NON-UNIQUENESS AND PRESCRIBED ENERGY  
FOR THE CONTINUITY EQUATION  

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Abstract. In this note we provide new non-uniqueness examples for the continuity equation by constructing infinitely many weak solutions with prescribed energy.

1. Introduction

In this paper we consider the continuity equation for a bounded scalar function $u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ with a bounded divergence-free vector field $b: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$:

$$\partial_t u + \text{div}(ub) = 0, \quad (1)$$

$$\text{div } b = 0. \quad (2)$$

This equation appears in various problems of mathematical physics, in particular fluid mechanics and kinetic theory. In the smooth setting (and assuming suitable integrability) the energy

$$\mathcal{E}(t) := \int_{\mathbb{R}^d} u^2(t,x) \, dx$$

of the solution $u$ is conserved:

$$\frac{d}{dt} \mathcal{E}(t) = 0. \quad (3)$$

Indeed, since $b$ is divergence-free, by multiplying $(1)$ with $u$, using the chain rule and integrating over $\mathbb{R}^d$ one immediately obtains (3).

In many applications one has to study $(1)$ in a nonsmooth setting. Roughly speaking, since $(1)$ is linear, the conservation of energy $(3)$ implies uniqueness of weak solutions to the corresponding initial-value problem for $(1)$. In fact, conservation of energy is a consequence of the so-called renormalization property, which was proved by [DL89] for any vector field $b$ with Sobolev regularity and later extended by Ambrosio [Amb] to the case when $b$ has bounded variation. We refer to [DL08, AC14] for a detailed review of recent results in this direction.

On the other hand, when the regularity of the vector field $b$ is too low, the conservation of energy $(3)$ fails in general. In a nonsmooth setting several counterexamples to the uniqueness, and therefore to the conservation of energy, are known, see [Aiz78, CLR03, Dep03, ABC14, ABC13]. A similar phenomenon occurs in the context of the Euler equations. For example, in the papers [Sch93, Shm97, DLS09] weak solutions of the Euler equations were constructed with compact support in space time.

In particular the example in [Dep03] gives a bounded vector field $b$ and a bounded scalar field $u$ which satisfy $(1)$ and $(2)$ such that

$$\mathcal{E}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0. \end{cases} \quad (4)$$

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In this paper, for any given nonnegative bounded function $E : \mathbb{R} \to \mathbb{R}$ which is continuous on an open interval and zero outside we construct infinitely many pairs $(b, u)$ satisfying (1) and (2), such that $E(t) = E(t)$ for a.e. $t$. Thus, in contrast with (4), we provide more general profiles for the energy. Our results are also connected to the chain rule problem for the divergence operator, see [ADLM07, BG14, CGSW].

We construct such pairs $(b, u)$ using the method of convex integration, and our techniques are similar to the ones used in [DLS09, Szé12]. The latter reference contains an appendix giving a general framework for convex integration, but for the problem at hand we need to consider a nonlinear constraint that depends on the points in the domain (as was the case e.g. in [DLS10], albeit in a different functional setting). For this reason we adapt the framework from [Szé12] to this more general situation (see §2). We then apply this abstract framework to the specific situation of the continuity equation (see §3).

Finally, let us mention [CFG11, Shv11, BLFNL], where results were obtained by convex integration, respectively, that yield as a byproduct counterexamples to the energy conservation for continuity equations. However, in these works the energy profile is always piecewise constant.

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2. Differential inclusions with non-constant nonlinear constraint

We start with the so-called Tartar framework (cf. e.g. [DLS09]). Consider a system of $m$ linear partial differential equations

\[ \sum_{i=1}^{D} A_i \partial_i z = 0 \]  \hspace{1cm} (5)

in an open set $\mathcal{D} \subset \mathbb{R}^D$, where $A_i$ are constant $m \times n$ matrices and $z : \mathcal{D} \to \mathbb{R}^n$. Consider a nonlinear constraint

\[ z(y) \in K_y \]  \hspace{1cm} (6)

for a.e. $y$ in $\mathcal{D}$, where $K_y \subset \mathbb{R}^n$ is a compact set for any $y \in \mathcal{D}$.

