)$.

Remark 1. By relying on standard arguments, we get that any weak solution of (8) satisfies, for every $\varphi \in \mathcal{C}_c^\infty([0,T[\times\mathbb{R}^d)$,

$$\int_0^T \int_\Omega u_\varepsilon\big(\partial_t\varphi + b\,\nabla\varphi\big)\,dxdt - \varepsilon\int_0^T\int_\Omega \nabla u_\varepsilon \nabla\varphi\,dxdt + \int_0^T\int_\Omega (c\,u\,\varphi + f\,\varphi)\,dxdt$$
$$= -\int_\Omega \varphi(0,x)\bar{u}\,dx - \int_0^T\int_{\partial\Omega} \bar{g}\varphi\big[\mathrm{Tr}\,b\big]^- d\mathcal{H}^{d-1}dt + \int_0^T\int_{\partial\Omega} u_\varepsilon\varphi\big[\mathrm{Tr}\,b\big]^+ d\mathcal{H}^{d-1}dt. \tag{12}$$

The following lemma provides an existence and uniqueness result for (8).

Lemma 3.2. *Assume that b and c verify the same assumptions as in the statement of Theorem 1.1 and assume moreover that $\bar{g} \in L^\infty(\Gamma^-)\cap L^2(\Gamma^-)$, $\bar{u} \in L^\infty(\Omega)\cap L^2(\Omega)$ and $f \in L^2(]0,T[\times\Omega)\cap L^\infty(]0,T[\times\Omega)$. Then for any given $\varepsilon > 0$ problem (8) admits a unique weak solution, in the sense of Definition 3.1.*

The proof of Lemma 3.2 follows by a classical Faedo-Galerkin method (see for instance [14, §9.3.2]).

3.2. **A priori estimates.** In this section we establish the estimates we need to study the convergence $\varepsilon \to 0^+$ of the family $\mathbf{u}_\varepsilon$ solving (8).

Lemma 3.3. *Let $\mathbf{u}_\varepsilon$ be the weak solution of problem (8). Then*

$$\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)}$$
$$\le\Big(\|\bar{u}\|^2_{L^2} + \int_{\Gamma^-} \bar{g}^2\mathrm{Tr}b\,d\mathcal{H}^{d-1}ds + \|f\|^2_{L^2}\Big)\exp\Big((\|\mathrm{div}\,b\|_{L^\infty} + 2\|c\|_{L^\infty} + 1)t\Big) \tag{13}$$

for every $t \in]0,T[$. Also,

$$\int_0^T\int_\Omega |\sqrt{\varepsilon}\nabla u_\varepsilon(t,x)|^2 dxdt \le C, \tag{14}$$

where C is a constant only depending on T, $\|b\|_{L^\infty}$, $\|\mathrm{div}b\|_{L^\infty}$, $\|\bar{g}\|_{L^2}$, $\|\bar{u}\|_{L^2}$, $\|c\|_{L^\infty}$ and $\|f\|_{L^2}$.

Proof. First, we recall (see e.g. [14, Theorem 7.22]) that

$$\int_0^t \langle \dot{\mathbf{u}}_\varepsilon(s), \mathbf{u}_\varepsilon(s)\rangle ds = \frac{1}{2}\Big(\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)} - \|\mathbf{u}_\varepsilon(0)\|^2_{L^2(\Omega)}\Big) \quad \forall\, t \in [0,T]. \tag{15}$$

Next, we suitably choose the test functions in (4) and we use the density of the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $W^{1,1}(\Omega)$ and the continuity of the trace operator from $W^{1,1}(\Omega)$ onto $L^1(\partial\Omega)$, obtaining that for $\mathcal{L}^1$-a.e. $t \in]0,T[$

$$\int_\Omega w\,\mathrm{div}\,b(t,\cdot)\,dx + \int_\Omega b(t,\cdot)\cdot\nabla w\,dx = \int_{\partial\Omega} w\,\mathrm{Tr}\,b(t,\cdot)\,d\mathcal{H}^{d-1} \quad \forall\, w \in W^{1,1}(\Omega). \tag{16}$$

