POLARIZATIONS AND NULLCONE OF REPRESENTATIONS OF REDUCTIVE GROUPS

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Abstract. The paper starts with the following simple observation. Let V be a representation of a reductive group G, and let $f_1, f_2, \ldots, f_n$ be homogeneous invariant functions. Then the polarizations of $f_1, f_2, \ldots, f_n$ define the nullcone of $k \leq m$ copies of V if and only if every linear subspace L of the nullcone of V of dimension $\leq m$ is annhilated by a one-parameter subgroup (shortly a 1-PSG). This means that there is a group homomorphism $\lambda\colon \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)x = 0$ for all $x \in L$.

This is then applied to many examples. A surprising result is about the group SL_2 where almost all representations V have the property that all linear subspaces of the nullcone are annihilated. Again, this has interesting applications to the invariants on several copies.

Another result concerns the n-qubits which appear in quantum computing. This is the representation of a product of n copies of SL_2 on the n-fold tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Here we show just the opposite, namely that the polarizations never define the nullcone of several copies if $n \geq 3$.

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1. Linear subspaces of the nullcone

In this paper we study finite dimensional complex representations of a reductive algebraic group G. It is a well-known and classical fact that the nullcone $\mathcal{N}_V$ of such a representation V plays a fundamental role in the geometry of the representation. Recall that $\mathcal{N}_V$ is defined to be the union of all G-orbits in V containing the origin 0 in their closure. Equivalently, $\mathcal{N}_V$ is the zero set of all non-constant homogeneous G-invariant functions on V.

In a previous paper [KrW06] we have seen that certain linear subspaces of the nullcone play a central role for understanding its irreducible components. In this paper we will discuss arbitrary linear subspaces of the nullcone $\mathcal{N}_V$ of a representation V of a reductive group G and show how they relate to questions about system of generators and systems of parameter for the invariants.

We first recall the definition of a *polarization* of a regular function $f \in \mathcal{O}(V)$. For $k \geq 1$ and arbitrary parameters $t_1, \ldots, t_k$ we write

$$f(t_1v_1 + t_2v_2 + \cdots + t_kv_k) = \sum_{i_1,i_2,\ldots,i_k} P_{i_1,\cdots,i_k} f(v_1, \ldots, v_k) \cdot t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k}. \tag{1}$$

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Then the regular functions $P_{i_1,\cdots,i_k}f$ defined on the sum $V^{\oplus k}$ of k copies of the original representation V are called *polarizations* of f. Here are a few well-known and easy facts.

(a) If f is homogeneous of degree d then $P_{i_1,\cdots,i_k}f$ is multihomogeneous of multidegree $(i_1,\cdots,i_k)$ and thus $i_1+\cdots+i_k=d$ unless $P_{i_1,\cdots,i_k}f=0$.
(b) If f is G-invariant then so are the polarizations.
(c) For a subset $A\subset\mathcal{O}(V)$ the algebra $\mathbb{C}[PA]\subset\mathcal{O}(V^{\oplus k})$ generated by the polarizations Pa, $a\in A$, contains all polarizations Pf for $f\in\mathbb{C}[A]$.

It is easily seen from examples that, in general, the polarizations of a system of generators do not generate the invariant ring of more than one copy (see [Sch07]). However, we might ask the following question.

Main Question. *Given a set of invariant functions $f_1,\ldots,f_m$ defining the nullcone of a representation V, when do the polarizations define the nullcone of a direct sum of several copies of V?*

From now on let G denote a connected reductive group. An important tool in the context is the Hilbert-Mumford criterion which says that a vector $v\in V$ belongs to the nullcone $\mathcal{N}_V$ if and only if there is a one-parameter subgroup (abbreviated: 1-PSG) $\lambda^*\colon\mathbb{C}^*\to G$ such that $\lim_{t\to 0}\lambda(t)v=0$ ([Kr85, Kap. II]). We will say that a 1-PSG λ *annihilates a subset* $S\subset V$ if $\lim_{t\to 0}\lambda(t)v=0$ for all $v\in S$.

Proposition 1. *Let V be a representation of G and let $f_1,f_2,\ldots,f_r$ be homogeneous invariants defining the nullcone $\mathcal{N}_V$. For every integer $m\geq 1$ the following statements are equivalent:*

(i) *Every linear subspace $L\subset\mathcal{N}_V$ of dimension $\leq m$ is annihilated by a 1-PSG of G.*
(ii) *The polarizations Pf_i define the nullcone of $V^{\oplus k}$ for all $k\leq m$.*

Proof. By the very definition (1), the polarizations $P_{i_1,\cdots,i_k}f_i$ vanish in a tuple $(v_1,\ldots,v_k)\in V^{\oplus k}$ if and only if the linear span $\langle v_1,\ldots,v_k\rangle$ consists of elements of the nullcone $\mathcal{N}_V$. □

A first application is the following result about commutative reductive groups.

Proposition 2. *Let D be a commutative reductive group and let V be a representation of D. Assume that $\mathcal{O}(V)^D$ is generated by the homogeneous invariants $f_1,\ldots,f_r$. Then the polarizations Pf_i define the nullcone of $V^{\oplus k}$ for any number k of copies of V.*

Proof. The represention V has a basis $(v_1,\ldots,v_n)$ consisting of eigenvectors of D, i.e., there are characters $\chi_i\in X(D)$ $(i=1,\ldots,n)$ such that $hv_i=\chi_i(h)\cdot v_i$ for all $h\in D$. Denote by $x_1,\ldots,x_n$ the dual basis so that $\mathcal{O}(V)=\mathbb{C}[x_1,\ldots,x_n]$. It is well-known that the invariants are generated by the invariant monomials in the x_i. Hence, the nullcone is a union of linear subspaces: $\mathcal{N}_V=\bigcup_j L_j$, where L_j is spanned by a subset of the basis $(v_1,\ldots,v_n)$. If $v\in L_j$ is a general element, i.e. all coordinates are non-zero, and if $\lim_{t\to 0}\lambda(t)v=0$, then λ also annihilates the subspace L_j. Thus every linear subspace of $\mathcal{N}_V$ is annihilated by a 1-PSG. □

Remark 1. The example of the representation of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t(x,y):=(tx,t^{-1}y)$ shows that the polarizations of the invariants do not generate the ring of invariants of more than one copy of $\mathbb{C}^2$.

