ON THE TOPOLOGIES ON IND-VARIETIES AND RELATED IRREDUCIBILITY QUESTIONS
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IMMANUEL STAMPFLI

Abstract. In the literature there are two ways of endowing an affine ind-variety with a topology. One possibility is due to Shafarevich and the other to Kambayashi. In this paper we specify a large class of affine ind-varieties where these two topologies differ. We give an example of an affine ind-variety that is reducible with respect to Shafarevich’s topology, but irreducible with respect to Kambayashi’s topology. Moreover, we give a counter-example of a supposed irreducibility criterion given in [Sha81] which is different from the counter-example given by Homma in [Kam96]. We finish the paper with an irreducibility criterion similar to the one given by Shafarevich.

0. Introduction. In the 1960s, in [Sha66], Shafarevich introduced the notion of an infinite-dimensional variety and infinite-dimensional group. In this paper, we call them ind-variety and ind-group, respectively. His motivation was to explore some naturally occurring groups that allow a natural structure of an infinite-dimensional analogue to an algebraic group (such as the group of polynomial automorphisms of the affine space). More precisely, he defined an ind-variety as the successive limit of closed embeddings

\[ X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots \]

of ordinary algebraic varieties \(X_n\) and an ind-group as a group that carries the structure of an ind-variety compatible with the group structure. We denote the limit of \(X_1 \hookrightarrow X_2 \hookrightarrow \ldots\) by \(\lim\ X_n\) and call \(X_1 \hookrightarrow X_2 \hookrightarrow \ldots\) a filtration. If all \(X_n\) are affine, then \(\lim\ X_n\) is called affine. For example, one can define a filtration on the group of polynomial automorphisms of the affine space via the degree of an automorphism. Further examples of ind-groups are \(\text{GL}_n(k[t])\), \(\text{SL}_n(k[t])\), etc., where the filtrations are given via the degrees of the polynomial entries of the matrices (for properties of these filtrations in case \(n = 2\) see [Sha04]). Fifteen years after his first paper [Sha66], Shafarevich wrote another paper with the same title [Sha81], where he gave more detailed explanations of some statements of his first paper. Moreover, he endowed an ind-variety \(\lim\ X_n\) with the weak topology induced by the topological spaces \(X_1 \subseteq X_2 \subseteq \ldots\). Later Kambayashi defined (affine) ind-varieties in [Kam96] and [Kam03] via a different approach. Namely, he defined an affine ind-variety as a certain spectrum of a so-called pro-affine algebra (see Section 1 for the definition). This pro-affine algebra is then the ring of regular functions on the affine ind-variety. With this approach Kambayashi introduced a topology in a natural way on an affine ind-variety. Namely, a subset is closed

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if it is the zero-set of some regular functions on the affine ind-variety. In analogy to the Zariski topology defined on an ordinary affine variety, we call this topology again Zariski topology. In this paper, we call the weak topology on an affine ind-variety ind-topology to prevent confusion, as the weak topology is finer than the Zariski topology. The Zariski topology and the ind-topology differ in general. For example, it follows from Exercise 4.1.E, IV. in [Kum02] that these topologies differ on the infinite-dimensional affine space $A^\infty = \lim_{\rightarrow} A^n$ (see Example 1). The aim of this paper is to specify classes of affine ind-varieties where these topologies differ or coincide, and to study questions concerning the irreducibility of an affine ind-variety (with respect to these topologies).

This paper is organized as follows. We give some basic definitions and notations in Section 1. In the next section we describe a large class of ind-varieties where the two topologies differ. The main result of this paper is the following

**Theorem A.** Let $X = \lim_{\rightarrow} X_n$ be an affine ind-variety. If there exists $x \in X$ such that $X_n$ is normal or Cohen-Macaulay at $x$ for infinitely many $n$, and the local dimension of $X_n$ at $x$ tends to infinity, then the ind-topology and the Zariski topology are different.

This theorem follows from a more general statement given in Proposition 1 (see also Remark 1). As a corollary to this theorem we get

**Corollary B.** Let $X = \lim_{\rightarrow} X_n$ be an affine ind-variety such that $X_n$ is normal for infinitely many $n$. Then the ind-topology and the Zariski topology coincide if and only if for all $x \in X$ the local dimension of $X_n$ at $x$ is bounded for all $n$.

This corollary follows from a more general statement given in Corollary 6. As a contrast to Theorem A, we show in Proposition 7 that the two topologies coincide if $X = \lim_{\rightarrow} X_n$ is “locally constant” with respect to the Zariski topology. More precisely we prove

**Proposition C.** If $X = \lim_{\rightarrow} X_n$ is an affine ind-variety such that every point has a Zariski open neighbourhood $U$ with $U \cap X_n = U \cap X_{n+1}$ for all sufficiently large $n$, then the ind-topology and the Zariski topology coincide.

Section 3 contains an example of an affine ind-variety that is reducible with respect to the ind-topology, but irreducible with respect to the Zariski topology. This is the content of Example 4.

