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Free $\mathbb{C}^+$-actions on affine threefolds

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Abstract. We study algebraic actions of the additive group $\mathbb{C}^+$ on an affine threefold X and prove a smoothness property for the quotient morphism $\pi\colon X \to X/\!\!/\mathbb{C}^+$. Then, following Shulim Kaliman, we give a proof of the conjecture that every free $\mathbb{C}^+$-action on affine 3-space $\mathbb{C}^3$ is a translation.

1. The main result

These notes are about algebraic actions of the additive group $\mathbb{C}^+$ on complex affine threefolds and, in particular, about the following conjecture which was recently proved by Shulim Kaliman [**Ka04**].

Conjecture. *Every free $\mathbb{C}^+$-action on $\mathbb{C}^3$ is a translation, i.e. there is a closed subset $S \subset \mathbb{C}^3$ such that the natural morphism $\mathbb{C}^+ \times S \xrightarrow{\sim} \mathbb{C}^3$, $(s, x) \mapsto s \cdot x$, is an isomorphism.*

In fact, Kaliman proves a more general statement. Together with some results from the joint paper [**KaS04**] with Saveliev one has now the following general theorem.

Theorem (Kaliman-Saveliev). *Every free $\mathbb{C}^+$-action on a smooth contractible affine threefold X is a translation.*

For more information about this type of problems and additional background we refer to the survey [**Kr96**]. In particular, the corresponding result in dimension 2 goes back to Rentschler [**Re68**] who classified the $\mathbb{C}^+$-actions on $\mathbb{C}^2$, and the conjecture is wrong in dimension 4 as shown by Winkelmann in [**Wi90**].

The text is organized as follows. In the next two sections we describe an approach to the problem in a more general setting and work out the algebraic "obstructions" for actions to be locally trivial. This part is partially based on discussions I had with Dave Finston and James Deveney some time ago where we developed a strategy to prove the conjecture using the smoothness of the quotient morphism

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$\pi\colon \mathbb{C}^3 \to \mathbb{C}^3/\!\!/\mathbb{C}^+$. Looking again at their paper [**DeF00**] I realized that the smoothness of π can be proved under more general assumptions (see Proposition 1 in §2).

In the last section, using this smoothness property of the quotient map and some topological considerations due to Kaliman we are able to give a short proof of the Theorem above, where we assume, in addition, that the quotient $X/\!\!/\mathbb{C}^+$ is smooth (see the Main Theorem in §4).

It should be pointed out that most results here generalize to free actions of a unipotent group U of dimension n on an $(n+2)$-dimensional affine variety with the appropriate properties.

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2. The basic setup

Our base field is the field $\mathbb{C}$ of complex numbers. Let X be an affine variety of dimension 3 with a non-trivial action of the additive group $\mathbb{C}^+$. Assume that X is *factorial*, i.e. that the coordinate ring $\mathcal{O}(X)$ is a unique factorization domain. Denote by $X/\!\!/\mathbb{C}^+$ the *algebraic quotient*. This means that $X/\!\!/\mathbb{C}^+$ is the affine variety with coordinate ring $\mathcal{O}(X)^{\mathbb{C}^+}$, the *invariant ring* under $\mathbb{C}^+$, which is finitely generated by a well-known result of Zariski [**Za54**]. Moreover, $\pi = \pi_X\colon X \to X/\!\!/\mathbb{C}^+$ denotes the *quotient morphism* defined by the inclusion $\mathcal{O}(X)^{\mathbb{C}^+} \subset \mathcal{O}(X)$.

Lemma 1. *The quotient $Y := X/\!\!/\mathbb{C}^+$ is a normal and factorial surface. If $C \subset Y$ is an irreducible curve, then the inverse image $\pi^{-1}(C)$ is reduced and irreducible. Moreover, $\pi\colon X \to Y$ is equidimensional, i.e. every non-empty fiber has dimension* 1*, and $Y \setminus \pi(X)$ is finite.*

Proof. The first part is well-known. It essentially uses the fact that the group $\mathbb{C}^+$ has no characters.

An irreducible closed curve $C \subset Y$ is defined by an irreducible function $f \in \mathcal{O}(Y)$ which remains irreducible when considered as an element in $\mathcal{O}(X)$. Thus the schematic inverse image of C is reduced and irreducible.

Assume that a fiber $\pi^{-1}(y)$ has an irreducible component Z of dimension 2. Then Z is defined by an irreducible function $f \in \mathcal{O}(X)$. Since Z is stable under $\mathbb{C}^+$ the function f is invariant. If h is any other invariant, then h is constant on Z, $h|_Z \equiv c$, and so $h - c$ is a multiple of f. But this implies that the invariant ring is $\mathbb{C}[f]$, contradicting the fact that $\dim Y = 2$.