For the study of linear subspaces of the nullcone the following result turns out to be useful.

Proposition 3. *If there is a linear subspace L of $\mathcal{N}_V$ of a certain dimension d, then there is also a B-stable linear subspace of $\mathcal{N}_V$ of the same dimension where B is a Borel subgroup of G.*

Proof. The set of linear subspaces of the nullcone of a given dimension d is easily seen to form a closed subset Z of the Grassmanian $\mathrm{Gr}_d(V)$. Since Z is also stable under G it has to contain a closed G-orbit. Such an orbit always contains a point which is fixed by B, and this point corresponds to a B-stable linear subspace of V of dimension d. □

2. Some examples

Let us give some instructive examples.

Example 1 (Orthogonal representations)**.** Consider the standard representation of SO_n on $V = \mathbb{C}^n$. Then a subspace $L \subset V$ belongs to the nullcone if and only if L is *totally isotropic* with respect to the quadratic form q on V. Then V can be decomposed in the form $V = V_0 \oplus (L \oplus L')$ such that $q|_{V_0}$ is non-degenerate, L' is totally isotropic and $L \oplus L'$ is the orthogonal complement of V_0. It follows that the 1-PSG λ of $\mathrm{GL}(V)$ given by

$$\lambda(t)v := \begin{cases} t \cdot v & \text{for } v \in L, \\ t^{-1} \cdot v & \text{for } v \in L', \\ v & \text{for } v \in V_0, \end{cases}$$

belongs to SO_n and annihilates L. Therefore, the polarizations of q define the nullcone of any number of copies of $\mathbb{C}^n$. Here the polarizations of q are given by the quadratic form q applied to each copy of V in $V^{\oplus m}$ and the associated bilinear form $\beta(v,w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$ applied to each pair of copies in $V^{\oplus m}$.

Of course, this result is also an immediate consequence of the First Fundamental Theorem for O_n or SO_n (see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]).

Example 2 (Conjugacy classes of matrices)**.** Let GL_3 act on the 3×3-matrices $M_3(\mathbb{C})$ by conjugation and consider the following two matrices:

$$J := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

It is easy to see that $sJ + tN$ is nilpotent for all $s, t \in \mathbb{C}$. However, JN is a non-zero diagonal matrix and so there is no 1-PSG which annihilates the two-dimensional subspace $L := \langle J, N \rangle$ of the nullcone of M_3. It follows that the polarizations of the functions $X \mapsto \operatorname{tr} X^k$ $(1 \le k \le 3)$ do not define the nullcone of two and more copies of M_3.

The polarizations for two copies are the following 9 homogeneous invariant functions defined for $(A, B) \in M_3 \oplus M_3$:

$$\operatorname{tr} A,\ \operatorname{tr} B,\ \operatorname{tr} A^2,\ \operatorname{tr} AB,\ \operatorname{tr} B^2,\ \operatorname{tr} A^3,\ \operatorname{tr} A^2B,\ \operatorname{tr} AB^2,\ \operatorname{tr} B^3.$$

(Use the fact that $\operatorname{tr} ABA = \operatorname{tr} A^2B$ etc.) It is an interesting fact that these 9 functions define a subvariety Z of $M_3 \oplus M_3$ of codimension 9 and so the nullcone of $M_3 \oplus M_3$ is an irreducible component of Z. However, the invariant ring of $M_3 \oplus M_3$

has dimension 10 (= 18−8) and so a system of parameters must contain 10 elements. It was shown by TERANISHI [Te86] that one obtains a system of parameters by adding the function $\operatorname{tr} ABAB$, and a system of generators by adding, in addition, the function $\operatorname{tr} ABA^2B^2$.

Conjecture. *The polarizations of the functions $X \mapsto \operatorname{tr} X^j$ $(j = 1, \dots, n)$ for two copies of M_n define a subvariety Z of codimension $\frac{n^2+3n}{2}$ which is a set-theoretic complete intersections and has the nullcone as an irreducible component.*
(Note that the number of polarizations of these n functions is $2+3+\cdots+(n+1) = \frac{n^2+3n}{2}$ and that this number is also equal to the codimension of the nullcone (see [KrW06, Example 2.1]).

Remark 2. It has been shown by GERSTENHABER [Ge58] that a linear subspace L of the nilpotent matrices $\mathcal{N}$ in M_n of maximal possible dimension $\binom{n}{2}$ (see Proposition 3) is conjugate to the nilpotent upper triangular matrices, hence annihilated by a 1-PSG. Jointly with JAN DRAISMA and JOCHEN KUTTLER we have generalized this result to arbitrary semisimple Lie algebras, see [DKK06].

Example 3 (Symmetric matrices, see [KrW06, Example 2.4])**.** Consider the representation of $G := \mathrm{SO}_4$ on $S_0^2(\mathbb{C}^4)$, the space of trace zero symmetric 4×4-matrices. This is equivalent to the representation of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $V_2 \otimes V_2$ where V_2 is the space of quadratic forms in 2 variables. The invariant ring is a polynomial ring generated by the functions $f_i := \operatorname{tr} X^i$, $2 \leq i \leq 4$. A direct calculation shows that every two-dimensional subspace of the nullcone is annihilated by a 1-PSG. This implies that the polarizations of the functions f_2, f_3, f_4 define the nullcone for two copies of $S_0^2(\mathbb{C}^4)$. Since the number of polarizations is $12 = 3 + 4 + 5$ which is the dimension of the invariant ring (i.e. of the quotient $(S_0^2(\mathbb{C}^4) \oplus S_0^2(\mathbb{C}^4))/\!\!/ \mathrm{SO}_4)$, we see that these 12 polarizations form a system of parameters. (This completes the analysis given in [WaW00].)