In the last section we give a counter-example to Proposition 1 in [Sha81] (see Example 5). The content of the proposition is: an ind-variety $X = \lim_{\rightarrow} X_n$ is irreducible with respect to the ind-topology if and only if the set of irreducible components of all $X_n$ is directed under inclusion. One can see that the latter condition is equivalent to the existence of a filtration $X'_1 \hookrightarrow X'_2 \hookrightarrow \ldots$ where each $X'_n$ is irreducible and $\lim_{\rightarrow} X'_n = X$. In [Kam96], Homma gave a counter-example to that supposed irreducibility criterion. But in contrast to his counter-example, the number of irreducible components of $X_n$ in our counter-example is bounded for all $n$. We finish the paper with the following irreducibility criterion. The proposition follows from Proposition 8.
Proposition D. Let \( X = \lim_{\rightarrow} X_n \) be an affine ind-variety where the number of irreducible components of \( X_n \) is bounded for all \( n \). Then \( X \) is irreducible with respect to the ind-topology (Zariski topology) if and only if there exists a chain of irreducible subvarieties \( X'_1 \subseteq X'_2 \subseteq \ldots \) in \( X \) (i.e., \( X'_n \) is an irreducible subvariety of some \( X_m \)) such that \( \bigcup_n X'_n \) is dense in \( X \) with respect to the ind-topology (Zariski topology).

1. Definitions and notation. Throughout this paper we work over an uncountable algebraically closed field \( k \). We use the definitions and notation of Kambayashi in [Kam03] and Kumar in [Kum02]. Let us recall them briefly. A pro-affine algebra is a complete and separated commutative topological \( k \)-algebra such that \( 0 \) admits a countable base of open neighbourhoods consisting of ideals. Let \( A \) be a pro-affine algebra and let \( \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots \) be a base for \( 0 \in A \) as mentioned above. Let \( A_n = A/\mathfrak{a}_n \) and let \( \mathcal{Spm}(A) \) be the set of closed maximal ideals of \( A \). Then we have

\[
A = \lim_{\leftarrow} A_n \quad \text{and} \quad \mathcal{Spm}(A) = \bigcup_{n=1}^{\infty} \mathcal{Spm}(A_n)
\]

(cf. 1.1 and 1.2 in [Kam03]).

Definition 1. An affine ind-variety is a pair \((\mathcal{Spm}(A), A)\) where \( A \) is a pro-affine algebra such that \( A/\mathfrak{a}_n \) is reduced and finitely generated for some countable base of ideals \( \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \ldots \) of \( 0 \in A \). We call \( A \) the coordinate ring of the affine ind-variety and the elements of \( A \) regular functions. Two ind-varieties are called isomorphic if the underlying pro-affine algebras are isomorphic as topological \( k \)-algebras. Such an isomorphism induces then a bijection of the spectra.

One can construct affine ind-varieties in the following way. Consider a filtration of affine varieties, i.e., a countable sequence of closed embeddings of affine varieties

\[
X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \ldots
\]

Let \( X = \bigcup_{n=1}^{\infty} X_n \) as a set and let \( \mathcal{O}(X) := \lim_{\rightarrow} \mathcal{O}(X_n) \). We endow \( \mathcal{O}(X) \) with the topology induced by the product topology of \( \prod_n \mathcal{O}(X_n) \), where \( \mathcal{O}(X_n) \) carries the discrete topology for all \( n \). Then \((\mathcal{Spm}(\mathcal{O}(X)), \mathcal{O}(X))\) is an affine ind-variety and there is a natural bijection \( X \to \mathcal{Spm}(\mathcal{O}(X)) \) induced by the bijections \( X_n \to \mathcal{Spm}(\mathcal{O}(X_n)) \). In the following, we denote this ind-variety by \( \lim_{\rightarrow} X_n \). In fact, every affine ind-variety can be constructed in this way (up to isomorphy). Two filtrations \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \) and \( X'_1 \hookrightarrow X'_2 \hookrightarrow \ldots \) induce the same affine ind-variety (up to isomorphy) if and only if there exists a bijection

\[
f: \bigcup_{n=1}^{\infty} X_n \to \bigcup_{n=1}^{\infty} X'_n
\]

with the following property: for every \( i \) there exists \( j_i \) and for every \( j \) there exists \( i_j \), such that \( f|_{X_i}: X_i \to X_{j_i} \) and \( f^{-1}|_{X_j}: X_j \to X_{i_j} \) are closed embeddings of affine varieties. Such filtrations are called equivalent.

2. Topologies on affine ind-varieties. So far we have not established any topology on the set \( \mathcal{Spm}(A) \) of an affine ind-variety \((\mathcal{Spm}(A), A)\). As mentioned in the introduction there are two ways to introduce a topology on the set \( \mathcal{Spm}(A) \). The first possibility is due to Shafarevich [Sha66], [Sha81] and we call it the ind-topology. A subset \( Y \subseteq \mathcal{Spm}(A) \) is closed in this topology if and only if \( A \cap \mathcal{Spm}(A_n) \) is a closed subset of \( \mathcal{Spm}(A_n) \) for all \( n \). One can easily check that this topology does
Lemma 1. Let \( \rho : Z \to A \) be a finite surjective morphism, then there exists a unique factorization domain or Cohen-Macaulay (and hence also if \( A \) is Gorenstein, locally a complete intersection or regular) (see Theorem 23.8 [Mat86]).

We will use the following lemmata to prove Proposition 1.