For any $y \in \mathcal{D}$ let $U_y := \mathrm{int} \ \mathrm{conv} \ K_y$, where with conv we denote the convex hull of the set $K_y$ and with int we denote its interior. Let $\mathcal{U} \subset \mathcal{D}$ be a bounded open set.

**Definition 1** (Subsolutions). We say that $z \in L^2(\mathcal{D})$ is a subsolution of (5), (6) if $z$ is a weak solution of (5) in $\mathcal{D}$, $z$ is continuous on $\mathcal{U}$, (6) holds for a.e. $y \in \mathcal{D} \setminus \mathcal{U}$ and

\[ z(y) \in U_y \]  \hspace{1cm} (7)

for any $y \in \mathcal{U}$.

**Definition 2** (Localized plane waves/wave cone). A set $\Lambda \subset \mathbb{R}^n$ is called wave cone if there exists a constant $C > 0$ such that for any $\bar{z} \in \Lambda$ there exists a sequence $w_k \in C_0^\infty(B_1(0); \mathbb{R}^n)$ solving (5) in $\mathbb{R}^D$ such that

- $\mathrm{dist}(w_k(x), [-\bar{z}, \bar{z}]) \to 0$ for all $x \in B_1(0)$ uniformly as $k \to \infty$,
- $w_k \to 0$ in $L^2$ as $k \to \infty$,
- $\int |w_k|^2 \ dy > C|\bar{z}|^2$ for all $k \in \mathbb{N}$.
In the above definition we denoted the segment with endpoints \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) with \([x, y] := \text{conv}\{x, y\}\). The functions \( w_k \) are called \textit{localized plane waves}. We make the following assumptions:

\textbf{Assumption 1.} \textit{There exists a wave cone} \( \Lambda \) \textit{dense in} \( \mathbb{R}^n \).

Let \( \mathcal{K} \) denote the set of all compact subsets of \( \mathbb{R}^n \), endowed with the Hausdorff metric \( d_H \). It is well-known that \( \mathcal{K} \) is a complete metric space.

\textbf{Assumption 2 (Continuity of the nonlinear constraint).} \textit{The map} \( f : \mathcal{U} \ni y \mapsto K_y \in \mathcal{K} \) \textit{is continuous and bounded in the Hausdorff metric.}

Our main abstract result is the following:

\textbf{Theorem 1.} \textit{Suppose that Assumptions 1 and 2 hold. Suppose that} \( z_0 \) \textit{is a subsolution of} (5), (6). \textit{Then there exist infinitely many weak solutions} \( z \in L^2(\mathcal{D}) \) \textit{of} (5) \textit{which agree with} \( z_0 \) \textit{a.e. on} \( \mathcal{D} \setminus \mathcal{U} \) \textit{and satisfy} (6) \textit{for a.e.} \( y \in \mathcal{D} \).

2.1. \textbf{Geometric preliminaries.} The next lemma shows that compact subsets of the interior of the convex hull of a compact set \( K \) are stable with respect to sufficiently small perturbations of \( K \) in the Hausdorff metric.

\textbf{Lemma 1.} \textit{Let} \( K \subset \mathbb{R}^n \) \textit{be a compact set. Then for any compact set} \( C \subset \text{int conv} K \) \textit{there exists} \( \varepsilon > 0 \) \textit{such that for any compact set} \( K' \subset \mathbb{R}^n \) \textit{with} \( d_H(K, K') < \varepsilon \) \textit{we have} \( C \subset \text{int conv} K' \).

\textbf{Figure 1.} An illustration of Lemma 1 in the case when \( K = \{L, M, N\} \) and \( K' = \{L', M', N'\} \).