These examples show that there are two basic questions in this context:

Question 1. *What are the linear subspaces of the nullcone of a representation V?*

Question 2. *Given a linear subspace $U \subset \mathcal{N}_V$ of the nullcone of a representation V, is there a 1-PSG which annihilates U?*

We now give a general construction where we get a negative answer to Question 2 above. Denote by $\mathbb{C}^2 = \mathbb{C}e_0 \oplus \mathbb{C}e_1$ the standard representation of SL_2.

Proposition 4. *Let V be a representation of a reductive group H. Consider the representation $W := \mathbb{C}^2 \otimes V$ of $G := \mathrm{SL}_2 \times H$.*

(a) *For every $v \in V$ the subspace $\mathbb{C}^2 \otimes v$ belongs to the nullcone $\mathcal{N}_W$.*
(b) *If $v \in V \setminus \mathcal{N}_V$ then there is no 1-PSG λ of G such that* $\lim_{t\to 0} \lambda(t)w = 0$ *for all $w \in \mathbb{C}^2 \otimes v$.*

Proof. (1) Clearly, $e_0 \otimes v \in \mathcal{N}_W$ for any $v \in V$. Hence $\{g\, e_0 \otimes v \mid g \in \mathrm{SL}_2\} \subset N_W$, and the claim follows since $\mathbb{C}^2 \otimes v = \{g\, e_0 \otimes v \mid g \in \mathrm{SL}_2\} \cup \{0\}$.

(2) Assume that $\lim_{t\to 0} \lambda(t)w = 0$ for all $w \in \mathbb{C}^2 \otimes v$. Write $v = \sum v_j$ such that $\lambda(t)v_j = t^j \cdot v_j$ and choose $f \in \mathbb{C}^2$ such that $\lambda(t)f = t^s \cdot f$ where $s \leq 0$. Since $v \notin \mathcal{N}_V$ there exists a $k \leq 0$ such that $v_k \neq 0$. Then $\lambda(t)(f \otimes v_k) = t^{s+k} \cdot (f \otimes v_k)$ which leads to a contradiction since $s + k \leq 0$. □

Corollary 1. *If the representation V admits non-constant G-invariants, then the polarizations of the invariants of $W := \mathbb{C}^2 \otimes V$ do not define the nullcone of 2 or more copies of W.*

Corollary 2. *For $n \geq 3$ the polarizations of the invariants of the n-qubits $Q_n := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (n factors) under $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ do not define the nullcone of two or more copies of Q_n.*

3. General polarizations

For our applications we have to generalize the notion of polarization introduced in Section 1. Let V be a finite dimensional vector space and $f \in \mathcal{O}(V^{\oplus k})$ a (multihomogeneous) regular function on k copies of V. Fixing $m \geq k$ and using parameters t_{ij}, $1 \leq i \leq k$, $1 \leq j \leq m$ where $m \geq k$ we write, for $(v_1, v_2, \ldots, v_m) \in V^{\oplus m}$,

$$f(\sum_j t_{1j} v_j, \sum_j t_{2j} v_j, \cdots, \sum_j t_{kj} v_j) = \sum_A t^A P_A f(v_1, v_2, \ldots, v_m). \tag{2}$$

where $A = (a_{ij})$ runs through the $k \times m$-matrices with non-negative integers a_{ij} and $t^A := \prod_{ij} t_{ij}^{a_{ij}}$. The regular (multihomogeneous) functions $P_A f \in \mathcal{O}(V^{\oplus m})$ obtained in this way are again called *polarizations* of f. As before, if V is a representation of G and f a G-invariant function, then so are the polarizations $P_A f$. The next lemma is an immediate consequence of the definition.

Lemma 1. *Let $f \in \mathcal{O}(V^{\oplus k})$, $v_1, \ldots, v_m \in V$ where $m \geq k$ and denote by $U := \langle v_1, v_2, \ldots, v_m \rangle \subset V$ the linear span of $v_1, \ldots, v_m$. Then the following two statements are equivalent.*

(i) *f vanishes on $U^{\oplus m} \subset V^{\oplus m}$.*
(ii) *$P_A f(v_1, \ldots, v_m) = 0$ for all polarizations $P_A f$ of f.*

Let us go back to the general situation of a representation of a connected reductive group G on a vector space V. Denote by $\mathcal{L}_V$ the set of linear subspaces of V which are annihilated by a 1-PSG of G and which are maximal under this condition, and by $\mathcal{M}_V$ the set of all maximal linear subspaces of the nullcone $\mathcal{N}_V$ of V.

We can regard $\mathcal{L}_V$ and $\mathcal{M}_V$ as closed G-stable subvarieties of the Grassmannian $\mathrm{Gr}(V) = \bigcup_{1 \leq d \leq \dim V} \mathrm{Gr}_d(V)$. We have seen in [KrW06] that $\mathcal{L}_V$ consists of a finite number of closed orbits. In particular, $\dim \mathcal{L}_V \leq \dim G/B$.