Lemma 2. Let \( Z \) and \( Y \) be affine varieties and assume that there exists a closed embedding \( Z \to Y \). If \( f : Z \to A^{\dim Z} \) is a finite surjective morphism, then there exists a finite surjective morphism \( g : Y \to A^{\dim Y} \) such that \( g|_Z = \psi \circ f \), where \( \psi : A^{\dim Y} \to \psi(A^{\dim Y}) \) is given by \( \psi(v) = (v,0) \).

Proof. Let \( A := \mathcal{O}(Z) \), \( B := \mathcal{O}(Y) \) and let \( \psi : B \to A \) be the surjective homomorphism induced by \( Z \to Y \). Further, let \( f_1, \ldots, f_n \) be the coordinate functions of \( f \). By assumption \( k[f_1, \ldots, f_n] \subseteq A \) is an integral extension and \( f_1, \ldots, f_n \) are algebraically independent. Choose generators \( b_1, \ldots, b_l \) of the \( k \)-algebra \( B \) such that \( \psi(b_i) = f_i \) for \( i = 1, \ldots, n \). For every \( j = n+1, \ldots, l \) there exists a monic polynomial \( p_j \in k[b_1, \ldots, b_n][T] \) such that \( h_j := p_j(b_j) \in \ker(\psi) \), since \( k[f_1, \ldots, f_n] \subseteq A \) is integral. Thus,

\[
k[b_1, \ldots, b_n, h_{n+1}, \ldots, h_l] \subseteq B
\]

is an integral extension. If \( b_1, \ldots, b_n, h_{n+1}, \ldots, h_l \) are algebraically independent, then we are done. Otherwise, there exists a non-zero polynomial \( f(X_1, \ldots, X_l) \) with coefficients in \( k \) such that \( f(b_1, \ldots, b_n, h_{n+1}, \ldots, h_l) = 0 \). Exactly the same as in the proof of Lemma 2, §33 [Mat86] one can see that there exist \( c_1, \ldots, c_{l-1} \in k \) such that \( h_l \) is integral over \( k[b'_1, \ldots, b'_n, h'_{n+1}, \ldots, h'_{l-1}] \), where \( b'_i := b_i - c_i h_l \) and \( h'_{l} := h_l - c_l h_l \). Thus,

\[
k[b'_1, \ldots, b'_n, h'_{n+1}, \ldots, h'_{l-1}] \subseteq B
\]

is an integral extension. By induction there exists \( m \) with \( n \leq m < l \) and algebraically independent elements \( b''_1, \ldots, b''_n, h''_{n+1}, \ldots, h''_m \in B \) such that \( B \) is integral over \( k[b'_{1}, \ldots, b'_n, h''_{n+1}, \ldots, h''_m] \) and \( b''_1 - b_1, h''_l \in \ker(\psi) \). This proves the lemma. \( \square \)
Remark 2. From an iterative use of the lemma above we can deduce the following. For every affine ind-variety $X = \lim_n X_n$ there exists a surjective map of the underlying sets $X \rightarrow \mathbb{A}^\infty$ such that the restriction to every $X_n$ yields a finite surjective morphism $X_n \rightarrow \mathbb{A}^{\dim X_n}$.

Lemma 3. We assume that $\text{char}(k) = 0$. Let $Y$ be an irreducible affine variety and let $X$ be an affine scheme of finite type over $k$ that is reduced in an open dense subset. If $f: X \rightarrow Y$ is a dominant morphism, then there exists an open dense subset $U \subseteq Y$ such that $f^{-1}(u)$ is reduced in an open dense subset for all $u \in U$.

Proof. Without loss of generality, one can assume that $f$ is flat and surjective (see Theorem 14.4 (Generic freeness) [Eis95]). Since $X$ is reduced in an open dense subset, there exists an open dense subset $X' \subseteq X$ such that all fibres of $f|_{X'}: X' \rightarrow Y$ are reduced (see Corollary 10.7, Ch. III (Generic smoothness) and Theorem 10.2, Ch. III [Har77]; here we use $\text{char}(k) = 0$). Let $K := X \setminus X'$ be endowed with the reduced induced closed subscheme structure of $X$ and let $g := f|_{K}: K \rightarrow Y$. If $g$ is not dominant, then the fibres of $f$ over an open dense subset are reduced and we are done. Hence we can assume that $g$ is dominant. Again according to Theorem 24.1 [Mat86] there exists an open dense subset $U \subseteq Y$ such that $g|_{g^{-1}(U)}: g^{-1}(U) \rightarrow U$ is flat and surjective. Thus, we have for all $u \in U$ and $x \in g^{-1}(u)$

$$\dim_x g^{-1}(u) = \dim_x g^{-1}(U) - \dim_u U < \dim_x X - \dim_u Y = \dim_x f^{-1}(u).$$

It follows that $f^{-1}(u) \setminus g^{-1}(u)$ is a reduced open dense subscheme of $f^{-1}(u)$ for all $u \in U$. This implies the lemma. \hfill $\Box$

According to Ex. 11.10 [Eis95] we have the following criterion for reducedness of a Noetherian ring.

Lemma 4. A Noetherian ring $A$ is reduced if and only if

(R0) the localization of $A$ at each prime ideal of height 0 is regular,

(S1) $A$ has no embedded associated prime ideals.