Finally, if $Y \setminus \pi(X)$ is not finite, then $C := \overline{Y \setminus \pi(X)}$ is a curve and so $F := \pi^{-1}(C)$ is of dimension 2. This implies that $\pi(F)$ is of dimension 1 which contradicts the fact that $\pi(F)$ is contained in the finite subset $C \cap \pi(X)$ of Y. □

Proposition 1. *Let $x \in X \setminus X^{\mathbb{C}^+}$ and assume that X is smooth in x and Y is smooth in $\pi(x)$. Then π is smooth in x.*

Proof. (a) We first claim that there are only finitely many $y \in Y \setminus \pi(X^{\mathbb{C}^+})$ such that the fiber $\pi^{-1}(y)$ is not reduced. Otherwise there is an irreducible curve $C \not\subset \pi(X^{\mathbb{C}^+})$ such that $\pi^{-1}(y)$ is not reduced for almost all $y \in C$. Since $F := \pi^{-1}(C)$ is reduced and irreducible by Lemma 1 this implies that almost all fibers of the morphism $\pi|_F : F \to C$ are non-reduced. On the other hand the morphism $\pi|_F$ is smooth on a dense open set which is stable under $\mathbb{C}^+$. It follows that almost all (schematic) fibers of $\pi|_F$ are smooth, hence reduced, and we end up with a contradiction.

(b) Now let $x_0 \in X \setminus X^{\mathbb{C}^+}$ such that X is smooth in x_0 and Y is smooth in $y_0 := \pi(x_0)$, and assume that π is not smooth in x_0. We will show that this leads to a contradiction. We can find an irreducible closed surface $S \subset X$ containing x_0 and smooth in x_0 such that $T_{x_0}X = T_{x_0}S \oplus T_{x_0}(\mathbb{C}^+ x_0)$, i.e. S is transversal to the $\mathbb{C}^+$-orbit through x_0. It follows that there is an open dense subset $S' \subset S$ containing x_0 such that the following holds: (i) S' is smooth and $\pi(S') \subset Y$ is open and smooth; (ii) S' is transversal to the orbits $\mathbb{C}^+ x$ for all $x \in S'$. In particular, $S' \cap X^{\mathbb{C}^+} = \emptyset$.

It follows that $\pi|_{S'}$ is smooth in $x \in S'$ if and only if π is smooth in x. It is well-known that the set of points of S' where $\pi|_{S'}$ is not smooth is either empty or a curve, hence does not contain isolated points. By construction, $\pi|_{S'}$ is not smooth in x_0 and, by (a), the point $x_0 \in S'$ is an isolated point with this property. Thus we end up with a contradiction which finally proves the proposition. □

Corollary 1. *If X and Y are both smooth and the $\mathbb{C}^+$-action fixed point free then the quotient morphism $\pi : X \to Y$ is smooth.*

Remark 1. In case $X = \mathbb{C}^3$ this is proved in [**DeF00**] by a direct argument. In this situation the quotient is $\mathbb{C}^2$ by a result of Miyanishi [**Mi80**].

Remark 2. It is shown in [**KaS04**] that for any smooth contractible affine threefold X the quotient $X /\!\!/ \mathbb{C}^+$ is smooth. Thus the quotient map is smooth in this case. (Here we use the fact that a contractible affine smooth threefold is factorial.)

It is helpful to compare the results of this and the subsequent sections with the following example of a non-free $\mathbb{C}^+$-action on $\mathbb{C}^3$.

Example. Consider the $\mathbb{C}^+$-action on $\mathbb{C}^3$ given by the nilpotent derivation $\delta := x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$. Then the quotient $\mathbb{C}^3 /\!\!/ \mathbb{C}^+$ is isomorphic to $\mathbb{C}^2$ and the quotient morphism is given by

$$\pi : \mathbb{C}^3 \to \mathbb{C}^2, \quad (x, y, z) \mapsto (x, xz + y^2).$$

The fixed point set $(\mathbb{C}^3)^{\mathbb{C}^+}$ is the z-axis $\{x = y = 0\} \subset \mathbb{C}^3$. It equals $\pi^{-1}(0)$ and coincides with the points where π is not smooth. The fibers $\pi^{-1}((0, v))$ for $v \neq 0$ consist of two orbits, and $\pi : \mathbb{C}^3 \setminus \{x = 0\} \to \mathbb{C}^2 \setminus \{u = 0\}$ is a trivial $\mathbb{C}^+$-bundle.

3. Configuration of "bad" curves

From now on we make the following assumptions:

X is an affine, factorial and smooth variety of dimension 3 with a fixed point free $\mathbb{C}^+$-action such that the quotient $Y := X /\!\!/ \mathbb{C}^+$ is smooth.

It then follows from Proposition 1 that the quotient morphism $\pi : X \to Y$ is smooth. Consider the following subset $E \subset Y$:

$$E := \{y \in Y \mid \pi^{-1}(y) \text{ is not an orbit}\}.$$

Observe that E contains the finite set $Y \setminus \pi(X)$.