Proposition 5. *Let $k < m$ be positive integers and assume that the invariant functions $f_1, \ldots, f_n \in \mathcal{O}(V^{\oplus k})^G$ define the nullcone $\mathcal{N}_{V^{\oplus k}}$. If every linear subspace $U \subset \mathcal{N}_V$ with $k < \dim U \leq m$ is annihilated by a 1-PSG, then the polarizations $P_A f_i$ define the nullcone $\mathcal{N}_{V^{\oplus m}}$ of $V^{\oplus m}$.*

Proof. Assume that for a given $v = (v_1, \ldots, v_m)$ we have $P_A f_i(v_1, \ldots, v_m) = 0$ for all polarizations $P_A f_i$. Define $U := \langle v_1, \ldots, v_m \rangle$. By the lemma above $U^{\oplus k}$ belongs to the nullcone of $V^{\oplus k}$, hence $U \subset \mathcal{N}_V$. If $\dim U > k$, then by assumption U is annihilated by a 1-PSG and so $(v_1, \ldots, v_m) \in \mathcal{N}_{V^{\oplus m}}$.

If $\dim U \leq k$, then, after possible rearrangement of $\{v_1, \ldots, v_m\}$, we can assume that $U = \langle v_1, \ldots, v_k \rangle$. Since $(v_1, \ldots, v_k) \in \mathcal{N}_{V^{\oplus k}}$, by assumption, it follows again that U is annihilated by a 1-PSG. $\square$

Example 4. For the standard representation of SL_n on $V := \mathbb{C}^n$ there are no invariants for less than n copies, and $\mathcal{O}(V^{\oplus n})^{\mathrm{SL}_n} = \mathbb{C}[\det]$. Therefore, the determinants $\det(v_{i_1} v_{i_2} \cdots v_{i_n})$ define the nullcone on any number of copies of V. In fact, one knows that they even generate the ring of invariants, by the so-called "First Fundamental Theorem for SL_n" (see [Pro07, 11.1.2]).

Example 5. For the standard representation of Sp_{2n} on $V := \mathbb{C}^{2n}$ there are no invariants on one copy, and $\mathcal{O}(V \oplus V)^{\mathrm{Sp}_{2n}} = \mathbb{C}[f]$ where $f(u, v)$ is the skew form defining $\mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$. As in the orthogonal case (see Example 1), one easily sees that every linear subspace of the nullcone is annihilated by a 1-PSG. Hence, the skew forms $f_{ij} = f(v_i, v_j)$ define the nullcone of any number of copies of V. Again, the "First Fundamental Theorem" shows that these invariants even generate the invariant ring (see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]).

Example 6 (see Example 2)**.** Applying the proposition to the case of the adjoint representation of GL_n on the matrices M_n we get the following result. *If the invariants $f_1, \ldots, f_k$ define the nullcone of $\binom{n}{2} - 1$ copies of M_n, then the polarizations $P_A f_i$ define the nullcone of any number of copies of M_n.*

For $n = 3$ this implies (see Example 2) that the traces $\{\operatorname{tr} A_i, \operatorname{tr} A_i A_j, \operatorname{tr} A_i A_j A_k, \operatorname{tr} A_i A_j A_k A_\ell\}$ define the nullcone of any number of copies of M_3.

Let m_V denote the maximal dimension of a linear subspace of the nullcone $\mathcal{N}_V$.

Corollary 3. *If $f_1, \ldots, f_n \in \mathcal{O}(V^{\oplus m_V})^G$ define the nullcone $\mathcal{N}_{V^{\oplus m_V}}$, then the polarizations $P_A f_i$ define the nullcone of any number of copies of V.*

4. Nullcone of several copies of binary forms

In this section we study the invariants and the nullcone of representations of the group SL_2. We denote by $V_n := \mathbb{C}[x, y]_n$ the binary forms of degree n considered as a representation of SL_2. Recall that in this setting the form $y^n \in V_n$ is a highest weight vector with respect to the standard Borel subgroup $B \subset \mathrm{SL}_2$ of upper triangular matrices.

The main result of this section is the following.

Theorem 4. *Consider the irreducible representation V_n of SL_2. Assume that $n > 1$ and that the homogeneous invariant functions $f_1, f_2, \ldots, f_m \in \mathcal{O}(V_n)^{\mathrm{SL}_2}$ define the nullcone of V_n. Then the polarizations of the f_i's for any number N of copies of V_n define the nullcone of $V_n^{\oplus N}$.*

The following result is a main step in the proof.

Lemma 2. *Let $h_1, h_2 \in V_n$ be two non-zero binary forms. Assume that every non-zero linear combination $\alpha h_1 + \beta h_2$ has a linear factor of multiplicity $> \frac{n}{2}$. Then h_1 and h_2 have a common linear factor of multiplicity $> \frac{n}{2}$.*

Proof. We can assume that h_1 and h_2 are linearly independent. Fix a number $k \in \mathbb{N}$ such $\frac{n}{2} < k \leq n$ and define the following subsets of $V_n \oplus V_1$:

$$Y_k := \{(f, \ell) \in V_n \oplus V_1 \mid \ell^k \text{ divides } f\}.$$

This is a closed subset of $V_n \oplus V_1$, because $Y_k = \mathrm{SL}_2 \cdot (W \oplus \mathbb{C}y)$ where $W := \bigoplus_{i=k}^n \mathbb{C}x^{n-i}y^i$, and $W \oplus \mathbb{C}y$ is a B-stable linear subspace of $V_n \oplus V_1$. Moreover,

Y_k is stable under the action of $\mathbb{C}^*$ by scalar multiplication on V_1. Therefore, the quotient $Y_k \setminus (W \times \{0\})/\mathbb{C}^*$ is a vector bundle $p\colon \mathcal{V}_k \to \mathbb{P}(V_1)$, namely the subbundle of the trivial bundle $V_n \times \mathbb{P}(V_1)$ whose fiber over $[\ell]$ is the subspace $\ell^k \cdot V_{n-k} \subset V_n$. It is clear that this vector bundle can be identified with the associated bundle $\mathrm{SL}_2 \times^B W \to \mathrm{SL}_2/B = \mathbb{P}^1$.