One can see that condition (R0) is satisfied for a Noetherian ring $A$ if Spec($A$) is reduced in an open dense subset. Thus we get the following

Lemma 5. Let $X$ be a Noetherian affine scheme that is reduced in an open dense subset. If $\mathcal{O}_{X,x}$ satisfies (S1) for a point $x \in X$, then $\mathcal{O}_{X,x}$ is reduced.

Now we have the preliminary results to prove Proposition 1. The strategy is as follows. First we construct $0 \neq f_n \in \mathcal{O}(X_n)$ such that $f_n(x) = 0$, $f_n|_{X_{n-1}} = f_{n-1}^2$ and $\mathcal{O}_{X_n,x}/f_n\mathcal{O}_{X_n,x}$ is reduced. The main part of the proof is devoted to showing the reducedness and for that matter we use the condition (S2) of the local ring $\mathcal{O}_{X_n,x}$. Then we define $Y := \bigcup_n V_{X_n}(f_n)$. It follows that $Y$ is closed in $X$ with respect to the ind-topology. Afterwards, we prove that $Y$ is not closed in $X$ with respect to the Zariski topology. For that purpose, we take $\varphi = (\varphi_n) \in \mathcal{O}(X) = \lim \mathcal{O}(X_n)$ that vanishes on $Y$, and we show that $\varphi_n$ vanishes also on all irreducible components of $X_n$ passing through $x$. The latter we deduce from the fact that

$$\varphi_n = \varphi_{n+1}|_{X_n} \in f_{n+1}|_{X_n}\mathcal{O}_{X_n,x} = f_n^2\mathcal{O}_{X_n,x}$$

for all $i \geq 0$ and Krull’s Intersection Theorem.
Proof of Proposition 1. For the sake of simpler notation, we assume that \( \mathcal{O}_{X_n,x} \) satisfies \((S_2)\) and \( \dim_x X_n = n \) for all \( n \). Let \( X'_n \) be the union of all irreducible components of \( X_n \) containing \( x \) and let \( W_n \) be the union of all irreducible components of all \( X_i \) with \( i \leq n \), not containing \( x \) and of strictly smaller dimension than \( n \). Then, \( X'_1 \cup W_1 \hookrightarrow X'_2 \cup W_2 \hookrightarrow \ldots \) is an equivalent filtration of \( X \) to \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \), since \( \dim_x X_n \rightarrow \infty \). Thus, we can further impose that \( \dim_x X_n = \dim X'_n \) and \( \dim_p X_n < \dim X_n \) for all \( p \not\in X'_n \). As \( \mathcal{O}_{X_n,x} \) satisfies \((S_2)\), it follows from Corollary 5.10.9 \cite{gro65} that \( X'_n \) is equidimensional.

Now, we construct the \( 0 \neq f_n \in \mathcal{O}(X_n) \). From Lemma 2 it follows that we can choose algebraically independent elements \( x_1, \ldots, x_n \in \mathcal{O}(X_n) \) such that \( \mathcal{O}(X_n) \) is finite over \( k[x_1, \ldots, x_n] \) and \( x_n \) restricted to \( X_{n-1} \) is zero. We can assume that the finite morphism \( X_n \to \mathbb{A}^n \) induced by \( k[x_1, \ldots, x_n] \subseteq \mathcal{O}(X_n) \) sends \( x \) to \( 0 \in \mathbb{A}^n \). Since \( \dim_p X_n < \dim X_n \) for all \( p \not\in X'_n \) and \( X'_n \) is equidimensional, it follows that

\[
(*) \quad k[x_1, \ldots, x_n] \hookrightarrow \mathcal{O}(X_n) \to \mathcal{O}(K) \text{ is injective}
\]

for all irreducible components \( K \) of \( X'_n \). Let us define

\[
f_1 := c_1 x_1 \quad \text{and} \quad f_{n+1} := f_n^2 + c_{n+1} x_{n+1},
\]

where \( c_1, c_2, \ldots \in k \), not equal to zero. It follows that \( f_n(x) = 0 \) and \( f_{n+1}|_{X_n} = f_n^2 \). The aim is to prove that \( c_1, c_2, \ldots \in k \) can be chosen such that not all are equal to zero and \( \mathcal{O}_{X_n,x}/f_n \mathcal{O}_{X_n,x} \) is reduced for \( n > 1 \). Consider the morphism

\[
\psi_n : Z_n \to \mathbb{A}^n,
\]

where \( Z_n \) is the affine scheme with coordinate ring

\[
S_n := \mathcal{O}(X'_n)[c_1, \ldots, c_n]/(f_n)
\]

and \( \psi_n \) is the restriction of the canonical projection \( X'_n \times \mathbb{A}^n \to \mathbb{A}^n \) to the closed subscheme \( Z_n \). If \( (c_1, \ldots, c_n) \) is fixed, then \( \mathcal{O}_{X_n,x}/f_n \mathcal{O}_{X_n,x} \) is the local ring of the fibre \( \psi_n^{-1}(c_1, \ldots, c_n) \) in the point \((x, c_1, \ldots, c_n) \in Z_n \). For that reason we will study the fibres of the morphism \( \psi_n : Z_n \to \mathbb{A}^n \). We claim that \( Z_n \) is reduced in an open dense subset for \( n > 1 \). To prove this claim, we mention first that

\[
(S_n)_{x_n} \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_n]/(f_n^2 - c_n x_n) \simeq \mathcal{O}(X'_n)_{x_n}[c_1, \ldots, c_{n-1}]
\]

is reduced. Let \( R_n := k[x_1, \ldots, x_n][c_1, \ldots, c_n]/(f_n) \). It follows that the morphisms \( \text{Spec}(S_n) \to \text{Spec}(R_n) \) and \( \text{Spec}(S_n/(x_n)) \to \text{Spec}(R_n/(x_n)) \) are both finite and surjective. As \( \dim R_n/(x_n) < \dim R_n \) for \( n > 1 \) we get \( \dim S_n/(x_n) < \dim S_n \). Since \( X'_n \) is equidimensional one can deduce from \((*)\) that \( Z_n \) is equidimensional. Hence, \( \text{Spec}((S_n)_{x_n}) \subseteq Z_n \) is an open dense reduced subscheme.