Proposition 2. *If E is finite, then X is a principal $\mathbb{C}^+$-bundle over $\pi(X)$. In particular, $E = Y \setminus \pi(X)$.*

Proof. Consider the fiber product

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y \end{array}$$

where the horizontal π is considered as (a smooth) base change and so the $\mathbb{C}^+$-action on Z is such that μ is equivariant. In addition, π_Z is the quotient morphism since almost all fibers are orbits, by assumption (use [**Kr85, II.3.4 Lemma and Bemerkung**]). Clearly, π_Z (and hence Z) is smooth. Moreover, there is an equivariant morphism

$$\varphi: \mathbb{C}^+ \times X \to Z$$

such that $\mu \circ \varphi_Z = \mathrm{pr}_X$, the projection onto X, and $\mu \circ \varphi: \mathbb{C}^+ \times X \to X$ is given by the $\mathbb{C}^+$-action on X. Define $\tilde{E} := \pi^{-1}(E) \subset X$. Since, by construction, every fiber of π over $Y \setminus E$ is an orbit, the same holds for the fibers of π_Z over $X \setminus \tilde{E}$, and so φ induces a bijective morphism

$$\varphi': \mathbb{C}^+ \times X \setminus \tilde{E} \xrightarrow{\sim} Z \setminus \pi_Z^{-1}(\tilde{E}).$$

Since E is finite we see that $\operatorname{codim}_X \tilde{E} \geq 2$ and so $\operatorname{codim}_Z \pi_Z^{-1}(\tilde{E}) \geq 2$. Hence φ is an isomorphism (see [**Kr85, II.3.4**]). Thus the morphism $\pi: X \to \pi(X)$ becomes a trivial $\mathbb{C}^+$-bundle after a smooth surjective base change, and the claim follows. □

For a (complex) variety Z we denote by $\chi(Z)$ its *Euler characteristic* with respect to the usual $\mathbb{C}$-topology. We refer to [**KrP85, Appendix**] for some basic properties of χ with respect to complex algebraic varieties and for more details.

Corollary 2. *If E is finite then the following statements are equivalent:*

(i) $\chi(X) = \chi(Y)$;
(ii) *π is surjective;*
(iii) *$\pi: X \to Y$ is a trivial $\mathbb{C}^+$-bundle;*
(iv) *The $\mathbb{C}^+$-action is a translation, i.e. $X \simeq \mathbb{C}^+ \times S$ for some closed $S \subset X$.*

Proof. Since X is a locally trivial $\mathbb{C}^+$-bundle over $\pi(X)$ we have $\chi(X) = \chi(\pi(X))$. Moreover, $\chi(\pi(X)) = \chi(Y) - d$ where d is the cardinality of $Y \setminus \pi(X)$ (see [**KrP85, Appendix**]). This proves the equivalence of (i) and (ii).

The assertion (iii) follows from (ii) since every principal $\mathbb{C}^+$-bundle over an affine variety is trivial, and (iii) obviously implies (iv). Finally, if $X \simeq \mathbb{C}^+ \times S$ then the quotient morphism $\pi: X \to Y$ corresponds to the projection $\mathbb{C}^+ \times S \to S$, and so π is surjective. Hence (ii). □

The second part of Proposition 2 generalizes to the following results which states that the only possible isolated points of $\overline{E}$ are in $Y \setminus \pi(X)$.

Corollary 3. *If the set E is not finite then $\overline{E \cap \pi(X)}$ is equidimensional of dimension 1 and the induced morphism*

$$X' := X \setminus \pi^{-1}(\overline{E}) \to \pi(X')$$

is a principal $\mathbb{C}^+$-bundle.

PROOF. Let $C \subset \overline{E \cap \pi(X)}$ be the union of the 1-dimensional components. Then C is defined by an invariant function $f \in \mathcal{O}(X)^{\mathbb{C}^+} = \mathcal{O}(Y)$. Define the principal open sets $X_f \subset X$ and $Y_f \subset Y$ as usual. Then $X_f = X \setminus \pi^{-1}(C)$ is a $\mathbb{C}^+$-stable affine open subvariety of X. Moreover, $\pi|_{X_f}: X_f \to Y_f$ is the quotient morphism (see Remark 3 below). Since $E \cap Y_f$ is finite we can apply Proposition 2 and conclude that $\pi|_{X_f}: X_f \to \pi(X_f)$ is a principal $\mathbb{C}^+$-bundle. Thus $\pi(X_f) \cap E = \emptyset$. In addition, $\pi(X) \cap Y_f = \pi(X_f)$, because $X_f = \pi^{-1}(Y_f)$. Hence

$$\emptyset = E \cap \pi(X_f) = E \cap \pi(X) \cap Y_f = \overline{E \cap \pi(X)} \cap Y_f,$$