\textit{Proof.} Since \( \text{int conv} K \) is open, for any point \( x \in C \) there exists a simplex \( S_x \) with vertices \( \{v_i\}_{i=1..n+1} \subset \text{conv} K \) such that \( x \) belongs to the inner open simplex

\[ I_x := \left\{ \sum_{i=1}^{n+1} \lambda_i v_i \mid \lambda_i \in \left( \frac{1}{2(n+1)}, \frac{2}{n+1} \right), \sum_{i=1}^{n+1} \lambda_i = 1, i = 1..n+1 \right\}. \]

Since \( C \) is a compact set and the inner simplices \( \{I_x\}_{x \in C} \) cover \( C \) we can extract a finite subcover \( \{I_{x_k}\}_{k=1..m} \) of \( C \). Let us fix \( k \in 1..m \) and consider the simplex \( S := S_{x_k} \) with vertices \( \{v_i\}_{i=1..n+1} \subset \text{conv} K \). Let \( I := I_{x_k} \) denote the corresponding inner simplex.
If $\varepsilon < \text{dist}(\partial I, \partial S)$ then for any points $v_i' \in B_\varepsilon(v_i)$, $i = 1..n + 1$ one has

$$I \subset \text{conv}\{v_1', v_2', \ldots, v_{n+1}'\}. \quad (8)$$

Observe that for any $\varepsilon > 0$ and $i = 1..n + 1$ the ball $B_\varepsilon(v_i)$ contains a point $v_i' \in \text{conv} K'$. Indeed, by Caratheodory’s theorem $v_i = \sum_{j=1}^{n+1} \mu_j z_j$ for some $z_j \in K$ and $\mu_j \in [0, 1]$ with $\sum_{j=1}^{n+1} \mu_j = 1$. Since $d_H(K, K') < \varepsilon$ there exist points $z_j' \in K'$ such that $z_j' \in B_\varepsilon(z_j)$, where $j = 1..n + 1$. Let

$$v_i' := \sum_{j=1}^{n+1} \mu_j z_j',$$

then $|v_i - v_i'| \leq \sum_{j=1}^{n+1} \mu_j |z_j - z_j'| < \varepsilon$. Hence by (8) we have $I \subset \text{conv}\{v_1', v_2', \ldots, v_{n+1}'\}$ provided that $\varepsilon$ is small enough. But $v_i' \in \text{conv} K'$, hence $I \subset \text{conv} K'$. Since $I$ is open we can also write $I \subset \text{int} \text{conv} K'$.

Since we have finitely many simplices, we can choose $\varepsilon > 0$ in such a way that the inclusion $I_{x_k} \subset \text{int} \text{conv} K'$ holds for any $k = 1..m$ (provided that $d_H(K, K') < \varepsilon$). Then

$$C \subset \bigcup_{k=1..m} I_{x_k} \subset \text{int} \text{conv} K'. \quad \square$$

We will also need the following elementary lemma:

**Lemma 2.** Suppose that $z \in C(\mathcal{U} : \mathbb{R}^n)$ where $\mathcal{U} \subset \mathbb{R}^D$ is an open set. Suppose that for any $y \in \mathcal{U}$ we have a compact set $K_y \subset \mathbb{R}^n$ and the function $y \mapsto K_y$ is continuous in the Hausdorff metric. Then the function $F: y \mapsto \text{dist}(z(y), K_y)$ is continuous on $\mathcal{U}$.

**Proof.** One can prove directly that the function $(z, K) \mapsto \text{dist}(z, K)$ is continuous on $\mathbb{R}^n \times \mathcal{K}$. The function $y \mapsto (z(y), K_y)$ is continuous in view of the assumptions. Hence the function $F$ is continuous as a composition of continuous functions. \qed

In general the distance from a point $z$ to a compact set $K$ does not control from below the distance from $z$ to the boundary of $\text{conv} K$. However the following lemma shows that there exists a segment inside $\text{int} \text{conv} K$ with midpoint $z$ and length controlled from below by $\text{dist}(z, K)$:

**Lemma 3** (Geometric lemma). Let $K \subset \mathbb{R}^n$ be a compact set. For any $z \in \text{int} \text{conv} K$ there exists $\bar{z} \in \mathbb{R}^n$ such that

- $[z - \bar{z}, z + \bar{z}] \subset \text{int} \text{conv} K$
- $|\bar{z}| \geq \frac{1}{2n} \text{dist}(z, K)$

(This is exactly Lemma 5.3 from [DLS12].)