We now choose $v := \mathbf{u}_\varepsilon(s)$ as a test function in (11), we integrate over $]0,t[$ and we use (16) with $w = u_\varepsilon^2(s,\cdot)$. After straightforward computations, we arrive at

$$\begin{aligned}\|\mathbf{u}_\varepsilon(t)\|_{L^2}^2 + 2\varepsilon \int_0^t\int_\Omega |\nabla u_\varepsilon|^2 dxds + \int_0^t\int_{\partial\Omega} u_\varepsilon^2 \big[\mathrm{Tr}\, b\big]^+ d\mathcal{H}^{d-1}ds \\ + \int_0^t\int_{\partial\Omega} (u_\varepsilon - \bar g)^2 \big[\mathrm{Tr}\, b\big]^- d\mathcal{H}^{d-1}ds \\ = \|\bar u\|_{L^2}^2 - \int_0^t\int_\Omega u_\varepsilon^2 \mathrm{div}\, b\, dxds + 2\int_0^t\int_\Omega c\, u_\varepsilon^2 dxds \\ + \int_0^t\int_{\partial\Omega} \bar g^2 \big[\mathrm{Tr}\, b\big]^- d\mathcal{H}^{d-1}ds + 2\int_0^t\int_\Omega f\, u_\varepsilon\, dxds \\ \le \|\bar u\|_{L^2}^2 + (\|\mathrm{div}\, b\|_{L^\infty} + 2\|c\|_{L^\infty} + 1)\int_0^t \|\mathbf{u}_\varepsilon(s)\|_{L^2}^2 ds \\ + \int_0^t\int_{\partial\Omega} \bar g^2 \big[\mathrm{Tr}\, b\big]^- d\mathcal{H}^{d-1}ds + \|f\|_{L^2}^2.\end{aligned}$$

Hence, by relying on the Gronwall Lemma we get (13) and then (14). □

We now establish a maximum principle

Lemma 3.4. *Let $\mathbf{u}_\varepsilon$ be the weak solution of problem (8), then*

$$\|u_\varepsilon\|_{L^\infty} \le \Big(M + \|f\|_{L^\infty} T\Big) \exp\Big((\|\mathrm{div}\, b\|_{L^\infty} + \|c\|_{L^\infty})T\Big), \tag{17}$$

where

$$M := \max\{\|\bar g\|_{L^\infty}, \|\bar u\|_{L^\infty}\}. \tag{18}$$

Also,

$$\|u_\varepsilon [\mathrm{Tr}\, b]^+\|_{L^\infty} \le \Big(M + \|f\|_{L^\infty} T\Big)\|b\|_{L^\infty} \exp\Big((\|\mathrm{div}\, b\|_{L^\infty} + \|c\|_{L^\infty})T\Big). \tag{19}$$

Proof. The argument is divided into four steps.

Step 1. we introduce some preliminary notation and remarks.

i) We set $B := \|\mathrm{div}\, b\|_{L^\infty} + \|c\|_{L^\infty}$.

ii) We define the function m_ε by setting

$$m_\varepsilon(t,x) := \Big(\big[u_\varepsilon(t,x) + (M + \|f\|_{L^\infty} t)e^{Bt}\big]^-\Big)^2.$$

By a slight abuse of notation, in the following we denote by m_ε' the function $2\big[u_\varepsilon(t,x) + (M + \|f\|_{L^\infty} t)e^{Bt}\big]^-$. Note that, for $\mathcal{L}^1$-a.e. $t \in]0,T[$, $m_\varepsilon \in L^1(\Omega)$ and $m_\varepsilon' \in V = W^{1,2}(\Omega)$ since

$$\big[u_\varepsilon + (M + \|f\|_{L^\infty} t)e^{Bt}\big]^- \le \max\{-u_\varepsilon, 0\}.$$

In the following we also use the formula

$$\begin{aligned}u_\varepsilon m_\varepsilon' &= \Big(u_\varepsilon + (M + \|f\|_{L^\infty} t)e^{Bt}\Big)m_\varepsilon' - Me^{Bt}m_\varepsilon' - \|f\|_{L^\infty}\, t\, e^{Bt}\, m_\varepsilon' \\ &= 2m_\varepsilon - Me^{Bt}m_\varepsilon' - \|f\|_{L^\infty}\, t\, e^{Bt}\, m_\varepsilon'.\end{aligned} \tag{20}$$

iii) We choose a sequence of smooth cut-off functions $\{\psi_n\}_{n\in\mathbb{N}}$. More precisely, we require that, for every $n \in \mathbb{N}$, $\psi_n \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and

$$0 \le \psi_n(x) \le 1, \quad |\nabla\psi_n(x)| \le 1/n \quad \forall\, x \in \mathbb{R}^d \quad \text{and} \quad \psi_n \equiv 1 \text{ on } B_n(0).$$