and the claim follows. □

REMARK 3. Let X be an irreducible affine variety with a $\mathbb{C}^+$-action. Then we have $\mathrm{Quot}(\mathcal{O}(X)^{\mathbb{C}^+}) = \mathrm{Quot}(\mathcal{O}(X))^{\mathbb{C}^+}$. In fact, let $r \in \mathrm{Quot}(\mathcal{O}(X))^{\mathbb{C}^+}$ be an invariant rational function. Then $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid qr \in \mathcal{O}(X)\}$ is a $\mathbb{C}^+$-stable ideal in $\mathcal{O}(X)$ and so $\mathfrak{a}^{\mathbb{C}^+} \neq 0$ which implies that $r \in \mathrm{Quot}(\mathcal{O}(X)^{\mathbb{C}^+})$. More generally, if $f \in \mathcal{O}(X)^{\mathbb{C}^+}$ is an invariant function then $(\mathcal{O}(X)_f)^{\mathbb{C}^+} = (\mathcal{O}(X)^{\mathbb{C}^+})_f$.

Assume now that E is not finite and let C_i be an irreducible component of $C = \overline{E \cap \pi(X)}$ of dimension 1. Then $S_i := \pi^{-1}(C_i)$ is irreducible and reduced of dimension 2, $\pi|_{S_i}: S_i \to C_i$ is smooth (Proposition 1) and there are infinitely many fibers of $\pi|_{S_i}$ which have more than one component. Thus we get a factorization

$$\pi|_{S_i} = \varphi_i \circ \pi_{S_i}: S_i \to S_i/\!\!/\mathbb{C}^+ \to C_i$$

where $\varphi_i: S_i/\!\!/\mathbb{C}^+ \to C_i$ has degree > 1. It also follows that the fibers of the quotient morphism π_{S_i} are reduced, hence smooth. Now we have the following general result.

LEMMA 2. *Let S_0 be an irreducible affine surface with a non-trivial $\mathbb{C}^+$-action.*
(a) *The quotient $S_0/\!\!/\mathbb{C}^+$ is an irreducible curve and $\pi: S_0 \to S_0/\!\!/\mathbb{C}^+$ is surjective.*
(b) *There is a dense open set $U \subset S_0/\!\!/\mathbb{C}^+$ and an isomorphism $\mathbb{C}^+ \times U \xrightarrow{\sim} \pi^{-1}(U)$ over U.*
(c) *Assume that the action is fixed point free and that the fibers of π are reduced. Then the following are equivalent:*
(i) *S_0 is smooth.*
(ii) *S_0 is normal.*
(iii) *$S_0/\!\!/\mathbb{C}^+$ is normal, hence smooth.*
If (i), (ii) *or* (iii) *holds then π is smooth.*

PROOF. (a) Assume that π is not surjective. Then there is a rational function r on $S_0/\!\!/\mathbb{C}^+$ which is defined on $\pi(S_0)$ but not on $S_0/\!\!/\mathbb{C}^+$. Then $\pi^*(r) \in \mathcal{O}(S_0)^{\mathbb{C}^+}$ and so $r \in \mathcal{O}(S_0/\!\!/\mathbb{C}^+)$, contradicting the assumption.

(b) This is a general fact: For every non-trivial $\mathbb{C}^+$-action on an irreducible affine variety Z there is an invariant function $f \neq 0$ and a closed subset $T \subset Z_f$ such that the action induces an isomorphism $\mathbb{C}^+ \times T \xrightarrow{\sim} Z_f$.

(c) We clearly have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). By assumption, the fibers of π are reduced and are finite unions of orbits, hence smooth. As a consequence, we have $T_x(\mathbb{C}^+x) = \mathrm{Ker}(d\pi)_x$ for every $x \in S_0$, and so S_0 is smooth in x if and only if $S_0/\!\!/\mathbb{C}^+$ is smooth in $\pi(x)$, and in this case π is smooth in x. □

Let us go back to the situation above where $C = \overline{E \cap \pi(X)} = \bigcup_i C_i$ and $S_i = \pi^{-1}(C_i)$, and consider the factorization as above:

$$S_i \xrightarrow{\pi_{S_i}} D_i := S_i /\!\!/ \mathbb{C}^+ \xrightarrow{\varphi_i} C_i.$$

LEMMA 3. *In the following commutative diagram, let the vertical maps η_{S_i} and η_{C_i} denote the normalizations. Then the outer square is Cartesian, $\tilde{S}_i$ is smooth, the morphism $\tilde{\varphi}_i : \tilde{S}_i /\!\!/ \mathbb{C}^+ \to \tilde{C}_i$ is smooth and η is birational.*

$$\begin{array}{ccccc}
\tilde{S}_i & \xrightarrow{\pi_{\tilde{S}_i}} & \tilde{S}_i /\!\!/ \mathbb{C}^+ & \xrightarrow{\tilde{\varphi}_i} & \tilde{C}_i \\
\downarrow{\scriptstyle \eta_{S_i}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta_{C_i}} \\
S_i & \xrightarrow{\pi_{S_i}} & S_i /\!\!/ \mathbb{C}^+ & \xrightarrow{\varphi_i} & C_i
\end{array}$$