2.2. Convex integration. The following lemma is the main building block of the convex integration scheme:

**Lemma 4** (Perturbation lemma). Suppose that Assumptions 2 and 3 hold and $z$ is a subsolution of (5) and (6) such that

$$\int_\mathcal{U} \text{dist}^2(z(y), K_y) \, dy = \varepsilon > 0.$$

Then there exists $\delta = \delta(\varepsilon) > 0$ and a sequence $\{z_k\}_{k \in \mathbb{N}}$ of subsolutions of (5) and (6) such that

- $z_k = z$ on $\mathcal{D} \setminus \mathcal{U}$ for any $k \in \mathbb{N}$
- $\int_\mathcal{U} |z - z_k|^2 \, dy \geq \delta$ for any $k \in \mathbb{N}$
- $z_k \rightharpoonup z$ in $L^2(\mathcal{U})$ as $k \to \infty$. 
**Proof.** Step 1. Let \( y \in \mathcal{U} \). Since \( z(y) \in U_y \) we can apply Lemma 3 to obtain \( \bar{z}_*(y) \) such that

\[
[z(y) - \bar{z}_*(y), z(y) + \bar{z}_*(y)] \subset U_y,
\]

\[
|\bar{z}_*(y)| \geq \frac{1}{2n} \operatorname{dist}(z(y), K_y),
\]

Since \( \Lambda \) is dense in \( \mathbb{R}^n \) and \( U_y \) is open we can find \( \bar{z}(y) \in \Lambda \) such that

\[
[z(y) - \bar{z}(y), z(y) + \bar{z}(y)] \subset U_y,
\]

\[
|\bar{z}(y)| \geq \frac{1}{4n} \operatorname{dist}(z(y), K_y).
\]

Due to (9) there exists \( \rho(y) > 0 \)

\[
[z(y) - \bar{z}(y), z(y) + \bar{z}(y)] + B_{2\rho(y)}(0) \subset U_y.
\]

Hence using Assumption 2, Lemma 1 and the continuity of \( z \) we can find \( R(y) > 0 \) such that

\[
[z(x) - \bar{z}(y), z(x) + \bar{z}(y)] + B_\rho(y)(0) \subset U_x
\]

for all \( x \in B_{R(y)}(y) \subset \mathcal{U} \). Moreover, in view of Lemma 2 we can choose \( R(y) \) in such a way that

\[
\operatorname{dist}(z(x), K_x) \leq 2 \operatorname{dist}(z(y), K_y)
\]

for all \( x \in B_{R(y)}(y) \).

Using Assumption 1 for any fixed \( y \in \mathcal{U} \) we can construct a sequence \( \{w_{y,k}\}_{k \in \mathbb{N}} \subset C_0^\infty(B_1(0)) \) such that

- \( w_{y,k}(x) \in [-\bar{z}(y), \bar{z}(y)] + B_{\rho(y)}(0) \) for all \( x \in B_1(0) \) and \( k \in \mathbb{N} \),
- \( w_{y,k} \to 0 \) in \( L^2 \) as \( k \to \infty \),
- \( \int |w_{y,k}|^2 \, dx > C |\bar{z}(y)|^2 \) for all \( k \in \mathbb{N} \).

**Step 2.** Let \( \varepsilon := \int_\mathcal{U} \operatorname{dist}^2(z(y), K_y) \, dy \). The balls \( \{B_r(y) \mid y \in \mathcal{U}, \, r \in (0, R(y))\} \) cover \( \mathcal{U} \), so using Vitali’s covering theorem (see e.g. [Bog07], Theorem 5.5.2) and the absolute continuity of the Lebesgue integral we can find finitely many points \( \{y_i\}_{i=1..N} \subset \mathcal{U} \) and radii \( r_i \in (0, R(y_i)) \) such that

\[
\sum_{i=1}^N \int_{B_{r_i}} \operatorname{dist}^2(z(y), K_y) \, dy > \frac{1}{2} \varepsilon,
\]

where the balls \( B_i := B_{r_i}(y_i) \) are pairwise disjoint.