In the previous expression, $B_n(0)$ denotes the ball of radius n and center at 0 in $\mathbb{R}^d$.

iv) Finally, we observe that for every $n \in \mathbb{N}$ and $\mathcal{L}^1$−a.e. $t \in]0, T[$

$$\int_0^t \langle \dot{\mathbf{u}}_\varepsilon(s) + B\Big(M + \|f\|_{L^\infty} s\Big)e^{Bs} + \|f\|_{L^\infty} e^{Bs}, m'_\varepsilon \psi_n \rangle\, ds = \int_\Omega m_\varepsilon(t,x)\psi_n(x)\, dx - \int_\Omega \Big(\big[\bar{u} + M)\big]^-\Big)^2 \psi_n(x)\, dx.$$

This formula can be established by relying on an approximation argument, see for example the analysis in the book by Lions and Magenes [13, Section 2.2].

Step 2. we use equation (11). First, we observe that by applying (16) with $w = vu_\varepsilon(t,\cdot)$ we infer that equation (11) implies

$$\begin{aligned}\langle \dot{\mathbf{u}}_\varepsilon, v\rangle + &\int_\Omega u_\varepsilon v \operatorname{div} b\, dx + \int_\Omega v\, b \cdot \nabla u_\varepsilon\, dx + \varepsilon \int_\Omega \nabla u_\varepsilon \nabla v\, dx \\ &+ \int_{\partial\Omega} u_\varepsilon v [\operatorname{Tr} b]^- d\mathcal{H}^{d-1} - \int_\Omega c\, u\, v dx \\ &= \int_{\partial\Omega} \bar{g} v [\operatorname{Tr} b]^- d\mathcal{H}^{d-1} + \int_\Omega f\, v\, dx \qquad \forall\, v \in V \text{ and } \mathcal{L}^1 - a.e.\ t \in]0, T[. \end{aligned} \tag{21}$$

Next, we fix $n \in \mathbb{N}$ and we apply (16) with $w = m_\varepsilon(t,\cdot)\psi_n$, obtaining

$$\int_\Omega m_\varepsilon \psi_n \operatorname{div} b\, dx + \int_\Omega m'_\varepsilon \psi_n\, b \cdot \nabla u_\varepsilon\, dx = \int_{\partial\Omega} m_\varepsilon \psi_n \operatorname{Tr} b\, d\mathcal{H}^{d-1} - \int_\Omega m_\varepsilon\, b \cdot \nabla\, \psi_n dx. \tag{22}$$

Hence, by plugging $v := m'_\varepsilon(t,\cdot)\psi_n$ as a test function in (21) and by using (20) and (22) we obtain

$$\begin{aligned}&\langle \dot{\mathbf{u}}_\varepsilon, m'_\varepsilon \psi_n\rangle + \int_\Omega m_\varepsilon \psi_n \operatorname{div} b\, dx - \int_\Omega M e^{Bt} m'_\varepsilon \psi_n \operatorname{div} b\, dx \\ &- \int_\Omega \|f\|_{L^\infty} t\, e^{Bt}\, m'_\varepsilon\, \psi_n \operatorname{div} b\, dx + \int_{\partial\Omega} m_\varepsilon \psi_n [\operatorname{Tr} b]^+ dx + 2\varepsilon \int_\Omega \chi_\varepsilon |\nabla u_\varepsilon|^2 \psi_n\, dx \\ &\quad - \int_\Omega 2 m_\varepsilon \psi_n\, c\, dx + \int_\Omega M e^{Bt} m'_\varepsilon \psi_n\, c\, dx + \int_\Omega \|f\|_{L^\infty} t\, e^{Bt}\, m'_\varepsilon\, \psi_n\, c\, dx + \mathcal{R}_n \\ &= \int_{\partial\Omega} \psi_n [\operatorname{Tr} b]^- \Big(m_\varepsilon + m'_\varepsilon \big(\bar{g} + (M + \|f\|_{L^\infty} t)e^{Bt} - (u_\varepsilon + (M + \|f\|_{L^\infty} t)e^{Bt})\big)\Big)\, d\mathcal{H}^{d-1} \\ &\quad + \int_\Omega f\, m'_\varepsilon \psi_n, \end{aligned} \tag{23}$$