PROOF. Since $\pi|_{S_i} : S_i \to C_i$ is smooth the fiber product $\tilde{S}_i := S_i \times_{C_i} \tilde{C}_i$ is a smooth irreducible surface and so the induced (finite and birational) morphism $\tilde{S}_i \to S_i$ is the normalization. Therefore, the quotient $\tilde{S}_i /\!\!/ \mathbb{C}^+$ is normal, hence smooth, and the composition $\tilde{S}_i \xrightarrow{\pi_{\tilde{S}_i}} \tilde{S}_i /\!\!/ \mathbb{C}^+ \to \tilde{C}_i$ is smooth. If follows that $\tilde{S}_i /\!\!/ \mathbb{C}^+ \to \tilde{C}_i$ is smooth (because $\pi_{\tilde{S}_i}$ is surjective by Lemma 2(a)). Since π_{S_i} is the trivial $\mathbb{C}^+$-bundle over a dense open set (Lemma 2(b)) we see that η is birational. □

REMARK 4. We believe that $\eta : \tilde{S}_i /\!\!/ \mathbb{C}^+ \to S_i /\!\!/ \mathbb{C}^+$ is finite, i.e. that η is the normalization map.

For a general variety Z with irreducible components $Z = \bigcup_i Z_i$ we denote by $\tilde{Z}$ the disjoint union of the normalizations $\tilde{Z}_i$ of the components Z_i and will call the canonical morphism $\eta_Z : \tilde{Z} \to Z$ the normalization of Z. Now Lemma 3 immediately implies the following result.

LEMMA 4. *In the following diagram where η_S and η_C are the normalizations the outer square is cartesian, $\tilde{S}$ is smooth and the morphisms $\pi_{\tilde{S}}$ and $\tilde{\varphi}$ are both smooth.*

$$\begin{array}{ccccc}
\tilde{S} & \xrightarrow{\pi_{\tilde{S}}} & \tilde{D} := \tilde{S} /\!\!/ \mathbb{C}^+ & \xrightarrow{\tilde{\varphi}} & \tilde{C} \\
\downarrow{\scriptstyle \eta_S} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta_C} \\
S & \xrightarrow{\pi_S} & D := S /\!\!/ \mathbb{C}^+ & \xrightarrow{\varphi} & C
\end{array}$$

4. Topological considerations (following Shulim Kaliman)

We follow here the ideas of Shulim Kaliman [**Ka04**] to give a proof of the conjecture of §1, based on some topological considerations. In fact, it suffices to assume that the ambient variety X is smooth and contractible and that the quotient Y is smooth. (The latter is even automatically the case as shown in the subsequent paper [**KaS04**].)

MAIN THEOREM. *Let X be a 3-dimensional smooth affine variety with a free $\mathbb{C}^+$-action. Assume that X is factorial, that $H_2(X) = H_3(X) = 0$ and that $Y := X/\!\!/\mathbb{C}^+$ is smooth. Then the action is a translation, i.e. X is $\mathbb{C}^+$-isomorphic to $\mathbb{C}^+ \times Y$.*

We use the notation introduced in the previous section. Clearly, there is a finite subset $\mathcal{F} \subset C$ such that the following holds:
1. $C' := C \setminus \mathcal{F}$ is smooth,
2. $D' := D \setminus \varphi^{-1}(\mathcal{F})$ is smooth,
3. $\varphi' := \varphi|_{D'}: D' \to C'$ is unramified and surjective,
4. $S' := S \setminus \pi^{-1}(\mathcal{F})$ is smooth and $\pi_{S'}: S' \to D'$ is a trivial $\mathbb{C}^+$-bundle.

Put $Y' := Y \setminus \mathcal{F}$, $X' := X \setminus \pi^{-1}(\mathcal{F})$, $Y^0 := Y \setminus C$ and $X^0 := X \setminus S$. Then $X^0 \subset X' \subset X$ and $Y^0 \subset Y' \subset Y$, and $\pi^0: X^0 \to \pi(X^0) \subset Y^0$ is a principal $\mathbb{C}^+$-bundle by Proposition 2. We get the following exact homology sequence where the vertical maps are induced by π:

$$\begin{array}{ccccccccccc}
\longrightarrow & H_{j+1}(X^0) & \longrightarrow & H_{j+1}(X') & \longrightarrow & H_{j+1}(X',X^0) & \longrightarrow & H_j(X^0) & \longrightarrow & H_j(X') & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_{j+1}(Y^0) & \longrightarrow & H_{j+1}(Y') & \longrightarrow & H_{j+1}(Y',Y^0) & \longrightarrow & H_j(Y^0) & \longrightarrow & H_j(Y') & \longrightarrow \cdots
\end{array}$$

In the sequel we will frequently use the Thom isomorphism in the following form (see [**Do72, VIII.11**]).