For each \( i = 1..N \) let us introduce the scaled and translated perturbations \( w_{i,k}(x) := w_{y_i,k}(\frac{x-y_i}{r_i}) \). These functions belong to \( C_0^\infty(B_i) \) and satisfy

(i) \( w_{i,k}(x) \in [-\bar{z}(y_i), \bar{z}(y_i)] + B_{\rho(y_i)}(0) \) for all \( x \in B_i, \, k \in \mathbb{N}, \, i = 1..N \);

(ii) \( w_{i,k} \to 0 \) in \( L^2 \) as \( k \to \infty \) (for each fixed \( i = 1..N \));

(iii) \( \int |w_{i,k}|^2 \, dx > C |\bar{z}(y_i)|^2 \mathcal{L}^D(B_i) \) for all \( k \in \mathbb{N} \).

In view of (i) and (11) we have \( z(x) + w_{i,k}(x) \in U_x \) for all \( x \in \mathcal{U} \) and \( i = 1..N \), hence \( z + w_{i,k} \in X_0 \). Since the balls \( B_i \) are pairwise disjoint the function

\[
z_k := z + \sum_{i=1}^N w_{i,k}
\]

also belongs to \( X_0 \).
Using successively (iii), (10), (12) and (13) we obtain:
\[
\int_{\mathcal{W}} |z - z_k|^2 \, dy = \sum_{i=1}^{N} \int_{B_i} |w_{i,k}(y)|^2 \, dy \overset{(iii)}{\geq} C \sum_{i=1}^{N} |\tilde{z}(y_i)|^2 \mathcal{L}^D(B_i) \geq \frac{C}{16n^2} \sum_{i=1}^{N} \text{dist}^2(z(y_i), K_{y_i}) \mathcal{L}^D(B_i) = \frac{C}{16n^2} \sum_{i=1}^{N} \int_{B_i} \text{dist}^2(z(y_i), K_{y_i}) \, dx \geq \frac{C}{32n^2} \sum_{i=1}^{N} \int_{B_i} \text{dist}^2(z(x), K_x) \, dx \overset{(13)}{=} \frac{C}{64n^2} \varepsilon.
\]

It remains to observe that since $N$ is finite and the points $y_i$ are fixed we have $z_k \to z$ in $L^2$ as $k \to \infty$. \hfill \Box

2.3. Proof of Theorem \ref{Theorem 1} We are now ready to prove our main abstract theorem.

Proof of Theorem \ref{Theorem 1}. Let $X_0$ denote a set of all subsolutions of (5) and (6) which agree with $z_0$ on $\mathcal{D} \setminus \mathcal{W}$. Let $X$ be the closure of $X_0$ in the weak topology of $L^2(\mathcal{W})$, endowed with the corresponding induced weak topology. Clearly any $z \in X$ solves (5) and satisfies (2) a.e. on $\mathcal{D} \setminus \mathcal{W}$.

For any $z \in X$ let us define
\[
I(z) := \int_{\mathcal{W}} |z(y)|^2 \, dy.
\]
This functional is a Baire-1 function on $X$. Indeed, for any $j \in \mathbb{N}$ let
\[
I_j(z) := \int_{\{y \in \mathcal{W} \mid \text{dist}(y, \partial U) > 1/j\}} |(\omega_{1,j} * z)(y)|^2 \, dy
\]
where for any $\varepsilon > 0$ we denote by $\omega_{\varepsilon}() = \varepsilon^{-D} \omega(\cdot/\varepsilon)$ the standard convolution kernel. Then for any $j \in \mathbb{N}$ the functional $I_j$ is continuous on $X$, and for any $z \in X$ we have $I_j(z) \to I(z)$ as $j \to \infty$.

In view of Assumption 2, $X$ is a bounded subset of $L^2(\mathcal{W})$. Since the weak topology is metrizable on the norm-bounded subsets of $L^2(\mathcal{W})$, we can consider $X$ as a complete metric space with some metric $d_X$.

Then by Baire category theorem (see also Theorem 7.3 from [Oxt80]) the set
\[
Y := \{z \in X \mid I(z) \text{ is continuous at } z\}
\]
is residual in $X$ (and hence is infinite). We claim that $z \in Y$ implies $J(z) = 0$, where
\[
J(z) := \int_{\mathcal{W}} \text{dist}^2(z(y), K_y) \, dy.
\]
Indeed, suppose that $J(z) = \varepsilon > 0$ for some $z \in Y$. Let $z_j \in X_0$ be a sequence such that $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$. Since $I$ is continuous at $z$ this implies that $I(z_j) \to I(z)$ and consequently $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$.