Since \( \{x\} \times \mathbb{A}^n \) is contained in \( Z_n \), it follows that \( \psi_n \) is surjective. For \( n > 1 \) there exists an open dense subset \( U_n \subseteq \mathbb{A}^n \) such that

\[
\psi_n|_{\psi_n^{-1}(U_n)} : \psi_n^{-1}(U_n) \to U_n
\]

is surjective and flat, and every fibre is reduced in an open dense subset (see Lemma 3 and Theorem 24.1 \cite{mat86}). With the aid of \((*)\) it follows that \( f_n \) is an \( \mathcal{O}_{X_n,x}\)-regular sequence for every choice \( (c_1, \ldots, c_n) \in U_n \). Since \( \mathcal{O}_{X_n,x} \) satisfies \((S_2)\), we get from Corollary 5.7.6 \cite{gro65} that \( \mathcal{O}_{X_n,x}/f_n \mathcal{O}_{X_n,x} \) satisfies \((S_1)\). But as \( \psi_n^{-1}(c_1, \ldots, c_n) \) is reduced in an open dense subset, it follows from Lemma 5 that it is reduced in the point \((x, c_1, \ldots, c_n) \). Hence, for \( n > 1 \) it follows that \( \mathcal{O}_{X_n,x}/f_n \mathcal{O}_{X_n,x} \)
is reduced if we choose \((c_1, \ldots, c_n) \in U_n\). For \(i \geq n\) let \(\pi_i^U : \mathbb{A}^i \to \mathbb{A}^n\) be the projection onto the first \(n\) components. As the field \(k\) is uncountable, one can choose inductively

\[
0 \neq c_1 \in \bigcap_{i \geq 1} \pi_i^U(U_i), \quad (c_1, \ldots, c_n, c_{n+1}) \in \bigcap_{i \geq n+1} \pi_i^U(U_i) \cap \{(c_1, \ldots, c_n)\} \times \mathbb{A}^1.
\]

Hence, \((c_1, \ldots, c_n) \in U_n\) for all \(n > 1\) and not all \(c_1, c_2, \ldots\) are equal to zero. This finishes the construction of the \(f_n\).

Let us define \(Y := \bigcup_n V_{X_n}(f_n)\). Since \(f_{n+1}|_{X_n} = f_n^2\) for all \(n\), \(Y\) satisfies i). Take any \(\varphi = (\varphi_n) \in \lim \mathcal{O}(X_n)\) that vanishes on \(Y\). We claim that \(\varphi|_{X'} = 0\), where \(X' := \bigcup_n X_n\). It is enough to prove that \(\varphi_n = 0\) in \(\mathcal{O}_{X_n, x}\). Since \(\varphi_m|_{y_m} = 0\) and \(\mathcal{O}_{X_n,x}/f_m \mathcal{O}_{X_n,x}\) is reduced, it follows that \(\varphi_m \in f_m \mathcal{O}_{X_n,x}\). Using \(f_{m+1}|_{X_n} = f_m^2\) again, we get by induction

\[
\varphi_n = \varphi_{n+1}|_{X_n} \in f_m^{2^i} \mathcal{O}_{X_n,x} \quad \text{for all } i \geq 0, \ n > 1.
\]

But according to Krull’s Intersection Theorem (see Theorem 8.10 [Mat86]), we have \(f_{n+1}|_{X_n} = 0\), hence \(\varphi_n = 0\) in \(\mathcal{O}_{X_n,x}\). Since \(f_n|_{X'} \neq 0\) (cf. (*)), we get \(X' \cup Y \supseteq Y\). Thus \(Y\) satisfies ii) according to the aforementioned claim. \(\square\)

The following example is a special case of the construction in the proof of Proposition 1. We mention it here, since we will use it in future examples.

**Example 1** (See Ex. 4.1.E, IV. in [Kum02]). Let \(f_n \in k[x_1, \ldots, x_n] = \mathcal{O}(\mathbb{A}^n)\) be recursively defined as

\[
f_1 := x_1, \quad f_{n+1} := f_n^2 + x_{n+1}.
\]

Then \(\bigcup_n V_{\mathbb{A}^n}(f_n)\) is a proper closed subset of the infinite-dimensional affine space \(\mathbb{A}^\infty = \lim \mathbb{A}^n\) with respect to the ind-topology, but it is dense in \(\mathbb{A}^\infty\) with respect to the Zariski topology.