Now consider the following subset of $\mathbb{C}^2 \times \mathbb{P}(V_1)$

$$\mathcal{L}_k := \{((\alpha, \beta), [\ell]) \in \mathbb{C}^2 \times \mathbb{P}(V_1) \mid \ell^k \text{ divides } \alpha h_1 + \beta h_2\}.$$

$\mathcal{L}_k$ is the inverse image of $\mathcal{V}_k$ under the morphism $\varphi\colon \mathbb{C}^2 \times \mathbb{P}(V_1) \to V_n \times \mathbb{P}(V_1)$ given by $((\alpha, \beta), [\ell]) \mapsto (\alpha h_1 + \beta h_2, [\ell])$, and so $\mathcal{L}_k$ is a closed subvariety of $\mathbb{C}^2 \times \mathbb{P}(V_1)$. Since φ is a closed immersion we can identify $\mathcal{L}_k$ with a closed subvariety of the $\mathcal{V}_k$.

If two linearly independent members f_1, f_2 of the family $\alpha h_1 + \beta h_2$ have the same linear factor ℓ of multiplicity $\geq k$, then all the members of the family have this factor and we are done. Otherwise, the morphism $p\colon \mathcal{L}_k \to \mathbb{P}(V_1)$ induced by the projection is surjective and the fibers are lines of the form $\mathbb{C}f \times \{[\ell]\}$. Hence $\mathcal{L}_k$ is a subbundle of $\mathcal{V}_k$. It follows from the construction of $\mathcal{L}_k$ as a subbundle of the trivial bundle of rank 2 that $\mathcal{L}_k$ is isomorphic to $\mathcal{O}(-1)$. The following Lemma 3 shows that this bundle cannot occur as a subbundle of $\mathrm{SL}_2 \times^B W \to \mathrm{SL}_2/B = \mathbb{P}^1$ provided that $n > 1$. □

Remark 3. It was shown by Matthias Bürgin in his thesis (see [Bü06]) that the following generalization of Lemma 2 holds. *Let $f, h \in \mathbb{C}[t]$ be two polynomials and k an integer ≥ 2. Assume that every linear combination $\lambda f + \mu h$ has a root of multiplicity $\geq k$. Then f and h have a common root of multiplicity $\geq k$.*

Lemma 3. *Denote by V_n^+ the B-stable subspace of V_n consisting of positive weights. Then we have*

$$\mathrm{SL}_2 \times^B V_n^+ \simeq \begin{cases} \mathcal{O}(-k)^k & \text{if } n = 2k-1, \\ \mathcal{O}(-k-1)^k & \text{if } n = 2k. \end{cases}$$

Proof. If M is a B-module we denote by $M(i)$ the module obtained from M by tensoring with the character $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mapsto t^i$. If $\mathcal{V}(M) := \mathrm{SL}_2 \times^B M$ then $\mathcal{V}(M(i)) = \mathcal{V}(M)(-i)$. With this notation we have the following isomorphisms as B-modules:

$$V_{2k-1}^+ \simeq V_{k-1}(k) \quad \text{and} \quad V_{2k}^+ \simeq V_{k-1}(k+1).$$

Since $\mathrm{SL}_2 \times^B V_m$ is the trivial bundle of rank $m+1$ the claim follows. □

Now we can give the proof of our Main Theorem of this section.

Proof of Theorem 4. Let $h = (h_1, h_2, \ldots, h_N) \in V_n^N$ an n-tupel of forms such that all polarizations of all f_i vanish on h. This implies that $f_i(\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_N h_N) = 0$ for all $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{C}^N$ and all i's. It follows that $\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_N h_N$ belongs to the nullcone of V_n for all $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{C}^N$, hence they all have a linear factor ℓ of multiplicity $> \frac{n}{2}$. Using Lemma 2 above, an easy induction shows that the h_i must have a common linear factor ℓ of multiplicity $> \frac{n}{2}$. Thus h belongs to the nullcone of V_n^N. □

From the proof above we immediately get the following generalization of our Theorem 4.

Theorem 5. *Consider the representation* $V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_k}$ *of* SL_2*, where* $1 < n_1 < n_2 < \cdots < n_k$*. Assume that the multihomogeneous invariant functions* $f_1, f_2, \ldots, f_m \in \mathcal{O}(V)^{\mathrm{SL}_2}$ *define the nullcone of* V*. Then the polarizations of the* f_i*'s to the representation* $\tilde{V} = V_{n_1}^{N_1} \oplus V_{n_2}^{N_2} \oplus \cdots \oplus V_{n_k}^{N_k}$ *for any* k*-tuple* $(N_1, N_2, \ldots, N_k)$ *define the nullcone of* $\tilde{V}$*.*

Remark 4. One can also include the case $n_1 = 1$ by either assuming that $N_1 = 1$ or by adding the invariants $[i, j]$ of $V_1^{N_1}$ to the set of polarizations. (Recall that $[i, j](\ell_1, \ldots, \ell_N) := [\ell_i, \ell_j] := \alpha_i\beta_j - \alpha_j\beta_i$ where $\ell_i = \alpha_i x + \beta_i y \in V_1$.) Since the covariants $\mathcal{O}(V)^U$ can be identified with the invariants $\mathcal{O}(V \oplus V_1)$ the theorem above has some interesting consequences for covariants.