Then in view of Assumption 2, we have $J(z_j) \to J(z)$ as $j \to \infty$ and hence without loss of generality we can assume that $J(z_j) > \varepsilon/2$ for all $j \in \mathbb{N}$.

Applying Lemma 3 to $z_j$ for each $j \in \mathbb{N}$ we can find $\tilde{z}_j \in X_0$ such that $d_X(\tilde{z}_j, z_j) < 2^{-j}$ and $\int_{\mathcal{W}} |\tilde{z}_j - z_j|^2 \, dy \geq \delta > 0$, where $\delta = \delta(\varepsilon)$ is independent of $j$.

Since $d_X(\tilde{z}_j, z) \leq d_X(\tilde{z}_j, z_j) + d_X(z_j, z) \to 0$ as $j \to \infty$ we also have $\tilde{z}_j \to z$ in $L^2$. Since $z$ is a point of continuity of $I$ we also have $z_j \to z$ in $L^2(\mathcal{W})$ as $j \to \infty$. But then $\tilde{z}_j - z_j \to 0$ in $L^2(\mathcal{W})$, which contradicts the construction of $\tilde{z}_j$. \hfill \Box
3. Application to the continuity equation

In this section we apply our abstract framework to the case of the continuity equation.

**Theorem 2.** Suppose that $d \geq 2$. Let $E: \mathbb{R} \to \mathbb{R}$ be a non-negative bounded function which is continuous on some bounded open interval $I \subset \mathbb{R}$ and vanishes on $\mathbb{R} \setminus I$. Then there exist infinitely many bounded, compactly supported $u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $b: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy (1) and (2) in sense of distributions and such that

$$\int_{\mathbb{R}^2} u^2(t,x) \, dx = E(t) \quad \text{for a.e. } t \in I.$$

**Remark 1.** It is well-known that a representative of $u$ can be chosen such that the map $t \mapsto u(t, \cdot)$ is continuous with values in $L^2$ equipped with the weak topology. Then the question arises whether the assertion in the theorem holds even for every, and not just almost every, time $t$. We expect this to be true: indeed this should follow by methods similar to those of [DLS10]. We will however not pursue this question further in this article.

**Remark 2.** When $d = 2$ and $f$ is a characteristic function of an interval, the statement of Theorem 2 essentially follows from the example constructed in [Dep03]. This particular case of Theorem 2 was also proved in [Gus11] using the convex integration method.

**Remark 3.** A similar problem can be addressed for more general equation of the form $\text{div}(uB) = 0$ instead of (1). For this equation the problem is stated as follows: given a distribution $g$ is it possible to construct compactly supported bounded functions $u: \mathbb{R}^n \to \mathbb{R}$, $B: \mathbb{R}^n \to \mathbb{R}^n$ such that $\text{div}(uB) = 0$, $\text{div}B = 0$, $\text{div}(u^2B) = g$? This is related to the so-called chain rule problem [ADLM07]. When $n = 2$ such a construction is not possible for $g \neq 0$ in view of [BG14], while for $n \geq 3$ it is possible and is obtained using convex integration and rank-2 laminates in [CGSW].

Let us put the continuity equation in the framework of the previous section. Fix a bounded open set $\Omega \subset \mathbb{R}^d$. Let $\mathcal{D} := I \times \Omega$ and

$$F(t,x) := \frac{E(t)}{\mathcal{L}^d(\Omega)} 1_{\Omega}(x),$$

where $1_{\Omega}$ denotes the characteristic function of $\Omega$.

We consider equations (1) and (2) as a linear system

$$\partial_t u + \text{div}_x m = 0,$$

$$\text{div}_x b = 0$$

in $\mathcal{D} := \mathbb{R} \times \mathbb{R}^d$ with $u: \mathcal{D} \to \mathbb{R}$, $m: \mathcal{D} \to \mathbb{R}^d$ and $b: \mathcal{D} \to \mathbb{R}^d$ such that $z := (u, m, b)$ satisfies the constraint

$$z(y) \in K_y := \begin{cases} \{ (u, m, b) \mid m = ub, \ |b| = 1, \ u^2 = F(y) \} & \text{if } y \in \mathcal{W} \\ 0 & \text{if } y \in \mathcal{D} \setminus \mathcal{W} \end{cases} \quad \text{for a.e. } y = (x,t) \in \mathcal{D}.$$

Suppose that $z = (u, m, b) \in L^\infty(\mathcal{D})$ satisfies (14) and (15) in sense of distributions and moreover (16) holds a.e. in $\mathcal{D}$. Then the couple $(u, b)$ satisfies the assertion of Theorem 2.