Let \(G\) be the group of polynomial automorphisms of the affine space \(\mathbb{A}^n\), where \(n\) is a fixed number \(\geq 2\). We prove in the next example that the ind-topology and the Zariski topology on \(G\) differ if we consider \(G\) as an affine ind-variety via the filtration given by the degree of an automorphism.

**Example 2.** First, we define on \(G\) a filtration of affine varieties (via the degree). Let \(E\) be the set of polynomial endomorphisms of the affine space \(\mathbb{A}^n\) and let \(E_d\) be the subset of all \(\varphi \in E\) of degree \(\leq d\). Denote by \(U_d \subseteq E_d\) the subset of all \(\varphi \in E_d\) such that \(\text{Jac}(\varphi) \in k^*\). One can see that \(U_d \subseteq E_d\) is a locally closed subset and it inherits the structure of an affine variety from \(E_d\). With Corollary 0.2 [Kam79] and the estimate of the degree of the inverse of an automorphism due to Gabber (see Corollary 1.4 in [BCW82]) one can deduce that \(G_d \subseteq U_d\) is a closed subset. Thus \(G_d\) is locally closed in \(E_d\) and it inherits the structure of an affine variety from \(E_d\).

Moreover, one can see that \(G_d\) is closed in \(G_{d+1}\). In the following, we consider \(G\) as an affine ind-variety via the filtration \(G_1 \hookrightarrow G_2 \hookrightarrow \ldots\) of affine varieties.

We claim that the ind-topology and the Zariski topology on \(G\) differ. Consider the subset

\[
M := \{(x_1 + p, x_2, \ldots, x_n) \in G \mid p \in k[x_n]\} \subseteq G.
\]

It is closed in \(G\) with respect to the ind-topology. We consider \(M\) as an affine ind-variety via \(M := \varprojlim M \cap G_d\) and thus \(M \simeq \mathbb{A}^\infty\) as affine ind-varieties. According to Example 1 there exists a proper subset \(Y \subseteq M\) that is closed with respect to
the ind-topology, but it is dense in $M$ with respect to the Zariski topology. Hence, every regular function on $G$ vanishing on $Y$, vanishes also on $M$. This implies the claim.

Remark 3. A similar argument as in Example 2 shows that the ind-topology and the Zariski topology differ on $\text{GL}_n(k[t])$ and also on $\text{SL}_n(k[t])$.

Next, we give a corollary to Proposition 1 which implies Corollary B. Before we state the corollary, we introduce the following notation. For any ind-variety $X = \varinjlim X_n$ we choose connected components $X^n_i$ of $X_n$, $i = 1, \ldots, k_n$, such that

$$X_n = \bigcup_{i=1}^{k_n} X^n_i \quad \text{and} \quad X^n_i \subseteq X^n_{i+1} \text{ for all } i = 1, \ldots, k_n$$

(it can be that $X^n_i = X^n_j$ for $i \neq j$). We remark that the decomposition of an ind-variety into connected components is the same for the ind-topology and the Zariski topology.

Corollary 6. We assume that $\text{char}(k) = 0$. Let $X = \varinjlim X_n$ be an affine ind-variety such that for $i$ fixed, the number of irreducible components of $X^n_i$ is bounded for all $n$. Moreover, assume that $O(X_n)$ satisfies ($S_2$) for infinitely many $n$. Then the following statements are equivalent:

i) The ind-topology and the Zariski topology on $X$ coincide.

ii) For all $x \in X$ the local dimension of $X_n$ at $x$ is bounded for all $n$.

iii) Every connected component of $X$ is contained in some $X_n$.

Proof. Every connected component of $X$ is equal to some $X^i := \bigcup_{n=1}^{\infty} X^n_i$.

i) $\Rightarrow$ ii): This follows from Proposition 1.

ii) $\Rightarrow$ iii): As $X^n_i$ satisfies ($S_2$) and is connected, $X^n_i$ is equidimensional (see Corollary 5.10.9 [Gro65]). Thus, $X^n_i = X^n_{i+1}$ for $n$ large enough, as the number of irreducible components of $X^n_i$ is bounded for all $n$. Thus, $X^i \subseteq X_n$ for some $n$.

iii) $\Rightarrow$ i): This follows from the fact that every connected component of $X$ is closed and open with respect to the Zariski topology. □

As a contrast to Proposition 1, the two topologies coincide if the affine ind-variety is “locally constant” with respect to the Zariski topology. The following proposition coincides with Proposition C.

Proposition 7. Let $X = \varinjlim X_n$ be an affine ind-variety. Assume that every $x \in X$ has a Zariski open neighbourhood $U_x \subseteq X$ such that $U_x \cap X_n = U_x \cap X_{n+1}$ for all sufficiently large $n$. Then the two topologies on $X$ coincide.

Proof. Let $Y \subseteq X$ be a closed subset with respect to the ind-topology. One can see that $Y \cap U_x$ is closed in $U_x$ with respect to the Zariski topology for all $x \in X$. This proves that $Y$ is closed in $X$ with respect to the Zariski topology. □

The following example is an application of the proposition above. We construct a proper ind-variety (i.e., it is not a variety) such that the ind-topology and the Zariski topology coincide and moreover, it is connected.