Example 7 (Covariants of V_3^N)**.** The covariants of V_3^N can be identified with the invariants of $V_3^N \oplus V_1$. The case $N = 1$ is well-known and classical: $\mathcal{O}(V_3 \oplus V_1)^{\mathrm{SL}_2} = \mathbb{C}[h, f_{1,3}, f_{2,2}, f_{3,3}]$, where h is the discriminant of V_3 and the $f_{i,j}$ are bihomogenous invariants of degree (i, j) corresponding to $V_3 \subset \mathcal{O}(V_3)_1$, $V_2 \subset \mathcal{O}(V_3)_2$ and $V_1 \subset \mathcal{O}(V_3)_3$. Recall that an embedding $V_n \subset \mathcal{O}(V_3)_d$ defines a covariant $\varphi\colon V_3 \to V_n$ of degree d and thus an invariant $f_{d,n}\colon (f, \ell) \mapsto [\varphi(f), \ell^n]$ where the bracket $[\cdot, \cdot]$ denotes the invariant bilinear form on $V_n \times V_n$.

It is easy to see that $h, f_{1,3}, f_{2,2}$ form a system of parameters, i.e. define the nullcone of $V_3 \oplus V_1$. Therefore, their polarizations (in the variables of V_3) define the nullcone of $V_3^N \oplus V_1$ for any $N \geq 1$. Therefore, we always have a system of parameters in degree 4 and thus can easily calculate the Hilbert series for small N, e.g.:

$$\mathrm{Hilb}_{V_3^2 \oplus V_1} = \frac{h_2}{(1-t^2)(1-t^4)^6} \qquad \text{and} \qquad \mathrm{Hilb}_{V_3^3 \oplus V_1} = \frac{h_3}{(1-t^2)^3(1-t^4)^8}$$

where

$$h_2 := 1 + 6t^4 + 13t^6 + 12t^8 + 13t^{10} + 6t^{12} + t^{16}$$

and

$$h_3 := 1 + 24t^4 + 62t^6 + 177t^8 + 300t^{10} + 320t^{12} + 300t^{14} + 177t^{16} + 62t^{18} + 24t^{20} + t^{24}$$

For the calculation we use the fact (due to Knop [Kn89]) that the degree of the Hilbert series is $\leq -\dim V$ and that the numerator is palindromic since the invariant ring is Gorenstein. The Theorem of Weyl implies that the covariants for V_3^N are obtained from those of V_3^3 by polarization. Since the representation is symplectic they are even obtained from V_3^2 by polarization (see Schwarz [Sch87]).

5. Generators and system of parameters for the invariants of 3-qubits

Lemma 4. *Consider the polynomial ring* $\mathbb{C}[a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}]$ *in the coefficients of a quadratic form in 3 variables and put*

$$d := \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Then the elements $\{a_{11}-a_{22},\ a_{22}-a_{33},\ a_{12}, a_{13}, a_{23}, d\}$ *form a homogeneous system of parameters.*

Proof. The proof is easy: One simply shows that the zero set of these functions is the origin. □

Let us now consider N copies of the standard representation $\mathbb{C}^n$ of the complex orthogonal group $\mathrm{O}_n = \mathrm{O}_n(\mathbb{C})$: $W := \mathbb{C}^N \otimes \mathbb{C}^n$. The first fundamental theorems for O_n and SO_n tells us that the invariants under O_n are generated by the quadratic invariants $\sum_{\nu=1}^n x_{i\nu}x_{j\nu}$ $(1 \le i \le j \le N)$ and that for SO_n we have to add the $n \times n$ minors of the matrix $(x_{i\nu})$. In terms of representation theory this means the following. We have (by CAUCHY'S formula)

$$S^2\mathbb{C}^N \otimes \mathbb{C} \subset S^2(\mathbb{C}^N \otimes \mathbb{C}^n) \qquad \text{and} \qquad \bigwedge^n \mathbb{C}^N \otimes \mathbb{C} \subset S^n(\mathbb{C}^N \otimes \mathbb{C}^n),$$

where $\mathbb{C}$ denotes the trivial representation of SO_n, and these subspaces form a generating system for $S(\mathbb{C}^N \otimes \mathbb{C}^n)^{\mathrm{SO}_n}$.

As before we denote by V_m the irreducible representation of SL_2 of dimension $m+1$. We apply the above first to the the case of three copies of the irreducible 3-dimensional representation V_2 of SL_2: $W = \mathbb{C}^3 \otimes V_2$. Then the subspaces $S^2\mathbb{C}^3 \otimes V_0$ and $\bigwedge^3 \mathbb{C}^3 \otimes V_0$ form a minimal generating system for the SL_2-invariants. Thus we get 6 generators in degree 2 and one generator in degree 3.

Now we consider the space $\mathbb{C}^3 \otimes V_2$ as a representation of $\mathrm{SO}_3 \times \mathrm{SL}_2$ and denote the 6 quadratic generators by $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}$ with the obvious meaning. Then the cubic generator q satisfies the relation

$$q^2 = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Moreover, the space $S^2\mathbb{C}^3$ decomposes under SO_3 into the direct sum of two irreducible representations

$$S^2\mathbb{C}^3 = S_0^2\mathbb{C}^3 \oplus \mathbb{C}$$

where $\mathbb{C}$ is the trivial representation. In terms of coordinates, $\mathbb{C}$ is spanned by $a_{11} + a_{22} + a_{33}$ and $S_0^2\mathbb{C}^3$ by $\{a_{11} - a_{22}, a_{22} - a_{33}, a_{12}, a_{13}, a_{23}\}$. With Lemma 6 above we therefore have the following result.

Proposition 6. *Consider the representation* $W := \mathbb{C}^3 \otimes V_2$ *of* $\mathrm{SO}_3 \times \mathrm{SL}_2$*. Then the 5-dimensional subspace* $S_0^2(\mathbb{C}^3) \otimes V_0 \subset S^2(W)$ *together with the 1-dimensional subspace* $\bigwedge^3 \mathbb{C}^3 \otimes V_0 \subset S^3(W)$ *form a homogeneous system of parameters for the invariants* $S(W)^{\mathrm{SL}_2}$.