Thom Isomorphism. *Let Z be a smooth variety and $A \subset Z$ a closed smooth subvariety of codimension d. Put $Z^0 := Z \setminus A$. Then there is an isomorphism*

$$H_j(Z, Z^0) \xrightarrow{\sim} H_{j-2d}(A).$$

In particular, $H_j(Z, Z^0) = 0$ for $j < 2d$ and so $H_j(Z^0) \xrightarrow{\sim} H_j(Z)$ for $j < 2d - 1$.

Recall that the homology of an affine variety Z vanishes above $\dim Z$ (see [**Mi63**]). In particular, $H_j(Y) = H_j(Y^0) = 0$ for $j > 2$ and $H_j(X) = H_j(X^0) = 0$ for $j > 3$. Since $\pi^{-1}(\mathcal{F})$ consists of a finite number of $\mathbb{C}^+$-orbits we obtain $H_i(\pi^{-1}(\mathcal{F})) = 0$ for $i > 0$ and therefore get, from the Thom isomorphism and the exact sequence of the pair (X, X'),

$$H_i(X') \xrightarrow{\sim} H_i(X) \text{ for } i = 0,1,2 \quad \text{and} \quad H_j(X') = H_j(X) = 0 \text{ for } j \geq 4,$$

together with an exact sequence

$$0 \to H_0(\pi^{-1}(\mathcal{F})) \to H_3(X') \to H_3(X) \to 0.$$

Similarly, $H_i(Y') \xrightarrow{\sim} H_i(Y)$ for $i = 0,1,2$, $H_j(Y') = H_j(Y) = 0$ for $j \geq 4$ and $H_0(\mathcal{F}) \simeq H_3(Y')$. Moreover, $H_j(X^0) \xrightarrow{\sim} H_j(\pi(X^0))$ for all j since $X^0 \to \pi(X^0)$ is a principal $\mathbb{C}^+$-bundle. Hence $H_2(X^0) \xrightarrow{\sim} H_2(\pi(X)) \xrightarrow{\sim} H_2(Y^0)$, because $Y^0 \setminus \pi(X^0)$ is a finite set. Finally, $H_3(X', X^0) \xrightarrow{\sim} H_1(S') \simeq H_1(D')$ and $H_3(Y', Y^0) \simeq H_1(C')$.

Now assume that $H_2(X) = 0$. Then we get the following commutative diagram with exact rows where θ is induced by $\varphi: D \to C$:

$$(*) \qquad \begin{array}{ccccccccc}
H_3(X^0) & \longrightarrow & H_3(X') & \xrightarrow{\rho_X} & H_1(D') & \xrightarrow{\mu_X} & H_2(X^0) & \longrightarrow & 0 \\
\downarrow & & \delta' \downarrow & & \theta \downarrow & & \simeq \downarrow & & \downarrow \\
0 & \longrightarrow & H_3(Y') & \xrightarrow{\rho_Y} & H_1(C') & \xrightarrow{\mu_Y} & H_2(Y^0) & \longrightarrow & H_2(Y')
\end{array}$$

It is well-known and easy to see that the normalization $\eta_D\colon \tilde{D} \to D$ induces an injection $H_1(\tilde{D}) \hookrightarrow H_1(D)$ so that the two natural maps

$$\iota_D\colon H_1(D') \to H_1(D) \quad \text{and} \quad \iota_{\tilde{D}}\colon H_1(D') \to H_1(\tilde{D})$$

have the same kernel. Moreover, $\iota_{\tilde{D}}$ is surjective. Clearly, the same holds for the curve C and so, using again the Thom Isomorphism, we get the following diagram with exact rows:

$$(**)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H_0(\tilde{D}\setminus D') & \longrightarrow & H_1(D') & \xrightarrow{\iota_{\tilde{D}}} & H_1(\tilde{D}) & \longrightarrow & 0 \\ & & \theta_0\big\downarrow & & \theta\big\downarrow & & \tilde{\theta}\big\downarrow & & \\ 0 & \longrightarrow & H_0(\tilde{C}\setminus C') & \longrightarrow & H_1(C') & \xrightarrow{\iota_{\tilde{C}}} & H_1(\tilde{C}) & \longrightarrow & 0 \end{array}$$

In other words, the kernel of the homomorphism $\iota_{\tilde{C}}$ and hence of ι_C is generated by simple loops around the points of $\tilde{C} \setminus C'$. The crucial point is to show that $\tilde{\theta}$ is an isomorphism.