Let us check the assumption of Theorem 2.

**Lemma 5.** Suppose that $A, B \subset \mathbb{R}^n$ are compact sets and $r > 0$ is such that

- for any $z \in A$ there exists $z' \in B \cap B_r(z)$
Lemma 7.
Proof. Suppose that \( d_H(A, B) \geq r \). Then without loss of generality we can assume that there exists \( z \in A \) such that for any \( z' \in B \) we have \( z \notin B_r(z') \). But then the ball \( B_r(z) \) cannot contain any point of \( B \), which leads to a contradiction. \( \square \)

Lemma 6. If \( F: \mathcal{U} \to \mathbb{R} \) is continuous, bounded and non-negative then the map \( y \mapsto K_y \) is continuous and bounded (w.r.t. \( d_H \)) on \( \mathcal{U} \).

Proof. Let \( f(y) := \sqrt{F(y)} \). Let us fix \( y \in \mathcal{U} \). For any \( \varepsilon > 0 \) let \( \delta > 0 \) be such that \( |f(y) - f(y')| < \varepsilon \) for any \( y' \in B_\delta(y) \subset \mathcal{U} \). Let us prove that \( d_H(K_y, K_{y'}) < 2\varepsilon \) for all \( y' \in B_\delta(y) \).

For any \( z \in K_y \) there exist \( \sigma \in \{ \pm 1 \} \) and \( b \in \mathbb{R}^d \) with \( |b| = 1 \) such that \( z = (\sigma f(y), \sigma f(y)b, b) \). Then \( z' := (\sigma f(y'), \sigma f(y')b, b) \) belongs to \( K_{y'} \) and \( |z - z'| \leq 2|f(y) - f(y')| \). Hence there exists \( z' \in K_{y'} \cap B_{2\varepsilon}(z) \).

Similarly, for any \( z' \in K_{y'} \) there exist \( \sigma \in \{ \pm 1 \} \) and \( b \in \mathbb{R}^d \) with \( |b| = 1 \) such that \( z' = (\sigma f(y'), \sigma f(y')b, b) \). Then \( z := (\sigma f(y), \sigma f(y)b, b) \) belongs to \( K_y \) and \( |z - z'| \leq 2|f(y) - f(y')| \). Hence there exists \( z \in K_y \cap B_{2\varepsilon}(z') \).

Therefore by Lemma 5 we have \( d_H(K_y, K_{y'}) < 2\varepsilon \). \( \square \)

Lemma 7. Assumption [ ] holds for the system [ ]–[ ].

Proof. Let \( \phi: \varnothing \to \mathbb{R} \) be a non-negative smooth function such that \( 0 \leq \phi \leq 1 \) on \( \varnothing \), \( \phi = 0 \) on \( \varnothing \setminus B_1(0) \) and \( \phi = 1 \) on \( B_{1/2}(0) \).

Part 1. Suppose that \( d > 2 \). Let us show that Assumption [ ] holds with \( \Lambda = \mathbb{R}^{2d+1} \). Fix \( \bar{u} \in \mathbb{R} \), \( \bar{m} \in \mathbb{R}^d \) and \( \bar{b} \in \mathbb{R}^d \) and let \( \bar{z} = (\bar{u}, \bar{m}, \bar{b}) \). Since \( d > 2 \) there exists a unit vector \( n \in \mathbb{R}^d \) such that \( n \cdot \bar{m} = n \cdot \bar{b} = 0 \). Denote \( \bar{n} = (0, n) \), \( \bar{a} = (\bar{u}, \bar{m}) \). For any \( k \in \mathbb{N} \) define \( \bar{a}_k: \varnothing \to \mathbb{R}^{d+1} \) by

\[
\bar{a}_k(y) := \bar{a}(\bar{n} \cdot \nabla_y (\phi \Pi_k)) - \bar{a}(\bar{n} \cdot \nabla_y (\phi \Pi_k))
\]

where \( y = (t, x) \) and

\[
\Pi_k(y) := \frac{\sin(k\bar{n} \cdot y)}{k}.
\]