We want to apply this to the invariants of two copies of 3-qubits, i.e. to the representation

$$V := V_1 \otimes V_1 \otimes V_1 \oplus V_1 \otimes V_1 \otimes V_1 = \mathbb{C}^2 \otimes V_1 \otimes V_1 \otimes V_1$$

of $G := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. We consider this as a representation of $\mathrm{SL}_2 \times \mathrm{SO}_4$:

$$V = \mathbb{C}^2 \otimes V_1 \otimes \mathbb{C}^4$$

where $\mathbb{C}^4$ is the standard representation of SO_4. As a representation of SO_4 this is the direct sum of 4 copies of the standard representation. Therefore, the SO_4 invariants are generated by $S^2(\mathbb{C}^2 \otimes V_1) \otimes V_0 \subset S^2(V)$ and $\bigwedge^4(\mathbb{C}^2 \otimes V_1) \otimes V_0 \subset S^4(V)$, i.e. we have ten generators in degree 2 and one generator q_4 in degree 4. Moreover, the induced morphism

$$\pi_1 : V \to S^2(\mathbb{C}^2 \otimes V_1)$$

is surjective (and homogeneous of degree 2), and π_1 is the quotient map under O_4. The generator q_4 is invariant under the full group G. The 10-dimensional representation $S^2(\mathbb{C}^2 \otimes V_1)$ decomposes under SL_2 in the form

$$S^2(\mathbb{C}^2 \otimes V_1) = S^2(\mathbb{C}^2) \otimes V_2 \oplus \bigwedge^2 \mathbb{C}^2 \otimes V_0 = \mathbb{C}^3 \otimes V_2 \oplus \mathbb{C} \otimes V_0.$$

Thus there is G-invariant q_2 in degree 2 given by the second summand. We have seen above that the SL_2-invariants of $\mathbb{C}^3 \otimes V_2$ are generated by six invariants in degree 2 and one in degree 3, represented by the subspaces $S^2(\mathbb{C}^3) \otimes V_0 \subset S^2(\mathbb{C}^3 \otimes V_2)$ and $\bigwedge^3 \mathbb{C}^3 \otimes V_0 \subset S^3(\mathbb{C}^3 \otimes V_2)$. This proves the first part of the following theorem. The second part is an immediate consequence of Proposition 6 above.

Theorem 6. *The* $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$*-invariants of* $(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)^{\oplus 2}$ *are generated by one invariant* q_2 *in degree 2, seven invariants* $p_1, \dots, p_6, q_4$ *in degree 4 and one invariant* q_6 *in degree 6. A homogeneous system of parameters for the invariant ring is given by* $q_2, p_1, \dots, p_5, q_6$ *where* $p_1, \dots, p_5$ *span the subspace* $S^2_0(\mathbb{C}^3) \otimes V_0$ *stable under* SO_3 *acting on* $\mathbb{C}^3$.

Remark 5. The generating invariants have the following bi-degrees: $\deg q_2 = (1,1)$, $\deg q_4 = (2,2)$, $\deg q_6 = (3,3)$, and the bi-degrees of the p_i's are $(4,0)$, $(3,1)$, $(2,2)$, $(2,2)$, $(1,3)$, $(0,4)$.

Remark 6. The invariants of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ under $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ are generated by one invariant p of degree 4. It is given by the consecutive quotient maps

$$p : \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^4 \xrightarrow{/\,\mathrm{SO}_4} S^2\mathbb{C}^2 = V_2 \xrightarrow{/\,\mathrm{SL}_2} \mathbb{C}.$$

The nullcone $p^{-1}(0)$ is irreducible of dimension 7 and contains a dense orbit, namely the orbit of $v_0 := e_1 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_1$. (In fact, it easy to see, by HILBERT's criterion, that v_0 is in the nullcone; moreover, the annihilator of v_0 in $\mathrm{Lie}\, G$ has dimension 2, hence Gv_0 is an orbit of dimension 7.) Therefore, all fibres of p are irreducible (of dimension 7) and contain a dense orbit. More precisely, we have the following result. (We use the notation $e_{ijk} := e_i \otimes e_j \otimes e_k$.)

Proposition 7. *The nullcone of* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *contains six orbits, the origin* $\{0\}$*, the orbit* Ge_{111} *of the highest weight vector which is of dimension 4, the dense orbit* $G(e_{110} + e_{101} + e_{011})$ *of dimension 7, and the three orbits of the elements* $e_{100} + e_{010}$*,* $e_{010} + e_{001}$*,* $e_{001} + e_{100}$ *which are of dimension 5 and which are permuted under the symmetric group* $\mathcal{S}_3$ *permuting the three factors in the tensor product.*

Proof. The weight vector e_{ijk} has weight

$$\varepsilon_{ijk} := ((-1)^{i+1}, (-1)^{j+1}, (-1)^{k+1}) \in \mathbb{Z}^3 = X(\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*)$$

and so the set X_V of weights of $V := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ consists of the vertices of a cube in $\mathbb{R}^3$ centered in the origin. There are four maximal unstable subset of X_V in the sense of [KrW06, Definition 1.1], up to the action of the Weyl group, namely the set of vertices of the three faces of the cube containing the highest weight ε_{111}, and the set $\{\varepsilon_{111}, \varepsilon_{011}, \varepsilon_{101}, \varepsilon_{110}\}$. The corresponding maximal unstable subspaces of V are (see [KrW06, Definition 1.2]):

$$\begin{aligned} W_1 &:= \langle e_{111}, e_{110}, e_{101}, e_{100}\rangle \\ W_2 &:= \langle e_{111}, e_{110}, e_{011}, e_{010}\rangle \\ W_3 &:= \langle e_{111}, e_{011}, e_{101}, e_{001}\rangle \\ U &:= \langle e_{111}, e_{011}, e_{101}, e_{110}\rangle. \end{aligned}$$

It follows that the nullcone is given as a union

$$\mathcal{N}_V = GU \cup GW_1 \cup GW_2 \cup GW_3.$$

The subspace U is stabilized by $B \times B \times B$ whereas $W_1 = e_1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is stable under $B \times \mathrm{SL}_2 \times \mathrm{SL}_2$, and similarly for W_2 and W_3. Since the spaces W_i are not stable under G we get $\dim GW_i = \dim W_i + 1 = 5$, and so $GU = \mathcal{N}_V$.