where χ_ε is the characteristic function of the set where $u_\varepsilon + (M + \|f\|_{L^\infty} t)e^{Bt} \leq 0$ and

$$\mathcal{R}_n := - \int_\Omega m_\varepsilon b \cdot \nabla \psi_n\, dx + \varepsilon \int_\Omega m'_\varepsilon \nabla u_\varepsilon \cdot \nabla \psi_n\, dx.$$

Step 3. we establish an estimate on the time integral of the first and third term in (23). We fix $\tau \in]0, T[$ and we integrate in time to get

$$\begin{aligned}
&\int_0^\tau \langle \dot{\mathbf{u}}_\varepsilon, m'_\varepsilon \psi_n \rangle \, dt - \int_0^\tau \int_\Omega m'_\varepsilon M e^{Bt} \psi_n \operatorname{div} b \, dx dt - \int_0^\tau \int_\Omega \|f\|_{L^\infty} \, t \, e^{Bt} \, m'_\varepsilon \psi_n \operatorname{div} b \, dx \, dt \\
&\qquad + \int_0^\tau \int_\Omega M \, e^{Bt} \, m'_\varepsilon \, \psi_n \, c \, dx dt + \int_0^\tau \int_\Omega \|f\|_{L^\infty} \, t \, e^{Bt} \, m'_\varepsilon \, \psi_n \, c \, dx - \int_0^\tau \int_\Omega f \, m'_\varepsilon \psi_n \, dx \\
&\quad = \int_0^\tau \langle \dot{\mathbf{u}}_\varepsilon + B(M e^{Bt} + \|f\|_{L^\infty} t + \|f\|_{L^\infty} e^{Bt}, m'_\varepsilon \psi_n \rangle \, dt \\
&\quad\quad - \int_0^\tau \int_\Omega m'_\varepsilon \psi_n M e^{Bt} \big(B + \operatorname{div} b - c \big) \, dx dt \\
&\quad\quad - \int_0^\tau \int_\Omega \|f\|_{L^\infty} \, t \, e^{Bt} \, (B + \operatorname{div} b - c) m'_\varepsilon \psi_n \\
&\quad\quad\; - \int_0^\tau \int_\Omega \|f\|_{L^\infty} e^{Bt} \, m'_\varepsilon \, \psi_n \, dx \, dt - \int_0^\tau \int_\Omega f \, m'_\varepsilon \psi_n \, dx \\
&\quad = \int_\Omega m_\varepsilon(\tau, x) \psi_n(x) \, dx - \int_\Omega \big([\bar{u} + M]^- \big)^2 \psi_n \, dx \\
&\quad\quad - \int_0^\tau \int_\Omega m'_\varepsilon \psi_n M e^{Bt} \big(B + \operatorname{div} b - c \big) dx dt \\
&\quad\quad - \int_0^\tau \int_\Omega \|f\|_{L^\infty} \, t \, e^{Bt} (B + \operatorname{div} b - c) \, m'_\varepsilon \, \psi_n \, dx \, dt \\
&\quad\quad - \int_0^\tau \int_\Omega \Big(\|f\|_{L^\infty} e^{Bt} + f \Big) \, m'_\varepsilon \, \psi_n \, dx \, dt \\
&\quad \geq \int_\Omega m_\varepsilon(\tau, x) \psi_n(x) \, dx.
\end{aligned} \tag{24}$$

To get the last inequality, we have used that $\bar{u} + M \geq 0$, $B + \operatorname{div} b - c \geq 0$, $\|f\|_{L^\infty} e^{Bt} + f \geq 0$ and that $m'_\varepsilon \leq 0$.