Proposition 3. *Assume $H_2(X) = H_3(X) = 0$. Then $C \subset \pi(X)$. In particular, $\varphi\colon D \to C$ and $\tilde{\varphi}\colon \tilde{D} \to \tilde{C}$ are both surjective.*

Proof. If $C \not\subset \pi(X)$ then there is a point $y_0 \in \mathcal{F}$ which is not in $\pi(X)$ (by assumption (3) on the set $\mathcal{F}$, see above). The isomorphism $H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y')$ maps the point y_0 to the class ω given by a 3-sphere in Y' around y_0. The image $\gamma_C := \rho_Y(\omega) \in H_1(C')$ is the sum of simple loops $\gamma_i \subset \Gamma_i$ around y_0 in the different analytic branches Γ_i of C through y_0. These loops can be lifted to D' which implies that there is an element $\gamma_D \in H_1(D')$ such that $\theta(\gamma_D) = m\gamma_C$ with an integer $m > 0$, see the diagram $(*)$ above. It follows that $\mu_X(\gamma_D) = 0$ and so $\gamma_D = \rho_X(\omega_X)$ for some $\omega_X \in H_3(X')$ such that $\delta'(\omega_X) = m\omega$. Now we have the commutative diagram

$$\begin{array}{ccc} H_3(X') & \xrightarrow{\tau_X} & H_3(X) = 0 \\ \delta'\big\downarrow & & \delta\big\downarrow \\ H_3(Y') & \xrightarrow{\tau_Y} & H_3(\pi(X)) \end{array}$$

By construction, $\tau_Y(\omega) \neq 0$ since $y_0 \notin \pi(X)$, and so $\tau_Y(\delta'(\omega_X)) \neq 0$ which leads to a contradiction. Thus $\pi\colon X \to Y$ is surjective, as well as $\varphi\colon D \to C$. Now Lemma 4 implies that $\tilde{\varphi}\colon \tilde{D} \to \tilde{C}$ is surjective, too. $\square$

Remark 5. It follows from the diagram $(**)$ above and Proposition 3 that the homomorphism $\theta\colon H_1(D') \to H_1(C')$ maps the kernel of $\iota_{\tilde{D}}$ surjectively onto the kernel of $\iota_{\tilde{C}}$, because $\tilde{D} \setminus D' \to \tilde{C} \setminus C'$ is surjective.

Remark 6. In case $X = \mathbb{C}^3$ it was shown by Bonnet in [**Bo02**] that the quotient map π is surjective which implies the result above in this case.

Proposition 4. *Assume that $H_2(X) = H_3(X) = 0$. Then, with the notation above,*

$$\rho_X(H_3(X')) \subset \operatorname{Ker}(\iota_D\colon H_1(D') \to H_1(D)),$$
$$\rho_Y(H_3(Y')) \subset \operatorname{Ker}(\iota_C\colon H_1(C') \to H_1(C))$$

(see the diagram $(*)$ above).

PROOF. Let us first look at the map ρ_Y and recall that $Y' = Y \setminus \mathcal{F}$. The isomorphism $H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y')$ says that $H_3(Y')$ is generated by small spheres around the points of $\mathcal{F}$. Thus we get a sequence of maps

$$H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y') \xrightarrow{\rho_Y} H_3(Y', Y^0) \xrightarrow{\sim} H_1(C') \xrightarrow{\iota_C} H_1(C)$$

and we have to show that the composition is zero. For every $y \in \mathcal{F}$ we choose a small open neighborhood $U_y \subset Y$ such that $U_y \cap C$ is contractible. Put $U := \bigcup_y U_y$, $U' := U \cap Y'$ and $U^0 := U \cap Y^0 = U \setminus C$. The functoriality of the Thom isomorphism with respect to open embeddings (see [**Do72, VIII.11**]) implies the following commutative diagram (were we use $U \setminus U' = Y \setminus Y' = \mathcal{F}$)

$$\begin{array}{ccccccc}
H_0(U \setminus U') & \longrightarrow & H_3(U') & \longrightarrow & H_1(C' \cap U) & \longrightarrow & H_1(C \cap U) = 0 \\
\| & & \downarrow & & \downarrow & & \downarrow \\
H_0(\mathcal{F}) & \xrightarrow{\simeq} & H_3(Y') & \xrightarrow{\rho_Y} & H_1(C') & \xrightarrow{\iota_C} & H_1(C)
\end{array}$$

and the claim follows.

For X and the map ρ_X we get a similar diagram (using $H_3(X) = 0$):

$$\begin{array}{ccccccccc}
H_0(X \setminus X') & \xrightarrow{\simeq} & H_3(X') & \xrightarrow{\rho_X} & H_3(X', X^0) & \xrightarrow{\simeq} & H_1(S') & \longrightarrow & H_1(S) \\
 & & & & & & \simeq\downarrow & & \downarrow \\
 & & & & & & H_1(D') & \xrightarrow{\iota_D} & H_1(D)
\end{array}$$