The group $\mathrm{SL}_2 \times \mathrm{SL}_2$ has three orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2$, the dense orbit of $e_1 \otimes e_0 + e_0 \otimes e_1$, the highest weight orbit of $e_1 \otimes e_1$, and $\{0\}$. This shows that $\overline{G(e_{110} + e_{101})} = GW_1$ and that $GW_1 \setminus G(e_{110} + e_{101}) \subset \overline{Ge_{111}}$, and similarly for W_2 and W_3. One also sees that the elements $e_{110} + e_{101}$, $e_{110} + e_{011}$, and $e_{101} + e_{011}$ represent three different orbits of dimension 5, all containing the highest weight orbit in their closure. In fact, $GW_1 = \{ge_1 \otimes v \mid g \in G \text{ and } v \in \mathbb{C}^2 \otimes \mathbb{C}^2\}$, and so $ge_1 \otimes v$ is not in W_2 except if v is a multiple of $ge_1 \otimes ge_1$. In particular, $GW_1 \cap GW_2 \cap GW_3 = \overline{Ge_{111}}$.

Finally, it is easy to see that $(B \otimes B \otimes B)v_0 = \mathbb{C}^* e_{110} \times \mathbb{C}^* e_{101} \times \mathbb{C}^* e_{011} \times \mathbb{C} e_{111}$. Hence, $\overline{Gv_0} = GU = \mathcal{N}_V$ and $\mathcal{N}_V \setminus Gv_0 \subset GW_1 \cup GW_2 \cup GW_3$. □

Proposition 8. *The invariants in degree* 4 *of any number of copies of Q_3 define the nullcone. In particular, for any $N \geq 1$ there is a system of parameters of $Q_3^{\oplus N}$ in degree 4.*

Proof. We can identify $\mathbb{C}^N \otimes Q_3$, as a representation of ${\mathrm{SL}_2}^3$, with $\mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$, as a representation of $\mathrm{SL}_2 \times \mathrm{SO}_4$. The quotient of $\mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ by O_4 is given by

$$\pi \colon \mathbb{C}^N \otimes \mathbb{C}^2 \otimes \mathbb{C}^4 \to S^2(\mathbb{C}^N \otimes \mathbb{C}^2),$$

where the image of π is the closed cone of symmetric matrices of rank ≤ 4 (First Fundamental Theorem for the orthogonal group, see [GoW98, Theorem 4.2.2] or [Pro07, 11.2.1]). This means that the O_4-invariants are generated by the obvious quadratic invariants. Moreover, the morphism π is SL_2-equivariant.

As a representation of SL_2 we have

$$S^2(\mathbb{C}^N \otimes \mathbb{C}^2) = S^2(\mathbb{C}^N) \otimes V_2 \,\oplus\, \textstyle\bigwedge^2 \mathbb{C}^N \otimes \mathbb{C}$$

where V_2 is the 3-dimensional irreducible representation of SL_2 corresponding to the standard representation of SO_3, and $\mathbb{C}$ denotes the trivial representation. Again, consider this as a representation of O_3. Then the O_3-invariants are generated by the quadratic (and the linear) invariants. Summing up we see that the invariant ring

$$(\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{O}_4})^{\mathrm{O}_3}$$

is generated by the elements of degree 2 and 4. By construction,

$$(\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{O}_4})^{\mathrm{O}_3} \subset (\mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{SL}_2 \times \mathrm{SL}_2})^{\mathrm{SL}_2} = \mathcal{O}(\mathbb{C}^n \otimes Q_3)^{\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2}$$

and the latter is a finite module over the former. Therefore, both quotients have the same nullcone and so the nullcone is defined by invariants in degree 2 and 4. □

Remark 7. The representation $Q_3 \oplus Q_3$ has one invariant of degree 2 and eight invariants of degree 4. Since the dimension of the quotient is 7 it follows that there is a system of parameters for the invariant ring consisting of seven invariants of degree 4. A priori it is not clear that there is also a system of parameters consisting of one invariant of degree 2 and six invariants of degree 4 as suggested by the HILBERT series which has the form

$$\operatorname{Hilb}_{Q_3 \oplus Q_3} = \frac{1 + t^4 + t^6 + t^{10}}{(1-t^2)(1-t^4)^6}.$$

However, the analysis above shows that in case of 2 copies of Q_3 we obtain the following composition of quotient maps

$$\pi \colon Q_3 \oplus Q_3 \xrightarrow{\pi_1} S^2\mathbb{C}^2 \otimes V_2 \oplus \mathbb{C} \xrightarrow{\pi_2} S^2S^2\mathbb{C}^2 \oplus \mathbb{C}$$

where π_1 is the quotient by O_4 and π_2 the quotient by O_3. Since both morphisms π_1 and π_2 are surjective in this case it follows that the zero fiber $\mathcal{N}$ of π is defined by the quadratic invariant and six invariants of degree 4. As we remarked above the (reduced) zero fiber of π is the nullcone of $Q_3 \oplus Q_3$ with respect to $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, hence these seven invariants form a homogeneous system of parameters for the ring of invariants.

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