Step 4. we conclude the proof of Lemma 3.4. First, we point out that the convexity of the function $z \mapsto ([z]^-)^2$ implies that

$$\begin{aligned}
m_\varepsilon + m'_\varepsilon & \Big(\bar{g} + (M + \|f\|_{L^\infty} t) e^{Bt} - (u_\varepsilon + (M + \|f\|_{L^\infty} t) e^{Bt}) \Big) \\
& \leq \Big(\big[\bar{g} + (M + \|f\|_{L^\infty} t) e^{Bt} \big]^- \Big)^2 = 0.
\end{aligned}$$

Next, we observe that

$$\int_{\partial \Omega} m_\varepsilon \psi_n [\operatorname{Tr} b]^+ \, d\mathcal{H}^{d-1} + 2\varepsilon \int_\Omega \chi_\varepsilon |\nabla u_\varepsilon|^2 \psi_n dx \geq 0.$$

Hence, by time integrating (23) and combining (24) with the above observations we obtain

$$\int_\Omega m_\varepsilon(\tau, x) \psi_n(x) dx + \int_0^\tau \int_\Omega m_\varepsilon \psi_n (\operatorname{div} b - 2c) \, dx dt + \int_0^\tau \mathcal{R}_n dt \leq 0. \tag{25}$$

Next, we observe that, by recalling estimates (13), (14) and the bounds on ψ_n, we get

$$\lim_{n \to +\infty} \int_0^\tau \mathcal{R}_n dt \to 0.$$

Hence, by letting $n \to +\infty$ in (25) and applying the Gronwall Lemma, we conclude that $u_\varepsilon(\tau, x) + e^{B\tau}(M + \|f\|_{L^\infty} \tau) \geq 0$ for $\mathcal{L}^{d+1}$-a.e. $(\tau, x) \in \Omega$. By using an analogous argument, we obtain that $u_\varepsilon(\tau, x) \leq e^{B\tau}(M + \|f\|_{L^\infty} \tau)$ for $\mathcal{L}^{d+1}$-a.e.

$(\tau, x) \in]0, T[\times\Omega$ and this concludes the proof of (17). Finally, by combining (17) with (23) we get (19). □

3.3. **Conclusion of the proof of Theorem 1.1.** We proceed in two steps.
Step A. we prove Theorem 1.1 under the additional assumptions $\bar g \in L^2(\Gamma^-)$, $\bar u \in L^2(\Omega)$ and $f \in L^2(]0, T[\times\Omega)$. By applying Lemma 3.2 and recalling Remark 1, we deduce that, for every $\varepsilon > 0$, there is a function $\{u_\varepsilon\}$ satisfying (12). By relying on Lemma 3.4, we infer that the families $\{u_\varepsilon\}$ and $\{u_\varepsilon[\mathrm{Tr}\, b]^+\}$ are weakly-$*$ compact in $L^\infty(]0, T[\times\Omega)$ and in $L^\infty(\Gamma^{0+})$, respectively. By recalling the a priori estimate (14), we can pass to the limit in (12) and obtain a couple (u, β) satisfying (7).
Step B. we remove the assumptions $\bar g \in L^2(\Gamma^-)$, $\bar u \in L^2(\Omega)$ and $f \in L^2(]0, T[\times\Omega)$.
Given a function $\bar u \in L^\infty(\Omega)$, we can construct a sequence $\{\bar u_k\}_{k\in\mathbb{N}}$ such that

$$\|\bar u_k\|_{L^\infty} \le \|\bar u\|_{L^\infty}; \quad \bar u_k \in L^2 \text{ for every } k; \quad \bar u_k \to \bar u \text{ for } \mathcal{L}^d\text{-a.e. } x \in \Omega. \tag{26}$$

For example, we can take $\bar u_k := \bar u \mathbf{1}_{\Omega_k}$, where $\{\Omega_k\}_{k\in\mathbb{N}}$ is a sequence of open bounded sets invading Ω. We analogously construct sequences $\{\bar g_k\}_{k\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ approaching $\bar g$ and f, respectively. We term $\{u_k, \beta_k\}_{k\in\mathbb{N}}$ the corresponding sequence of functions satisfying (7), constructed as in **Step A.**. Note that by recalling (17), (19) and (26) we infer that the sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ are weakly-$*$ compact in $L^\infty(]0, T[\times\Omega)$ and in $L^\infty(\Gamma^{0+})$, respectively. Hence, by extracting converging subsequences we conclude the proof of Theorem 1.1.

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