Example 3. Let $L_n$ be defined as

$$L_n := V_{\mathbb{A}^n}(x_1 - 1, x_2 - 1, \ldots, x_{n-1} - 1) \subseteq \mathbb{A}^n.$$
Remark that $L_n \cap L_{n+1} = \{(1, \ldots, 1)\} \subseteq \mathbb{A}^n$ for all $n$ and $L_n \cap L_m = \emptyset$ for all $n, m$ with $|n - m| \geq 2$. Let $X := \lim X_n$ where $X_n := L_1 \cup \ldots \cup L_n \subseteq \mathbb{A}^n$. It follows that $X \subseteq \mathbb{A}^\infty$ is a closed connected subset in the ind-topology. We claim that the ind-topology and the Zariski topology on $X$ coincide. According to the proposition above it is enough to show that $X$ is “locally constant” with respect to the Zariski topology. Let $x \in X$. Then there exists $N$ such that $x \in L_N$, but $x \notin L_{N+1}$. Let $U_x := X \setminus V_{\mathbb{A}^\infty}(f_1, \ldots, f_N) \subseteq X$ where $f_i \in \mathcal{O}(\mathbb{A}^\infty)$ is given by

$$f_i|_{\mathbb{A}^n} = x_i - 1 \in k[x_1, \ldots, x_n] \quad \text{for all } n \geq N.$$ 

Thus, $U_x \subseteq X$ is a Zariski open neighbourhood of $x$. Moreover, for all $n > N$ we have $L_n \subseteq V_{\mathbb{A}^\infty}(f_1, \ldots, f_N)$. Hence we have $U_x \cap X_n = U_x \cap X_{n+1}$ for all $n \geq N$.

As remarked before Corollary 6, connectedness of an affine ind-variety is the same for both topologies. But this is no longer true for irreducibility as we will see in the next section (see Example 4).

3. Irreducibility via the coordinate ring. It is well known that an affine variety $X$ is irreducible if and only if the coordinate ring $\mathcal{O}(X)$ is an integral domain. This statement remains true for affine ind-varieties endowed with the Zariski topology. The proof is completely analogous to the proof for affine varieties. In the case of the ind-topology it is still true that $\mathcal{O}(X)$ is an integral domain if $X$ is irreducible, as the ind-topology is finer than the Zariski topology. But the converse is in general false. In the following we give an example of an affine ind-variety $X$, which is reducible in the ind-topology, but its coordinate ring $\mathcal{O}(X)$ is an integral domain and thus it is irreducible in the Zariski topology.

**Example 4.** Throughout this example we work in the ind-topology. Let $g_n \in k[x_1, \ldots, x_n]$ be defined as

$$g_n := x_1 + \ldots + x_n,$$

and let $f_n$ be defined as in Example 1. By construction, $f_n$ and $g_n$ are irreducible polynomials. The affine ind-variety $X := \lim (V_{\mathbb{A}^\infty}(f_n) \cup V_{\mathbb{A}^\infty}(g_n))$ decomposes into the proper closed subsets $\bigcup_n V_{\mathbb{A}^\infty}(f_n)$ and $\bigcup_n V_{\mathbb{A}^\infty}(g_n)$ and thus $X$ is reducible. We claim that $\mathcal{O}(X) = \lim k[x_1, \ldots, x_n]/(f_n, g_n)$ is an integral domain. Assume towards a contradiction that there exist $(\varphi_n), (\psi_n) \in \prod_{n=1}^{\infty} k[x_1, \ldots, x_n]$ such that $(\varphi_n)$ and $(\psi_n)$ define non-zero elements in $\mathcal{O}(X)$, but $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$. By definition, there exists $\alpha_n \in k[x_1, \ldots, x_n]$ such that

$$(*) \quad \varphi_{n+1}(x_1, \ldots, x_n, 0) = \varphi_n + f_n g_n \alpha_n \quad \text{for all } n.$$ 

Since $(\varphi_n \psi_n)$ defines zero in $\mathcal{O}(X)$, it follows that $f_n g_n$ divides $\varphi_n \psi_n$ for $n > 0$. Hence we can assume without loss of generality that $f_n$ divides $\varphi_n$ for infinitely many $n$. Eq. $(*)$ and the definition of $f_{n+1}$ show that $f_n$ divides $\varphi_n$ for all $n$. Since $(\varphi_n) \neq 0$ in $\mathcal{O}(X)$ there exists $N > 1$ such that $g_N$ does not divide $\varphi_N$. Let $\rho_n \in k[x_1, \ldots, x_n]$ such that $\varphi_n = f_n \rho_n$. It follows that $g_N$ does not divide $\rho_N$, in particular $\rho_N \neq 0$. According to $(*)$ and the definition of $f_{n+1}$ we have

$$(**) \quad \rho_n = f_n \cdot \rho_{n+1}(x_1, \ldots, x_n, 0) - g_n \cdot \alpha_n \quad \text{for all } n.$$ 

Since $g_N$ does not divide $\rho_N$ it follows that there exists $p \in \mathbb{A}^N$ with $g_N(p) = 0$ and $\rho_N(p) \neq 0$. Let $\gamma_n : \mathbb{A}^1 \to \mathbb{A}^n$ be the curve defined by $\gamma_n(t) = (p, 0, \ldots, 0) + \cdots$
(t, −t, 0, . . . , 0) for n ≥ N. Since \( g_n(\gamma_n(t)) = 0 \) it follows from (**) that \( \rho_n(\gamma_n(t)) = f_n(\gamma_n(t))\rho_{n+1}(\gamma_{n+1}(t)) \). This implies

\[
0 \neq \rho_N(\gamma_N(t)) = \left( \prod_{i=N}^{n-1} f_i(\gamma_i(t)) \right) \cdot \rho_n(\gamma_n(t)) \quad \text{for all } n \geq N.
\]

Since \( f_i(\gamma_i(t)) \) is a polynomial of degree \( 2^{i-1} \) for all \( i \geq N \), it follows that the polynomial \( \rho_N(\gamma_N(t)) \) is of unbounded degree, a contradiction.