Observe that

\[
\text{div}_y \bar{a}_k = (\bar{a} \cdot \nabla_y)(\bar{n} \cdot \nabla_y)(\phi \Pi_k) - (\bar{n} \cdot \nabla_y)(\bar{a} \cdot \nabla_y)(\phi \Pi_k) = 0.
\]

Let \( (u_k, m_k) \) denote the components of \( \bar{a}_k \), then by the equation above we have \( \partial_t u_k + \text{div}_x m_k = 0 \).

Similarly let

\[
b_k(t, x) := \bar{b}(n \cdot \nabla_x (\phi \Pi_k)) - n(\bar{b} \cdot \nabla_x (\phi \Pi_k))
\]

Then arguing as above \( \text{div} b_k = 0 \).

Now we introduce \( w_k := (u_k, m_k, b_k) \). Then

\[
w_k(y) = \bar{z} \phi \cos(k\bar{n} \cdot y) + f\Pi_k
\]

where \( f \) does not depend on \( k \) and vanishes on \( B_{1/2}(0) \).

On the other hand,

\[
\int_\varnothing |w_k|^2 dy \geq \int_{B_{1/2}(0)} |w_k|^2 dy = \int_{B_{1/2}(0)} |\bar{z}|^2 \cos^2(k\bar{n} \cdot y) dy = \int_{B_{1/2}(0)} |\bar{z}|^2 \frac{1 + \cos(2k\bar{n} \cdot y)}{2} dy \geq \frac{|\bar{z}|^2}{4} |B_{1/2}(0)|
\]

provided that \( k \) is sufficiently large.
Part 2. Suppose that $d = 2$ and fix $\tilde{z} = (\tilde{u}, \tilde{m}, \tilde{b}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ with $\tilde{u} \neq 0$. Let us look for a localized plane wave in the following form:

$$w_k = (a_k, b_k)$$

with

$$a_k(y) = \nabla y \times \left( \phi A \sin(kn \cdot y) \right)$$

$$b_k(t, x) = \nabla^\perp x \left( \phi \sin(kn \cdot (t, x)) \right)$$

where $n = (n_t, n_x) \in \mathbb{R} \times \mathbb{R}^2$ and $A \in \mathbb{R}^3$ are to be chosen and $k \in \mathbb{N}$. Then, by construction

$$\text{div}_y a_k = 0, \quad \text{div}_x b_k = 0.$$ 

Then, we get

$$w_k = \hat{z} \phi \cos(kn \cdot y) + f \frac{\sin(kn \cdot y)}{k}$$

where $\hat{z} = (A \times n, n_x^\perp)$ and $f$ does not depend on $k$ and vanishes on $B_{1/2}(0)$.

In order to have $\hat{z} = \bar{z}$ the vectors $A$ and $n$ must satisfy

$$A \times n = (\bar{u}, \bar{m}),$$

$$n_x^\perp = \bar{b}.$$ 

From the second equation we immediately obtain that $n_x = -\bar{b}^\perp$. Since $\bar{u} \neq 0$ there exists $n_t$ such that $n \perp (\bar{u}, \bar{m})$. Then, we can always find $A$ such that the first equation is satisfied. It remains to observe that the estimate (17) holds also in the considered case. We thus have verified Assumption 1 for $\Lambda = \mathbb{R}^3 \setminus \{\bar{u} = 0\}$. □

Proof of Theorem 2. By symmetry of $K_y$ for any $y \in \mathcal{U}$ we have $0 \in \text{int conv } K_y$. On the other hand $K_y = \{0\}$ for any $y \in \mathcal{D} \setminus \mathcal{U}$. Therefore $u \equiv 0$, $m \equiv 0$ and $b \equiv 0$ is a subsolution of (14)–(16). Then the result follows from Lemma 2, Lemma 7 and Theorem 1. □

References