We have to show that the composition $H_0(X \setminus X') \to H_1(D)$ is zero. This time we choose for every orbit $O \subset X \setminus X'$ a small open neighborhood $U_O \subset X$ with the property that the image V_O of $S \cap U_O$ in D is contractible. Setting $U := \bigcup_O U_O$, $U' := U \cap X'$, $U^0 := U \cap X^0$ and $V = \bigcup V_O$ we obtain, as above, the commutative diagram

$$\begin{array}{ccccccccc}
H_0(U \setminus U') & \longrightarrow & H_3(U') & \longrightarrow & H_1(S' \cap U) & \longrightarrow & H_1(S \cap U) & \longrightarrow & H_1(V) = 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_0(X \setminus X') & \xrightarrow{\simeq} & H_3(X') & \xrightarrow{\rho_X} & H_1(S') & \longrightarrow & H_1(S) & \longrightarrow & H_1(D)
\end{array}$$

Hence, the horizontal composition of the maps is zero and the claim follows. □

PROOF OF THE MAIN THEOREM. We look again at the diagram $(**)$ and claim that $\tilde{\theta}$ is an isomorphism. It follows from Proposition 4 that $\iota_{\tilde{D}}$ factors through μ_X and $\iota_{\tilde{C}}$ through μ_Y (see the diagram $(*)$), inducing the diagram

$$\begin{array}{ccccccc}
H_1(D') & \xrightarrow{\mu_X} & H_2(X^0) & \xrightarrow{\bar{\iota}_D} & H_1(\tilde{D}) & \longrightarrow & 0 \\
\theta\downarrow & & \simeq\downarrow & & \tilde{\theta}\downarrow & & \\
H_1(C') & \xrightarrow{\mu_Y} & H_2(Y^0) & \xrightarrow{\bar{\iota}_C} & H_1(\tilde{C}) & \longrightarrow & 0
\end{array}$$

where the composition of the horizontal maps are $\iota_{\tilde{D}}$ and $\iota_{\tilde{C}}$, respectively. Since θ maps the kernel of $\iota_{\tilde{D}}$ surjectively onto the kernel of $\iota_{\tilde{C}}$ (Remark 5) we see that the isomorphism $H_2(X^0) \xrightarrow{\sim} H_2(Y^0)$ maps the kernel of $\bar{\iota}_D$ onto the kernel of $\bar{\iota}_C$. Hence $\tilde{\theta}$ is an isomorphism, too.

By definition, $H_1(\tilde{C}) = \bigoplus_i H_1(\tilde{C}_i)$ and $H_1(\tilde{D}) = \bigoplus_i H_1(\tilde{D}_i)$ and so the (unramified) morphisms $\tilde{D}_i \to \tilde{C}_i$ induce an isomorphisms $H_1(\tilde{D}_i) \xrightarrow{\sim} H_1(\tilde{C}_i)$ for all i. The next well-known lemma shows that $\tilde{\varphi}$ is an isomorphism which contradicts our assumption that the set E is not finite. Therefore, by Proposition 2, the morphism $\pi\colon X \to \pi(X)$ is a principal $\mathbb{C}^+$-bundle. The claim follows from Corollary 2 once we show that $\pi(X) = Y$. But this follows from the exact homology sequence for the pair $(Y, \pi(X))$ and the Thom Isomorphism:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_4(Y) & \longrightarrow & H_4(Y,\pi(X)) & \longrightarrow & H_3(\pi(X)) & \longrightarrow & \cdots \\ & & \| & & \simeq\downarrow & & \| & & \\ & & 0 & & H_0(Y \setminus \pi(X)) & & 0 & & \end{array}$$

□

LEMMA 5. *Let C_0, D_0 be irreducible affine smooth curves and $\varphi\colon D_0 \to C_0$ an unramified surjective morphism. Assume that φ induces an isomorphism $H_1(D_0) \xrightarrow{\sim} H_1(C_0)$. Then φ is an isomorphism.*

PROOF. Assume that the morphism φ is of degree $d > 1$. Let $\mathcal{F} \subset C_0$ be the (finite) subset of those points y where $\#\varphi^{-1}(y) < d$. Then we obtain the following relation between the Euler characteristics of the curves C_0 and D_0:

$$\chi(D_0) - \#\varphi^{-1}(\mathcal{F}) = d\,(\chi(C_0) - \#\mathcal{F}).$$

By assumption, $\chi(D_0) = \chi(C_0)$ and $\#\varphi^{-1}(\mathcal{F}) \leq d\,\#\mathcal{F}$ and so $(d-1)\chi(C_0) \geq 0$. This implies that C_0 and D_0 are both isomorphic to $\mathbb{C}$ or to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The first case cannot occur since there are no unramified morphisms $\mathbb{C} \to \mathbb{C}$ of degree > 1. In the second case there is, up to isomorphism only one morphism $\mathbb{C}^* \to \mathbb{C}^*$ of degree d, namely $z \mapsto z^d$, and this one induces the map $d \cdot \mathrm{Id}$ on $H_1(\mathbb{C}^*) \simeq \mathbb{Z}$ which is not an isomorphism. □

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