4. Irreducibility via the filtration. One would like to give a criterion for connectedness or irreducibility in terms of the filtration \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \) of the affine ind-variety. In the case of connectedness Shafarevich gave a nice description via the filtration (see Proposition 2 [Sha81]) and Kambayashi gave a proof for it (see Proposition 2.4 [Kam96]) (the proof works in both topologies, as connectedness of an affine ind-variety is the same for both topologies). In the case of irreducibility, things look different.

If we start with a filtration \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \) of irreducible affine varieties, then one can see that \( \lim X_n \) is an irreducible affine ind-variety in both topologies. Likewise one can ask if every irreducible affine ind-variety is obtained from a filtration of irreducible affine varieties. One can see that the latter property is equivalent to the following condition: the set \( K \) of all irreducible components of all \( X_n \) is directed under inclusion for some (and hence every) filtration \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \). Shafarevich claims in [Sha81] that the latter condition is equivalent to the irreducibility of \( X \) in the ind-topology. But Homma gave in [Kam96] a counter-example \( X \) to this statement. For every filtration \( X_1 \hookrightarrow X_2 \hookrightarrow \ldots \) of Homma’s counter-example \( X \) the number of irreducible components of \( X_n \) tends to infinity if \( n \to \infty \). Here we give another counter-example. Namely, we construct an irreducible affine ind-variety \( X = \lim X_n \) (irreducible with respect to both topologies) such that \( X \) is not directed, but \( X_n \) consists of exactly two irreducible components for \( n > 1 \).

**Example 5.** Let us define \( g_n \in k[x_1, \ldots, x_n] \) recursively by

\[
g_1 := (x_1 - 1), \quad g_{n+1} := (x_1 - (n + 1)) \cdot g_n - x_{n+1}.
\]

By construction every \( g_n \) is an irreducible polynomial. Let \( Y_n := V_{\mathbb{A}^n}(g_n) \subseteq \mathbb{A}^n \). It follows that \( Y_n \subseteq Y_{n+1} \) for all \( n \). Let further \( Z_n := V_{\mathbb{A}^n}(x_2, \ldots, x_n) \subseteq \mathbb{A}^n \) and \( X_n := Y_n \cup Z_n \). It follows that \( X_n \subseteq X_{n+1} \) is a closed subset for all \( n \). Let \( X := \lim X_n \). We get

\[
Y_n \cap Z_n = V_{\mathbb{A}^n}(g_n, x_2, \ldots, x_n) = V_{\mathbb{A}^n}(\prod_{i=1}^{n}(x_1 - i), x_2, \ldots, x_n) = \{ e_1, 2e_1, \ldots, ne_1 \},
\]

where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{A}^n \). The set \( X \) defined above is not directed and \( X_n \) decomposes in two irreducible components for \( n > 1 \). It remains to show that \( X \) is irreducible with respect to the ind-topology, as in that case \( X \) is also irreducible in the Zariski topology. As \( Y_n \) is irreducible for all \( n \), it follows that \( Y = \bigcup_n Y_n \) is irreducible. Since

\[
Z_m \subseteq \bigcup_{n=1}^{\infty} Y_n \cap Z_n \subseteq Y \subseteq X \quad \text{for all } m,
\]

we have \( X = \overline{Y} \), where the closure is taken in the ind-topology. Since \( Y \) is irreducible, as a consequence \( X \) is also irreducible.
We conclude this paper with a criterion for the irreducibility of an affine ind-variety $X = \varprojlim X_n$ where the number of irreducible components of $X_n$ is bounded for all $n$. Unfortunately we need for this criterion also information about the closure of a subset in the “global” object $X$ and not only about the filtration $X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ itself. The following proposition implies Proposition D.

**Proposition 8.** Let $X = \varprojlim X_n$ be an affine ind-variety such that the number of irreducible components of $X_n$ is bounded by $l$ for all $n$. Then $X$ is irreducible in the ind-topology (Zariski topology) if and only if for all $n$ there exists an irreducible component $F_n$ of $X_n$ such that $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_n F_n$ is dense in $X$ with respect to the ind-topology (Zariski topology).

**Proof.** One can read the proof either with respect to the ind-topology or with respect to the Zariski topology. Let $X = \varprojlim X_n$ be irreducible. For all $n$ let us write $X_n = X_1^n \cup \ldots \cup X_l^n$ where $X_i^n$ is an irreducible component of $X_n$ and for all $n$ we have $X_i^n \subseteq X_{i+1}^n$ (it can be that $X_i^n = X_j^n$ for $i \neq j$). Thus, one gets $X = \bigcup_n X_1^n \cup \ldots \cup \bigcup_n X_l^n$.

Since $X$ is irreducible the claim follows. The converse of the statement is clear. $\square$

**References**


